Dehn Filling and Asymptotically Hyperbolic Manifolds

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Abstract of the Dissertation

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In this thesis, we extend Anderson's higher-dimensional Dehn filling construction to a large class of infinite-volume hyperbolic manifolds. This gives an infinite family of topologically distinct asymptotically hyperbolic Einstein manifolds with the same conformal infinity. The construction involves finding a sequence of approximate solutions to the Einstein equations and then perturbing them to exact ones.

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Chapter 1

Introduction

The main goal of this thesis is to prove that it is possible to obtain infinitely many topologically distinct asymptotically hyperbolic Einstein(AHE) metrics by closing cusps on certain infinite-volume hyperbolic n-manifolds($n \geq 3$). We will start by defining AHE metrics and discussing some of their properties and structure, and then we will explain how our result fits into this picture. Following this we will provide a brief outline of the proof of our main result.

Definition 1. *Let $\bar{M}$ be a compact manifold with boundary. A smooth function ρ is said to be a defining function for ∂M iff*

$$\rho : \bar{M} \longrightarrow [0, \infty) \tag{1.1}$$

satisfies $\rho(p) = 0$ iff $p \in \partial M$ and $d\rho \neq 0$ on ∂M.

Then we have:

Definition 2. *A complete metric g on $M = int(\bar{M})$ is said to be conformally compact iff there exists a defining function ρ for ∂M such that $\bar{g} = \rho^2 g$ extends to a metric on $\bar{M}$.*

The canonical example of a conformally compact manifold is the Poincaré model of hyperbolic space. In this case, $\bar{g}$ is the flat metric on the ball, and $\rho(x) = \frac{1-|x|^2}{2}$. In what follows, quantities with a bar over them will be measured with respect to the compactified metric $\bar{g}$, and ones without bars will be measured with respect to the metric g on $M = int\bar{M}$.

Definition 3. *Consider a conformally compact metric $\bar{g}$ on $\bar{M}$. If there exists a defining function ρ such that $|\bar{\nabla}\rho| \equiv 1$ on ∂M, then we say that (M, g) is asymptotically hyperbolic.*

The reason for this terminology is that in this case the sectional curvatures of (M, g) tend uniformly to -1. (cf. [And2])

Asymptotically hyperbolic(AH) manifolds are a natural class of non-compact manifolds to work with because they have a nice structure at infinity; their curvature tends toward a constant, and via the compactification $\bar{g}$ they have a "boundary metric" at infinity. Since this metric is determined by the choice of the function ρ, it actually only makes sense to speak of a boundary conformal class. This conformal class is known as the conformal infinity of the complete manifold (M, g).

Given an AH manifold, it is natural to want to put a canonical AH metric on it. In two or three dimensions, the natural choice is a hyperbolic metric. In higher dimensions, however, hyperbolic metrics generalize in two ways: to hyperbolic metrics and to negatively curved Einstein metrics(constant negative Ricci curvature.) If $n > 3$, the curvature tensor has more components than the metric, so prescribing sectional curvature becomes more difficult. On the other hand, the Ricci tensor has the same rank as the metric, so Einstein metrics are natural candidates to be canonical metrics. Accordingly, we define:

Definition 4. *An asymptotically hyperbolic manifold with constant Ricci curvature is called an asymptotically hyperbolic Einstein(AHE) manifold.*

There has been a great deal of interest in AHÉ metrics recently due to applications to physics. Physicists are particularly interested in the correspondence between AHE metrics and their conformal infinities(c.f. [Bek], [Wit], [And5]). This correspondence can be thought of as a geometric Dirichlet problem(although a priori the topology of the filling manifold could be undetermined.)

Anderson has worked extensively on this correspondence in dimension 4([And2], [And3], [And6], [And5]). In this case, there is generally not a bijective correspondence between AHE metrics and their conformal infinities, but under certain geometric conditions on the boundary, there are only finitely many AHE manifolds bounded by the given conformal class. This is not always the case, however; in [And2], Anderson constructs infinitely many AHE metrics bounded by a fixed conformal class. He then shows that in dimension 4, any such collection has a limit point which is an AHE manifold with cusps(i.e. an Einstein metric whose ends are either conformally compact or of finite volume.) Furthermore, this limit point has the same conformal infinity as all the elements in the set.

Under some fairly natural conditions on these limit manifolds, it is possible to show(still in 4 dimensions) that these limit manifolds are actually hyperbolic. This suggests a natural question: given a hyperbolic manifold whose ends are either conformally compact or cusps, is there a sequence of AHE metrics with the same same conformal infinity converging toward it? This is indeed the case in three dimensions(where AHE metrics are hyperbolic,) although the methods used in this case come from hyperbolic geometry, and cannot be applied to Einstein manifolds. Recently, Anderson developed a cusp closing technique for finite-volume hyperbolic manifolds([And1]), generalizing Thurston's Dehn Filling result to higher dimensions. His construction leads to infinite families of topologically distinct compact Einstein manifolds.

Our main result applies Anderson's cusp closing construction to generate a host of AHE metrics with the same conformal infinity:

Theorem 5. *Let (N^n, g), $(n > 2)$ be a complete geometrically finite hyperbolic manifold, all of whose cusps have toric cross sections. Then it is possible to close the cusps to obtain infinitely many metrically distinct AHE manifolds, all of which have the same conformal infinity as the original one. If the hyperbolic manifold N^n is nonelementary, then this procedure gives infinitely many homotopy types. If $n > 3$ these AHE metrics are non-hyperbolic.*

One can generate a large class of manifolds satisfying the hypotheses of the theorem by taking Maskit combination of complete hyperbolic cusps along hyperplanes. (c.f. [Kvt].)

The proof of the main result follows Anderson's proof in [And1]: we construct a sequence of approximate solutions to the Einstein equations(i.e. metrics whose Ricci curvature is tending toward some constant) and then perturb the metrics to an exact solution. This basic gluing procedure is quite common in geometric analysis, c.f. for example [Tau], [Kap], [MPU], [MPa].

The choice of approximate solutions is the first stumbling block in this procedure. Although in theory it is easy to prescribe Ricci curvature, since the Ricci tensor is of the same rank as the metric, in practice it is extremely difficult, since we must find explicit solutions to a coupled system of nonlinear PDEs. The construction of the approximate solutions requires the use of a very special (explicit) family of metrics, as we shall see below.

The perturbation argument which gives the Einstein metrics has two parts. We will be using a functional Φ, and metrics which satisfy $\Phi(g) = 0$ will be Einstein. Then we will have a sequence of approximate solutions g_n such that $\Phi(g_n) \longrightarrow 0$. We can represent this by the the graph in Figure (1.1).

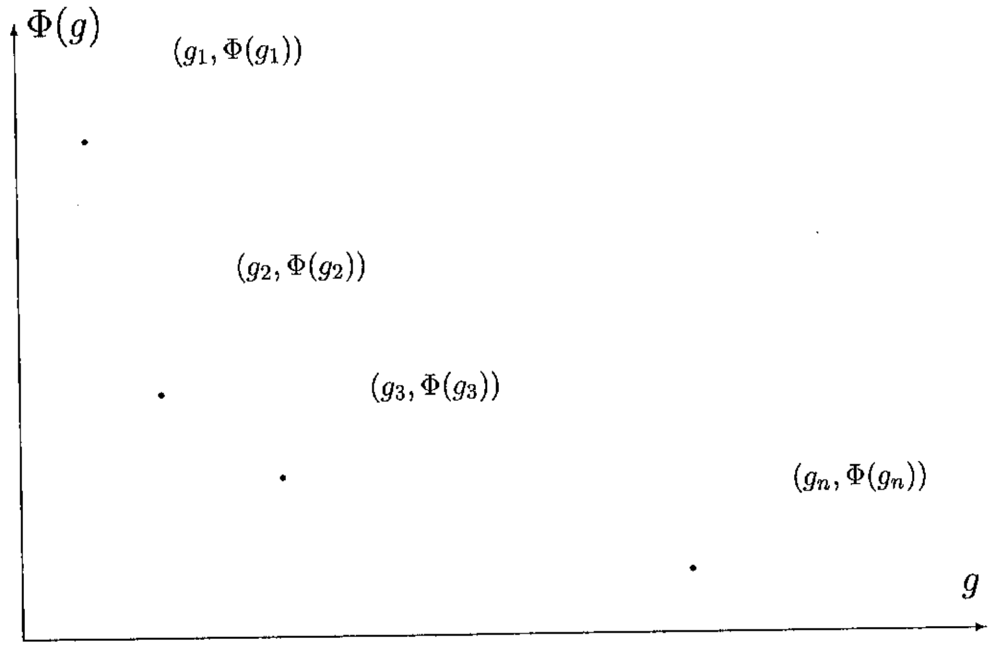

Figure 1.1: A sequence of approximate solutions

It turns out that sequences of approximate solutions degenerate, so we cannot use a limiting argument to obtain our exact solution. On the other hand, the linearization of Φ at each metric g_n is invertible. Thus, we could hope to use the inverse function theorem to invert Φ in a neighborhood of $\Phi(g_n)$. Since $\Phi(g_n) \longrightarrow 0$, we can hope that for n large enough, one of these neighborhoods will contain 0, which will give us a metric g such that $\Phi^{-1}(0) = g$. Invertibility of Φ near $\Phi(g_n)$ is not enough to insure this however. We could have a situation in which the region on which Φ is invertible shrinks as $n \longrightarrow \infty$, so 0 never lies in this region. Such a case is represented in Figure (1.2), where the disks represent the maximal region on which it is possible to invert Φ.

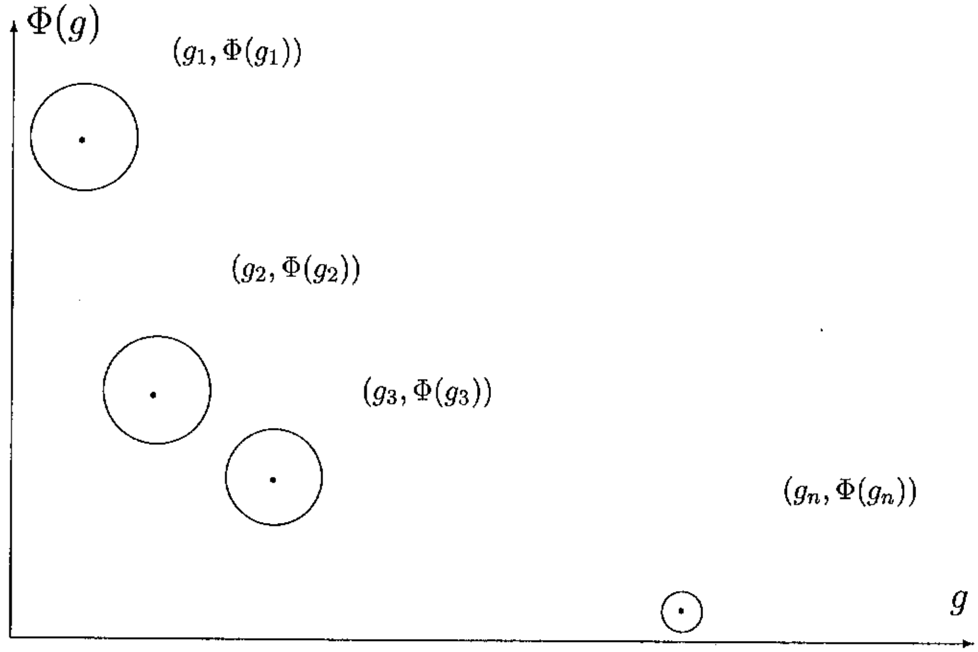

Figure 1.2: Φ invertible, but not on balls of uniform size.

Thus, we need to get a control on the size of the balls on which we can invert. In this case, the picture will look like Figure (1.3), and so for n large enough we can perturb g_n to a metric satisfying $\Phi(g) = 0$. We obtain this uniform control by bounding the linearization of Φ, as Anderson does in

[And1].

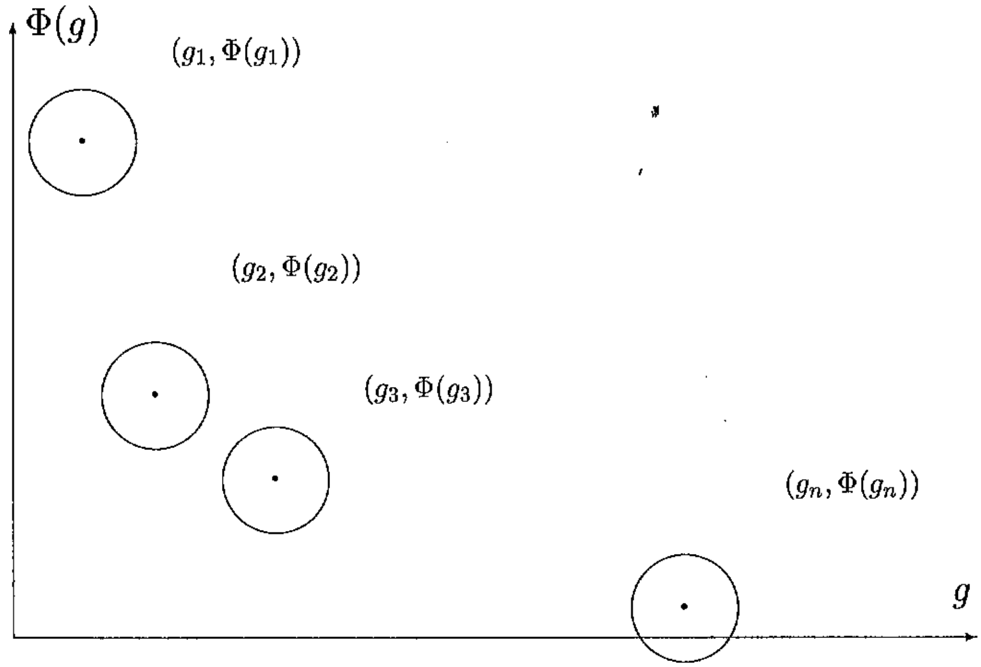

Figure 1.3: Φ invertible on balls of uniform size.

These figures are somewhat misleading since, as we shall see, the topology of the approximate solutions M_i will actually be varying. Furthermore, the balls on which we will be inverting will actually be shrinking, and we will only have a lower bound on their radius. Nonetheless, the idea is the same as in these figures. Note also that all we need for this argument is uniform surjectivity, rather than uniform invertibility, but as we will see, we get the invertibility for free, which makes things easier.

The main difference between our construction and Anderson's is that our approximate solutions are noncompact, due to the presence of the expanding ends. This introduces some small difficulties in our analysis, but by construction we have very strong control over the expanding ends, since our metrics have fixed conformal infinities. As it turns out, the bulk of the argument is identical to Anderson's.

Let us now set some of our notation and conventions. From here on, all manifolds will be assumed to be complete and AH, unless otherwise stated. Pointwise norms and inner products will be denoted by $|h|$ and (f, h) respectively, while global ones will be denoted by $||h||$ and $\langle f, h \rangle$. K, ric, z and s will represent the sectional, Ricci, trace-free Ricci and scalar curvatures. $inj(M)$ will denote the injectivity radius of M. n will be reserved for the dimension the manifold M, and will always be strictly greater than 2. The curvature operator is defined as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \tag{1.2}$$

for any three X, Y, $Z \in TM$. Our Laplacians will have negative spectrum, so $\Delta_{S^1} = -\frac{d^2}{d\theta^2}$. This is the so-called "Geometer's Laplacian." We will often drop subscripts to improve readability if this will not lead to any confusion.

This thesis will be organized as follows: chapter 2 will cover background material on the operator and function spaces we will be using. In chapter 3 we construct our approximate solutions and discuss some of their properties, and then in chapter 4 we obtain a uniform control over the operator $D\Phi$ on all the approximate solutions. Finally, in chapter 5, we wrap things up by perturbing our approximate solutions to exact ones and provide examples of manifolds on which one can perform this cusp-closing construction.

Chapter 2

Background

In this chapter, we will discuss some of the analytical tools which will be needed later. We will start by discussing the operator which we will be using. For now, we will assume that our operators are defined on the space of smooth(C^∞) symmetric positive-definite bilinear forms on M, i.e. the space of all metrics on M, which we will denote by $S_0^2(M)$. We will denote the space of smooth symmetric bilinear forms on M by $S^2(M)$. This is isomorphic to the tangent space to $S_0^2(M)$ at some fixed metric g in the sense that infinitesimal deformations of g can represented by elements of $S^2(M)$.

Since we are looking for Einstein metrics, the most natural choice for the operator Φ which we are studying would seem to be the trace-free Ricci curvature

$$z = ric - \frac{s}{n}g. \tag{2.1}$$

We would then try to solve the equation

$$z = 0 \tag{2.2}$$

by inverting z near 0. The operator z is natural because it is the negative of the gradient of the total scalar curvature functional restricted to metrics with a fixed volume form, (c.f [Bes],) and is also invariant under rescalings and changes of coordinates(i.e. under the action of the group of diffeomorphisms.) For our purposes, however, these last two features are actually drawbacks; they ensure that z maps large subspaces onto points. Thus, z is far from being invertible. Furthermore, the kernel which arises from rescalings and diffeomorphisms is not particularly interesting to us, since we are looking for Einstein metrics; once we find one of them, we already know that pullbacks and rescalings of it will also be Einstein.

We can remove the scale invariance by considering the equation

$$ric_g + (n-1)g = 0 \tag{2.3}$$

Note that if g satisfies this equation, then $s_g = -n(n-1)$, just like hyperbolic n-space. One should also note that this equation is simpler than the previous one in that we have replaced $\frac{s}{n}$, which is nonlinear in the partial derivatives of g, by the constant $(n-1)$.

We still have to remove the diffeomorphism invariance of this equation. We do this by taking a slice complementary to the action of the diffeomorphism group. Recall(c.f. [Bes]) that infinitesimally, $T_{g_0}S_0^2(M) \cong S^2(M)$ splits orthogonally with respect to the L^2 product

$$T_{g_0}S_0^2(M) = \text{Im}(\delta^*) \bigoplus \text{Ker}(\delta), \tag{2.4}$$

where the first term is tangent to the action of the group of diffeomorphisms. Here δ^* is the symmetrized covariant derivative, and the divergence δ is its formal adjoint. Thus, it would be natural to fix some reference metric and then work with the equations

$$ric_g + (n-1)g = 0 \tag{2.5}$$

$$\delta_{g_0} g = 0 \tag{2.6}$$

where g is near some reference metric g_0.

For reasons which will become clear below, we will actually choose to work with the system

$$ric_g + (n-1)g = 0 \tag{2.7}$$

$$B_{g_0} g = 0, \tag{2.8}$$

where B_{g_0} is the Bianchi operator associated to the metric g_0:

$$B_{g_0} : S^2(M) \longrightarrow \Omega^1(M) \tag{2.9}$$

$$B_{g_0}(h) = \delta_{g_0} h + \frac{1}{2} dtr_{g_0} h \tag{2.10}$$

(note that the differential Bianchi identity implies that $B_g(ric_g) = 0$ for any metric g.)

The reason that we use this system is that we can combine the operator

$ric + (n-1)g$ and the Bianchi operator into a single map

$$\Phi_{g_0} : S_0^2(M) \longrightarrow S^2(M) \tag{2.11}$$

$$g \longmapsto ric_g + (n-1)g + \delta_g^* B_{g_0}(g) \tag{2.12}$$

Now the linearization of Φ at g_0 is

$$D_{g_0}\Phi(h) = \frac{1}{2}\Delta_L h + (n-1)h, \tag{2.13}$$

(c.f. [Biq1],) where is the Lichnerowicz Laplacian Δ_L is defined as

$$\Delta_L : S^2(M) \longrightarrow S^2(M) \tag{2.14}$$

$$\Delta_L h = D^* D h + ric \circ h + h \circ ric - 2R(h), \tag{2.15}$$

and R is the action of the curvature tensor on $S^2(M)$:

$$Rh(X,Y) = tr(((W,Z) \mapsto h(R(X,W)Y,Z)) \tag{2.16}$$

$$= \sum_{i=1}^{n} h(R(X,e_i)Y,e_i) \tag{2.17}$$

where $\{e_i\}$ is an orthonormal basis with respect to the metric at which we are linearizing. Composition of symmetric bilinear forms is defined by using the metric to identify them with elements of $\mathrm{Hom}(TM,TM)$.

$D_{g_0}\Phi$ is clearly elliptic, and so Φ is elliptic near g_0. This means that locally its linearization has a finite dimensional kernel, so we have effectively eliminated the kernel arising from the action of the diffeomorphism group.

Of course, this is all a waste of time unless there is some connection between the operator Φ_{g_0} and Einstein metrics, something which is not *a priori* obvious. The following lemma deals with this issue. Let $p \in M$ be some fixed point, and let $r(x) = \text{dist}(p, x)$.

Lemma 6. *([Biq1]) Let (M, g) be AH, with $ric_g < 0$. If $\Phi_{g_0}(g) = 0$ and $\lim_{r\to 0} |B_{g_0}(g)| = 0$, then $ric_g = -(n-1)g$. In other words, g is Einstein with scalar curvature $-n(n-1)$.*

Proof: By construction of Φ, we only need to show that if $\Phi_{g_0}(g) = 0$ and $\lim_{r\to 0} |B_{g_0}(g)| = 0$, then $B_{g_0} g = 0$.

By the differential Bianchi identity we have that

$$B_g(ric_g) = 0 \tag{2.18}$$

so applying B_g to $\Phi(g) = 0$ gives

$$B_g \delta_g^* B_{g_0}(g) = 0 \tag{2.19}$$

Now set $\gamma = B_{g_0}(g)$. Using the Weitzenböck formula (cf. [Biq2])

$$B_g \delta_g^* = \frac{1}{2}\left(D_g^* D_g - ric_g\right), \tag{2.20}$$

we get

$$(D_g^* D_g - ric_g)\gamma = 0 \tag{2.21}$$

Taking the pairing of this with γ and taking the trace will give

$$\Delta_g |\gamma|^2 \leq 0 \tag{2.22}$$

Since $|\gamma| \longrightarrow 0$ as $r \longrightarrow \infty$, we can invoke the maximum principle to conclude that

$$\gamma = B_{g_0}(g) \equiv 0 \tag{2.23}$$

and so $\Phi(g) = ric_g + (n-1)g = 0$. ∎

We must now discuss the function spaces on which we will do our analysis. We cannot use the spaces $S^2(M)$, because we will need elliptic estimates and various compactness properties. Thus, we can use either Sobolev spaces or Hölder spaces. We can define the latter locally, which makes it easier to control them over a large class of manifolds which have similar local geometry.

Recall that on a bounded open domain $\Omega \subset \mathbb{R}^n$, one defines the (k,α)-Hölder norm of a smooth function f to be

$$\|f\|_{k,\alpha} = \sum_{|\beta| \leq k} \sup_{p \in \Omega} |\partial^\beta f| + \max_{|\beta|=k} \left(\sup_{p,q \in \Omega} \frac{|\partial^\beta f(p) - \partial^\beta f(q)|}{|p-q|^\alpha} \right) \tag{2.24}$$

The space $C^{k,\alpha}(\Omega)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the (k,α)-Hölder norm. These spaces have the desired analytical properties. In particular, if $\alpha' < \alpha$, then the inclusion of $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha'}(\Omega)$ is compact, and given an elliptic operator L, we can obtain Schauder estimates for it, which give us strong control over L.

We will need similar properties for the analysis below, so we will have to define Hölder spaces of bilinear forms on a Riemannian manifold. We will do this in local coordinates, so that the corresponding properties of Hölder spaces on a Euclidean domain will follow immediately. Our definition will not be intrinsic, but we have the following result which allows us to compare these

Hölder spaces on large families of manifolds:

Proposition 7. *([And4]) Let $Q > 1$, $C, i_0 > 0$, $k \in \mathbb{N}$, $0 < \alpha < 1$. Then there exists $\rho_0 > 0$ such that if $\|\nabla^{k-1} ric\|_{L^\infty} \leq C$ and $i(M) \geq i_0$, then for any $x \in M$, the ball $B(x, \rho_0)$ has harmonic coordinates in which we have*

$$Q^{-1} I \leq g \leq QI \tag{2.25}$$

$$\sum_{1\leq|\beta|\leq k} \rho_0^{|\beta|} \sup_{y\in B(x,\rho_0)} |\partial^\beta g_{ij}(y)| + \tag{2.26}$$

$$\sum_{|\beta|=k} \rho_0^{k+\alpha} \sup_{y_1,y_2\in B(x,\rho_0)} \frac{|\partial^\beta g_{ij}(y_1) - \partial^\beta g_{ij}(y_2)|}{|y_1 - y_2|^\alpha} \leq Q - 1 \tag{2.27}$$

Here, we will fix Q sufficiently close to 1 for the rest of this paper. We shall call the above coordinates $C^{k,\alpha}$-harmonic coordinates. Now, given $C > 0, i_0 > 0$, we can define (k, α)-Hölder norms on the class of manifolds with $\|\nabla^{k-1} ric\|_{L^\infty} < C$ and $i(M) > i_0$. Start by choosing a locally finite collection of balls $B(x_i, \rho_0)$ with (k, α)-harmonic coordinates which cover M such that the balls $B\left(x_i, \frac{\rho_0}{4}\right)$ are disjoint. (This is possible because the Ricci curvature is bounded.) Then define the (k, α)-Hölder norm of $h \in S^2(M)$

$$\|h\|_{k,\alpha} = \sup_{x_i} \left\{ \sum_{1\leq|\beta|\leq k} \rho_0^{|\beta|} \sup_{y\in B(x_i,\rho_0)} |\partial^\beta h_{ij}(y)| \right. \tag{2.28}$$

$$\left. + \sum_{|\beta|=k} \rho_0^{k+|\alpha|} \sup_{y_1,y_2\in B(x_i,\rho_0)} \frac{|\partial^\beta h_{ij}(y_1) - \partial^\beta h_{ij}(y_2)|}{|y_1 - y_2|^\alpha} \right\} \tag{2.29}$$

where the supremum is taken over all the balls $B(x_i, \rho_0)$. We can then define the $C^{k,\alpha}$ topology on the space of metrics near g_0 by setting the norm of a metric g near g_0 to be

$$\|g - g_0\|_{k,\alpha}. \tag{2.30}$$

From here on out i_0, C, $k > 2$ and α will all be fixed. The reader may be concerned that we are only defining our operator Φ and our function spaces near some base metric g_0. This is not an issue, since we are using a perturbation argument, and thus will only be working in a neighborhood of the metric we wish to perturb.

Now, it is clear that on open bounded sets $\Omega \subset M$, we will have that the inclusion of $C^{k,\alpha}(\Omega)$ into $C^{k,\alpha'}(\Omega)$ is compact for $\alpha' < \alpha$. Furthermore, by our control of the metric in these coordinates we will have interior Schauder estimates for the elliptic operator $L = D_{g_0}\Phi$ on bounded sets $\Omega \subset M$. (c.f. [GiT].)

Our analysis will take place on manifolds approximating the hyperbolic manifold whose cusps we will be closing. Thus we need to take into account two types of noncompact behavior: moving down the cusps makes the injectivity radius arbitrarily small, and moving out into the expanding end makes the volume tend toward infinity. We will deal with these issues separately. In both cases, we will need infinitely many coordinate charts, so to be able to get uniform control over the entire manifold, we shall need to have coordinate charts which are "uniformly similar" in some sense; i.e. they must be defined on balls of approximately the same size and local geometry.

The problem that we will encounter from the injectivity radius tending toward zero is that our coordinate charts will have to be made arbitrarily small as we move down the cusp. On the other hand, since the geometry is hyperbolic(or, as we shall see, very close to it,) we can lift to a large enough cover, and then calculate the norm on this cover.

Thus, we take Anderson's definition from [And1]:

Definition 8. *We say that a manifold has uniformly bounded local covering geometry if, given some fixed constant $i_0 > 0$, any ball $B(x, i_0)$ has a finite cover $\bar{B}(\bar{x}, i_0)$ with diameter less than 1 and $i(\bar{x}) \geq i_0$.*

Then, define the modified $C^{k,\alpha}(M)$ norm $\tilde{C}^{k,\alpha}(M)$ to be the $C^{k,\alpha}(M)$ norm, with the norm being evaluated in (k, α)-harmonic coordinates on a large enough cover if the injectivity radius is less than i_0.

Now, let us define the main space which we will be working with:

Definition 9. *Let $S^{k,\alpha}$ be the completion of S^2 with respect to the $\tilde{C}^{k,\alpha}(M)$ norm.*

Even though the space $S^{k,\alpha}$ is well-defined for our noncompact manifolds, it is too large for our purposes, since it includes many forms over whose asymptotic behavior we have very little control. Furthermore, we do not want to change the conformal infinity of our approximate solution when we perturb it to an exact solution, so we want our perturbation to vanish at infinity. Both of these considerations lead us to the following definition:

Definition 10. *Let ρ be a geodesic defining function, and let $r(x) = \log\left(\frac{2}{\rho}\right)$. For $\delta > 0$, we define the δ-weighted Hölder space $S_\delta^{k,\alpha}(M)$ to be $\{u = e^{-\delta r} u_0 | u_0 \in S^{k,\alpha}(M)\}$. If $u \in S_\delta^{k,\alpha}(M)$, define $\|u\|_{k,\alpha,\delta} = \|u_0\|_{k,\alpha} = \|e^{\delta r} u\|_{k,\alpha}$.*

Note that although this norm depends on our choice of ρ, the space $S_\delta^{k,\alpha}$ does not. ρ is a geodesic defining function iff $|\bar{\nabla}\rho| \equiv 1$ in a neighborhood of ∂M. Such functions have the property that if $r = \log\left(\frac{2}{\rho}\right)$, then

$$|\nabla r| = |\bar{\nabla}\rho| \equiv 1 \tag{2.31}$$

in some neighborhood of ∂M. Thus r is a distance function outside some compact set. It always possible to construct such defining functions for AH metrics(c.f. [GL].)

Before we state some analytic properties of these spaces, we will need a technical lemma:

Lemma 11. *There is some constant $C_{\beta,\delta} > 0$ such that*

$$|\partial^\beta(e^{\pm\delta r} f)| \leq C_{\beta,\delta} e^{\pm\delta r}\left(\sum_{|\gamma|\leq|\beta|} |\partial^\gamma f|\right) \tag{2.32}$$

Proof: First, note that we are not taking any inner products, since $\partial^\beta(e^{\pm\delta r} f)$ is not a tensor, but rather an expression calculated in local coordinates. The lemma will follow from the product rule provided we can bound

$$|\partial_i e^{\pm\delta r}| \leq \left|\delta e^{\pm\delta r}\frac{\partial r}{\partial x_i}\right| \tag{2.33}$$

But

$$\left|\frac{\partial r}{\partial x_i}\right| \leq \sqrt{\sum_j \left(\frac{\partial r}{\partial x_j}\right)^2} \leq \sqrt{Q|\nabla r|} = \sqrt{Q}, \tag{2.34}$$

where Q is the fixed constant we used to define our harmonic coordinates. ∎

This lemma allows us to interchange differentiation and multiplication by our scaling factor. Let us now adapt various standard results on Hölder spaces into our present context. The following results are taken from [Lee].

Theorem 12. *([Lee]) Let $\delta' < \delta$ and $\alpha' < \alpha$. Then the inclusion $S^{k,\alpha}_{\delta}(M) \to S^{k,\alpha'}_{\delta'}(M)$ is compact.*

Proof: Say that we have a set $\{h_\gamma\}$ in $S^{k,\alpha'}_{\delta'}(M)$ such that

$$||h_\gamma||_{k,\alpha,\delta} \leq C \tag{2.35}$$

Then we wish to find an $h \in S^{k,\alpha'}_{\delta'}(M)$ such that we can take a subsequence (h_i) with

$$h_i \longrightarrow h \quad \text{in } S^{k,\alpha'}_{\delta'} \tag{2.36}$$

Now note that by definitions of our norms,

$$||h_i||_{k,\alpha,\delta} = ||e^{\delta r} h_i||_{k,\alpha} \tag{2.37}$$

By [GiT], given any bounded set $\Omega_0 \subset M$, the inclusion $S^{k,\alpha}(\Omega_0) \longrightarrow S^{k,\alpha'}(\Omega_0)$ is compact. Thus, by a diagonal argument, we can construct an h such that

$$e^{\delta r} h_i \longrightarrow e^{\delta r} h \quad \text{in } S^{k,\alpha'}(\Omega) \tag{2.38}$$

for all bounded sets Ω. The problem is that we cannot necessarily conclude uniform convergence on the whole manifold M. To do do this, we will use the decay properties of these forms. Now, note that there is some $A > 0$ such that $||h_i - h||_{k,\alpha',\delta} \leq A$, since $S^{k,\alpha}_{\delta}(M) \subset S^{k,\alpha'}_{\delta}(M)$. Say we are given $\epsilon > 0$. Let

$\Omega_N = \{x \in M | r(x) < N\}$, and let Γ_N be an open set containing $M - \Omega_N$. Then choose $N(\epsilon)$ large enough that

$$\|e^{-(\delta-\delta')r}\|_{S^{k,\alpha'}(\Gamma_N)} \leq \frac{\epsilon}{2A} \tag{2.39}$$

Then if $i > N(\epsilon)$, we have

$$\begin{aligned}
\|h - h_i\|_{k,\alpha',\delta'} &= \|e^{\delta' r}(h - h_i)\|_{k,\alpha'} &(2.40)\\
&= \|e^{-(\delta-\delta')r}e^{\delta r}(h - h_i)\|_{k,\alpha'} &(2.41)\\
&\leq \|e^{-(\delta-\delta')r}e^{\delta r}(h - h_i)\|_{S^{k,\alpha'}(\Omega_N)} &(2.42)\\
&\quad + \|e^{-(\delta-\delta')r}e^{\delta r}(h - h_i)\|_{S^{k,\alpha'}(\Gamma_N)} &(2.43)\\
&\leq \|e^{-(\delta-\delta')r}\|_{S^{k,\alpha'}(\Omega_N)}\|e^{\delta r}(h - h_i)\|_{S^{k,\alpha'}(\Omega_N)} &(2.44)\\
&\quad + \|e^{-(\delta-\delta')r}\|_{S^{k,\alpha'}(\Gamma_N)}\|(h - h_i)\|_{S^{k,\alpha'}_{\delta}(\Gamma_N)} &(2.45)\\
&\leq C_1\|e^{\delta r}(h - h_i)\|_{S^{k,\alpha'}(\Omega_N)} + \frac{\epsilon}{2} &(2.46)
\end{aligned}$$

where C_1 depends only on N. And we are done, since $e^{\delta r}h_i$ converges to $e^{\delta r}h$ in $S^{k,\alpha'}(\Omega_N)$. ∎

We will be using Bochner technique arguments, so we shall need to use forms which are square-integrable. The following lemma describes under which conditions this occurs.

Lemma 13. *([Lee]) Let (M, g) be asymptotically hyperbolic. If $\delta > \frac{n-1}{2}$ then $S^{k,\alpha}_{\delta}(M) \subset L^2(S^2(M))$.*

Proof: Let $u \in S^{k,\alpha}_{\delta}$. Then by definition, there exists $u_0 \in S^{k,\alpha}(M)$ such that

$$u = u_0 e^{-\delta r} \tag{2.47}$$

In particular, u_0 is bounded.

To show that $\|u\|_{L^2}$ converges, all we need to do is to check that

$$\int_{r_0}^{\infty} \int_{r^{-1}(p)} |u(p)|^2 dV < \infty \tag{2.48}$$

for some $r_0 > 0$.

Since r is a distance function outside a compact set, we can deduce from the Gauss lemma that $dV = e^{(n-1)r} d\bar{V}$ outside this compact set, where $d\bar{V}$ is the volume form for the metric $\bar{g}$ on the compact manifold $\bar{M}$. Then we will be done if we can bound the integral

$$\int_{r_0}^{\infty} \int_{r^{-1}(p)} |u_0(p)|^2 e^{-2\delta r} e^{(n-1)r} d\bar{V} \leq C(\bar{g}) \sup |u_0|^2 \int_{r_0}^{\infty} e^{(n-1-2\delta)r} dr \tag{2.49}$$

This last integral is finite as long as $\delta > \frac{n-1}{2}$. ∎

Let us now discuss some mapping properties of the operator

$$\Phi : S_\delta^{k,\alpha}(M) \longrightarrow S_\delta^{k-2,\alpha}(M) \tag{2.50}$$

Proposition 14. *([Lee]) Φ is well-defined as an operator from $S_\delta^{k,\alpha}$ into $S_\delta^{k-2,\alpha}$.*

Proof: The only thing we need to check is that the exponential decay property, since it is clear that Φ maps forms in $S^{k,\alpha}(M)$ into $S^{k-2,\alpha}(M)$. But by Lemma 11, we have

$$\begin{aligned}\|\Phi(h)\|_{k,\alpha,\delta} &= \|e^{\delta r}\Phi(h)\|_{k,\alpha} &(2.51)\\ &\leq C_0\|\Phi(e^{\delta r}h)\|_{k,\alpha} &(2.52)\\ &\leq C_1\|e^{\delta r}h\|_{k,\alpha} &(2.53)\\ &\leq C_1\|h\|_{k,\alpha,\delta} &(2.54)\end{aligned}$$

∎

We will also need elliptic estimates for these operators. In analogy to the weighted Hölder norms, define the δ-weighted L^∞ norm to be $\|h\|_{L^\infty_\delta} = \|e^{\delta r}h\|_{L^\infty}$.

Proposition 15. *([Lee]) Let $L_g = D_g\Phi_{g_0}$ be the linearization of Φ_{g_0} at g, where $\|g - g_0\|_{k,\alpha,\delta} < \epsilon_0$, and $\epsilon_0 > 0$ is chosen such that L_g is elliptic. Then there is some constant Λ_0, depending on k, α, ϵ_0 and δ such that we have the following estimate*

$$\|h\|_{k,\alpha,\delta} \leq \Lambda_0(\|Lh\|_{k-2,\alpha,\delta} + \|h\|_{L^\infty_\delta}) \tag{2.55}$$

Proof: By definition of L, we have uniform $(k-2,\alpha)$-control over its coefficients via our (k,α)-control over the metric g. Then, by definition of our Hölder spaces, we get the estimates

$$\|h\|_{k,\alpha} \leq \Lambda_0(\|Lh\|_{k-2,\alpha} + \|h\|_{L^\infty}) \tag{2.56}$$

from the Schauder interior estimates on the balls $B(x_i, \rho_0)$. (c.f. [GiT])

Using lemma 11 as in the previous proposition allows us the conclude the same estimate for the δ-weighted spaces. ∎

We end this section by quoting a key result of Biquard's on the behavior of the operator $L = D_{g_0}\Phi_{g_0}$. We know that L is elliptic, but since we are working on noncompact manifolds M, it does not follow that L is Fredholm. Thus if we do not choose our function spaces carefully, $\ker(L)$ or $\operatorname{coker}(L)$ could well be infinite-dimensional. Biquard's results give appropriate conditions to guarantee that L is Fredholm. (Also see [Lee].)

Proposition 16. *([Biq1]) Let (M^n, g_0) be an asymptotically hyperbolic manifold. If $\delta \in (0, n-1)$, then*

$$L : S^{k,\alpha}_{\delta}(M^n) \longrightarrow S^{k-2,\alpha}_{\delta}(M^n) \tag{2.57}$$

is Fredholm. Furthermore, L is an isomorphism iff $\ker_{L^2}(L) = 0$.

Chapter 3

Construction of Approximate Solutions

In this chapter, we construct our approximate solutions, and discuss some of their topological properties. Topologically, filling in cusps just amounts to attaching a solid torus to each cusp end. Metrically, we must truncate the cusp at some finite distance before attaching the solid torus. This is not too hard, assuming that we do not require anything of the filling manifolds. But we want our filled manifolds(our approximate solutions) to have Ricci curvature close to a constant. This turns out to be much more difficult.

We will be filling each cusp separately, so we will only need to explain the procedure on one of them. All of our cusps look like

$$g_C = \rho^{-2}d\rho^2 + \rho^2 g_{T^{n-1}}; \qquad \rho_0 > \rho > 0 \tag{3.1}$$

Note that as $\rho \to 0$, the T^{n-1}'s are collapsing. Without loss of generality, we can assume that $\rho_0 \geq 1$ by rescaling the ρ parameter. This will give us a metric of the same form, but with a rescaled T^{n-1}. Let us cut off the cusp at the torus $\rho = 1$. Then we are faced with the task of attaching something to boundary torus T_0 in such a way that the metric on the glued manifold is smooth. Note that T_0's metric is the flat metric $g_{T^{n-1}}$.

For this construction to work, we will need to use a sequence of filling manifolds which are hyperbolic near their boundary, and whose trace-free Ricci curvature tends toward zero. We will use members of a family of AHE metrics on $D^2 \times T^{n-2}$. We can obtain our filling manifolds by truncating these at some fixed distance, and then perturbing the metric near the boundary to make it hyperbolic. The perturbations will get smaller as we go further and further out, since the manifold is AH. We will start by discussing these filling manifolds.

Consider the following metric on $D^2 \times T^{n-2}$;

$$g_{BH} = (V(r))^{-1}dr^2 + V(r)d\theta^2 + r^2 g_{T^{n-2}} \tag{3.2}$$

where $g_{T^{n-2}}$ is an arbitrary flat metric on the $(n-2)$-torus.

We will specify the range of the (r, θ) parameters and the exact form of $V(r)$ below, but first let us calculate the curvatures of these metrics in terms of the function $V(r)$. We will start by setting up an orthonormal basis for these metrics: let $e_1 = \sqrt{V}\partial_r$, $e_2 = \frac{1}{\sqrt{V}}\partial_\theta$, and $e_j = \frac{1}{r}\partial_{\phi_j}$, where the ∂_{ϕ_j}, $3 \leq j \leq n$ are an orthonormal basis for the T^{n-2}. A straightforward calculation(performed at the end of this chapter) shows that the e_i diagonalize the curvature tensor, and that the corresponding sectional curvatures are

$$K_{12} = -\frac{V''}{2} \tag{3.3}$$

$$K_{1j} = K_{2j} = -\frac{V'}{2r} \qquad j > 2 \tag{3.4}$$

$$K_{ij} = -\frac{V}{r^2} \qquad i, j > 2 \tag{3.5}$$

$$\tag{3.6}$$

Now, let

$$V(r) = r^2 - 2mr^{3-n} \tag{3.7}$$

Using the same basis as above, we have

$$K_{12} = -1 + \frac{(n-3)(n-2)m}{r^{n-1}} \tag{3.8}$$

$$K_{1j} = K_{2j} = -1 - \frac{(n-3)m}{r^{n-1}} \tag{3.9}$$

$$K_{ij} = -1 + \frac{2m}{r^{n-1}} \tag{3.10}$$

where once again i, j are assumed to be greater than 2.

Another straightforward calculation shows that this metric is Einstein with scalar curvature $-n(n-1)$ and asymptotically hyperbolic. (We have yet to specify the range of the r parameter, but it is clear that the metric is well-defined for large enough r, so it makes sense to speak of its asymptotic properties.)

If $m = 0$, we get a hyperbolic cusp metric

$$g_C = r^{-2}dr^2 + r^2 g_{S^1 \times T^{n-2}} \tag{3.11}$$

This metric will be complete if we let $r \in (0, \infty)$.

On the other hand, if $m > 0$, we get a nontrivial Einstein metric. These metrics are called T^{n-2} Anti-deSitter Black Hole metrics. They will be complete provided we let $r \geq r_+ = (2m)^{\frac{1}{n-1}}$ and $0 \leq \theta < \beta_m = \frac{4\pi}{(n-1)r_+}$. (c.f. [And2].) Note that the locus $\{r = r_+\}$ is a flat totally geodesic T^{n-2}. By analogy with the core geodesics in hyperbolic Dehn surgery(c.f. [Thu],) we call this a core torus.

Now recall that we introduced these manifolds because we want to glue them into a cusp. They have the correct topological and local geometric properties to work. The first choice we could make would be to cut off one of the black hole metrics above at some large r, and then perturb it to make it hyperbolic near the boundary. The problem is that we cannot fix the global geometry near the boundary; although we can choose the metric on the T^{n-2}, the boundary metric will necessarily be the product of this flat metric and a large S^1, since the size of the S^1 factor is determined by r.

To resolve this difficulty, we will exploit the large isometry group of these metrics to take a quotient with the desired boundary. Below, we shall use the term "black hole metric" to refer to any metric on $D^2 \times T^{n-2}$ which has the same universal cover as g_{BH}.

Proposition 17. *Suppose we have an $S^1 \times \mathbb{R}^{n-2}$-invariant metric on $D^2 \times \mathbb{R}^{n-2}$. Let T_0 be some flat $(n-1)$-torus, and let $\sigma \subset T_0$ be a simple closed geodesic such that*

$$L(\sigma) = L(\partial D^2). \tag{3.12}$$

Then $\exists\Gamma_0 \subset$ *Isom*$(D^2 \times \mathbb{R}^{n-2})$ *such that*

$$\frac{(D^2 \times \mathbb{R}^{n-2})}{\Gamma_0} \simeq M_{fill} \tag{3.13}$$

is a solid torus with $\partial M_{fill} = T_0$.

Proof: We have

$$T_0 = \mathbb{R}^{n-1}/\Gamma, \tag{3.14}$$

where Γ is some $(n-1)$-dimensional group of translations of $\mathbb{R}^{n-1}$. Since σ is closed and simple, the translation induced by σ is a generator for Γ. We can find elements $\gamma_i \in \mathbb{R}^{n-1}$ such that the set $\{\sigma, \gamma_1, ..., \gamma_{n-2}\}$ forms a set of generators for Γ. Let us denote the subgroup of $\pi_1(T_0)$ generated by σ by $\langle\sigma\rangle$. Then since $\pi_1(T_0)$ is Abelian, $\langle\sigma\rangle$ is normal, which implies that the covering map

$$p : \mathbb{R}^{n-1} \longrightarrow T_0 \tag{3.15}$$

splits as $p = p_1 \circ p_2$, where

$$p_2 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}/\langle\sigma\rangle \simeq S^1 \times \mathbb{R}^{n-2} \tag{3.16}$$

and

$$p_1 : S^1 \times \mathbb{R}^{n-2} \longrightarrow (S^1 \times \mathbb{R}^{n-2})/\Gamma_0 \simeq T_0 \tag{3.17}$$

where $\Gamma_0 = \Gamma/\langle\sigma\rangle$

Now, say we have an $S^1 \times \mathbb{R}^{n-2}$-invariant metric on $D^2 \times \mathbb{R}^{n-2}$, and that the length of ∂D^2 is $L(\sigma)$.

We will use the above remark to construct a quotient of this metric with boundary T_0. We can describe this quotient in terms of coordinates $(r, \theta, \phi_3, ..., \phi_{n-2})$, where the ϕ_i's are the standard coordinates on $\mathbb{R}^{n-2}$. Define an isometric action of Γ_0 on $D^2 \times \mathbb{R}^{n-2}$ by keeping r fixed and acting on the $S^1 \times \mathbb{R}^{n-2}$ coordinates. The boundary of this quotient will certainly be T_0, and it is clear that there are no fixed points on $r = 0$ since no element maps σ to itself except the identity, so the quotient is indeed a manifold. ∎

Note that the reason that this works is that we were able to split off the $\langle\sigma\rangle$ from the rest of Γ, and then fill it in with a disk. The basic point is that when we project the other generators onto the core $\mathbb{R}^{n-2}$, they cannot be zero, or else they would be parallel to σ. If one thinks about the three-dimensional case, one can picture the universal cover as being a tubular neighborhood of a geodesic in hyperbolic space. Then σ would be the boundary of a disk perpendicular to the core geodesic. One can obtain the torus T^2 by taking the quotient of the cylinder by $\langle\gamma\rangle$, where γ is some composition of a translation and a rotation. The only way that this action will not extend to the core geodesic is if γ has no translation component. But this is impossible if the quotient of the boundary is to be a torus.

Now we will get an appropriate metric on this quotient. All we need is an $S^1 \times \mathbb{R}^{n-2}$-invariant metric on $D^2 \times \mathbb{R}^{n-2}$. Since we want an Einstein metric, we will take the universal cover of the T^{n-2}-black hole metrics, slightly altered near the boundary. It turns out that the value of m is irrelevant to the local geometry of these(c.f. [And2],) so we will set $m = 1$.

Let

$$R = \frac{L(\sigma)}{\beta_1} \tag{3.18}$$

and define

$$\widetilde{g_{fill}} = V(r)^{-1}dr^2 + V(r)d\theta^2 + r^2 g_{Eucl}; r \in [r_+, R), \theta \in [0, \beta_1) \tag{3.19}$$

where $V(r) = r^2 - \frac{2\chi(r)}{r^{n-3}}$, $\chi(r) = 1$ for $r < R-2$, $\chi(r)' = 0$ for $r > R-1$.

Now note that for $r > R-1$,

$$\widetilde{g_{fill}} = r^{-2}dr^2 + r^2\left(d\theta^2 + g_{Eucl}\right) \tag{3.20}$$

By taking the quotient, we will get the metric

$$g_{fill} = r^{-2}dr^2 + r^2\frac{g_{T_0}}{R^2} \tag{3.21}$$

on $r > R-1$.

By the change of coordinates

$$\rho = \frac{r}{R} \tag{3.22}$$

we get a metric which is identically equal near its boundary to the hyperbolic cusp metric which we are trying to fill in. In the event that we have k toric cusps, we can cut off each one, and perform this procedure on a geodesic σ^i in each boundary torus. We then obtain a manifold (M_σ, g_σ), where σ is the ordered k-tuple of geodesics $(\sigma^i)_{1\leq i\leq k}$.

We say that M_σ is a Dehn filling of N, again by analogy with the three-dimensional case. The size of the Dehn filling is defined to be

$$|\sigma| = \min_i L(\sigma^i) = \min_i (R_{\sigma^i}\beta_1) \tag{3.23}$$

This is well-defined, since we fix the boundary tori at the beginning.

We can see explicitly from (1.3)-(1.5) that

$$\begin{aligned} \|\Phi_{g_\sigma}(g_\sigma)\|_{k-2,\alpha,\delta} &= \|ric_{g_\sigma} + (n-1)g_\sigma\|_{k-2,\alpha,\delta} &(3.24)\\ &= C\left\|\chi'' + \frac{\chi'}{r} + \frac{\chi}{r^2}\right\|_{k-2,\alpha,\delta} &(3.25)\\ &\leq C_1\frac{1}{R^{n-1}} &(3.26)\end{aligned}$$

Note that one must be careful in the last line, since we are bounding Hölder norms, in which derivatives are calculated with respect to harmonic coordinates, and not with respect to the coordinate r. But r is related to the geodesic coordinate s by

$$r = O(e^s) \tag{3.27}$$

for large s, so we can establish the bound with respect to s, and then translate back into terms of R.

Thus, we have the following proposition:

Proposition 18. *Let (N, g) be a geometrically finite hyperbolic manifold, whose k cusps all have toric cross-sections. Then for any k-tuple σ of geodesics in these cusp cross-sections, it is possible to construct a manifold (M_σ, g_σ) such that*

$$\|\Phi_{g_\sigma}(g_\sigma)\|_{k-2,\alpha,\delta} = O(|\sigma|^{1-n}) \tag{3.28}$$

These (M_σ, g_σ) *are AH and have the same conformal infinity as* (N, g).

In three dimensions, we no longer have these black hole metrics. In fact, the only candidate for the glued-in metric is a quotient of hyperbolic space. But there are many hyperbolic 3-manifolds bounded by 2-tori; we may take the quotient of a tubular neighborhood of a geodesic γ in $\mathbf{H}^3$ by a cyclic group of loxodromic transformations which fix γ. To fill in a cusp, we must alter the metric near the boundary torus so that the second fundamental form of the solid torus agrees with that of the truncated cusp. Thus, the filled manifold is not hyperbolic.

Then the above proposition still holds in this case.

We end this chapter with a few remarks on the topology of the M_σ's. We start with the Gromov-Thurston 2π-theorem. (The proof is due to Bleiler and Hodgson, c.f.[BlH])

Proposition 19. *(Gromov-Thurston, [GrT]) If* $|\sigma| > 2\pi$*, then* M_σ *admits a nonpositively curved metric, such that the core tori are totally geodesic and whose sectional curvature is strictly negative off the core tori.*

Proof: By proposition 17, all we need to do is to construct a nonpositively curved $S^1 \times \mathbb{R}^{n-2}$-invariant metric on $D^2 \times \mathbb{R}^{n-2}$ which is hyperbolic near the boundary, is flat exactly on the core $\mathbb{R}^{n-2}$, and such that $L(\partial D^2) = L(\sigma)$. We will construct these metrics explicitly.

Consider the following metric on $D^2 \times \mathbb{R}^{n-2}$;

$$\widetilde{g_{2\pi}} = dr^2 + f^2(r)d\theta^2 + h^2(r)g_{Eucl}; \qquad 0 < r < 1; 0 \le \theta < 2\pi \tag{3.29}$$

Let $(\phi_3, ..., \phi_n)$ be the standard coordinates on $\mathbb{R}^{n-2}$. If we choose the orthonormal basis $e_1 = \partial_r, e_2 = \frac{\partial_\theta}{f}, e_j = \frac{\partial_{\phi_j}}{h}$, $j > 2$, then we can calculate the curvatures of this metric in the same fashion as for the black hole metrics to obtain

$$K_{12} = -\frac{f''}{f} \tag{3.30}$$

$$K_{1i} = -\frac{h''}{h}; \quad i > 2 \tag{3.31}$$

$$K_{2j} = -\frac{h'f'}{hf}; \quad i > 2 \tag{3.32}$$

$$K_{ij} = -\left(\frac{h'}{h}\right)^2; \quad i, j > 2 \tag{3.33}$$

We want to find positive functions f and h which are increasing, strictly convex, and such that $f, h \neq 0$ away from $r = 0$. We must also, however, ensure that the metric is hyperbolic near $r = 1$, and complete at $r = 0$. We can choose

$$f(r) = L(\sigma)e^{r-1} \tag{3.34}$$

$$h(r) = e^{r-1} \tag{3.35}$$

on $[1 - \epsilon_1, 1]$ and

$$f(r) = 2\pi \sinh(r) \tag{3.36}$$

$$h(r) = \cosh(r) \tag{3.37}$$

on $[0, \epsilon_2]$

In this case

$$h'(0) = 0 \tag{3.38}$$

$$f'(0) = 2\pi \tag{3.39}$$

Note that

$$f'(0) = \lim_{r \to 0} \frac{1}{r} \int_0^{2\pi} f(r) dr \tag{3.40}$$

measures the cone angle over the locus $r = 0$, so the metric is complete.

The problem we face now is to extend the definitions of these two functions in such a way as to ensure that f and h are strictly convex and increasing on $[0, 1]$. In general this will not be possible, since the convexity will force the function to have greater than linear growth. If $f(\epsilon_2) = 2\pi \sinh \epsilon_2$ and $f(\epsilon_1) = L(\sigma)e^{\epsilon_1 - 1}$ are too close, we will not be able to connect them by a strictly convex and increasing function.

But if $L(\sigma) > 2\pi$, one can use a bump function to construct such an f explicitly. ∎

Thus, we have:

Proposition 20. *For $|\sigma| > 2\pi$, M_σ is a $K(\pi, 1)$, and every noncyclic Abelian subgroup of $\pi_1(M_\sigma)$ is carried by one of the core tori.*

Proof: (Adapted from Theorem 6.3.9 of [Pet].) The first statement follows directly from the fact that M_σ admits a metric of nonpositive sectional curvature. To prove the second one, consider the action of $\pi_1(M_\sigma)$ by isometries on $\widetilde{M_\sigma}$, the universal cover of M_σ, where $\widetilde{M_\sigma}$ is equipped with the lifted metric.

Let $\alpha \in \pi_1(M_\sigma)$, and let $p \in \widetilde{M_\sigma}$ satisfy

$$d(p, \alpha(p)) = \inf_{x \in \widetilde{M_\sigma}} d(x, \alpha(x)) \tag{3.41}$$

Such a p exists because M_σ's is injectivity radius is nonzero, and is large on the expanding ends of M_σ. Thus any minimizing sequence for the function $d(x, \alpha(x))$ must project down to some compact subset of M_σ.

Since $\widetilde{M_\sigma}$ is simply connected and of nonpositive curvature, there is a unique geodesic γ joining $p = \gamma(0)$ and $\alpha(p) = \gamma(t_0)$. Furthermore, α must fix γ. To show this, all we need to prove is that $\alpha^2(p)$ lies on γ. If it does not, then let $x = \gamma(\frac{t_0}{2})$ be the midpoint of the segment $\overline{p\alpha(p)}$ of γ. By assumption, the three points x, $\alpha(x)$ and $\alpha(p)$ cannot all lie on the same geodesic, so we must have

$$\begin{aligned} d(x, \alpha(x)) &< d(x, \alpha(p)) + d(\alpha(p), \alpha(x)) && (3.42) \\ &= \frac{t_0}{2} + d(p, x) && (3.43) \\ &= t_0 && (3.44) \\ &= d(p, \alpha(p)), && (3.45) \end{aligned}$$

which is impossible, since p minimizes $d(x, \alpha(x))$.

If $\alpha, \beta \in \pi_1(M_\sigma)$ commute, then $\alpha \circ \beta(p) = \beta \circ \alpha(p)$, so β sends the geodesic segment $\overline{p\alpha(p)}$ to the geodesic segment $\overline{\beta(p)\beta(\alpha(p))}$. Furthermore, we can see that α must map the unique geodesic joining p to $\beta(p)$ to the unique one joining $\alpha(p)$ and $\beta(\alpha(p))$. This gives a geodesic quadrilateral whose angles add up to 2π. By the Topogonov theorem, this quadrilateral must lie in a flat submanifold, so we are done. ∎

We will also need the following result:

Proposition 21. *Let N admit a nonelementary geometrically finite hyperbolic metric. Then N admits only finitely many classes of homotopy equivalencies.*

Proof: Let

$$F : N \longrightarrow N \tag{3.46}$$

be a homotopy equivalence. Given a hyperbolic metric g on N, we can deform F so that it fixes the conformal infinity of (N, g). Then by Sullivan rigidity(c.f. [Kvt]), F can be represented by an isometry. But by [Rat2] generic nonelementary geometrically finite hyperbolic metrics have only finitely many isometries. ∎

We may now prove that our Dehn fillings give rise to infinitely many homotopy classes.

Proposition 22. *Let N admit a geometrically finite nonelementary hyperbolic metric, all of whose cusps are tori. Let M_σ be obtained from N by a Dehn filling. Then there are only finitely many other Dehn fillings which have the same homotopy type as M_σ.*

Proof: We will simply adapt Anderson's proof from [And1] to the infinite volume case. The idea of the proof is to show that a homotopy equivalence

$$F : M_\sigma \longrightarrow M_{\sigma'} \tag{3.47}$$

leads either to a nontrivial homotopy equivalence of the hyperbolic manifold N, or to a nontrivial isomorphism of the Dehn filling data. Since there are only finitely many members in either of these classes, we can conclude that there are only finitely many M_σ's in any homotopy class.

By Seifert-Van Kampen, we have that

$$\pi_1(M_\sigma) = \frac{\pi_1(N)}{\langle \cup \sigma_i \rangle} \tag{3.48}$$

By proposition 20, we know that for $|\sigma|$ sufficiently large, the only (conjugacy classes) of noncyclic Abelian subgroups of $\pi_1(M_\sigma)$ are carried by the core tori. Now, say that we have a homotopy equivalence

$$F : M_\sigma \longrightarrow M_{\sigma'} \tag{3.49}$$

Then F_* must permute the cyclic subgroups carried by the essential tori. This in turn implies that F must map neighborhoods of these tori onto each other. We can then use F to define a map G from the original hyperbolic manifold N to itself such that G fixes the conformal infinity of N, interchanges the cusps of N and such that G_* is an isomorphism of $\pi_1(N)$. Then again by Sullivan rigidity, G is a homotopy equivalence, of which there are only finitely many by the previous proposition. Now the homotopy equivalence F must also preserve the Dehn filling data, so necessarily we must have

$$F_*\langle \sigma_i \rangle = \langle \sigma_j' \rangle \tag{3.50}$$

But given a cyclic group, there are only two elements which can generate it. Thus, there are only finitely many Dehn-filled manifolds homotopy equivalent to M_σ. ∎

The theorem does not hold if we drop the hypothesis that (N, g) is nonelementary; one can construct infinitely many nonisometric black hole metrics on the solid torus $D^2 \times T^{n-2}$ with the same conformal infinity T^{n-1}(c.f. [And2].) These can be thought of as Dehn fillings of the complete hyperbolic cusps

$$g_C = r^{-2}dr^2 + r^2 g_{T^{n-1}} \tag{3.51}$$

Now, as promised above, we will perform the calculation of the curvature of the metric

$$g_{BH} = (V(r))^{-1}dr^2 + V(r)d\theta^2 + r^2 g_{T^{n-2}} \tag{3.52}$$

To simplify our calculations, we will use $f(r)$ to denote $\sqrt{V(r)}$. Then we have

$$e_1 = f\partial_r \tag{3.53}$$

$$e_2 = \frac{\partial_\theta}{f} \tag{3.54}$$

$$e_j = \frac{\partial_{\phi_j}}{r}; \quad j > 2 \tag{3.55}$$

where the ∂_{ϕ_j} form an orthonormal frame for the T^{n-2}.

By direct calculation, we get that

$$[e_2, e_1] = f' e_2 \tag{3.56}$$

$$[e_j, e_1] = \frac{f}{r} e_j; \quad j > 2 \tag{3.57}$$

Since the frame (e_j) is orthonormal, we have

$$(\nabla_{e_i} e_j, e_k) = \frac{1}{2}\left(([e_i, e_j], e_k) - ([e_i, e_k], e_j) - ([e_j, e_k], e_i)\right) \tag{3.58}$$

Thus we obtain that if $j, k > 2$,

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_j = 0 \tag{3.59}$$

$$\nabla_{e_2} e_1 = f' e_2 \tag{3.60}$$

$$\nabla_{e_j} e_1 = \frac{f}{r} e_j \tag{3.61}$$

$$\nabla_{e_2} e_j = \nabla_{e_j} e_2 = 0 \tag{3.62}$$

$$\nabla_{e_2} e_2 = -f' e_1 \tag{3.63}$$

$$\nabla_{e_j} e_k = -\delta_j^k \frac{f}{r} e_1 \tag{3.64}$$

Now that we have all the necessary covariant derivatives, it is now completely straightforward to calculate the various curvatures;

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \tag{3.65}$$

By inspection, we see that $R(e_i, e_j)e_k = 0$ if i, j and k are all distinct. Since $R(X, Y)Z = -R(Y, X)Z$, this means that the frame(e_i) diagonalizes the curvature tensor, and that we only have to perform the four following calculations to know the entire curvature tensor.

$$
\begin{aligned}
R(e_1,e_2)e_1 &= \nabla_{e_2}\nabla_{e_1}e_1 - \nabla_{e_1}\nabla_{e_2}e_1 - \nabla_{[e_2,e_1]}e_1 && (3.66)\\
&= -\nabla_{e_1}(f'e_2) - f'\nabla_{e_2}e_1 && (3.67)\\
&= -(e_1 f')\,e_2 - f'\nabla_{e_1}e_2 - f'\nabla_{e_2}e_1 && (3.68)\\
&= -ff''e_2 - (f')^2 e_2 && (3.69)\\
&= -\frac{V''}{2}e_2 && (3.70)
\end{aligned}
$$

We also have if $j > 2$,

$$
\begin{aligned}
R(e_1,e_j)e_1 &= \nabla_{e_j}\nabla_{e_1}e_1 - \nabla_{e_1}\nabla_{e_j}e_1 - \nabla_{[e_j,e_1]}e_1 && (3.71)\\
&= -\nabla_{e_1}\left(\frac{f}{r}e_j\right) - \nabla_{\frac{f}{r}e_j}e_1 && (3.72)\\
&= -e_1\left(\frac{f}{r}\right)e_j - \frac{f}{r}\nabla_{e_1}e_j - \frac{f}{r}\nabla_{e_j}e_1 && (3.73)\\
&= -f\left(\frac{f}{r}\right)'e_j - \left(\frac{f}{r}\right)^2 e_j && (3.74)\\
&= -\frac{V'}{2r}e_j && (3.75)
\end{aligned}
$$

and

$$
\begin{aligned}
R(e_j,e_2)e_j &= \nabla_{e_2}\nabla_{e_j}e_j - \nabla_{e_j}\nabla_{e_2}e_j - \nabla_{[e_2,e_j]}e_j && (3.76)\\
&= -\nabla_{e_2}\left(\frac{f}{r}e_1\right) && (3.77)\\
&= -f'\left(\frac{f}{r}\right)e_2 && (3.78)\\
&= -\frac{V'}{2r}e_2 && (3.79)
\end{aligned}
$$

Finally, assume that $j, k > 2$ and are distinct. Then

$$\begin{aligned} R(e_j, e_k)e_j &= \nabla_{e_k}\nabla_{e_j}e_j - \nabla_{e_j}\nabla_{e_k}e_j - \nabla_{[e_k,e_j]}e_j && (3.80) \\ &= -\nabla_{e_k}\left(\frac{f}{r}e_1\right) && (3.81) \\ &= -\left(\frac{f}{r}\right)^2 e_k && (3.82) \\ &= -\frac{V}{r^2}e_k && (3.83) \end{aligned}$$

Since

$$K_{ij} = (R(e_i, e_j)e_i, e_j), \tag{3.84}$$

we are done.

Chapter 4

Control of Inverse

In this section, we will perform the analysis which will allow us to invert our operators Φ_{g_σ}. To do this, we will need to invert the linear operators $L_\sigma = 2D_{g_\sigma}\Phi_{g_\sigma}$ and get some kind of uniform control on the behavior of the inverses. We cannot get an absolute uniform bound on their operator norms, but we can make sure that they do not grow too fast with respect to the length of the Dehn surgery σ. We prove via a contradiction argument that there is some Λ independent of σ such that

$$||h||_{k,\alpha,\delta} \leq \Lambda \log|\sigma| ||L_\sigma(h)||_{k-2,\alpha,\delta} \tag{4.1}$$

for all $h \in S_\delta^{k,\alpha}$, provided $k \geq 3$ and that σ is weakly balanced, in the following sense:

Definition 23. *We say that $\sigma = (\sigma^1, \sigma^2, ..., \sigma^q)$ is weakly balanced if it satisfies the following inequality*

$$\max_j L(\sigma^j) \leq |\sigma|^{c_0} = \left(\min_j L(\sigma^j)\right)^{c_0} \tag{4.2}$$

for some fixed $c_0 > 0$.

We will motivate the presence of the $\log|\sigma|$ term and the weak balancing condition below. The estimate above shows that $\ker(L_\sigma) = 0$, so by Biquard's Theorem we get that

$$L_\sigma^{-1} : S_\delta^{k-2,\alpha} \longrightarrow S_\delta^{k,\alpha} \tag{4.3}$$

is well-defined and

$$\|L_\sigma^{-1} f\|_{k,\alpha,\delta} \leq \Lambda \log|\sigma| \|f\|_{k-2,\alpha,\delta} \tag{4.4}$$

The proof of estimate (4.1) is essentially identical to Anderson's proof in the finite-volume case([And1]). The only additional difficulty that we face here is that $S_\delta^{k,\alpha}$ functions are not necessarily in L^2, but by choosing $\delta > \frac{n-1}{2}$ we can avoid this problem. In this section, any unlabeled norms will be assumed to be L^2 norms and $\delta \in \left(\frac{n-1}{2}, n-1\right)$ will be assumed to be fixed.

Proposition 24. *Let (M_σ, g_σ) be a sequence of approximate solutions. Then there exists a constant independent of σ such that*

$$\|h\|_{k,\alpha,\delta} \leq \Lambda \log|\sigma| \|L_\sigma(h)\|_{k-2,\alpha,\delta} \tag{4.5}$$

Proof: We work by contradiction, so we will have to take limits. This leads to some difficulty, since there is no uniform bound on the diameter of the M_σ's, and so the limits will not be uniquely defined. On the other hand, all of the limits are Einstein, which gives them extra structure which we will exploit.

Let us set up the contradiction. If there is no Λ such that

$$\|h\|_{k,\alpha,\delta} \leq \Lambda \log|\sigma| \|L_\sigma(h)\|_{k-2,\alpha,\delta}, \tag{4.6}$$

for all σ, then necessarily there is a sequence $h_i \in S_\delta^{k,\alpha}(M_{\sigma_i}, g_{\sigma_i})$ such that

$$\|h_i\|_{k,\alpha,\delta} = 1 \tag{4.7}$$

but

$$\log|\sigma_i| \|L_i(h_i)\|_{k-2,\alpha,\delta} \longrightarrow 0 \tag{4.8}$$

where we have replaced the subscript σ_i by an i.

By the Schauder estimates, we know that there is a Λ_0 independent of σ such that

$$\|h\|_{k,\alpha,\delta} \leq \Lambda_0(\|L_\sigma(h)\|_{k-2,\alpha,\delta} + \|h\|_{L_\delta^\infty}) \tag{4.9}$$

for all $h \in S_\delta^k(M_\sigma)$. Thus, under our assumption,

$$\|h_i\|_{L_\delta^\infty} \geq \Lambda_0^{-1} > 0 \tag{4.10}$$

Therefore, showing that $\|h_i\|_{L_\delta^\infty} \to 0$ will give us a contradiction.

The most natural way of looking at M_σ is to see it as being made up of two distinct pieces: the original hyperbolic manifold N and the collection of black hole metrics with which we are filling in the cusps of N. Our strategy is to take the limit of the h_i's, which will lead to infinitesimal Einstein deformations of each piece. We will get our contradiction by showing that there can be no nontrivial deformations.

We will spend most of our time working on the filled regions of the cusps, i.e. the ends which are close to the black hole metrics. We will use the variables r to refer to the r-variable in our parametrization of the black hole metrics, and R will refer to the gluing region, as seen from the black hole metrics. Note that, by construction, $R = \frac{L(\sigma)}{\beta_1}$. Our analysis will take place on each black hole region separately, so there will be no risk of confusion.

For the region associated to the kth cusp, we have the relation:

$$R_i^k = \beta_1 L(\sigma_i^k) \geq \beta_1 |\sigma_i| \tag{4.11}$$

To begin, note that we have the following Weitzenböck formula(c.f. [Bes]):

$$\delta dh + d\delta h = D^* Dh - Rh + h \circ ric \tag{4.12}$$

where d is the exterior derivative on vector-valued one-forms induced by the connection, and δ is its formal adjoint.

From this, we get that

$$\begin{aligned} Lh &= D^* Dh - 2Rh + h \circ ric + ric \circ h + 2(n-1)h && (4.13) \\ &= \delta dh + d\delta h - Rh + ric \circ h + 2(n-1)h && (4.14) \end{aligned}$$

We will work primarily with this form of L, since we can prove stronger positivity properties with it.

Now, by construction, we have that on M_i,

$$ric + (n-1)g = \tau(r) \tag{4.15}$$

where τ is supported on the region of the black hole metric $(R_i - 2, R_i - 1)$ and satisfies

$$|\tau| \leq C R_i^{1-n} \tag{4.16}$$

Thus,

$$Lh = \delta dh + d\delta h - Rh + (n-1)h + \tau \circ h \tag{4.17}$$

Now, consider

$$\langle Lh, h\rangle = \langle \delta dh, h\rangle + \langle d\delta h, h\rangle - \langle Rh, h\rangle + (n-1)\|h\|^2 + \langle \tau \circ h, h\rangle \tag{4.18}$$

Recall that we have $Lh \to 0$ and h bounded, and we want to show that $h \to 0$. This should be possible as long as all the terms on the right hand side are positive or tend toward 0. Integration by parts will work on the first two terms, so we need to get a handle on the term $\langle Rh, h\rangle$. We will do this by controlling the pointwise norm (Rh, h). First, we will break it up into three pieces. Let $h = h_0 + \frac{(trh)g}{n}$ (so h_0 is the trace-free part of h.)

Lemma 25.

$$(Rh, h) = (Rh_0, h_0) + \mu_i(r) + O\left((trh)^2\right) \tag{4.19}$$

Where $\mu_i(r)$ is supported in the black hole region and is $O\left(R_i^{-(n-1)}\right)$

Proof: To begin, note that

$$\begin{aligned}
(Rh, h) &= \left(R\left(h_0 + \frac{(trh)}{n}g\right), h_0 + \frac{(trh)g}{n}\right) && (4.20)\\
&= (Rh_0, h_0) + \frac{trh}{n}(Rg, h_0) + \frac{trh}{n}(Rh_0, g) && (4.21)\\
&\quad + \frac{(trh)^2}{n^2}(ric, g) && (4.22)
\end{aligned}$$

Now, we have that $(ric, g) = s$, which is uniformly bounded. Furthermore, if we take an orthonormal frame (e_j) which diagonalizes the curvature tensor,

$$Rh(e_j, e_k) = \sum_l h(R(e_j, e_l)e_k, e_l) \tag{4.23}$$

Thus,

$$\begin{aligned}
(Rg, h_0) &= (g, Rh_0) && (4.24)\\
&= tr(Rh_0) && (4.25)\\
&= \sum_{j,k} h_0(R(e_j, e_k)e_j, e_k) && (4.26)\\
&= \sum_{j,k} K_{jk} h_0(e_k, e_k) && (4.27)\\
&= -(n-1)tr(h_0) + \mu_i && (4.28)\\
&= \mu_i && (4.29)
\end{aligned}$$

Where $\mu_i(r) = \sum_{j,k}(\delta_j^k + K_{jk})h_0(e_k, e_k) = O\left(R_i^{-(n-1)}\right)$ since h_0 is uniformly bounded. ∎

The following lemma allows us to control (Rh_0, h_0).

Lemma 26. *(Besse) Let*

$$a = \sup_{\{h_0 | tr h_0 = 0\}} \frac{(Rh_0, h_0)}{|h_0|^2} \tag{4.30}$$

be the largest eigenvalue of R acting on S_0^2. Then

$$a < (n-2)K_{max} - ric_{min} \tag{4.31}$$

Proof: Choose some trace-free η such that $R\eta = a\eta$. Since we are working pointwise, we may assume that we are using an orthonormal frame in which η is diagonal, and that $\eta(e_1, e_1) = \eta_{11} = \sup |\eta_{jj}| = 1$.

We have that $R\eta = a\eta$, so

$$\begin{aligned}
a &= a\eta_{11} = R\eta(e_1, e_1) \\
&= \sum_i \eta(R(e_1, e_i)e_1, e_i) \\
&= \sum_{i\neq 1} \eta_{ii} K_{1i} \\
&= \sum_{i\neq 1} \eta_{ii} K_{max} - \sum_{i\neq 1} \eta_{ii}(K_{max} - K_{1i}) \\
&\leq K_{max}(tr\eta - \eta_{11}) + \eta_{11} \sum_{i\neq 1}(K_{max} - K_{1i}) \\
&= (n-2)K_{max} - \sum_{i\neq 1} K_{1i} \\
&\leq (n-2)K_{max} - ric_{min}
\end{aligned}$$

∎

Lemma 27. *As $i \to \infty$, we have $\|trh_i\|_{L^2} \longrightarrow 0$*

Proof: Recall that

$$L(h) = D^*Dh - 2R(h) + ric \circ h + h \circ ric + 2(n-1)h \qquad (4.32)$$

We have that $trR(h) = (ric, h)$ and $trD^*Dh = \Delta trh$. Thus,

$$\begin{aligned}
trL(h) &= \Delta trh - 2(ric, h) + 2(n-1)trh + \frac{2s}{n}trh \\
&= \Delta(trh) - 2(z, h) + 2(n-1)trh
\end{aligned}$$

Thus,

$$\Delta(trh) + 2(n-1)trh = trL(h) + 2(z,h) \tag{4.33}$$

Integrating both sides against trh gives us

$$\begin{aligned}
\|trh\|^2 &\leq \langle \Delta trh, trh\rangle + 2(n-1)\|trh\|^2 &(4.34)\\
&= \langle trL(h) + 2(z,h), trh\rangle &(4.35)\\
&\leq \|trL(h) + 2(z,h)\| \cdot \|trh\| &(4.36)\\
&\leq (\|trL(h)\| + 2\|z\| \cdot \|h\|)\|trh\|, &(4.37)
\end{aligned}$$

Thus,

$$\|trh\| \leq \|trL(h)\| + 2\|z\| \cdot \|h\| \tag{4.38}$$

which tends to 0, since $\|z\|$ and $\|Lh\| \longrightarrow 0$ (because $L^\infty_\delta \subset L^2$), and $\|h\|$ is bounded. ∎

Now, let $\mathcal{U}_\rho = \{x | r(x) < \rho\}$ be a tubular neighborhood of the totally geodesic core T^{n-1}'s, and let $M_i^\rho = M_i - \mathcal{U}_\rho$. We will fix $\rho \leq R_i = \inf_k R_i^k$ below. Note that $\mathcal{U}_\rho$ has q connected components, where q is the number of cusps of N.

By Lemmas 25 and 26, on M_i^ρ.

$$\begin{aligned}
(R(h),h) &= (R(h_0),h_0)+\mu_i+O\left((trh)^2\right) && (4.39)\\
&\leq ((n-2)K_{max}-ric_{min})\,|h|^2+\mu_i+C_1|trh|^2 && (4.40)\\
&= \left(-(n-2)+(n-1)+O(R_i^{1-n})\right)|h|^2 && (4.41)\\
&\quad +\mu_i+C_1|trh|^2 && (4.42)\\
&\leq \left(1+C_2R_i^{1-n}\right)|h|^2+\mu_i+C_1|trh|^2 && (4.43)
\end{aligned}$$

Now consider

$$\begin{aligned}
\int_{M^\rho}(Lh,h)dV &= \int_{M^\rho}(\delta dh,h)+(d\delta h,h)-(Rh,h) && (4.44)\\
&\quad +(n-1)|h|^2+(\tau\circ h,h)\;dV && (4.45)\\
&= \int_{M^\rho}|\delta h|^2+|dh|^2-(Rh,h)+(n-1)|h|^2 && (4.46)\\
&\quad +(\tau\circ h,h)\;dV+\int_{\partial\mathcal{U}_\rho}Q(h,\partial h)dA && (4.47)
\end{aligned}$$

Here, $Q(h,\partial h)$ is the boundary term from the integration by parts. It is a fixed quadratic polynomial in h and its derivatives. Choose some $\epsilon>0$. By our estimates on (Rh,h), and assuming that i is large enough that $|\tau_i|<\epsilon$ and

$$1+C_2R_i^{1-n}\leq\frac{n}{2} \qquad (4.48)$$

we get that this last quantity is

$$\geq \int_{M^\rho} -\left(\frac{n}{2}+\epsilon\right)|h|^2 + (n-1)|h|^2\, dV - C_1 \int_{M^\rho} (trh)^2\, dV \quad (4.49)$$

$$-\int_{M^\rho} \mu_i\, dV + \int_{\partial\mathcal{U}_\rho} Q(h, \partial h)\, dA \quad (4.50)$$

Now, μ_i is $O(R_i^{1-n})$ and supported on a region of bounded volume(the black hole region) and by the previous lemma $\|trh_i\|_{L^2} \longrightarrow 0$, so we may choose i large enough that the previous quantity is

$$\geq \int_{M^\rho} -\left(\frac{n}{2}+\epsilon\right)|h|^2 + (n-1)|h|^2\, dV - 2\epsilon + \int_{\partial\mathcal{U}_\rho} Q(h, \partial h) dA \quad (4.51)$$

$$= \left(\frac{n}{2}-1-\epsilon\right)\int_{M^\rho} |h|^2\, dV + \int_{\partial\mathcal{U}_\rho} Q(h, \partial h) dA - 2\epsilon \quad (4.52)$$

We also have that

$$\int_{M^\rho} (Lh, h) dV \leq \frac{1}{2}\left(\int_{M^\rho} |Lh|^2 dV + \int_{M^\rho} |h|^2 dV\right) \quad (4.53)$$

and

$$\left|\int_{\partial\mathcal{U}_\rho} Q(h, \partial h) dA\right| \leq CVol(\partial\mathcal{U}_\rho) \quad (4.54)$$

since we have $C^{k,\alpha}$ control over the h_i's.

Combining all this information gives

$$\frac{1}{2}\int_{M^\rho} |Lh|^2\, dV + CVol(\partial\mathcal{U}_\rho) \geq \left(\frac{n}{2}-\epsilon-\frac{3}{2}\right)\int_{M^\rho} |h|^2\, dV - 2\epsilon \quad (4.55)$$

where we can make $\epsilon > 0$ arbitrarily small by making i large.

By assumption, $\|Lh\|_{L^\infty_\delta} \longrightarrow 0$, so we have that $\int_{M^\rho} |Lh|^2 dV \longrightarrow 0$. Remark that $inj(\partial\mathcal{U}_\rho) = O\left(\frac{\rho}{R_i}\right)$. Thus, if we choose a sequence ρ_i such that $\frac{\rho_i}{R_i} \longrightarrow 0$, we'll obtain that $\int_{M^{\rho_i}} |h|^2 dV$ tends to 0. Since $S_\delta^{k,\alpha} \subset L^2$, this gives us uniform convergence of the h_i's on any set whose injectivity radius remains bounded below. Thus, by a diagonal argument, we can find some sequence r_i such that h_i converges uniformly to 0 on M^{r_i} and $\frac{r_i}{R_i} \longrightarrow 0$.

Now, let us examine what is happening on each component of $\mathcal{U}_{r_i}$, the complement of the set M^{r_i}. By construction, the core tori are collapsing to points, so any neighborhood of these tori is degenerating to a line segment. Since we want to have a nice limit, we lift everything to finite covers in order to "unwrap the collapse."(c.f. [And2].) Choose a sequence of points p_i in a core torus. Since $inj(T(r)) = O\left(\frac{r}{R_i}\right)$, by lifting $(\mathcal{U}_{r_i}, g_i, p_i)$ to an $\left[\frac{R_i}{r_i}\right]$-fold cover, where [ ] is the greatest integer part function, we will get a sequence of manifolds whose injectivity radius is bounded away from 0.

By definition of the $\tilde{C}^{k,\alpha}$ norm, the $C^{k,\alpha}$-norm of these manifolds is bounded, so we get a limit manifold (M_{BH}, g_{BH}). Clearly the h_i's lift to the finite covers too, so we get lifted forms $\tilde{h}_i$. These $\tilde{h}_i$ satisfy

$$\|\tilde{h}_i\|_{k,\alpha,\delta} \leq C \tag{4.56}$$

on the lifted manifolds $(\tilde{\mathcal{U}}_{r_i}, \tilde{g}_i)$. Thus, given $\alpha' < \alpha$, and $\delta' < \delta$, we can extract a subsequence to get a limit $\tilde{h} \in S_{\delta'}^{k,\alpha'}$. Note that $\tilde{h}$ must be T^{n-1} invariant and satisfy $L\tilde{h} = 0$.

If $|r_i - r_+|$ is uniformly bounded, the torus $T(r_i)$ stays within a fixed distance of the core torus for all i. Thus, the core torus can always see the region on which h_i is tending uniformly to 0. We can therefore take a pointed limit based at $p_i \in T(r_i)$, and conclude that $h = 0$ if $r > r_i$. By analyticity of infinitesimal Einstein deformations, this leads us to conclude that h is identically 0, giving us our contradiction. Thus, we shall assume that $r_i \longrightarrow \infty$. This gives us that the limiting manifolds (M_{BH}, g_{BH}) are complete. We are going to work at an infinite distance from the conformal infinity of M_σ, so we can drop the weight factor δ. It will also be understood that we are working on the lifted manifolds, so we will suppress the tildes.

At this point, we would like to say that since $h_i(r_i) \longrightarrow 0$ and $r_i \longrightarrow \infty$, this forces our limiting sequence to have

$$\lim_{r \to \infty} ||h|| = 0 \tag{4.57}$$

We could then apply the following results to get our contradiction:

Proposition 28. *h is tangent to the space of T^{n-1}-invariant AHE metrics on M*

Proof: This is nontrivial, since spaces of AHE metrics are infinite-dimensional, and so vector fields do not necessarily integrate. By [And3], however, we know that if the AHE manifold (M, g) has a C^2 conformal compactification, then infinitesimal deformations do indeed integrate. The function $\rho = r^{-1}$ is certainly 0 exactly at the conformal infinity of (M_{BH}, g_{BH}), and with respect to the compactified metric,

$$|d\rho|^2_{\bar{g}} = \left|\frac{dr}{r^2}\right|^2_{\bar{g}} = \frac{V(r)}{r^2} \tag{4.58}$$

is nonzero on the boundary, so ρ is a defining function. Near the boundary, the compactified metric is

$$\bar{g} = \frac{dr^2}{r^2 V(r)} + \frac{V(r)}{r^2} d\theta^2 + g_{T^{n-2}} \tag{4.59}$$

One may replace r by the coordinate s, where

$$\frac{ds}{dr} = \frac{1}{r\sqrt{V(r)}} \tag{4.60}$$

To get

$$\bar{g} = ds^2 + F(s) d\theta^2 + g_{T^{n-2}} \tag{4.61}$$

and a short calculation shows that F is C^2 up to the boundary. ∎

Proposition 29. *([And1], c.f. also [BB]) Let g be a complete T^{n-1}-invariant AHE metric on the solid torus $D^2 \times T^{n-2}$. Then g is a black hole metric.*

Proof: All such metrics are covered by $S^1 \times \mathbb{R}^{n-2}$-invariant metrics on $D^2 \times \mathbb{R}^{n-2}$. All we need to do is to show that this cover is the universal covering of the black hole metrics. The analysis (relegated to the end of the chapter) shows that up to rescaling, there is only one negative scalar curvature $S^1 \times \mathbb{R}^{n-2}$-invariant Einstein metric on $D^2 \times \mathbb{R}^{n-2}$, which must therefore be the black hole metric. ∎

Proposition 30. *Let g_t be a curve of complete T^{n-1}-invariant AHE metric on the solid torus $D^2 \times T^{n-2}$. Then g_t is completely determined by g_0 and the curve γ_t consisting of the conformal infinities of the g_t's.*

Proof: By the previous proposition, all the g_t's are covered by $(D^2 \times$

$\mathbb{R}^{n-2}, \tilde{g}_{BH})$. Thus,

$$g_t = \frac{\tilde{g}_{BH}}{\Gamma_t} \tag{4.62}$$

where $\Gamma_t \subset Isom(g_{BH})$ is isomorphic to $\mathbb{Z}^{n-2}$. Consider the "defining function" $\rho = r^{-1}$ for $\tilde{g}_{BH}$(note that $(D^2 \times \mathbb{R}^{n-2}, \rho^2 \tilde{g}_{BH})$ is not a compact manifold with boundary.)

Then the conformal infinity of $(D^2 \times \mathbb{R}^{n-2}, \rho^2 \tilde{g}_{BH})$ is a flat $S^1 \times \mathbb{R}^{n-2}$. The action of Γ_t commutes with multiplication by ρ, so the conformal infinity of g_t is the quotient of the flat $S^1 \times \mathbb{R}^{n-2}$ by Γ_t. Conversely, the conformal infinity γ_t of g_t also determines the group $G_t = \Gamma_t + v_t$ up to conjugacy, where

$$\gamma_t = \frac{\mathbb{R}^{n-1}}{G_t} \tag{4.63}$$

Given an initial g_0, we can identify v_0 with the S^1 in the universal cover. This determines v_t for $t > 0$, and thus Γ_t determines g_t. ∎

Combining these propositions gives us the following corollary:

Corollary 31. *A T^{n-1}-invariant AHE metric on a solid torus has no non-trivial infinitesimal deformations for which*

$$\lim_{r \to \infty} |h(r)| = 0 \tag{4.64}$$

Proof: Since it integrates, it is tangent to curve of deformations. From the invariance, we know that this curve must lie in the space of black hole metrics. Since $\lim_{r\to\infty} |h(r)| = 0$ it must be a curve which fixes conformal infinities. Thus it is a constant curve, and so $h \equiv 0$. ∎

Unfortunately(for us,) we cannot apply these propositions directly; the problem comes from the fact that the h_i's tend to h uniformly on compact subsets, and we cannot relate the rate of convergence to the size of the compact set. Thus, we cannot know that r_i is included in each set. Consider the following example: Let $I_k = [-k, k]$, and

$$\begin{aligned} f_k : I_k &\longrightarrow [-1, 1] \\ x &\mapsto \frac{x}{k} \end{aligned}$$

Clearly, $\|f_k'(x)\|_{L^\infty} \longrightarrow 0$ as $k \longrightarrow \infty$, so one would expect that the f_k's are tending toward a constant function. This constant may, however, depend on the basepoint x_k; let $x_k = \alpha k$, where $-1 < \alpha < 1$. Then the (pointed) Gromov-Hausdorff limit of the triple $(I_k, x_k, f_k(x))$ will be the triple $(\mathbb{R}, 0, \alpha)$. Thus the limit of the f_k is indeed a constant function, but the constant depends on the choice of the basepoints x_k.

The issue here is that the manifolds I_k are converging to their limit faster than the f_k are converging to theirs, so the convergence cannot be made uniform on the whole set.

More precisely, we have that

$$\begin{aligned} |f_k(x) - f_k(y)| &\leq \left| \int_x^y f_k'(s)\, ds \right| && (4.65) \\ &\leq \|f_k'\|_{L^\infty} \left| \int_x^y ds \right| && (4.66) \\ &\leq \|f_k'\|_{L^\infty} |x - y| && (4.67) \end{aligned}$$

Therefore, if we choose points x_k, y_k whose distance is increasing fast enough, we cannot conclude that $f_k(x_k)$ and $f_k(y_k)$ have the same limit

On the other hand, if we require that $\|f_k'\|_{L^\infty} \longrightarrow 0$ more rapidly than any two points can separate, say by demanding that $diam(I_k)\|f_k'\|_{L^\infty} \longrightarrow 0$, then we can get that

$$\begin{aligned} |f_k(x) - f_k(y)| &\leq \|f_k'\|_{L^\infty}|x-y| & (4.68) \\ &\leq diam(I_k)\|f_k'\|_{L^\infty} \longrightarrow 0 & (4.69) \end{aligned}$$

no matter which basepoints we take.

Let us now attempt to adapt this argument to the operator L. As we will see below, on T^{n-1}-invariant deformations h, the components of $Lh = 0$ are asymptotic to Euler equations. For the components $(Lh)_{1j}$, this equation has a nontrivial 0th order term, so all of its solutions either blow up or decay to 0. Its other components are asymptotic to equations of the form

$$Lf = r^2 f'' + nrf' = 0 \tag{4.70}$$

which have constant solutions. This leads to a problem for us, since we could have the same situation as above; even though $f(r_i) \to 0$ and f tends to a constant, we cannot conclude that $f \longrightarrow 0$ everywhere. This is where the $\log(R_i)$ factor comes in.

By use of an integrating factor([BdP]), we may rewrite this as

$$f(r) = \int \frac{1}{r^n} \int s^{n-2} Lf(s)\, ds\, dr \tag{4.71}$$

Thus,

$$
\begin{aligned}
|f(r_1) - f(r_2)| &= \left| \int_{r_1}^{r_2} \frac{1}{r^n} \int_{r_+}^{r} s^{n-2} Lf(s)\, ds\, dr \right| && (4.72)\\
&\le \frac{\|Lf\|_{L^\infty}}{n-1} \left| \int_{r_1}^{r_2} r^{-1} + O\left(r^{1-n}\right)\, dr \right| && (4.73)\\
&\le \frac{\|Lf\|_{L^\infty}}{n-1} \left| \log\left(\frac{r_2}{r_1}\right) + C_0 \right| && (4.74)\\
&\le C_1 \|Lf\|_{L^\infty} \log(R_i) \longrightarrow 0 && (4.75)
\end{aligned}
$$

Our situation is a bit more delicate, since we are not working exactly with this operator, but with perturbations of it. Furthermore, since we need to use the rate at which the h_i's converge, we cannot just work with limits, but must rather get precise bounds on how things behave asymptotically.

We now want to analyze the system of ODE's that these deformations must satisfy. The proof of the following proposition consists of a long calculation, and is therefore relegated to the end of this chapter.

Proposition 32. *Say we have a black hole metric g. Let $e_1 = \sqrt{V}\partial_r$, $e_2 = \frac{1}{\sqrt{V}}\partial_{\phi_j}$, and $e_j = \frac{1}{r}\partial_{\phi_j}$, where the ∂_{ϕ_j}'s, $3 \le j \le n$ form an orthonormal basis* *for the core torus. Then if h is $S^1 \times T^{n-2}$-invariant, $Lh = 2D_g\Phi_g h$ is given by*

$$
\begin{aligned}
(Lh)_{11} &= Ah_{11} + h_{11}\left(\frac{(V')^2}{2V} + \frac{2(n-2)V}{r^2}\right) && (4.76)\\
&\quad -h_{22}\left(\frac{(V')^2}{2V} + 2K_{12}\right) && (4.77)\\
&\quad -2\sum_{k>2}\left(\left(\frac{V}{r^2}\right) + K_{1k}\right)h_{kk} && (4.78)\\
(Lh)_{22} &= Ah_{22} + h_{22}\frac{(V')^2}{2V} - h_{11}\left(\frac{(V')^2}{2V} + 2K_{12}\right) - 2\sum_{k>2} K_{2k}h_{kk} && (4.79)\\
(Lh)_{12} &= Ah_{12} + h_{12}\left(\frac{(V')^2}{V} + \frac{2(n-2)V}{r^2} + 2K_{12}\right) && (4.80)
\end{aligned}
$$

where

$$Ah_{ij} = \left(-Vh''_{ij} - \left(V' + \frac{n-2}{r}V\right)h'_{ij}\right) \tag{4.81}$$

If $j > 2$, we have

$$
\begin{aligned}
(Lh)_{jj} &= Ah_{jj} + \frac{2V}{r^2}(h_{jj} - h_{11}) - 2\sum_{k\neq j} K_{kj}h_{kk} && (4.82)\\
(Lh)_{1j} &= Ah_{1j} + h_{1j}\left(\frac{(V')^2}{4V} + \frac{(n+1)V}{r^2} + 2K_{1j}\right) && (4.83)\\
(Lh)_{2j} &= Ah_{2j} + h_{2j}\left(\frac{(V')^2}{4V} + \frac{V}{r^2} + 2K_{2j}\right) && (4.84)
\end{aligned}
$$

and finally, if $i, j > 2$, we get

$$(Lh)_{ij} = Ah_{ij} + h_{ij}\left(\frac{(V')^2}{2V} + 2K_{ij}\right) \tag{4.85}$$

This system seems unmanageable for the black hole metrics, but we can get around this by noting the following two facts:

Proposition 33. *Let g_C be a complete hyperbolic cusp metric*

$$g_C = r^{-2}dr^2 + g_{T^{n-1}} \tag{4.86}$$

where $g_{T^{n-1}}$ is an arbitrary flat metric on the torus with orthonormal basis ∂_{ϕ_j}, $2 \leq j \leq n$. Let $\left(e_1 = r\partial_r, e_j = \frac{\partial_{\phi_j}}{r}; j \geq 2\right)$ form an orthonormal frame for g_C. Then if h is T^{n-1}-invariant, $L_C h = L_{g_C} h$ is given by the following formulas.

$$(Lh)_{11} = Ah_{11} + 2(n-1)h_{11} \tag{4.87}$$

and if $j, k > 2$,

$$(Lh)_{jj} = Ah_{jj} + 2trh - 2h_{11} \tag{4.88}$$

$$(Lh)_{1j} = Ah_{1j} + nh_{1j} \tag{4.89}$$

$$(Lh)_{jk} = Ah_{jk} \tag{4.90}$$

where $A = -r^2\partial_r^2 - nr\partial_r$.

Proof: Set $V(r) = r^2$ and $K_{jk} = -1 + \delta_j^k$ above. ∎

Now that we have the much simpler form of L for g_C, we must relate it to the corresponding operator on the black hole metrics. We will use C^k estimates instead of Hölder ones, since they are easy to establish, and we do not need the stronger norms; we already have the existence of the limit form h. The C^k norms will be calculated in the same harmonic coordinates as the $S_\delta^{k,\alpha}$ ones.

Proposition 34. *If $\|h\|_{C^2}$ is bounded and T^{n-1}-invariant, then*

$$\|L_C h - L_{BH} h\|_{L^\infty} = O(r^{-(n-1)}) \tag{4.91}$$

Proof: This can be seem by inspection of the formulas, remarking that

$$V(r) = r^2 - \frac{2}{r^{n-3}} \tag{4.92}$$

Thus, if K_{jk} represent the sectional curvatures of the metric,

$$|1 + K_{jk}| = O(r^{-(n-1)}) \tag{4.93}$$

$$\left|1 - \frac{(V')^2}{4V}\right| = O(r^{-(n-1)}) \tag{4.94}$$

$$\left|1 - \frac{V}{r^2}\right| = O(r^{-(n-1)}) \tag{4.95}$$

∎

Thus, if h is T^{n-1}-invariant and bounded in C^2, we get the following systems of equations for $Lh = 0$:

$$Ah_{11} + 2(n-1)h_{11} = u_{11} \tag{4.96}$$

$$Ah_{jj} + 2trh - 2h_{11} = u_{jj} \tag{4.97}$$

$$Ah_{1j} + nh_{1j} = u_{1j} \tag{4.98}$$

$$Ah_{ij} = u_{ij} \tag{4.99}$$

where $i, j > 1$, and $|u| = O(r^{-(n-1)})$

Recall that we want to prove that

$$\lim_{r\to\infty} |h(r)| = 0 \tag{4.100}$$

The above system is uncoupled, apart from the h_{11} and the trh in the equation for the diagonal terms. We will begin by showing that both of these terms are $O(r^{-(n-1)})$.

The equation for h_{11} is a nonhomogeneous Euler equation, with indicial equation

$$\alpha(\alpha - 1) + n\alpha - 2(n-1) = 0 \tag{4.101}$$

Which has roots

$$\alpha_1 = \frac{-(n-1) - \sqrt{(n-1)^2 + 8(n-1)}}{2} \tag{4.102}$$

$$\alpha_2 = \frac{-(n-1) + \sqrt{(n-1)^2 + 8(n-1)}}{2} \tag{4.103}$$

$$\tag{4.104}$$

Note that

$$\alpha_1 \;<\; -(n-1) \tag{4.105}$$

$$0 \;<\; \alpha_2 \tag{4.106}$$

Then, by variation of parameters, the general solution is

$$h_{11} = c_1 r^{\alpha_1} + c_2 r^{\alpha_2} + \frac{1}{\alpha_2 - \alpha_1}\left(r^{\alpha_1}\int u_{11}(r^{-1-\alpha_1})\,dr - r^{\alpha_2}\int u_{11}(r^{-1-\alpha_2})\,dr\right) \tag{4.107}$$

It is clear that the first, third and fourth terms are $O(r^{-(n-1)})$. Since h_{11} is bounded, $c_2 = 0$. Thus,

$$h_{11} = O\left(r^{-(n-1)}\right) \tag{4.108}$$

We can obtain an equation for trh by adding those for the $(Lh)_{jj}$'s. This gives us the equation

$$A(trh) + 2(n-1)trh = \sum_{i=1} u_{ii} \tag{4.109}$$

which is of exactly the same form as that for h_{11}. Thus, we can conclude that $trh = O\left(r^{-(n-1)}\right)$.

These two results give us that the equation for h_{ij}, with neither i or j equal to 1, is

$$Ah_{ij} = O\left(r^{-(n-1)}\right) \tag{4.110}$$

We will deal with this in a second, but first, we must examine the equation for h_{1j}, $j > 1$;

$$r^2 h_{1j} + nrh_{1j} - nh_{1j} = u_{1j} \tag{4.111}$$

This is once again a nonhomogeneous Euler equation, with indicial equation

$$\alpha^2 + (n-1)\alpha - n = 0 \tag{4.112}$$

This gives the two roots

$$\alpha_1 = \frac{-(n-1) + \sqrt{(n-1)^2 + 4n}}{2} = 1 \tag{4.113}$$

$$\alpha_2 = \frac{-(n-1) - \sqrt{(n-1)^2 + 4n}}{2} = -n \tag{4.114}$$

Thus, the general solution is

$$h_{1j} = c_1 r + c_2 r^{-n} + \frac{-1}{n+1}\left(r\int u_{1j}(r^{-2})\,dr - r^{-n}\int u_{1j}(r^{n+1})\,dr\right) \quad (4.115)$$

Once again, since $u_{1j} = O(r^{-(n-1)})$ and h_{1j} is bounded, we see that $h_{1j} = O(r^{-(n-1)})$.

Thus, all that is left is to examine the equations for h_{ij}, $i, j > 1$. Although the equation that they satisfy seems to be the simplest of the ones that we have looked at, the h_{ij}'s are in fact the most subtle case. As we mentioned above, the issue is that the equation $Ah_{ij} = 0$ is an Euler equation with no 0th order term. Therefore, there are constant solutions, which neither blow up nor go to zero as $r \to \infty$.

Since we want to invoke the rate at which Lh_i tends toward 0, we will now be working with the h_i's rather than h. Thus, we will need to quantify the rate at which the h_i's are converging to their T^{n-1}-invariant limit.

Let

$$\hat{h}_i(r) = \frac{1}{A(T(r))}\int_{T(r)} h_i(r,x)dA \quad (4.116)$$

be the average of h over the torus $T(r)$. Then we have

Proposition 35. $\|h_i - \hat{h}_i(r)\|_{C^2} = O\left(\frac{r}{R_i}\right)$

Proof: We have

$$\sup_{x\in T(r)} |h_i(r,x) - \hat{h}_i(r)| \leq \frac{1}{A(T(r))}\int_{T^{n-1}(r)} |h_i(r,x) - \hat{h}_i(r)|\, dA \quad (4.117)$$

Now, we know that h_i is the lift of a form defined on a torus of diameter

$O\left(\frac{r}{R_i}\right)$. Since $\|h_i\|_{k,\alpha} \leq C$, we know that the integrand must be less than C times the diameter of the base torus, so it is also $O\left(\frac{r}{R_i}\right)$. We have assumed that $k \geq 3$, so we can repeat this for the first and second derivatives of h_i. ∎

We know that on our unwrapped black hole metrics, we have

$$L(\hat{h}) = L_C(\hat{h}) + O(r^{-(n-1)}) \tag{4.118}$$

Finally, by assumption,

$$\begin{aligned} L(h) &= o\left(\frac{1}{\log|\sigma|}\right) && (4.119) \\ &= o\left(\frac{1}{\log\left(\min_k L(\sigma^k)\right)}\right) && (4.120) \\ & && (4.121) \end{aligned}$$

where the minimum in the last line is taken over all the cusps of $(M_{\sigma_i}, g_{\sigma_i})$. The problem here is that we need to get a bound involving the R_i of the particular cusp end we are working on. The hypothesis that σ is weakly balanced implies that

$$\log\left(\max_k L(\sigma_i^k)\right) \leq c_0 \log|\sigma_i| \tag{4.122}$$

which implies that

$$L(h_i) = o\left(\frac{1}{\log(R_i)}\right) \tag{4.123}$$

Putting this all together gives us

$$
\begin{aligned}
L_C(\hat{h}_i) &= L_{BH}(\hat{h}_i) + O\left(r^{-(n-1)}\right) && (4.124)\\
&= L_{BH}(h_i) + O\left(r^{-(n-1)}\right) + O\left(\frac{r}{R_i}\right) && (4.125)\\
&= o\left(\frac{1}{\log R_i}\right) + O\left(r^{-(n-1)}\right) + O\left(\frac{r}{R_i}\right) && (4.126)
\end{aligned}
$$

Now, dropping the i's and the hats, we see that if $a, b > 1$

$$r^2 h''_{ab} + nrh'_{ab} = e_{ab} \tag{4.127}$$

where

$$e_{ab} = O\left(\frac{r}{R_i}\right) + O\left(r^{-(n-1)}\right) + o\left(\frac{1}{\log R_i}\right) \tag{4.128}$$

Recall that we know that $\lim_{i\to\infty} h_i(r_i) = 0$. We want to show that this is true for any sequence $\rho_i \leq r_i$ with $\rho_i \to \infty$.

Using an integrating factor, we get

$$
\begin{aligned}
|h_{ab}(r_i) - h_{ab}(\rho_i)| &\leq \int_{\rho_i}^{r_i} \frac{1}{r^n} \int_{r+}^{r} |e_{ab}(s)| s^{n-2}\, ds dr && (4.129)\\
&\leq \int_{\rho_i}^{r_i} \frac{1}{r^n} \left(C_1 \int_{r+}^{r} \frac{s^{n-1}}{R_i}\, ds + C_2 \int_{r+}^{r} s^{-1}\, ds \right. && (4.130)\\
&\qquad \left. + c_i \int_{r+}^{r} \frac{s^{n-2}}{\log R_i}\, ds \right) dr && (4.131)
\end{aligned}
$$

where $c_i \to 0$. Then this is

$$\leq \ C_3 \frac{r_i}{R_i} + C_4 r_i^{-(n-1)} + c_i \frac{\log r_i}{\log R_i} \tag{4.132}$$

Thus, we can conclude that

$$\lim_{r\to\infty} |h(r)| = 0 \tag{4.133}$$

So by corollary (31) we finally have our contradiction, and therefore the main estimate. ∎

Finally, to finish this chapter, we extend the invertibility of $L_g = 2D_g\Phi_{g_\sigma}$ to a neighborhood of our approximate solution. Below, $B(x,\epsilon)$ will refer to a ball in the $S_\delta^{k,\alpha}$-topology.

Proposition 36. *There exist $\epsilon > 0, \Lambda > 0$ such that for all σ large enough and weakly balanced, the operator L_g is invertible on the ball $B(g_\sigma, \epsilon)$, and for all $f \in \Phi(B(g_\sigma, \epsilon))$, we have that*

$$\|(L_g)^{-1} f\|_{k,\alpha,\delta} \leq \Lambda \log|\sigma| \|f\|_{k-2,\alpha,\delta} \tag{4.134}$$

Proof: If not, there is a sequence of g_i's and σ_i's with $\|g_i - g_{\sigma_i}\|_{k,\alpha,\delta} \to 0$, and a sequence $h_i \in S_\delta^{k,\alpha}(M_i, g_i)$ such that

$$\|h_i\|_{k,\alpha,\delta} = 1 \tag{4.135}$$

but

$$\log|\sigma_i| \|L_{g_i} h_i\|_{k-2,\alpha,\delta} \longrightarrow 0 \tag{4.136}$$

But then we can repeat the proof of the previous proposition to obtain a contradiction. ∎

Now, as an appendix to this chapter, we will finish the uniqueness proof for the black hole metrics and perform the calculation of the equations satisfied by their T^{n-1}-invariant infinitesimal deformations.

Rest of Proof of Prop 29: Let g be a complete $S^1 \times \mathbb{R}^{n-2}$-invariant Einstein metric on $D^2 \times \mathbb{R}^{n-2}$ with scalar curvature $s = -n(n-1)$. Then we can take coordinates $(r, \phi_1, ..., \phi_{n-1})$ such that

$$g = dr^2 + \sum_{i=1}^{n-1} f_i^2(r) d\phi_i^2 \tag{4.137}$$

This choice of coordinates is somewhat awkward, since ϕ_1 is a circular coordinate, while all the other ϕ_i's range over all real numbers. Nonetheless, this choice will keep our formulas simple. Note that we can assume each factor has a warped product form because the intrinsic metric on each one is flat. We will associate the index 0 to the r coordinate.

As in our proof of the 2π-theorem, it is easy to see that in these coordinates

$$K_{0j} = -\frac{f_j''}{f_j} \tag{4.138}$$

and if $j, k > 0$, $j \neq k$,

$$K_{jk} = -\frac{f_j' f_k'}{f_j f_k} \tag{4.139}$$

Then $ric = -(n-1)g$ leads to the equations

$$\sum_{j=1}^{n-1} \frac{f_j''}{f_j} = n - 1 \tag{4.140}$$

and if $k > 0$,

$$\frac{f_k''}{f_k} + \sum_{j|j\neq k} \frac{f_j' f_k'}{f_j f_k} = n - 1 \tag{4.141}$$

Let us define $v_i = \frac{f_i'}{f_i}$ and $u = \sum v_i$. Then

$$v_i' = \frac{f_i''}{f_i} - \left(\frac{f_i'}{f_i}\right)^2 \quad (4.142)$$

$$= \frac{f_i''}{f_i} - v_i^2 \quad (4.143)$$

$$(4.144)$$

and so

$$u' = \sum_j \left(\frac{f_j''}{f_j} - v_j^2\right) \quad (4.145)$$

and the equations for $k > 0$ become

$$v_k' + uv_k = n - 1 \quad (4.146)$$

If we add all of the equations for $k > 0$, we get

$$u' + u^2 = (n-1)^2 \quad (4.147)$$

This equation has the trivial solutions $u = \pm(n-1)$. In this case, we find that the v_k's satisfy the equation

$$v_k' \pm (n-1)v_k = (n-1) \quad (4.148)$$

which has the solution

$$v_k = \pm 1 + c_k e^{\mp(n-1)r} \quad (4.149)$$

Since $v_k = (\log f_k)'$, this gives us

$$\log f_k = \pm r \mp \frac{c_k}{n-1} e^{\mp r} + A_k \tag{4.150}$$

Now, recall that we want the ϕ_1 coordinates to bound a disk, so we must have $\lim_{r \to 0} f_1(r) = 0$. But no matter how we choose A_1 or c_1, we cannot make this happen. Thus $u \neq \pm(n-1)$.

Assuming u is not identically $n-1$, we get

$$\frac{du}{(n-1)^2 - u^2} = dr \tag{4.151}$$

Partial fractions or a hyperbolic substitution lead to

$$u = (n-1)\coth\left((n-1)(r - r_0)\right) \tag{4.152}$$

We may set $r_0 = 0$ by allowing the r coordinate to range from r_+ to ∞, where r_+ will be specified below. Then we get the following equations for each $k > 0$:

$$v_k' + u v_k = (n-1) \tag{4.153}$$

or

$$v_k' + (n-1)\coth\left((n-1)r\right) v_k = (n-1) \tag{4.154}$$

Using an integrating factor, this becomes

$$\left(\sinh\left((n-1)r\right) v_k\right)' = (n-1)\sinh\left((n-1)r\right) + c_k \tag{4.155}$$

or

$$v_k = \coth\left((n-1)r\right) + c_k \operatorname{csch}\left((n-1)r\right) \tag{4.156}$$

As above, $v_k = (\log(f_k))'$, so integration gives

$$\begin{aligned} \log(f_k) &= \frac{1}{n-1}\log\left(\sinh\left((n-1)r\right)\right) & (4.157)\\ &-\frac{c_k}{n-1}\log\left|\operatorname{csch}\left((n-1)r\right) + \coth\left((n-1)r\right)\right| + A_k & (4.158)\end{aligned}$$

Finally, we get

$$f_k = B_k \left(\frac{\sinh^{1-c_k}\left((n-1)r\right)}{\left(1+\cosh\left((n-1)r\right)\right)^{c_k}}\right)^{\frac{1}{n-1}} \tag{4.159}$$

The B_k's can be absorbed into the $d\phi$'s by rescaling, so they all give equivalent metrics.

We want the ϕ_1 factor to bound a disk, so we must have $f_1 \to 0$ as $r \to r_+$. This can only occur if $r_+ = 0$ and $c_1 < -1$. For the metric to be smooth, we must also have that $f_1'(0)$ is well-defined.

$$f_1'(r) = v_1(r) f_1(r) \tag{4.160}$$

which is equal to

$$\left(\coth((n-1)r) + c_1 \operatorname{csch}((n-1)r)\right)\left(\frac{\sinh^{1-c_1}((n-1)r)}{1+\cosh((n-1)r))^{c_1}}\right)^{\frac{1}{n-1}}, \tag{4.161}$$

so for this to stay bounded as $r \to 0$, the exponent of the $\sinh((n-1)r)$ must be equal to 1. This gives us

$$\frac{1-c_1}{n-1} = 1 \tag{4.162}$$

so $c_1 = -(n-2)$. To determine the other c_k's, note that all the curvatures must be bounded at $r = 0$ and

$$\begin{aligned}
K_{1k} &= v_1 v_k && (4.163)\\
&= \coth^2((n-1)r) && (4.164)\\
&\quad +(c_1+c_k)\coth((n-1)r)\operatorname{csch}((n-1)r) && (4.165)\\
&\quad\quad +c_1 c_k \operatorname{csch}^2((n-1)r) && (4.166)\\
&= \frac{\cosh^2((n-1)r)+(c_1+c_k)\cosh((n-1)r)+c_1c_k}{\sinh^2((n-1)r} && (4.167)
\end{aligned}$$

Now, one can see by taking a Maclaurin series expansion that necessary condition for this to be bounded is that

$$1 + c_1 + c_k + c_1 c_k = 0 \tag{4.168}$$

or

$$c_k(c_1+1) = -(c_1-1) \tag{4.169}$$

Since $c_1 \neq 1$, we get that $c_k = -1$ if $k > 1$. Now, we have proved that if there exists an $S^1 \times \mathbb{R}^{n-2}$-invariant Einstein metric on $D^2 \times \mathbb{R}^{n-2}$ with $s = -n(n-1)$, it must be unique. Thus it must be isometric to $\tilde{g}_{BH}$. ∎

Proof of Proposition 32: Recall that

$$Lh = D^*Dh - 2Rh \tag{4.170}$$

where

$$D^*Dh = -tr(D(Dh)) = -\sum_k \left(\nabla_{e_k}\nabla_{e_k}h - \nabla_{\nabla_{e_k}e_k}h\right) \tag{4.171}$$

and

$$Rh(X,Y) = \sum_k h(R(X,e_k)Y, e_k) \tag{4.172}$$

We will also need the following covariant derivatives: if $j,k > 2$ and $f = \sqrt{V}$, we have

$$\nabla_{e_2}e_j = \nabla_{e_j}e_2 = 0 \tag{4.173}$$

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = \nabla_{e_1}e_j = 0 \tag{4.174}$$

$$\nabla_{e_2}e_2 = -f'e_1 \tag{4.175}$$

$$\nabla_{e_j}e_k = -\delta_j^k \frac{f}{r} e_1 \tag{4.176}$$

$$\nabla_{e_2}e_1 = f'e_2 \tag{4.177}$$

$$\nabla_{e_j}e_1 = \frac{f}{r} e_j \tag{4.178}$$

where

$$f = \sqrt{V} \tag{4.179}$$

Therefore,

$$\nabla_{e_2} e_j^* = \nabla_{e_j} e_2^* = 0 \tag{4.180}$$

$$\nabla_{e_1} e_1^* = \nabla_{e_1} e_2^* = \nabla_{e_1} e_j^* = 0 \tag{4.181}$$

$$\nabla_{e_2} e_2^* = f' e_1^* \tag{4.182}$$

$$\nabla_{e_j} e_k^* = \delta_j^k \frac{f}{r} e_1^* \tag{4.183}$$

$$\nabla_{e_2} e_1^* = -f' e_2^* \tag{4.184}$$

$$\nabla_{e_j} e_1^* = -\frac{f}{r} e_j^* \tag{4.185}$$

We will calculate D^*Dh explicitly using the Leibniz rule; let

$$h = \sum_{a,b} h_{ab} e_a^* e_b^* \tag{4.186}$$

so

$$D^*Dh = \sum_{a,b} D^*D\left(h_{ab} e_a^* e_b^*\right) \tag{4.187}$$

We have that

$$D^*D\left(h_{ab}e_a^*e_b^*\right) = -\sum_k \left(\nabla_{e_k}\nabla_{e_k}\left(h_{ab}e_a^*e_b^*\right)\right) + \sum_k \nabla_{\nabla_{e_k}e_k}\left(h_{ab}e_a^*e_b^*\right) \tag{4.188}$$

$$= -\sum_k \left((e_k e_k h_{ab})e_a^*e_b^* + 2(e_k h_{ab})\nabla_{e_k}(e_a^*e_b^*) + h_{ab}\nabla_{e_k}\nabla_{e_k}(e_a^*e_b^*)\right) \tag{4.189}$$

$$-\left(f' + (n-2)\frac{f}{r}\right)\nabla_{e_1}\left(h_{ab}e_a^*e_b^*\right) \tag{4.190}$$

Since the h_{ab}'s only depend on r, this gives

$$= -(e_1 e_1)h_{ab}e_a^* e_b^* - 2(e_1 h_{ab})\nabla_{e_1}(e_a^* e_b^*) - \sum_k (h_{ab}\nabla_{e_k}\nabla_{e_k}(e_a^* e_b^*)) \quad (4.191)$$

$$- \left(f' + (n-2)\frac{f}{r}\right)((e_1 h_{ab})e_a^* e_b^* + h_{ab}\nabla_{e_1}(e_a^* e_b^*)) \quad (4.192)$$

Since $\nabla_{e_1} e_a^* = 0 \ \ \forall a$, we get

$$- \left((e_1 e_1 h_{ab}) + \left(f' + (n-2)\frac{f}{r}\right)(e_1 h_{ab})\right) e_a^* e_b^* \quad (4.193)$$

$$-h_{ab}\sum_k ((\nabla_{e_k}\nabla_{e_k} e_a^*)e_b^* + 2(\nabla_{e_k} e_a^*)(\nabla_{e_k} e_b^*) + e_a^*(\nabla_{e_k}\nabla_{e_k} e_b^*)) \quad (4.194)$$

The first term is

$$= \ -\left(f\partial_r(f\partial_r h_{ab}) + \left(f' + (n-2)\frac{f}{r}\right) f\partial_r h_{ab}\right) e_a^* e_b^* \quad (4.195)$$

$$= \ -\left(f^2 h''_{ab} + ff'h'_{ab} + ff'h'_{ab} + (n-2)\frac{f^2}{r}h'_{ab}\right) e_a^* e_b^* \quad (4.196)$$

$$= \ A(h_{ab})e_a^* e_b^* \quad (4.197)$$

where

$$A = -f^2\partial_r^2 - \left(2ff' + (n-2)\frac{f^2}{r}\right)\partial_r \quad (4.198)$$

To calculate the second part of $D^*D(h_{ab}e_a^* e_b^*)$, we must consider different cases corresponding to different values of a and b.

$$\sum_k \nabla_{e_k}\nabla_{e_k} e_1^* = -(f')^2 e_1^* - \sum_{k>3}\left(\frac{f}{r}\right)^2 e_1^* \tag{4.199}$$

$$= -\left((f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right) e_1^* \tag{4.200}$$

$$\sum_k (\nabla_{e_k}\nabla_{e_k} e_2^*) = -(f')^2 e_2^* \tag{4.201}$$

and finally, if $j > 3$, this gives us

$$\sum_k (\nabla_{e_k}\nabla_{e_k} e_j^*) = -\left(\frac{f}{r}\right)^2 e_j^* \tag{4.202}$$

Thus, exploiting the symmetry of h, we get

$$D^*D(h_{11}e_1^*e_1^*) = (Ah_{11})e_1^*e_1^* \tag{4.203}$$

$$-2h_{11}\left(\sum_k (\nabla_{e_k}\nabla_{e_k} e_1^*)e_1^* + (\nabla_{e_k} e_1^*)(\nabla_{e_k} e_1^*)\right) \tag{4.204}$$

$$= (Ah_{11})e_1^*e_1^* \tag{4.205}$$

$$+2h_{11}\left((f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right) e_1^*e_1^* \tag{4.206}$$

$$-2h_{11}\left((f')^2 e_2^*e_2^* + \left(\frac{f}{r}\right)^2 \sum_{k>2} e_k^*e_k^*\right) \tag{4.207}$$

$$D^*D(h_{22}e_2^*e_2^*) = (Ah_{22})e_2^*e_2^* \tag{4.208}$$

$$-2h_{22}\left(\sum_k (\nabla_{e_k}\nabla_{e_k} e_2^*)e_2^* + (\nabla_{e_k} e_2^*)(\nabla_{e_k} e_2^*)\right) \tag{4.209}$$

$$= (Ah_{22})e_2^*e_2^* + 2h_{22}(f')^2 e_2^*e_2^* - 2h_{22}(f')^2 e_1^*e_1^* \tag{4.210}$$

and if $j > 2$,

$$\begin{aligned}
D^*D(h_{jj}e_j^*e_j^*) &= (Ah_{jj})e_j^*e_j^* && (4.211)\\
&\quad -2h_{jj}\left(\sum_k(\nabla_{e_j}\nabla_{e_k}e_j^*)e_j^* + (\nabla_{e_k}e_j^*)(\nabla_{e_k}e_j^*)\right) && (4.212)\\
&= (Ah_{jj})e_j^*e_j^* + 2h_{jj}\left(\frac{f}{r}\right)^2 e_j^*e_j^* && (4.213)\\
&\quad -2h_{jj}\left(\frac{f}{r}\right)^2 e_1^*e_1^* && (4.214)
\end{aligned}$$

For the off-diagonal terms, we have

$$\begin{aligned}
D^*D(h_{12}e_1^*e_1^*) &= (Ah_{12})e_1^*e_2^* && (4.215)\\
&\quad -h_{12}\sum_k((\nabla_{e_k}\nabla_{e_k}e_1^*)e_2^* && (4.216)\\
&\quad +2(\nabla_{e_k}e_1^*)(\nabla_{e_k}e_2^*) + e_1^*(\nabla_{e_k}\nabla_{e_k}e_2^*)) && (4.217)\\
& && (4.218)
\end{aligned}$$

which gives us

$$(Ah_{12})e_1^*e_2^* + h_{12}\left((f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right)e_1^*e_2^* + 2h_{12}(f')^2e_1^*e_2^* + h_{12}(f')^2e_1^*e_2^* \tag{4.219}$$

or equivalently

$$\left((Ah_{12}) + h_{12}\left(4(f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right)\right)e_1^*e_2^* \tag{4.220}$$

If $j > 2$, we get

$$
\begin{aligned}
D^*D(h_{1j}e_1^*e_j^*) &= (Ah_{1j})e_1^*e_j^* && (4.221)\\
&\quad -h_{1j}\sum_k((\nabla_{e_k}\nabla_{e_k}e_1^*)e_j^* + 2(\nabla_{e_k}e_1^*)(\nabla_{e_k}e_j^*) && (4.222)\\
&\quad +e_1^*(\nabla_{e_k}\nabla_{e_k}e_j^*)) && (4.223)\\
&= (Ah_{1j})e_1^*e_j^* && (4.224)\\
&\quad +h_{1j}\left((f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right)e_1^*e_j^* && (4.225)\\
&\quad +2h_{12}\left(\frac{f}{r}\right)^2 e_1^*e_j^* + h_{1j}\left(\frac{f}{r}\right)^2 e_1^*e_j^* && (4.226)
\end{aligned}
$$

which is equal to

$$
\left((Ah_{1j}) + h_{1j}\left((f')^2 + (n+1)\left(\frac{f}{r}\right)^2\right)\right)e_1^*e_j^* \qquad (4.227)
$$

and

$$
\begin{aligned}
D^*D(h_{2j}e_2^*e_j^*) &= (Ah_{2j})e_2^*e_j^* && (4.228)\\
&\quad -h_{2j}\sum_k((\nabla_{e_k}\nabla_{e_k}e_2^*)e_j^* + 2(\nabla_{e_k}e_2^*)(\nabla_{e_k}e_j^*) && (4.229)\\
&\quad +e_2^*(\nabla_{e_k}\nabla_{e_k}e_j^*)) && (4.230)\\
&= (Ah_{2j})e_2^*e_j^* + h_{2j}\left((f')^2 + \left(\frac{f}{r}\right)^2\right)e_2^*e_j^* && (4.231)
\end{aligned}
$$

Finally, if $i, j > 2, i \neq j$, we have

$$
D^*D(h_{ij}e_i^*e_j^*) = \left((Ah_{ij}) + 2h_{ij}\left(\frac{f}{r}\right)^2\right)e_i^*e_j^* \qquad (4.232)
$$

The reader will be glad to know that the calculation of Rh is much easier;

$$Rh(e_j, e_j) = \sum_k h(R(e_j, e_k)e_j, e_k) \tag{4.233}$$

$$= \sum_k K_{kj} h_{kk} \tag{4.234}$$

with the convention that $K_{jj} = 0$, and if $i \neq j$,

$$Rh(e_i, e_j) = \sum_k h(R(e_i, e_k)e_j, e_k) \tag{4.235}$$

$$= h(R(e_i, e_j)e_j, e_j) = -K_{ij} h_{ij} \tag{4.236}$$

We are now in a position to write out all of the components of Lh.

$$(Lh)_{11} = Ah_{11} + 2h_{11}\left((f')^2 + (n-2)\left(\frac{f}{r}\right)^2\right) \tag{4.237}$$

$$-2h_{22}\left((f')^2 + K_{12}\right) - 2\sum_{k>2}\left(\left(\frac{f}{r}\right)^2 + K_{1k}\right) h_{kk} \tag{4.238}$$

$$(Lh)_{22} = Ah_{22} + 2h_{22}(f')^2 - 2h_{11}\left((f')^2 + K_{12}\right) \tag{4.239}$$

$$-2\sum_{k>2} K_{2k} h_{kk} \tag{4.240}$$

$$(Lh)_{12} = Ah_{12} + h_{12}\left(4(f')^2 + (n-2)\left(\frac{f}{r}\right)^2 + 2K_{12}\right) \tag{4.241}$$

If $j > 2$, we have

$$(Lh)_{jj} = Ah_{jj} + 2h_{jj}\left(\frac{f}{r}\right)^2 - 2h_{11}\left(\frac{f}{r}\right)^2 - 2\sum_{k\neq j} K_{kj}h_{kk} \quad (4.242)$$

$$(Lh)_{1j} = Ah_{1j} + h_{1j}\left((f')^2 + (n+1)\left(\frac{f}{r}\right)^2 + 2K_{1j}\right) \quad (4.243)$$

$$(Lh)_{2j} = Ah_{2j} + h_{2j}\left((f')^2 + \left(\frac{f}{r}\right)^2 + 2K_{2j}\right) \quad (4.244)$$

and finally, if $i, j > 2$, we get

$$(Lh)_{ij} = Ah_{ij} + 2h_{ij}\left((f')^2 + K_{ij}\right) \quad (4.245)$$

Replacing f by $\sqrt{V}$ and noting that $(\sqrt{V})' = \frac{V'}{2\sqrt{V}}$ gives us Proposition 32

∎

Chapter 5

Conclusions

In this chapter, we conclude the construction of the AHE metrics on the M_σ's.

Proposition 37. *If $|\sigma|$ is large enough and σ is weakly balanced, then the manifolds M_σ admit an asymptotically hyperbolic Einstein manifold with the same conformal infinity as N.*

Proof: Let $\delta \in \left(\frac{n-1}{2}, n-1\right)$. There is some $\epsilon > 0$ such that on the ball $B(g_\sigma, \epsilon)$, the map

$$\Phi_{g_\sigma} : S_\delta^{k,\alpha} \longrightarrow S_\delta^{k-2,\alpha} \tag{5.1}$$

has an invertible linearization and

$$\|(D\Phi)^{-1} f\|_{k,\alpha,\delta} \leq \Lambda \log|\sigma| \|f\|_{k-2,\alpha,\delta} \tag{5.2}$$

Thus, by the inverse function theorem([Dieu]), Φ is invertible on $B(g_\sigma, \epsilon)$, and maps $B(g_\sigma, \epsilon)$ surjectively onto some $\mathcal{U} \subset S_\delta^{k-2,\alpha}$ containing $\Phi_{g_\sigma}(g_\sigma)$. All we need to show is that $0 \in \mathcal{U}$. To do this, we will need a lower bound on the diameter of $\mathcal{U}$.

Let $B(\Phi(g_\sigma),\gamma) \subset \mathcal{U}$. By our control on $(D\Phi)^{-1}$ we know that Φ^{-1} is Lipschitz with Lipschitz constant $\Lambda \log|\sigma|$. Therefore,

$$\Phi^{-1}(B(\Phi(g_\sigma),\gamma)) \subseteq B(g_\sigma, \Lambda \log|\sigma|\gamma) \tag{5.3}$$

Thus if we choose $\gamma = \frac{\epsilon}{\Lambda \log|\sigma|}$, we will obtain that $B(\Phi(g_\sigma),\gamma) \subseteq Im(\Phi)$. All that is left to do is to make sure that $0 \in B(\Phi(g_\sigma),\gamma)$ for σ large enough. But $\|\Phi(g_\sigma)\|_{k-2,\alpha,\delta} = O(|\sigma|^{1-n})$, so

$$\|\Phi(g_\sigma) - 0\|_{k-2,\alpha,\delta} \leq C|\sigma|^{1-n} \leq \frac{\epsilon}{\Lambda}\left(\frac{1}{\log|\sigma|}\right) \tag{5.4}$$

for $|\sigma|$ large enough, since ϵ and Λ are fixed. ∎

Abusing notation slightly, let us denote these AHE metrics by g_σ. Then we have that, for any sequence of points p_σ which remain within a bounded distance of a gluing torus

$$\lim_{|\sigma|\to\infty}(M_\sigma, g_\sigma, p_\sigma) = (N, g) \tag{5.5}$$

in any of the $S^{k,\alpha}_\delta$ topologies, where (N,g) is our original hyperbolic manifold.

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