A New Construction of Anti-self-dual 4-Manifolds

A Dissertation, Presented

by

Dan Moraru

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2004

Stony Brook University

The Graduate School

Dan Moraru

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Claude LeBrun
Professor, Department of Mathematics
Dissertation Director

Michael Anderson
Professor, Department of Mathematics
Chairman of Dissertation

Sorin Popescu
Associate Professor, Department of Mathematics

Martin Roček
Professor, Department of Physics
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation

A New Construction of Anti-self-dual 4-Manifolds

by

Dan Moraru

Doctor of Philosophy

in

Mathematics

Stony Brook University

2004

This dissertation describes a new construction of anti-self-dual metrics on four-manifolds. These metrics are characterized by the property that their twistor spaces project as affine line bundles over surfaces.

To any affine bundle with the appropriate sheaf of local translations we associate a solution of a second order partial differential equations system $D_2V = 0$ on a 5-dimensional manifold $\mathbf{Y}$. The solution V and its differential completely determine an anti-self-dual conformal structure on an open set in $\{V = 0\}$. This generalizes a previous construction of hyperkähler metrics introduced by Lindström and Roček.

We show how our construction applies in the specific case of conformal structures for which the twistor space $\mathcal{Z}$ has $\dim\left|-\frac{1}{2}K_{\mathcal{Z}}\right| \geq 2$, projecting thus over $\mathbb{CP}_2$ with twistor lines mapping onto plane conics. We give the precise form of the differential equations $D_2V = 0$ on the space of conics and we construct defining functions V for standard examples of metrics in this family.

Învăţătorilor Cornelia şi Ioan Opriş

Contents

Acknowledgements

This thesis would not have been possible without the support, advice and patience of all the ones who guided me in my quest.

The first on this list is my advisor, Claude LeBrun, who has led me through many interesting areas of geometry, patiently waiting for me to choose the ones that most interested me. He has taught me mathematics and above all, how to do mathematics. I have learned a lot under his guidance, and I have also enjoyed it.

The teachings and encouragement of Sorin Popescu have also contributed greatly, and were much appreciated. We had many interesting conversations regarding this thesis, and other mathematical and non-mathematical subjects.

I am grateful to Martin Roček for useful discussions concerning his work on hyperkähler metrics.

A special "thank you" is due to my former mathematics professors in Romania, especially Petre Simion, who started me on this trip almost 20 years ago.

I would also like to thank my friends for their support throughout these years, in Stony Brook, Ionut, Ioana, Rares, Olga, Rodrigo, and at home in Romania or around the world Maria, Adi, Nicu, Donatela and Toni.

I thank you my relatives, for being always close to me at least in heart if not in person.

Chapter 1

Introductory Remarks

The present work provides an explicit construction of a family of *anti-self-dual* conformal structures on four manifolds. In this introductory chapter we collect the basic definitions and results on anti-self-dual manifolds and their twistor spaces and we sketch the structure of this paper.

1.0.1

The choice of a conformal class of metrics on an oriented 4-dimensional manifold induces a splitting of the bundle of 2-forms into *self-dual* and *anti-self-dual* 2-forms:

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

In consequence the curvature operator $\mathcal{R}$ splits into

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W_- + \frac{s}{12} \end{array} \right)$$

where W_+ and W_- are the self-dual and respectively anti-self-dual components of the Weyl curvature. By definition, a metric is anti-self-dual if W_+ vanishes.

Twistor theory founded by Penrose [20] relates the study of anti-self-dual metrics to 3-dimensional complex geometry. The natural settings for this theory involve conformal classes of *complex-riemannian* four-manifolds.

Definition 1.0.1. *Let* X *be a complex manifold. A complex-riemannian metric* $\mathfrak{g}$ *on* X *is a holomorphic section of* $\odot^2 T^*\mathrm{X}$ *which is non-degenerate at any point.*

In local complex coordinates a complex-riemannian metric can be written as

$$\mathfrak{g} = \sum \mathfrak{g}_{ij} dz^i dz^j$$

with $\mathfrak{g}_{ij}$ holomorphic functions, $\det(\mathfrak{g}_{ij}) \neq 0$.

Such metrics arise naturally as analytic continuations of real-analytic pseudo-riemannian metrics and conversely, real metrics are obtained by restricting to the fixed-point set of an anti-holomorphic involution (complex conjugation) $\alpha : \mathrm{X} \to \mathrm{X}$ satisfying $\alpha^*\mathfrak{g} = \overline{\mathfrak{g}}$.

Two complex-riemannian metrics $\mathfrak{g}$ and $\mathfrak{h}$ are conformally equivalent if there exits a non-vanishing holomorphic function f such that $\mathfrak{h} = f \cdot \mathfrak{g}$.

In contrast to the real case, for an arbitrary chosen complex manifold the existence of a complex-riemannian metric is not automatic. This fact switches our interest to locally defined conformal equivalence classes of metrics.

Definition 1.0.2. *A* holomorphic conformal structure *on* X *is a holomorphic line subbundle* $\mathfrak{G}$ *of* $\odot^2 T^*\mathrm{X}$ *such that any non-vanishing local section in* $\mathfrak{G}$ *is a local complex-riemannian metric. Such a section is called a* local representative metric *of the conformal structure.*

A holomorphic tangent vector v is *null* with respect to the conformal structure $\mathfrak{G}$ if $\mathfrak{g}(v,v)=0$ for any local metric $\mathfrak{g}$ in $\mathfrak{G}$. All null vectors in $T_x\mathbf{X}$ are forming the *null cone* at $x\in\mathbf{X}$. By projectivization of the null cone at every point we obtain a holomorphic bundle $\coprod_{x\in\mathbf{X}}\mathfrak{Q}_x$ of nonsingular quadrics in $\mathbb{P}(T\mathbf{X})$ which determines uniquely the conformal structure.

We will assume in the following that $dim_{\mathbb{C}}\mathbf{X}=4$. In this case $\mathfrak{Q}_x$ is a quadric in $\mathbb{P}(T_x\mathbf{X})\cong\mathbb{P}^3$, so we obtain two families of null 2-planes in $T_x\mathbf{X}$, namely the α-planes and the β-planes. The distinction between the two can be made by the choice of an orientation on $\mathbf{X}$, which in complex-riemannian settings amounts precisely to an orthogonal direct sum decomposition of two forms into self-dual and anti-self-dual. An α-plane will be a null plane whose second exterior power is self-dual (similarly for a β-plane).

The distinction can be expressed also in relation to the locally defined *Spin* bundles $\mathbb{S}^+=\mathcal{O}^A$ and $\mathbb{S}^-=\mathcal{O}^{A'}$. We will henceforth assume that $\mathbf{X}$ admits a spin structure. In terms of the formula

$$T\mathbf{X}=\mathbb{S}^+\otimes\mathbb{S}^-$$

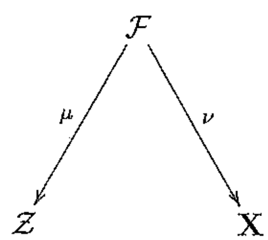
a null vector is given by a simple product $v^{AA'}=\omega^A\pi^{A'}$. An α-plane will be a plane of the form $\omega^A\otimes\mathbb{S}^-$ with $\omega^A\in\mathbb{S}^+\backslash\{0\}$ fixed, while a β-plane is of the form $\mathbb{S}^+\otimes\pi^{A'}$ with $\pi^{A'}\in\mathbb{S}^-\backslash\{0\}$ fixed [20].

1.0.2

A maximal totally geodesic isotropic surface in $\mathbf{X}$ whose tangent space at any point is an α-plane is called an α-surface.

The family of α-planes associated to a conformal structure is *integrable* if for any $x \in \mathbf{X}$ and for any α-plane in $T_x\mathbf{X}$ there exists an α-surface with the given tangent space at x. In terms of curvature, the family of α-planes is integrable if and only if the oriented conformal structure is anti-self-dual [20, 2].

Under these conditions, one defines $\mathcal{Z}$ - the *twistor space* of $\mathbf{X}$ - as the space of α-surfaces. Let $\mathcal{F} = \mathbb{P}(\mathbb{S}_+)$ denote the projective spin bundle. It can be proved [2] that $\mathcal{Z}$ is a 3-dimensional complex manifold such that μ in the double fibration



(1.1)

is a holomorphic map of maximal rank. Here μ naturally associates to a projective class $[\omega_A] \in \mathbb{P}(\mathbb{S}_+)_x$, the α-surface through x with tangent space $\omega^A \otimes \mathbb{S}^-$.

For any fixed point $x \in \mathbf{X}$, the set of α-surfaces through x will form a *complex twistor line* $L_x \cong \mathbb{P}_1$ with normal bundle $N_{L_x|\mathcal{Z}} \cong \mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)$.

We also point out the existence on $\mathcal{Z}$ of a naturally defined line bundle $\mathcal{O}_{\mathcal{Z}}(-1)$ of autoparallel tangent spinors, obtained as the direct image through μ of the universal bundle on $\mathcal{F} = \mathbb{P}(\mathbb{S}_+)$. Following [17], one obtains this way a fourth root of the canonical bundle of $\mathcal{Z}$. The notation derives from the fact that $\mathcal{O}_{\mathcal{Z}}(-1)$ restricts with degree -1 on the twistor lines. Even though the twistor space can be constructed without assuming that $\mathbf{X}$ is spin, the existence [2] of the $\mathcal{O}_{\mathcal{Z}}(-1)$ line bundle requires and is in fact equivalent to this condition.

1.0.3

One important aspect of this construction is that it is reversible [20]. Suppose that we start with $\mathcal{Z}$ a complex 3-manifold containing a rational curve $L \cong \mathbb{P}_1$ with normal bundle $N_L \cong \mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)$. We have that $H^1(L, N_L) = 0$ and $H^0(L, N_L) \cong \mathbb{C}^4$, so, following Kodaira's work [12], L belongs to a locally complete family $\mathcal{F} = \coprod_{x \in \mathbf{X}} L_x$ of lines in $\mathcal{Z}$ parametrized by a complex 4-dimensional manifold $\mathbf{X}$ with tangent space $T_x\mathbf{X}$ canonically isomorphic to $H^0(L_x, N_{L_x})$. Note that $H^1(L, End(N_L)) = 0$, so the normal bundle N_L is infinitesimally rigid [13] and thus we can assume that the normal bundle of any of the deformed lines L_x is still isomorphic to $\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)$.

Let $K_{\mathcal{Z}}$ denote the canonical bundle of $\mathcal{Z}$. The adjunction formula gives us that the restriction of $K_{\mathcal{Z}}$ to the line L is $K_{\mathcal{Z}}|_L \cong \mathcal{O}_{\mathbb{P}_1}(-4)$ so there exists at least locally (on a neighborhood of L) a fourth root line bundle $K_{\mathcal{Z}}^{-1/4}$ whose restriction to L is $\mathcal{O}_{\mathbb{P}_1}(1)$. Let $\mathbb{S}_x^- = H^0(L_x, K_{\mathcal{Z}}^{-1/4})$ and $\mathbb{S}_x^+ = H^0(L_x, N_{L_x} \otimes K_{\mathcal{Z}}^{1/4})$. The Künneth isomorphism

$$H^0(L_x, N_{L_x}) \cong H^0(L_x, K_{\mathcal{Z}}^{-1/4}) \otimes H^0(L_x, N_{L_x} \otimes K_{\mathcal{Z}}^{1/4})$$

determines the conformal structure on $\mathbf{X}$. The null vectors at x are given by those sections in $H^0(L_x, N_{L_x})$ that vanish somewhere on L_x. The twistor space coresponding to this conformal structure on $\mathbf{X}$ will be in general a neighborhood of the line L in $\mathcal{Z}$.

1.0.4

In order to recover the differential geometry of X from the holomorphic structure of $\mathcal{Z}$ one has to study the double fibration (1.1). This way one obtains the Penrose correspondence between holomorphic objects on $\mathcal{Z}$ and conformal properties of X.

First thing to observe is that a biholomorphism of $\mathcal{Z}$ corresponds to a conformal isometry of X, so any holomorphic vector field on $\mathcal{Z}$ relates to a conformal Killing field on X.

A more subtle but extremely important occurrence of the Penrose correspondence is the relation between the cohomology of $\mathcal{O}_{\mathcal{Z}}(m) \equiv K_{\mathcal{Z}}^{-m/4}$ and solutions of conformally invariant differential equations on X (see [7]). Let $\mathbb{S}_+^m$ and $\mathbb{S}_-^m$ denote the m-th symmetric powers of the spin bundles. Let

$$\mathrm{D}_m : \Gamma(\mathbb{S}_+^m) \longrightarrow \Gamma(\mathbb{S}_+^{m-1} \otimes \mathbb{S}_-)$$

be the Dirac operator and

$$\overline{\mathrm{D}}_m : \Gamma(\mathbb{S}_+^m) \longrightarrow \Gamma(\mathbb{S}_+^{m+1} \otimes \mathbb{S}_-)$$

be the Penrose twistor operator. These can be defined by composing the Riemannian connection of a metric in the conformal class

$$\nabla : \Gamma(\mathbb{S}_+^m) \longrightarrow \Gamma(\mathbb{S}_+^m \otimes T^*\mathrm{X}) = \Gamma(\mathbb{S}_+^m \otimes \mathbb{S}_- \otimes \mathbb{S}_+)$$

with the projections in the decomposition:

$$\mathbb{S}_+^m \otimes \mathbb{S}_- \otimes \mathbb{S}_+ = (\mathbb{S}_+^{m-1} \otimes \mathbb{S}_-) \oplus (\mathbb{S}_+^{m+1} \otimes \mathbb{S}_-)$$

In local coordinates

$$\mathrm{D}_m \psi_{B_1 \ldots B_m} = \nabla^{B_1 A'} \psi_{B_1 \ldots B_m}$$

and

$$\overline{\mathrm{D}}_m \psi_{B_1 \ldots B_m} = \nabla^{A'}_{(A} \psi_{B_1 \ldots B_m)}$$

For $m = 0$

$$\mathrm{D}_0 = \nabla^* \nabla + \frac{1}{6} R$$

will denote the conformal Laplacian acting on scalars. Then one has:

Theorem 1.0.3. *[7] For any $m \geq 0$,*

$$H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(m)) \cong Ker \overline{\mathrm{D}}_m$$

and

$$H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(-m-2)) \cong Ker \mathrm{D}_m$$

1.0.5

Sections 2.1 and 2.2 of our paper include descriptions of two constructions that provided the motivation for our work. We describe the classical construction of anti-self-dual metrics admitting an S^1-symmetry, with the twistor space given as the total space of a $c_1 = 0$ line bundle over a complex surface. We also review the generalized Legendre transform construction of hyperkähler metrics for which the standard holomorphic projection $\mathcal{Z} \to \mathbb{CP}_1$ factorizes through the line bundle $\mathcal{O}(4)$ over $\mathbb{CP}_1$.

Section 2.3 contains a general method of producing anti-self-dual complex conformal structures on certain hypersurfaces in the space $\mathbf{Y}$ of deformations of rational curves embedded with normal bundle $\mathcal{O}(4)$ in a complex surface S. Any affine line bundle $\mathcal{E} \to S$ with the appropriate sheaf of local translations determines a solutions of a second order differential equations system $D_2V = 0$ on $\mathbf{Y}$. Any solution V defines a complex conformal structure on an open set in $(V = 0)$ with the twistor space given by the total space of $\mathcal{E}$. The resulting metrics are characterized by the fact that they admit Killing spinors of valence (3,1), i.e. non-zero sections $\psi \in \Gamma(\mathbb{S}_+^3 \otimes \mathbb{S}_-)$ satisfying the equation:

$$\nabla_{(A}^{(A'}\psi_{BCD)}^{B')} = 0$$

In Section 3.1 we prove that our construction is indeed an extension of the generalized Legendre transform.

In Section 3.2 we show how our method applies to the case where the twistor space projects over $\mathbb{CP}_2$ with twistor lines mapping onto plane conics. The resulting conformal structures live on hypersurfaces in the space of nondegenerate plane conics $\mathbf{Y} = \{A = (a_{ij})_{0\le i,j\le 2} : A = A^t, \det A = 1\}$. We show that any solution

$V(A)$ of the differential system

$$\triangle V = \left(Trace\left(A \cdot \frac{\partial}{\partial A}\right)\right)^2 V - V = 0$$
$$\Box V = \left(A \cdot \frac{\partial}{\partial A}\right)^2 V = 0$$

defines such a metric. Here $\frac{\partial}{\partial A}$ denotes the operator $\frac{\partial}{\partial A} = \left(\frac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial a_{ij}}\right)_{ij}$. We also construct the defining functions $V(A)$ for the standard metrics on S^4 and $\overline{\mathbb{CP}}_2$.

Chapter 2

General Construction

In this chapter we will describe a method of constructing anti-self-dual conformal structures admitting Killing spinors, i.e. non-zero sections $\psi \in \Gamma(\mathbb{S}_+^m \otimes \mathbb{S}_-)$ satisfying the equation:

$$\nabla_{(A}^{(A'}\psi_{B_1\ldots B_m)}^{B')} = 0$$

We start by reviewing the case $m = 1$ and then the hyperkähler case.

2.1 Classical Construction

Our entire work will be guided by the construction of anti-self-dual conformal structures from a 3-dimensional Einstein-Weyl geometry. We will just summarize here the basic ideas. For more details and specific applications one can see [9, 19, 11, 16].

One starts with a complex 3-manifold $\mathbf{Y}$ endowed with a conformal structure $[h]$ and a torsion-free affine connection $\mathbb{D}$ satisfying the compatibility condition:

$$\mathbb{D}h = -2\nu \otimes h$$

for some 1-form ν and the curvature constraint on R_{ij} - the Ricci tensor of $\mathbb{D}$:

$$R_{(ij)} = \lambda h_{ij}$$

for some function λ. The *minitwistor space* of $(\mathbf{Y}, [h], \mathbb{D})$ is the complex surface $\mathcal{S}$ of totally geodesic null hypersurfaces in $\mathbf{Y}$. For any point $y \in \mathbf{Y}$, the surfaces passing through y define a rational curve $C \subset \mathcal{S}$ of self-intersection 2.

As in the anti-self-dual 4-dimensional case, this construction is reversible [9]. Starting with any complex surface $\mathcal{S}$ containing a self-intersection 2 rational curve C, we get an Einstein-Weyl structure on the space of deformations of C in $\mathcal{S}$.

The next ingredient in the construction is a holomorphic line bundle $\mathcal{E}$ on $\mathcal{S}$ with vanishing Chern class. The total space $\mathcal{Z}$ of the line bundle $\mathcal{E}$ can play the role of the twistor space of an anti-self-dual conformal structure. To see this, we need to produce a family of lines in $\mathcal{Z}$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

The vanishing of $c_1(\mathcal{E})$ makes the restriction of $\mathcal{E}$ to rational curves in $\mathcal{S}$ trivial. The normal bundle in $\mathcal{Z}$ of any holomorphic section $\widetilde{C} = \sigma(C)$, $\sigma \in H^0(C, \mathcal{E}|_C)$ will be determined by an extension on C:

$$0 \to \mathcal{E}|_C \to \sigma^* N_{\widetilde{C}|\mathcal{Z}} \to N_{C|\mathcal{S}} \to 0$$

where we implicitly use that the normal bundle of $\widetilde{C}$ in $\mathcal{E}|_C$ can be identified through σ^* with $\mathcal{E}|_C$.

The extension class will associate to any curve C a value in

$$H^1(C, N_C^* \otimes \mathcal{E}|_C) \cong H^1(\mathbb{P}_1, \mathcal{O}(-2)) \cong \mathbb{C}$$

defining thus a function V on $\mathbf{Y}$. As long as $V \neq 0$, the normal bundle of $\widetilde{C}$ is $N_{\widetilde{C}} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, so indeed the total space of $\mathcal{E}$ is a twistor space with the twistor lines given by holomorphic sections $\widetilde{C}$.

Another way of recovering the function V is through the Penrose correspondence by regarding $\mathcal{E}$ as an element of $H^1(\mathcal{S}, \mathcal{O})$. This approach is particularly useful in proving that V is not an arbitrary function. It has to satisfy the equation

$$d * (d + \nu)V = 0 \tag{2.1}$$

where $*$ denotes the Hodge operator on $\mathbf{Y}$. This will allow us to think of $F = *(d + \nu)V$ as the curvature of a circle bundle $\mathbf{X} \xrightarrow{\pi} \mathbf{Y}$ and write $\pi^* F = d\theta$, with θ a connection 1-form.

The anti-self-dual conformal structure we are looking for will live on $\mathbf{X}$ and will be given by

$$g = \pi^*(V \cdot h) + V^{-1} \cdot \theta^2$$

An important example is obtained by starting with the complex euclidean 3-space and its coresponding minitwistor space $\mathcal{S} = T\mathbb{P}_1$. Equation (2.1) becomes $\Delta V = 0$, so one can use the harmonic function

$$V = \sum_{j=1}^{k+1} \frac{1}{r_j}$$

where r_j are distance functions from $k + 1$ fixed points in $\mathbb{C}^3$. The resulting conformal classes will contain the gravitational multi-instantons metrics of Gibbons and Hawking [6].

2.2 Generalized Legendre Transform

Another construction that plays an important role in our work is the generalized Legendre transform introduced by Lindström and Roček in [18].

Its main purpose is describing hyperkähler (anti-self-dual, Ricci flat) metrics with twistor space admitting a holomorphic projection onto line bundles over $\mathbb{P}_1$.

This method was successfully used in [10] and [4] to obtain information on the D_k ALE hyperkähler metrics proved to exist by Kronheimer [14, 15] and to construct explicit formulas for D_k ALF hyperkähler metrics.

We will briefly record here the main features of this construction.

First we will recall that the hyperkähler condition on the metric has important repercussions at twistor space level. Given a 4-dimensional riemannian manifold (X, g) with $W_+ = 0$ and $r = 0$, the $\mathbb{S}^+$ and Λ^+ bundles are flat, so there exists a holomorphic projection $p : \mathcal{Z} \to \mathbb{P}_1$ with fibers diffeomorphic to X. Any $\zeta \in \mathbb{P}_1$ determines a parallel complex structure $I(\zeta)$ on $X \cong p^{-1}(\zeta)$, so the metric g is Kähler with respect to each of them.

The hyperkähler condition also implies the existence of a holomorphic section ω of $\Lambda^2 T_p^*(2)$, a twisted two-form along the fibers of p whose restriction defines the holomorphic symplectic structure on each fiber. Determining ω is always an important step in recovering the metric g from the twistor space.

The generalized Legendre transform deals with the case where the holomorphic projection $p : \mathcal{Z} \to \mathbb{P}_1$ factors through an even degree line bundle over $\mathbb{P}_1$. We are mainly interested in the case of degree 4.

$$\mathcal{Z} \to \mathcal{O}(4) \to \mathbb{P}_1$$

The antipodal map on $\mathbb{P}_1$ induces a complex conjugation on the space of holomorphic sections of $\mathcal{O}(4)$. Starting with an affine cover $\{U, \tilde{U}\}$ of $\mathbb{P}_1$ with local coordinates $\{\zeta, \tilde{\zeta}\}$ related by $\tilde{\zeta} = \zeta^{-1}$ on $U \cap \tilde{U}$, a "real" section (invariant to the complex conjugation) is given by a pair $\{\eta, \tilde{\eta}\}$ consisting of

$$\eta(\zeta) = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4$$

a polynomial of degree 4 with $w \in \mathbb{R}$ and $\tilde{\eta}(\tilde{\zeta}) = \zeta^{-4}\eta(\zeta)$.

We will use $\eta(\zeta)$ as a coordinate on X, holomorphic with respect to $I(\zeta)$. A second holomorphic coordinate $\{\chi, \tilde{\chi}\}$ will be defined such that the holomorphic two-form ω is given by

$$\omega(\zeta) = d\eta \wedge d\chi = \zeta^2 d\tilde{\eta} \wedge d\tilde{\chi}$$

This implies that the relation between χ and $\tilde{\chi}$ amounts to defining a function $\hat{f}(\eta, \zeta)$ and a contour C in $\mathbb{P}_1$ such that

$$\oint_0 \frac{\chi}{\zeta^{m-2}} d\zeta = \oint_\infty \frac{\tilde{\chi}}{\zeta^m} d\zeta + \oint_C \frac{\hat{f}}{\zeta^m} d\zeta$$

From $\hat{f}$ one defines G such that $\partial G / \partial \eta = \hat{f} / \zeta^2$ and then the function of coefficients of $\eta(\zeta)$:

$$F(z, v, w, \bar{v}, \bar{z}) = \frac{1}{2\pi i} \oint_C \frac{G(\eta, \zeta)}{\zeta^2} d\zeta \qquad (2.2)$$

Notice that F trivially satisfies the system of differential equations

$$
\begin{gathered}
F_{zw} = F_{vv} \qquad\qquad -F_{z\bar{v}} = F_{vw} \\
F_{v\bar{z}} = -F_{w\bar{v}} \qquad\qquad F_{w\bar{z}} = F_{\bar{v}\bar{v}} \\
F_{z\bar{z}} = -F_{v\bar{v}} = F_{ww}
\end{gathered} \tag{2.3}
$$

where the subscripts denote partial derivatives.

The generalized Legendre transform produces a formula for the Kähler potential $K = K(z, \bar{z}, u, \bar{u})$ in terms of F. It is obtained by imposing the constraint $F_w = 0$ and then performing a Legendre transform with respect to v and $\bar{v}$:

$$
K(z, \bar{z}, u, \bar{u}) = F(z, v, w, \bar{v}, \bar{z}) - vu - \bar{v}\bar{u},\ u = F_v,\ \bar{u} = F_{\bar{v}} \tag{2.4}
$$

For future use, we will write here the expression of the metric in the nonholomorphic coordinates $z, \bar{z}, v, \bar{v}$ modulo the restriction to the hypersurface $F_w = 0$.

Let $\mathbb{F}$ denote the matrix of partial derivatives:

$$
\mathbb{F} = \begin{pmatrix} F_{vv} & F_{vw} & F_{v\bar{v}} \\ F_{wv} & F_{ww} & F_{w\bar{v}} \\ F_{\bar{v}v} & F_{\bar{v}w} & F_{\bar{v}\bar{v}} \end{pmatrix}
$$

and $\mathbb{G}$ its inverse:

$$
\mathbb{F}^{-1} = \mathbb{G} = \begin{pmatrix} G^{vv} & G^{vw} & G^{v\bar{v}} \\ G^{wv} & G^{ww} & G^{w\bar{v}} \\ G^{\bar{v}v} & G^{\bar{v}w} & G^{\bar{v}\bar{v}} \end{pmatrix}
$$

Differentiating the constraints in (2.4) gives

$$\begin{aligned}
\left(\frac{\partial}{\partial z}\begin{pmatrix} v & w & \bar{v} \end{pmatrix}\right) \cdot \mathbb{F} &= -\frac{\partial}{\partial z}\begin{pmatrix} F_v & F_w & F_{\bar{v}} \end{pmatrix} \\
\left(\frac{\partial}{\partial \bar{z}}\begin{pmatrix} v & w & \bar{v} \end{pmatrix}\right) \cdot \mathbb{F} &= -\frac{\partial}{\partial \bar{z}}\begin{pmatrix} F_v & F_w & F_{\bar{v}} \end{pmatrix} \\
\left(\frac{\partial}{\partial u}\begin{pmatrix} v & w & \bar{v} \end{pmatrix}\right) \cdot \mathbb{F} &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\
\left(\frac{\partial}{\partial \bar{u}}\begin{pmatrix} v & w & \bar{v} \end{pmatrix}\right) \cdot \mathbb{F} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Using these relations we get the following formulas for the components of the metric in terms of second derivatives of F:

$$\begin{aligned}
K_{u\bar{u}} &= -\frac{\partial v}{\partial \bar{u}} = -G^{v\bar{v}} \\
K_{u\bar{z}} &= -\frac{\partial v}{\partial \bar{z}} = \sum_{a\in\{v,w,\bar{v}\}} F_{\bar{z}a}G^{av} \\
K_{z\bar{z}} &= F_{z\bar{z}} - \sum_{a,b\in\{v,w,\bar{v}\}} F_{za}G^{ab}F_{b\bar{z}}
\end{aligned}$$

Following [4] the expression of the metric can be simplified to:

$$ds^2 = \frac{1}{\beta}\left(dzd\bar{z} + (\alpha dz + \beta du)(\bar{\alpha}d\bar{z} + \bar{\beta}d\bar{u})\right) \tag{2.5}$$

where

$$\alpha = \sum_{a\in\{v,w,\bar{v}\}} F_{za}G^{a\bar{v}}, \;\; \beta = -G^{v\bar{v}} \tag{2.6}$$

We want to emphasize here that the main components in this construction are the function $\hat{f}$ (or equivalently G) and the contour C in (2.2) which define the transition from χ to $\tilde{\chi}$. By carefully choosing these elements one obtains important examples of hyperkähler metrics of type D_k.

2.3 Affine Line Bundles as Twistor Spaces

In trying to extend these constructions we will look for anti-self-dual metrics with twistor spaces that project as affine line bundle over surfaces.

2.3.1

Let A denote the group of affine transformations of $\mathbb{C}$, $z \to az + b$, $a \neq 0$. It projects onto the group of rotations $\mathbb{C}^*$, with kernel the group of translations $\mathbb{C}$:

$$0 \to \mathbb{C} \to A \xrightarrow{\rho} \mathbb{C}^* \to \mathbb{1} \qquad (2.7)$$

Given a complex manifold $\mathcal{S}$, let $\mathcal{A}_{\mathcal{S}}$ denote the sheaf of germs of A-valued holomorphic function on $\mathcal{S}$. A holomorphic affine line bundle $\mathcal{E}$ over $\mathcal{S}$ is a fiber bundle with fiber $\mathbb{C}$ and transition functions given by local sections of $\mathcal{A}_{\mathcal{S}}$.

Let such an affine line bundle $\mathcal{E}$ be given by transition functions $z_i = a_{ij}z_j + b_{ij}$ with respect to a certain cover $(U_i)_i$ of $\mathcal{S}$. Using the surjective map ρ in (2.7), we can naturally associate to $\mathcal{E}$ an isomorphism class of line bundles $[\mathcal{L}] = \rho(\mathcal{E})$ represented by the transition functions $\{a_{ij}\}$.

On the other hand, if we choose a fixed line bundle $\mathcal{L} \in [\mathcal{L}]$, we can obtain any affine line bundle $\mathcal{E}$ with $\rho(\mathcal{E}) = [\mathcal{L}]$ in the following way. Let

$$0 \to \mathcal{L} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\pi} \mathcal{O} \to 0 \qquad (2.8)$$

be an extension of the trivial line bundle through $\mathcal{L}$ and define $\mathcal{E} = \pi^{-1}(1)$. We can choose local trivializations on $\mathcal{M}$ such that in coordinates $\iota(z) = (z, 0)$ and $\pi(z, w) = w$. With respect to these local frames, the transition functions of $\mathcal{M}$ are of the form

$$g_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}$$

where a_{ij} define $\mathcal{L}$, while b_{ij} are arbitrary. The restriction of the linear transformations g_{ij} to $\mathcal{E} = \pi^{-1}(1)$ are just the affine transition functions $z_i = a_{ij}z_j + b_{ij}$. Notice that under this description, $\mathcal{L}$ acts upon $\mathcal{E}$ through local translations.

The space of equivalence classes of extensions (2.8) can be identified with $H^1(\mathcal{S}, \mathcal{L})$. Recall that two extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \| & = & \downarrow\wr & = & \| & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

are equivalent if there exists an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ making the above diagram commutative.

Thus, fixing a line bundle $\mathcal{L}$, the space of affine bundles with translation sheaf isomorphic to $\mathcal{L}$, equipped with a fixed isomorphism to $\mathcal{L}$, is given by $H^1(\mathcal{S}, \mathcal{L})$. The element $0 \in H^1(\mathcal{S}, \mathcal{L})$ coresponds to the trivial extension in (2.8) and this generates an improper affine bundle $\mathcal{E}$ equivalent to $\mathcal{L}$ with the trivial affine structure.

2.3.2

We will restrict from now on to the case of a complex surface $\mathcal{S}$. We choose a fixed line bundle $\mathcal{L}$ on $\mathcal{S}$ such that $H^1(\mathcal{S},\mathcal{L}) \neq 0$ and $\mathcal{E} \in H^1(\mathcal{S},\mathcal{L}) - \{0\}$ a proper affine bundle with the sheaf of local translations equal to $\mathcal{L}$.

Let also $C \subset \mathcal{S}$ be a rational curve of self-intersection n. C belongs to a family of curves parametrized by a $(n+1)$-dimensional manifold $\mathbf{Y}$ such that $T_y\mathbf{Y}$ is canonically isomorphic to $H^0(C_y, N_{C_y})$. We are mainly interested in a local construction and the $\mathcal{O}_{\mathbb{P}_1}(n)$ line bundle is infinitesimally rigid, so we can assume that $N_{C_y} \cong \mathcal{O}_{\mathbb{P}_1}(n)$ for every curve in the family.

By restricting to a neighborhood of C in $\mathcal{S}$ we can also assume the existence of a line bundle $\mathcal{O}_{\mathcal{S}}(1) = K_{\mathcal{S}}^{-\frac{1}{n+2}}$ on $\mathcal{S}$ whose restriction to C has degree 1. We will write $\mathcal{O}_{\mathcal{S}}(k) = \otimes^k \mathcal{O}_{\mathcal{S}}(1)$.

Using this we can define on $\mathbf{Y}$ the equivalent of *Spin* bundles on a 4-manifold, that is an $SL(2,\mathbb{C})$ bundle $\mathbb{S}$ with fiber $\mathbb{S}_y = H^0(C_y, \mathcal{O}(1))$. We will also use the alternative notation $\mathbb{S} = \mathcal{O}^A$. This bundle comes naturally equipped with a skew-symmetric, complex bilinear form $\varepsilon = \varepsilon_{AB}$, given on each fiber $\mathbb{S}_y = H^0(C_y, \mathcal{O}(1))$ by

$$\varepsilon(l, l') = \varepsilon(l^0 s + l^1 t,\ l'^0 s + l'^1 t) = l^0 l'^1 - l^1 l'^0$$

The form ε_{AB} provides a natural isomorphism between $\mathbb{S} = \mathcal{O}^A$ and its dual $\mathbb{S}^* = \mathcal{O}_A$ and will be used repeatedly in raising and lowering indices.

The holomorphic tangent bundle on $\mathbf{Y}$ can then be written as the n-th symmetric power of $\mathbb{S}$:

$$T\mathbf{Y} = Sym^n \mathbb{S}$$

We notice here the existence of a nondegenerate symmetric bilinear form Q on $T\mathbf{Y}$ that will play a role in our construction. In "spinorial" notations:

$$Q(\eta, \psi) = \eta^{AB..D} \cdot \psi_{AB..D}$$

2.3.3

We want to investigate the possibility of presenting the total space of the affine line bundle $\mathcal{E}$ as the desired twistor space with twistor lines given by sections $\widetilde{C}_y$ of the restrictions $\mathcal{E}|_{C_y}, y \in Y$.

There are two conditions on $\mathcal{E}$ to be satisfied, these sections need to exist and to have normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. We will address first the second restriction. Notice that the normal bundle of a presumptive section $\widetilde{C}_y$ in $\mathcal{E}|_{C_y}$ can be identified with $\mathcal{L}|_{C_y}$, so the isomorphism class of the normal bundle of $\widetilde{C}_y$ in the total space $\mathcal{Z}$ of $\mathcal{E}$ will be determined by an extension on C_y:

$$0 \to \mathcal{L}|_{C_y} \to N_{\widetilde{C}_y|\mathcal{Z}} \to N_{C_y|S} \to 0 \tag{2.9}$$

To get at least the right degree, we need to impose the condition $\deg(\mathcal{L}|_{C_y}) = 2-n$, so we will naturally choose $\mathcal{L} \cong \mathcal{O}_S(2-n)$.

This makes the existence of sections in $\mathcal{E}|_{C_y}$ no longer automatic for $n \geq 4$. In this case, by fixing a consistent isomorphism $K_{C_y} = \mathcal{O}_S(-2)|_{C_y}$ for any $y \in \mathbf{Y}$, we have

$$H^1(C_y, \mathcal{L}) = H^1(C_y, K_{C_y} \otimes \mathcal{O}_S(4-n)) = \mathbb{C}^{n-3}$$

so by restricting $\mathcal{E}$ to C_y for any $y \in \mathbf{Y}$ we get a function $V : \mathbf{Y} \longrightarrow \mathbb{C}^{n-3}$

well defined up to a multiplicative constant. Notice that the zero set of V does not depend on the above chosen isomorphism.

The affine bundle $\mathcal{E}|_{C_y}$ needs to be improper (isomorphic to $\mathcal{L}|_{C_y}$) for a section $\widetilde{C}_y$ to exist and this condition defines precisely the 4-dimensional subvariety $\{V = 0\} \subset \mathbf{Y}$. For any x with $V(x) = 0$ we have $\mathcal{E}|_{C_x} \cong \mathcal{L} \cong \mathcal{O}_{\mathbb{P}_1}(2-n)$, so there exists exactly one section $\widetilde{C}_x$ of $\mathcal{E}|_{C_x}$, namely the zero section.

2.3.4

In addition to the equation $V(x) = 0$ that insures the existence of the cross section $\widetilde{C}_x$ of $\mathcal{E}|_{C_x}$, we need to make sure that the normal bundle to $\widetilde{C}_x$ given by the extension (2.9) is the desired $\mathcal{O}(1) \oplus \mathcal{O}(1)$. We will concentrate on the special case $n = 4$ which interests us the most and we will express the necessary and sufficient conditions in terms of the defining function $V : \mathbf{Y} \longrightarrow \mathbb{C}$.

In this case the isomorphism class of the normal bundle $N_{\widetilde{C}_x}$ is determined by the extension class of

$$0 \to \mathcal{L}|_{C_x} \to N_{\widetilde{C}_x} \to N_{C_x} \to 0 \tag{2.10}$$

as an element in

$$\begin{aligned} H^1(C_x, N^*_{C_x} \otimes \mathcal{L}) =& H^0(C_x, N_{C_x} \otimes \mathcal{L}^* \otimes K_C)^* \\ =& H^0(C_x, N_{C_x} \otimes \mathcal{O}_S(2) \otimes \mathcal{O}_S(-2))^* \\ =& H^0(C_x, N_{C_x})^* \\ =& T^*_x\mathbf{Y} \end{aligned}$$

Following [19] we will prove:

Lemma 2.3.1. *At any point $x \in \mathbf{Y}$ with $V(x) = 0$ the extension class of (2.10) is determined by the restriction of $\mathcal{E}$ to the first formal neighborhood of C_x and with respect to the above identity it is given by $dV_x \in T_x^*\mathbf{Y}$.*

If $\mathcal{I}$ denotes the ideal sheaf of holomorphic functions on $\mathcal{S}$ vanishing on C_x we have the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_\mathcal{S}/\mathcal{I}^2 \to \mathcal{O}_{C_x} \to 0$$

with $\mathcal{I}/\mathcal{I}^2 = N^*_{C_x}$ the conormal bundle and $\mathcal{O}_\mathcal{S}/\mathcal{I}^2 = \mathcal{O}^{(1)}_{C_x}$ the sheaf of holomorphic functions on the first formal neighborhood of C_x in $\mathcal{S}$. After tensoring with $\mathcal{O}_\mathcal{S}(-2)$ we get the long exact sequence:

$$\begin{array}{ccccccc} 0 & \to & H^1(C_x, N^*_{C_x}(-2)) & \xrightarrow{\iota} & H^1(C_x, \mathcal{O}^{(1)}_{C_x}(-2)) & \xrightarrow{\tau} & H^1(C_x, \mathcal{O}_{C_x}(-2)) \\ & & dV_x & \longrightarrow & \mathcal{E}|_{C_x^{(1)}} & \longrightarrow & \mathcal{E}|_{C_x} = 0 \end{array}$$

Here ι is an isomorphism onto the kernel of τ that maps dV_x to $\mathcal{E}|_{C_x^{(1)}}$ as long as $V(x) = 0$.

Proof of 2.3.1. Let $(U_i)_{i\in I}$ be a covering of $\mathcal{S}$, (ζ_i, w_i) coordinates on U_i such that our curve C_x is defined by $w_i = 0$ and let the transition functions of $\mathcal{E}$ on $U_i \cap U_j$ be $z_i = a_{ij} \cdot z_j + b_{ij}$. We can assume that $b_{ij}(\zeta_j, 0) = 0$ as we have that $\mathcal{E}|_{C_x} = \mathcal{O}_\mathcal{S}(-2)|_{C_x}$. The restriction of $\mathcal{E}$ to the first formal neighborhood of C_x will then be given by $\dfrac{\partial b_{ij}}{\partial w_j}(\zeta_j, 0)$.

At the same time, computing modulo $\dfrac{\partial}{\partial \zeta_i}$ and then restricting to the zero section

$\widetilde{C}_x$, we have that

$$\frac{\partial}{\partial z_j} = a_{ij} \cdot \frac{\partial}{\partial z_i} + \left(\frac{\partial a_{ij}}{\partial w_j} \cdot z_j + \frac{\partial b_{ij}}{\partial w_j} \right) \cdot \frac{\partial}{\partial w_i}$$
$$= a_{ij}(\zeta_j, 0) \cdot \frac{\partial}{\partial z_i} + \frac{\partial b_{ij}}{\partial w_j}(\zeta_j, 0) \cdot \frac{\partial}{\partial w_i}$$

so the transition matrix of $N_{\widetilde{C}_x}$ on $U_i \cap U_j \cap C_x$ will be

$$A_{ij}(\zeta_i, 0) = \begin{pmatrix} a_{ij} & \frac{\partial b_{ij}}{\partial w_j}(\zeta_j, 0) \\ 0 & \frac{\partial w_i}{\partial w_j}(\zeta_j, 0) \end{pmatrix}$$

and so indeed the extension class of (2.10) is also given by $\frac{\partial b_{ij}}{\partial w_j}(\zeta_j, 0)$. ■

2.3.5

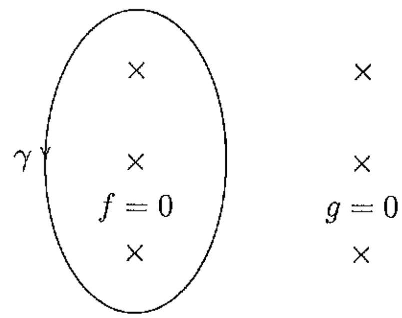
Over $\mathbb{P}_1$ any vector bundle splits as a sum of line bundles, so the normal bundle defined by (2.10) will have the form

$$N_{\widetilde{C}_x} \cong \mathcal{O}_{\mathbb{P}_1}(a) \oplus \mathcal{O}_{\mathbb{P}_1}(b)$$

with $-2 \leq a, b \leq 4$ and $a + b = 2$. The only possibilities are: $\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)$, $\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(2)$, $\mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(3)$ and the trivial extension $\mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(4)$. We will investigate them case by case.

The trivial extension coresponds to $dV_x = 0$, while in all the other cases the isomorphism class of $N_{\widetilde{C}_x}$ is determined by the projective point $[dV_x] \in \mathbb{P}T^*_x\mathbf{Y}$.

If $N_{\widetilde{C}_x} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, then the map $N_{\widetilde{C}_x} \longrightarrow N_{C_x}$ in (2.10) is given by a pair of sections $f, g \in H^0(C_x, \mathcal{O}_S(3)|_{C_x})$ with disjoint sets of roots.



The extension class of (2.10) can be identified in this case with the linear map $H^0(C_x, N_{C_x}) \to \mathbb{C}$ given by the integral

$$\eta \to \int_\gamma \frac{\eta}{f \cdot g} \tag{2.11}$$

along a path γ that separates the roots of f and g.

The case $N_{\tilde{C}_x} \cong \mathcal{O}(-1) \oplus \mathcal{O}(3)$ follows the same pattern, but this time with $f \in H^0(C_x, \mathcal{O}_S(5)|_{C_x})$ and $g \in H^0(C_x, \mathcal{O}_S(1)|_{C_x})$, while in the $N_{\tilde{C}_x} \cong \mathcal{O} \oplus \mathcal{O}(2)$ case we have $f \in H^0(C_x, \mathcal{O}_S(4)|_{C_x})$ and $g \in H^0(C_x, \mathcal{O}_S(2)|_{C_x})$.

The distinction becomes much clearer when expressed in terms of the bilinear symmetric form

$$Q(\eta, \psi) = \eta^{ABCD} \cdot \psi_{ABCD}$$

Invariantly, Q is the unique quadratic form vanishing on sections $\eta \in H^0(C_x, N_{C_x})$ admitting a triple zero.

In the $N_{\tilde{C}_x} \cong \mathcal{O}(-1) \oplus \mathcal{O}(3)$ case the kernel of (2.11) is $\{g \cdot q \;:\; q \in H^0(C_x, \mathcal{O}_S(3)|_{C_x})\}$ for some $g \in H^0(C_x, \mathcal{O}_S(1)|_{C_x})$ and its orthogonal complement with respect to Q is just the line generated by $g^4 \in H^0(C_x, \mathcal{O}_S(4)|_{C_x})$. This implies that the set of extension classes for which $N_{\tilde{C}_x} \cong \mathcal{O}(-1) \oplus \mathcal{O}(3)$ is a completely determined rational normal curve Δ_x in $\mathbb{P}T^*_x\mathbf{Y}$.

If we write in coordinates $dV(\eta) = V_{ABCD}\eta^{ABCD}$, we have that $[dV_x] \in \Delta_x$ if and only if there exists a nonzero $g^A \in \mathbb{S}_x$ such that $V_{ABCD}g^A \equiv 0$. Thus

$$[dV_x] \in \Delta_x \quad \Leftrightarrow \quad rank \begin{pmatrix} V_{0000} & V_{0001} & V_{0011} & V_{0111} \\ V_{0001} & V_{0011} & V_{0111} & V_{1111} \end{pmatrix} = 1$$

$$\Leftrightarrow \qquad \varepsilon^{AE}V_{ABCD}V_{EFGH} = 0$$

Following a similar approach one can also verify that $N_{\widetilde{C}_x} \cong \mathcal{O} \oplus \mathcal{O}(2)$ if and only if $[dV_x] \in \Sigma_x - \Delta_x$, where Σ_x denotes the secant variety of Δ_x. In this case $\ker dV_x = \{f \cdot c + g \cdot q : c \in \mathbb{C}, q \in H^0(C_x, \mathcal{O}_S(2)|_{C_x})\}$ for some fixed $f \in H^0(C_x, \mathcal{O}_S(4)|_{C_x})$ and $g \in H^0(C_x, \mathcal{O}_S(2)|_{C_x})$.

Its orthogonal complement with respect to Q is given by the $span\langle g_1^4, g_2^4\rangle \cap f^\perp$ for some decomposition $g = g_1g_2$ with $g_i \in H^0(C_x, \mathcal{O}_S(1)|_{C_x})$. This implies that indeed, as f and g vary, $[dV_x]$ covers the secant variety Σ_x.

Writing as before in coordinates:

$$[dV_x] \in \Sigma_x \Leftrightarrow \begin{vmatrix} V_{0000} & V_{0001} & V_{0011} \\ V_{0001} & V_{0011} & V_{0111} \\ V_{0011} & V_{0111} & V_{1111} \end{vmatrix} = 0$$

This analysis gives us the following

Proposition 2.3.2. *The normal bundle of $\widetilde{C}_x$ in $\mathcal{Z}$ is:*

$\mathcal{O}(-2) \oplus \mathcal{O}(4)$ *if* $dV_x = 0$

$\mathcal{O}(-1) \oplus \mathcal{O}(3)$ *if* $dV_x \neq 0$ *and* $[dV_x] \in \Delta_x$

$\mathcal{O} \oplus \mathcal{O}(2)$ *if* $dV_x \neq 0$ *and* $[dV_x] \in \Sigma_x - \Delta_x$

$\mathcal{O}(1) \oplus \mathcal{O}(1)$ *otherwise.*

2.3.6

This result on the normal bundle $N_{\widetilde{C}_x}$ lets us conclude that the four-dimensional manifold

$$\mathbf{X} = \{x \in \mathbf{Y} : V(x) = 0,\ dV_x \neq 0,\ [dV_x] \notin \Sigma_x\}$$

comes equipped with a natural anti-self-dual conformal structure $[\mathfrak{g}]$. In order to completely describe this structure, we need to specify the null cone in each tangent space.

Following our previous discussion, for each $x \in \mathbf{X}$ we have a pair of sections $f, g \in H^0(C_x, \mathcal{O}_S(3)|_{C_x})$ with no common roots such that

$$T_x\mathbf{X} = \ker dV_x = \{f \cdot l_1 + g \cdot l_2 : l_1, l_2 \in H^0(C_x, \mathcal{O}_S(1)|_{C_x})\}$$

A tangent vector $\eta = f \cdot l_1 + g \cdot l_2$ will be null with respect to $[\mathfrak{g}]$, if l_1 and l_2 vanish at the same point on C_x.

The conformal structure is in fact completely determined by dV. The differential defines a map $dV^\# : Sym^3\mathbb{S} \to \mathbb{S}^*$ whose kernel consists of $h \in Sym^3\mathbb{S}$ such that $h \cdot l \in \ker dV$ for any $l \in \mathbb{S}$.

Writing in coordinates, if $dV(\eta) = V_{ABCD}\eta^{ABCD}$, then

$$\ker dV^\# = \{h : V_{ABCD}h^{BCD} = 0\}$$

and we can see that it has dimension two as long as

$$\varepsilon^{AE} V_{ABCD} V_{EFGH} \neq 0$$

This condition is satisfied under our hypothesis because it means precisely $[dV_x] \notin \Delta_x$.

The null cone of $[\mathfrak{g}]$ will then be given by

$$\ker dV^{\#} \cdot \mathbb{S} = \{\eta = h \cdot l : h \in \ker dV^{\#}, l \in \mathbb{S}\}$$

2.3.7

We will give a better description of the conformal class $[\mathfrak{g}]$ by showing that it is the restriction to $\mathbf{X}$ of a conformal structure defined on the 5 dimensional manifold $\mathbf{Y}$.

We will construct in fact a family of symmetric bilinear forms $\mathfrak{G}$ on $\mathbf{Y}$ such that they vanish on the null cone of $[\mathfrak{g}]$ in $T\mathbf{X} = \ker dV$. These forms will differ from each other by multiples of dV.

We will then show that as long as $[dV_y] \notin \Sigma_y$, a generic bilinear form $\mathfrak{G}$ in this family is nondegenerate on $T\mathbf{Y}_y$, defining thus an actual conformal structure on a neighborhood of $y \in \mathbf{Y}$. So for any point $x \in \mathbf{X}$ we will obtain a family of conformal structures on a neighborhood of x in $\mathbf{Y}$ which equal $[\mathfrak{g}]$ when restricted to $\mathbf{X}$.

Let $\mathfrak{K} \in Sym^2(Sym^2\mathbb{S}^*)$ denote the image of $dV \in Sym^4\mathbb{S}^*$ through

$$Sym^4\mathbb{S}^* \quad \to \quad \wedge^2(Sym^3\mathbb{S}^*) \quad \xrightarrow{\simeq} \quad Sym^2(Sym^2\mathbb{S}^*)$$

$$V_{ABCD} \quad \to \quad \varepsilon^{DH} V_{ABCD} V_{EFGH} \quad \to \quad \varepsilon^{CG} \varepsilon^{DH} V_{ABCD} V_{EFGH}$$

We define then the symmetric bilinear form $\mathfrak{G} \in Sym^2(Sym^4\mathbb{S}^*)$ on $\mathbf{Y}$ as

$$\mathfrak{G}_{ABCD}{}^{EFGH} = -\mathfrak{K}_{(ABCD)}\mathfrak{K}^{(EFGH)} + \mathfrak{K}_{(AB}{}^{(EF}\mathfrak{K}_{CD)}{}^{GH)} + \mathfrak{K}_{(AB}{}^{(GH}\mathfrak{K}_{CD)}{}^{EF)}$$

This formula is not unique. It can be modified by any term of the form

$$\Gamma_{ABCD}{}^{EFGH} = \frac{1}{2}\left(V_{ABCD}W^{EFGH} + W_{ABCD}V^{EFGH}\right)$$

with $W \in Sym^4\mathbb{S}^*$.

Continuing to write our formulas in spinorial indices can become cumbersome at this moment, so we will temporarily change the notations as follows. A tangent vector η to $\mathbf{Y}$ is a homogeneous polynomial of degree 4 in two variables $\eta = \eta_1 \cdot s^4 + \eta_2 \cdot s^3 t + \eta_3 \cdot s^2 t^2 + \eta_4 \cdot s t^3 + \eta_5 \cdot t^4$. If $dV(\eta) = \sum V_i \eta_i$ we will denote by σ the determinant:

$$\sigma = \begin{vmatrix} V_1 & V_2 & V_3 \\ V_2 & V_3 & V_4 \\ V_3 & V_4 & V_5 \end{vmatrix}$$

and by $\delta_1, ..., \delta_6$ the minors in $\begin{pmatrix} V_1 & V_2 & V_3 & V_4 \\ V_2 & V_3 & V_4 & V_5 \end{pmatrix}$:

$$\delta_1 = \begin{vmatrix} V_1 & V_2 \\ V_2 & V_3 \end{vmatrix}, \delta_2 = \begin{vmatrix} V_1 & V_3 \\ V_2 & V_4 \end{vmatrix}, \delta_3 = \begin{vmatrix} V_1 & V_4 \\ V_2 & V_5 \end{vmatrix}$$

$$\delta_4 = \begin{vmatrix} V_2 & V_3 \\ V_3 & V_4 \end{vmatrix}, \delta_5 = \begin{vmatrix} V_2 & V_4 \\ V_3 & V_5 \end{vmatrix}, \delta_6 = \begin{vmatrix} V_3 & V_4 \\ V_4 & V_5 \end{vmatrix}$$

In these coordinates dV belongs to the rational normal curve Δ if and only if $\delta_j = 0, \forall j$, while $\sigma = 0$ characterizes the points for which dV lies on the secant variety Σ.

The components of $\mathfrak{K} \in Sym^2(Sym^2\mathbb{S}^*)$ are

$$\mathfrak{K}_{0000} = 2\delta_1, \mathfrak{K}_{0001} = \delta_2, \mathfrak{K}_{0111} = \delta_5, \mathfrak{K}_{1111} = 2\delta_6$$

$$\mathfrak{K}_{0011} = \delta_3 - \delta_4, \mathfrak{K}_{0101} = 2\delta_4$$

and this makes our previous choice of a family of quadratic forms equivalent to

$$\begin{aligned} \mathfrak{G}(\eta, \eta) =& \delta_1^2\eta_1^2 + \delta_1\delta_4\eta_2^2 + \delta_4^2\eta_3^2 + \delta_4\delta_6\eta_4^2 + \delta_6^2\eta_5^2 \\ &+ \delta_1\delta_2\eta_1\eta_2 + \delta_2\delta_4\eta_2\eta_3 + \delta_4\delta_5\eta_3\eta_4 + \delta_5\delta_6\eta_4\eta_5 \\ &+ (\delta_2^2 - 2\delta_1\delta_4)\eta_1\eta_3 + (\delta_5^2 - 2\delta_4\delta_6)\eta_3\eta_5 \\ &+ \delta_4(\delta_3 - \delta_4)\eta_2\eta_4 + (\delta_3^2 + \delta_4^2 - 2\delta_2\delta_5)\eta_1\eta_5 \\ &+ (\delta_2(\delta_3 - \delta_4) - \delta_1\delta_5)\eta_1\eta_4 + (\delta_5(\delta_3 - \delta_4) - \delta_2\delta_6)\eta_2\eta_5 \end{aligned} \tag{2.12}$$

with the ambiguity given by

$$\Gamma_W(\eta,\eta) = \sum V_i\eta_i \sum W_i\eta_i$$

Claim 1. Any quadratic form in the family $\mathfrak{G} + \Gamma_W$ vanishes on the null cone of $[\mathfrak{g}]$.

We will verify this through a direct computation. Let η be a vector in the null cone of $[\mathfrak{g}]$. We can write $\eta = h \cdot l$ with h in the kernel of $du^{\#}$ and if we expand $h = h_1 \cdot s^3 + h_2 \cdot s^2 t + h_3 \cdot st^2 + h_4 \cdot t^3$ this condition implies

$$\begin{aligned} k_1 &= \delta_1 h_2 + \delta_2 h_3 + \delta_3 h_4 &= 0 \\ k_2 &= -\delta_1 h_1 + \delta_4 h_3 + \delta_5 h_4 &= 0 \\ k_3 &= \delta_2 h_1 + \delta_4 h_2 - \delta_6 h_4 &= 0 \\ k_4 &= \delta_3 h_1 + \delta_5 h_2 + \delta_6 h_3 &= 0 \end{aligned}$$

On the other hand one can verify, using the relation $\delta_1\delta_6 - \delta_2\delta_5 + \delta_3\delta_4 = 0$, that the following identity holds for any $\eta = h \cdot l \in Sym^3\mathbb{S} \cdot \mathbb{S}$:

$$\begin{aligned} \mathfrak{G}(\eta,\eta) =& k_1 \left((h_1\delta_2 + h_2\delta_4) l_1^2 + h_1 \frac{\delta_3+\delta_4}{2} l_1 l_2 \right) \\ & - k_2 \left((h_1\delta_1 - h_3\delta_4) l_1^2 + (h_1\delta_2 - h_3\delta_5 - h_2\frac{\delta_3+\delta_4}{2}) l_1 l_2 + h_1\delta_4 l_2^2 \right) \\ & - k_3 \left(h_4\delta_4 l_1^2 + (h_4\delta_5 - h_2\delta_2 - h_3\frac{\delta_3+\delta_4}{2}) l_1 l_2 + (h_4\delta_6 - h_2\delta_4) l_2^2 \right) \\ & + k_4 \left(h_4 \frac{\delta_3+\delta_4}{2} l_1 l_2 + (h_3\delta_4 + h_4\delta_5) l_2^2 \right) \end{aligned}$$

and so $\mathfrak{G}$ and the whole family $\mathfrak{G} + \Gamma_W$ vanish on $\ker du^{\#} \cdot \mathbb{S}$ $\diamondsuit$

Claim 2. Given any point $y \in \mathbf{Y}$ such that $[dV_y] \notin \Sigma_y$ (or equivalently $\sigma \neq 0$), the set of nondegenerate quadratic forms coresponding to

$$\{W \in Sym^4\mathbb{S}_y^* : \mathfrak{G} + \sigma\Gamma_W \text{ is nondegenerate on } T\mathbf{Y}_y\}$$

is open, non-empty.

Again a direct computation gives:

$$\begin{aligned}\det(\mathfrak{G} + \sigma\Gamma_W) = 2\sigma^6\big(&\delta_6 W_1 W_3 + \delta_5 W_2 W_3 + \delta_4 W_1 W_5 + \delta_2 W_3 W_4 + \delta_1 W_3 W_5 \\ &- \delta_5 W_1 W_4 + (\delta_3 - \delta_4) W_2 W_4 - \delta_2 W_2 W_5 \\ &- \delta_6 W_2^2 - \delta_3 W_3^2 - \delta_1 W_4^2 + \delta_5 W_2 - (\delta_3 + \delta_4) W_3 + \delta_2 W_4 - \delta_4\big)\end{aligned}$$

Thus $\mathfrak{G} + \sigma\Gamma_W$ is nondegenerate as long as $\sigma \neq 0$ and W avoids an algebraically closed subset in $Sym^4\mathbb{S}_y^*$. $\diamondsuit$

2.3.8

The defining function V plays a central role throughout the entire construction. In the following we will see that V is not an arbitrary function on $\mathbf{Y}$; it has to satisfy a certain second order partial differential equations system.

This type of information on V can be obtained through studying the double fibration:

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathcal{S} & & \mathbf{Y} \end{array} \tag{2.13}$$

where $\mathcal{F} = \{(s, y) \in \mathcal{S} \times \mathbf{Y} | s \in C_y\}$. The fibration $\mathcal{F} \xrightarrow{\nu} \mathbf{Y}$ in (2.13) is just the projectivization of the dual bundle $\mathbb{S}^* = \mathcal{O}_A$. We want to understand the relation between the space of classes of equivalence of affine line bundles with a fixed sheaf of local translations on $\mathcal{S}$ and families of sections in bundles over $\mathbf{Y}$. More precisely we want to express the elements of $H^1(\mathcal{S}, \mathcal{O}_\mathcal{S}(-2))$ as solutions of differential equations on sections in tensorial powers of $\mathbb{S}$. This is possible due to the relatively simple topology of the fibers of μ and ν.

The first goal is to pull-back the cohomology $H^1(\mathcal{S}, \mathcal{O}_\mathcal{S}(-2))$ from $\mathcal{S}$ to $\mathcal{F}$. We will denote by $\mu^{-1}\mathcal{O}_\mathcal{S}(-2))$ the topological pull-back of $\mathcal{O}_\mathcal{S}(-2)$, the sheaf of germs of sections in the bundle $\mu^*\mathcal{O}_\mathcal{S}(-2)$ that are locally constant along the fibres of μ.

As long as we assume that the fibres of μ in (2.13) are connected and simply connected and μ is surjective of maximal rank, a result of Buchdahl [3] guarantees that the canonical morphism

$$H^1(\mathcal{S}, \mathcal{O}_\mathcal{S}(-2)) \to H^1(\mathcal{F}, \mu^{-1}\mathcal{O}_\mathcal{S}(-2))$$

is an isomorphism.

The next step is to understand the interpretation of $H^1(\mathcal{F}, \mu^{-1}\mathcal{O}_\mathcal{S}(-2))$ down on $\mathbf{Y}$. To do this we need to write a resolution of $\mu^{-1}\mathcal{O}_\mathcal{S}(-2)$ on $\mathcal{F}$, such that we are able to compute the direct images through ν of all the sheaves involved.

We define the sheaf of germs of relative holomorphic 1-forms Ω^1_μ as the quotient

$$\mu^*\Omega^1_S \to \Omega^1_{\mathcal{F}} \to \Omega^1_\mu \to 0$$

and the relative k-forms as

$$\Omega^k_\mu = \wedge^k \Omega^1_\mu$$

Locally they are just k-forms along the fibers of μ, parametrized by the variables transverse to the fibers. We also define the sheaf of $\mu^*\mathcal{O}_S(-2)$-valued relative k-forms as

$$\Omega^k_\mu(-2) = \Omega^k_\mu \otimes \mu^*\mathcal{O}_S(-2)$$

The exterior differentiation naturally restricts to a differentiation along the fibers operator $d_\mu : \Omega^k_\mu \to \Omega^{k+1}_\mu$. This operator annihilates the transition functions of $\mu^*\mathcal{O}_S(-2)$, so we obtain also an extension $d_\mu : \Omega^k_\mu(-2) \to \Omega^{k+1}_\mu(-2)$.

Using again the fact that μ is of maximal rank, we obtain the needed resolution of $\mu^{-1}\mathcal{O}_S(-2)$, *the relative de Rham exact sequence* on $\mathcal{F}$:

$$0 \to \mu^{-1}\mathcal{O}_S(-2) \xrightarrow{d_\mu} \Omega^0_\mu(-2) \xrightarrow{d_\mu} \Omega^1_\mu(-2) \xrightarrow{d_\mu} \Omega^2_\mu(-2) \xrightarrow{d_\mu} \Omega^3_\mu(-2) \xrightarrow{d_\mu} \Omega^4_\mu(-2) \to 0$$

The *hypercohomology* spectral sequence associated to this resolution

$$E_1^{p,q} = H^q(\mathcal{F}, \Omega^p_\mu(-2)) \Longrightarrow H^{p+q}(\mathcal{F}, \mu^{-1}\mathcal{O}_S(-2)) \qquad (2.14)$$

will compute the desired cohomology group.

On the other hand, the fibers of ν are rational curves, so all the direct images

$R^s\nu_*(\Omega^p_\mu(-2))$ vanish for $s \geq 2$ and are coherent sheaves, even locally free for $s = 0, 1$. If $\mathbf{Y}$ is sufficiently topologically simple - in our local problems $\mathbf{Y}$ can be assumed Stein - then the Leray spectral sequence assures us that the only non-zero entries in (2.14) are

$$H^0(\mathcal{F}, \Omega^p_\mu(-2)) = H^0(\mathbf{Y}, \nu_*(\Omega^p_\mu(-2)))$$

and

$$H^1(\mathcal{F}, \Omega^p_\mu(-2)) = H^0(\mathbf{Y}, R^1\nu_*(\Omega^p_\mu(-2)))$$

In order to compute these we need to express $\Omega^p_\mu(-2)$ in terms of pull-backs of bundles from $\mathcal{S}$ and $\mathbf{Y}$ commonly denoted

$$\mathcal{O}_{A...E}(k) := \nu^*\mathcal{O}_{A..E} \otimes \mu^*\mathcal{O}_{\mathcal{S}}(k)$$

We notice first that

$$\Omega^1_\mu = \Omega^1_{\mathcal{F}}/\mu^*\Omega^1_{\mathcal{S}}$$

fits into

$$0 \to \mathcal{N}^* \to \nu^*\Omega^1_{\mathbf{Y}} \to \Omega^1_\mu \to 0$$

where $\mathcal{N}^*$ is the conormal bundle of $\mathcal{F}$ in $\mathcal{S} \times \mathbf{Y}$. The left part of this sequence is just the tautological embedding

$$0 \to \mathcal{O}(-4) \to \mathcal{O}_{(ABCD)}$$

so we get

$$\Omega^1_\mu \cong \mathcal{O}_{(ABC)}(1)$$

By taking exterior powers

$$\Omega^2_\mu \cong \wedge^2(\mathcal{O}_{(ABC)}(1)) \cong \mathcal{O}(2) \oplus \mathcal{O}_{(ABCD)}(2)$$

We have then:

$$\begin{aligned} \nu_*(\Omega^0_\mu(-2)) &= 0 & R^1\nu_*(\Omega^0_\mu(-2)) &= \mathcal{O} \\ \nu_*(\Omega^1_\mu(-2)) &= 0 & R^1\nu_*(\Omega^1_\mu(-2)) &= 0 \\ \nu_*(\Omega^2_\mu(-2)) &= \mathcal{O} \oplus \mathcal{O}_{(ABCD)} & R^1\nu_*(\Omega^2_\mu(-2)) &= 0 \end{aligned}$$

In the end, the $E_1^{p,q}$ term in the spectral sequence (2.14) is

q				
	0	0	0	
	$H^0(\mathbf{Y},\mathcal{O})$	0	0	
	0	0	$H^0(\mathbf{Y},\mathcal{O}\oplus\mathcal{O}_{(ABCD)})$	p

so

$$H^1(\mathcal{S},\mathcal{O}_\mathcal{S}(-2)) \cong \ker D_2 : H^0(\mathbf{Y},\mathcal{O}) \to H^0(\mathbf{Y},\mathcal{O}\oplus\mathcal{O}_{(ABCD)})$$

for a second order differential operator D_2. This gives us the equation $D_2V = 0$ satisfied by the function V previously defined. The precise form of D_2 depends on the particular geometry of the 5-manifold $\mathbf{Y}$.

2.3.9

The purpose of this subsection is to understand the nature of anti-self-dual conformal structures with twistor space projecting as affine bundles over surfaces. As before, let $\mathcal{S}$ be a complex surface, $C \subset \mathcal{S}$ a rational curve of self-intersection 4, $\mathcal{L} = \mathcal{O}_{\mathcal{S}}(-2)$ a line bundle on $\mathcal{S}$ with $\mathcal{L}^{\otimes 3} = K_{\mathcal{S}}$ and $\mathcal{Z} \xrightarrow{p} \mathcal{S}$ the total space of an affine bundle $\mathcal{E} \in H^1(\mathcal{S}, \mathcal{L})$.

The exact sequence:

$$0 \to p^*\mathcal{O}_{\mathcal{S}}(-2) \to T\mathcal{Z} \to p^*T\mathcal{S} \to 0$$

gives

$$K_{\mathcal{Z}} \cong p^*K_{\mathcal{S}} \otimes p^*\mathcal{O}_{\mathcal{S}}(2) = p^*\mathcal{O}_{\mathcal{S}}(-4)$$

so we can write as expected

$$p^*\mathcal{O}_{\mathcal{S}}(1) = K_{\mathcal{Z}}^{-1/4} = \mathcal{O}_{\mathcal{Z}}(1)$$

The embedding of the vertical bundle $\mathcal{O}_{\mathcal{Z}}(-2)$ into the tangent bundle of $\mathcal{Z}$ will give us a non-vanishing section Ψ of $T\mathcal{Z} \otimes \mathcal{O}_{\mathcal{Z}}(2)$. We need to understand the implications this has on the ASD conformal structure.

We will recall first [17] the description of the holomorphic tangent bundle on the twistor space in terms of bundles of solutions to conformally invariant differential equations. For $\alpha \in \mathcal{Z}$ an α-surface, the fiber of $\mathcal{O}_{\mathcal{Z}}(-1)$ at α consists of autoparallel spinors along α:

$$\omega^A \nabla_{AA'} \omega_B = 0$$

One also defines the rank 3 Jacobi-spinor bundle $\mathfrak{E} \to \mathcal{Z}$ with fiber at α given by the equation:

$$\omega_A \nabla^{A(A'} \pi^{B')} = 0 \tag{2.15}$$

where ω_A is an autoparallel spinor along α. With this notations the holomorphic tangent bundle to $\mathcal{Z}$ is given by

$$T\mathcal{Z} \cong \mathfrak{E} \otimes \mathcal{O}_{\mathcal{Z}}(1)$$

A tangent vector $v \in T_\alpha \mathcal{Z}$ defines a Jacobi vector field $J_v^{AA'}$ along α (as long as we choose a foliation through null geodesics) and this produces a Jacobi-spinor through $\pi^{A'} = J_v^{AA'} \omega_A$. This clearly depends on the autoparallel spinor ω_A and ends up defining the aforementioned isomorphism.

A non-vanishing global section Ψ of $T\mathcal{Z} \otimes \mathcal{O}_{\mathcal{Z}}(2) \cong \mathfrak{E} \otimes \mathcal{O}_{\mathcal{Z}}(3)$ will define a non-zero section $\psi \in \Gamma(\mathbb{S}_+^3 \otimes \mathbb{S}_-)$ such that $\pi^{A'} = \psi^{A'ABC} \omega_A \omega_B \omega_C$ is a Jacobi-spinor, so

$$\omega_A \nabla^{A(A'} \psi^{B')BCD} \omega_B \omega_C \omega_D = 0$$

for any autoparallel ω. Thus

$$\omega_A \omega_B \omega_C \omega_D \nabla^{A(A'} \psi^{B')BCD} = 0$$

so ψ satisfies the Killing equation:

$$\nabla_{(A}^{(A'} \psi_{BCD)}^{B')} = 0 \tag{2.16}$$

Reciprocally, any non-trivial solution of (2.16) determines an $\mathcal{O}_{\mathcal{Z}}(2)$-valued vector field on the twistor space $\mathcal{Z}$. The projection along its flow exhibits $\mathcal{Z}$ locally as an affine line bundle over the factor space $\mathcal{S}$ with the twistor lines mapping over rational curves with normal bundle $\mathcal{O}(4)$ in $\mathcal{S}$.

Chapter 3

Examples

3.1 Generalized Legendre Transform Revisited

We will show in this section how the generalized Legendre transform fits in our settings and then prove that our approach gives the correct conformal class.

The role of the complex surface $\mathcal{S}$ will be played by the total space of the $\mathcal{O}(4)$ bundle over $\mathbb{P}_1$, so the 5-dimensional manifolds $\mathbf{Y}$ is just the linear space of sections $\eta \in H^0(\mathbb{P}_1, \mathcal{O}(4)) = \mathbb{C}^5$, $\eta = \eta_1 + \eta_2\zeta + \eta_3\zeta^2 + \eta_4\zeta^3 + \eta_5\zeta^4$. By η we will denote either the coordinate in the fiber of $\mathcal{O}(4)$, a section in $H^0(\mathbb{P}_1, \mathcal{O}(4))$ or a tangent vector to $\mathbf{Y} = \mathbb{C}^5$, the difference being clear from the context.

To keep consistency with the notations in [10] we choose a cohomology class in $H^1(\mathcal{S}, \mathcal{O}_\mathcal{S}(-2))$ with Čech representative $\hat{f}(\eta, \zeta)/\zeta^2$ on $\zeta \neq 0, \infty$ and define

$$V = F_w = \frac{1}{2\pi i}\oint_C \frac{\hat{f}(\eta, \zeta)}{\zeta^2} d\zeta$$

We obtain a holomorphic function on $\mathbf{Y}$, solution of the system of second order

PDE's:

$$
\begin{aligned}
V_{13} &= V_{22} & V_{14} &= V_{23} \\
V_{25} &= V_{34} & V_{35} &= V_{44} \\
& V_{15} = V_{24} = V_{33}
\end{aligned}
\tag{3.1}
$$

The subscripts denote partial derivatives: $V_1 = \partial V/\partial \eta_1$, $V_{13} = \partial^2 V/\partial \eta_1 \partial \eta_3$ and so on. We will also continue using the notations introduced in (2.3.7).

By restricting the quadratic form $\mathfrak{G}$ (2.12) to the 4-dimensional manifold $\mathbf{X} = \{V = F_w = 0\}$ we can obtain a formula for the complex conformal class $\mathfrak{g}$ in terms of the η_1, η_2, η_4, η_5 coordinates only. The elimination of η_3 is done by imposing in (2.12) the linear dependence $dV = 0$. A straightforward computation gives:

$$
\begin{aligned}
V_3^2 \mathfrak{g} = & \left(-V_1 V_3 \delta_2^2 + (V_3\delta_1 + V_1\delta_4)^2\right) \eta_1^2 + \left(V_3^2 \delta_1 \delta_4 - (V_3\delta_2 - V_2\delta_4) V_2 \delta_4\right) \eta_2^2 \\
& + \left(-V_1 V_3 \delta_2 \delta_4 + (V_3\delta_1 - V_2\delta_2) V_3 \delta_2 + 2(V_3\delta_1 + V_1\delta_4) V_2\delta_4\right) \eta_1\eta_2 \\
& + \left(-V_3 V_5 \delta_4 \delta_5 + (V_3\delta_6 - V_4\delta_5) V_3 \delta_5 + 2(V_3\delta_6 + V_5\delta_4) V_4\delta_4\right) \eta_4\eta_5 \\
& + \left(V_3^2 \delta_4 \delta_6 - (V_3\delta_5 - V_4\delta_4) V_4 \delta_4\right) \eta_4^2 + \left(-V_3 V_5 \delta_5^2 + (V_3\delta_6 + V_5\delta_4)^2\right) \eta_5^2 \\
& + \left((-V_3 V_4 \delta_2 + V_3^2 \delta_3 - V_3^2 \delta_4)\delta_2 + (2V_4\delta_4 - V_3\delta_5)(V_3\delta_1 + V_1\delta_4)\right) \eta_1\eta_4 \\
& + \left((-V_2 V_3 \delta_5 + V_3^2 \delta_3 - V_3^2 \delta_4)\delta_5 + (2V_2\delta_4 - V_3\delta_2)(V_5\delta_4 + V_3\delta_6)\right) \eta_2\eta_5 \\
& + \left(2V_2 V_4 \delta_4 + V_3^2(\delta_3 - \delta_4) - V_3(V_4\delta_2 + V_2\delta_5)\right) \delta_4 \eta_2 \eta_4 \\
& + \big(-V_3 V_5 \delta_2^2 + V_3^2 \delta_3^2 - V_1 V_3 \delta_5^2 + V_1 V_5 \delta_4^2 + (V_3^2 + V_1 V_5)\delta_4^2 \\
& \qquad + 2V_3 V_5 \delta_1 \delta_4 - 2V_3^2 \delta_2 \delta_5 + 2V_1 V_3 \delta_4 \delta_6\big) \eta_1 \eta_5
\end{aligned}
$$

Using repeatedly the relations

$$\begin{aligned} V_3\delta_1 + V_1\delta_4 &= V_2\delta_2 \qquad & V_3\delta_2 - V_2\delta_4 &= V_4\delta_1 \\ V_3\delta_6 + V_5\delta_4 &= V_4\delta_5 \qquad & V_3\delta_5 - V_4\delta_4 &= V_2\delta_6 \end{aligned}$$

trivially derived from the matricial equality

$$\begin{pmatrix} V_1 & V_2 & V_3 \\ V_2 & V_3 & V_4 \\ V_3 & V_4 & V_5 \end{pmatrix}^{-1} = \frac{1}{\sigma}\begin{pmatrix} \delta_6 & -\delta_5 & \delta_4 \\ -\delta_5 & \delta_3 + \delta_4 & \delta_2 \\ \delta_4 & \delta_2 & \delta_1 \end{pmatrix}$$

we can simplify the above formula to:

$$\begin{aligned} V_3^2\mathfrak{g} = &- \delta_1(\delta_2\eta_1 + \delta_4\eta_2)^2 - \delta_6(\delta_4\eta_4 + \delta_5\eta_5)^2 \\ &+ \big(2V_2V_4\delta_4 + V_3^2(\delta_3 - \delta_4) - V_3(V_4\delta_2 + V_2\delta_5)\big) \cdot \\ &\quad \big(\delta_4\eta_2\eta_4 + \delta_2\eta_1\eta_4 + \delta_5\eta_2\eta_5\big) \\ &+ \big(-V_3V_5\delta_2^2 + V_3^2\delta_3^2 - V_1V_3\delta_5^2 + V_1V_5\delta_4^2 + (\delta_3 - \delta_4 + 2V_2V_4)\delta_4^2 \\ &\quad + 2V_3V_5\delta_1\delta_4 - 2V_3^2\delta_2\delta_5 + 2V_1V_3\delta_4\delta_6\big)\eta_1\eta_5 \end{aligned}$$

For the last stubborn coefficients

$$\begin{aligned} \sigma &= - V_2\delta_5 + V_3(\delta_3 + \delta_4) - V_4\delta_2 \\ V_3\sigma &= \delta_1\delta_6 - \delta_4^2 \end{aligned} \tag{3.2}$$

give:

$$2V_2V_4\delta_4 + V_3^2(\delta_3 - \delta_4) - V_3(V_4\delta_2 + V_2\delta_5) =$$
$$2V_2V_4\delta_4 + V_3^2(\delta_3 - \delta_4) - V_3^2(\delta_3 + \delta_4) + V_3\sigma =$$
$$2\delta_4^2 + V_3\sigma = \delta_1\delta_6 + \delta_4^2$$

while

$$V_1\delta_5 + V_3\delta_2 = V_2(\delta_3 + \delta_4) \qquad V_3\delta_5 + V_5\delta_2 = V_4(\delta_3 + \delta_4)$$

helps simplify the $\eta_1\eta_5$ coefficient:

$$- V_3V_5\delta_2^2 + V_3^2\delta_3^2 - V_1V_3\delta_5^2 + V_1V_5\delta_4^2 =$$
$$V_3^2\delta_3^2 + V_1V_5\delta_4^2 - V_2V_4(\delta_3 + \delta_4)^2 + \delta_2\delta_5(V_1V_5 + V_3^2) =$$
$$- \delta_3^2\delta_4 - 2V_2V_4\delta_3\delta_4 + \delta_4^2\delta_3 + \delta_2\delta_5(\delta_3 - \delta_4 + 2V_2V_4) =$$
$$(\delta_2\delta_5 - \delta_3\delta_4)(\delta_3 - \delta_4 + 2V_2V_4) = \delta_1\delta_6(\delta_3 - \delta_4 + 2V_2V_4)$$

We also notice that

$$2V_3V_5\delta_1\delta_4 - 2V_3^2\delta_2\delta_5 + 2V_1V_3\delta_4\delta_6 =$$
$$2\left(2\delta_1\delta_4\delta_6 + V_4^2\delta_1\delta_4 + V_2^2\delta_4\delta_6 - (V_4\delta_1 + V_2\delta_4)(V_2\delta_6 + V_4\delta_4)\right) =$$
$$4\delta_1\delta_4\delta_6 - 2V_2V_4(\delta_1\delta_6 + \delta_4^2)$$

so we end up with the relatively simple expression:

$$
\begin{aligned}
V_3^2 \mathfrak{g} = & - \delta_1(\delta_2\eta_1 + \delta_4\eta_2)^2 - \delta_6(\delta_4\eta_4 + \delta_5\eta_5)^2 \\
& + (\delta_1\delta_6 + \delta_4^2)(\delta_4\eta_2\eta_4 + \delta_2\eta_1\eta_4 + \delta_5\eta_2\eta_5) \\
& + \big(4\delta_1\delta_4\delta_6 + (\delta_1\delta_6 + \delta_4^2)(\delta_3 - \delta_4)\big)\, \eta_1\eta_5
\end{aligned}
\tag{3.3}
$$

The antipodal map on $\mathbb{CP}_1$ induces on $\mathbf{Y}$ a natural complex conjugation with a real slice given by:

$$
Y_{\mathbb{R}} = \{\eta \in \mathbf{Y} : \eta_1 = \bar{\eta}_5 = z,\ \eta_2 = -\bar{\eta}_4 = v,\ \eta_3 = w \in \mathbb{R}\}
$$

By restricting to $Y_{\mathbb{R}}$, V becomes a real valued real analytic function and the minors δ_i satisfy

$$
\delta_1 = \begin{vmatrix} V_z & V_v \\ V_v & V_w \end{vmatrix} = \bar{\delta}_6,\ \delta_2 = \begin{vmatrix} V_z & V_w \\ V_v & -V_{\bar{v}} \end{vmatrix} = -\bar{\delta}_5
$$

$$
\delta_3 = \begin{vmatrix} V_z & -V_{\bar{v}} \\ V_v & V_{\bar{z}} \end{vmatrix} \in \mathbb{R},\ \delta_4 = \begin{vmatrix} V_v & V_w \\ V_w & -V_{\bar{v}} \end{vmatrix} \in \mathbb{R}
$$

The restriction of (3.3) to $X_{\mathbb{R}} = \mathbf{X} \cap Y_{\mathbb{R}}$ gives then the following formula for the anti-self-dual conformal class in terms of the coordinates $z, \bar{z}, v, \bar{v}$:

$$
\begin{aligned}
g = -\frac{1}{V_w^2}\Big(& \delta_1(\delta_2 dz + \delta_4 dv)^2 + \bar{\delta}_1(\bar{\delta}_2 d\bar{z} + \delta_4 d\bar{v})^2 \\
& + (\delta_1\bar{\delta}_1 + \delta_4^2)(\delta_2 dz d\bar{v} + \bar{\delta}_2 dv d\bar{z} + \delta_4 dv d\bar{v}) \\
& - \big(4\delta_1\bar{\delta}_1\delta_4 + (\delta_1\bar{\delta}_1 + \delta_4^2)(\delta_3 - \delta_4)\big)\, dz d\bar{z}\Big)
\end{aligned}
\tag{3.4}
$$

We will show next that the hyperkähler metric obtained through the Legendre transform (2.4) belongs to this conformal class.

The partial derivatives matrix $\mathbb{F}$ and its inverse $\mathbb{G}$ can be expressed in terms of $dV = dF_w$ using the PDE system (2.3) satisfied by F:

$$\mathbb{F} = \begin{pmatrix} F_{vv} & F_{vw} & F_{v\bar{v}} \\ F_{wv} & F_{ww} & F_{w\bar{v}} \\ F_{\bar{v}v} & F_{\bar{v}w} & F_{\bar{v}\bar{v}} \end{pmatrix} = \begin{pmatrix} V_z & V_v & -V_w \\ V_v & V_w & V_{\bar{v}} \\ -V_w & V_{\bar{v}} & V_{\bar{z}} \end{pmatrix} \tag{3.5}$$

$$\mathbb{G} = \begin{pmatrix} G^{vv} & G^{vw} & G^{v\bar{v}} \\ G^{wv} & G^{ww} & G^{w\bar{v}} \\ G^{\bar{v}v} & G^{\bar{v}w} & G^{\bar{v}\bar{v}} \end{pmatrix} = \mathbb{F}^{-1} = \frac{1}{\sigma} \begin{pmatrix} \bar{\delta}_1 & \bar{\delta}_2 & -\delta_4 \\ \bar{\delta}_2 & \delta_3 + \delta_4 & \delta_2 \\ -\delta_4 & \delta_2 & \delta_1 \end{pmatrix} \tag{3.6}$$

The coefficients α, β in the formula for the hyperkähler metric (2.5) become:

$$\alpha = \frac{1}{\sigma}(V_z\delta_2 - V_v\delta_1 - F_{zv}\delta_4), \quad \beta = \frac{\delta_4}{\sigma}$$

Also, using again (2.3), we can write:

$$du = dF_v = F_{zv}dz + V_z dv + V_v dw - V_w d\bar{v} - V_{\bar{v}} d\bar{z}$$

so

$$\alpha dz + \beta du = \frac{1}{\sigma}\left((V_z\delta_2 - V_v\delta_1)dz + \delta_4(V_z dv + V_v dw - V_w d\bar{v} - V_{\bar{v}} d\bar{z})\right)$$

The linear dependence $dV = 0$ eliminates dw to give:

$$\sigma(\alpha dz + \beta du)V_w = \delta_1\delta_2 dz + \delta_1\delta_4 dv + \delta_4^2 d\bar{v} + \bar{\delta}_2\delta_4 d\bar{z} \tag{3.7}$$

hence the metric becomes:

$$\begin{aligned} ds^2 &= \frac{1}{\beta}\Big(dzd\bar{z} + (\alpha dz + \beta du)\cdot\overline{(\alpha dz + \beta du)}\Big) \\ &= \frac{1}{\sigma\delta_4 V_w^2}\Big(\sigma^2 V_w^2 dzd\bar{z} + (\delta_1\delta_2 dz + \delta_1\delta_4 dv + \delta_4^2 d\bar{v} + \bar{\delta}_2\delta_4 d\bar{z})\cdot \\ &\qquad (\delta_2\delta_4 dz + \delta_4^2 dv + \delta_4\bar{\delta}_1 d\bar{v} + \bar{\delta}_1\delta_2 d\bar{z})\Big) \\ &= \frac{1}{\sigma\delta_4 V_w^2}\Big(\sigma^2 V_w^2 dzd\bar{z} + \delta_2\bar{\delta}_2(\delta_1\bar{\delta}_1 + \delta_4^2)dzd\bar{z} \\ &\qquad + \delta_4\delta_1(\delta_2 dz + \delta_4 dv)^2 + \delta_4\bar{\delta}_1(\bar{\delta}_2 d\bar{z} + \delta_4 d\bar{v})^2 \\ &\qquad + \delta_4(\delta_1\bar{\delta}_1 + \delta_4^2)(\delta_2 dzd\bar{v} + \bar{\delta}_2 dvd\bar{z} + \delta_4 dvd\bar{v})\Big) \end{aligned}$$

But squaring (3.2) gives

$$\begin{aligned} \sigma^2 V_w^2 &= (\delta_1\bar{\delta}_1 - \delta_4^2)^2 \\ &= (\delta_1\bar{\delta}_1 + \delta_4^2)^2 - 4\delta_1\bar{\delta}_1\delta_4^2 \\ &= -\delta_2\bar{\delta}_2(\delta_1\bar{\delta}_1 + \delta_4^2) - \delta_4\left(4\delta_1\bar{\delta}_1\delta_4 + (\delta_1\bar{\delta}_1 + \delta_4^2)(\delta_3 - \delta_4)\right) \end{aligned}$$

and thus we indeed verified that the hyperkähler metric ds^2 belongs to the conformal family $[g]$ defined by (3.4).

3.2 Space of Plane Conics

We will illustrate in detail how the general construction described in Section 2.3 works when the twistor space projects as an affine line bundle over the projective plane. More precisely, let $\mathcal{S}$ be the complement of a finite set in $\mathbb{CP}_2$ and let $\mathbf{Y}$ be the space of nondegenerate plane conics sitting in $\mathcal{S}$. We can identify $\mathbf{Y}$ as an open set in

$$\mathrm{SL}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{C}) = \{A = (a_{ij})_{0\leq i,j\leq 2} : A = A^t, \det A = 1\}$$

Indeed, any conic is defined by a unique symmetric matrix A of determinant 1 through the equation $z \cdot A \cdot z^t = 0$, where $z = [z_0 : z_1 : z_2]$ denote the homogeneous coordinates on $\mathbb{CP}_2$. An element $g \in \mathrm{SL}(3,\mathbb{C})$ acts naturally on symmetric matrices $A \to g \cdot A \cdot g^t$ and the stabilizer of the identity is $\mathrm{SO}(3,\mathbb{C})$.

We will consider the affine bundles $\mathcal{E}$ over $\mathcal{S} \subset \mathbb{CP}_2$ coresponding to the tautological $\mathcal{O}_{\mathbb{P}_2}(-1)$. This line bundle restricts to any conic with degree -2, so it plays the role of $\mathcal{L}$ from the previous sections. Any cohomology class in $H^1(\mathcal{S}, \mathcal{O}_{\mathbb{P}_2}(-1))$ defines through integration a holomorphic function on $\mathbf{Y}$.

We want to determine the precise form of the system of differential equations $D_2V = 0$ satisfied by all these functions V. From the general case we know that D_2 is a second degree operator with values in the $\mathrm{SL}(2,\mathbb{C})$ bundle $\mathcal{O} \oplus \mathcal{O}_{(ABCD)} = \mathcal{O} \oplus T^*\mathbf{Y}$ and of course we expect it to be $\mathrm{SL}(3,\mathbb{C})$ invariant. We will determine D_2 by constructing a sufficiently large family of solutions.

3.2.1

Let $\mathcal{S} = \mathbb{CP}_2 - [1:0:0]$ with the acyclic covering $\mathcal{U} = \{U_1, U_2\}$, where $U_1 = (z_1 \neq 0)$ and $U_2 = (z_2 \neq 0)$. A cohomology class in $H^1(\mathcal{S}, \mathcal{O}_{\mathbb{P}_2}(-1))$ will be determined by a holomorphic function f on $U_1 \cap U_2$ homogeneous of degree -1. In order to integrate f along a conic $(A = 0)$ we need to construct a consistent isomorphism $K_A \cong \mathcal{O}_{\mathbb{P}_2}(-1)|_A$. In practice this is done by choosing an $\mathcal{O}_{\mathbb{P}_2}(1)$-valued 1-form θ satisfying

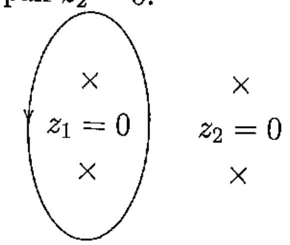
$$\theta \wedge dA = z_0 dz_1 \wedge dz_2 - z_1 dz_0 \wedge dz_2 + z_2 dz_0 \wedge dz_1 \tag{3.8}$$

namely

$$\theta = \frac{z_1 dz_2 - z_2 dz_1}{A_{z_0}} = \frac{-z_0 dz_2 + z_2 dz_0}{A_{z_1}} = \frac{z_0 dz_1 - z_1 dz_0}{A_{z_2}}$$

Using the fact that $dA|_A = 0$ we deduce that the restriction of θ to any conic is uniquely determined by equation (3.8), so θ coherently defines an "integration along conics" that will associate to any element in $H^1(\mathcal{S}, \mathcal{O}_{\mathbb{P}_2}(-1))$ a solution of $D_2 V = 0$.

We will consider first the "baby case" of integrating $f_1(z) = \dfrac{z_0}{z_1 z_2}$ along the standard conic $(z_0^2 + z_1^2 + z_2^2 = 0)$ coresponding to $A = Id$. This will reduce to computing a contour integral $\oint \dfrac{z_0}{z_1 z_2} \cdot \theta$ along a path that separates the pair of points $z_1 = 0$ on the conic from the pair $z_2 = 0$.



$\zeta - plane$

In affine coordinates $(z_0, z_1, 1)$ on U_0 the conic is given by $(z_0^2 + z_1^2 + 1 = 0)$ and can be $2:1$ parameterized by:

$$z_0 = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right), \quad z_1 = \frac{1}{2i}\left(\zeta + \frac{1}{\zeta}\right)$$

The contour integral would become:

$$\oint \frac{z_0}{z_1 z_2} \cdot \theta = \oint \frac{z_0}{z_1 z_2} \cdot \left(-\frac{dz_1}{2z_0}\right) = -\frac{1}{2}\oint \frac{dz_1}{z_1} = -\frac{1}{4}\oint \frac{\zeta^2 - 1}{\zeta(\zeta^2+1)} d\zeta = -\pi i$$

3.2.2

We will extend this computation to a general conic $(A = 0)$ showing that the solution of $D_2 u = 0$ coresponding to the cohomology class $f_1(z) = \dfrac{z_0}{z_1 z_2}$ is

$$V_1(A) = \frac{1}{a_{00}}$$

up to multiplying with $-\pi i$. This solution is of course defined on the space $\mathbf{Y}$ of conics not passing through $[1:0:0]$.

In order to parametrize a general conic we will use the fact that any symmetric matrix can be written as the product of an upper triangular matrix with its transpose. Thus, given a conic $(A = 0)$, let

$$N = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 \\ 0 & 0 & c_1 \end{pmatrix}$$

such that

$$\det N = 1 \text{ and } N \cdot N^t = A^{-1}. \tag{3.9}$$

This will determine a 2 : 1 map from $\mathbb{CP}_1$ onto the conic:

$$(z_0, z_1, z_2)^t = N \cdot (s^2 - t^2, 2st, i(s^2 + t^2))^t \tag{3.10}$$

where $[s : t]$ denote homogeneous coordinates on $\mathbb{CP}_1$.

With respect to this parametrization we have

$$\theta = \frac{z_1 dz_2 - z_2 dz_1}{A_{z_0}} = i(tds - sdt)$$

so, restricting to the affine chart $(t = 1)$:

$$V_1(A) = \oint \frac{z_0}{z_1 z_2} \cdot \theta = \frac{1}{2} \oint \frac{z_0(s)}{z_1(s) i c_1 (s^2 + 1)} i ds = \frac{1}{2} 2\pi i \frac{a_1}{b_1 c_1} = \frac{\pi i}{a_{00}}$$

In all of these computations we repeatedly used condition (3.9). The difference in sign between the above result and the "baby case" comes from changing the integration contour to enclose the pair of points $(z_2 = 0)$, instead of $(z_1 = 0)$.

Through similar computations we can obtain other solutions of the equation $D_2 V = 0$ coresponding to the cohomology classes given by $f_p(z) = \frac{z_0^p}{z_1^p z_2}$ on $U_1 \cap U_2$:

$$V_p(A) = \frac{\pi i}{a_{00}^p} \sum_{k=0}^{[\frac{p-1}{2}]} (-1)^{p+k+1} \begin{pmatrix} p \\ 2k+1 \end{pmatrix} a_{01}^{p-2k-1} b_{22}^k$$

where we denote by $B = (b_{ij}) = A^{-1}$ the inverse matrix of A, so in particular

$b_{22} = a_{00}a_{11} - a_{01}^2$. We will use:

$$V_2(A) = -\frac{2\pi i a_{01}}{a_{00}^2}$$
$$V_3(A) = \frac{\pi i}{a_{00}^3}(3a_{01}^2 - b_{22}) = \frac{\pi i}{a_{00}^3}(4a_{01}^2 - a_{00}a_{11})$$

Also by integrating $g_p(z) = \frac{z_0^{p+1}}{z_1^p z_2^2}$ along conics we can get another series of solutions

$$W_p(A) = \pi i(p+1)\frac{b_{02}}{a_{00}^p}\sum_{k=0}^{[\frac{p-1}{2}]}(-1)^{p+k+1}\begin{pmatrix} p \\ 2k+1 \end{pmatrix} a_{01}^{p-2k-1} b_{22}^{k-1}$$
$$+\pi i p\frac{b_{12}}{a_{00}^{p+1}}\sum_{k=0}^{[\frac{p}{2}]}(-1)^{p+k+1}\begin{pmatrix} p+1 \\ 2k+1 \end{pmatrix} a_{01}^{p-2k} b_{22}^{k-1},$$

of which we are most interested in

$$W_2(A) = \frac{2\pi i}{a_{00}^3}(4a_{01}a_{02} - a_{00}a_{12})$$

3.2.3

However, solutions of a more interesting flavor can be obtained by using $f(z) = \frac{1}{\sqrt{z_1 z_2}}$. In this case we end up with an elliptic integral that evaluates a period of the elliptic curve realized as a ramified cover branched at the four points $(z_1 = 0), (z_2 = 0)$ on the conic. We will write this as a function of the cross-ratio λ of the given points.

Using parametrization (3.10) and fixing a Möbius transform that takes $(z_1 = 0)$

to $(0,1)$ and $(z_2 = 0)$ to (λ, ∞), we can express λ in terms of the matrix A

$$\lambda = \frac{1}{2} + \frac{1}{2}\frac{b_{12}}{\sqrt{b_{11}b_{22}}} \tag{3.11}$$

and we can compute

$$\begin{aligned} V_f(A) = \oint \frac{1}{\sqrt{z_1 \cdot z_2}} \cdot \theta &= \frac{1}{2} \oint \frac{ds}{\sqrt{(b_2(s^2+1) - 2ib_1 s) \cdot c_1(s^2+1)}} \\ &= \frac{\pi}{2(b_{11}b_{22})^{1/4}} \cdot I(\lambda) \end{aligned}$$

with

$$I(\lambda) = \oint \frac{ds}{\sqrt{s(s-1)(s-\lambda)}} \tag{3.12}$$

3.2.4

Our main goal in performing these computations is to determine the second degree operator D_2 on the space of conics such that $D_2 V_f = 0$ for any f.

We will verify directly that a plausible candidate is the pair $(\triangle, \square)$ given by

$$\triangle = \left(Trace\left(A \cdot \frac{\partial}{\partial A} \right) \right)^2 - 1 \tag{3.13}$$

and

$$\square = \frac{\partial}{\partial A} \cdot A \cdot \frac{\partial}{\partial A} \tag{3.14}$$

with

$$\frac{\partial}{\partial A} = \left(\frac{1}{2}(1+\delta_{ij}) \frac{\partial}{\partial a_{ij}} \right)_{ij}$$

Explicitly

$$\triangle V = \sum_{i\leq j} a_{ij}\frac{\partial V}{\partial a_{ij}} + \sum_{i,j,k,l} a_{ij}a_{kl}\frac{\partial^2 V}{\partial a_{ij}\partial a_{kl}} - V$$

and

$$\begin{aligned}\Box_{00}V =& a_{00}\frac{\partial^2 V}{\partial a_{00}^2} + a_{01}\frac{\partial^2 V}{\partial a_{00}\partial a_{01}} + a_{02}\frac{\partial^2 V}{\partial a_{00}\partial a_{02}} + \\ &+\frac{1}{4}a_{11}\frac{\partial^2 V}{\partial a_{01}^2} + \frac{1}{2}a_{12}\frac{\partial^2 V}{\partial a_{01}\partial a_{02}} + \frac{1}{4}a_{22}\frac{\partial^2 V}{\partial a_{02}^2} + 2\frac{\partial V}{\partial a_{00}}\end{aligned}$$

$$\begin{aligned}\Box_{01}V =& \frac{1}{2}a_{00}\frac{\partial^2 V}{\partial a_{00}\partial a_{01}} + a_{01}\frac{\partial^2 V}{\partial a_{00}\partial a_{11}} + \frac{1}{2}a_{02}\frac{\partial^2 V}{\partial a_{00}\partial a_{12}} + \\ &+\frac{1}{4}a_{01}\frac{\partial^2 V}{\partial a_{01}\partial a_{01}} + \frac{1}{2}a_{11}\frac{\partial^2 V}{\partial a_{01}\partial a_{11}} + \frac{1}{4}a_{12}\frac{\partial^2 V}{\partial a_{01}\partial a_{12}} + \\ &+\frac{1}{4}a_{02}\frac{\partial^2 V}{\partial a_{02}\partial a_{01}} + \frac{1}{2}a_{12}\frac{\partial^2 V}{\partial a_{02}\partial a_{11}} + \frac{1}{4}a_{22}\frac{\partial^2 V}{\partial a_{02}\partial a_{12}} + \frac{\partial V}{\partial a_{01}}\end{aligned}$$

with all the other entries obtained through cyclical permutations.

These verifications are just a prelude to a proof that $(\triangle, \Box)$ are indeed the right operators.

We will start with $V_1(A) = \dfrac{1}{a_{00}}$. We have

$$\begin{aligned}\triangle V_1 &= \left(a_{00}\frac{\partial}{\partial a_{00}} + a_{00}^2\frac{\partial^2}{\partial a_{00}^2}\right)\left(\frac{1}{a_{00}}\right) - \frac{1}{a_{00}} = 0 \\ \Box_{00}V_1 &= \left(a_{00}\frac{\partial^2}{\partial a_{00}^2} + 2\frac{\partial}{\partial a_{00}}\right)\left(\frac{1}{a_{00}}\right) = 0\end{aligned}$$

and all the other entries vanish as there are no more terms involving only $\dfrac{\partial}{\partial a_{00}}$.

For $V_2(A) = \frac{a_{01}}{a_{00}^2}$ the only possibly non-zero entries are:

$$\triangle V_2 = \left(a_{00}\frac{\partial}{\partial a_{00}} + a_{01}\frac{\partial}{\partial a_{01}} + a_{00}^2\frac{\partial^2}{\partial a_{00}^2} + 2a_{00}a_{01}\frac{\partial^2}{\partial a_{00}a_{01}} \right) \left(\frac{a_{01}}{a_{00}^2} \right) - \frac{a_{01}}{a_{00}^2} = 0$$

and

$$\Box_{00}V_2 = \left(a_{00}\frac{\partial^2}{\partial a_{00}^2} + a_{01}\frac{\partial}{\partial a_{00}}\frac{\partial}{\partial a_{01}} + 2\frac{\partial}{\partial a_{00}} \right) \left(\frac{a_{01}}{a_{00}^2} \right) = 0$$

$$\Box_{01}V_2 = \left(\frac{1}{2}a_{00}\frac{\partial}{\partial a_{00}}\frac{\partial}{\partial a_{01}} + \frac{\partial}{\partial a_{01}} \right) \left(\frac{a_{01}}{a_{00}^2} \right) = 0$$

Similar computations show that $V_3(A)$ and $W_2(A)$ are solutions of $(\triangle, \Box)$.

3.2.5

In order to prove that

$$V_f(A) = \frac{1}{(b_{11}b_{22})^{1/4}} \cdot I(\lambda)$$

is also a solution of (3.13) and (3.14) we will have to use repeatedly the Picard-Fuchs equation [5] satisfied by $I(\lambda)$:

$$\lambda(\lambda-1)\frac{d^2 I}{d\lambda^2} + (2\lambda-1)\frac{dI}{d\lambda} + \frac{1}{4}I = 0 \tag{3.15}$$

We will also use MAPLE to get:

$$\begin{aligned}\Box_{00}V_f &= \frac{1}{4(b_{11}b_{22})^{1/4}}\frac{a_{00}}{b_{22}}\left(\frac{a_{00}}{4b_{11}b_{22}}I'' - \frac{b_{12}}{\sqrt{b_{11}b_{22}}}I' - \frac{1}{4}I \right) \\ &= \frac{1}{4(b_{11}b_{22})^{1/4}}\frac{a_{00}}{b_{22}}\left(\lambda(1-\lambda)I'' - (2\lambda-1)I' - \frac{1}{4}I \right) = 0\end{aligned}$$

and so on.

3.2.6

As a result of these rather lengthy computations we get the following:

Proposition 3.2.1. *Let S be the complement of a finite set in $\mathbb{CP}_2$ and $\mathbf{Y}$ the space of nondegenerate plane conics sitting in S. Then integration along conics gives an isomorphism*

$$H^1(S, \mathcal{O}_{\mathbb{P}_2}(-1)) \cong Ker\triangle \cap Ker\square$$

where $\triangle$ and $\square$ are the second order operators on $\mathbf{Y}$

$$\begin{aligned} \triangle &= \left(Trace\left(A \cdot \frac{\partial}{\partial A}\right)\right)^2 - 1 \\ \square &= \left(A \cdot \frac{\partial}{\partial A}\right)^2 \end{aligned} \tag{3.16}$$

Proof. Following the discussion in (2.3.8) we need to verify that, at 2-jet level, the kernel of $(\triangle, \square)$ coincides with the space of 2-jets of functions obtained through integration along conics $V(A) = \langle A = 0, \omega \rangle$. Given the equivariance of the two operators, it is enough to verify this claim in $A_0 = Id \in \mathbf{Y}$.

The space of 2-jets at the identity on the space of determinant 1 symmetric matrices is

$$\mathcal{J}^2_{A_0}\mathbf{Y} = \langle 1, da_{ij}, da_{ij}da_{kl} : i \leq j, k \leq l \rangle / \langle \sum a_{ii}, a_{ij} \sum a_{ii} \rangle$$

We notice that $Ker\square$ intersects $\langle \sum a_{ii}, a_{ij} \sum a_{ii} \rangle$ along $\langle \sum a_{ii} - 2(\sum a_{ii})^2 \rangle$ and this is not included in $Ker\triangle$. We obtain that $Ker\triangle \cap Ker\square \cap \mathcal{J}^2_{A_0}\mathbf{Y}$ is 15 dimensional so our proof would be complete if we exhibit 15 linearly independent elements in $\mathcal{J}^2_{A_0}\mathbf{Y}$ given as 2-jets of functions obtained through integration which are also solutions of $(\triangle, \Box)$.

These are:

- $1 - da_{ii} + 2(da_{ii})^2$ coresponding to functions of type V_1;
- $da_{ij} - 2da_{ii}da_{ij}, i \neq j$ from functions of type V_2;
- $2 - da_{ii} - da_{jj} + 2da_{ii}da_{jj} + 8(da_{ij})^2, i \neq j$ from $V_3 + 3V_1$;
- $da_{ij} - 2da_{kk}da_{ij} - 4da_{ik}da_{jk}, \{i,j,k\} = \{1,2,3\}, i < j$ from W_2 type integrals.

■

3.2.7

A natural question we will answer next is: What are the defining features of anti-self-dual metrics on hypersurfaces in the space of plane conics, corresponding to solutions of (3.16)?

We notice that the line bundle $p^*\mathcal{O}_{\mathbb{P}_2}(1)$ defining the projection $\mathcal{Z} \xrightarrow{p} \mathbb{CP}_2$ of the twistor space over the projective plane coincides with the square root of the anticanonical bundle of $\mathcal{Z}$.

$$p^*\mathcal{O}_{\mathbb{P}_2}(1) = \mathcal{O}_{\mathcal{Z}}(2) = K_{\mathcal{Z}}^{-1/2} \tag{3.17}$$

We deduce that any metric emerging from our construction admits a net of *fundamental divisors*:

$$\dim \left| -\frac{1}{2}K_{\mathcal{Z}} \right| \geq 2 \tag{3.18}$$

Through Penrose correspondence, such divisors are in bijection with self-dual 2-forms $\omega \in \Gamma(\mathbb{S}^2_+) = \Gamma(\Lambda^+)$ satisfying the Penrose twistor equation $\overline{\mathrm{D}}_2\omega = 0$ or equivalently

$$\nabla\omega = d\omega$$

Following [21] we deduce that our conformal class contains a 2-sphere of Kähler, scalar flat metrics.

Conversely, the existence of a 3-dimensional space of fundamental divisors implies the existence of a map $\mathcal{Z} \overset{p}{\dashrightarrow} \mathbb{CP}_2$ as above, with $p^*\mathcal{O}_{\mathbb{P}_2}(1) = K_{\mathcal{Z}}^{-1/2}$. There are two mutually exclusive cases to be considered. Either the image of p lies on a curve in $\mathbb{CP}_2$, or, by shrinking $\mathcal{Z}$ if necessary, we can assume that the differential p_* is of maximal rank on $\mathcal{Z}$.

In the first case, using that $K_{\mathcal{Z}}^{-1/2}|_L \cong \mathcal{O}_{\mathbb{P}_1}(2)$ we deduce that the image of p can only be a plane conic, thus p becomes a map $\mathcal{Z} \longrightarrow \mathbb{P}_1$ given by the linear system $K_{\mathcal{Z}}^{-1/4}$, so the conformal class contains a locally hyperkähler metric.

In the second case, the exact sequence

$$0 \longrightarrow \ker p_* \longrightarrow T\mathcal{Z} \longrightarrow p^*T\mathbb{CP}_2 \longrightarrow 0$$

gives

$$\ker p_* \cong (K_{\mathcal{Z}})^{-1} \otimes p^*K_{\mathbb{P}_2} \cong p^*\mathcal{O}_{\mathbb{P}_2}(-1) \cong K_{\mathcal{Z}}^{1/2}$$

so $\mathcal{Z}$ can be seen as an affine bundle over $\mathbb{CP}_2$ with translation sheaf $\mathcal{O}_{\mathbb{P}_2}(-1)$ and thus, excluding the hyperkähler case, any conformal structure satisfying (3.18) can be obtained through our construction.

3.2.8

We remark here that any anti-self-dual metric with positive scalar curvature on S^4, $\overline{\mathbb{CP}_2}$, $\overline{\mathbb{CP}_2}\#\overline{\mathbb{CP}_2}$ and $\overline{\mathbb{CP}_2}\#\overline{\mathbb{CP}_2}\#\overline{\mathbb{CP}_2}$ satisfies the condition (3.18), so it can be obtained from appropriate solutions of (3.16).

Indeed, assuming that the anti-self-dual manifold **X** is compact, the Euler characteristic of $K_{\mathcal{Z}}^{-1/2}$ can be expressed in terms of the signature of **X** [8]:

$$\chi(K_{\mathcal{Z}}^{-1/2}) = 2(5 + \tau_{\mathbf{X}})$$

Moreover, Hitchin's vanishing theorem [7] shows that $H^2(\mathcal{Z}, K_{\mathcal{Z}}^{-1/2}) = 0$ under the positive scalar curvature assumption, so as long as $\tau_{\mathbf{X}} \geq -3$ the space of fundamental divisors is at least 4 dimensional.

We will show next how the standard metric on S^4 and the Fubini-Study metric on $\overline{\mathbb{CP}_2}$ adapt to our construction. In both cases the coresponding solutions V of (3.16) will not be unique, as they depend on the choice of a 3-dimensional subspace of $H^0(\mathcal{Z}, K_{\mathcal{Z}}^{-1/2})$.

For the conformally flat case, the twistor space is $\mathcal{Z} = \mathbb{CP}_3$ with canonical bundle $K_{\mathcal{Z}} = \mathcal{O}_{\mathbb{CP}_3}(-4)$, so the map $\mathbb{CP}_3 \stackrel{p}{\dashrightarrow} \mathbb{CP}_2$ will be given by a choice of three linearly independent quadrics on $\mathbb{CP}_3$, say

$$\begin{aligned} z_0 &= x_0 x_1 \\ z_1 &= x_0^2 + x_2^2 \\ z_2 &= x_1^2 + x_3^2 \end{aligned}$$

A vertical section $\Psi \in TZ \otimes \mathcal{O}_Z(2) = T\mathbb{CP}_3(2)$ can be written as

$$\Psi = \sum \Psi_i(x) \frac{\partial}{\partial x_i}$$

with $\Psi_i(x)$ homogeneous polynomials of degree 3 satisfying

$$x_0\Psi_1 + x_1\Psi_0 = 0$$

$$x_0\Psi_0 + x_2\Psi_2 = 0$$

$$x_1\Psi_1 + x_3\Psi_3 = 0$$

and so, up to a multiplicative constant we have

$$\Psi = x_0x_2x_3\frac{\partial}{\partial x_0} - x_1x_2x_3\frac{\partial}{\partial x_1} - x_0^2x_3\frac{\partial}{\partial x_2} + x_1^2x_2\frac{\partial}{\partial x_3}$$

In order to define a cohomology class representing an affine line bundle we need to restrict to neighborhoods of a fixed twistor line and of the coresponding plane conic. We will choose the line $L_0 = (x_2 = x_3 = 0)$ projecting over $A_0 = (z_0^2 = z_1z_2)$.

Given any plane conic A close to A_0 the function $V(A)$ will measure the obstruction for A to be the image of a twistor line close to L_0. $V(A)$ will then be given by a contour integral on A

$$V(A) = \oint t(z_0, z_1, z_2)\theta$$

where the function $t(z_0, z_1, z_2)$ is the "distance" between the planes $x_2 = 0$ and $x_3 = 0$ along the fiber $p^{-1}([z_0, z_1, z_2])$. This distance function is given implicitly by the equation

$$\frac{dx}{dt} = \Psi(x)$$

which, using the fiber constraints, can be simplified to

$$\frac{dx_0}{dt} = x_0\sqrt{z_1 - x_0^2}\sqrt{z_2 - \frac{z_0^2}{x_0^2}}$$

Substituting $\tau = x_0^2$ we get:

$$t(z_0, z_1, z_2) = \frac{1}{2\sqrt{z_2}}\int_{z_1}^{z_0^2/z_2} \frac{d\tau}{\sqrt{\tau(z_1-\tau)(\tau - z_0^2/z_2)}}$$

In the case of the Fubini-Study metric on $\overline{\mathbb{CP}_2}$, the twistor space is known to be the flag manifold of points on lines in $\mathbb{CP}_2$

$$\mathcal{Z} = \{(x,y) \in \mathbb{CP}_2 \times \mathbb{CP}_2^* : x \in y\} = \{([x_i],[y^j]) \in \mathbb{CP}_2 \times \mathbb{CP}_2^* : \sum_{i=0}^{2} x_i y^i = 0\}$$

The canonical bundle is the restriction to $\mathcal{Z}$ of the line bundle on $\mathbb{CP}_2 \times \mathbb{CP}_2^*$ of bidegree $(-2,-2)$, thus $K_{\mathcal{Z}}^{-1/2} = \mathcal{O}(1,1)|_{\mathcal{Z}}$, so the map $\mathcal{Z} \overset{p}{\dashrightarrow} \mathbb{CP}_2$ can be chosen of the form

$$\begin{aligned} z_0 &= x_0 y^1 \\ z_1 &= x_1 y^2 \\ z_2 &= x_2 y^0 \end{aligned}$$

defined on the complement of three points in $\mathcal{Z}$.

A section $\Psi \in T\mathcal{Z} \otimes K_{\mathcal{Z}}^{-1/2}$ can be written as an $\mathcal{O}(1,1)$ valued vector field on

$\mathbb{CP}_2 \times \mathbb{CP}_2^*$ vanishing on $x_0y^0 + x_1y^1 + x_2y^2$. Let

$$\Psi = \sum \Psi_i(x,y)\frac{\partial}{\partial x_i} + \sum \Phi^l(x,y)\frac{\partial}{\partial y^l}$$

with $\Psi_i(x,y) = \Psi_{ij}^{kl}x_kx_ly^j \in \mathcal{O}(2,1)$ and $\Phi^l(x,y) = \Phi_{ij}^{kl}x_ky^iy^j \in \mathcal{O}(1,2)$. Notice that Ψ_{ij}^{kl} is symmetric only in its upper indices, while Φ_{ij}^{kl} in its lower ones. The condition that Ψ vanishes on $x_0y^0 + x_1y^1 + x_2y^2$ is equivalent to

$$\Psi_{ij}^{kl} + \Psi_{ji}^{kl} + \Phi_{ij}^{kl} + \Phi_{ij}^{lk} = 0$$

The condition that Ψ is tangent to the fibers of p imposes further linear restrictions on the coefficients Ψ_{ij}^{kl} and Φ_{ij}^{kl}. After solving this system of linear equations, modulo the Euler vector $\sum x_i\frac{\partial}{\partial x_i} - y^i\frac{\partial}{\partial y^i}$ and up to a multiplicative constant, we get to the unique

$$\begin{aligned}\Psi =& x_2y^2\left(x_0\frac{\partial}{\partial x_0} - y^1\frac{\partial}{\partial y^1}\right) + \\ & x_0y^0\left(x_1\frac{\partial}{\partial x_1} - y^2\frac{\partial}{\partial y^2}\right) + \\ & x_1y^1\left(x_2\frac{\partial}{\partial x_2} - y^0\frac{\partial}{\partial y^0}\right)\end{aligned}$$

The equations

$$\frac{dx_i}{dt} = \Psi_i(x,y) \qquad \frac{dy^l}{dt} = \Phi^l(x,y)$$

determining the vertical distance function $t(z_0, z_1, z_2)$ can then be reduced to

$$\frac{dx_0}{dt} = \frac{x_0x_2}{x_1}z_1 \qquad \frac{dx_1}{dt} = \frac{x_0x_1}{x_2}z_2 \qquad \frac{dx_2}{dt} = \frac{x_1x_2}{x_0}z_0$$

where x_i are constrained by

$$\frac{x_0}{x_2}z_2 + \frac{x_1}{x_0}z_0 + \frac{x_2}{x_1}z_1 = 0$$

Under the natural substitution $a = \frac{x_0}{x_2}z_2$, $b = \frac{x_1}{x_0}z_0$ and $c = \frac{x_2}{x_1}z_1$ the equations become:

$$\frac{da}{dt} = z_0z_1z_2\left(\frac{1}{b} - \frac{1}{c}\right)$$
$$\frac{db}{dt} = z_0z_1z_2\left(\frac{1}{c} - \frac{1}{a}\right)$$
$$\frac{dc}{dt} = z_0z_1z_2\left(\frac{1}{a} - \frac{1}{b}\right)$$

with $a + b + c = 0$ and $abc = z_0z_1z_2$.

Eliminating b and c we end up with

$$\frac{da}{dt} = \sqrt{a^4 - 4az_0z_1z_2}$$

Recall that the Fubini-Study conformal class is given by the family of twistor lines

$$L_{m,l} = \{(x, y) \in \mathcal{Z} : x \in l, m \in y\}$$

indexed by pairs $(m, l) \in \mathbb{CP}_2 \times \mathbb{CP}_2^*$ with $m \notin l$. We will restrict to the neighborhoods of the line $L_0 = (x_0 = x_2, y_0 = y_1)$ and of its image through p, that is the double line $A_0 = (z_0 = z_2)$.

Notice that the fiber $p^{-1}(z_0, z_1, z_2)$ intersects $x_0 = x_2$ at $a = z_2$ and $y_0 = y_1$ at

$a = z_0$ so the vertical distance function between $x_0 = x_2$ and $y_0 = y_1$ is

$$t(z_0, z_1, z_2) = \int_{z_0}^{z_2} \frac{da}{\sqrt{a^4 - 4az_0z_1z_2}}$$

A solution of (3.16) will be obtained by integrating $t(z_0, z_1, z_2)$ on appropriate contours on plane conics close to A_0.

Bibliography

[1] M. F. ATIYAH, *Complex fibre bundles and ruled surfaces*, Proc. Lond. Math. Soc. 5 (1955), pp. 407–434.

[2] M. F. ATIYAH, N. J. HITCHIN AND I. M. SINGER, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A 362 (1978), pp. 425–461.

[3] N. BUCHDAHL, *On the relative de Rham sequence*, Proc. Amer. Math. Soc. 87 (1983), pp. 363–366.

[4] S. A. CHERKIS AND N. J. HITCHIN, *Gravitational Instantons of Type D_k*, e-print, hep-th/0310084.

[5] R. FUCHS, *Über lineare homogene Differentialgleichungen zweiter Ordnung mit im endlich gelegene wesentlich singulären Stellen*, Math. Ann. 63 (1907), pp. 301–321.

[6] G. W. GIBBONS AND S. W. HAWKING, *Gravitational multi-instantons*, Phys. Lett. 78B (1978), pp. 430–432.

[7] N. J. HITCHIN, *Linear field equations on self-dual spaces*, Proc. Roy. Soc. London Ser. A 370 (1980), pp. 173–191.

[8] N. J. HITCHIN, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) 43 (1981), no. 1, pp. 133–150.

[9] N. J. HITCHIN, *Complex manifolds and Einstein's equations*, Lecture Notes in Math. 970, Springer, Berlin, (1982), pp. 73–99.

[10] I. T. IVANOV AND M. ROČEK, *Supersymmetric σ-models, twistors, and the Atiyah-Hitchin metric*, Comm. Math. Phys. 182 (1996), pp. 291–302.

[11] P. E. JONES AND K. P. TOD, *Minitwistor spaces and Einstein-Weyl spaces*, Classical Quantum Gravity 2 (1985), pp. 565–577.

[12] K. KODAIRA, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. (2) 75 (1962), pp. 146–162.

[13] K. KODAIRA AND D. C. SPENCER , *On deformations of complex analytic structures. I, II.*, Ann. of Math. (2) 67 (1958), pp. 328–466.

[14] P. KRONHEIMER, *The Construction Of ALE Spaces As Hyper-Kähler Quotient*, J. Diff. Geom. 29 (1989), pp. 665–683.

[15] P. KRONHEIMER, *A Torelli Type Theorem For Gravitational Instantons*, J. Diff. Geom. 29 (1989), pp. 685–697.

[16] C. R. LEBRUN, *Explicit self-dual metrics on $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$*, J. Diff. Geom. 34 (1991), pp. 223–253.

[17] C. R. LEBRUN, *Thickenings and gauge fields*, Classical Quantum Gravity 3 (1986), pp. 1039–1059.

[18] U. LINDSTRÖM AND M. ROČEK, *New hyper-Kähler metrics and new supermultiplets*, Comm. Math. Phys. 115 (1988), pp. 21–29.

[19] H. PEDERSEN, *Einstein metrics, spinning-top motions, and monopoles*, Math. Ann. 274 (1986), pp. 35–59.

[20] R. PENROSE, *Nonlinear gravitons and curved twistor theory*, General Relativity and Gravitation 7 (1976), pp. 31–52.

[21] M. PONTECORVO, *On twistor spaces of anti-self-dual Hermitian surfaces*, Trans. Amer. Math. Soc. 331 (1992), no. 2, pp. 653–661.