Eulerian and Lagrangian stability of fluid motions

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1 Introduction

In this thesis, we study stability of inviscid fluids in the Lagrangian sense. That is, we look at how paths of particles diverge from each other under a small perturbation of the initial velocity field. The more usual question of stability is in the Eulerian sense, when one looks at how velocities at each point diverge from a steady state under a small perturbation. We demonstrate, for certain restricted types of flows as well as for some very particular examples, the connection between these two forms of stability. In particular, we observe that in many cases, Eulerian stability is incompatible with exponential Lagrangian instability, although it is compatible with polynomial Lagrangian instability.

Several of the most important features of fluid stability, especially in the incompressible case, are actually general properties of Lie groups with one-sided invariant metrics. In finite dimensions, we can perform many computations on such groups explicitly. For example, the geodesic equation linearized about a steady solution is just a differential equation with constant coefficients, whose solution may always be found explicitly. In addition, the Jacobi equation in this case can always be written as two decoupled first-order equations, and this makes them even easier to study.

There is a simple criterion for the Eulerian stability of a steady flow on a Lie group; we verify that it captures nearly all instances of Eulerian stability on a three-dimensional group. On the other hand, we compile some of the known criteria for Lagrangian stability and demonstrate that they are much less broadly applicable. Since these results all involve constant-coefficient differential equations, we are able to avoid much technical machinery and prove most results quite directly.

We finally list many examples of Jacobi fields and curvature tensors which behave unlike what one might intuitively expect. For example, we have Jacobi fields which grow exponentially despite having a strictly positive curvature tensor (the Rauch comparison theorem only bounds the Jacobi field up to the first conjugate point). We have geodesics along which the curvatures are both positive and negative, and such that their Jacobi fields can oscillate, grow exponentially, or even grow polynomially. (Surprisingly, directions of negative curvature do not necessarily imply instability, due to the possibility of rotating between directions of positive and negative curvature.) We conclude in the end that, with the exception of curvature tensors which are negative-definite along a geodesic, curvature is not a useful way to predict the growth of Jacobi fields.

The theory of geodesics on Lie groups is of interest primarily because incompressible fluid mechanics happens to fall into this category. V. Arnol'd showed in 1965 that incompressible fluid flows are geodesics on the group of volume-preserving diffeomorphisms $\mathcal{D}_\mu$ when this group is provided with a right-invariant kinetic energy metric. He conjectured that when the sectional curvature in a plane section containing the tangent vector is negative, one should have exponential divergence of geodesics. Since the group of volume-preserving diffeomorphisms has many sections with negative curvature, one expects that most fluid flows should be highly unstable, with small perturbations in the initial conditions leading to exponential growth of errors. As an application, he used a simplified model of weather prediction, in which weather is viewed as the small perturbations of a steady fluid flow (the tradewind current). On the basis of this exponential growth conjecture, he computed that weather on the Earth would be impossible to predict more than several weeks in advance.

Many studies after this fundamental observation focused on rather difficult computations of curvature; in particular, finding the plane sections in which it is positive or negative. We summarize some of these results, including Rouchon's Theorem that the only geodesics in $\mathcal{D}_\mu$ with nonnegative sectional curvature in all planes containing the tangent vector are the isometries. Thus we cannot expect to find many stable fluid flows by looking for those with positive sectional curvature. We also compute explicitly the solution of the Jacobi equation for isometries on constant-curvature discs, and find that most solutions grow linearly in time. Interestingly, if the disc is flat, all solutions grow linearly, while if the disc has positive or negative curvature, some solutions are bounded in time.

Arnol'd also proved, using a very different method, an Eulerian stability criterion for two-dimensional steady incompressible flows. This criterion is an extension of the Eulerian stability test for finite-dimensional Lie groups, and says that under certain conditions, all perturbations of the velocity field of a steady flow remain bounded. Misiołek pointed out that there were fluid flows with negative curvature in all planes containing the tangent vector but which satisfied the Arnol'd stability criterion. This suggested that it was possible for the velocity field of a fluid to remain close to its steady state under small perturbations, but for the fluid particle paths themselves to diverge exponentially from each other. In this thesis, we show that for certain classes of steady two-dimensional flows, this is impossible: if the velocity field is stable in the Eulerian sense, then the particle paths diverge at most quadratically in time, not exponentially. (In many cases, the particle paths diverge only linearly in time.)

We also demonstrate that even for the geodesics that one expects to be most unstable (the ones with nonpositive curvature in every plane section containing the tangent vector), it is still possible for the solutions of the Jacobi equation to grow only linearly. We demonstrate this explicitly for the case of Couette flow, which appears in applications as a steady state solution of the viscous Euler equations. Thus on $\mathcal{D}_\mu$, the sign of the curvature does not necessarily imply anything about the growth of solutions to the Jacobi equation. So we cannot in general expect to use curvature to determine whether incompressible fluid flows are stable in the Lagrangian sense.

For a compressible barotropic fluid filling up a flat manifold, one expects the Hessian of the potential energy on the diffeomorphism group to play the same role in Lagrangian stability as the curvature of the volume-preserving diffeomorphism group does for incompressible fluids. In contrast with curvature on $\mathcal{D}_\mu$, we find that the positivity of the Hessian depends not on the velocity field of the fluid, but only on the density function on the manifold. We derive formulas for the gradient and Hessian with a technique involving Lie derivatives which easily generalizes to many other situations.

We next give a precise criterion on the density function for the Hessian of the potential energy to be nonnegative. Unfortunately, In the infinite-dimensional case, the Hessian is unbounded, so the proof that positivity of the Hessian implies stability (even locally) does not carry over. The obstacle is that the set of conjugate points is dense along any curve which represents the motion of a compressible fluid.

In one space dimension, for example on a circle, the Hessian is always nonnegative, and is in fact strictly positive on a subspace of codimension one. For the case of a constitutive law of the form $p = A\rho^3$, we can write down a very explicit solution of the linearized Euler equations, and thus examine the boundedness of Jacobi fields. We find that for initial conditions lying in a codimension-one subspace, the Jacobi fields are bounded; the remainder grow linearly in time.

We also compute the solution of the Jacobi equation for some steady flows in two dimensions: a uniform motion on a torus and a rigid rotation on a disc. In the first case, the Hessian of the potential is nonnegative; in the second, it is strictly positive and bounded away from zero. We find a precise criterion on the initial conditions, in each case, to ensure that the Jacobi fields are all bounded in time. We find that compressible fluid flows seem more stable under perturbations than corresponding incompressible fluid flows, in the sense that for two steady flows with the same velocity field, the compressible one generally has a wider range of initial conditions that lead to bounded Jacobi fields.

We end by suggesting natural possibilities for further study. Firstly, we can study higher-dimensional incompressible and compressible flows, which seem to feature different phenomena of stability than one- or two-dimensional flows. Secondly, we can also study more general potential functions, such as those corresponding to the motion of incompressible elastic solids, or to surface tension for incompressible fluids with a free boundary. Finally, we can apply methods of fluid mechanics to the diffeomorphism groups of general Riemannian manifolds and hope to say something about their geometry.

2 Stability in a finite-dimensional Lie group

In studying Lagrangian stability of incompressible fluids, we must work with the geometry of the group of volume-preserving diffeomorphisms. Explicit computations in this group are quite difficult, partly because the diffeomorphism group is infinite-dimensional, and partly because the formulas all involve a nonlocal operator (the solution operator of a Dirichlet or Neumann problem). Thus it is difficult to test conjectures with explicit examples. However, many of the results that are known about the group of volume-preserving diffeomorphisms are actually special cases of more general results on Lie groups, of any dimension. It is therefore helpful to study those features of stability theory which are true on finite-dimensional Lie groups, since there are many well-known examples.

The presentation here is unusual in that it is focused on the Jacobi equation along a stationary geodesic; this is the primary question in the stability of fluids, but in the general theory of one-sided invariant metrics on Lie groups it has not held great interest. We will follow Milnor [9] in notation and in the emphasis on explicit computations in three dimensions, where a rich source of examples can already be found.

The results presented here find application apart from fluid dynamics. For example, along a stationary geodesic, the Jacobi equation is very easy to solve explicitly, as we demonstrate. We thus immediately have many examples of Jacobi fields for which the curvature operator has both positive and negative eigenvalues; the simplest examples in symmetric spaces involve curvature of only one sign.

Let G be a Lie group of dimension n, and let $\mathfrak{g}$ be its Lie algebra. Suppose that G is equipped with a left-invariant metric $\langle \cdot, \cdot \rangle$. Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{g}$, which we also identify with the left-invariant vector fields on G. Denote the structure constants by α_{ijk}, so that

$$[E_i, E_j] = \sum_{k=1}^{n} \alpha_{ijk} E_k$$

If X and Y are both left-invariant vector fields, then so is the covariant derivative $\nabla_X Y$. The connection is given by the standard formula

$$\langle \nabla_{E_i} E_j, E_k \rangle = \tfrac{1}{2}(\alpha_{kij} + \alpha_{kji} + \alpha_{ijk})$$

See Milnor [9] for more details.

Any curve $\gamma\colon (-\epsilon, \epsilon) \to G$ in the Lie group with $\gamma(0) = \mathrm{id}$ corresponds to a curve $X\colon (-\epsilon, \epsilon) \to \mathfrak{g}$ in the Lie algebra, by the formula

$$\frac{d\gamma}{dt} = DL_{\gamma(t)} X(t) \tag{1}$$

where $DL_g\colon \mathfrak{g} \to T_g G$ denotes the differential of the left-translation map $L_g\colon G \to G$ at the identity. When G is the configuration space of a mechanical system, the curve γ in the group G is called the Lagrangian picture of the system, while the curve X in the algebra $\mathfrak{g}$ is called the Eulerian picture of the system. The equation (1), an ordinary differential equation on G, always has a unique solution with $\gamma(0) = \mathrm{id}$, so we can always go from one to the other.

Suppose $Y(t)$ is a time-dependent vector in $\mathfrak{g}$; then $DL_{\gamma(t)}Y(t)$ is a vector field along γ. The covariant derivative of Y along γ is given by the following formula:

$$\frac{D}{dt}DL_{\gamma(t)}Y(t) = DL_{\gamma(t)}\left(\frac{dY}{dt} + \nabla_{X(t)}Y(t)\right) \tag{2}$$

In particular, the geodesic equation $\frac{D}{dt}\frac{d\gamma}{dt} = 0$ is equivalent to the Euler equation on $\mathfrak{g}$:

$$\frac{dX}{dt} + \nabla_X X = 0 \tag{3}$$

The study of linear stability is focused on the linearized geodesic equation, which is derived as follows. Suppose $\gamma(s,t)$ is a family of geodesics depending on a parameter s; let $X(s,t)$ denote the corresponding family of curves in $\mathfrak{g}$. Let $Z(t) = \frac{\partial X(s,t)}{\partial s}\big|_{s=0}$ and let $Y(t)$ be defined such that $\frac{\partial \gamma(s,t)}{\partial s}\big|_{s=0} = DL_{\gamma(t)}Y(t)$. If we differentiate the equations (1) and (3) with respect to s, and set $s = 0$, we obtain the equations

$$\begin{aligned} \frac{dY}{dt} + [X,Y] &= Z \\ \frac{dZ}{dt} + \nabla_X Z + \nabla_Z X &= 0 \end{aligned} \tag{4}$$

Note that the second equation does not involve Y at all; thus the equations are decoupled. By plugging the first of equations (4) into the second, we obtain the usual Jacobi equation for Y:

$$\left(\tfrac{d}{dt} + \nabla_X\right)^2 Y(t) + R(Y,X)X = 0 \tag{5}$$

but equations (4) are obviously easier to solve explicitly.

One complication that arises in the study of stability theory, especially for Lagrangian stability, is that $Y = (at+b)X$ is always a solution of the Jacobi equation. Thus, to obtain boundedness of all Jacobi fields, we should look only among those fields that are orthogonal to X. Fortunately, as is well-known, a Jacobi field such that $Y(0)$ and $\frac{DY}{dt}(0)$ are both orthogonal to X will remain orthogonal to X for all time.

There are two notions of stability for a geodesic on a Lie group with left-invariant metric. Our goal is to clarify the relationship between the two, in the simplest cases.

Definition 2.1. *Let X be a solution of the Euler equation (3).*

- *X is called* stable in the Eulerian sense *if every solution $Z(t)$ of the equation*

$$\dot{Z}(t) + \nabla_{X(t)}Z(t) + \nabla_{Z(t)}X(t) = 0$$

remains bounded for all time.

- *X is called* stable in the Lagrangian sense *if every solution $Y(t)$ of the Jacobi equation (or of equations (4)) with $Y(0) = 0$ and $\dot{Y}(0)$ orthogonal to X remains bounded for all time.*

The simplest case is when one is dealing with a stationary vector X; that is, a time-independent solution of Euler's equation. Such a vector satisfies the equation $\nabla_X X = 0$, or, writing $X = \sum_{i=1}^{n} X^i E_i$, the n equations

$$\sum_{i,j}^{n} X^i X^j \alpha_{kij} = 0, \qquad k = 1, \ldots, n \tag{6}$$

Such stationary solutions correspond to those tangent vectors for which the geometric (Riemannian) exponential map coincides with the algebraic (Lie group) exponential map. In this case, the equations (4) have constant coefficients, and we can do everything quite explicitly. The following construction makes things a bit more convenient.

Proposition 2.2. *If X is a stationary solution of the Euler equation, then the linearized geodesic equations (4) can be decoupled into a set of equations on the 1-dimensional space spanned by X and the $(n-1)$-dimensional space perpendicular to X.*

Proof. We already know that $Z = aX$ and $Y = (at + b)X$ are solutions of equations (4) for any constants a and b. We need to show that the operators $Z \mapsto \nabla_Z X + \nabla_X Z$ and $Y \mapsto [X, Y]$ map the orthogonal complement of X to itself, i.e. that if Z and Y are orthogonal to X, then so are $\nabla_Z X + \nabla_X Z$ and $[X, Y]$. We have

$$\begin{aligned} \langle \nabla_Z X + \nabla_X Z, X \rangle &= \langle \nabla_X Z, X \rangle \\ &= -\langle Z, \nabla_X X \rangle \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle [X, Y], X \rangle &= \langle \nabla_X Y - \nabla_Y X, X \rangle \\ &= -\langle Y, \nabla_X X \rangle \\ &= 0 \end{aligned}$$

So the equations (4) are actually equations on the orthogonal complement of X. $\square$

If a basis of $\mathfrak{g}$ is chosen so that $X = E_1$, then we define $(n-1) \times (n-1)$ matrices A and B and corresponding operators by the formulas

$$A_{ij} \equiv \langle A(E_i), E_j \rangle = \langle \nabla_X E_i, E_j \rangle \qquad\qquad B_{ij} \equiv \langle B(E_i), E_j \rangle = \langle \nabla_{E_i} X, E_j \rangle$$

for $2 \leq i, j \leq n$. Since $\langle \nabla_X U, V\rangle + \langle \nabla_X V, U\rangle = 0$ by left-invariance, A is an antisymmetric matrix. The equations (4) become

$$\begin{aligned} \frac{dY}{dt} + (A - B)Y = Z \\ \frac{dZ}{dt} + (A + B)Z = 0 \end{aligned} \tag{7}$$

The explicit solution of these equations, with $Z(0) = Z_0$ and $Y(0) = 0$, is

$$\begin{aligned} Z(t) &= e^{-t(A+B)} Z_0 \\ Y(t) &= \left(\int_0^t e^{(s-t)(A-B)} e^{-s(A+B)}\, ds \right) Z_0 \end{aligned} \tag{8}$$

Note that it is very rare for A and B to commute, so this formula generally does not simplify.

3 Eulerian stability theory

The following theorem is simple and follows naturally from standard principles of ordinary differential equations.

Theorem 3.1. *Let X be a stationary solution of the Euler equation. X is stable in the Eulerian sense iff both of the following hold.*

- *All the eigenvalues of $(A + B)$ have nonnegative real part.*
- *For any eigenvalue λ of $(A + B)$ with vanishing real part and multiplicity k, the eigenspace has dimension k. (In other words, nondegenerate Jordan blocks of $(A+B)$ correspond to eigenvalues with strictly negative real part only.)*

Proof. The general solution of $dZdt + (A + B)Z = 0$ is a sum of terms of the form $t^j e^{-\lambda t}$, where λ is an eigenvalue of $(A + B)$ and j is an integer between 0 and $(k - 1)$, with k the size of the largest Jordan block corresponding to λ. If the real part of λ is positive, then $t^j e^{-\lambda t}$ is always bounded in time; if the real part of λ is negative, then $t^j e^{-\lambda t}$ is always unbounded in time. If the real part of λ vanishes, then the solutions are of the form t^j or $t^j e^{i|\lambda|t}$, and both of these are bounded for all time if and only if j is always zero; that is, if the size of the corresponding Jordan block is exactly one. If each term in the solution is bounded for all time, then the general solution is bounded as well for all time. □

Corollary 3.2. *If the left-invariant vector field generated by X is divergence-free (for example if G is a unimodular group), then X is stable in the Eulerian sense iff $(A+B)$ is diagonalizable and all eigenvalues of $(A+B)$ are imaginary or zero.*

Proof. Choosing a basis such that $X = E_1$, the trace of B is

$$\mathrm{Tr}\ (B) = \sum_{i=1}^{n} \langle \nabla_{E_i} X, E_i \rangle = \mathrm{div}\, X = -\mathrm{Tr}\ (\mathrm{ad}\, X)$$

So if $\mathrm{div}\, X = 0$, then $\mathrm{Tr}\ (B) = 0$. More generally, if G is unimodular, then $\mathrm{Tr}\ (\mathrm{ad}\, U) = 0$ for any U and in particular for X.

The consequence is that $\mathrm{Tr}\ (A + B) = 0$, and therefore if any eigenvalue of $(A + B)$ has nonvanishing real part, there must be an eigenvalue with negative real part. Thus, for stability, all eigenvalues must have vanishing real part. By Theorem 3.1, $(A + B)$ must be diagonalizable. □

The following criterion for Eulerian stability is by far the simplest, but is rarely satisfied.

Corollary 3.3. *If the left-invariant vector field generated by X is a Killing field (for example if the given metric on G is bi-invariant) then X is stable in the Eulerian sense.*

Proof. The condition that X be Killing is that

$$\langle \nabla_U X, V \rangle + \langle \nabla_V X, U \rangle = 0$$

for any U and V, which is equivalent to the condition that B be antisymmetric. In that case, $(A + B)$ is also antisymmetric and so $i(A + B)$ is Hermitian. Thus $i(A + B)$ has a basis of eigenvectors with real eigenvalues, and so $(A + B)$ has a basis of eigenvectors with imaginary eigenvalues. □

If the given metric is bi-invariant, then every vector in $\mathfrak{g}$ is stationary, since the metric exponential map and the group exponential map are identical. Thus on a group with bi-invariant metric, every geodesic is Eulerian stable.

It is, unfortunately, often difficult to check whether the eigenvalues of $(A+B)$ are real or imaginary, and there are not many simple criteria known for the general case. The most useful one is given by the following theorem. The criterion is originally due to Arnol'd [1], but the proof we give is a simpler one due to Barston [4]. It is a special case of a general technique presented by Barston.

Theorem 3.4. *The stationary geodesic vector X is stable in the Eulerian sense if the symmetric matrix*

$$\tfrac{1}{2}(B - B^T)(B^T - A) \tag{9}$$

is either positive definite or negative definite.

Proof. Let us write $B = S + U$, where S is symmetric and U is antisymmetric, for convenience. Since the curvature $R(Y, X)X$ is symmetric and is equal to $(A + B)(A - B) + A^2 = BA - AB - B^2$, the antisymmetric part of $BA - AB -$

B^2 must vanish. Since the symmetric part is $SA - AS - S^2 - U^2$ and the antisymmetric part is $UA - AU - SU - US$, we must have

$$UA - AU - SU - US = 0 \tag{10}$$

The matrix $U(S-U-A)$ is symmetric, by equation (10). It is also a constant of motion of the system $\frac{dZ'}{dt} + (S-U-A)Z' = 0$, since

$$\begin{aligned}
&\frac{d}{dt}\langle Z', U(S-U-A)Z'\rangle \\
&\quad = \left\langle \frac{dZ'}{dt}, U(S-U-A)Z' \right\rangle + \left\langle Z', U(S-U-A)\frac{dZ'}{dt} \right\rangle \\
&\quad = -\langle (S-U-A)Z', U(S-U-A)Z'\rangle - \langle Z', U(S-U-A)(S-U-A)\rangle \\
&\quad = -\langle Z', \big[(S+U+A)U(S-U-A) + U(S-U-A)(S-U-A)\big]Z'\rangle \\
&\quad = 0
\end{aligned}$$

since $(S+U+A)U + U(S-U-A) = 0$ by equation (10).

Now since the eigenvalues and Jordan form of $(B^T - A) = (B+A)^T$ are exactly the same as those of $(B+A)$, the system $dZ'dt + (B^T - A)Z' = 0$ is stable iff the system $dZdt + (B+A)Z = 0$ is stable. If $U(S-U-A)$ is sign-definite (say positive definite for example) then $C = \langle Z', U(S-U-A)Z'\rangle > \epsilon\langle Z', Z'\rangle$ for some positive constants C and ϵ. So the solutions of $\frac{dZ'}{dt} + (B^T - A)Z' = 0$ are bounded for all time, and therefore so are the solutions of the linearized Euler equation $\frac{dZ}{dt} + (B+A)Z = 0$. □

In terms of the structure constants α_{ijk}, and in a basis where $X = E_1$, the operator $(B - B^T)(B^T - A)$ is given by

$$\langle (B - B^T)(B^T - A)E_i, E_j\rangle = \sum_{k=2}^{n} \alpha_{ki1}(\alpha_{j1k} + \alpha_{jk1}) \tag{11}$$

3.1 Lagrangian stability theory

The following criterion for Lagrangian stability is simple to prove but often difficult to check in practice.

Theorem 3.5. *The geodesic generated by X is stable in the Lagrangian sense if and only if the eigenvalues of both $(A+B)$ and $(A-B)$ are all purely imaginary or zero, and the matrix*

$$Q = \begin{pmatrix} (A-B) & -I \\ 0 & (A+B) \end{pmatrix} \tag{12}$$

is diagonalizable.

Proof. Since the trace of the matrix Q is $2\operatorname{Tr} A = 0$, if any eigenvalues of Q have nonzero real part, then some eigenvalue has negative real part, and hence the equation

$$\frac{d}{dt}\begin{pmatrix} Y \\ Z \end{pmatrix} + \begin{pmatrix} (A-B) & -I \\ 0 & (A+B) \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix} = 0$$

has exponentially growing solutions.

If the matrix Q is not diagonalizable, then there are solutions which grow linearly in time, and hence the system is unstable. If Q is diagonalizable with purely imaginary eigenvalues, then clearly the system is stable. $\square$

Corollary 3.6. *If the stationary vector X is stable in the Lagrangian sense, then it is stable in the Eulerian sense.*

Proof. We can determine the eigenvectors of $Q = \left(\begin{smallmatrix} A-B & -I \\ 0 & A+B \end{smallmatrix}\right)$ in terms of the eigenvectors of $(A-B)$ and $(A+B)$. First, any eigenvector of Q is of the form $\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)$. The equation

$$\begin{pmatrix} A-B & -I \\ 0 & A+B \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}$$

implies the two equations

$$(A-B)U - V = \lambda U$$
$$(A+B)V = \lambda V$$

In particular, if V is nonzero then V is an eigenvector of $(A+B)$ and λ is an eigenvalue of $(A+B)$. If Q is diagonalizable, then there must be enough eigenvectors of Q to span the $(n-1)$-dimensional space $\{0\} \times X^{\perp} \subset \mathfrak{g} \times \mathfrak{g}$; $(n-1)$ of these vectors form a basis, and the second components of each form a basis of eigenvectors of $(A+B)$. Since all of the eigenvalues of Q are zero or imaginary, so are the eigenvalues of $(A+B)$. Thus X is stable in the Eulerian sense by Theorem 3.1. $\square$

The easiest criteria to prove Lagrangian stability are direct methods that do not rely on the splitting (7). A general summary of the simple criteria is given in Walker [17]. The following theorem is simple to verify and yields strong results.

Theorem 3.7. *If the operator $Y \mapsto \nabla_{[X,Y]}X + \nabla_X[X,Y]$ is positive definite on the orthogonal complement of X in $\mathfrak{g}$, then X is stable in the Lagrangian sense.*

Proof. Using the formula $\frac{DY}{dt} = \frac{dY}{dt} + AY$, we see that the Jacobi equation is

$$\frac{d^2Y}{dt^2} + 2A\frac{dY}{dt} + A^2Y + R(Y,X)X = 0 \tag{13}$$

Computing the inner product of equation (13) with $\frac{dY}{dt}$, we obtain

$$\frac{d}{dt}\left(\left\langle \frac{dY}{dt}, \frac{dY}{dt}\right\rangle + \langle A^2Y, Y\rangle + \langle R(Y,X)X, Y\rangle\right) = 0$$

Since $Y(0) = 0$ and $\dot{Y}(0) = Z_0$, we have

$$\left\langle \frac{dY}{dt}, \frac{dY}{dt} \right\rangle + \langle A^2Y, Y\rangle + \langle R(Y,X)X, Y\rangle = \langle Z_0, Z_0\rangle \tag{14}$$

If Y is orthogonal to X and $Y \mapsto A^2Y + R(Y,X)X$ is positive definite on the orthogonal complement of X, then there is some $\varepsilon > 0$ such that $\langle A^2Y, Y\rangle + \langle R(Y,X)X, Y\rangle > \epsilon\langle Y, Y\rangle$. Thus by equation (14)

$$\langle Y, Y\rangle < \frac{1}{\varepsilon}\langle Z_0, Z_0\rangle$$

Computing $R(Y,X)X + A^2Y$, we obtain

$$\begin{aligned} R(Y,X)X + A^2Y &= \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X + \nabla_X\nabla_X Y \\ &= \nabla_{[X,Y]}X + \nabla_X[X,Y] \end{aligned}$$

The latter expression is usually easier to compute than the curvature. □

In coordinates for which $X = E_1$, the expression $\nabla_{[X,Y]}X + \nabla_X[X,Y]$ is given by

$$\langle \nabla_{[X,E_i]}X + \nabla_X[X,E_i], E_j\rangle = \sum_{k=1}^{n} \alpha_{1ik}(\alpha_{jk1} + \alpha_{j1k})$$

Notice that since A^2 is nonpositive, positive-definiteness of this expression implies positive-definiteness of the curvature. As we will see, positivity of the curvature alone does not in general imply stability.

However, there is one condition under which positivity of curvature is equivalent to stability.

Theorem 3.8. *If $(\nabla_X R)(Y,X)X = 0$ for all Y, and if for some $\varepsilon > 0$, the inequality $\langle R(Y,X)X, Y\rangle > \varepsilon\langle Y, Y\rangle$ holds for every Y orthogonal to X, then X is stable in the Lagrangian sense.*

Proof. Recall that the covariant derivative of a tensor is defined so that

$$\begin{aligned} \big(\nabla_X R\big)(Y,X)X &= \nabla_X\Big(R(Y,X)X\Big) \\ &\qquad - R(\nabla_X Y, X)X - R(Y, \nabla_X X)X - R(Y,X)\nabla_X X \\ &= \nabla_X\big(R(Y,X)X\big) - R(\nabla_X Y, X)X \end{aligned}$$

since $\nabla_X X = 0$. So

$$\nabla_X\big(R(Y,X)X\big) = R(\nabla_X Y, X)X$$

If R_X denotes the linear operator $Y \mapsto R(Y,X)X$, then this formula is equivalent to $AR_X = R_X A$. Note that it is very rare for a symmetric matrix to commute with an antisymmetric matrix: in this case, one can check that it happens iff there is an orthonormal basis of eigenvectors of R_X such that R_X and A take the form

$$R_X = \begin{pmatrix} r_1 I_1 & 0 & \cdots & 0 \\ 0 & r_2 I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k I_k \end{pmatrix} \qquad A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$$

with each I_j an identity matrix and each A_j an antisymmetric matrix of the same dimension as I_j. In particular, if R_X is proportional to the identity matrix, this condition is satisfied for any A.

The Jacobi equation (13) may be rewritten as

$$e^{-tA}\frac{d^2}{dt^2}\big(e^{tA}Y(t)\big) + R_X Y = 0$$

Multiplying by e^{tA} and letting $V(t) = e^{tA}Y(t)$, we have

$$\frac{d^2V}{dt^2} + e^{tA}R_X e^{-tA}V(t) = 0$$

and since R_X and A commute,

$$\frac{d^2V}{dt^2} + R_X V = 0$$

Computing the inner product with $\frac{dV}{dt}$, we obtain

$$\frac{d}{dt}\left(\left\langle \frac{dV}{dt}, \frac{dV}{dt}\right\rangle + \langle R(V,X)X, V\rangle\right) = 0$$

Since $V(0) = 0$ and $\dot{V}(0) = Z_0$, we have

$$\left\langle \frac{dV}{dt}, \frac{dV}{dt}\right\rangle + \langle R(V,X)X, V\rangle = \langle Z_0, Z_0\rangle$$

and since $\langle R(V,X)X, V\rangle > \varepsilon\langle V, V\rangle = \varepsilon\langle Y, Y\rangle$, we have

$$\langle Y, Y\rangle < \frac{1}{\varepsilon}\langle Z_0, Z_0\rangle$$

and X is stable in the Lagrangian sense. □

Note that the condition of the theorem applies if the left-invariant metric on G is actually bi-invariant. In this case, the curvature is automatically nonnegative, but the hypothesis that the curvature is actually strictly positive in planes containing X is stronger.

There is also a simple criterion for instability: if the curvature operator $R_X \colon Y \mapsto R(Y,X)X$ is nonpositive, then X must be unstable in the Lagrangian sense, and all Jacobi fields grow at least linearly. In fact, if R_X is negative definite, then all Jacobi fields grow exponentially. These facts are simple consequences of the Rauch comparison theorem. Note that the Rauch comparison theorem can be used to guarantee instability, but not to guarantee stability: positive curvature only gives one boundedness up to the first conjugate point, and not necessarily anything beyond that. We will see examples for which the curvature is positive in all directions along the geodesic but Jacobi fields still grow exponentially, as well as those for which the curvature is sometimes negative yet all Jacobi fields remain bounded.

When the criteria given above are not satisfied, there are several more complicated "parameter-dependent" criteria for stability; some are given in Walker [17]. However, it is easier to look for criteria using the splitting (7). The following is helpful and quite simple. Note that G admits a bi-invariant metric iff it is the direct product of a compact group and a commutative group, by Lemma 7.5 of Milnor [9].

Theorem 3.9. *If G admits a bi-invariant metric, then all solutions of the homogeneous equation*

$$\frac{dY}{dt} + [X, Y] = 0$$

are bounded in time.

Proof. The main observation is that the equation depends not on the metric but only on the differential geometric structure of the Lie group. Thus the eigenvalues of $Y \mapsto [X, Y]$ do not depend on which metric is being used, and so one might as well use a bi-invariant metric.

Under a bi-invariant metric $\langle\langle \cdot, \cdot \rangle\rangle$, the operator $Y \mapsto [X, Y]$ is antisymmetric, since if $\widetilde{\nabla}$ denotes the bi-invariant connection,

$$\langle\langle [X, Y], Y \rangle\rangle = \langle\langle \widetilde{\nabla}_X Y, Y \rangle\rangle - \langle\langle \widetilde{\nabla}_Y X, Y \rangle\rangle = -\langle\langle X, \widetilde{\nabla}_Y Y \rangle\rangle = 0$$

because $\widetilde{\nabla}_Y Y = 0$ for any Y. Therefore the eigenvalues of $Y \mapsto [X, Y]$ must be purely imaginary and $Y \mapsto [X, Y]$ must be diagonalizable. So all solutions are bounded in time. □

If the conditions of both Theorem 3.4 and Theorem 3.9 are satisfied, then we can at least say that $(A + B)$ and $(A - B)$ are both diagonalizable with imaginary eigenvalues. To then conclude Lagrangian stability requires knowing whether the matrix Q defined by (12) is diagonalizable, which is more difficult. However, we can at least conclude the following.

Theorem 3.10. *If $(A+B)$ and $(A-B)$ are each diagonalizable with imaginary eigenvalues, then all Jacobi fields grow at most linearly.*

Proof. Choose a basis in which $(A - B)$ is diagonal with imaginary eigenvalues, and let P be the matrix of this basis. Then $P^{-1}(A - B)P \equiv iD$ is purely imaginary and diagonal, and therefore

$$\begin{aligned}
\left(P^{-1}(A-B)P\right)^* &= -P^{-1}(A-B)P \\
P^*(A-B)^T P^{-1*} &= -P^{-1}(A-B)P \\
(A-B)^T (PP^*)^{-1} &= -(PP^*)^{-1}(A-B) \\
\left((PP^*)^{-1}(A-B)\right)^T &= -\left((PP^*)^{-1}(A-B)\right)
\end{aligned}$$

That is, if M is the symmetric, positive definite matrix $M = (PP^*)^{-1}$, then $M(A-B)$ is antisymmetric.

The solution of $\frac{dY}{dt} + (A-B)Y = Z$ is given by

$$Y(t) = \int_0^t e^{(s-t)(A-B)} Z(s)\, ds = \int_0^t P e^{i(s-t)D} P^{-1} Z(s)\, ds$$

We then have

$$\begin{aligned}
\langle Y(t), MY(t)\rangle &= \langle P^{-1}Y(t), P^{-1}Y(t)\rangle \\
&= \int_0^t \int_0^t \left\langle e^{i(s-t)D} P^{-1} Z(s), e^{i(r-t)D} P^{-1} Z(r)\right\rangle dr\, ds \\
&\leq \Big[\int_0^t \sqrt{\left\langle e^{i(s-t)D} P^{-1} Z(s), e^{i(s-t)D} P^{-1} Z(s)\right\rangle}\, ds\Big]^2 \\
&= \Big[\int_0^t \sqrt{\langle P^{-1}Z(s), P^{-1}Z(s)\rangle}\, ds\Big]^2 \\
&= \Big[\int_0^t \sqrt{\langle Z(s), MZ(s)\rangle}\, ds\Big]^2
\end{aligned}$$

If $\mu_1, \ldots, \mu_n$ denote the n positive eigenvalues of M in increasing order, then $\langle Z, MZ\rangle \leq \mu_n \langle Z, Z\rangle$ and $\langle Y, Y\rangle \leq \frac{1}{\mu_1}\langle Y, MY\rangle$ for all Z and Y. By assumption, $\langle Z(t), Z(t)\rangle$ is bounded for all time by some number c^2. Thus we have

$$\langle Y(t), Y(t)\rangle \leq \frac{\mu_n}{\mu_1} c^2 t^2$$

so that $Y(t)$ grows at most linearly in time for large t. □

It seems likely that under the hypotheses of the theorem, Jacobi fields will typically be bounded, and only in degenerate cases will they grow linearly; the theorem shows that this is the worst degeneracy that can occur. For example, if G admits a bi-invariant metric and the quadratic form (9) is positive-definite, then under small perturbations of the metric, both these conditions are still satisfied. One can conjecture that there exists a perturbation of the metric which makes the system matrix Q from equation (12) diagonalizable, and thus leads to boundedness of all Jacobi fields.

In any case, one should consider the case in which Jacobi fields grow linearly to be different in nature from the case in which they grow exponentially; if

G admits a bi-invariant metric, then the latter can happen only when there is exponential growth in the solutions of the linearized Euler equation. On the other hand, if G does not admit a bi-invariant metric, then it is possible for exponential growth to be introduced in passing to the Lagrangian point of view.

Of course, for practical purposes one should note that all of these results on linear stability do not necessarily tell one much about the solutions of the genuine Euler equation under a perturbation; there are well-known examples for which it is quite possible to have stability in the linearized equations but instability in the actual nonlinear equation. We must leave this question aside in this research, however, as the linear problem already presents many subtleties.

3.2 Examples in three dimensions

We can illustrate how effective the various stability criteria are with very explicit examples in three dimensions. In addition, we can illustrate how curvature often fails to predict stability or instability in the way one intuitively expects.

Suppose that G is a three-dimensional unimodular Lie group (that is, every left-invariant vector field is divergence-free). Then, as given in Milnor [9], there is a basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ and numbers $\{\lambda_1, \lambda_2, \lambda_3\}$ such that

$$[E_1, E_2] = \lambda_3 E_3, \qquad [E_2, E_3] = \lambda_1 E_1, \qquad [E_3, E_1] = \lambda_2 E_2 \tag{15}$$

or more concisely, in terms of the structure constants,

$$\alpha_{ijk} = \mathrm{sgn}(ijk)\lambda_k \tag{16}$$

where $\mathrm{sgn}(ijk)$ is 1 if the permutation (ijk) is even, -1 if the permutation is odd, and 0 if any two of i, j, and k are equal.

We then have the following formula for the covariant derivative:

$$\langle \nabla_{E_i} E_j, E_k \rangle = \tfrac{1}{2}\mathrm{sgn}(ijk)(\lambda_j - \lambda_i + \lambda_k) \tag{17}$$

The equation $\nabla_X X = 0$ is

$$\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} X^i X^j \mathrm{sgn}(ijk)(\lambda_j - \lambda_i + \lambda_k) = 0$$

for all k from 1 to 3. Writing these equations out, we obtain

$$\begin{aligned}(\lambda_3 - \lambda_2)X^2X^3 &= 0\\ (\lambda_1 - \lambda_3)X^3X^1 &= 0\\ (\lambda_2 - \lambda_1)X^1X^2 &= 0\end{aligned}$$

If the λ's are all distinct, then the only solutions are those with two of the X's vanishing and the third one nonzero. So we can assume $X^1 \neq 0$ and $X^2 = X^3 = 0$. If two λ's are the same, it is possible to have other solutions.

Suppose the coordinates are labeled so that $\lambda_1 = \lambda_2$, but λ_3 is distinct from either one. Then either $X^3 \neq 0$ and $X^1 = X^2 = 0$ (in which case we can switch the numbering so that $X^1 \neq 0$ and the others vanish) or X^1 and X^2 are both nonvanishing and $X^3 = 0$. In the latter case, we can rotate in the X^1-X^2 plane so that $X^1 \neq 0$ and $X^2 = 0$. If all three λ's are the same, then every choice of X's yields a solution and we can rotate axes so that X is parallel to E_1. In all cases, then, we can assume that X is parallel to E_1. Since the norm of X is conserved, we might as well assume that $\langle X, X\rangle = 1$ and that $X = E_1$.

The linearized Euler equations (7) become

$$\frac{d}{dt}\begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} + \begin{pmatrix} 0 & \lambda_1 - \lambda_3 \\ \lambda_2 - \lambda_1 & 0 \end{pmatrix}\begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} = 0 \tag{18}$$

$$\frac{d}{dt}\begin{pmatrix} Y^2 \\ Y^3 \end{pmatrix} + \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_3 & 0 \end{pmatrix}\begin{pmatrix} Y^2 \\ Y^3 \end{pmatrix} = \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{19}$$

By applying Theorem 3.1 directly, we have the following result.

Proposition 3.11. *The stationary solution $X = E_1$ is stable in the Eulerian sense if and only if one of the following holds:*

- $\lambda_3 > \lambda_1$ *and* $\lambda_2 > \lambda_1$
- $\lambda_3 < \lambda_1$ *and* $\lambda_2 < \lambda_1$
- $\lambda_3 = \lambda_2 = \lambda_1$

Proof. The characteristic equation of $(A + B)$ is

$$r^2 + (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1) = 0$$

$(A + B)$ is certainly diagonalizable if $(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1) > 0$, for then it has two distinct imaginary eigenvalues. In case $\lambda_3 = \lambda_1$ or $\lambda_2 = \lambda_1$, the matrix has two repeated roots. If one of these equalities holds but not the other, then the Jordan form of $(A+B)$ is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and the system is not stable. However, if both equalities hold, then $(A + B) \equiv 0$ and the system is stable. $\square$

We can also apply Theorem 3.5 directly, and obtain the following result.

Proposition 3.12. *The stationary solution $X = E_1$ is stable in the Lagrangian sense if and only if it is stable in the Eulerian sense, $\lambda_1 \neq 0$, and in addition one of the following holds:*

- $\lambda_2 > 0$ *and* $\lambda_3 > 0$
- $\lambda_2 < 0$ *and* $\lambda_3 < 0$
- $\lambda_2 = \lambda_3 = 0$

Proof. First, the characteristic polynomial of $(A - B)$ is

$$r^2 + \lambda_2\lambda_3 = 0$$

So we must have $\lambda_2\lambda_3 \geq 0$. If $\lambda_2\lambda_3 > 0$, then $(A - B)$ has two distinct eigenvalues. If the four eigenvalues of Q are distinct, then of course Q always has a diagonal basis. The eigenvalues of $(A + B)$ and $(A - B)$ will be distinct unless

$$(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1) = \lambda_2\lambda_3$$

that is, unless $\lambda_1 = 0$ or $\lambda_1 = \lambda_2 + \lambda_3$. If $\lambda_1 = 0$, we can verify that $Q = \left(\begin{smallmatrix} (A-B) & -I \\ 0 & (A+B) \end{smallmatrix}\right)$ never has a basis of eigenvectors. If, on the other hand, $\lambda_1 = \lambda_2 + \lambda_3$ then Q always has a basis of eigenvectors. (The computations are easy but slightly tedious.)

The other degenerate case is when one of λ_2 or λ_3 vanishes. Again one can verify that if only one vanishes, then Q is not diagonalizable, but if both vanish, then Q is diagonalizable. □

Now we can check to see how effective each of the stability criteria proposed in the previous sections are in this case. Theorem 3.4 tells us that X is stable in the Eulerian sense if the quadratic form

$$(B - B^T)(B^T - A) = \begin{pmatrix} \lambda_1(\lambda_3 - \lambda_1) & 0 \\ 0 & \lambda_1(\lambda_2 - \lambda_1) \end{pmatrix}$$

is either positive-definite or negative-definite. This happens iff $\lambda_1 \neq 0$ and $(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) > 0$. Thus the only instances of stability not indicated by this criterion are those when $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \lambda_1$. Thus we can expect the criterion for Eulerian stability is in general quite strong.

For Lagrangian stability, the situation is not so promising. The criterion given by Theorem 3.7 says that X is stable in the Lagrangian sense if the quadratic form

$$(B + A)(B - A) = \begin{pmatrix} \lambda_3(\lambda_1 - \lambda_3) & 0 \\ 0 & \lambda_2(\lambda_1 - \lambda_2) \end{pmatrix}$$

is positive definite. This happens iff one of the following holds:

- $\lambda_3 > 0$, $\lambda_2 > 0$, $\lambda_1 > \lambda_3$, and $\lambda_1 > \lambda_2$
- $\lambda_3 < 0$, $\lambda_2 < 0$, $\lambda_1 < \lambda_3$, and $\lambda_1 < \lambda_2$

Although this criterion is sometimes useful, it clearly misses many of the actual parameter values for which E_1 is Lagrangian stable.

The criterion given by Theorem 3.9 is slightly more useful. The only three-dimensional groups which admit bi-invariant metrics are the ones for which all λ's are positive, all are negative, or all are zero. This covers many of the cases in Proposition 3.12, except that λ_1 in general need not be of the same sign as λ_2 and λ_3. Of course, Theorem 3.9 does not actually tell us whether we have stability in the Lagrangian sense; we still have to check whether the system matrix Q is diagonalizable in each case.

The curvature operator is easiest to compute from the formula

$$R(Y,X)X = -\nabla_X \nabla_X Y + \nabla_X [X,Y] + \nabla_{[X,Y]} X$$

(which is valid whenever $\nabla_X X = 0$) since we already know the last two terms. We have

$$\begin{aligned} R(E_2,E_1)E_1 &= \Big[\tfrac{1}{4}(\lambda_1-\lambda_2-\lambda_3)^2 + \lambda_3(\lambda_1-\lambda_3)\Big]E_2 \\ R(E_3,E_1)E_1 &= \Big[\tfrac{1}{4}(\lambda_1-\lambda_2-\lambda_3)^2 + \lambda_2(\lambda_1-\lambda_2)\Big]E_3 \end{aligned} \tag{20}$$

Using this formula, we can demonstrate some seemingly paradoxical relations between curvature and stability.

Example 3.13. *There are Lie groups for which the curvature is both positive and negative in sections containing the tangent vector along a steady geodesic, and such that the Jacobi fields all have either*

- *no growth in time,*
- *exponential growth in time, or*
- *polynomial growth in time.*

For the first example, let $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$. Then the curvature matrix $Y \mapsto R(Y,E_1)E_1$ is

$$R_X = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

The stationary vector E_1 satisfies the conditions of Propositions 3.11 and 3.12 and is therefore stable in the Lagrangian sense; all Jacobi fields oscillate sinusoidally.

For the second example, let $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = 2$. The curvature matrix is

$$R_X = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}$$

and the Jacobi fields grow exponentially on the order of $e^{\sqrt{2}t}$.

For the third example, let $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 2$. The curvature matrix is

$$R_X = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$$

The equations are then

$$\begin{aligned} \dot{Z}^2 &= 2Z^3 & \qquad \dot{Y}^2 &= Z^2 \\ \dot{Z}^3 &= 0 & \qquad \dot{Y}^3 &= -2Y^2 + Z^3 \end{aligned}$$

and the solution with $Z^2(0) = p$, $Z^3(0) = q$, $Y^2(0) = 0$, and $Y^3(0) = 0$ is

$$\begin{aligned} Z^2(t) &= p + 2qt & Y^2(t) &= pt + qt^2 \\ Z^3(t) &= q & Y^3(t) &= qt - pt^2 - \tfrac{2}{3}qt^3 \end{aligned}$$

and this gives perhaps the only known example of a Jacobi field that grows like t^3.

As a result of these examples, we cannot say that negative curvature in one direction will necessarily imply exponential instability, even in finite-dimensional Lie groups. (Arnol'd conjectured in 1965 that it did for the infinite-dimensional group of volume-preserving diffeomorphisms.) It is only negative curvature in all directions that yields such results.

Example 3.14. *There is a Lie group for which the curvature operator is positive-definite along a steady geodesic, and for which almost all Jacobi fields grow exponentially.*

Let $\lambda_1 = 6$, $\lambda_2 = -1$, and $\lambda_3 = 1$. Then there is a Lagrangian instability which results in Jacobi fields with $Y(0) = 0$ growing like e^t, except for those with initial conditions such that $Z^2(0) + Z^3(0) = 0$. But the curvature operator is positive-definite:

$$R_X = \begin{pmatrix} 14 & 0 \\ 0 & 2 \end{pmatrix}$$

The reason for this is that the Rauch comparison theorem only yields bounds on Jacobi fields up to the first conjugate point. Beyond that, it says nothing about growth, and so it is entirely possible for the amplitude to grow exponentially even when the Jacobi field vanishes intermittently.

In conclusion, we have found that the relationship between curvature and Lagrangian stability is much more subtle than was anticipated in Arnol'd's original research, for general Lie groups. We have also shown while there are simple criteria which guarantee Eulerian stability, the known criteria for Lagrangian stability are generally not exhaustive. One rigorous criterion is that the curvature $R(Y, X)X$ be strictly larger than the second covariant derivative along the geodesic, $-\nabla_X \nabla_X Y$, but this is rarely satisfied. Another criterion is topological and algebraic in nature, but only guarantees sublinear growth of Jacobi fields. In general it seems easier to distinguish between polynomial and nonpolynomial growth of Jacobi fields than between boundedness and unboundedness. We will find this to be especially true in infinite dimensions, when studying the stability of incompressible fluid flows.

4 Riemannian geometry of the diffeomorphism group

The results in the preceding section apply only to finite-dimensional Lie groups. In fluid mechanics, the main topic of interest is infinite-dimensional Lie groups.

We expect many of the results to remain the same, however, although the proofs may change somewhat. Almost all of the computations remain essentially the same, however.

In the remainder of this research, we are interested in the motion of a fluid which fills up a fixed domain M. We consider only the case where M is a compact manifold, possibly with boundary. For many purposes, it is sufficient to consider only the velocity of a fluid element at each point of the domain, and study how this field of velocities evolves in time. (This is the Eulerian, or spatial, point of view.) We, however, are interested in the motion of each individual fluid particle. (This is the Lagrangian, or material, point of view.)

One way to think of this motion is to consider the particles separately: for each point in the domain, we have a path through the domain. Then the motion of the fluid is determined by a map from M to $C(\mathbb{R}, M)$. An alternative way is to consider the motion of the fluid as determined by a map from $\mathbb{R}$ to $C(M, M)$, or into $\mathcal{D}(M)$ if we require the maps to be nonsingular (as we generally will). The advantage of this more global approach is that if we view $\mathcal{D}(M)$ as the configuration space, then the fluid motion is simply a path in this space. We can then view the motion as being a standard Newtonian mechanics problem on an infinite-dimensional space, and use analogies from finite-dimensional Newtonian mechanics to understand the fluid.

The advantage of this over the alternative Eulerian approach is that one is working with what are essentially ordinary differential equations on an infinite-dimensional space, rather than partial differential equations on a finite-dimensional space. In addition, we can then use notions such as positive curvature and convexity of potential energy to study the stability of such motion.

For "dust," i.e. a fluid with no internal forces, the motion of the fluid will be a geodesic through the space $\mathcal{D}(M)$, if the kinetic energy in $\mathcal{D}(M)$ is defined in the usual way as the integral of the kinetic energy of each particle. This is, of course, equivalent to stating that each individual particle simply follows a geodesic in M. All geometrical information about $\mathcal{D}(M)$ comes in a natural way from the geometry of M; for example, nonnegative curvature throughout M implies nonnegative curvature throughout $\mathcal{D}(M)$. Similarly, given a potential energy function on M, we get a natural potential energy function on $\mathcal{D}(M)$ by integration. Convexity of the potential energy on M implies convexity of the integrated potential on $\mathcal{D}(M)$. Again, the solution of Newton's equation on $\mathcal{D}(M)$ is determined completely by the solution to Newton's equation on M.

4.1 Differential geometry of $\mathcal{D}(M)$

Although, in practice, we are generally interested in the diffeomorphisms of a single space, it is helpful conceptually to distinguish the domain and range of the maps. The domain is considered the "home" of the fluid particles, and the mass of any collection of particles is defined there. The range is the physical space in which the particles are moving, and the Riemannian metric (along with the Riemannian volume form) is defined there. The advantage of this approach is that conservation of mass is automatic, and the equation of continuity is then unnecessary. It is this which enables us to consider a fluid flow as a path in infinite-dimensional space, rather than simply as a solution to a nonlinear partial differential equation.

So let N be a compact orientable manifold, without boundary, and let ν be a volume form on N. N will be considered the home of the particles; the mass of a subset $\Omega \subset N$ is defined to be $\int_\Omega \nu$. Let M be a compact orientable manifold, without boundary. Let $g = \langle \cdot, \cdot \rangle$ be a Riemannian metric on M, and let μ denote the corresponding volume form. We will suppose that N and M are diffeomorphic, but we will retain the distinction in notation. The case in which M and N have a boundary is of practical interest, but various complications arise when one carries over the ideas. We will discuss this point in more detail later.

The space of C^∞ diffeomorphisms $\eta\colon N \to M$ is denoted by $\mathcal{D}(N, M)$ or simply $\mathcal{D}$. For technical reasons, it is sometimes convenient to expand this space to include those diffeomorphisms that are only H^s, with H^s inverses. (Here we assume $s > n/2 + 1$, so that the maps will all be at least C^1.) The space of H^s diffeomorphisms is denoted by $\mathcal{D}^s(N, M)$. $\mathcal{D}^s(N, M)$ is a Hilbert manifold with the topology given by the Sobolev H^s norm. $\mathcal{D}(N, M)$ is an inverse-limit-Hilbert (ILH) manifold, in the terminology of Omori [12]; its topology is the Frechet topology generated by all the Sobolev metrics for $s > n/2 + 1$.

For technical aspects and rigorous proofs of this background material, see Ebin-Marsden [7]. Here we are interested mainly in certain formulas and explicit solutions of certain differential equations, so we will merely summarize the known results given there. In particular, we will assume everything is C^∞.

It will be very useful to define the density first, as it appears frequently throughout fluid dynamics. Given an orientation-preserving diffeomorphism $\eta\colon N \to M$, $(\eta^{-1})^*\nu$ is a volume form on M. It is thus a multiple of μ, the Riemannian volume form on M, and we define the density $\rho\colon M \to \mathbb{R}$ to be the map satisfying

$$(\eta^{-1})^*\nu = \rho\mu$$

Clearly ρ is always positive, by our restrictions, and depends on the diffeomorphism η. (Physically, the mass density changes as the particles move through space, although the masses of particles themselves never change.)

The tangent space to the manifold $\mathcal{D}$ at a particular η is thought of as the set of infinitesimal displacements in $\mathcal{D}$ starting at η. If we interpret what this means for individual points, we see that an infinitesimal displacement should be a map which takes a point $p \in N$ to an infinitesimal displacement in M, based at $\eta(p)$. In other words, it is a map of the form $X \circ \eta$, where X is a vector field on M. Summarizing, we have

$$T_\eta \mathcal{D}(N, M) = \left\{ X \circ \eta \,\middle|\, X \text{ is a vector field on } M \right\}$$

Loosely speaking, tangent vectors in $T_\eta \mathcal{D}$ correspond to vector fields on the physical space.

A natural question is then what vector fields on $\mathcal{D}$ look like. In general, they are very difficult to describe explicitly. However, recall that for practical purposes in Riemannian geometry, we only need to be able to find a single vector field which coincides with a particular vector at one point. This we can do easily. Given any vector field X on M, we define a right-invariant vector field $\mathbf{X}$ on $\mathcal{D}(N, M)$ by the formula

$$\mathbf{X}_\eta = X \circ \eta$$

This field is right-invariant since composition with η is both the right translation on the group and the differential of the right translation. Its usefulness arises from the fact that we can do virtually any computation on $\mathcal{D}$ with right-invariant vector fields, and this simplifies things greatly. Left translations, on the other hand, are pretty much useless: the right-translation of $X \circ \mathrm{id}$ is $X \circ \eta$, while the left-translation is $\eta_* X$, which is far too complicated to work with, either theoretically or in computations.

Since M is compact, any vector field X has a one-parameter flow ϕ_t defined for all time t. Now if X is a vector field on M, and $\mathbf{X}$ is its right-invariant extension to $\mathcal{D}$, then the flow of $\mathbf{X}$ is a one-parameter family $\mathbf{\Phi}_t \colon \mathcal{D} \to \mathcal{D}$ given by the easily-verified formula

$$\mathbf{\Phi}_t(\eta) = \phi_t \circ \eta$$

and also defined for all t.

An easy way to obtain examples of functions on $\mathcal{D}$ is to begin with a function $f \colon M \to \mathbb{R}$, and define $F \colon \mathcal{D}(N, M) \to \mathbb{R}$ by the formula

$$F(\eta) = \int_N f \circ \eta \, \nu$$

Note that we can rewrite this formula in terms of integration on the physical space M as

$$F(\eta) = \int_M f\rho \, \mu$$

(This is one instance where keeping the domain and range separate clarifies the situation. Natural physical functions will always be defined on the physical space M, but integration is more natural over N; thus the density appears in most physically-constructed integrals.)

Now the function $\mathbf{X}F$ is simple enough to compute: we find

$$(\mathbf{X}F)(\eta) = \int_N Xf \, \nu \tag{21}$$

In a similar fashion, we can compute the Lie bracket of two right-invariant vector fields $\mathbf{X}$ and $\mathbf{Y}$ from the flow definition. We find that

$$[\mathbf{X}, \mathbf{Y}]_\eta = [X, Y] \circ \eta \tag{22}$$

4.2 The metric and covariant derivative

The metric $\mathbf{g} = \langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{D}(N, M)$ is defined so that $\frac{1}{2}\mathbf{g}(X \circ \eta, X \circ \eta)$ is the kinetic energy corresponding to a configuration η of particles having velocities defined by X. Thus it is

$$\mathbf{g}(X \circ \eta, X \circ \eta) = \int_N g(X, X) \circ \eta \; \nu = \int_M \rho g(X, X) \; \mu$$

Clearly then

$$\mathbf{g}(X \circ \eta, Y \circ \eta) = \int_N g(X, Y) \circ \eta \; \nu$$

We observe that the metric is not left-invariant, or even right-invariant. By a change of variables, we have $\mathbf{g}(X \circ \eta, Y \circ \eta) = \mathbf{g}(X \circ \xi, Y \circ \xi)$ if and only if the densities corresponding to η and ξ are the same, and they will not be unless $\eta^{-1} \circ \xi$ happens to be a volume-preserving diffeomorphism.

An important point is that this metric is only L^2; the distance function defined by this metric on $\mathcal{D}$ or even $\mathcal{D}^s$ does not generate the corresponding topology. The topology generated by this metric is too weak to be useful for studying classical solutions of the equations of fluid mechanics, in which we need everything to have at least one derivative. Because of this, we cannot expect global theorems of Riemannian geometry to be valid for this situation, at least not in the usual form or with the usual proofs.

The existence of the covariant derivative is already somewhat tricky to prove, although getting a formula for it is much easier, at least for right-invariant vector fields. If $\mathbf{X}$ and $\mathbf{Y}$ are right-invariant, then the usual derivation, using equations (21) and (22), yields the following formula in terms of X and Y:

$$\left(\nabla_{\mathbf{X}} \mathbf{Y}\right)_\eta = \nabla_X Y \circ \eta \tag{23}$$

See Bao-Lafontaine-Ratiu [3] for details.

This formula is nice, but it is a bit too specialized since we want to compute covariant derivatives of fields which are not necessarily right-invariant. Although we won't need the most general formula for the covariant derivative of a vector field, we will need the most general formula for the derivative of a vector field along a curve. Fortunately this is fairly simple. Let $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{D}(N, M)$ be a curve in $\mathcal{D}$. We can write a general vector field along γ in the form $Y(t) \circ \gamma(t)$, where $Y(t)$ is a time-dependent vector field on M.

To compute $\frac{\mathbf{DY}}{dt}$, we let $X(t)$ denote the vector field on M such that $\frac{d\gamma}{dt} = X(t) \circ \gamma(t)$.

Proposition 4.1. *The covariant derivative of $t \mapsto Y(t) \circ \gamma(t)$ along γ is*

$$\frac{\mathbf{DY}}{dt} = \frac{\partial Y}{\partial t} + \nabla_{X(t)} Y(t) \tag{24}$$

See Bao-Lafontaine-Ratiu [3] for a proof. Note the similarity to equation (2); in this case, the right-translations are done implicitly.

With this formula we can characterize geodesics on $\mathcal{D}(N, M)$. Suppose $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{D}(N, M)$ is a curve. Then for each $p \in N$, $\gamma_p\colon (-\epsilon, \epsilon) \to M$ defined by $\gamma_p(t) = \gamma(t)(p)$ is a curve in M. Covariant differentiation along γ is very simply related to covariant differentiation along γ_p. Let Y be a time-dependent vector field along γ. Define a vector field Y_p along γ_p by the formula $Y_p(t) = Y(t, \gamma_p(t))$. We can easily check that the covariant derivative of Y_p along γ_p is

$$\frac{DY_p}{dt} = \frac{\partial Y_p}{\partial t} + \nabla_{d\gamma_p/dt} Y_p$$

and this formula obviously agrees with the infinite-dimensional analogue from Proposition 4.1.

The consequence of this is that if γ is a geodesic in $\mathcal{D}(N, M)$, i.e. a curve for which the covariant derivative of the tangent vector field is zero, then γ_p is a geodesic in M for each $p \in N$, and the converse is obviously true as well. So for each $\eta \in \mathcal{D}(N, M)$ and each vector $X \circ \eta \in T_\eta \mathcal{D}(N, M)$, we find the geodesic through η in direction $X \circ \eta$ by simply following, for each p, the geodesic through $\eta(p)$ in direction $X_{\eta(p)}$. This is as one would expect: since the dust metric represents particles moving with no internal or external forces, one expects the collection to move in the same way as the individual particles would. It is, however, somewhat surprising that the exponential map for the dust metric has the same smoothness properties as in finite dimensions, even when the topology of $\mathcal{D}$ is given by the Frechet topology or a Sobolev H^s norm. See Ebin-Marsden [7] for details.

Since M is compact without boundary, it is geodesically complete: any geodesic extends for all time. However, $\mathcal{D}(N, M)$ is not geodesically complete. The reason is that if we extend geodesics from every point, some pair of geodesics will typically intersect, say, at time t_c. (One can easily visualize this when M is a circle: if two particles are moving counterclockwise, each with constant velocity, and one is moving faster than the other, then they must eventually collide. In the language of physics, this is one example of a shock.) Once the geodesics intersect, $\gamma(t_c)$ cannot be a diffeomorphism, since it is not 1-1. The space $C^\infty(N, M)$ with the same metric *is* geodesically complete, since we don't need to worry about whether the maps are invertible, but then the density is no longer defined and the physical application breaks down.

Finally, we note that we can rewrite the geodesic equation in a more familiar form, using formula (23). If $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{D}(N, M)$ is a geodesic, let X be the time-dependent vector field such that $\dot\gamma(t) = X(t) \circ \gamma(t)$. Then the geodesic equation is $\frac{\mathbf{D}}{dt}\frac{d\gamma}{dt} = 0$, which can also be written as

$$\frac{\partial X}{\partial t} + \nabla_X X = 0 \tag{25}$$

Note that there is no explicit dependence on γ, and thus we can think of this as simply being an equation in the space of vector fields on M. In fact, this is the more typical way to think of such equations.

Equation (25) is generally known as the pressureless Euler equation. Note that this is a partial differential equation in n dimensions, although the solution (in terms of geodesics on M) is obtained by thinking of it as an ordinary differential equation in infinite dimensions. For explicit computations, Euler equations are usually easier; but for theoretical purposes, it is sometimes preferable to consider the ordinary differential equation on $T\mathcal{D}$, since ODE's in any dimension are nearly always simpler than PDE's.

The point of view in which one takes, as the fundamental objects of study, the velocity field and equations like (25), is called the "Eulerian," or sometimes "spatial," approach. Our approach, in which the fundamental objects are the paths of particles (or, collectively, the path in the diffeomorphism group), is called the "Lagrangian," or sometimes "material" approach. The connection between the two comes from simply integrating the velocity field to obtain the particle paths.

4.3 The curvature

Since the formulas for the covariant derivative are so simple, we should not be surprised that the curvature of $\mathcal{D}$ turns out to be very simple as well. The curvature is defined in terms of vector fields, but since it is tensorial, it depends only on the values of those vector fields at a particular point. Thus, when computing curvature, we can assume the fields are right-invariant, and use equation (23) to evaluate it.

Let $\eta \in \mathcal{D}(N, M)$. Let X, Y, and Z be vector fields on M, and let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ denote the corresponding right-invariant vector fields on $\mathcal{D}(N, M)$. Then the curvature $\mathbf{R}$ at η is given by

$$\begin{aligned} \mathbf{R}(\mathbf{X}_\eta, \mathbf{Y}_\eta)\mathbf{Z}_\eta &= \nabla_{\mathbf{X}_\eta} \nabla_{\mathbf{Y}} \mathbf{Z} - \nabla_{\mathbf{Y}_\eta} \nabla_{\mathbf{X}} \mathbf{Z} - \nabla_{[\mathbf{X}, \mathbf{Y}]_\eta} \mathbf{Z} \\ &= \big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\big) \circ \eta \\ &= R(X, Y) Z \circ \eta \end{aligned} \tag{26}$$

So we find that not only does the curvature depend only on the vector fields X, Y, and Z, as must be true for any manifold, but that the curvature vector field at a point depends only on the vector fields at that point. This is more than we could generally expect of tensors on $\mathcal{D}$.

So far everything that we have computed on $\mathcal{D}(N, M)$ has turned out to be essentially the same, at least for right-invariant fields, as the corresponding objects on M. The only thing that is not quite the same is the sectional curvature, which ends up being slightly more subtle. It is still easy to compute, though. Let X and Y be vector fields on M, $\eta \in \mathcal{D}(N, M)$, and ρ the density function corresponding to η. Then we have

$$
\begin{aligned}
&\mathbf{K}(X\circ\eta, Y\circ\eta) = \\
&\quad= \frac{\langle\langle \mathbf{R}(Y\circ\eta, X\circ\eta)X\circ\eta, Y\circ\eta\rangle\rangle}{\langle\langle X\circ\eta, X\circ\eta\rangle\rangle\,\langle\langle Y\circ\eta, Y\circ\eta\rangle\rangle - \langle\langle X\circ\eta, Y\circ\eta\rangle\rangle\,\langle\langle X\circ\eta, Y\circ\eta\rangle\rangle} \\
&\quad= \frac{\int_M \rho\langle R(Y,X)X, Y\rangle\,\mu}{\big(\int_M \rho\langle X,X\rangle\,\mu\big)\big(\int_M \rho\langle Y,Y\rangle\,\mu\big) - \big(\int_M \rho\langle X,Y\rangle\,\mu\big)^2} \\
&\quad= \frac{\int_M \rho K(X,Y)\big(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2\big)\,\mu}{\big(\int_M \rho\langle X,X\rangle\,\mu\big)\big(\int_M \rho\langle Y,Y\rangle\,\mu\big) - \big(\int_M \rho\langle X,Y\rangle\,\mu\big)^2}
\end{aligned}
\tag{27}
$$

Clearly if the sectional curvature on M is everywhere zero, then so is the sectional curvature on $\mathcal{D}(N,M)$. However, even if the sectional curvature on M is nowhere zero, there are always directions in which the sectional curvature of $\mathcal{D}(N,M)$ is zero. To see this, let X be arbitrary, and let Y be of the form $Y = fX$, for some nonconstant function f. Then $K(X,Y) \equiv 0$ pointwise, since these fields are pointwise dependent. However, as tangent vectors in $T\mathcal{D}$, they are independent; the denominator in equation (27) is nonzero. So $\mathbf{K}(X\circ\eta, f\circ\eta\, Y\circ\eta) = 0$.

What this means is that in general, useful curvature bounds on M do not give useful curvature bounds on $\mathcal{D}(N,M)$. Strict positivity of curvature on M, for example, yields only nonnegativity of curvature on $\mathcal{D}(N,M)$. This point is discussed more thoroughly, with explicit examples, in [3].

Lastly, we write down the Jacobi equation on $\mathcal{D}(N,M)$. Generally the Jacobi equation for a vector field Y along a geodesic γ is

$$\frac{D}{dt}\frac{DY}{dt} + R(Y,\dot\gamma)\dot\gamma = 0$$

Letting $\dot\gamma = X(t)\circ\gamma$, writing this out using formula (24), and eliminating explicit dependence on γ, we obtain

$$\left(\frac{\partial}{\partial t} + \nabla_X\right)\left(\frac{\partial Y}{\partial t} + \nabla_X Y\right) + R(Y,X)X = 0 \tag{28}$$

The relation to the geodesic equation (25) is somewhat unclear from this expression, though. The relationship is as follows. Given a solution X of equation (25), the linearized Euler equation is easily seen to be

$$\frac{\partial Z}{\partial t} + \nabla_X Z + \nabla_Z X = 0 \tag{29}$$

If we now define Y by the equation

$$\frac{\partial Y}{\partial t} + [X,Y] = Z \tag{30}$$

—this is the linearization of the equation $\dot{\gamma}(t) = X(t) \circ \gamma(t)$—we find that Y satisfies the Jacobi equation. Conversely, if Y satisfies the Jacobi equation and we define Z by equation (30), then Z satisfies the linearized Euler equation (29). So the second-order Jacobi equation (28) is equivalent to the two first-order equations (29) and (30).

At the moment this does not seem to help us. The Jacobi equation is equivalent to the Jacobi equations for Y_p along geodesics γ_p, for all p. So in this case, we would simply solve the Jacobi equation (28) by solving ordinary differential equations in M. On the other hand, we note that we have split the usual Jacobi equation into two first-order equations, which are decoupled from each other. Translating back to M, we obtain two decoupled first-order ordinary differential equations along each integral curve of X, which yield the usual Jacobi fields. The consequences of this fact are potentially very interesting, and we would expect to be able to use this to analyze ordinary Riemannian manifolds. However, this takes us too far afield for the moment, and we will leave the topic for future research.

This same trick also works when we use internal forces and study real fluids. The equation (30) in general relates the linearized Euler equation, commonly used for stability analysis, with the curvature. If it is easier to compute the curvature than to solve the linearized Euler equation, as is typical, then we might expect this approach to yield more results than the usual stability analysis. In the rest of this paper, we will discuss why this approach does not work very often, either in incompressible or compressible fluid stability studies, and why it is generally much easier to study the two equations (29) and (30) than to study the full Jacobi equation (28). We also resolve at least one paradox that has caused confusion in previous literature, namely the fact that fluid flows can be "stable" in the Eulerian sense and "unstable" in the Lagrangian sense.

5 Stability of the motion of an incompressible fluid

A fluid is called incompressible if the volume of any collection of fluid particles remains unchanged as the fluid moves. Since the mass of any such collection is already unchanged throughout the motion, this implies that the density of the fluid must be constant in time (although not necessarily in space). Symbolically, we have the formula

$$\gamma(t)^*\mu = \gamma(0)^*\mu$$

for all t, if γ is the path in $\mathcal{D}(N, M)$ of an incompressible fluid.

Thus the condition of incompressibility implies that the motion is constrained to move through the space of diffeomorphisms which preserve the volume form. It is typical to assume that in fact the density is uniform throughout the fluid (in other words, constant in both time and space). Such a fluid is called homogeneous. In this case, we have that

$$\gamma(t)^*\mu = \nu$$

For a homogeneous incompressible fluid, therefore, there is no advantage in considering N and M separately, since the distinction between the two volume forms μ and ν is irrelevant. Therefore, in keeping with convention, we will simply work with a single manifold M and the group of diffeomorphisms $\mathcal{D} = \mathcal{D}(M)$ from M to itself. When we discuss compressible flows, we will again distinguish between M and N.

We denote by $\mathcal{D}_\mu(M)$ the set of diffeomorphisms of M which preserve the volume form μ:

$$\mathcal{D}_\mu(M) = \left\{ \eta \in \mathcal{D}(M) \,\middle|\, \eta^*\mu = \mu \right\}$$

Then clearly the notion of incompressibility is a constraint that the diffeomorphism lies in $\mathcal{D}_\mu$. Important results of Ebin and Marsden [7] show that $\mathcal{D}_\mu$ is actually a submanifold of $\mathcal{D}$, in the ILH topology or even in the H^s Sobolev topology, for $s > n/2+1$. In fact, the same is true if M has a smooth boundary ∂M; the diffeomorphisms then must map the boundary ∂M to itself. So we may extend the discussion to manifolds which have a boundary.

Now that we have our configuration space, the volume-preserving diffeomorphisms $\mathcal{D}_\mu(M)$ of a possibly bounded manifold M, we can state the minimization principle which enables us to determine the motion of the fluid. Since we have a metric and a kinetic energy on $\mathcal{D}(M)$ already, we can simply restrict this to $\mathcal{D}_\mu$ to get a metric and kinetic energy there as well. We then require that the motion of an incompressible fluid minimize the length of the path in $\mathcal{D}_\mu$. In other words, we are studying geodesics in $\mathcal{D}_\mu$, and therefore the submanifold geometry of $\mathcal{D}_\mu$.

The equations for incompressible fluid flow were known and understood long before Riemannian geometry or the theory of diffeomorphism groups. The fact that they could be derived by a minimum-action principle using only the constraint of volume-preserving was shown by Ehrenfest in his thesis, in the early part of the twentieth century. This method was not used for other purposes, however, until Arnol'd rederived the formulas in 1965 in a paper called "Sur la ge'ome'trie diffe'rentielle des groupes de Lie de dimension infinie et ses applications 'a l'hydrodynamique des fluides parfaits," using the Lie group perspective and an analogy with the Euler equations for a rigid body. He also proposed that negative curvature of the group $\mathcal{D}_\mu$ should imply exponential divergence of fluid paths. An excellent and up-to-date reference for this approach is Arnol'd-Khesin [2].

Subsequent research has been focused mainly on computations of curvature, yet there are very few results on the implications for Lagrangian stability. Much more is currently known about Eulerian stability of incompressible steady flows, and this is also due to Arnol'd, but through quite different methods. The connection between the two methods has not been entirely clear, despite the fact that both were essentially discovered by the same person. We hope to illuminate this connection somewhat here, at least for some special classes of fluid flows.

5.1 Differential geometry of $\mathcal{D}_\mu(M)$

The group of volume-preserving diffeomorphisms of a manifold M, possibly with boundary ∂M, consists of those satisfying

$$\eta^*\mu = \mu$$

This description is not very convenient, and it is in fact much easier to describe the tangent space $T\mathcal{D}_\mu$.

If $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{D}_\mu(M)$ is a curve, then we must have

$$\gamma(t)^*\mu = \mu \tag{31}$$

for all t. Let us define a time-dependent vector field X on M by the formula

$$\dot\gamma(t) = X(t) \circ \gamma(t)$$

Differentiating equation (31) with respect to time, we obtain

$$0 = \frac{d}{dt}(\gamma(t)^*\mu) = \gamma(t)^*(\mathcal{L}_{X(t)}\mu) \tag{32}$$

Here $\mathcal{L}_X$ is the Lie derivative of the volume form μ in direction X. This formula is a generalization to the time-dependent case of the more well-known definition of the Lie derivative in the direction of a time-independent vector field.

In addition, since any diffeomorphism of M to itself must take the boundary ∂M to itself, we know that the vector fields $X(t)$ must all be tangent to the boundary. In other words, if $\hat n$ denotes the unit normal vector field on ∂M, then $\langle X, \hat n\rangle\big|_{\partial M} = 0$.

Because of these facts, the tangent space at a diffeomorphism $\eta \in \mathcal{D}_\mu$ is

$$T_\eta\mathcal{D}_\mu(M) = \big\{X \circ \eta \,\big|\, \mathcal{L}_X\mu = 0,\ \langle X, \hat n\rangle\big|_{\partial M} = 0\big\}$$

Since we know that $\mathcal{L}_X\mu = \operatorname{div} X\,\mu$, this implies that the tangent space to $\mathcal{D}_\mu(M)$ at the identity consists of all divergence-free vector fields which are tangent to the boundary.

5.2 The metric on $\mathcal{D}_\mu$; orthogonal projection

The metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{D}_\mu$ is simply the metric inherited from the space $\mathcal{D}$. Specifically, it is

$$\langle\langle X \circ \eta, Y \circ \eta\rangle\rangle = \int_M \langle X, Y\rangle \circ \eta\ \mu$$

for vector fields X and Y and a volume-preserving diffeomorphism η. Changing variables, and using the fact that $\eta^*\mu = \mu$, we obtain

$$\langle\langle X \circ \eta, Y \circ \eta\rangle\rangle = \int_M \langle X, Y\rangle\ \mu$$

In other words, the inner product does not depend on the diffeomorphism η. Thus, on the group of volume-preserving diffeomorphisms, the metric is right-invariant. (It is still not left-invariant, though.) This results in the simplification of a number of formulas, even if we don't use Lie-theoretic methods explicitly.

One of the most important formulas in the study of incompressible fluids is the following decomposition: if X is any vector field on a manifold M, possibly with boundary ∂M, then we can write

$$X = U + \nabla f \tag{33}$$

where $\operatorname{div} U = 0$, U is tangent to the boundary, and ∇f is the gradient of some function f.

We demonstrate this as follows. If we compute the divergence of these fields, we obtain $\Delta f = \operatorname{div} X$. If we impose the Neumann boundary condition $\langle \nabla f, \hat{n} \rangle = \langle X, \hat{n} \rangle$, then we have a Neumann problem for f which has a unique solution (up to an arbitrary constant, which we can specify by requiring $\int_M f \, \mu = 0$, for example). Then $U := X - \nabla f$ is divergence-free, and by construction its normal component on the boundary is zero.

The value of this construction is that the two components are orthogonal in the metric $\langle\langle \cdot, \cdot \rangle\rangle$: if V is divergence-free and tangent to the boundary, then for any function g

$$\begin{aligned} \int_M \langle V, \nabla g \rangle \, \mu &= \int_M \operatorname{div}(gV) \, \mu - \int_M g \operatorname{div} V \, \mu \\ &= \int_{\partial M} g \langle V, \hat{n} \rangle \, \iota_{\hat{n}} \mu \\ &= 0 \end{aligned} \tag{34}$$

Therefore the orthogonal projection of an arbitrary vector field X onto the space $T_e\mathcal{D}_\mu$ of divergence-free vector fields tangent to the boundary is

$$\mathbf{P}(X) = X - \nabla f \tag{35}$$

where f is defined to be the solution of the Neumann problem above.

There is an alternative formula for the orthogonal projection, which uses Hodge theory on forms, and is especially useful in two dimensions. We will use the notation of Schwarz [15], a good reference especially for the Hodge theory of forms in Sobolev topologies.

If X is a vector field, then lowering indices using the metric g yields a 1-form $X^\flat$. There is then a Hodge decomposition

$$X^\flat = dq + \delta\omega + d\epsilon + \phi \tag{36}$$

Here q is a 0-form (i.e., a function) which vanishes on the boundary, ω is a 2-form whose normal component vanishes on the boundary (and we may assume that $d\omega = 0$), ϵ is a harmonic function with $\Delta\epsilon = 0$, and ϕ is a Neumann field (i.e., a harmonic field with vanishing normal component). (Recall that a harmonic field α is one satisfying $d\alpha = 0$ and $\delta\alpha = 0$.) The decomposition is orthogonal in the metric $\langle\langle\cdot,\cdot\rangle\rangle$.

Since $\operatorname{div} V = -\delta V^\flat$ and $df = (\nabla f)^\flat$ for any vector field V and function f, we see that by equation (34), the orthogonal projection of the term $dq + d\epsilon$ is zero. So we have

$$\mathbf{P}(X) = (\delta\omega + \phi)^\sharp$$

This is not so helpful without formulas for ω and ϕ, however.

Computing d of both sides of (36), we obtain $dX^\flat = d\delta\omega$. Noting that $d\omega = 0$, this is equivalent to

$$dX^\flat = (d\delta + \delta d)\omega$$

and therefore ω is the solution of $\triangle_H \omega = dX^\flat$ (where $\triangle_H = d\delta + \delta d$ is the Hodge Laplacian) satisfying the condition $\mathbf{n}\omega = 0$ on the boundary. This does not completely specify ω, since ω is unique only up to a choice of a Neumann field, but it does completely specify $\delta\omega$, which is all we really need. Unfortunately, 2-forms are not terribly convenient to work with in general, and this is why this approach is mainly preferred in 2 dimensions.

When $n = 2$, 2-forms are equivalent to functions: any 2-form ω can be written as $-\star\psi$, where ψ is a function on M. The condition $\mathbf{n}\omega = 0$ translates into $\psi = 0$ on the boundary (see Schwarz [15], Section 1.2). So the term $\delta\omega$ is $-\delta(\star\psi) = \star d\psi$. We denote the vector field corresponding to this 1-form as $\operatorname{sgrad}\psi$. The function ψ solves the Dirichlet problem $\Delta\psi = -\triangle_H \psi = \star\triangle_H \omega = \star dX^\flat = \operatorname{curl} X$. (We interpret the curl of a vector field in 2 dimensions as a function, and we consider it to be defined on a general manifold by the relation above.) Thus we find that the projection of X onto the space of divergence-free vector fields tangent to the boundary is

$$\mathbf{P}(X) = \operatorname{sgrad}\psi + W \tag{37}$$

where ψ is defined by $\Delta\psi = \operatorname{curl} X$ and $\psi\big|_{\partial M} = 0$, while W satisfies $\operatorname{curl} W = 0$ and $\operatorname{div} W = 0$.

The other component of the Hodge decomposition, as yet undetermined, is the Neumann field ϕ (corresponding to the vector field W). The space of Neumann 1-fields $\mathcal{H}^1_N$ is finite-dimensional, and is in fact determined completely by the topology of M. (The same is true if M has no boundary.) Thus it is fairly easy to compute the projection of X onto this finite-dimensional subspace. If $\{W_k, 1 \le k \le K\}$ is an orthogonal basis of this subspace, then

$$\mathbf{P}(X) = \operatorname{sgrad}\psi + \sum_{k=1}^{K} \frac{\int_M \langle X, W_k\rangle\,\mu}{\int_M \langle W_k, W_k\rangle\,\mu} W_k$$

For the reader's convenience, we provide formulas for sgrad and curl on a general 2-dimensional Riemannian manifold with metric $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{smallmatrix}\right)$ and volume form $\mu = \sqrt{\det g}\,dx \wedge dy$. We have

$$\operatorname{sgrad} f = \frac{1}{\sqrt{\det g}}\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) \tag{38}$$

and

$$\operatorname{curl} X = \frac{1}{\sqrt{\det g}} \left(\frac{\partial}{\partial x}(g_{12}\, X^1 + g_{22}\, X^2) - \frac{\partial}{\partial y}(g_{11}\, X^1 + g_{12}\, X^2) \right) \tag{39}$$

The following are easily-verified consequences, for any f:

$$\begin{aligned} \langle \operatorname{sgrad} f, \operatorname{sgrad} f \rangle &= \langle \nabla f, \nabla f \rangle \\ \langle \operatorname{sgrad} f, \nabla f \rangle &= 0 \\ \operatorname{curl} \nabla f &= 0 \\ \operatorname{curl} \operatorname{sgrad} f &= \Delta f \end{aligned} \tag{40}$$

5.3 The covariant derivative and the geodesic equation on $\mathcal{D}_\mu$

Since $\mathcal{D}_\mu$ is a submanifold of $\mathcal{D}$, and the metric on $\mathcal{D}_\mu$ is inherited from that on $\mathcal{D}$, we know that the covariant derivative $\widetilde{\nabla}$ on $\mathcal{D}_\mu$ is the projection of the covariant derivative ∇ on $\mathcal{D}$.

For vector fields X and Y in $T_e\mathcal{D}_\mu$, define a function p^{XY} by the following conditions:

$$\begin{aligned} \Delta p^{XY} &= -\operatorname{div}(\nabla_X Y) \\ \langle \nabla p^{XY}, \hat{n} \rangle &= -\langle \nabla_X Y, \hat{n} \rangle \\ \int_M p^{XY}\, \mu &= 0 \end{aligned} \tag{41}$$

The first two problems specify a Neumann problem for p^{XY}, which has a unique solution up to an arbitrary constant; the last equation simply specifies the constant.

Suppose $\mathbf{X}$ and $\mathbf{Y}$ are right-invariant vector fields on $\mathcal{D}_\mu$, with $\mathbf{X}_\eta = X \circ \eta$ and $\mathbf{Y}_\eta = Y \circ \eta$ for any $\eta \in \mathcal{D}_\mu$. Then by equation (35), we have the following formula for the covariant derivative of a right-invariant vector fields $\mathbf{Y}$ in direction X:

$$\widetilde{\nabla}_{\mathbf{X}_\eta} \mathbf{Y} = \mathbf{P}(\nabla_{\mathbf{X}_\eta} \mathbf{Y}) = \mathbf{P}(\nabla_X Y \circ \eta) = \mathbf{P}(\nabla_X Y) \circ \eta = (\nabla_X Y + \nabla p^{XY}) \circ \eta \tag{42}$$

It should be noted that the equation $\mathbf{P}(U \circ \eta) = \mathbf{P}(U) \circ \eta$ is not entirely obvious, and is true only because the metric is right-invariant. For the remainder of this section, we will omit explicit mention of the diffeomorphism η since our formulas will always be right-invariant. Thus, computations at an arbitrary diffeomorphism η are done by right-translating all quantities to the identity e, performing the computation there, then right-translating back to η. This procedure will be implicit in what we do from now on.

The formula (42) is true in any dimension, and is quite useful especially when studying the curvature of $\mathcal{D}_\mu$. Recall that the second fundamental form $\mathbf{B}(\mathbf{X}, \mathbf{Y})$ is defined to be the component of $\nabla_{\mathbf{X}} \mathbf{Y}$ which is perpendicular to $T_e\mathcal{D}_\mu$. It actually depends only on the values of $\mathbf{X}$ and $\mathbf{Y}$ at a particular diffeomorphism, so we have $\mathbf{B}(\mathbf{X},\mathbf{Y})_e = \mathbf{B}(\mathbf{X}_e,\mathbf{Y}_e) = \mathbf{B}(X,Y)$ for divergence-free vector fields X and Y on M. By formula (42), we have

$$\mathbf{B}(X,Y) = \nabla p^{XY} \tag{43}$$

It follows from this, and can be shown directly, that the function p^{XY} is symmetric as an operator in X and Y.

Before continuing, we derive another formula for the covariant derivative, in terms of the action on 1-forms. Let $\alpha = X^\flat$ and let β be the 1-form defined by $\beta^\sharp = \mathbf{P}(\nabla_X X)$. Let Z be any vector field in $T_e\mathcal{D}_\mu$, and let $\gamma = Z^\flat$. Then we have, for any such γ,

$$\begin{aligned}
\int_M \langle\beta,\gamma\rangle\,\mu &= \int_M \langle\mathbf{P}(\nabla_X X), Z\rangle\,\mu \\
&= \int_M \langle\nabla_X X, Z\rangle\,\mu \\
&= \int_M X\langle X, Z\rangle\,\mu + \int_M \langle X,[Z,X]\rangle\,\mu - \tfrac{1}{2}\int_M Z\langle X,X\rangle\,\mu
\end{aligned}$$

and the first and third integrals on the last line vanish by formula (34).

Now we use the fact that

$$\langle X,[Z,X]\rangle = \alpha([Z,X]) = X\alpha(Z) - Z\alpha(X) - d\alpha(Z,X)$$

and obtain

$$\int_M \langle\beta,\gamma\rangle\,\mu = -\int_M d\alpha(Z,X)\,\mu$$

where we again used formula (34) to eliminate two integrals. Now since $d\alpha(Z,X) = \langle\gamma\wedge\alpha, d\alpha\rangle$, we have (using the definition of the Hodge star operator)

$$\begin{aligned}
\int_M \langle\beta,\gamma\rangle\,\mu &= -\int_M \langle\gamma\wedge\alpha, d\alpha\rangle\,\mu \\
&= -\int_M \gamma\wedge\alpha\wedge\star d\alpha \\
&= (-1)^n\int_M \langle\gamma, \star(\alpha\wedge\star d\alpha)\rangle\,\mu
\end{aligned}$$

Thus, since this is true for any γ in the kernel of δ, we must have (by the Hodge decomposition)

$$\beta = (-1)^n \star(\alpha\wedge\star d\alpha) + dq \tag{44}$$

for some function q.

Knowing $\mathbf{P}(\nabla_X X)$ for any X, we can obtain $\mathbf{P}(\nabla_X Y) + \mathbf{P}(\nabla_Y X)$ by polarization. And since $\mathbf{P}(\nabla_X Y) - \mathbf{P}(\nabla_Y X) = \mathbf{P}([X,Y]) = [X,Y]$, we can easily obtain a general formula for $\mathbf{P}(\nabla_X Y)$.

Now we look at this formula in the special case $n = 2$. When $n = 2$, $d\alpha$ is a 2-form, so $\star d\alpha$ is a 0-form. Therefore, $\alpha\wedge\star d\alpha = (\star d\alpha)\alpha$, and so

$$\beta = (\star d\alpha)\star\alpha + dq$$

Computing d of the right-hand side, we obtain

$$\begin{aligned} d((\star d\alpha) \star \alpha) &= d(\star d\alpha) \wedge \star\alpha + (\star d\alpha) \wedge d \star \alpha \\ &= \langle d(\star d\alpha), \alpha \rangle \\ &= X(\star d\alpha) \\ &= X(\text{curl } X) \end{aligned}$$

Using equation (37), we have

$$\mathbf{P}(\nabla_X X) = \text{sgrad } (\Delta_0^{-1} X(\text{curl } X)) + W \tag{45}$$

for some harmonic vector field W, tangent to the boundary, with curl $W = 0$ and $\operatorname{div} W = 0$.

Now the geodesic equation on $\mathcal{D}_\mu$ is obtained in exactly the same way as on $\mathcal{D}$ in equation (25). A geodesic γ on $\mathcal{D}_\mu$ satisfies the equation

$$\frac{\widetilde{\mathbf{D}}}{\partial t} \frac{d\gamma}{dt} = 0$$

The covariant derivative $\frac{\widetilde{\mathbf{D}}}{\partial t}$ along a curve in $\mathcal{D}_\mu$ is simply the projection of the covariant derivative in $\mathcal{D}$. So we have

$$\mathbf{P}\left(\frac{\mathbf{D}}{\partial t} \frac{d\gamma}{dt} \right) = 0$$

Now, if $\dot{\gamma}(t) = X(t) \circ \gamma(t)$, then using Proposition 4.1, we have

$$\mathbf{P}\left(\frac{\partial X}{\partial t} + \nabla_X X \right) = 0$$

Since the metric and volume form are independent of time, the projection of the first term is simply $\frac{\partial X}{\partial t}$, while the projection of the second term is obtained from equation (42). We get

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla p^{XX} \tag{46}$$

This is the Euler equation for incompressible fluid flows. The function p^{XX} is called the pressure of the incompressible flow X.

In two dimensions, we can use formula (37) to rewrite the geodesic equation in a much more convenient form. We obtain

$$\frac{\partial}{\partial t} \text{curl } X + X(\text{curl } X) = 0 \tag{47}$$

If $X = \text{sgrad } f + W$ where f is a function which vanishes on the boundary and W is a vector field with $\operatorname{div} W = 0$ and curl $W = 0$, then this equation becomes

$$\frac{\partial}{\partial t} \Delta f + \{f, \Delta f\} + W(\Delta f) = 0 \tag{48}$$

where the Poisson bracket $\{f, g\}$ is a function defined by

$$\{f, g\} = \star(df \wedge dg)$$

Of course, we still need an equation for W, the Neumann-field component of the velocity field X, but the best we can do in general is

$$\frac{\partial W}{\partial t} = -\Pi_{\mathcal{H}^1_N}(\nabla_X X)$$

where $\Pi_{\mathcal{H}^1_N}$ denotes orthogonal projection onto the finite-dimensional space of Neumann fields. For special cases we can work this out explicitly.

A fluid motion γ is called steady if the corresponding vector field is independent of time, i.e. $\dot{\gamma}(t) = X \circ \gamma(t)$. From equation (46), we see that a steady incompressible fluid flow satisfies the equation

$$\nabla_X X = -\nabla p^{XX} \tag{49}$$

In two dimensions, using equation (47), we find that a steady flow satisfies

$$X(\text{curl } X) = 0 \tag{50}$$

5.4 Curvature formulas for $\mathcal{D}_\mu$

Up to this point we have essentially been rederiving well-known equations in the general Riemannian case, from our basic hypothesis that flows of incompressible fluids are geodesics on $\mathcal{D}_\mu$. The Euler equations had been derived well before the geometric interpretation was understood, from physical arguments. The fact that they are equivalent to geodesic equations, however, makes curvature a natural object to study.

In finite dimensions, the curvature along a geodesic tells us about the stability of small perturbations along it. If the curvature is strictly negative, then small perturbations grow exponentially; if the curvature is strictly positive, then small perturbations are bounded (at least up to the first conjugate point). We would expect something similar to be valid in the infinite-dimensional case of $\mathcal{D}_\mu$. Thus we compute several formulas for the curvature, in order to figure out when it is positive and when it is negative.

The simplest formula for the curvature of $\mathcal{D}_\mu$ comes from the theory of isometric immersions. By Gauss' Equation for isometric immersions, the curvature $\widetilde{\mathbf{R}}$ of $\mathcal{D}_\mu$ is related to the curvature $\mathbf{R}$ of $\mathcal{D}$ through of the second fundamental form $\mathbf{B}$. Using equation (43), we have, for vector fields X, Y, Z, W in $T_e\mathcal{D}_\mu$,

$$\begin{aligned}
\langle\langle \widetilde{\mathbf{R}}(X,Y)Z, W\rangle\rangle &= \langle\langle \mathbf{R}(X,Y)Z, W\rangle\rangle + \\
&\qquad \langle\langle \mathbf{B}(X,W), \mathbf{B}(Y,Z)\rangle\rangle - \langle\langle \mathbf{B}(X,Z), \mathbf{B}(Y,W)\rangle\rangle \\
&= \int_M \langle R(X,Y)Z, W\rangle \,\mu + \\
&\qquad \int_M \langle \nabla p^{XW}, \nabla p^{YZ}\rangle \,\mu - \int_M \langle \nabla p^{XZ}, \nabla p^{YW}\rangle \,\mu
\end{aligned} \tag{51}$$

We can rewrite this formula to remove the implicit dependence on W, to get an equation for $\widetilde{\mathbf{R}}(X,Y)Z$ by itself. To do this, we integrate the middle term by parts, using the definition (41) and formula (34).

$$\begin{aligned}
\int_M \langle \nabla p^{XW}, \nabla p^{YZ}\rangle\,\mu &= \int_M \operatorname{div}(p^{YZ}\nabla p^{XW})\,\mu - \int_M p^{YZ}\Delta p^{XW}\,\mu \\
&= \int_{\partial M} p^{YZ}\langle \nabla p^{XW}, \hat{n}\rangle\,\iota_{\hat{n}}\mu + \int_M p^{YZ}\operatorname{div}(\nabla_X W)\,\mu \\
&= -\int_{\partial M} p^{YZ}\langle \nabla_X W, \hat{n}\rangle\,\iota_{\hat{n}}\mu + \int_M \operatorname{div}(p^{YZ}\nabla_X W)\,\mu \\
&\qquad\qquad - \int_M \langle \nabla p^{YZ}, \nabla_X W\rangle \\
&= -\int_M X\langle \nabla p^{YZ}, W\rangle\,\mu + \int_M \langle \nabla_X \nabla p^{YZ}, W\rangle\,\mu \\
&= \int_M \langle \nabla_X \nabla p^{YZ}, W\rangle\,\mu
\end{aligned}$$

We get a similar formula by integrating the last term in equation (51) by parts.

Using these formulas in equation (51), we get

$$\begin{aligned}
\int_M \langle \widetilde{\mathbf{R}}(X,Y)Z, W\rangle\,\mu &= \int_M \langle R(X,Y)Z, W\rangle\,\mu \\
&\quad + \int_M \langle \nabla_X \nabla p^{YZ}, W\rangle\,\mu - \int_M \langle \nabla_Y \nabla p^{XZ}, W\rangle\,\mu
\end{aligned} \tag{52}$$

Therefore, the curvature $\widetilde{\mathbf{R}}$ itself is

$$\widetilde{\mathbf{R}}(X,Y)Z = \mathbf{P}\Big(R(X,Y)Z + \nabla_X \nabla p^{YZ} - \nabla_Y \nabla p^{XZ}\Big) \tag{53}$$

Rouchon [14] first derived this formula using a rather different method, using the Jacobi equation on $\mathcal{D}_\mu$. He used it to derive a necessary and sufficient condition for sectional curvature to be positive for all planes containing a given direction. We will present his result in the next section.

We obtain a very different formula for the curvature by using a more direct method. This formula fits more naturally with the alternative formulas given above in two dimensions for the covariant derivative. We are primarily interested in the sectional curvature, which involves only $\langle\langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\rangle$. Using the definition of the curvature, we have

$$\widetilde{\mathbf{R}}(Y,X)X = \widetilde{\nabla}_Y \widetilde{\nabla}_X X - \widetilde{\nabla}_X \widetilde{\nabla}_Y X - \widetilde{\nabla}_{[Y,X]} X$$

Computing the inner product, we get

$$
\begin{aligned}
&\langle\langle \widetilde{\mathbf{R}}(Y,X)X,Y\rangle\rangle = \\
&\qquad = \int_M \langle \widetilde{\nabla}_Y \widetilde{\nabla}_X X, Y\rangle\,\mu - \int_M \langle \widetilde{\nabla}_X \widetilde{\nabla}_Y X, Y\rangle\,\mu - \int_M \langle \widetilde{\nabla}_{[Y,X]} X, Y\rangle\,\mu \\
&\qquad = \int_M Y\langle \widetilde{\nabla}_X X, Y\rangle\,\mu - \int_M \langle \widetilde{\nabla}_X X, \widetilde{\nabla}_Y Y\rangle\,\mu - \int_M X\langle \widetilde{\nabla}_Y X, Y\rangle\,\mu \\
&\qquad\qquad + \int_M \langle \widetilde{\nabla}_Y X, \widetilde{\nabla}_X Y\rangle\,\mu - \int_M \langle \widetilde{\nabla}_{[Y,X]} X, Y\rangle\,\mu \\
&\qquad = -\int_M \langle \mathbf{P}(\nabla_X X), \mathbf{P}(\nabla_Y Y\rangle\,\mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_X Y\rangle\,\mu \\
&\qquad\qquad + \int_M \langle \mathbf{P}(\nabla_{[X,Y]} X), Y\rangle\,\mu \\
&\qquad = -\int_M \langle \mathbf{P}(\nabla_X X), \mathbf{P}(\nabla_Y Y\rangle\,\mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X\rangle\,\mu \\
&\qquad\qquad + \int_M \langle \mathbf{P}(\nabla_Y X), [X,Y]\rangle\,\mu + \int_M \langle \mathbf{P}(\nabla_{[X,Y]} X), Y\rangle\,\mu
\end{aligned}
$$

Now notice that in the last line, the two projection operators are actually unnecessary, because $[X,Y]$ and Y are already in $T_e\mathcal{D}_\mu$. This enables us to write

$$
\begin{aligned}
\langle\langle \widetilde{\mathbf{R}}(Y,X)Y,X\rangle\rangle &= -\int_M \langle \mathbf{P}(\nabla_X X), \mathbf{P}(\nabla_Y Y\rangle\,\mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X\rangle\,\mu \\
&\qquad + \int_M \langle \nabla_Y X, [X,Y]\rangle\,\mu + \int_M \langle \nabla_{[X,Y]} X, Y\rangle\,\mu \\
&= -\int_M \langle \mathbf{P}(\nabla_X X), \mathbf{P}(\nabla_Y Y\rangle\,\mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X\rangle\,\mu \\
&\qquad + \int_M \mathcal{L}_X g(Y,[X,Y])\,\mu
\end{aligned}
\tag{54}
$$

The last formula is valid since

$$\mathcal{L}_X g(U,V) = \langle \nabla_U X, V\rangle + \langle \nabla_V X, U\rangle$$

A simpler version of this formula was first derived by Misiołek [10], under the assumption that $\mathcal{L}_X g = 0$, though the technique is the same.

5.5 The sign of the curvature

We would intuitively expect to be able to predict the motion of fluid particles if the curvature is nonnegative along the geodesic defined by the fluid flow. This is important especially when studying the motion of particles (such as pollutants) moving along with the fluid; for such a purpose, Eulerian stability theory is less directly helpful.

Arnol'd was the first to compute curvature explicitly for the torus $\mathbb{T}^2$. (The computation is described in Arnol'd and Khesin [2].) He showed that the curvature was often negative. Lukatsky [8] developed several explicit formulas for the curvature in various cases, and gave criteria for positive or negative curvature for Euclidean space. Rouchon [14] proved that when M is a body in $\mathbb{R}^3$, the only vectors X in $\mathcal{D}_\mu(M)$ with $\mathbf{K}(X,Y) \geq 0$ for all Y are Killing fields on M. His method is basically two-dimensional and local, so it is easily generalized as demonstrated below. Misiołek was the first to show in full generality that $\mathbf{K}(X,Y) \geq 0$ when X is a Killing field. Theorem 5.3 below unifies these results.

Before demonstrating the theorem, however, we first perform a simplification of formula (52). We are interested in the sectional curvature, so we compute $\langle\langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\rangle$. Here we assume X is the velocity field of a (possibly nonsteady) incompressible fluid flow at a particular time, while Y is some arbitrary divergence-free vector field linearly independent of X. Combining formulas (52) and (51) (in particular, performing our simplification on the middle but not the last term of formula (51)), we obtain

$$\begin{aligned}\int_M \langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\, \mu = \int_M \langle R(Y,X)X, Y\rangle\, \mu \\ + \int_M \langle \nabla_Y \nabla p^{XX}, Y\rangle\, \mu - \int_M \langle \nabla p^{XY}, \nabla p^{XY}\rangle\, \mu\end{aligned} \tag{55}$$

Note that the first two terms involve integrals of quantities bilinear in Y. This will be very useful, as we can then analyze these linear operators pointwise in each tangent space. The last term is nonpositive, and we can rewrite it using the orthogonal decomposition $\nabla_Y X = \mathbf{P}(\nabla_Y X) - \nabla p^{XY}$:

$$\begin{aligned}\int_M \langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\, \mu = \int_M \langle R(Y,X)X, Y\rangle\, \mu + \int_M \langle \nabla_Y \nabla p^{XX}, Y\rangle\, \mu \\ - \int_M \langle \nabla_Y X, \nabla_Y X\rangle\, \mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\, \mu\end{aligned} \tag{56}$$

Now the first three terms of equation (56) are bilinear in Y. We can write

$$\int_M \langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\, \mu = \int_M \langle Y, A(Y)\rangle\, \mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\, \mu$$

where the tensor A is given in index notation by the following formula:

$$A^i_j = R_{jkl}{}^i X^k X^l + \nabla_j \nabla^i p^{XX} - g_{kl} \nabla_j X^k \nabla^i X^l \tag{57}$$

This formula involves the Hessian of the pressure function p^{XX} corresponding to the fluid flow. Although we don't have an explicit formula for the Hessian of the pressure, we do have an explicit formula for the Laplacian, which is the trace of the Hessian. (See the definition in formula (41).) This suggests computing the trace of the tensor A. The result is quite illuminating, if one can tolerate the index debauchery.

$$
\begin{aligned}
A_i{}^i &= R_{ikl}{}^i X^k X^l + \nabla_i \nabla^i p^{XX} - g_{kl}\, g^{mi} \nabla_i X^k \nabla_m X^l \\
&= R_{ikl}{}^i X^k X^l - \nabla_i (X^j \nabla_j X^i) - \nabla_i X^k \nabla^i X_k \\
&= R_{ikl}{}^i X^k X^l - \nabla_i X^j \nabla_j X^i - X^j \nabla_i \nabla_j X^i - \nabla_i X^j \nabla^i X_j \\
&= R_{ikl}{}^i X^k X^l - \nabla_i X^j \nabla_j X^i - X^j (\nabla_j \nabla_i X^i + R_{ijk}{}^i X^k) - \nabla_i X^j \nabla^i X_j \\
&= -\nabla_i X^j \nabla_j X^i - \nabla_i X^j \nabla^i X_j \\
&= -g^{ik}\, g^{jl} \nabla_i X_j \nabla_l X_k - g^{ik}\, g^{jl} \nabla_i X_j \nabla_k X_l
\end{aligned}
$$

To simplify this further, let $S_{ij} = \nabla_i X_j + \nabla_j X_i$, and $\omega_{ij} = \nabla_i X_j - \nabla_j X_i$. S_{ij} are the components of the deformation tensor $\mathcal{L}_X g$, while ω_{ij} are the components of the vorticity 2-form $dX^\flat$. We find

$$
\begin{aligned}
A_i{}^i &= -g^{ik}\, g^{jl} (\nabla_i X_j) S_{kl} \\
&= -\tfrac{1}{2} g^{ik}\, g^{jl} (S_{ij} + \omega_{ij}) S_{kl} \\
&= -\tfrac{1}{2} g^{ik}\, g^{jl} S_{ij} S_{kl}
\end{aligned}
$$

since

$$
g^{ik}\, g^{jl} \omega_{ij} S_{kl} = g^{jl}\, g^{ik} \omega_{ji} S_{lk} = -g^{ik}\, g^{jl} \omega_{ij} S_{kl},
$$

so this term vanishes.

At any point p, we may choose coordinates such that $g^{ij}\big|_p = \delta^{ij}$. Then at p,

$$
A_i{}^i = -\tfrac{1}{2} \sum_{i=1}^n \sum_{j=1}^n S_{ij}^2
$$

In particular, if $S_{ij} \neq 0$ at any point, then Tr $A < 0$ at that point, and thus A must have at least one negative eigenvalue at that point. Using

$$
\langle\langle \widetilde{\mathbf{R}}(Y, X)X, Y \rangle\rangle = \int_M \langle Y, A(Y) \rangle \mu + \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X) \rangle\, \mu \tag{58}
$$

we see that the curvature will be negative as long as we can find a Y that is concentrated near an eigenvector of A with negative eigenvalue, and such that the positive second term is very small. The method below is due to Rouchon.

Lemma 5.1. *If A and B are two linear transformations of $\mathbb{R}^n$, with A symmetric and Tr $B = 0$, then there is a $w \in \mathbb{R}^n$ such that $\langle w, w \rangle = 1$, $\langle w, Aw \rangle \leq \frac{1}{n}$ Tr A, and $\langle w, Bw \rangle = 0$.*

Proof. Let S be the symmetric part of B; then clearly $\langle w, Sw \rangle = \langle w, Bw \rangle$ and Tr $S =$ Tr B. Thus we can assume B is symmetric.

Choose an orthogonal basis $\{v_i\}$ such that $B = \mathrm{diag}(\lambda_1 \cdots \lambda_n)$, with $\lambda_1 + \cdots + \lambda_n = 0$ by assumption. Consider the set of 2^n vectors

$$
J = \left\{ \frac{1}{\sqrt{n}} (\epsilon_1 v_1 + \cdots + \epsilon_n v_n) \;\Big|\; \epsilon_i^2 = 1,\ \forall\, i \right\} = \left\{ \frac{1}{\sqrt{n}} \boldsymbol{\epsilon} \;\Big|\; \boldsymbol{\epsilon} \in \{-1, 1\}^n \right\}
$$

Then for each $w \in J$, we have $\langle w, Bw \rangle = \sum_{i=1}^{n} \lambda_i \epsilon_i^2 = 0$.

We compute the sum of $\langle w, Aw \rangle$ over all $w \in J$:

$$\sum_{w \in J} \langle w, Aw \rangle = \frac{1}{n} \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^n} \sum_{i,j=1}^{n} a_{ij} \epsilon_i \epsilon_j = \frac{1}{n} \sum_{i,j=1}^{n} a_{ij} \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^n} \epsilon_i \epsilon_j \tag{59}$$

The sum $\sum_{\boldsymbol{\epsilon} \in \{-1,1\}^n} \epsilon_i \epsilon_j$ is easy to work out. If $i \neq j$, it is $2^{n-2}((-1)(-1) + (-1)(1) + (1)(-1) + (1)(1)) = 0$. If $i = j$, then it is $2^{n-1}((-1)^2 + (1)^2) = 2^n$. So

$$\sum_{w \in J} \langle w, Aw \rangle = \frac{1}{n} \sum_{i=1}^{n} a_{ii} 2^n = \frac{2^n}{n} \text{Tr } A$$

and therefore the average value of $\langle w, Aw \rangle$ over J is $\frac{1}{n} \text{Tr } A$. So at least one element of J must have $\langle w, Aw \rangle \leq \frac{1}{n} \text{Tr } A$. □

Theorem 5.2. *Let M be any manifold, with or without boundary. If X is any divergence-free vector field, tangent to the boundary, such that $(\mathcal{L}_X g)(p) \neq 0$ at some point p, then there is a divergence-free vector field Y, with support in a neighborhood of p, such that $\langle\langle \widetilde{\mathbf{R}}(Y, X)X, Y \rangle\rangle < 0$.*

Proof. If $\mathcal{L}_X(g)$ is not zero at p, then Tr $A < 0$ as we computed above. By Lemma 5.1, we can find a vector $W \in T_pM$ such that $\langle W, W \rangle = 1$, $\langle W, AW \rangle < 0$, and $\langle W, \nabla_W X \rangle = 0$. Let $V = \nabla_W X \in T_pM$, and $a = ||V||$. Choose normal coordinates in a neighborhood Ω of p such that $\partial_1\big|_p = W$, $\partial_2\big|_p = \frac{1}{a}V$, $g_{ij}\big|_p = \delta_{ij}$, and $\Gamma_{ij}^k\big|_p = 0$. (Thus we are simply using Gaussian normal coordinates, rotated so that the first two directions point the way we want.)

Let ϵ be a small positive number. Let $\psi\colon [0,\infty) \to [0,\infty)$ be a C^∞ function positive on $[0,1)$ and zero elsewhere. Let $\xi\colon M \to [0,\infty)$ be defined in coordinates by

$$\xi(x_1, x_2, x_3, \ldots, x_n) = \psi\left(\left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x_3}{\epsilon}\right)^2 + \cdots + \left(\frac{x_n}{\epsilon}\right)^2\right)$$

on Ω and 0 elsewhere on M. Note that the dependence on x_2 is different from the dependence on x_1 and $x_3, \ldots, x_n$. Let Ω_0 be the inverse image of $(0,\infty)$ under ξ. Ω_0 is open, and $x \in \Omega_0$ implies $x_1 < \epsilon, x_2 < \epsilon^2, x_3 < \epsilon, \ldots, x_n < \epsilon$.

Define Y to be

$$Y = \frac{1}{\sqrt{\det g}}\, \epsilon^2 \left(-\partial_2 \xi\, \partial_1 + \partial_1 \xi\, \partial_2\right)$$

Then clearly $\operatorname{div} Y = 0$, and Y is zero outside Ω_0.

By our use of normal coordinates, we have $g_{ij} = \delta_{ij} + O(|x|^2)$ near p. So inside Ω_0, $g_{ij} = \delta_{ij} + O(\epsilon^2)$. Thus, $\det g = 1 + O(\epsilon^2)$, and $\sqrt{\det g} = 1 + O(\epsilon^2)$ as well. So we can write Y as

$$Y = \frac{2\psi'(\rho^2)}{1 + O(\epsilon^2)} \left(-\frac{1}{\epsilon^2} x_2\, \partial_1 + x_1\, \partial_2\right)$$

where $\rho^2 = \left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x^3}{\epsilon}\right)^2 + \cdots + \left(\frac{x^n}{\epsilon}\right)^2$. The term $\frac{1}{\epsilon^2}x_2$ is $o(1)$ in Ω_0, while the term x_1 is $O(\epsilon)$ in Ω_0. So to first order in ϵ, we can write

$$Y = -2\psi'(\rho^2)\frac{1}{\epsilon^2}x_2\,\partial_1 + O(\epsilon)$$

Then we compute $\nabla_Y X$, only to lowest order in ϵ:

$$\begin{aligned}
\nabla_Y X &= -2\psi'(\rho^2)\frac{x_2}{\epsilon^2}\nabla_{\partial_1}X + O(\epsilon) \\
&= -2\psi'(\rho^2)\frac{x_2}{\epsilon^2}(\nabla_W X + O(\epsilon)) + O(\epsilon) \\
&= -2\psi'(\rho^2)\frac{ax_2}{\epsilon^2}\frac{\partial}{\partial x^2} + O(\epsilon)
\end{aligned}$$

We now construct a function which gives, to first order in ϵ, an approximation of the function p^{XY} used for computing the projection $\mathbf{P}(\nabla_Y X)$. Let

$$\sigma(x_1, \ldots, x_n) = a\epsilon^2\psi(\rho^2)$$

Then

$$\begin{aligned}
\nabla\sigma &= 2a\psi'(\rho^2)\left(x_1\partial_1 + \frac{x_2}{\epsilon^2}\partial_2 + x_3\partial_3 + \cdots + x_n\partial_n\right) + O(\epsilon^2) \\
&= 2a\psi'(\rho^2)\frac{x_2}{\epsilon^2}\partial_2 + O(\epsilon)
\end{aligned}$$

Therefore we have

$$\mathbf{P}(\nabla_Y X) = \mathbf{P}(\nabla_Y X + \nabla\sigma) = \mathbf{P}(O(\epsilon))$$

and so

$$\int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\,\mu = \int_{\Omega_0}\langle O(\epsilon), O(\epsilon)\rangle\,\mu = O(\epsilon^2)\mathrm{vol}(\Omega_0) = O(\epsilon^{n+3})$$

On the other hand, the other term in equation (58) is of a lower order than this. Since

$$Y = -2\psi'(\rho^2)\frac{x_2}{\epsilon^2}\partial_1 + O(\epsilon) = -2\psi'(\rho^2)\frac{x_2}{\epsilon^2}W + O(\epsilon)$$

we have

$$A(Y) = -2\psi'(\rho^2)\frac{x_2}{\epsilon^2}A(W) + O(\epsilon)$$

Therefore

$$\langle Y, A(Y)\rangle = 4(\psi'(\rho^2))^2\frac{x_2^2}{\epsilon^4}\langle W, A(W)\rangle + O(\epsilon)$$

and

$$\int_M \langle Y, A(Y)\rangle\,\mu = \langle W, A(W)\rangle\int_{\Omega_0} 4(\psi'(\rho^2))^2\frac{x_2^2}{\epsilon^4}\,\mu + O(\epsilon)$$

This quantity is an integral of a function which is $o(1)$ over a volume Ω_0, and therefore it is $o(\mathrm{vol}(\Omega_0)) = o(\epsilon^{n+1})$. In particular it is of lower order than the term $\int_M\langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\,\mu$. Since $\langle W, A(W)\rangle$ was chosen to be strictly negative, we find that by taking ϵ sufficiently small, $\langle\langle \widetilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle < 0$.

$\square$

The converse to Theorem 5.2 is much easier, and consists merely of showing that the curvature is nonnegative when X is a Killing field. We use formula (54). First, we show that if X is Killing, then it is always a steady solution of the Euler equations. We know that if X is Killing, then $\mathcal{L}_X g(U,V) = 0$ for any vector fields U and V, and that this is equivalent to

$$\langle \nabla_U X, V\rangle + \langle \nabla_V X, U\rangle = 0$$

Now we have, for any vector field Y,

$$\begin{aligned}\langle \nabla_X X, Y\rangle &= -\langle \nabla_Y X, X\rangle \\ &= -\tfrac{1}{2} Y\langle X, X\rangle \\ &= \langle -\tfrac{1}{2}\nabla\langle X, X\rangle, Y\rangle\end{aligned}$$

Therefore,

$$\nabla_X X = -\tfrac{1}{2}\nabla\langle X, X\rangle$$

so X is a solution of the Euler equation with $p^{XX} = \frac{1}{2}\langle X, X\rangle$. In particular, we have $\mathbf{P}(\nabla_X X) = 0$.

Thus, the first and third terms of formula (54) vanish, and we have

$$\langle\langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle\rangle = \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\, \mu$$

which is obviously nonnegative. (As we will see, it is often zero in many directions, and in certain situations it is zero in all directions.)

Thus we have the following general criterion for a geodesic in $\mathcal{D}_\mu$ to have nonnegative curvature in all plane sections along it which contain the tangent vector.

Theorem 5.3. *If $\gamma\colon (-\epsilon,\epsilon) \to \mathcal{D}_\mu$ is a geodesic, with a possibly time-dependent velocity field X, then the sectional curvature $\widetilde{\mathbf{K}}(X,Y)$ is nonnegative for every divergence-free vector field Y if and only if $\gamma(t)$ is an isometry for all t.*

The implication of this is that we cannot expect to use the Rauch Comparison Theorem to determine whether a fluid motion is stable (in the sense of having all solutions of the Jacobi equation remaining bounded). The best we can achieve is an estimate of growth linear in time, and even this is only in the case of isometries. Thus true Lagrangian stability seems impossible for incompressible fluids.

An alternative question is which fluid flows have nonpositive curvature in all directions. If one could actually get strictly negative curvature in all directions (bounded away from zero, for example), then the Rauch Comparison Theorem would tell us that all solutions of the Jacobi equation grow exponentially. Such flows would, for example, be useful in mixing, since the particles in the fluid would tend to separate from each other very quickly.

From formula (55),

$$\int_M \langle \widetilde{\mathbf{R}}(Y,X)X, Y\rangle \,\mu = \int_M \langle R(Y,X)X, Y\rangle \,\mu + \int_M \langle \nabla_Y \nabla p^{XX}, Y\rangle \,\mu - \int_M \langle \nabla p^{XY}, \nabla p^{XY}\rangle \,\mu$$

we see that a sufficient condition that the curvature be nonpositive for all directions Y is that the first two terms be nonpositive in Y. In other words, at every point, the operator $Y \mapsto R(Y,X)X + \nabla_Y \nabla p^{XX}$ should be nonpositive. This happens, in particular, when the pressure function p^{XX} is constant and the curvature of M is nonpositive. This was first noticed by Misiołek [10], who called such flows "pressure-constant."

In fact, it seems that all of the known examples of fluid flows with nonpositive sectional curvature in every direction are pressure-constant flows on flat spaces. Arnol'd's original example (cited in Arnol'd-Khesin [2], Theorem IV.3.4) is a steady zero-pressure flow on the torus $\mathbb{T}^2$, while the example of Nakamura-Hattori-Kambe [11] is a steady zero-pressure flow on the three-torus $\mathbb{T}^3$.

It is natural to ask whether there are any other fluid flows for which the curvature of the diffeomorphism group is nonpositive in every section containing the flow field, with either p^{XX} or $R(\cdot,X)X$ nonzero. Although one can obtain some results on nonpositive curvature using formula (54), there is nothing as complete as Rouchon's Theorem 5.2 for nonnegative curvature. We leave this question for future research.

5.6 General properties of the Jacobi equation

The Jacobi equation is the linearization of the geodesic equation. It is a second-order equation, whose initial conditions are usually given as $Y(0) = 0$ and $\frac{DY}{dt} = V_0$. With these initial conditions, solutions to the Jacobi equation describe the approximate behavior of geodesics which start at the same point as a given one, with an infinitesimal angle between them. This will be the case we are interested in when studying geodesics in $\mathcal{D}_\mu$: geodesics will always start at the identity, but the tangent vectors will be different (corresponding to limited knowledge of the initial velocity field).

In general the Jacobi equation on any Riemannian manifold may be written as

$$\frac{D^2Y}{dt^2} + R(Y,X)X = 0 \tag{60}$$

where X is the (time-dependent) tangent vector to the geodesic. If the sectional curvature is a constant κ_0, then solutions to the Jacobi equation will generally look like

$$Y(t) = \begin{cases} \dfrac{1}{\sqrt{\kappa_0}} \sin\left(\sqrt{\kappa_0}\,t\right) V_0 & \kappa_0 > 0 \\ t\,V_0 & \kappa_0 = 0 \\ \dfrac{1}{\sqrt{|\kappa_0|}} \sinh\left(\sqrt{|\kappa_0|}\,t\right) V_0 & \kappa_0 < 0 \end{cases}$$

If the sectional curvature is not constant, but is bounded above or below by some constant, then the Rauch comparison theorem tells us that solutions to the Jacobi equation are bounded above or below by one of the corresponding constant-curvature solution (at least until the first conjugate point).

In the case of incompressible fluid dynamics, the solutions to the Jacobi equation approximately represent the propagation of small errors in the initially-measured velocity field of the fluid. If the curvature is everywhere negative, then we expect exponential growth of solutions, and therefore that small errors are propagated exponentially. Thus prediction of the paths of individual particles is impossible for long times, because of inevitable errors in the initial conditions. Arnol'd famously used this principle to demonstrate heuristically why weather prediction is impossible: since the curvature of the diffeomorphism group is negative in many directions, errors will typically grow exponentially, and after a couple of weeks one's predictions are meaningless. (See [2] for a recent exposition of these ideas.)

However, without strict curvature bounds, one cannot make this conclusion rigorous. The only known tool for using curvature along the geodesic to estimate rates of geodesic spreading is the Rauch Comparison Theorem. But this theorem requires uniform curvature bounds in every direction along the geodesic. As we have seen in the previous section, uniform bounds are nearly impossible to find on the diffeomorphism group. The only geodesics with everywhere nonnegative curvature along them are the isometries, and these always have directions of zero curvature along them. So the best lower bound one can obtain on curvature is zero, and even this is rare.

In addition, there are no known examples of fluid flows where curvature is negative in all directions (or bounded away from zero). So although there are some steady flows where the curvature is nonpositive in all directions (such as the pressure-constant flows of Misiołek [10]), even this only gives us an upper bound of zero on the curvature. So instead of guaranteeing exponential growth of solutions of the Jacobi equation, the Rauch Theorem only guarantees linear growth of solutions. Thus, comparison methods cannot be relied upon to prove exponential growth of solutions.

However, we are fortunate in that the metric on $\mathcal{D}_\mu$ is right-invariant. The implication of this is that the nominally second-order Jacobi equation can be split into two first-order linear differential equations, one of which is uncoupled from the other, just as on finite-dimensional Lie groups. In the simplest cases, we can solve these equations explicitly. One of these linear equations is the well-known linearized Euler equation, which has been studied extensively. We simply use the known explicit solutions of this equation to obtain explicit solutions of the Jacobi equation. The solutions we present have the peculiar property that even when the curvature is everywhere nonpositive or everywhere nonnegative, the solutions to the Jacobi equation grow linearly in time. Without having many other explicit solutions, it is difficult to tell whether this is a general phenomenon or whether it is unique to these examples. However, it does tell us that negative curvature in most directions does not necessarily imply any kind of exponential growth of small perturbations, and thus gives a counterexample to Arnol'd's argument on weather prediction.

5.6.1 Splitting of the Jacobi equation

The reason the Jacobi equation can be decomposed into first-order equations is because the same is true of the geodesic equation itself. Recall that the geodesic equation

$$\mathbf{P}\left(\frac{\mathbf{D}}{\partial t}\frac{\partial \eta}{\partial t}\right) = 0$$

is equivalent to the two equations

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= X \circ \eta \\ \frac{\partial X}{\partial t} &+ \mathbf{P}(\nabla_X X) = 0 \end{aligned} \tag{61}$$

The reason the second equation does not involve η explicitly is precisely because of the right-invariance of the metric on $\mathcal{D}_\mu$.

If we consider a family of solutions to the equations above, depending on a parameter s, then we can differentiate the equations above with respect to s. Suppose that time-dependent vector fields Y and Z are defined so that

$$\begin{aligned} \frac{\partial \eta(t,s)}{\partial s}\Big|_{s=0} &= Y(t) \circ \eta(t,0) \\ \frac{\partial X(t,s)}{\partial s}\Big|_{s=0} &= Z(t) \end{aligned}$$

Then the field $Y(t)$ is the variation of the geodesic η, while $Z(t)$ is the variation of the velocity field $X(t)$.

The Euler equation, the second equation in formula (61), is easy to differentiate with respect to s, and we obtain at $s = 0$ the linearized Euler equation

$$\frac{\partial Z}{\partial t} + \mathbf{P}(\nabla_Z X) + \mathbf{P}(\nabla_X Z) = 0 \tag{62}$$

The first equation in (61) is a bit more delicate because of the composition. We have

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial \eta}{\partial t} &= \frac{\partial}{\partial s}(X \circ \eta) \\ \frac{\partial}{\partial t}(Y \circ \eta) &= \frac{\partial}{\partial s}(X \circ \eta) \\ \left(\tfrac{\partial Y^k}{\partial t} + X^j \partial_j Y^k\right) \circ \eta\, \partial_k &= \left(Z^k + Y^j \partial_j X^k\right) \circ \eta\, \partial_k \end{aligned}$$

Thus, taking $s = 0$ and composing with η^{-1}, we get

$$\frac{\partial Y}{\partial t} + [X, Y] = Z \tag{63}$$

We now compare these equations with the usual Jacobi equation, obtained by applying the formulas (42) and (53) in the formula (60). We have $\frac{DY}{\partial t} = \frac{\partial Y}{\partial t} + \nabla_X Y$, and so

$$\frac{\widetilde{\mathbf{D}}Y}{\partial t} = \mathbf{P}\left(\frac{\mathbf{D}Y}{\partial t}\right) = \frac{\partial Y}{\partial t} + \nabla_X Y + \nabla p^{XY}$$

Thus the second covariant derivative is

$$\begin{aligned}\frac{\mathbf{D}}{\partial t}\mathbf{P}\left(\frac{\mathbf{D}Y}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial Y}{\partial t} + \nabla_X Y + \nabla p^{XY}\right) + \nabla_X\left(\frac{\partial Y}{\partial t} + \nabla_X Y + \nabla p^{XY}\right) \\ &= \frac{\partial^2 Y}{\partial t^2} + \frac{\partial}{\partial t}(\nabla_X Y) + \nabla\left(\frac{\partial p^{XY}}{\partial t}\right) + \nabla_X\left(\frac{\partial Y}{\partial t}\right) \\ &\qquad + \nabla_X\nabla_X Y + \nabla_X\nabla p^{XY}\end{aligned}$$

Projecting this onto the space of divergence-free vector fields, we get

$$\frac{\widetilde{\mathbf{D}}}{\partial t}\frac{\widetilde{\mathbf{D}}Y}{\partial t} = \frac{\partial^2 Y}{\partial t^2} + \mathbf{P}\left(\frac{\partial}{\partial t}\nabla_X Y + \nabla_X\frac{\partial Y}{\partial t} + \nabla_X\nabla_X Y + \nabla_X\nabla p^{XY}\right)$$

From formula (53) we recall

$$\widetilde{\mathbf{R}}(Y,X)X = \mathbf{P}\left(R(Y,X)X + \nabla_Y\nabla p^{XX} - \nabla_X\nabla p^{XY}\right)$$

Thus the Jacobi equation is

$$\frac{\partial^2 Y}{\partial t^2} + \mathbf{P}\left(\frac{\partial}{\partial t}\nabla_X Y + \nabla_X\frac{\partial Y}{\partial t} + \nabla_X\nabla_X Y + R(Y,X)X + \nabla_Y\nabla p^{XX}\right) = 0 \quad (64)$$

Note that what we have effectively done here is to right-translate the vector field Y, which is properly thought of as a vector field along the geodesic η, back to the identity. This is how we obtained the explicit formula for the covariant derivative $\frac{\widetilde{\mathbf{D}}Y}{\partial t}$. It is more typical when analyzing the Jacobi equation to instead parallel-transport the vector field Y back to the starting location. For theoretical purposes this is useful, but it can almost never be done explicitly, and certainly not on $\mathcal{D}_\mu$.

We have separately derived the linearized geodesic equations

$$\begin{aligned}\frac{\partial Y}{\partial t} + [X,Y] &= Z \\ \frac{\partial Z}{\partial t} + \mathbf{P}(\nabla_X Z) + \mathbf{P}(\nabla_Z X) &= 0\end{aligned} \quad (65)$$

and if we use the first equation here to eliminate Z from the second equation, it is easy to check that we end up with equation (64).

5.6.2 Asymptotic growth of Jacobi fields

The second equation in (65) is fairly well-understood, at least for simple types of steady flows X. On the other hand, the full Jacobi equation does not appear to have been studied with the goal of finding explicit solutions. Thus simply solving the first equation in (65), given a solution Z of the second equation, already gives us a good deal of new information about the geometry of $\mathcal{D}_\mu$.

For a steady flow X, we can choose coordinates in some neighborhood such that $X = \frac{\partial}{\partial x^n}$. Then the first equation becomes the n equations

$$\frac{\partial Y^k}{\partial t} + \frac{\partial Y^k}{\partial x^n} = Z^k$$

for the components Y^k of Y. These are thus uncoupled first-order PDEs, and their solution can be written down explicitly. With the initial condition $Y(0) = 0$, we get

$$Y^k(t, x^1, \ldots, x^{n-1}, x^n) = \int_0^t Z^k(s, x^1, \ldots, x^{n-1}, x^n + s - t)\, ds$$

What this shows is that for steady flows, Lagrangian stability analysis (that is, of the solutions of the Jacobi equation) is essentially no more difficult than Eulerian stability analysis (that is, of the solutions of the linearized Euler equation). Misiołek [10] pointed out that the two forms of stability were generally distinct, in the sense that flows which are stable in the Eulerian sense could have everywhere nonpositive sectional curvature along them. He conjectured that such flows actually had exponential growth of Jacobi fields, so that they were exponentially unstable in the Lagrangian sense. However, the precise connection between the two has not been studied in detail.

We can obtain some rigorous information about the growth of Jacobi fields if X is a steady rotational flow on a two-dimensional annulus with rotationally symmetric metric. In this case, the metric is of the form $ds^2 = dr^2 + \varphi^2(r)\, d\theta^2$ and the volume element is of the form $\mu = \varphi(r)\, dr\, d\theta$. The fluid is described by the steady vector field $X = u(r)\, \partial_\theta$. We can then compute a bound on $||Y||$ in terms of $||Z||$, and if the norm of Z is bounded (i.e. if X is a stable fluid flow in the Eulerian sense) then Y cannot grow exponentially.

Theorem 5.4. *Suppose $X = u(r)\, \partial_\theta$ is a steady flow on a two-dimensional manifold M defined by the inequality $a \le r \le b$ with metric $ds^2 = dr^2 + \varphi^2(r)\, d\theta^2$. Let $A = \sup\limits_{a \le r \le b} |\varphi(r) u'(r)|$. Then if Y and Z are solutions of equation (65) with $Y(0) = 0$, then*

$$||Y(t)|| \le \int_0^t \sqrt{2\big(1 + A^2 (t-s)^2\big)}\, ||Z(s)||\, ds$$

Here $||U||$ denotes the L^2 norm of a vector field U. In particular, if the L^2 norm of Z is bounded, then the L^2 norm of Y grows at most quadratically in time.

Proof. Given a solution $Z = Z^1(t,r,\theta)\,\partial_r + Z^2(t,r,\theta)\,\partial_\theta$ of equation (62), we can write equation (63) in components as

$$\begin{aligned}\frac{\partial Y^1}{\partial t} + u(r)\frac{\partial Y^1}{\partial \theta} &= Z^1\\ \frac{\partial Y^2}{\partial t} + u(r)\frac{\partial Y^2}{\partial \theta} &= Z^2 + u'(r)Y^1\end{aligned}$$

The first of these equations is easily solved, with initial condition $Y^1(0)$:

$$Y^1(t,r,\theta) = \int_0^t Z^1\big(s,r,\theta + u(r)(s-t)\big)\,ds \tag{66}$$

The solution of the second equation can be written in terms of Y^1:

$$\begin{aligned}Y^2(t,r,\theta) = \int_0^t Z^2\big(s,r,\theta + u(r)(s-t)\big)\,ds\\ + u'(r)\int_0^t Y^1\big(s,r,\theta+u(r)(s-t)\big)\,ds\end{aligned} \tag{67}$$

Inserting equation (66) into equation (67) and simplifying, we obtain

$$\begin{aligned}Y^2(t,r,\theta) = \int_0^t Z^2\big(s,r,\theta + u(r)(s-t)\big)\,ds\\ + u'(r)\int_0^t (t-s)Z^1\big(s,r,\theta+u(r)(s-t)\big)\,ds\end{aligned} \tag{68}$$

Now we simply compute the norm of $Y(t)$ using these formulas:

$$\begin{aligned}&\int_M \langle Y(t), Y(t)\rangle\,\mu\\ &\quad= \int_M \int_0^t \int_0^t \Big[Z^1\Big(s,r,\theta+u(r)(s-t)\Big)\Big]\Big[Z^1\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)\Big]\\ &+\varphi^2(r)\Big[Z^2\Big(s,r,\theta+u(r)(s-t)\Big)+u'(r)(t-s)Z^1\Big(s,r,\theta+u(r)(s-t)\Big)\Big]\\ &\cdot\Big[Z^2\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)+u'(r)(t-\sigma)Z^1\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)\Big]\,d\sigma\,ds\,\mu\end{aligned}$$

First we interchange the order of integration.

$$\begin{aligned}&\int_M \langle Y(t), Y(t)\rangle\,\mu\\ &\quad= \int_0^t \int_0^t \int_M \Big[Z^1\Big(s,r,\theta+u(r)(s-t)\Big)\Big]\Big[Z^1\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)\Big]\\ &+\varphi^2(r)\Big[Z^2\Big(s,r,\theta+u(r)(s-t)\Big)+u'(r)(t-s)Z^1\Big(s,r,\theta+u(r)(s-t)\Big)\Big]\\ &\cdot\Big[Z^2\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)+u'(r)(t-\sigma)Z^1\Big(\sigma,r,\theta+u(r)(\sigma-t)\Big)\Big]\,\mu\,d\sigma\,ds\end{aligned}$$

Now we use the Schwarz inequality on the volume integral, so that we can separate the s and σ parts.

$$||Y(t)||^2 \leq \int_0^t \int_0^t \xi(\sigma)\xi(s)\, d\sigma\, ds \tag{69}$$

where

$$\xi(s)^2 = \int_M \Big[Z^1\Big(s, r, \theta + u(r)(s-t)\Big)\Big]^2$$
$$+ \varphi^2(r)\Big[Z^2\Big(s, r, \theta + u(r)(s-t)\Big) + u'(r)(t-s)Z^1\Big(s, r, \theta + u(r)(s-t)\Big)\Big]^2 \mu \tag{70}$$

The integrand in (69) is a function of s multiplied by a function of σ, and therefore the double integral is simply a product of two integrals, which are identical. Taking the square root of both sides, we have

$$||Y(t)|| \leq \int_0^t \xi(s)\, ds \tag{71}$$

Now we change variables in the expression $\xi(s)$ and let $\psi = \theta + u(r)(s-t)$. The limits remain from 0 to 2π, and the volume element is also unchanged, so we obtain

$$\xi(s)^2 = \int_M \big[Z^1(s, r, \psi)\big]^2 + \varphi^2(r)\big[Z^2(s, r, \psi) + u'(r)(t-s)Z^1(s, r, \psi)\big]^2 \mu \tag{72}$$

Now we want to bound the integrand by $\langle Z, Z\rangle$:

$$\begin{aligned}
\big[Z^1(s)\big]^2 + \varphi^2\big[Z^2(s) + u'(t-s)Z^1(s)\big]^2 & \\
&\leq [Z^1(s)]^2 + 2\varphi^2[Z^2(s)]^2 + 2\varphi^2 u'^2(t-s)^2[Z^1(s)]^2 \\
&\leq 2\big(1 + \varphi^2 u'^2(t-s)^2\big)\big[Z^1(s)^2 + \varphi^2 Z^2(s)^2\big] \\
&\leq 2\big(1 + A^2(t-s)^2\big)\big[Z^1(s)^2 + \varphi^2 Z^2(s)^2\big] \\
&= 2\big(1 + A^2(t-s)^2\big)\langle Z, Z\rangle
\end{aligned}$$

Inserting this into equation (72), we get

$$\xi(s)^2 \leq 2\big(1 + A^2(t-s)^2\big) \int_M \langle Z, Z\rangle\, \mu$$

and therefore from equation (71) we obtain

$$||Y(t)|| \leq \int_0^t \sqrt{2\big(1 + A^2(t-s)^2\big)}\, ||Z(s)||\, ds$$

as desired.

So if $||Z(s)|| \leq C$ for all s, then

$$\begin{aligned}
||Y(t)|| &\leq C \int_0^t \sqrt{2\big(1 + A^2(t-s)^2\big)}\, ds \\
&= \frac{C}{\sqrt{2}} \left(t\sqrt{1 + A^2t^2} + \frac{1}{A} \ln\big(At + \sqrt{1 + A^2t^2}\big) \right)
\end{aligned}$$

For large values of t, this is $O(t^2)$, as was to be shown. $\square$

If the velocity field is not stable in the Eulerian sense, then the typical situation is that we have some number λ with positive real part and some eigenfield ζ such that the solution of the linearized Euler equation (62) is

$$Z(t, r, \theta) = e^{\lambda t} \zeta(r, \theta)$$

Expanding this in a Fourier series

$$Z(t, r, \theta) = e^{\lambda t} \sum_{n=-\infty}^{\infty} \zeta_n(r) e^{in\theta}$$

and using equations (66) and (68) to construct the Jacobi field, we obtain

$$\begin{aligned}
Y(t, r, \theta) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda + inu(r)} \bigg[&(e^{\lambda t} - e^{-inu(r)t}) \Big(\zeta_n^1(r)\, \partial_r + \zeta_n^2(r)\, \partial_\theta \Big) \\
&+ u'(r) \left(\frac{1}{\lambda}(e^{\lambda t} - 1) + \frac{1}{inu(r)} (e^{-inu(r)t} - 1) \right) \partial_\theta \bigg] e^{in\theta}
\end{aligned}$$

So in particular, the Jacobi field also grows exponentially in time at each point, as we expect.

The quadratic growth estimate of the L^2 norm in Theorem 5.4 is, in general, the best possible. To see this, we construct the following example. Let M be the torus $\mathbb{T}^2$, with a metric given by $ds^2 = dr^2 + \varphi^2(r)\, d\theta^2$ with $\varphi(r)$ a periodic function of r (e.g. $\varphi(r) = c + d\cos r$ for $|c| > |d|$). Let

$$X = \frac{1}{\varphi^2(r)} \frac{\partial}{\partial \theta}$$

and

$$Z = \frac{1}{\varphi(r)} \frac{\partial}{\partial r}$$

Then $\operatorname{div} X = 0$ and $\operatorname{div} Z = 0$, and since

$$\nabla_X Z + \nabla_Z X = 0$$

Z is a steady solution of the linearized Euler equation (62).

By equations (66) and (68), the corresponding Jacobi field is

$$Y(t,r,\theta) = \frac{t}{\varphi(r)}\,\frac{\partial}{\partial r} - \frac{t^2\varphi'(r)}{\varphi^4(r)}\,\frac{\partial}{\partial\theta} \tag{73}$$

Thus the Jacobi field grows quadratically in time at each point. We conjecture that this happens because Z is harmonic. In the case where Y and Z can be expressed as skew gradients, growth seems to be typically linear in time. (See Theorem 5.5 in the next section.)

Regardless, the important point is that growth is polynomial in time: we have ruled out exponential growth of the Jacobi fields unless the perturbed velocity field also grows exponentially in time. Arnol'd and Khesin [2] discussed the relation between Eulerian stability and Lagrangian stability (Chapter IV, Section 4). There, they conjectured that negativity of the curvature corresponds to exponential divergence of nearby fluid paths, even if the fluids are stable in the Eulerian sense. We have seen that for rotational flows in two dimensions, this is generally not true.

Arnol'd and Khesin used the example of a sinusoidal flow on a torus, for which the pressure is constant and the curvature is nonpositive in all directions, to demonstrate the impossibility of long-term weather prediction. However, it is known that a sinusoidal velocity profile of the form $X = \sin y\,\partial_x$ on a torus $T^2 = [0,a]\times[0,2\pi]$ is actually stable in the Eulerian sense if $a \le 2\pi$. (See Arnol'd-Khesin [2], Chapter II, Section 4.) Therefore on a "short" torus, nonpositive curvature gives at worst quadratic growth of perturbations. In such a situation it is clearly much easier to predict the fluid motion than if the growth were exponential.

5.7 Explicit solutions of the Jacobi equation

In certain special cases, we can solve the linearized Euler equation

$$\frac{\partial Z}{\partial t} + \nabla_X Z + \nabla_Z X = -2\nabla p^{XZ} \tag{74}$$

explicitly. Here we will work in two dimensions only, for simplicity.

We have seen that we can express any divergence-free vector field Z in the form $Z = \text{sgrad}\,h + W$, where h is a function which vanishes on the boundary of M, and W is a harmonic vector field, with $\operatorname{div} W = 0$ and curl $W = 0$. Computing the curl of equation (74), and using formulas (40), we get

$$\frac{\partial}{\partial t}\Delta h + \text{curl}\,(\nabla_Z X) + \text{curl}\,(\nabla_X Z) = 0 \tag{75}$$

We can make this more explicit if we assume that X is a rotational flow on a rotationally symmetric manifold. Then the metric is $ds^2 = dr^2 + \varphi^2(r)d\theta^2$, and the vector field can be written as $X = u(r)\,\frac{\partial}{\partial\theta}$. Let us assume for simplicity that M is an annulus. Thus it has a one-parameter family of harmonic vector fields tangent to the boundary, spanned by $W = \frac{1}{\varphi^2} \frac{\partial}{\partial\theta}$. We then have an explicit formula for Z:

$$Z = -\frac{1}{\varphi(r)} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial r} + \frac{1}{\varphi(r)} \frac{\partial h}{\partial r} \frac{\partial}{\partial\theta} + \frac{c}{\varphi^2(r)} \frac{\partial}{\partial\theta}$$

We compute the covariant derivative in equation (75):

$$\nabla_Z X = -\frac{u'(r)}{\varphi(r)} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial\theta} - \frac{\varphi'(r)u(r)}{\varphi^2(r)} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial\theta} \\ - \varphi'(r)u(r) \frac{\partial h}{\partial r} \frac{\partial}{\partial r} - c\frac{\varphi'(r)u(r)}{\varphi(r)} \frac{\partial}{\partial r}$$

The curl of this is

$$\operatorname{curl}\,(\nabla_Z X) = -\frac{1}{\varphi(r)} \frac{d^2}{dr^2}\big(\varphi(r)u(r)\big) \frac{\partial h}{\partial\theta} - u'(r) \frac{\partial^2 h}{\partial\theta\,\partial r}$$

We also have

$$\nabla_X Z = u(r)\frac{\partial Z}{\partial\theta} - \frac{\varphi'(r)u(r)}{\varphi^2(r)} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial\theta} - \varphi'(r)u(r)\frac{\partial h}{\partial r} \frac{\partial}{\partial r} - c\frac{\varphi'(r)u(r)}{\varphi(r)} \frac{\partial}{\partial r}$$

so that

$$\operatorname{curl}\,(\nabla_X Z) = u'(r) \frac{\partial^2 h}{\partial\theta\,\partial r} + u(r) \frac{\partial}{\partial\theta}\Delta h - \frac{1}{\varphi(r)} \frac{d}{dr}\big(\varphi'(r)u(r)\big) \frac{\partial h}{\partial\theta}$$

Putting these expressions in equation (75) and simplifying, we obtain

$$\frac{\partial}{\partial t}\Delta h + u(r)\frac{\partial}{\partial\theta}\Delta h - \frac{1}{\varphi(r)} \frac{d}{dr} \left[\frac{1}{\varphi(r)} \frac{d}{dr}\big(\varphi^2(r)u(r)\big) \right] \frac{\partial h}{\partial\theta} = 0 \tag{76}$$

This tells us how the skew-gradient part of Z evolves in time.

To find out how the harmonic part evolves, we compute the inner product of equation (74) with the harmonic field $W = \frac{1}{\varphi^2(r)} \frac{\partial}{\partial\theta}$. We get $\frac{dc}{dt} = 0$, since every other term vanishes (we obtain integrals of functions of the form $\frac{\partial J}{\partial\theta}$, which vanish since all of our functions are periodic). So $c(t) = c(0)$, and the harmonic part of Z always remains constant.

Now we demonstrate how to solve equation (63). If $Y = \operatorname{sgrad} g + \frac{b}{\varphi^2(r)} \frac{\partial}{\partial\theta}$ then we have

$$\begin{aligned}[X, Y] &= -\frac{u(r)}{\varphi(r)} \frac{\partial^2 g}{\partial\theta^2} \frac{\partial}{\partial r} + \frac{u(r)}{\varphi(r)} \frac{\partial^2 g}{\partial r\,\partial\theta} \frac{\partial}{\partial\theta} + \frac{u'(r)}{\varphi(r)} \frac{\partial g}{\partial\theta} \frac{\partial}{\partial\theta} \\ &= \operatorname{sgrad}\, \left(u(r)\frac{\partial g}{\partial\theta}\right)\end{aligned}$$

So we get the equation

$$\operatorname{sgrad}\, \left(\frac{\partial g}{\partial t} + u(r)\frac{\partial g}{\partial\theta}\right) = \operatorname{sgrad} h$$

or more simply,

$$\frac{\partial g}{\partial t} + u(r)\frac{\partial g}{\partial \theta} = h$$

So, if $Y(0) = 0$, the skew-gradient part is generally found from the formula

$$g(t, r, \theta) = \int_0^t h\big(s, r, \theta - u(r)(t - s)\big)\, ds \tag{77}$$

We obtain the equation for the harmonic part of Y by computing the inner product of equation (63) with the harmonic field $\frac{1}{\varphi^2}\,\partial_\theta$. We obtain the equation $\frac{db}{dt} = c(t) = c(0)$. Thus the harmonic part of Y always grows linearly: $b(t) = c(0)t$.

As for the skew-gradient part of Y, we can derive the following analogue of Theorem 5.4. It describes the growth of the H^{-1} norm of the Jacobi field, whereas Theorem 5.4 describes the growth of the L^2 norm.

Theorem 5.5. *If g is given by formula (77), then the L^2 norm of g (that is, the H^{-1} norm of Y) is bounded in terms of the L^2 norm of h (that is, the H^{-1} norm of Z) through*

$$\sqrt{\int_M g^2(t, r, \theta)\,\mu} \le \int_0^t \sqrt{\int_M h^2(s, r, \theta)\,\mu}\, ds \tag{78}$$

In particular, if the H^{-1} norm of Z is bounded, then the H^{-1} norm of Y grows at most linearly.

Proof. The proof uses the same technique as Theorem 5.4, but it is much simpler. Using equation (77), we have

$$\begin{aligned}
&\int_M g^2(t, r, \theta)\,\mu \\
&\quad= \int_0^t \int_0^t \int_M h\big(s, r, \theta - u(r)(t-s)\big)\, h\big(\sigma, r, \theta - u(r)(t-\sigma)\big)\,\mu\, ds\, d\sigma \\
&\quad\le \int_0^t \int_0^t \sqrt{\int_M h^2\big(s, r, \theta - u(r)(t-s)\big)\,\mu} \;\cdot \\
&\qquad\qquad\qquad \cdot \sqrt{\int_M h^2\big(\sigma, r, \theta - u(r)(t-\sigma)\big)\,\mu}\; ds\, d\sigma \\
&\quad\le \left[\int_0^t \sqrt{\int_M h^2\big(s, r, \theta - u(r)(t-s)\big)\,\mu}\; ds\right]^2 \\
&\quad\le \left[\int_0^t \sqrt{\int_M h^2\big(s, r, \psi\big)\,\mu}\; ds\right]^2
\end{aligned}$$

So obviously if

$$\sqrt{\int_M h^2\big(t,r,\psi\big)\,\mu} \le C$$

for all t, then

$$\sqrt{\int_M g^2(t,r,\theta)\,\mu} \le Ct$$

for all t. □

We can solve equation (76) explicitly in two cases: when X is a Killing field ($u(r) \equiv 1$), and when X is a Couette field ($u(r) = \frac{C}{\varphi^2(r)} \int \varphi(r)\,dr$). In these solutions, we find that the Jacobi field typically grows linearly, in the H^{-1} norm, the L^2 norm, and the L^∞ norm.

5.7.1 Killing field flow

We first consider the case where X is a Killing field. Then $u(r)$ is constant, and we can assume $u(r) = 1$. Then equation (76) simplifies to

$$\frac{\partial}{\partial t}\Delta h + \frac{\partial}{\partial \theta}\Delta h - 2\frac{\varphi''(r)}{\varphi(r)}\frac{\partial h}{\partial \theta} = 0$$

We can compute that for a rotationally invariant metric, the Gaussian curvature is given by $\kappa = -\frac{\varphi''(r)}{\varphi(r)}$. Thus this equation will be simplest when the curvature κ is a constant. So we have

$$\frac{\partial}{\partial t}\Delta h + \frac{\partial}{\partial \theta}\Delta h + 2\kappa\frac{\partial h}{\partial \theta} = 0 \tag{79}$$

The eigenfunctions of the Laplacian are of the form $\phi_{k,n}(r,\theta) = \psi_{k,n}(r)e^{in\theta}$, with $\Delta\phi_{k,n} = -\lambda_{k,n}\phi_{k,n}$. Suppose the curvature κ is constant. If we write

$$h(t,r,\theta) = \sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty} h_{k,n}(t)\phi_{k,n}(r,\theta)$$

then equation (79) decomposes into

$$-\lambda_{k,n}\frac{dh_{k,n}}{dt} - in\lambda_{k,n}h_{k,n} + 2in\kappa h_{k,n} = 0$$

whose solution is

$$h_{k,n}(t) = h_{k,n}(0)e^{in\left(\frac{2\kappa}{\lambda_{k,n}}-1\right)t}$$

So this tells us the skew-gradient part of Z for all time.

Now, to solve equation (63), we expand g in a series of eigenfunctions of the Laplacian,

$$g(t,r,\theta) = \sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty} g_{k,n}(t)\phi_{k,n}(r,\theta)$$

Therefore, using formula (77), the explicit solution with $g_{k,n}(0)=0$ is

$$g_{k,n}(t) = \begin{cases} \dfrac{\lambda_{k,n} h_{k,n}(0)}{2i\kappa n}\left(e^{in\left(\frac{2\kappa}{\lambda_{k,n}}-1\right)t} - e^{-int}\right) & \kappa \neq 0 \text{ and } n \neq 0 \\ h_{k,n}(0) t e^{-int} & \kappa = 0 \text{ or } n = 0 \end{cases}$$

So there is always at least one component that grows linearly (the one corresponding to $n=0$, that is, the vector fields that point in the same direction as X).

The squared L^2 norm of Y is

$$\int_M \langle Y, Y\rangle\, \mu = c(0)^2 t^2 \int_M \frac{1}{\varphi^2}\,\mu + t^2 \sum_{k=1}^{\infty} \lambda_{k,0} |h_{k,0}(0)|^2 \\ + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \lambda_{k,n} \left(1 - \cos \tfrac{2\kappa n}{\lambda_{k,n}} t\right) \left(\frac{\lambda_{k,n}}{\kappa n}\right)^2 |h_{k,n}(0)|^2$$

if $\kappa \neq 0$, and

$$\int_M \langle Y, Y\rangle\, \mu = t^2 \left(c(0)^2 \int_M \frac{1}{\varphi^2}\,\mu + \sum_{k=1}^{\infty} \lambda_{k,0} |h_{k,0}(0)|^2 + 2 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \lambda_{k,n} |h_{k,n}(0)|^2 \right)$$

if $\kappa = 0$. In particular, if the curvature is nonzero, then the norm of Y grows linearly with time unless $c(0) = 0$ and $h_{k,0} = 0$ for all k. If the curvature is zero, then the norm of every solution grows linearly. We know already that Y could grow at most linearly, using Theorem 5.4 with $A = 0$. But it is interesting that on spaces of positive or negative curvature, there are bounded solutions of the Jacobi equation along a rotation, while on flat space there are none.

The reason is that on a flat two-dimensional space, where $\varphi(r) = r$, the curvature along a rotation is actually zero in all directions. This is because $\nabla_Y X$ is always a gradient, so $\mathbf{P}(\nabla_Y X) = 0$ and thus all terms in formula (54) vanish. Thus the fact that every solution of the Jacobi equation grows linearly is expected there. It is interesting that in fact the curvature is greater along a rotation in hyperbolic space ($\kappa = -1$) than in flat space. This is the intuitive reason that some Jacobi fields in hyperbolic space remain bounded, while none in flat space do.

The important conclusion to draw from these computations is that even in the best possible case, when the curvature is positive in nearly all directions, the solutions to the linearized geodesic equations are still typically not bounded in time. Thus if one defines linear stability of a geodesic to mean boundedness of all solutions of the Jacobi equation (as Misiołek [10] did), then one could never actually expect any geodesic in $\mathcal{D}_\mu$ to be linearly stable.

On the other hand, even if all geodesics are unstable in this sense, one can still distinguish between differing rates of growth. In Arnol'd's model of the weather, small-scale weather is considered a time-dependent perturbation of a large-scale rotational steady flow, namely the tradewind currents around the earth. Clearly one could still predict the weather to a good precision if one knew that perturbations grew linearly in time, although predictions would be effectively impossible if errors grew exponentially in time. However, as we will see in the next section, flows with negative curvature in almost every direction can have the same order of growth as flows with positive curvature in almost every direction. Thus curvature computations in $\mathcal{D}_\mu$ must be considered of very limited usefulness when studying the stability of steady fluid flows, at least among rotational flows.

5.7.2 Couette flow

The equation (76) has two terms involving the Laplacian of h and the third term which is obviously of a very different nature. Couette flow can be considered as the special case in which the third term vanishes. The formula for Couette flow is thus found by solving the equation

$$\frac{1}{\varphi(r)}\frac{d}{dr}\left[\frac{1}{\varphi(r)}\frac{d}{dr}\big(\varphi^2(r)u(r)\big)\right] = 0$$

The solution is easy to write down:

$$u(r) = \frac{C}{\varphi^2(r)}\int \varphi(r)\,dr$$

The general formula involves two arbitrary constants: C is one, and the other is obtained from the indefinite integral. For example, if $\varphi(r) = 1$ (corresponding to a flat cylinder), then plane parallel Couette flow is $u(r) = Ar+B$. If $\varphi(r) = r$, then Couette flow is $u(r) = \frac{A}{r^2} + B$ (this is only defined if M is an annulus). We suppose for this section that M is the annulus $0 < a \le r \le b$.

For Couette flow, the linearized Euler equation is

$$\frac{\partial}{\partial t}\Delta h + u(r)\frac{\partial}{\partial \theta}\Delta h = 0$$

The explicit general solution to this equation was first given by Orr [13] for the case $\varphi(r) = 1$. We repeat his method; the only new addition is to integrate this solution to find Y.

First, we can immediately write the solution as

$$\Delta h(t,r,\theta) = F(r,\theta - u(r)t)$$

for some function F. Using the fact that $h\big|_{\partial M} = 0$, we can solve this Dirichlet problem for h in terms of F. We know that $F(r,\theta) = \Delta h(0,r,\theta)$. Let us expand h in a Fourier series in θ: then

$$h(t,r,\theta) = \sum_{n=-\infty}^{\infty} h_n(t,r)e^{in\theta}$$

Letting Δ_n denote the operator

$$\Delta_n f(r) = \frac{1}{\varphi(r)} \frac{d}{dr} \left(\varphi(r) \frac{df}{dr} \right) - \frac{n^2 f(r)}{\varphi^2(r)}$$

we have

$$\Delta h(t, r, \theta) = \sum_{n=-\infty}^{\infty} [\Delta_n h_n(t, r)] e^{in\theta}$$

Since

$$F(r, \theta - u(r)t) = \sum_{n=-\infty}^{\infty} F_n(r) e^{-inu(r)t} e^{in\theta}$$

we match up the Fourier coefficients to obtain

$$\Delta_n h_n(t, r) = e^{-inu(r)t} F_n(r) \tag{80}$$

with Dirichlet boundary conditions for $h_n(t, r)$. Since the Laplacian on M has a unique inverse, each of the operators Δ_n has a unique inverse. So we know we can always find a unique solution of equation (80). Explicitly, we can use the Green's function $G_n(r, \rho)$, given by

$$G_n(r, \rho) = \frac{1}{n \sinh\big(n\xi(b)\big)} \begin{cases} \sinh\Big(n\xi(\rho)\Big) \sinh\Big(\big[\xi(b) - \xi(r)\big]\Big) & \rho \le r \\ \sinh\Big(n\xi(r)\Big) \sinh\Big(\big[\xi(b) - \xi(\rho)\big]\Big) & \rho \ge r \end{cases} \tag{81}$$

so that the solution of the Dirichlet problem (80) is

$$h_n(t, r) = -\int_a^b G_n(r, \rho) e^{-inu(\rho)t} F_n(\rho) \varphi(\rho)\, d\rho$$

Using equation (77), we can get a useful formula for g_n:

$$\begin{aligned} g_n(t, r) &= \int_0^t e^{-inu(r)(t-s)} h_n(s, r)\, ds \\ &= -\int_0^t \int_a^b e^{-inu(r)(t-s)} G_n(r, \rho) e^{-inu(\rho)s} F_n(\rho) \varphi(\rho)\, d\rho\, ds \\ &= -e^{-inu(r)t} \int_a^b G_n(r, \rho) F_n(\rho) \varphi(\rho) \int_0^t e^{in[u(r)-u(\rho)]s}\, ds\, d\rho \\ &= -e^{-inu(r)t} \int_a^b G_n(r, \rho) F_n(\rho) \frac{e^{in[u(r)-u(\rho)]t} - 1}{in[u(r) - u(\rho)]} \varphi(\rho)\, d\rho \end{aligned}$$

so that finally we have

$$g_n(t, r) = \frac{1}{in} \int_a^b F_n(\rho) G_n(r, \rho) \frac{e^{-inu(r)t} - e^{-inu(\rho)t}}{u(r) - u(\rho)} \varphi(\rho)\, d\rho \tag{82}$$

If $n = 0$, this formula is not valid, and instead we have

$$g_0(t,r) = -t \int_a^b G_0(r,\rho) F_0(\rho) \varphi(\rho)\, d\rho$$

Although the explicit computation gets a bit messy, we remind ourselves that

$$Y = -\frac{1}{\varphi(r)} \frac{\partial g}{\partial \theta} \frac{\partial}{\partial r} + \frac{1}{\varphi(r)} \frac{\partial g}{\partial r} \frac{\partial}{\partial \theta} + \frac{c(0)t}{\varphi^2(r)} \frac{\partial}{\partial \theta}$$

We know already that the harmonic part, if any, grows linearly in time. Looking at the formula (82), we see that dependence on time is only through terms like $e^{in(\theta - u(r)t)}$. Differentiating with respect to θ will produce time-dependencies like $e^{in(\theta-u(r)t)}$, which is pointwise bounded in time. Thus the r-component of Y remains bounded in time. On the other hand, differentiating this term with respect to r produces time-dependencies like $u'(r) t e^{in(\theta - u(r)t)}$, which increases linearly with time at each point.

Thus we can say in general that for Couette flow, the Jacobi field Y always increases at most linearly with time at each point, and thus in the L^2 norm as well. This is in fact slower growth than the general quadratic estimate one has from Theorem 5.4, and leads one to expect that in fact linear growth may be the most typical case for the solutions of the Jacobi equation, when the solution of the linearized Euler equation is bounded.

In general the pattern seems to be that one can only obtain exponential growth in solutions of the Jacobi equation if one already obtains exponential growth in the solutions of the linearized Euler equation. Thus Eulerian stability analysis is equivalent to Lagrangian stability analysis for steady flows, at least in the special cases we have discussed. As we have found, even in the best cases one cannot expect that Lagrangian perturbations are bounded in time (i.e., stable in the sense of Lyapunov), unlike Eulerian perturbations. Therefore we should instead distinguish between polynomial and exponential rates of growth as our stability criterion for the Lagrangian motion of a fluid.

6 Stability of the motion of a compressible fluid

Once we interpret the Lagrangian motion of incompressible fluids as geodesics on the group of volume-preserving diffeomorphisms, it is natural to take a similar approach to studying the Lagrangian motion of compressible fluids. For barotropic fluids (that is, compressible fluids for which the internal forces depend only on the density of the fluid), the appropriate picture is of a Newtonian system on the group of all diffeomorphisms. As we have seen, it is easier to do computations on the group of all diffeomorphisms than on the group of volume-preserving diffeomorphisms, and thus in some sense it is easier to study the Lagrangian motion of compressible flows than of incompressible flows.

Much of the modern approach to the study of Lagrangian motion of compressible flow was initiated in the late 1970s. N. Smolentsev [16] used a potential energy function and a Maupertuis principle to describe the equations of compressible fluid mechanics as a geodesic equation on the group of diffeomorphisms. Here the metric is not the usual kinetic energy metric but the Jacobi metric, which is conformally equivalent to it. Geodesics of this metric correspond to solutions of Newton's equations after a reparametrization. The geometry of the Jacobi metric is fairly complicated; the curvature tensor, for example, is difficult to compute explicitly. It therefore seems preferable to simply study the Newton equation directly rather than to force the strict geometric analogy. Hence we will avoid the Maupertuis approach.

D. Ebin [6] derived the equations of barotropic compressible flow from a Lagrangian minimization principle. He showed that when the potential energy is multiplied by a sufficiently large parameter, the solution of the barotropic equations is close to the solution of the incompressible equations, in the C^1 sense. Thus positions and velocities of particles will be close in the two models; accelerations in general will not be close, however. (The same thing happens in finite-dimensional Newtonian systems; see [6] for an example.) Since the theory of Lagrangian linear stability involves the deviation of acceleration, we would not necessarily expect stability properties of compressible flow to be similar to those of incompressible flow.

In what follows we first present Ebin's derivation of the Euler equations of barotropic flow. Then we compute the Hessian of the potential energy function, which appears in the linearized Newton equation and plays the same role as the Riemann curvature tensor does for incompressible fluid mechanics. Next we show how the linearized equations for unsteady one-dimensional flow may be solved explicitly for a special case, and we compute a stability result for this case. Finally we study two-dimensional steady flows, and present explicit solutions of the linearized equations about steady plane parallel and rigid rotational flow.

6.1 Derivation of the Euler equation

Once again it will be convenient to keep in mind the distinction between the domain and range of the diffeomorphisms; that is, between the home of the particles and the physical space. We remind the reader that for any diffeomorphism $\eta\colon N \to M$, the density $\rho\colon M \to \mathbb{R}$ is defined by the formula

$$\rho\,\mu = (\eta^{-1})^*\nu$$

where μ is the volume form on M defined by the Riemannian metric and ν is the volume form on N which represents the masses of the particles. The Jacobian $J(\eta)\colon N \to \mathbb{R}$ of η is defined by the formula

$$J(\eta)\,\nu = \eta^*\mu$$

and we have the general formulas

$$J(\eta) = \frac{1}{\rho\circ\eta} \qquad \text{and} \qquad \rho = \frac{1}{J(\eta)\circ\eta^{-1}}$$

If $F\colon \mathbb{R} \to \mathbb{R}$ is any function of one variable, we define a potential energy function $\mathbf{Q}\colon \mathcal{D}(N, M) \to \mathbb{R}$ by the formula

$$\mathbf{Q}(\eta) = \int_N F\big(J(\eta)\big)\,\nu$$

The assumption that the potential energy depends only on the Jacobian of η is characteristic of fluid mechanics. By allowing a more general form of the potential energy, we could also incorporate many other situations, such as elastic motion of solids and fluids with surface tension. The computations we will perform would be essentially the same; we set up an integral on N and use the Lie derivative on M to determine formulas for the gradient and the Hessian.

We recall that the dust metric on $\mathcal{D}(N, M)$ is given by

$$\langle\langle \mathbf{U}, \mathbf{V}\rangle\rangle_\eta = \int_N \langle U \circ \eta, V \circ \eta\rangle\,\nu = \int_M \rho\,\langle U, V\rangle\,\mu$$

where $\mathbf{U}$ and $\mathbf{V}$ are tangent vectors in $T_\eta\mathcal{D}(N, M)$ which are given by $\mathbf{U} = U \circ \eta$ and $\mathbf{V} = V \circ \eta$ for some vector fields U and V on M.

The Lagrangian of barotropic fluid mechanics is then

$$L(\eta, \dot\eta) = \int_N \Big(\tfrac{1}{2}\langle\dot\eta, \dot\eta\rangle - F(J(\eta))\Big)\,\nu$$

and Newton's equation on $\mathcal{D}(N, M)$ is

$$\frac{\mathbf{D}}{dt}\frac{d\eta}{dt} = -\nabla\mathbf{Q}_{\eta(t)} \tag{83}$$

Proposition 6.1. *The gradient of* $\mathbf{Q}$ *is given by the formula*

$$\nabla\mathbf{Q}_\eta = -\frac{1}{\rho}\,\nabla F'\left(\frac{1}{\rho}\right) \tag{84}$$

Proof. Let X be a vector field on M, and ϕ_t be its flow on M. If M has a boundary, then we assume X is tangent to the boundary of M. Let $\mathbf{X}$ denote the vector field defined by $\mathbf{X}_\eta = X \circ \eta$. Let $\Phi_t\colon \mathcal{D}(N, M) \to \mathcal{D}(N, M)$ denote the flow of $\mathbf{X}$, which satisfies

$$\Phi_t(\eta) = \phi_t \circ \eta$$

Then

$$\langle\langle \nabla\mathbf{Q}_\eta, \mathbf{X}_\eta\rangle\rangle = \frac{d}{dt}\Big|_{t=0}\mathbf{Q}(\Phi_t(\eta))$$

So we have

$$
\begin{aligned}
\langle\langle \nabla \mathbf{Q}_\eta, \mathbf{X}_\eta \rangle\rangle &= \frac{d}{dt}\Big|_{t=0} \int_N F\big(J(\Phi_t(\eta))\big)\,\nu \\
&= \int_N F'\big(J(\Phi_0(\eta))\big)\ \frac{d}{dt}\Big|_{t=0} (J(\phi_t \circ \eta)\,\nu \\
&= \int_N F'\big(J(\eta)\big)\ \frac{d}{dt}\Big|_{t=0} \Big[\big(\phi_t \circ \eta\big)^* \mu\Big] \\
&= \int_N F'\big(J(\eta)\big)\ \eta^* \left[\frac{d}{dt}\Big|_{t=0} \phi_t^* \mu\right] \\
&= \int_N F'\big(J(\eta)\big)\ \eta^*[\mathcal{L}_X \mu] \\
&= \int_N F'\big(J(\eta)\big)\ \eta^*[\operatorname{div} X \mu] \\
&= \int_N F'\big(J(\eta)\big)\ \big(\operatorname{div} X \circ \eta\big)\ \eta^* \mu \\
&= \int_M F'\big(J(\eta)\big) \circ \eta^{-1}\ \operatorname{div} X\ \mu \\
&= \int_M F'\big(\tfrac{1}{\rho}\big) \operatorname{div} X\ \mu \\
&= \int_M \operatorname{div} \Big(F'\big(\tfrac{1}{\rho}\big) X\Big)\ \mu - \int_M \Big\langle X, \nabla F'\big(\tfrac{1}{\rho}\big)\Big\rangle\ \mu \\
&= -\int_M \rho \Big\langle X, \tfrac{1}{\rho} \nabla F'\big(\tfrac{1}{\rho}\big)\Big\rangle\ \mu
\end{aligned}
$$

The desired formula follows. □

Using this formula in equation (83), we obtain the Euler equation for barotropic compressible flow:

$$\frac{\mathbf{D}X}{dt} = \frac{1}{\rho} \nabla F' \left(\frac{1}{\rho}\right)$$

Using equation (24) and defining the function p by the formula $p(x) = -F'\big(\frac{1}{x}\big)$, we can write this function in the more usual form

$$\frac{\partial X}{\partial t} + \nabla_X X = -\frac{1}{\rho}\,\nabla p(\rho)$$

Defining a function h by the formula $h'(x) = \frac{1}{x}\,p'(x)$, we can also write this equation in the form

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla h(\rho) \tag{85}$$

For our purposes, this will be the most convenient form.

The function F must still be specified in order to have a complete system. Typically one specifies the pressure function p instead; in terms of p, F is given by the formula

$$F(x) = \int_1^{1/x} \frac{p(s)}{s^2}\, ds$$

For example, if the fluid is polytropic, i.e. if p is given by $p(x) = Ax^\gamma$, then

$$F(x) = \frac{A}{\gamma - 1}(x^{1-\gamma} - 1)$$

and the potential energy is given by

$$\mathbf{Q}(\eta) = \frac{A}{\gamma - 1}\left(\int_M \rho^\gamma\, \mu - \int_M \rho\, \mu\right)$$

In this case, the function h is given by

$$h(x) = \frac{A\gamma}{\gamma - 1} x^{\gamma - 1}$$

If we compute the time derivative of the equation

$$\eta_t^*(\rho\mu) = \nu$$

we obtain

$$\eta_t^*\left(\frac{\partial \rho}{\partial t}\mu\right) + \eta_t^*\Big(\mathcal{L}_X(\rho\mu)\Big) = 0$$

$$\eta_t^*\left(\frac{\partial \rho}{\partial t}\mu + X(\rho)\mu + \rho \operatorname{div} X \mu\right) = 0$$

Composing with $(\eta_t^{-1})^*$, we see that this equation implies

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho X) = 0 \tag{86}$$

This equation is called the equation of continuity and is generally considered one of the fundamental equations along with (85). In our approach, it is automatically satisfied and can be considered a consequence of the equation

$$\frac{d\eta}{dt} = X(t) \circ \eta(t) \tag{87}$$

6.2 Linearization of the Euler equation

In general, the procedure for linearizing the Newton equation on a Riemannian manifold is to construct a vector field Y along the curve. We assume that $Y = \frac{\partial}{\partial s}$ for some coordinate s, and that the curve parameter t can also be used as a coordinate in a neighborhood of the curve γ. Then since coordinate vector fields commute, we have

$$\frac{D}{\partial s}\frac{d\gamma}{dt} = \frac{DY}{\partial t}$$

If we compute the derivative of the Newton equation $\frac{D}{dt}\frac{d\gamma}{dt} = -\nabla U \circ \gamma(t)$ with respect to s, we then obtain

$$\frac{D}{\partial s}\frac{D}{\partial t}\frac{d\gamma}{dt} = -\frac{D}{\partial s}\nabla U \circ \gamma$$

Using the curvature operator to interchange the order of differentiation, and the fact that $\frac{D}{\partial s}V = \nabla_{\partial_s} V$ if V is the restriction of a vector field, we have

$$\frac{D^2 Y}{dt^2} + R(Y, \dot\gamma)\dot\gamma + \nabla_Y \nabla U \circ \gamma = 0 \tag{88}$$

Equation (88) is a generalization of the usual Jacobi equation on a Riemannian manifold, and we might call it the potential-Jacobi equation. One could expect that the role of the expression

$$\frac{\langle R(Y, \dot\gamma)\dot\gamma, Y\rangle + \langle \nabla_Y \nabla U, Y\rangle}{\langle Y, Y\rangle} \tag{89}$$

is analogous to the role of the sectional curvature in Riemannian geometry. Note that the Hessian of U is positive-definite iff U is convex, so on a flat manifold the expression (89) is positive iff the potential energy is convex. Since convexity of potential energy implies stability of stationary solutions, we might expect in general that it is also related to stability of nonstationary solutions.

To study the linearized Euler equation on the diffeomorphism group, we must first compute the Hessian of the barotropic potential $\mathbf{Q}$. To do this, we use the general formula

$$\langle \nabla_Y \nabla U, X\rangle = \langle \nabla_X \nabla U, Y\rangle = X\big(Y(U)\big) - \nabla_X Y(U)$$

and a technique similar to the proof of proposition 6.1 above.

Proposition 6.2. *At a vector $Y \circ \eta \in T_\eta \mathcal{D}(N, M)$, the Hessian of the barotropic potential $\mathbf{Q}$ is given by the formula*

$$\nabla_{Y\circ\eta} \nabla \mathbf{Q} = \big[-\nabla\big(h'(\rho)\, div\,(\rho Y)\big) + \nabla_Y \nabla h(\rho)\big] \circ \eta \tag{90}$$

Proof. Let X and Y be vector fields on M, tangent to the boundary of M. Let $\mathbf{X}$ and $\mathbf{Y}$ denote the right-invariant vector fields on $\mathcal{D}(N, M)$ defined by $\mathbf{X}_\eta = X \circ \eta$ and $\mathbf{Y}_\eta = Y \circ \eta$. Let ϕ_t denote the flow of X on M, and let Φ_t denote the flow of $\mathbf{X}$ on $\mathcal{D}(N, M)$, satisfying $\Phi_t(\eta) = \phi_t \circ \eta$. We have

$$\langle\langle \nabla_{\mathbf{Y}} \nabla \mathbf{Q}, \mathbf{X}\rangle\rangle = \langle\langle \nabla_{\mathbf{X}} \nabla \mathbf{Q}, \mathbf{Y}\rangle\rangle = \mathbf{X}\big(\mathbf{Y}(\mathbf{Q})\big) - \nabla_{\mathbf{X}}\mathbf{Y}(\mathbf{Q})$$

Since $(\nabla_{\mathbf{X}}\mathbf{Y})_\eta = \nabla_X Y \circ \eta$ by formula (23) for right-invariant vector fields, we already know what the second term is from proposition 6.1:

$$\nabla_{\mathbf{X}}\mathbf{Y}(\mathbf{Q}) = \int_M \rho\, \langle \nabla_X Y, \nabla h(\rho)\rangle\, \mu$$

We can rewrite this expression in terms of the Hessian of $h(\rho)$ as

$$\begin{aligned}
\nabla_{\mathbf{X}}\mathbf{Y}(\mathbf{Q}) &= \int_M \rho\,\langle \nabla_X Y, \nabla h(\rho)\rangle\,\mu \\
&= \int_M \rho\, X\big(Y(h(\rho))\big)\,\mu - \int_M \rho\,\langle Y, \nabla_X \nabla h(\rho)\rangle\,\mu \\
&= \int_M \rho\, X\big(h'(\rho) Y(\rho)\big)\,\mu - \int_M \rho\,\langle X, \nabla_Y \nabla h(\rho)\rangle\,\mu
\end{aligned}$$

Now we have to compute the term $\mathbf{XY}(\mathbf{Q})$. As we computed in proposition 6.1, we have

$$\mathbf{Y}_\eta(\mathbf{Q}) = \int_N F'\big(J(\eta)\big)\eta^*[\mathcal{L}_Y\mu]$$

Thus applying $\mathbf{X}$ to this function, we obtain

$$\begin{aligned}
\mathbf{X}_\eta\big(\mathbf{Y}(\mathbf{Q})\big) &= \frac{d}{dt}\Big|_{t=0} \mathbf{Y}_{\phi_t\circ\eta}(\mathbf{Q}) \\
&= \frac{d}{dt}\Big|_{t=0} \int_N F'\big(J(\phi_t\circ\eta)\big)\ (\phi_t\circ\eta)^*[\mathcal{L}_Y\mu] \\
&= \int_N \frac{d}{dt}\Big|_{t=0} F'\big(J(\phi_t\circ\eta)\big)\ \eta^*[\mathcal{L}_Y\mu] \\
&\qquad\qquad + \int_N F'\big(J(\eta)\big)\ \eta^*\left[\frac{d}{dt}\Big|_{t=0}\phi_t^*(\mathcal{L}_Y\mu)\right] \\
&= \int_N F''\big(J(\eta)\big)\ \frac{d}{dt}\Big|_{t=0} J(\phi_t\circ\eta)\ \eta^*[\mathcal{L}_Y\mu] \\
&\qquad\qquad + \int_N F'\big(J(\eta)\big)\ \eta^*[\mathcal{L}_X(\mathcal{L}_Y\mu)] \\
&= \int_N F''\big(J(\eta)\big)\ \operatorname{div} Y\circ\eta\ J(\eta)\ \frac{d}{dt}\Big|_{t=0} J(\phi_t\circ\eta)\,\nu \\
&\qquad\qquad + \int_N F'\big(J(\eta)\big)\ \eta^*\left[\big(X(\operatorname{div} Y) + \operatorname{div} X \operatorname{div} Y\big)\,\mu\right] \\
&= \int_N F''\big(J(\eta)\big)\ \operatorname{div} Y\circ\eta\ J(\eta)^2\ \operatorname{div} X\circ\eta\,\nu \\
&\qquad\qquad + \int_N F'\big(J(\eta)\big)\ \big(X(\operatorname{div} Y) + \operatorname{div} X \operatorname{div} Y\big)\circ\eta\ J(\eta)\,\nu \\
&= \int_M \left(\tfrac{1}{\rho}F''\big(\tfrac{1}{\rho}\big)\ \operatorname{div} Y \operatorname{div} X + F'\big(\tfrac{1}{\rho}\big)\ \big(X(\operatorname{div} Y) + \operatorname{div} X \operatorname{div} Y\big)\right)\mu
\end{aligned}$$

We can write $F'\big(\frac{1}{\rho}\big) = -p(\rho)$ and

$$\frac{1}{\rho}F''\left(\frac{1}{\rho}\right) + F'\left(\frac{1}{\rho}\right) = \rho p'(\rho) - p(\rho)$$

and obtain

$$\mathbf{X}_\eta\big(\mathbf{Y}(\mathbf{Q})\big) = \int_M \big[\rho p'(\rho) - p(\rho)\big]\operatorname{div} Y \operatorname{div} X\,\mu - \int_M p(\rho) X(\operatorname{div} Y)\,\mu$$

Then we integrate by parts to obtain

$$
\begin{aligned}
\mathbf{X}_\eta\big(\mathbf{Y}(\mathbf{Q})\big) &= -\int_M X\Big(\big[\rho p'(\rho) - p(\rho)\big]\operatorname{div} Y\Big)\,\mu - \int_M p(\rho) X(\operatorname{div} Y)\,\mu \\
&= -\int_M X\big(\rho p'(\rho) - p(\rho)\big)\operatorname{div} Y\,\mu - \int_M \rho p'(\rho) X(\operatorname{div} Y)\,\mu \\
&= -\int_M \rho p''(\rho) X(\rho)\operatorname{div} Y\,\mu - \int_M \rho p'(\rho) X(\operatorname{div} Y)\,\mu \\
&= -\int_M \rho\, X(\rho h'(\rho)\operatorname{div} Y)\,\mu
\end{aligned}
$$

Finally we combine these formulas to obtain

$$
\begin{aligned}
\langle\langle X\circ\eta, \nabla_{Y\circ\eta}\nabla\mathbf{Q}\rangle\rangle &= \mathbf{X}\big(\mathbf{Y}(\mathbf{Q})\big) - \nabla_{\mathbf{X}}\mathbf{Y}(\mathbf{Q}) \\
&= -\int_M \rho\, X\big(\rho h'(\rho)\operatorname{div} Y\big)\,\mu \\
&\qquad - \int_M \rho\, X\big(h'(\rho) Y(\rho)\big)\,\mu + \int_M \rho\,\langle X, \nabla_Y \nabla h(\rho)\rangle\,\mu \\
&= -\int_M \rho\, X\big(h'(\rho)\operatorname{div}(\rho Y)\big)\,\mu + \int_M \rho\,\langle X, \nabla_Y\nabla h(\rho)\rangle\,\mu \\
&= \int_M \rho\,\langle X, -\nabla\big(h'(\rho)\operatorname{div}(\rho Y)\big) + \nabla_Y\nabla h(\rho)\rangle\,\mu \\
&= \big\langle\big\langle X\circ\eta, \big[-\nabla\big(h'(\rho)\operatorname{div}(\rho Y)\big) + \nabla_Y\nabla h(\rho)\big]\circ\eta\big\rangle\big\rangle
\end{aligned}
$$

Since X was arbitrary, formula (90) follows. $\square$

The Hessian of the barotropic potential on $\mathcal{D}(N, M)$ is therefore quite simple; it is merely the Hessian of the function $h(\rho)$ on M, added to an nonnegative operator that looks like a Laplacian on the divergence-free vector fields. The following is easy to prove, using a simplified version of the technique in Rouchon's Theorem 5.2. The assumption on the pressure function is the standard one in compressible fluid mechanics.

Proposition 6.3. *Suppose the pressure function $\rho \mapsto p(\rho)$ is nondecreasing and the dimension of the manifold is $n \geq 2$. Then the Hessian of the barotropic potential $\mathbf{Q}$ is nonnegative iff the Hessian of the function $h(\rho)$ is nonnegative at every point.*

Proof. It is easy to check that the term $-\nabla\big(h'(\rho)\operatorname{div}(\rho Y)\big)$ is nonnegative; we have

$$
\begin{aligned}
\int_M \rho \, \langle -\nabla\big(h'(\rho)\operatorname{div}(\rho Y)\big), Y\rangle \, \mu \\
&- \int_M \rho \, Y\big(h'(\rho)\operatorname{div}(\rho Y)\big) \, \mu \\
&= -\int_M \operatorname{div}\Big(h'(\rho)\operatorname{div}(\rho Y)\,\rho Y\Big)\,\mu + \int_M h'(\rho)\,\big(\operatorname{div}(\rho Y)\big)^2\,\mu \\
&= \int_M \frac{1}{\rho}\,p'(\rho)\,\big(\operatorname{div}(\rho Y)\big)^2\,\mu
\end{aligned}
$$

and this is nonnegative since $p'(\rho) \geq 0$ for any function ρ. So if $\langle \nabla_Y \nabla h(\rho), Y\rangle$ is nonnegative for any $p \in M$ and any $Y \in T_pM$, then clearly

$$\langle\langle \nabla_{Y\circ\eta} \nabla \mathbf{Q}, Y \circ \eta\rangle\rangle \geq 0$$

for any vector field Y on M.

Now for the converse, we suppose that there is some point p and some vector $V \in T_pM$ such that $\langle \nabla_V \nabla h(\rho), V\rangle < 0$. As in the proof of Rouchon's Theorem 5.2, we construct a divergence-free vector field which is concentrated near p and is close to V, in coordinates.

So we again construct coordinates $x_1, x_2, \ldots, x_n$ in a neighborhood Ω of p, such that $\frac{\partial}{\partial x_1}\Big|_p = V$. Let ϵ be a small positive number and let $\psi\colon [0,\infty) \to [0,\infty)$ be a C^∞ function positive on $[0,1)$ and zero elsewhere. Let $\xi\colon M \to [0,\infty)$ be defined in coordinates by

$$\xi(x_1, x_2, x_3, \ldots, x_n) = \psi\left(\left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x_3}{\epsilon}\right)^2 + \cdots + \left(\frac{x_n}{\epsilon}\right)^2\right)$$

on Ω and 0 elsewhere on M.

Again, let Ω_0 be the inverse image of $(0,\infty)$ under ξ, so that $x \in \Omega_0$ implies $x_1 < \epsilon, x_2 < \epsilon^2, x_3 < \epsilon, \ldots, x_n < \epsilon$. Define Y to be

$$Y = \frac{1}{\rho\sqrt{\det g}}\,\epsilon^2\,(-\partial_2\xi\,\partial_1 + \partial_1\xi\,\partial_2)$$

Then $\operatorname{div}(\rho Y) = 0$, and Y is zero outside Ω_0. As before, we can write, to first order in ϵ,

$$Y = -\frac{2}{\rho}\,\psi'(\rho^2)\,\frac{x_2}{\epsilon^2}\,\partial_1 + O(\epsilon)$$

with $\rho^2 = \left(\frac{x_1}{\epsilon}\right)^2 + \left(\frac{x_2}{\epsilon^2}\right)^2 + \left(\frac{x^3}{\epsilon}\right)^2 + \cdots + \left(\frac{x^n}{\epsilon}\right)^2$

Inside Ω_0, we have

$$\langle \nabla_Y \nabla h(\rho), Y\rangle = \frac{4}{\rho^2}\,\psi'(\rho^2)^2\,\frac{x_2^2}{\epsilon^4}\,\langle \nabla_V \nabla h(\rho), V\rangle + O(\epsilon)$$

and therefore

$$\int_M \rho\langle \nabla_Y \nabla h(\rho), Y\rangle\, \mu = \int_{\Omega_0} \left[\frac{4}{\rho}\, \psi'(\rho^2)^2\, \frac{x_2^2}{\epsilon^4}\, \langle \nabla_V \nabla h(\rho), V\rangle + O(\epsilon)\right] \mu$$
$$= \langle \nabla_V \nabla h(\rho), V\rangle \int_{\Omega_0} \frac{4}{\rho}\, \psi'(\rho^2)^2\, \frac{x_2^2}{\epsilon^4}\, \mu + O(\epsilon^{n+2})$$

Since the quantity $\int_{\Omega_0} \frac{4}{\rho}\psi'(\rho^2)^2 \frac{x_2^2}{\epsilon^4}\, \mu$ is $o(\epsilon^{n+1})$ and $\langle \nabla_V \nabla h(\rho), V\rangle < 0$, it follows that

$$\int_M \rho\langle \nabla_Y \nabla h(\rho), Y\rangle\, \mu < 0$$

for sufficiently small ϵ. Since the term $\nabla(h'(\rho)\mathrm{div}\,(\rho Y))$ vanishes by construction, we know

$$\langle\langle \nabla_{Y\circ\eta} \nabla \mathbf{Q}, Y \circ \eta\rangle\rangle < 0$$

for sufficiently small ϵ as well. ☐

We may also be interested in the quantity that appears in equation (88),

$$\nabla_Y \nabla \mathbf{Q} + \mathbf{R}(Y, X)X$$

for some given vector field X. Once again we have an easily-verified criterion for nonnegativity of this expression.

Corollary 6.4. *Suppose the pressure function $\rho \mapsto p(\rho)$ is nondecreasing and the dimension of the manifold is $n \geq 2$. If X is a vector field on M and $\mathbf{X}$ the corresponding right-invariant vector field on $\mathcal{D}(N, M)$, then the operator*

$$\mathbf{Y} \mapsto \nabla_{\mathbf{Y}} \nabla \mathbf{Q} + \mathbf{R}(\mathbf{Y}, \mathbf{X})\mathbf{X}$$

is nonnegative iff the operator $Y \mapsto \nabla_Y \nabla h(\rho) + R(Y, X)X$ is nonnegative at each point.

Proof. The proof is exactly the same as the proof of Proposition 6.3 above, since $\mathbf{R}_\eta(\mathbf{Y}, \mathbf{X})\mathbf{X} = R(Y, X)X \circ \eta$. ☐

The construction above of divergence-free vector fields, which coincide with a given vector at a given point and vanish outside a neighborhood of that point, depends essentially on having two dimensions in order to use the skew-gradient construction. In one space dimension, things change drastically.

Proposition 6.5. *If $n = 1$ and the pressure function $\rho \mapsto p(\rho)$ is nondecreasing, then the Hessian of the barotropic potential energy $\mathbf{Q}$ is always nonnegative.*

Proof. The proof is based on a simple computation. Let us use θ as a coordinate on M, with the standard flat metric $\langle \partial_\theta, \partial_\theta\rangle = 1$. Write $Y = y(\theta)\, \partial_\theta$. Then we have

$$
\begin{aligned}
\nabla_Y \nabla \mathbf{Q} &= -\nabla\big(h'(\rho)\,\mathrm{div}\,(\rho Y)\big) + \nabla_Y \nabla h(\rho) \\
&= -\frac{d}{d\theta}\left(h'\big(\rho(\theta)\big)\frac{d}{d\theta}\big(\rho(\theta)y(\theta)\big)\right) + y(\theta)\frac{d^2}{d\theta^2}h\big(\rho(\theta)\big) \\
&= -h''(\rho)\frac{d\rho}{d\theta}\,\frac{d}{d\theta}(\rho y) - h'(\rho)\frac{d^2}{d\theta^2}(\rho y) + yh''(\rho)\frac{d\rho}{d\theta}\,\frac{d\rho}{d\theta} + yh'(\rho)\frac{d^2\rho}{d\theta^2} \\
&= -h''(\rho)\rho\,\frac{d\rho}{d\theta}\,\frac{dy}{d\theta} - 2h'(\rho)\,\frac{d\rho}{d\theta}\,\frac{dy}{d\theta} - h'(\rho)\rho\frac{d^2 y}{d\theta^2} \\
&= -\frac{dy}{d\theta}\,\frac{1}{\rho}\,\frac{d}{d\theta}\big(\rho^2 h'(\rho)\big) - \rho h'(\rho)\frac{d^2 y}{d\theta^2} \\
&= -\frac{dy}{d\theta}\,\frac{1}{\rho}\,\frac{d}{d\theta}\big(\rho p'(\rho)\big) - p'(\rho)\frac{d^2 y}{d\theta^2} \\
&= -\frac{1}{\rho}\frac{d}{d\theta}\left(\rho p'(\rho)\frac{dy}{d\theta}\right)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\langle\langle \nabla_Y \nabla \mathbf{Q}, Y\rangle\rangle &= -\int_M y(\theta)\frac{d}{d\theta}\left(\rho p'(\rho)\frac{dy}{d\theta}\right)\,d\theta \\
&= \int_M \rho p'(\rho)\left(\frac{dy}{d\theta}\right)^2\,d\theta \\
&\geq 0
\end{aligned}
$$

for any choice of functions y and ρ, as long as $p'(\rho) \geq 0$. $\square$

We have seen, of course, that the Jacobi equation for incompressible fluid flows may be split into two decoupled first-order equations. The natural analogue for compressible flows yields a similar kind of splitting, although the equations are no longer decoupled. We find that the potential-Jacobi equation (88) is equivalent to the two equations

$$
\begin{aligned}
\frac{\partial Y}{\partial t} + [X, Y] &= Z \\
\frac{\partial Z}{\partial t} + \nabla_Z X + \nabla_X Z &= \nabla\big(h'(\rho)\mathrm{div}\,(\rho Y)\big)
\end{aligned}
\tag{91}
$$

If we take the alternative approach of directly linearizing the Euler equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho X) &= 0 \\
\frac{\partial X}{\partial t} + \nabla_X X &= -\nabla h(\rho)
\end{aligned}
\tag{92}
$$

using the variations $Z = \frac{\partial X}{\partial s}\big|_{s=0}$ and $\sigma = \frac{\partial \rho}{\partial s}\big|_{s=0}$, we obtain the linearized Euler equations

$$
\begin{aligned}
\frac{\partial \sigma}{\partial t} + \operatorname{div}(\rho Z) + \operatorname{div}(\sigma X) = 0 \\
\frac{\partial Z}{\partial t} + \nabla_Z X + \nabla_X Z = -\nabla\big(h'(\rho)\sigma\big)
\end{aligned}
\tag{93}
$$

The two sets of equations are reconciled by the fact that

$$
\sigma = -\operatorname{div}(\rho Y) \tag{94}
$$

Because the linearized Lagrangian equations (91) are equivalent to the linearized Eulerian equations (93), under the transformation (94), we do not expect compressible flow to exhibit substantially different stability phenomena between the Eulerian and Lagrangian approach. The only difference that can occur is if the divergence-free part of Y grows in time.

We can prove an analogue of Theorem 5.4 in the compressible case. Suppose we have a rotational steady velocity field $X = u(r)\,\partial_\theta$ on a manifold with metric $ds^2 = dr^2 + \varphi^2(r)\,d\theta^2$. Then the density function is determined by the condition

$$
h\big(\rho(r)\big) = \int u^2(r)\varphi(r)\varphi'(r)\,dr
$$

up to some arbitrary constant. We can prove the following.

Theorem 6.6. *Let M be a two-dimensional manifold defined by the condition $a \le r \le b$. Suppose $X = u(r)\,\partial_\theta$, and the metric on M takes the form $ds^2 = dr^2 + \varphi^2(r)\,d\theta^2$. For a given vector field Z, let Y be the solution solution of the equation*

$$
\frac{\partial Y}{\partial t} + [X, Y] = Z
$$

with $Y(0) = 0$. Let $A = \sup\limits_{a \le r \le b} |\varphi(r)u'(r)|$. Then

$$
||Y(t)|| \le \int_0^t \sqrt{2\big(1 + A^2(t-s)^2\big)}\,||Z(s)||\,ds
$$

In particular, if $||Z(t)||$ is bounded, then Y grows at most quadratically.

Proof. The proof of this theorem is exactly the same as that of Theorem 5.4; the only difference is that the norm now involves an integral over volume element $\rho\,\mu$ rather than simply μ. But since ρ is a function of r alone, this does not change anything about the proof. □

Just as in the example after Theorem 5.4, we can find an example of a steady compressible flow for which Jacobi fields grow quadratically. Again, let

$$
X = \frac{1}{\varphi^2(r)} \frac{\partial}{\partial \theta}
$$

on the torus $\mathbb{T}^2$. In order for X to be a steady flow, we must have $\frac{d}{dr}h(\rho) = \frac{\varphi'(r)}{\varphi^3(r)}$ so that

$$h(\rho) = A - \frac{1}{2\varphi^2(r)}$$

Then the pair

$$Z = \frac{1}{\varphi(r)}\,\frac{\partial}{\partial r}, \qquad \sigma \equiv 0$$

is a solution to the linearized Euler equations (93). The explicit solution for Y is exactly the same as (73), so we see that potential-Jacobi fields can also grow quadratically in time.

There is one final thing to say in general, before studying specific examples. Suppose we ignore the equation of continuity, and the fact that the density function ρ evolves based on the velocities of the fluid particles. Let us instead assume that $\rho(t)$ is some given time-dependent function on M. Then the particles in a barotropic fluid move according to the law

$$\frac{D}{dt}\,\frac{d\eta}{dt} = -\nabla h\big(\rho(t)\big) \circ \eta \tag{95}$$

That is, they move as though they were individual Newtonian particles subject to the time-dependent potential energy $U = h(\rho)$.

If an individual particle in this surrogate Newtonian system (95) were perturbed, the perturbation $Y(t)$ would satisfy the potential-Jacobi equation

$$\frac{D^2Y}{dt^2} + R\big(Y(t), \dot\eta(t)\big)\dot\eta(t) + \nabla_{Y(t)}\nabla h\big(\rho(t)\big) = 0 \tag{96}$$

rather than the actual barotropic potential-Jacobi equation (88). We note that the "effective curvature" in the barotropic potential-Jacobi equation is identical to that of the surrogate system, except that it exceeds the latter by the term

$$-\nabla\big(h'(\rho)\mathrm{div}\,(\rho Y)\big)$$

which is a nonnegative operator of similar form to the wave operator.

So we intuitively expect that, in going from the surrogate Newtonian system (95) to the actual barotropic fluid, we have gained greater stability, since the effective curvature is larger. The problem with this line of thinking is that increasing curvature only reduces the size of Jacobi fields up to the first conjugate point; because of the wave operator, a Newtonian curve in $\mathcal{D}(N, M)$ with the barotropic potential would be expected to have infinitely many conjugate points, densely distributed along the curve. So standard techniques of comparison theory could not be hoped to apply here. The relationship, if any, between the stability properties of barotropic flow and the surrogate Newtonian flow remains unclear, although we will compare the two in the explicit examples we have below.

6.3 An explicit solution in one dimension

Unlike incompressible flow, which is trivial in one dimension (the only volume-preserving flows on the circle are isometries), compressible flow already exhibits many interesting features in one dimension. In very special cases, we can obtain explicit solutions of the Euler equations for barotropic flow, and therefore also for the linearized equations as well. We will illustrate with the simplest example, the case of a polytropic fluid where $\gamma = 3$.

We assume that N and M are diffeomorphic to the circle S^1, and let θ be the coordinate on M. If the velocity field is written as $X = x(\theta)\,\partial_\theta$, then the Euler equations (92) become

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho x) &= 0\\ \frac{\partial x}{\partial t} + x\,\frac{\partial x}{\partial \theta} &= -h'(\rho)\,\frac{\partial \rho}{\partial \theta}\end{aligned} \tag{97}$$

In the polytropic case, where $p(\rho) = A\rho^\gamma$, explicit solutions can be obtained in the special cases where

$$\gamma = \frac{2k+1}{2k-1}$$

for some positive integer k. The exact solution is written down implicitly in Courant-Friedrichs [5], Chapter III, Section 38. The simplest case is $k = 1$, corresponding to $\gamma = 3$. We may rescale ρ so as to assume that $A = \frac{1}{3}$, without changing the Euler equations (since the equation of continuity is linear in ρ). Thus in this case $h'(\rho) = \rho$, and the Euler equations can be written as

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial x}{\partial \theta} + x\,\frac{\partial \rho}{\partial \theta} &= 0\\ \frac{\partial x}{\partial t} + x\,\frac{\partial x}{\partial \theta} + \rho\,\frac{\partial \rho}{\partial \theta} &= 0\end{aligned}$$

If we add and subtract these equations, we obtain the two equations

$$\begin{aligned}\frac{\partial}{\partial t}(x+\rho) + (x+\rho)\,\frac{\partial}{\partial \theta}(x+\rho) &= 0\\ \frac{\partial}{\partial t}(x-\rho) + (x-\rho)\,\frac{\partial}{\partial \theta}(x-\rho) &= 0\end{aligned}$$

These equations are now decoupled, and the solution of each is given implicitly, in terms of the initial data $x_0(\theta) = x(0,\theta)$ and $\rho_0(\theta) = \rho(0,\theta)$, by the formulas

$$\begin{aligned}x(t,\theta) + \rho(t,\theta) &= x_0\Big(\theta - t\big[x(t,\theta) + \rho(t,\theta)\big]\Big) + \rho_0\Big(\theta - t\big[x(t,\theta) + \rho(t,\theta)\big]\Big)\\ x(t,\theta) - \rho(t,\theta) &= x_0\Big(\theta - t\big[x(t,\theta) - \rho(t,\theta)\big]\Big) - \rho_0\Big(\theta - t\big[x(t,\theta) - \rho(t,\theta)\big]\Big)\end{aligned} \tag{98}$$

In general, this is the best we can do; the solution for other values of γ quickly becomes much more complicated. It should also be pointed out that these equations generally can only be solved locally near $t = 0$. Singularities (shocks) are a characteristic feature of such equations when these solutions break down, and in fact are a subject of intense current research. We, however, will only be interested in the (possibly very short) interval of time before the first shock occurs, when the solutions remain smooth and classical.

Although we can write down and solve the linearized equations (93) in this case, it is easier to simply use the exact solution (98). Suppose we have a family of such solutions which depends smoothly on a parameter s. We differentiate equations (98) and set $s = 0$, using

$$z \equiv \frac{\partial x}{\partial s}\Big|_{s=0} \text{ and } \sigma \equiv \frac{\partial \rho}{\partial s}\Big|_{s=0}$$

We obtain the formulas

$$\begin{aligned} z + \sigma &= \frac{z_0\big(\theta - t[x+\rho]\big) + \sigma_0\big(\theta - t[x+\rho]\big)}{1 + t\Big({x_0}'\big(\theta - t[x+\rho]\big) + {\rho_0}'\big(\theta - t[x+\rho]\big)\Big)} \\ z - \sigma &= \frac{z_0\big(\theta - t[x-\rho]\big) - \sigma_0\big(\theta - t[x-\rho]\big)}{1 + t\Big({x_0}'\big(\theta - t[x-\rho]\big) - {\rho_0}'\big(\theta - t[x-\rho]\big)\Big)} \end{aligned} \tag{99}$$

This can be simplified. If we differentiate the first of equations (98) with respect to θ, we obtain

$$\frac{\partial}{\partial \theta}(x+\rho) = \Big({x_0}'\big(\theta - t[x+\rho]\big) + {\rho_0}'\big(\theta - t[x+\rho]\big)\Big)\left(1 - t\frac{\partial}{\partial \theta}(x+\rho)\right)$$

from which we find

$$\frac{1}{1 + t\Big({x_0}'\big(\theta - t[x+\rho]\big) + {\rho_0}'\big(\theta - t[x+\rho]\big)\Big)} = 1 - t\frac{\partial}{\partial \theta}(x+\rho)$$

The same trick can be performed on the second equation in (98). Inserting these expressions in (99), we get

$$\begin{aligned} z + \sigma &= \big(1 - t\tfrac{\partial}{\partial \theta}(x+\rho)\big)\Big(z_0\big(\theta - t[x+\rho]\big) + \sigma_0\big(\theta - t[x+\rho]\big)\Big) \\ z - \sigma &= \big(1 - t\tfrac{\partial}{\partial \theta}(x-\rho)\big)\Big(z_0\big(\theta - t[x-\rho]\big) - \sigma_0\big(\theta - t[x-\rho]\big)\Big) \end{aligned}$$

Then we can finally solve for z and σ to find

$$\begin{aligned} z = {}& \tfrac{1}{2}\big(1 - t\tfrac{\partial}{\partial \theta}(x+\rho)\big)\Big(z_0\big(\theta - t[x+\rho]\big) + \sigma_0\big(\theta - t[x+\rho]\big)\Big) \\ &+ \tfrac{1}{2}\big(1 - t\tfrac{\partial}{\partial \theta}(x-\rho)\big)\Big(z_0\big(\theta - t[x-\rho]\big) - \sigma_0\big(\theta - t[x-\rho]\big)\Big) \\ \sigma = {}& \tfrac{1}{2}\big(1 - t\tfrac{\partial}{\partial \theta}(x+\rho)\big)\Big(z_0\big(\theta - t[x+\rho]\big) + \sigma_0\big(\theta - t[x+\rho]\big)\Big) \\ &- \tfrac{1}{2}\big(1 - t\tfrac{\partial}{\partial \theta}(x-\rho)\big)\Big(z_0\big(\theta - t[x-\rho]\big) - \sigma_0\big(\theta - t[x-\rho]\big)\Big) \end{aligned} \tag{100}$$

We are interested in the Jacobi field $Y(t) = y(t, \theta)\, \partial_\theta$, defined by the equation (94), which on S^1 is

$$\frac{\partial}{\partial \theta}\big(\rho(t,\theta) y(t,\theta)\big) = -\sigma(t,\theta)$$

If we are mainly interested in the case where $Y(0) = 0$, as we have been so far, then we can assume that $\sigma_0 \equiv 0$, and we obtain the very simple formula

$$y(t,\theta) = \frac{1}{2\rho} \int_{\theta - t(x+\rho)}^{\theta - t(x-\rho)} z_0(\psi)\, d\psi \tag{101}$$

We therefore easily derive the following pointwise bound on Y in terms of the supremum of $||Z(0)||$. It depends on the condition that $\int_0^{2\pi} z_0(\theta)\, d\theta = 0$, which is equivalent to the requirement that the initial average velocity $\int_0^{2\pi} x_0(\theta)\, d\theta$ is known exactly. Since the average velocity $\int_0^{2\pi} x(t,\theta)\, d\theta$ is conserved by the Euler equations, this is not too drastic a requirement. If this requirement were not posed, Jacobi fields could grow linearly (for example, if $z_0(\theta) = 1$ then $y(t,\theta) = t$).

Proposition 6.7. *If $Y(0) = 0$ and $\int_0^{2\pi} z_0(\theta)\, d\theta = 0$, then for any t and θ,*

$$|y(t,\theta)| \leq \frac{1}{2\rho(t,\theta)} \int_0^{2\pi} |z_0(\psi)|\, d\psi$$

Proof. Since $\int_0^{2\pi} z_0(\theta)\, d\theta = 0$, we can write any integral of the form $\int_u^v z_0(\theta)\, d\theta$ as

$$\int_u^v z_0(\theta)\, d\theta = \int_{u \bmod 2\pi}^{v \bmod 2\pi} z_0(\theta)\, d\theta$$

Therefore we can say

$$\left|\int_u^v z_0(\theta)\, d\theta\right| = \left|\int_{u \bmod 2\pi}^{v \bmod 2\pi} z_0(\theta)\, d\theta\right| = \left|\int_{\theta_1}^{\theta_2} z_0(\theta)\, d\theta\right|$$

where $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$. Thus

$$\left|\int_u^v z_0(\theta)\, d\theta\right| \leq \int_{\theta_1}^{\theta_2} |z_0(\theta)|\, d\theta \leq \int_0^{2\pi} |z_0(\theta)|\, d\theta$$

Therefore

$$|y(t,\theta)| = \frac{1}{2\rho(t,\theta)} \left|\int_{\theta - t(x+\rho)}^{\theta - t(x-\rho)} z_0(\psi)\, d\psi\right| \leq \frac{1}{2\rho(t,\theta)} \int_0^{2\pi} |z_0(\psi)|\, d\psi$$

$\square$

This result gives us pointwise-boundedness of the Jacobi fields in terms of the L^1 norm of the initial perturbation. The drawback is of course that we don't necessarily have a bound on the function $1/\rho$. However, if the density actually approaches zero somewhere, then the flow has become singular, and the Lagrangian solution is no longer a diffeomorphism. So while it may continue to exist in a generalized sense, it is out of the realm of our approach. So we can say that as long as a solution exists in the diffeomorphism group, the minimum of ρ is bounded away from zero and thus the maximum of y is bounded. Given the fact that singularities are generic features of nonsteady flows, we cannot expect more than this.

We note that on the circle, the operator $Y \mapsto \nabla_Y \nabla \mathbf{Q}$ is nonnegative, by Proposition 6.5. $\nabla_Y \nabla \mathbf{Q}$ vanishes if and only if $Y = c\,\partial_\theta$ for some constant c; with the exception of this one-parameter family, $Y \mapsto \nabla_Y \nabla \mathbf{Q}$ is a strictly positive operator. We also note that except for a one-parameter family of solutions of the Jacobi equation, solutions to the Jacobi equation are bounded in the L^∞ norm, for as long as the solution exists in the diffeomorphism group. It seems plausible to guess that these phenomena are related: that positivity of the Hessian of the potential is related to the boundedness of Jacobi fields.

6.4 Explicit solutions in two dimensions

In higher dimensions, we are less interested in the stability of general fluid flows and more interested in the stability of steady flows, that is, those that satisfy the steady Euler equations

$$\begin{aligned} \nabla_X X &= -\nabla h(\rho) \\ \operatorname{div}(\rho X) &= 0 \end{aligned}$$

Because of the term $\nabla\big(h'(\rho)\operatorname{div}(\rho Y)\big)$ that appears in the Jacobi equation (91), it is very difficult to obtain solutions to it except in the simplest cases. We present two simple cases here: plane parallel flow at constant velocity, and rotational flow at constant angular velocity.

6.4.1 Plane parallel flow

Plane parallel flow occurs when $M = \mathbb{T}^2$, the flat torus. We suppose the steady flow is given by $\rho \equiv 1$ and $X = u\,\partial_x$, where u is constant. Since ρ is constant, we can write $p'(\rho) = c^2$ (c being the speed of sound).

Proposition 6.8. *On the torus $\mathbb{T}^2$, suppose $X = u\,\partial_x$ and $\rho \equiv 1$. If the initial data Z_0 is smooth and $\sigma_0 \equiv 0$, then the solutions Z and σ of the linearized Euler equations (93) are pointwise bounded for all time. Thus the flow is stable in the Eulerian sense.*

Proof. Letting $Z = f\,\partial_x + g\,\partial_y$, the linearized equations (93) take the form

$$\begin{aligned}\frac{\partial f}{\partial t} + u\,\frac{\partial f}{\partial x} + c^2\,\frac{\partial \sigma}{\partial x} &= 0\\ \frac{\partial g}{\partial t} + u\,\frac{\partial g}{\partial x} + c^2\,\frac{\partial \sigma}{\partial y} &= 0\\ \frac{\partial \sigma}{\partial t} + u\,\frac{\partial \sigma}{\partial x} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} &= 0\end{aligned}$$

We transform these equations into something simpler by making the coordinate change

$$\tau = t, \qquad \alpha = x - ut, \qquad \beta = y$$

We obtain the new equations

$$\begin{aligned}\frac{\partial f}{\partial \tau} + c^2\,\frac{\partial \sigma}{\partial \alpha} &= 0\\ \frac{\partial g}{\partial \tau} + c^2\,\frac{\partial \sigma}{\partial \beta} &= 0\\ \frac{\partial \sigma}{\partial \tau} + \frac{\partial f}{\partial \alpha} + \frac{\partial g}{\partial \beta} &= 0\end{aligned} \tag{102}$$

(The reason u has to be constant to obtain an explicit solution is that otherwise, the coordinate transformation would result in unwieldy equations.)

We readily find that σ satisfies the wave equation

$$\frac{\partial^2 \sigma}{\partial \tau^2} = c^2\,\left(\frac{\partial^2 \sigma}{\partial \alpha^2} + \frac{\partial^2 \sigma}{\partial \beta^2}\right)$$

Assuming that $\sigma(0) = 0$ (i.e. that the Jacobi field Y vanishes at $t = 0$), we obtain the general solution

$$\sigma(\tau, \alpha, \beta) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk} \sin\left(c\sqrt{j^2 + k^2}\tau\right) e^{ij\alpha} e^{ik\alpha}$$

for some constants a_{jk}. From this, we can integrate the first two of equations (102). Using the initial conditions

$$\begin{aligned}f(0, \alpha, \beta) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_{jk} e^{ij\alpha} e^{ik\beta}\\ g(0, \alpha, \beta) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_{jk} e^{ij\alpha} e^{ik\beta}\end{aligned}$$

we can obtain the general solution

$$
\begin{aligned}
f(\tau,\alpha,\beta) &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\Big[\frac{j}{j^2+k^2}\left(jf_{jk}+kg_{jk}\right)\cos\Big(c\sqrt{j^2+k^2}\tau\Big) \\
&\qquad\qquad + \frac{k}{j^2+k^2}\left(kf_{jk}-jg_{jk}\right)\Big]e^{ij\alpha}e^{ik\beta} \\
g(\tau,\alpha,\beta) &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\Big[\frac{k}{j^2+k^2}\left(jf_{jk}+kg_{jk}\right)\cos\Big(c\sqrt{j^2+k^2}\tau\Big) \\
&\qquad\qquad - \frac{j}{j^2+k^2}\left(kf_{jk}-jg_{jk}\right)\Big]e^{ij\alpha}e^{ik\beta}
\end{aligned}
\tag{103}
$$

Using $\tau = t$, $\alpha = x - ut$, and $\beta = y$, we can readily put these expressions in terms of the original variables. We see that the time-dependence of the coefficients is oscillatory. Since the initial data is smooth, these series converge absolutely, and thus Z^1, Z^2, and σ are all pointwise bounded in time.

□

However, the Jacobi fields generally grow linearly in time. Once again, we have a system which is stable in the Eulerian sense, but unstable in the Lagrangian sense (but not exponentially).

Proposition 6.9. *Under the circumstances above, all Jacobi fields with* $Y(0) = 0$ *grow pointwise linearly in time, unless curl* $(Z_0) = 0$*.*

Proof. To obtain the Jacobi field from equation (103), we use the first of equations (91), which in these coordinates is

$$\frac{\partial Y}{\partial \tau} = Z$$

Thus we have

$$
\begin{aligned}
Y(\tau,\alpha,\beta) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\Bigg[\bigg(&\frac{j}{c(j^2+k^2)^{3/2}}\left(jf_{jk}+kg_{jk}\right)\sin\Big(c\sqrt{j^2+k^2}\tau\Big) \\
&+ \frac{k}{j^2+k^2}\left(kf_{jk}-jg_{jk}\right)\tau\bigg)\partial_\alpha \\
+\bigg(&\frac{k}{c(j^2+k^2)^{3/2}}\left(jf_{jk}+kg_{jk}\right)\sin\Big(c\sqrt{j^2+k^2}\tau\Big) \\
&- \frac{j}{j^2+k^2}\left(kf_{jk}-jg_{jk}\right)\tau\bigg)\partial_\beta\Bigg]e^{ij\alpha}e^{ik\beta}
\end{aligned}
\tag{104}
$$

Recalling that $\tau = t$, equation (104) tells us that the Jacobi field grows linearly with time at each point of $\mathbb{T}^2$. The only exception is the case where $kf_{jk} = jg_{jk}$ for all j and k, which is equivalent to the condition that

$$\frac{\partial f_0}{\partial y} = \frac{\partial g_0}{\partial x}$$

or, in other words, curl $(Z_0) = 0$.

$\square$

By way of comparison, the Jacobi equation for the surrogate Newtonian system (95) is given in these coordinates by

$$\begin{aligned} \frac{\partial f}{\partial \tau} &= 0 \\ \frac{\partial g}{\partial \tau} &= 0 \\ \frac{\partial \sigma}{\partial \tau} + \frac{\partial f}{\partial \alpha} + \frac{\partial g}{\partial \beta} &= 0 \end{aligned}$$

The corresponding Jacobi field is

$$Y_{\mathrm{surr}}(\tau, \alpha, \beta) = \tau \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Big(f_{jk}\, \partial_\alpha + g_{jk}\, \partial_\beta \Big) e^{ij\alpha} e^{ik\beta}$$

which always grows linearly.

6.4.2 Rigid rotational flow

A more geometrically complicated flow arises from the motion of fluid in a disc, where the steady flow is a rigid rotation. We work with the polytropic model, where $p(\rho) = A\rho^\gamma$. We assume $\gamma = 2$, so that $h(\rho) = c^2\rho$, c being the (constant) speed of sound. This model arises as the simplest nonlinear approximation of the equations of shallow water waves on a surface, where the vertical motion is assumed to be negligible. See Courant and Friedrichs [5], Chapter I Appendix (Section 19) for a full derivation.

We consider, then, the motion of shallow water waves in the disc $D^2 = \{(r, \theta) \,\big|\, r \le 1\}$, equipped with the flat metric. We assume the steady velocity field is given by $X = \omega\, \partial_\theta$ for some constant angular velocity ω. Then the density is determined by the formula

$$\omega^2 \nabla_{\partial_\theta} \partial_\theta = -c^2 \nabla \rho$$

which can be solved to yield

$$\rho(r) = 1 + \frac{\omega^2 r^2}{2c^2} \tag{105}$$

under the assumption $\rho(0) = 1$.

Proposition 6.10. *If $X = \omega\, \partial_\theta$ with ρ given by (105) in the shallow water model, then the components of the field Z and the function σ remain pointwise bounded for all time. Thus, this steady flow is stable in the Eulerian sense.*

Proof. In polar coordinates, the linearized Euler equations (93) become

$$\begin{aligned} \frac{\partial Z^1}{\partial t} + \omega\,\frac{\partial Z^1}{\partial \theta} - 2\omega r Z^2 + c^2\,\frac{\partial \sigma}{\partial r} &= 0 \\ \frac{\partial Z^2}{\partial t} + \omega\,\frac{\partial Z^2}{\partial \theta} + \frac{2\omega}{r}\,Z^1 + \frac{c^2}{r^2}\,\frac{\partial \sigma}{\partial \theta} &= 0 \\ \frac{\partial \sigma}{\partial t} + \omega\,\frac{\partial \sigma}{\partial \theta} + \frac{1}{r}\,\frac{\partial}{\partial r}\big(r\rho(r)Z^1\big) + \rho(r)\,\frac{\partial Z^2}{\partial \theta} &= 0 \end{aligned} \tag{106}$$

The trick for simplifying this is to write

$$Z = \frac{1}{\rho}\,\big(\text{grad } f + \text{sgrad } g\big)$$

or more explicitly,

$$\begin{aligned} Z^1 &= \frac{1}{\rho}\,\left(\frac{\partial f}{\partial r} - \frac{1}{r}\,\frac{\partial g}{\partial \theta}\right) \\ Z^2 &= \frac{1}{\rho}\,\left(\frac{1}{r^2}\,\frac{\partial f}{\partial \theta} + \frac{1}{r}\,\frac{\partial g}{\partial r}\right) \end{aligned} \tag{107}$$

Using the change of coordinates $\tau = t,\ \psi = \theta - \omega t$, plugging equations (107) into (106), and performing some simplifications, we find that f and g satisfy the equations

$$\begin{aligned} \frac{\partial \Delta f}{\partial \tau} - 2\omega\Delta g + c^2 \text{div}\,(\rho\nabla\sigma) &= 0 \\ \frac{\partial \Delta g}{\partial \tau} + 2\omega\Delta f + \omega^2\,\frac{\partial \sigma}{\partial \psi} &= 0 \\ \frac{\partial \sigma}{\partial \tau} + \Delta f &= 0 \end{aligned} \tag{108}$$

This is almost a system with constant coefficients; the only problem is the operator $\sigma \mapsto \text{div}\,(\rho\nabla\sigma)$. However, we can simply expand all the functions in a series of eigenfunctions of the form $\phi(r)e^{in\psi}$ of this differential operator, where $\phi(r)$ satisfies the Sturm-Liouville equation

$$\frac{d}{dr}\left(r\rho(r)\,\frac{d\phi}{dr}\right) - \frac{n^2}{r}\,\rho(r)\phi(r) = -\lambda r\phi(r) \tag{109}$$

with the boundary conditions $\phi(1) = 0$ and $|\phi(0)| < \infty$.

We can construct the eigenfunctions fairly explicitly under the assumption that the flow is everywhere subsonic. If we expand $\phi(r)$ in a power series as

$$\phi(r) = \sum_{j=n}^{\infty} a_j r^j$$

then we quickly find, plugging formula (105) into equation (109) that a_j satisfies the recursive equation

$$(j^2 - n^2)a_j + \left(\frac{\omega^2}{2c^2}\left[j(j-2) - n^2\right] + \lambda\right) a_{j-2} = 0$$

This power series converges for $r \leq 1$, for any λ, as long as $\frac{\omega^2}{2c^2} < 1$. This criterion is equivalent to the condition that the speed of the fluid is less than the speed of sound, that is,

$$r\omega < c\sqrt{1 + \frac{\omega^2 r^2}{2c^2}}$$

for all $r \leq 1$. The eigenvalues λ are determined implicitly by the condition that $\phi(1) = 0$. (Note that if $\omega = 0$, this equation reduces to the standard Bessel equation and that the eigenvalues are therefore the squared zeroes of $J_n(r)$.) Of course, the eigenvalues and eigenfunctions exist even if the flow is supersonic, but this power series is nonconvergent in the supersonic region.

If we now write

$$\begin{aligned}
\Delta f &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} F_{kn}(\tau)\phi_{kn}(r)e^{in\psi} \\
\Delta g &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} G_{kn}(\tau)\phi_{kn}(r)e^{in\psi} \\
\sigma &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \sigma_{kn}(\tau)\phi_{kn}(r)e^{in\psi}
\end{aligned}$$

where $\operatorname{div}\Big(\rho(r)\nabla\big(\phi_{kn}(r)e^{in\psi}\big)\Big) = -\lambda_{kn}\phi_{kn}(r)e^{in\psi}$, then equations (108) become

$$\begin{aligned}
{F_{kn}}'(\tau) - 2\omega G_{kn}(\tau) - c^2\lambda_{kn}\sigma_{kn}(\tau) &= 0 \\
{G_{kn}}'(\tau) + 2\omega F_{kn}(\tau) + in\omega^2\sigma_{kn}(\tau) &= 0 \\
{\sigma_{kn}}'(\tau) + F_{kn}(\tau) &= 0
\end{aligned} \tag{110}$$

The characteristic equation of this system is

$$p^3 + (4\omega^2 + c^2\lambda_{kn})p - 2in\omega^3 = 0$$

If we let $p = iq$, we obtain an equation with real coefficients:

$$q^3 - (4\omega^2 + c^2\lambda_{kn})q + 2n\omega^3 = 0 \tag{111}$$

Notice that if $q = a + ib$ satisfies this equation, then so does $q = a - ib$; therefore if there are any non-real roots of equation (111), the coefficients F_{kn}, G_{kn}, and σ_{kn} must grow exponentially in τ. So a necessary and sufficient criterion for stability is that, for any n, the roots of equation (111) are real and distinct. (If they were real and equal, we would have linear or quadratic growth in time of F_{kn}, G_{kn}, and σ_{kn}.) Of course, for $n = 0$, it is obvious that the equation has three distinct real roots, so we can assume $n \neq 0$.

As is well-known, the roots of a cubic equation are real and distinct if and only if the discriminant is negative; in this case, the discriminant is

$$D = -\left(\frac{4\omega^2 + c^2\lambda_{kn}}{3}\right)^3 + \left(n\omega^3\right)^2$$

which is negative if and only if

$$\lambda_{kn} > \frac{\omega^2}{c^2}(3 \cdot n^{2/3} - 4) \tag{112}$$

Since we want this to be true for all k, it must of course be true for the smallest eigenvalue λ_{1n}.

By the Rayleigh principle, we have

$$\begin{aligned}
\lambda_{1n} &= \inf_{\phi \in C_0(0,1)} \frac{\int_0^1 \frac{n^2}{r}\left(1 + \frac{\omega^2}{2c^2}r^2\right)[\phi(r)]^2\,dr + \int_0^1 r\left(1 + \frac{\omega^2}{2c^2}r^2\right)[\phi'(r)]^2\,dr}{\int_0^1 r[\phi(r)]^2\,dr} \\
&= \inf_{\phi \in C_0(0,1)} \left(\frac{\int_0^1 \frac{n^2}{r}[\phi(r)]^2\,dr + \int_0^1 r\,[\phi'(r)]^2\,dr}{\int_0^1 r[\phi(r)]^2\,dr}\right. \\
&\qquad\qquad \left. + \frac{\omega^2}{2c^2}\,\frac{\int_0^1 n^2 r\,[\phi(r)]^2\,dr + \int_0^1 r^3\,[\phi'(r)]^2\,dr}{\int_0^1 r[\phi(r)]^2\,dr}\right) \\
&\geq \inf_{\phi \in C_0(0,1)} \frac{\int_0^1 \frac{n^2}{r}[\phi(r)]^2\,dr + \int_0^1 r\,[\phi'(r)]^2\,dr}{\int_0^1 r[\phi(r)]^2\,dr} + \frac{n^2\omega^2}{2c^2}
\end{aligned}$$

The first term on the right is the usual Rayleigh quotient for the Bessel function J_n, and so its minimum is the square of the first root of J_n. It is a standard result that this first eigenvalue is at least n^2, so that our estimate is

$$\lambda_{1n} > n^2\left(1 + \frac{\omega^2}{2c^2}\right) \tag{113}$$

To obtain stability, we must verify the inequality (112), and it is sufficient by (113) to verify the inequality

$$n^2\left(1 + \frac{\omega^2}{2c^2}\right) > \frac{\omega^2}{c^2}(3n^{2/3} - 4)$$

for all nonzero integers n. If we let $u = n^{2/3}$, the function

$$\chi(u) = \left(1 + \frac{\omega^2}{2c^2}\right)u^3 - \frac{\omega^2}{c^2}(3u - 4)$$

has a minimum at

$$u_0 = \frac{\omega}{c} \frac{1}{\sqrt{1 + \frac{\omega^2}{2c^2}}}$$

at which

$$\chi(u_0) = 2\,\frac{\omega^2}{c^2}\left(2 - \frac{\frac{\omega}{c}}{\sqrt{1 + \frac{\omega^2}{2c^2}}}\right)$$

and this is positive for any choice of ω and c.

Thus the solutions of equation (110) are oscillatory in time, and the functions Δf, Δg, and σ always remain pointwise bounded in time. From here, we can solve for the components Z^1 and Z^2 by performing spatial integrations, which do not affect the oscillatory time-dependence.

□

Proposition 6.11. *In the situation above, Jacobi fields with $Y(0) = 0$ grow pointwise linearly in time unless $\int_0^{2\pi}$ curl $(\rho Z_0)\, d\theta = 0$ for any r.*

Proof. Again, in our adapted coordinates, the first of equations (91) becomes

$$\frac{\partial Y}{\partial \tau} = Z$$

As we have seen, Z is a sum of terms of the form $e^{iq\tau}\phi(r)e^{in\psi}$ where q is a solution of the equation (111)

$$q^3 - (4\omega^2 + c^2\lambda_{kn})q + 2n\omega^3 = 0$$

Upon integrating with respect to τ, we will always obtain an τ-dependence of the form $\frac{1}{q}e^{iq\tau}$ unless $q = 0$. We see that $q = 0$ is a solution of equation (111) if and only if $n = 0$.

In case $n = 0$, the equations (110) become

$$\begin{aligned} F_{k0}{}'(\tau) - 2\omega G_{k0}(\tau) - c^2\lambda_{k0}\sigma_{k0}(\tau) &= 0 \\ G_{k0}{}'(\tau) + 2\omega F_{k0}(\tau) &= 0 \\ \sigma_{k0}{}'(\tau) + F_{k0}(\tau) &= 0 \end{aligned}$$

Letting $\xi = \sqrt{4\omega^2 + c^2\lambda_{k0}}$, we find that the general solution of these equations with $\sigma_{k0}(0) = 0$ is

$$\begin{aligned} F_{k0}(\tau) &= F_{k0}(0)\cos(\xi\tau) + \frac{2\omega}{\xi}G_{k0}(0)\sin(\xi\tau) \\ G_{k0}(\tau) &= \frac{1}{\xi^2}\,G_{k0}(0)\big(c^2\lambda_{k0} + 4\omega^2\cos(\xi\tau)\big) - \frac{2\omega}{\xi}F_{k0}(0)\sin(\xi\tau) \end{aligned}$$

Since Z is a sum of functions multiplied by $F_{k0}(\tau)$ and $G_{k0}(\tau)$, the only way the τ-integral of Z will not involve a power of τ is if the constant term in $G_{k0}(\tau)$ vanishes for each k, and this only happens if $G_{k0}(0) = 0$ for each k.

Now since

$$\Delta g_0 = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} G_{kn}(0)\phi_{kn}(r)e^{in\psi}$$

we have

$$\int_0^{2\pi} \Delta g_0 \, d\theta = 2\pi \sum_{k=1}^{\infty} G_{k0}(0)\phi_{k0}(r)$$

and so each $G_{k0}(0)$ vanishes if and only if $\int_0^{2\pi} \Delta g_0 \, d\theta$ does as well. By equation (107), $\Delta g_0 = \text{curl } (\rho Z)$, so the only way we can get all terms in Y to be oscillatory is if $\int_0^{2\pi} \text{curl } (\rho Z_0) \, d\theta$ vanishes for every r.

□

For the sake of comparison, we note that the surrogate potential-Jacobi equation (96) in this case is (using the adapted coordinates τ, r, ψ)

$$\begin{aligned} \frac{\partial Z^1}{\partial \tau} - 2\omega r Z^2 &= 0 \\ \frac{\partial Z^2}{\partial \tau} + \frac{2\omega}{r} Z^1 &= 0 \\ \frac{\partial \sigma}{\partial \tau} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho Z^1) + \rho \frac{\partial Z^2}{\partial \theta} &= 0 \end{aligned}$$

The general solution $Z(\tau)$ is

$$\begin{aligned} Z^1(\tau) &= Z^1(0)\cos(2\omega\tau) + rZ^2(0)\sin(2\omega\tau) \\ Z^2(\tau) &= -\frac{1}{r}Z^1(0)\sin(2\omega\tau) + Z^2(0)\cos(2\omega\tau) \end{aligned}$$

and thus the Jacobi field $Y(\tau)$ is

$$\begin{aligned} Y^1(\tau) &= \frac{1}{2\omega}\left(Z^1(0)\sin(2\omega\tau) - rZ^2(0)\cos(2\omega\tau)\right) \\ Y^2(\tau) &= \frac{1}{2\omega}\left(\frac{1}{r}Z^1(0)\cos(2\omega\tau) + Z^2(0)\sin(2\omega\tau)\right) \end{aligned}$$

Notice that the surrogate Jacobi fields are pointwise-bounded for any initial data, unlike the Jacobi fields in the actual barotropic problem. This suggests that the connection between stability in the barotropic system and the surrogate Newtonian system is subtle, if one exists at all.

We note also that the effective curvature is not only positive everywhere but actually bounded away from zero. Because $h(\rho) = 1 + \frac{\omega^2 r^2}{2c^2}$, the operator $Y \mapsto \nabla_Y \nabla h(\rho)$ is equal to $\frac{\omega^2}{c^2}$ times the identity, so the effective curvature is at least $\frac{\omega^2}{c^2}$.

This gives a final illustration of the paradoxical geometrical behavior of fluids. When the curvature along an incompressible flow is nonpositive in all directions, as for plane parallel Couette flow, the Jacobi fields can grow linearly.

When the effective curvature along a compressible flow is strictly positive in all directions, as in this example, Jacobi fields can still grow linearly. That two fluids with such seemingly opposite properties can have the exact same asymptotic behavior illustrates that there can be no simple connection between curvature and stability.

7 Conclusion and future directions

We have found that typically curvature cannot be used to determine asymptotic growth rates of Jacobi fields. In general, there is no short-cut to the study of Lagrangian stability in a fluid. Rather, we have to split the Jacobi equation into the linearized Euler equation and the linearized flow equation and study each one separately. But the advantage of this approach is that it naturally leads to a way of relating Lagrangian and Eulerian stability.

We have not found any fluids, incompressible or compressible, for which the Jacobi fields remain bounded in time under any norm. We conjecture that, in fact, there are none. Thus the search for Lagrangian stability in a fluid seems hopeless, unless we content ourselves with very specific initial conditions such as those in Proposition 6.11. It seems more practical to distinguish between the orders of growth of Jacobi fields. If prediction is the primary concern, then obviously polynomial growth of errors can be handled while exponential growth cannot. This, then, should be the criterion we seek from Lagrangian stability analysis.

In the examples which we have studied, it has been true that Eulerian stability leads to polynomial growth of Lagrangian perturbations, while Eulerian exponential instability leads to exponential growth of Lagrangian perturbations. It would be worth studying whether this is always the case: the difficulty is of course that even Eulerian stability theory is woefully incomplete beyond the class of rotational flows we have discussed here.

One is also curious whether the existence of the bi-invariant metric on $\mathcal{D}_\mu(M^2)$ defined by

$$\langle\langle \text{sgrad } f, \text{sgrad } g\rangle\rangle = \int_M f g\, \mu$$

in two dimensions is responsible for results like Theorem 5.4. Such a metric has no analogue in three dimensions; although a bi-invariant symmetric form has been constructed on $\mathcal{D}_\mu(M^3)$, there is no bi-invariant positive-definite form.

It is thus still an open question whether these types of results remain valid in three dimensions, or whether "purely Lagrangian instabilities" can be induced in those cases. (That is, exponential growth of the Jacobi field despite boundedness of the solution of the linearized Euler equation.) So far the only example of such an instability is given in Chapter II, Section 5 of Arnol'd-Khesin [2], and this works only in $\mathbb{R}^3$. The general question is what the nature of solutions of

$$\frac{\partial Y}{\partial t} + [X, Y] = 0$$

is on a compact manifold, with X a time-independent vector field. This question is probably not difficult but we have not been able to address it here.

It is to be hoped that such methods as presented here could be useful in other questions in continuum mechanics. For example, the motion of a fluid with free boundary and surface tension can be studied by methods similar to those presented here; the configuration space is the space of volume-preserving embeddings and a potential energy function is given by the surface area of the embedded boundary. We could also explore the motion of an elastic solid using such methods, since the configuration space is again $\mathcal{D}_\mu$ with a potential energy dependent not merely on the density.

One of the most intriguing directions for future research is quite separate from fluid mechanics entirely, despite the fact that the study of fluids inspired the idea. This is the splitting of the Jacobi equation on an ordinary Riemannian manifold, as described in Section 4.3. Basically the idea is to construct a right-invariant metric on $\mathcal{D}(M)$, such that a geodesic of $\mathcal{D}(M)$ corresponds to a family of geodesics on M. Then we can study any manifold by studying its diffeomorphism group, using techniques of Lie groups with right-invariant metrics. In addition, we can split the Jacobi equation along each geodesic individually, thus obtaining two first-order ordinary differential equations along each geodesic. One hopes this would simplify the study of Jacobi fields and perhaps tell us more about their asymptotic growth than the curvature does.

The relationship between Lagrangian and Eulerian stability in fluid mechanics is interesting because of its connection between the algebraic equations in the Lie algebra and the geometric equations in the Lie group. It also appears to be tractable, unlike the general Eulerian stability problem or the global existence problem. In addition, it seems possible that the study of this problem can yield interesting insights into pure geometry. Hopefully this research has contributed to the deeper understanding of this relationship.

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