Canonical variation and positive sectional curvature

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Abstract of the Dissertation,

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In this dissertation we display several metrics of positive sectional curvature as canonical variation metrics. We find some novel new metrics of positive curvature and are able to recast old examples in this framework.

Introduction

In the long history of differential geometry relatively few manifolds of positive sectional curvature have been discovered. Initial examples can be constructed in rather classical fashion as either spheres embedded in Euclidean space or as submersion metrics on Hopf bundles.

For the most part geometers have turned to representation theory as the source of examples. Berger [B1] classified the normal homogeneous spaces of positive sectional curvature. Those are the spaces G/K with the Riemannian submersion metric induced by taking a biinvariant metric on G. It is worth noting that is but a handful more examples here than above. Aloff, Wallach [AW], [W] and Berard Bergery [BB] discovered and classified the remaining homogeneous examples, modulo a confusion patched by by Wilking [Wi]. Again notable is that the new examples occur in few dimensions.

More recent years have seen the discovery of inhomogeneous examples arising from submersions metrics of free isometric group actions on Lie groups with left invariant metrics [E1-4],[Ba]. Again these have been in few dimensions; 7, 13 and 6. In odd dimensions the number of discoveries have been countably infinite and in even dimensions finite.

These examples all rely on the curvature nonnegativity of a biinvariant

Contents

metric on a compact Lie group and the curvature nondecreasing property of Riemannian submersions. (This property can be seen as a consequence of the structure equation of O'Neill [ON] or argued synthetically as Samelson [S].)

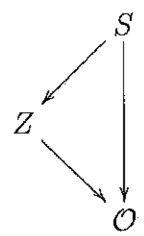
The left invariant homogeneous metrics of Wallach [W], [AW], [BB], are of considerable interest to us here. The metrics of Wallach can be understood to be the canonical variations of normal homogeneous metrics on homogeneous bundles over complex, quaternionic and Cayley projective planes.

This links into another important problem as to when a metric can be deformed to one with positive sectional curvature. Strake [St] addressed this to some degree in theory, though not a great deal has been done in practice in this regard, and the Wallach examples constituted the first real success with deformation. Some negative results in this direction have been obtained by Bourguignon, Deschamps and Sentenac [BDS] as well as more recently by Spatzier and Strake [SS]. We have more recently become aware or the work of Derdzinski, Chaves, Rigas [DCR], where they discuss matters closely related to those in this dissertation.

The first inhomogeneous metrics of positive curvature were constructed by Eschenburg on the orbit spaces of certain free isometric group actions (the biquotients). These examples were not far from the Wallach examples but were not canonical variations of the induced metric, since that does not sit well in the highly Lie algebraic treatments of Eschenburg [E1], [E2], [E3].

In this dissertation we break this mold a little and return to a more classical approach involving spheres in the search for examples. This approach draws on recent developments in the mathematics associated with the theory of strings.

Given a 3-Sasakian orbifold S one may construct an associated diagram of Riemannian submersions.



where Z, is the twistor space, S the Konishi bundle and $\mathcal{O}$ is a quaternionic Kähler orbifold.

Of particular interest in this article is the case where the base quaternionic Kähler orbifold is of positive sectional curvature. If we restrict to the case where $\mathcal{O}$ is a smooth manifold this class is extremely restricted. Berger [B2] showed that in dimensions $4n$ where $n \geq 2$ the only quaternionic Kähler manifolds of positive curvature are the quaternionic projective spaces $\mathbb{H}P^n$. In dimension 4, Hitchin showed that the only two quaternionic Kähler manifolds were $\mathbb{H}P^1 = S^4$, $\mathbb{C}P^2$ with their standard symmetric metrics which are both of positive curvature. The Konishi bundle over $\mathbb{H}P^n$ is S^{4n+3} with its standard metric. The Konishi bundle over $\mathbb{C}P^2$ is the more interesting case.

This bundle is a homogeneous 3-Sasakian manifold. Bielowski (cf [BG1]) showed that one could attain the normal homogeneous metric from the 3-Sasakian homogeneous metric by scaling the metric in the fibre SO(3) by a factor of $\frac{1}{2}$. 3-Sasakian bundles are fat, with totally geodesic positively curved spaces as fibres, in this case the base is a rank 1 symmetric space, so a theorem of Wallach paraphrased by Eschenburg shows scaling the normal homogeneous metric in the fibre with $t < 1$ gives positive sectional curvature. Now these particular homogeneous metrics with $t < 1$ are actually the normal homogeneous metrics of Wilking [Wi]. So scaling the original 3-Sasakian metric by $t < \frac{1}{2}$ will give positive sectional curvature on the Konishi bundle in this case.

This links into the celebrated hyperquotient construction [BGM]. An action of a Lie group G of 3-Sasakian isometries on a 3-Sasakian manifold S induces a 3-Sasakian moment map μ. The level set $N = \mu^{-1}(0)$ is a variety in S and N/G is a 3-Sasakian manifold with its submersion metric and its 3 vector fields defined by the projection of the restriction of the 3 vector fields of S. Taking the fundamental 3-foliation once again gives an orbifold fibration over a quaternionic Kähler orbifold.

In the case of regular circle actions, 0 is a regular value of the moment map, consequently N is a smooth manifold. In the case where the action is fixed point free the quotient N/G is a smooth manifold. In particular consider the case of a circle action on $\mathrm{S}^{4n+3} \subset \mathbb{H}^{n+1}$. If we take the action as just multiplication by a unit complex number on the left N becomes the Stiefel manifold of unitary 2-frames in complex $n+1$ space (not with the normal homogeneous metric) and the quaternionic orbifold is a Wolf space and is just the Grassmannian of complex 2-planes in $\mathbb{C}^{n+1}$ (standard symmetric space metric).

Now if one takes a canonical variation metric on $S^{4n+3} \to \mathbb{H}P^n$ to begin with N has a new metric and induces another metric on the 3-Sasakian manifold. This metric is just the canonical variation over the base quaternionic Kähler orbifold which is a weighted Grassmannian. So scaling the original metric down by $t < \frac{1}{2}$, in the $n = 2$ case will induce positive curvature on the homogeneous example and hence on quotients of deformations of $V_{3,2}^{\mathbb{C}} \to S \to \mathbb{C}P^2$ are sufficiently close in the equivariant Hausdorff sense [CFG]. That is a relatively cheap way to see inhomogeneous examples, these metrics are new albeit the spaces are not new since nearby the Aloff-Wallach space examples are known to have Eschenburg type metrics of positive curvature as well.

This situation is no coincidence. In this dissertation we prove the theorem.

Theorem 1 *Let $S \to \mathcal{O}$ be the canonical 3-foliation of a 3-Sasakian manifold S and $\mathcal{O}$ have positive sectional curvature, then S has a canonical variation metric of positive sectional curvature for sufficiently small t.*

Also we prove the proposition important for applications

Proposition 2 *Many of the quaternionic Kähler orbifold associated to the 3-Sasakian reduction of a circle action on S^{11} have positive sectional curvature.*

In particular this means many Konishi bundles over the Galicki-Lawson [GL] examples discussed in Boyer et al [BGM] have canonical variation metrics of positive sectional curvature.

There also is an equally interesting result in the case of Sasakian manifolds.

Theorem 3 *Let $S \to Z$ be the canonical foliation of a quasi-regular Sasakian manifold S and Z have positive sectional curvature, then S has a canonical variation metric of positive sectional curvature for sufficiently small t.*

In particular this puts familiar Sasakian metrics (up to homothety) of nonconstant positive sectional curvature on S^5 over $\mathbb{C}P^2$.

Chapter 1

The structure equation of a canonical variation

Let $\pi : M \to B$ be a Riemannian submersion (see [ON] for basic notions). Let X, Y and U, V be basic and vertical vector fields throughout. We have that the metric, $\langle , \rangle$ splits up as

$$\langle X + U, Y + V \rangle = \langle X, Y \rangle + \langle U, V \rangle$$

The canonical variation $\langle , \rangle_t$ is the Riemannian metric defined by

$$\langle X + U, Y + V \rangle_t = \langle X, Y \rangle + t\langle U, V \rangle$$

Let X_*, Y_* denote the fields on B corresponding to the basic fields X, Y, $\langle , \rangle_*$ the submersion metric induced on B and K, K^* be the unnormalised sectional curvatures of $\langle , \rangle$ and $\langle , \rangle_*$ respectively.

Let T and A denote the second fundamental form and O'Neill tensor of the submersion respectively.

Let $S_E F = \mathcal{H}(\nabla_E F - \nabla_X Y)$. Provided we restrict to E, F which project, we have S is a symmetric tensor.

The S tensor is given by

$$S_E F = T_U V + A_X V + A_Y U$$

Theorem 1.1 *The full curvature tensor, R_t, of the canonical variation, $\langle , \rangle_t$, of the Riemannian submersion, $\pi : M \to B$, is given by*

$$\begin{aligned} R_t(E,F,G,H) &= tR(E,F,G,H) + (1-t)R^*(X_*, Y_*, Z_*, Z'_*) \\ &\quad + t(1-t)(\langle S_E H, S_F G \rangle) - \langle S_E G, S_F H \rangle) \end{aligned}$$

Proof.

$$\begin{aligned} R_t(E,F,G,H) &= E\langle \nabla^t_F G, H \rangle_t - F\langle \nabla^t_E G, H \rangle_t \\ &\quad - \langle \nabla^t_F G, \nabla^t_E H \rangle_t + \langle \nabla^t_E G, \nabla^t_F H \rangle_t \\ &\quad - \langle \nabla^t_{[E,F]} G, H \rangle_t \end{aligned}$$

Note the following two important points

$$\nabla^t_E F = \nabla_E F + (t-1) S_E F$$

$$\begin{aligned} \langle E, F \rangle_t &= \langle X, Y \rangle + t\langle U, V \rangle \\ &= (1-t)\langle X, Y \rangle + t\langle X+U, Y+V \rangle \\ &= (1-t)\langle X, Y \rangle + t\langle E, F \rangle \end{aligned}$$

$$
\begin{aligned}
E\langle\nabla^t_F G, H\rangle_t &= E\langle\nabla_F G, H\rangle_t + (t-1)E\langle S_F G, Z'\rangle_t \\
&= E\langle\nabla_F G, H\rangle_t + (t-1)E\langle S_F G, Z'\rangle \\
&= tE\langle\nabla_F G, H\rangle + (1-t)E\langle\mathcal{H}\nabla_F G, Z'\rangle - (1-t)E\langle S_F G, Z'\rangle \\
&= tE\langle\nabla_F G, H\rangle + (1-t)E\langle\mathcal{H}\nabla_F G - S_F G, Z'\rangle \\
&= tE\langle\nabla_F G, H\rangle + (1-t)E\langle\mathcal{H}\nabla_Y Z, Z'\rangle \\
&= tE\langle\nabla_F G, H\rangle + (1-t)X\langle\mathcal{H}\nabla_Y Z, Z'\rangle \\
&= tE\langle\nabla_F G, H\rangle + (1-t)X_*\langle\nabla^*_{Y_*} Z_*, Z'_*\rangle_*
\end{aligned}
$$

$$
\begin{aligned}
\langle\nabla^t_F G, \nabla^t_E H\rangle_t &= \langle\nabla_F G + (t-1)S_F G, \nabla_E H + (t-1)S_E H\rangle_t \\
&= \langle\nabla_F G, \nabla_E H\rangle_t + (t-1)\langle S_F G, \nabla_E H\rangle_t \\
&\quad +(t-1)\langle\nabla_F G, S_E H\rangle_t + (t-1)^2\langle S_F G, S_E H\rangle_t \\
&= \langle\nabla_F G, \nabla_E H\rangle_t + (t-1)\langle S_F G, \mathcal{H}\nabla_E H\rangle \\
&\quad +(t-1)\langle\mathcal{H}\nabla_F G, S_E H\rangle + (t-1)^2\langle S_F G, S_E H\rangle \\
&= t\langle\nabla_F G, \nabla_E H\rangle + (1-t)\langle\mathcal{H}\nabla_F G, \mathcal{H}\nabla_E H\rangle \\
&\quad -(1-t)\langle S_F G, \mathcal{H}\nabla_E H\rangle - (1-t)\langle\mathcal{H}\nabla_F G, S_E H\rangle \\
&\quad +(1-t)^2\langle S_F G, S_E H\rangle \\
&= t\langle\nabla_F G, \nabla_E H\rangle + (1-t)\langle\mathcal{H}\nabla_F G - S_F G, \mathcal{H}\nabla_E H - S_E H\rangle \\
&\quad -(1-t)\langle S_F G, S_E H\rangle + (1-t)^2\langle S_F G, S_E H\rangle \\
&= t\langle\nabla_F G, \nabla_E H\rangle + (1-t)\langle\mathcal{H}\nabla_Y Z, \mathcal{H}\nabla_X Z'\rangle \\
&\quad -t(1-t)\langle S_F G, S_E H\rangle \\
&= t\langle\nabla_F G, \nabla_E H\rangle + (1-t)\langle\nabla^*_{Y_*} Z_*, \nabla^*_{X_*} Z'_*\rangle_* \\
&\quad -t(1-t)\langle S_F G, S_E H\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle \nabla^t_{[E,F]} G, H\rangle_t &= \langle \nabla_{[E,F]} G, H\rangle_t + (t-1)\langle S_{[E,F]} G, H\rangle_t \\
&= t\langle \nabla_{[E,F]} G, H\rangle + (1-t)\langle \mathcal{H}\nabla_{[E,F]} G, Z'\rangle \\
&\quad -(1-t)\langle S_{[E,F]} G, Z'\rangle \\
&= t\langle \nabla_{[E,F]} G, H\rangle + (1-t)\langle \mathcal{H}\nabla_{\mathcal{H}[E,F]} Z, Z'\rangle \\
&= t\langle \nabla_{[E,F]} G, H\rangle + (1-t)\langle \mathcal{H}\nabla_{\mathcal{H}[X,Y]} Z, Z'\rangle \\
&= t\langle \nabla_{[E,F]} G, H\rangle + (1-t)\langle \nabla^*_{[X_*,Y_*]} Z_*, Z'_*\rangle_*
\end{aligned}
$$

Collecting like terms we get the desired result.□

So for the unnormalised sectional curvature we get the following result.

Corollary 1.2 *The unnormalised sectional curvature, K_t, of the canonical variation, $\langle , \rangle_t$, of the Riemannian submersion, $\pi : M \longrightarrow B$, is given by*

$$K_t(E,F) = tK(E,F) + (1-t)K^*(X_*,Y_*) + t(1-t)(\langle S_E E, S_F F\rangle - |S_E F|^2)$$

Chapter 2

Previously known examples

Let us begin by reworking some know results in this framework. We will reprove the results of [W], [AW]. An elegant proof of these results has already been accomplished by Eschenburg using Riemannian submersions on normally homogeneous metrics on Lie groups [E4]. Eschenburg reposited the work of Wallach in the following theorem.

Theorem 2.1 *Let $H \subset K \subset G$ and Q be a biinvariant metric on $\mathfrak{g}$. Assume that the following three conditions are satisfied:*

∘ *G/K is a rank 1 symmetric space, i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ where $\mathfrak{m} = \mathfrak{k}^{\perp}$.*

∘ *The normal metric induced by Q on K/H has positive sectional curvature, i.e. for $\mathfrak{p} = \mathfrak{h}^{\perp} \cap \mathfrak{k}$ $[u, v] \neq 0$ for $u, v \in \mathfrak{p}$ linearly independent*

∘ *Fatness, i.e. $[x, u] \neq 0$ for $x \in \mathfrak{m}$, $u \in \mathfrak{p}$ not 0*

then the homogeneous metric on G/H given by $Q_t = tQ|_{\mathfrak{p}} + Q|_{\mathfrak{m}}$ has positive sectional curvature for and $0 < t < 1$. Moreover if in addition K/H is a symmetric space, Q_t with $1 < t < \frac{4}{3}$ has positive sectional curvature.

Eschenburg was able to prove the main part of the above theorem without calculation. The second part however can not be proven this way and requires a calculation.

The proof we shall give will be computational.

Proof. Throughout let the lower case letter denote the corresponding element of $\mathfrak{g}$.

$$\begin{aligned} K(E,F) &= \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[e,f]|_{\mathfrak{k}}|^2 + \tfrac{3}{4}|[e,f]_{\mathfrak{h}}|^2 \\ &= \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[x,y]+[u,v]|^2 + \tfrac{3}{4}|[e,f]_{\mathfrak{h}}|^2 \end{aligned}$$

$$K^*(X_*,Y_*) = \frac{1}{4}|[x,y]_{\mathfrak{m}}|^2 + |[x,y]_{\mathfrak{k}}|^2 = |[x,y]|^2$$

For a bundle with totally geodesic fibres $S_E F = A_X V + A_Y U$. The element of $\mathfrak{g}$ corresponding to $S_E F$ is hence

$$S_E F = \frac{1}{2}([x,v]+[y,u])$$

Hence $S_E E = [x,u]$ and $S_F F = [y,v]$.

$$\langle S_E E, S_F F\rangle - |S_E F|^2 = \langle [x,u],[y,v]\rangle - \frac{1}{4}|[x,v]-[u,y]|^2$$

$$\begin{aligned} \langle [x,u],[y,v]\rangle &= \langle x,[u,[y,v]]\rangle \\ &= -\langle x,[y,[v,u]]\rangle - \langle x,[v,[u,y]]\rangle \\ &= \langle x,[y,[u,v]]\rangle - \langle [x,v],[u,y]\rangle \\ &= \langle [x,y],[u,v]\rangle - \langle [x,v],[u,y]\rangle \end{aligned}$$

$$
\begin{aligned}
\langle S_E E, S_F F\rangle - |S_E F|^2 &= \langle [x,y],[u,v]\rangle - \langle [x,v],[u,y]\rangle - \tfrac{1}{4}|[x,v]-[u,y]|^2 \\
&= \langle [x,y],[u,v]\rangle - \tfrac{1}{4}|[x,v]+[u,y]|^2 \\
&= \langle [x,y],[u,v]\rangle - \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2
\end{aligned}
$$

$$
\begin{aligned}
K_t(E,F) &= tK(E,F) + (1-t)K^*(X_*,Y_*) \\
&\quad + t(1-t)(\langle S_E E, S_F F\rangle - |S_E F|^2) \\
&= t(\tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[x,y]+[u,v]|^2 + \tfrac{3}{4}|[e,f]^2_{\mathfrak{h}}) \\
&\quad + (1-t)|[x,y]|^2 + t(1-t)(\langle [x,y],[u,v]\rangle - \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2) \\
&= \tfrac{t^2}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{3t}{4}|[e,f]_{\mathfrak{h}}|^2 \\
&\quad + \tfrac{t}{4}|[x,y]+[u,v]|^2 + (1-t)|[x,y]|^2 + t(1-t)\langle [x,y],[u,v]\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{t}{4}|[x,y]+[u,v]|^2 + (1-t)|[x,y]|^2 + t(1-t)\langle [x,y],[u,v]\rangle \\
= {} & \tfrac{t}{4}(|[x,y]|^2 + 2\langle [x,y],[u,v]\rangle + |[u,v]|^2) + (1-t)|[x,y]|^2 + t(1-t)\langle [x,y],[u,v]\rangle \\
= {} & (\tfrac{t}{4} + (1-t))|[x,y]|^2 + \tfrac{t}{4}|[u,v]|^2 + (\tfrac{t}{2} + t(1-t))\langle [x,y],[u,v]\rangle \\
= {} & (1 - \tfrac{3t}{4})|[x,y]|^2 + \tfrac{t}{4}|[u,v]|^2 + t(\tfrac{3}{2} - t)\langle [x,y],[u,v]\rangle \\
= {} & ((1 - \tfrac{3t}{4}) - t(\tfrac{3}{2} - t)^2)|[x,y]|^2 + t|(\tfrac{3}{2} - t)[x,y] + \tfrac{1}{2}[u,v]|^2 \\
= {} & (1-t)^3|[x,y]|^2 + t|((1-t)[x,y] + \tfrac{1}{2}[e,f]_{\mathfrak{k}}|^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
K_t(E,F) &= \tfrac{t^2}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{3t}{4}|[e,f]_{\mathfrak{h}}|^2 \\
&\quad + (1-t)^3|[x,y]|^2 + t|((1-t)[x,y] + \tfrac{1}{2}[e,f]_{\mathfrak{k}}|^2
\end{aligned}
$$

If $0 < t < 1$ the end term is zero if and only if $[x,y] = [u,v] = 0$. But by two of the conditions this would make x, y linearly dependent and u, v linearly dependent. If e, f are linearly independent then this means $[e,f]_{\mathfrak{m}}$ is a nonzero multiple of $[x,u]$ which is nonzero. Hence the expression for $K_t(E,F)$

is positive.

Now assume that K/H is a symmetric space then $[u,v] \in \mathfrak{h}$. We rewrite $K(E,F)$ as

$$\begin{aligned} K(E,F) &= \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[e,f]|_{\mathfrak{p}}|^2 + |[e,f]_{\mathfrak{h}}|^2 \\ &= \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[x,y]_{\mathfrak{p}}|^2 + |[x,y]_{\mathfrak{h}} + [u,v]|^2 \end{aligned}$$

$$K^*(X_*,Y_*) = |[x,y]|^2 = |[x,y]_{\mathfrak{p}}|^2 + |[x,y]_{\mathfrak{h}}|^2$$

$$\begin{aligned} K_t(E,F) &= tK(E,F) + (1-t)K^*(X_*,Y_*) \\ &\quad + t(1-t)(\langle S_E E, S_F F\rangle - |S_E F|^2) \\ &= t(\tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2 + \tfrac{1}{4}|[x,y]_{\mathfrak{p}}|^2 + |[x,y]_{\mathfrak{h}} + [u,v]|^2) \\ &\quad + (1-t)(|[x,y]_{\mathfrak{p}}|^2 + |[x,y]_{\mathfrak{h}}|^2) \\ &\quad + t(1-t)(\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle - \tfrac{1}{4}|[e,f]_{\mathfrak{m}}|^2) \\ &= \tfrac{t^2}{4}|[e,f]_{\mathfrak{m}}|^2 + (1-\tfrac{3t}{4})|[x,y]_{\mathfrak{p}}|^2 \\ &\quad + t|[x,y]_{\mathfrak{h}} + [u,v]|^2 + (1-t)|[x,y]_{\mathfrak{h}}|^2 + t(1-t)\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle \end{aligned}$$

$$\begin{aligned} & t|[x,y]_{\mathfrak{h}} + [u,v]|^2 + (1-t)|[x,y]_{\mathfrak{h}}|^2 + t(1-t)\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle \\ =\ & t(|[x,y]_{\mathfrak{h}}|^2 + 2\langle |[x,y]_{\mathfrak{h}}, [u,v]\rangle + |[u,v]|^2) + (1-t)|[x,y]_{\mathfrak{h}}|^2 \\ & + t(1-t)\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle \\ =\ & |[x,y]_{\mathfrak{h}}|^2 + t|[u,v]|^2 + t(3-t)\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle \end{aligned}$$

$$\begin{aligned} K_t(E,F) = &\ \tfrac{t^2}{4}|[e,f]_{\mathfrak{m}}|^2 + (1-\tfrac{3t}{4})|[x,y]_{\mathfrak{p}}|^2 \\ & + |[x,y]_{\mathfrak{h}}|^2 + t|[u,v]|^2 + t(3-t)\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle \end{aligned}$$

Now suppose $1 < t < \frac{4}{3}$

$$
\begin{aligned}
t(3-t)|\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle| &\leq 2|\langle [x,y]_{\mathfrak{h}}, [u,v]\rangle| \\
&\leq |[x,y]_{\mathfrak{h}}|^2 + |[u,v]|^2 \\
&\leq |[x,y]_{\mathfrak{h}}|^2 + t|[u,v]|^2
\end{aligned}
$$

This gives positive curvature reasoning along the same lines.□

Theorem 2.2 *Let G/K be a rank 1 symmetric space and consider a free isometric biquotient action of U on G. is a commuting diagram of Riemannian submersions then the canonical variation on G, $\langle , \rangle_t$ of the metric $\langle , \rangle$, on G, induces nonnegative sectional curvature under submersion onto the biquotient $G//U$ for $0 < t < 1$.*

Let E, F be horizontal vector fields with respect to ρ and $e = x+u, f = y+v$ be the corresponding elements in the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ then the curvature vanishes only if $[u,v] = 0, [e,f] = 0$.

Proof.

The sectional curvature, K, of $\langle , \rangle$ is given by

$$
\begin{aligned}
K(E,F) &= \tfrac{t^2}{4}|[e,f]_{\mathfrak{m}}|^2 + 3|A_E F|_t^2 \\
&\quad +(1-t)^3|[x,y]|^2 + t|((1-t)[x,y] + \tfrac{1}{2}[e,f]_{\mathfrak{k}}|^2
\end{aligned}
$$

This completes the proof. □

This recaptures the examples of Eschenburg and Bazaikin. Note that if $t > 1$ that $(1-t)^3$ becomes negative and the conditions $[u,v] = 0, [e,f] = 0$ where e, f are horizontal are sufficient but not necessary for zero curvature. A priori the curvature might even become negative in places.

Eschenburg has illustrated that when $t < 1$ the Gromoll-Meyer sphere has positive curvature almost everywhere. When $t > 1$ his analysis also implicitly shows the existence of zero curvature plane. It would be interesting to know if the sectional curvature is almost positive for $t > 1$. Note the question is unresolved when $t = 1$ despite the prevalent misconception that Mandell showed this. Analogous statements are true of those Eschenburg spaces having zero curvature planes when $t < 1$.

Chapter 3

New metrics of positive curvature

To look at some new metrics it will be necessary to review some contact geometry, we apologise for any glaring omissions. Where we are lacking we refer the reader to the excellent survey of Boyer and Galicki [BG1].

Definition 3.1 *A Riemannian manifold S is said to be Sasakian if it has a unit Killing field ξ satisfying the equation*

$$R(X,\xi)Y = \langle \xi, Y\rangle X - \langle X, Y\rangle \xi$$

Given such a characteristic vector ξ we define a (1,1)-tensor ϕ to be given by $\phi(X) = \nabla_X \xi$ and the characteristic 1-form η to be given by $\eta(X) = \langle X, \xi\rangle$. Altogether we call the triple (ξ, η, ϕ) a Sasakian structure.

Notable here is Sasakian manifolds must be of odd dimension, since in even dimensions there are no nonvanishing vector fields [B2].

For calculations it is important to know the following rudimentary proposition

Proposition 3.2 *Let S be a Sasakian manifold with Sasakian structure (ξ, η, ϕ)*

and X, Y a pair of vector fields on S

$$
\begin{array}{ll}
\phi^2(X) = -X + \eta(X)\xi & \\
\phi(\xi) = 0 & \eta(\phi(X)) = 0 \\
\langle X, \phi(Y)\rangle + \langle \phi(X), Y\rangle = 0 & \langle \phi(X), \phi(Y)\rangle = \langle X, Y\rangle - \eta(X)\eta(Y) \\
d\eta(X, Y) = 2\langle \phi(X), Y\rangle & N_\phi(X, Y) = d\eta(X, Y) \otimes \xi
\end{array}
$$

where $N_\phi(X, Y) = [\phi(X), \phi(Y)] + \phi^2([X, Y]) - \phi([X, \phi(Y)]) - \phi([\phi(X), Y])$ is the Nijenhuis tensor.

A manifold is said to be 3-Sasakian if it has a triple of Sasakian structures, so characteristic vector fields ξ^1, ξ^2, ξ^3 are an orthonormal frame and induce a Lie algebra isomorphism when mapped to $i, j, k \in \mathfrak{sp}(1)$. Such manifolds are automatically Einstein and have positive constant scalar curvature. We have the following properties of the structure.

Proposition 3.3 *Let S be 3-Sasakian with 3-structures (ξ^n, η^n, ϕ^n) where $n = 1, 2, 3$ then*

$$
\begin{aligned}
&\eta^m(\xi^n) = \delta^{mn} \\
&\phi^m(\xi^n) = -\epsilon^{mnp}\xi^p \\
&\phi^m \circ \phi^n - \xi^m \otimes \eta^n = -\epsilon^{mnp}\phi^p - \delta^{mn}\mathrm{id}
\end{aligned}
$$

We calculate the curvature of the canonical variation of the 3-Sasakian foliation. We start by computing the curvature of the canonical variation of the 3-Sasakian foliation. We start by computing the sectional curvature of the

3-Sasakian metric.

$$\begin{aligned}
K(E,F) &= \langle R(E,F)F,E\rangle \\
&= \langle R(E,V)F,E\rangle + \langle R(E,Y)F,E\rangle \\
&= \langle R(E,V)F,E\rangle + \langle R(V,Y)F,E\rangle \\
&\quad +\langle R(X,Y)F,E\rangle \\
&= \langle R(E,V)F,E\rangle + \langle R(V,Y)F,E\rangle \\
&\quad +\langle R(X,Y)F,U\rangle + \langle R(X,Y)F,X\rangle \\
&= \langle R(E,V)F,E\rangle + \langle R(V,Y)F,E\rangle \\
&\quad +\langle R(X,Y)F,U\rangle + \langle R(X,Y)V,X\rangle \\
&\quad +\langle R(X,Y)Y,X\rangle \\
&= \langle R(E,V)F,E\rangle - \langle R(Y,V)F,E\rangle \\
&\quad +\langle R(F,U)X,Y\rangle - \langle R(X,V)X,Y\rangle \\
&\quad +\langle R(X,Y)Y,X\rangle \\
&= \langle V,F\rangle\langle E,E\rangle - \langle E,F\rangle\langle V,E\rangle - \langle V,F\rangle\langle Y,E\rangle + \langle Y,F\rangle\langle V,E\rangle \\
&\quad +\langle U,X\rangle\langle F,Y\rangle - \langle F,X\rangle\langle U,Y\rangle - \langle V,X\rangle\langle X,Y\rangle + \langle X,Y\rangle\langle V,Y\rangle \\
&\quad +K(X,Y) \\
&= |U|^2|V|^2 - \langle U,V\rangle^2 + |X|^2|V|^2 + |Y|^2|U|^2 - 2\langle U,V\rangle\langle X,Y\rangle \\
&\quad +K(X,Y)
\end{aligned}$$

Let $U = u_1\xi^1 + u_2\xi^2 + u_3\xi^3, V = v_1\xi^1 + v_2\xi^2 + v_3\xi^3$

$$S_E F = A_X V + A_Y U$$

$$\begin{aligned}
|S_E F|^2 &= |A_X V|^2 + |A_Y U|^2 + 2\langle A_X V, A_Y U\rangle \\
&= |X|^2|V|^2 + |Y|^2|U|^2 + 2\langle A_X V, A_Y U\rangle
\end{aligned}$$

$$S_E E = 2A_X U, S_F F = 2A_Y V$$

$$\langle S_E E, S_F F\rangle = 4\langle A_X U, A_Y V\rangle$$

$$\begin{aligned}
\langle A_X U, A_Y V\rangle &= \textstyle\sum_{m,n} u_m v_n \langle \phi^m(X), \phi^n(Y)\rangle \\
&= (\textstyle\sum_m u_m v_m)\langle X, Y\rangle + \sum_{m\neq n} u_m v_n \langle \phi^m(\phi^n(X)), Y\rangle \\
&= \langle U, V\rangle\langle X, Y\rangle - \langle \tfrac{1}{2}[U, V], A_X Y\rangle
\end{aligned}$$

$$\begin{aligned}
\langle S_E E, S_F F\rangle - |S_E F|^2 &= 4(\langle U, V\rangle\langle X, Y\rangle + \langle \tfrac{1}{2}[U, V], A_X Y\rangle) \\
&\quad -(|X|^2|V|^2 + |Y|^2|U|^2 + 2\langle U, V\rangle\langle X, Y\rangle \\
&\quad +2\langle \tfrac{1}{2}[U, V], A_X Y\rangle \\
&= 6\langle \tfrac{1}{2}[U, V], A_X Y\rangle \\
&\quad -(|X|^2|V|^2 + |Y|^2|U|^2 - 2\langle U, V\rangle\langle X, Y\rangle)
\end{aligned}$$

$$K(X, Y) = K^*(X_*, Y_*) - 3|A_X Y|^2$$

Substituting all of this into the curvature formula and collecting like terms we get

$$\begin{aligned}
K_t(E, F) &= t|\tfrac{1}{2}[U, V]|^2 + t^2(|X|^2|V|^2 + |Y|^2|U|^2 - 2\langle U, V\rangle\langle X, Y\rangle) \\
&\quad +6t(1-t)\langle \tfrac{1}{2}[U, V], A_X Y\rangle + K^*(X_*, Y_*) - 3t|A_X Y|^2 \\
&= t|\tfrac{1}{2}[U, V] + 3(1-t)A_X Y|^2 - 9t(1-t)^2|A_X Y|^2 \\
&\quad +t^2(|X|^2|V|^2 + |Y|^2|U|^2 - 2\langle U, V\rangle\langle X, Y\rangle) \\
&\quad +K^*(X_*, Y_*) - 3t|A_X Y|^2 \\
&= t|\tfrac{1}{2}[U, V] + 3(1-t)A_X Y|^2 \\
&\quad +t^2(|X|^2|V|^2 + |Y|^2|U|^2 - 2\langle U, V\rangle\langle X, Y\rangle) \\
&\quad +K^*(X_*, Y_*) - 3t(1 + 3(1-t)^2)|A_X Y|^2
\end{aligned}$$

We have $|A_X Y|^2 \leq |X \wedge Y|^2$ and for positively curved $\mathcal{O}$, $K^*(X_*, Y_*) \geq L|X \wedge Y|^2$. For sufficiently small t, $L - 3t(1 + 3(1-t)^2) > 0$ so for sufficiently small t the only way the last line can vanish is if X, Y are linearly dependent. But then $U \wedge V = 0$ so U, V are linearly dependent. The middle term vanishing in addition is what forces the linear dependence of E, F. We conclude that for sufficiently small t the canonical variation metric on S is of positive sectional curvature.

Consider 3-Sasakian manifold given by the Hopf fibration $\mathrm{S}^{11} \to \mathbb{H}P^2$. Consider an action on S^{11} by a free 3-Sasakian isometric circle action.

This moderate amount of information is adequate to conclude much about the curvature of the quaternionic orbifold given by the 3-foliation of the 3-Sasakian reduction, though interpreting the computations can be difficult. To illustrate how this works we restrict to the case where the Killing field ξ defines a Sasakian structure. This is a rather restrictive assumption, in fact among the examples of [BGM] it restricts us to $S(1,1,1) \to \mathbb{C}P^2$, but at the end we will see how the story differs when we remove it. Recall $N = \{p \in \mathrm{S}^{11} | \mu^n_p \equiv \langle \xi, \xi^n \rangle_p = 0, n = 1,2,3\}$, this is just a variety in S^{11}, so we can easily calculate normals to N.

$$\begin{aligned}
\langle \nabla \mu^n, E \rangle &= E\mu^n \\
&= E\langle \xi, \xi^n \rangle \\
&= \langle \nabla_E \xi, \xi^n \rangle + \langle \xi, \nabla_E \xi^n \rangle \\
&= -\langle \nabla_{\xi^n} \xi, E \rangle - \langle \nabla_\xi \xi^n, E \rangle \\
&= -\langle \nabla_\xi \xi^n, E \rangle - \langle \nabla_\xi \xi^n, E \rangle \\
&= \langle -2\phi^n(\xi), E \rangle
\end{aligned}$$

Hence $\nabla \mu^n = -2\phi^n(\xi)$.

$$\begin{aligned}\langle -2\phi^m(\xi), -2\phi^n(\xi)\rangle &= 4\langle \phi^m(\xi), \phi^n(\xi)\rangle \\ &= 4\delta^{mn}\end{aligned}$$

Hence $\{\phi^n(\xi)\}$ form an orthonormal basis for the normal bundle of N.

We now turn our attention to the second fundamental form.

$$\begin{aligned}\langle \nabla_E F, \phi^n(\xi)\rangle &= -\langle F, \nabla_E(\phi^n(\xi))\rangle \\ &= -\langle F, (\nabla_E \phi^n)(\xi) + \phi^n(\nabla_E \xi)\rangle \\ &= -\langle F, R(E, \xi^n)(\xi) + \phi^n(\nabla_E \xi)\rangle \\ &= -\langle \xi^n, \xi\rangle\langle E, F\rangle + \langle E, \xi\rangle\langle F, \xi^n\rangle - \langle \phi^n(\phi(E)), F\rangle \\ &= \langle E, \xi\rangle\langle F, \xi^n\rangle + \langle \phi(E), \phi^n(F)\rangle\end{aligned}$$

For E, F perpendicular to ξ, i.e. horizontal with respect to $N \to S$

$$T_E F = \sum_n \langle \phi(E), \phi^n(F)\rangle \phi^n(\xi)$$

Now consider X, Y orthonormal and horizontal to the 3-foliation

$$\begin{aligned}|T_X Y|^2 &= \textstyle\sum_n \langle \phi(X), \phi^n(Y)\rangle^2 \\ &= |\phi(X)|^2 - \langle \phi(X), Y\rangle^2 \\ &= 1 - \langle \phi(X), Y\rangle^2\end{aligned}$$

$$\begin{aligned}K_N(X,Y) &= K_{S^{11}}(X,Y) + \langle T_X X, T_Y Y\rangle - |T_X Y|^2 \\ &= 1 + \textstyle\sum_n \langle \phi(X), \phi^n(X)\rangle\langle \phi(Y), \phi^n(Y)\rangle + \langle \phi(X), Y\rangle^2 - 1 \\ &= \textstyle\sum_n \langle \phi(X), \phi^n(X)\rangle\langle \phi(Y), \phi^n(Y)\rangle + \langle \phi(X), Y\rangle^2\end{aligned}$$

$$
\begin{aligned}
\langle \nabla_E F, \xi \rangle &= -\langle F, \nabla_E \xi \rangle \\
&= -\langle \phi(E), F \rangle
\end{aligned}
$$

Hence $|A_X^S Y|^2 = \langle \phi(X), Y \rangle^2$.

$$
\begin{aligned}
K_S(X,Y) &= K_N(X,Y) + 3|A_X^S Y|^2 \\
&= \textstyle\sum_n \langle \phi(X), \phi^n(X) \rangle \langle \phi(Y), \phi^n(Y) \rangle + 4\langle \phi(X), Y \rangle^2
\end{aligned}
$$

$$
|A_X^{\mathcal{O}} Y|^2 = \sum_n \langle \phi^n(X), Y \rangle^2 = 1
$$

$$
\begin{aligned}
K_{\mathcal{O}}(X,Y) &= K_S(X,Y) + 3|A_X^{\mathcal{O}} Y|^2 \\
&= 3 + \textstyle\sum_n \langle \phi(X), \phi^n(X) \rangle \langle \phi(Y), \phi^n(Y) \rangle + 4\langle \phi(X), Y \rangle^2
\end{aligned}
$$

Put $x^n = \langle \phi(X), \phi^n(X) \rangle$ and $y^n = \langle \phi(Y), \phi^n(Y) \rangle$ then

$$
\sum_n \langle \phi(X), \phi^n(X) \rangle \langle \phi(Y), \phi^n(Y) \rangle = \sum_n x^n y^n
$$

Note

$$
\sum_n (x^n)^2 = |\phi(X)|^2 = 1, \sum_n (y^n)^2 = 1
$$

So finding the minimum of the second term is a simple constrained optimisation problem. The minimum occurs for $x^n = -y^n$ and makes the term -1.

Hence it follows that $K_{\mathcal{O}} \geq 2$.

The assumption that ξ was Sasakian was only imposing the additional condition that ξ is of unit length. Generally speaking this is not the case. The difficulty of the computations is not really increased by this but it does make estimates more difficult. In that case we get

$$\begin{aligned} K_{\mathcal{O}}(X,Y) &= 4 + \tfrac{1}{|\xi|^2}(\textstyle\sum_n(\langle\phi(X),\phi^n(X)\rangle\langle\phi(Y),\phi^n(Y)\rangle \\ &\quad -\langle\phi(X),\phi^n(Y)\rangle^2) + 3\langle\phi(X),Y\rangle^2) \end{aligned}$$

A subtle analysis is required to see when this is positive. The initial 4 however should provide a large amount of space for the other terms to misbehave without making the expression negative. If not all of possible orbifolds have positive sectional curvature a good portion of them do.

Now one of the main distinctions between the case where ξ is of unit length and of variable length is in what space $\phi(X)$ lies.

$$\begin{aligned} \langle\phi(X),\xi\rangle &= \langle\nabla_X\xi,\xi\rangle \\ &= \tfrac{1}{2}X(|\xi|^2) \end{aligned}$$

In the case where ξ is of constant length this is 0 and $\phi(X)$ lies in the horizontal $\mathcal{H}$ of the submersion $S \to \mathcal{O}$. Otherwise it is merely orthogonal to ξ^1, ξ^2, ξ^3.

To understand this better we should decompose $\mathcal{H}\phi(X)$.

$$|\mathcal{H}\phi(X)|^2 = \langle\phi(X),Y\rangle^2 + \sum_n \langle\phi(X),\phi^n(Y)\rangle^2$$

$$\begin{aligned} K_{\mathcal{O}}(X,Y) &= 4 + \tfrac{1}{|\xi|^2}(\textstyle\sum_n(\langle\phi(X),\phi^n(X)\rangle\langle\phi(Y),\phi^n(Y)\rangle \\ &\quad -|\mathcal{H}\phi(X)|^2 + 4\langle\phi(X),Y\rangle^2) \end{aligned}$$

Note that since ξ is Killing $\langle\nabla_X\xi, X\rangle = 0$. It follows that

$$|\mathcal{H}\phi(X)|^2 = \sum_n \langle\phi(X),\phi^n(X)\rangle^2$$

Note that $|\mathcal{H}\phi(X)|^2$ is actually independent of the tangent unit vector X. This can be seem from the symmetry of the curvature tensor. We thus have $|\mathcal{H}\phi(Y)|^2 = |\mathcal{H}\phi(X)|^2$. So we can rewrite the curvature

$$K_{\mathcal{O}}(X,Y) \;=\; 4 + \tfrac{1}{|\xi|^2}(-\tfrac{1}{2}\textstyle\sum_n(\langle\phi(X),\phi^n(X)\rangle - \langle\phi(Y),\phi^n(Y)\rangle)^2 + 4\langle\phi(X),Y\rangle^2)$$

The negative term is at worst when $\langle\phi(X),\phi^n(X)\rangle = -\langle\phi(Y),\phi^n(Y)\rangle$ In that case

$$K_{\mathcal{O}}(X,Y) \;=\; 4 + \tfrac{1}{|\xi|^2}(-2|\mathcal{H}\phi(X)|^2 + 4\langle\phi(X),Y\rangle^2)$$

It is possible now to see quite a broad range of the quaternionic Kähler orbifolds have positive sectional curvature.

Consider the circle action on S^{11} given by $z\circ(q_1,q_2,q_3) \equiv (z^{p_1}q_1, z^{p_2}q_2, z^{p_3}q_3)$. The Killing field is then $\xi = (ip_1q_1, ip_2q_2, ip_3q_3)$. In $\mathbb{H}^3$ the covariant derivative computed against the horizontal tangent vector $X = (x_1,x_2,x_3)$ is just $(ip_1x_1ip_2x_2, ip_3x_3)$. Since X is orthogonal to ξ, we immediately see the covariant derivative is orthogonal to (q_1,q_2,q_3). Hence $\phi(X) = (ip_1x_1ip_2x_2, ip_3x_3)$.

For now let us make a very crude estimate $|\phi(X)|^2 < \max\{p_i\}^2$, $|\xi|^2 > \min\{p_i\}^2$. Requiring $\sqrt{2}\min\{p_i\} > \max\{p_i\}$ will ensure

$$4 - \frac{2}{|\xi|^2}|\mathcal{H}\phi(X)|^2 > 0$$

and the curvature is positive.

Let us digress. The essential comparison is

$$4 - \frac{2}{|\xi|^2}|\mathcal{H}\phi(X)|^2 > 0$$

Note that this is only dependent on the point $p \in N$ and independent of the choice of horizontal unit tangent vector X. The requirement that $p \in N$ is just that p and ξ are orthogonal in the quaternionic Hermitian sense. The requirement that X is horizontal is just that the X is quaternionic Hermitian orthogonal to both p and ξ. This is quite a remarkable situation, unfortunately we were unable to complete this discussion by the defence date, promising though it is.

We now turn our attention to an Sasakian reduction in the case where ξ is Sasakian. Note that on the sphere our free circle action generates an associated characteristic vector field ξ defining a Sasakian structure on S^{11}. Note also the 3-Sasakian structure on S^{11} gives rise to a free action of Sp(1) and these two actions commute. This Sp(1) action acts by Sasakian isometries on S^{11} so we may preform a Sasakian reduction [GO]. The level set of the moment map of this reduction coincides with N by definition. Further quotienting N by the action of Sp(1) induces a Sasakian structure on the quotient $N/\mathrm{Sp}(1)$, where the characteristic field is given by the projection of the restriction of ξ. This now induces a Kähler structure on the orbifold of leaves, which coincides with $\mathcal{O}$, since the two actions commute.

So we make the observation that the base carries a metric which is simultaneously quaternionic Kähler, Kähler and of positive sectional curvature.

We now consider the canonical variation of a quasi-regular Sasakian manifold.

$$K(E,F) = \eta(V)^2|X|^2 + \eta(U)^2|Y|^2 - 2\eta(U)\eta(V)\langle X,Y\rangle + K(X,Y)$$

$$
\begin{aligned}
K(X,Y) &= K^*(X_*,Y_*) - 3|A_X Y|^2 \\
&= K^*(X_*,Y_*) - 3\langle \phi(X), Y\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
S_E F &= A_X V + A_Y U \\
&= \eta(V)\phi(X) + \eta(U)\phi(Y)
\end{aligned}
$$

$$
S_E E = 2\eta(U)\phi(X) \quad S_F F = 2\eta(V)\phi(Y)
$$

$$
\begin{aligned}
\langle S_E E, S_F F\rangle - |S_E F|^2 &= 4\langle \eta(U)\phi(X), \eta(V)\phi(Y)\rangle - |\eta(V)\phi(X) + \eta(U)\phi(Y)|^2 \\
&= -\eta(V)^2|X|^2 - \eta(U)^2|Y|^2 + 2\eta(U)\eta(V)\langle X, Y\rangle
\end{aligned}
$$

Substituting all of these into the canonical variation formula and collecting like terms we get

$$
\begin{aligned}
K_t(E,F) &= t^2(\eta(V)^2|X|^2 + \eta(U)^2|Y|^2 - 2\eta(U)\eta(V)\langle X,Y\rangle) \\
&\quad + K^*(X_*,Y_*) - 3t\langle \phi(X), Y\rangle^2
\end{aligned}
$$

Manifestly if $t < \frac{L}{3}$ then the above expression is positive unless of course E, F are linearly dependent in which case it is 0.

In particular we can conclude the Sasakian manifold $N/\mathrm{Sp}(1)$ has a canonical variation metric of positive sectional curvature over $\mathcal{O}$. This is hardly surprising as we merely are considering $\mathrm{S}^5 \to \mathbb{C}P^2$.

Concluding remarks

In their paper Boyer, Galicki and Mann [BGM] show that their examples are diffeomorphic to certain Eschenburg biquotients [E2].

To be precise the 3-Sasakian manifold $S(p_1, p_2, p_3)$ is defined to the the 3-Sasakian hyperquotient reduction of the circle action

$$z \circ (q_1, q_2, q_3) \equiv (z^{p_1} q_1, z^{p_2} q_2, z^{p_3} q_3)$$

on $\mathrm{S}^{11} \subset \mathbb{H}^3$.

Boyer, Galicki and Mann [BGM] show for positive $\{p_i\}$ that this space is diffeomorphic to the manifold attained by quotient by the circle action

$$z \circ A \equiv \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) A \mathrm{diag}(1, 1, z^{p_1+p_2+p_3})^{-1}$$

on SU(3).

Eschenburg [E2] shows using techniques from the theory of Lie algebras that a free circle action on SU(3) where one takes a suitable left invariant metric, induces positive sectional curvature on the quotient of the action

$$z \circ A \equiv \mathrm{diag}(z^{a_1}, z^{a_2}, z^{a_3}) A \mathrm{diag}(z^{b_1}, z^{b_2}, z^{b_3})^{-1}$$

provided that all b_j lie outside the interval $[\min\{a_i\}, \max\{a_i\}]$.

Note that if all p_i are positive then $S(p_1, p_2, p_3)$ has an Eschenburg metric of positive sectional curvature. Boyer, Galicki and Mann [BGM] show the Weyl 3-Sasakian isometries on S^{11} induce isometries between spaces with interchanged signs in p_i, so in fact all of their examples actually have positive curvature with an Eschenburg metric. So we can discover no new spaces of positive curvature among their examples.

We would now like to make some special remarks about $S(p_1, p_2, p_3)$ when $p_3 = p_1 + p_2$. We must admit these examples fall outside the rough range for which positive sectional curvature with the variation metric is established.

Let us hypothetically assume techniques of this dissertation induce positive curvature on such $\mathcal{O}$ then on the orbifold Z with a canonical variation metric on the orbifold submersion $Z \to \mathcal{O}$, since $S \to Z$ with the new variation metrics remains a Riemannian orbifold submersion.

Boyer and Galicki [BG2] state that the space $Z(p_1, p_2, p_3)$ is diffeomorphic to the quotient of the torus action

$$(z, w) \circ A \equiv \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) A \mathrm{diag}(w, \bar{w}, z^{p_1+p_2+p_3})^{-1}$$

on SU(3).

Note however that this action of SU(2) is taken on the left instead this is equivalent to

$$(z, w) \circ A \equiv \mathrm{diag}(w z^{p_1}, \bar{w} z^{p_2}, z^{p_3}) A \mathrm{diag}(1, 1, z^{p_1+p_2+p_3})^{-1}$$

Set $a = wz^{p_1}, b = \bar{w}z^{p_2}$ then $ab = z^{p_3}$ and $(ab)^2 = z^{p_1+p_2+p_3}$ This action is equivalent to the torus action

$$(a, b) \circ A \equiv \mathrm{diag}(a, b, ab)A\mathrm{diag}(1, 1, (ab)^2)^{-1}$$

This torus action is free and gives the twisted flag manifold of Eschenburg [E3-4],[Z]. The Eschenburg space $S(p_1, p_2, p_3)$ occurs as a circle bundle over it.

Similarly, $\mathcal{O}(p_1, p_2, p_3)$ is diffeomorphic to the quotient of the action of $\mathrm{U}(1) \times \mathrm{SU}(2)$ by

$$(z, B) \circ A \equiv \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3})A\mathrm{diag}(1, 1, z^{p_1+p_2+p_3})^{-1} \begin{pmatrix} A & \\ & 1 \end{pmatrix}^{-1}$$

If this action is allowed to pass from right to left the action is converted to a free action of $\mathrm{U}(2)$

$$C \circ A \equiv \begin{pmatrix} C & \\ & \mathrm{det}C \end{pmatrix} A\mathrm{diag}(1, 1, \mathrm{det}C^2)^{-1}$$

whose quotient is diffeomorphic to $\mathbb{C}P^2$. Hence $\mathcal{O}(p_1, p_2, p_3)$ is converted to $\mathbb{C}P^2$.

The spaces $M(p_1, p_2, p_3) = N(p_1, p_2, p_3)/\mathrm{Sp}(1)$ for positive p_1, p_2, p_3 are all diffeomorphic to S^5 [BGM]. The action on $\mathrm{SU}(3)$ is simply multiplication by $\mathrm{SU}(2)$ on the right and acting on the left leaves the quotient as S^5.

The physicists have a terminology for these pairs of fibrations in terms of their Kazula-Klein theories. The bases are said to be spontaneous compactifications (the idea being that crushing the fibres gives the base and crushing along distinct fibrations yields different bases).

Ziller [Z] has raised the question whether the twisted flag manifold has symplectic forms corresponding to the Euler class represented the Eschenburg spaces as a circle bundles in the hope for such or similar metrics to be deformable to metrics of positive sectional curvature.

This and related issues are currently being studied by LeBrun's student Răsdeaconu using techniques from algebraic geometry.

Of particular interest to Ziller was to induce natural metrics of positive sectional curvature on the Eschenburg spaces which are not of positive curvature with Eschenburg's natural metric by thinking of them as circle bundles over the twisted flag manifold.

It would be interesting to establish a link between these issues on the twisted flag manifold and the twistor spaces above.

Each twistor space has a natural Kähler-Einstein metric. One might hope the spontaneous compactification yields another metric on the Eschenburg space associated to a companion theory for the twisted flag manifold over $\mathbb{C}P^2$.

Certainly in dimension 4 quaternionic Kähler orbifolds are particularly good candidates to have metrics of positive sectional curvature and should the Konishi bundles over them be smooth manifolds, i.e. resolve the singularities, one has found 7-manifolds of positive sectional curvature. This provides further reason for the ample supply of metrics of positive sectional curvature in dimension 7.

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