Normal Subgroups of the Symplectomorphism Group

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Abstract of the Dissertation

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This thesis considers the group of symplectomorphisms of 2n-dimensional Euclidean space, a subgroup of the group of volume-preserving diffeomorphisms. Symplectomorphisms in this group can be generated by time-dependent Hamiltonian functions and, in the thesis, sub-collections of Hamiltonian functions are investigated as sources of normal subgroups of symplectomorphisms. An extension of the Hofer norm is introduced, and is used to show that some of these subgroups are proper subgroups. As the main result, this extended Hofer norm is used to show that another, rather unusual subgroup is also a proper subgroup. The result is interesting because a similarly-constructed subgroup of the group of volume-preserving diffeomorphisms has been shown to not be a proper subgroup in the cases where n is greater than or equal to 2. So the result distinguishes the structure of the group of symplectomorphisms from that of the group of volume-preserving diffeomorphisms.

To my parents

Contents

List of Figures

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There are many people that I wish to thank as I finish this degree; I will mention just a few here.

It was Greg Wetzel, over twenty years ago, who first inspired me to make teaching my goal. That goal, and Greg's friendship, have been constants in my life ever since.

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Chapter 1

Hamiltonian Symplectomorphisms and subgroups of $Symp\left(\mathbb{R}^{2n}, \omega_0\right)$

1.1 Introduction and Notation

In this paper, we discuss the *symplectomorphisms of* $\mathbb{R}^{2n}$, denoted $Symp\left(\mathbb{R}^{2n}\right)$. This set is the subset of the set *diffeomorphisms*, $Diff\left(\mathbb{R}^{2n}\right)$, consisting of those that preserve the *standard symplectic form*, $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. That is, $\phi \in Diff\left(\mathbb{R}^{2n}\right)$ such that $\phi^*\omega_0 = \omega_0$. Since ω_0^n is a volume form, any such ϕ will be an element of the set of *volume-preserving diffeomorphisms*, $Diff_{vol}\left(\mathbb{R}^{2n}\right)$, as well. With the operation of composition, each of the three sets we have mentioned is a group, and we have the sequence of subgroups, $Symp\left(\mathbb{R}^{2n}, \omega_0\right) < Diff_{vol}\left(\mathbb{R}^{2n}\right) < Diff\left(\mathbb{R}^{2n}\right)$. These are topological groups, with the compact-open topology. We begin by considering symplectomorphisms *generated by* functions.

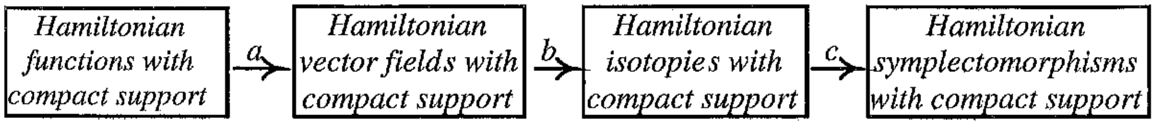

Figure 1.1: generation of symplectomorphisms from Hamiltonian functions

1.2 Hamiltonian Symplectomorphisms of Compact Support

Let C be the vector space of all compactly supported smooth maps $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$. C is called the set of *compactly supported (time dependent) Hamiltonian functions.* Such a function *generates* a *Hamiltonian Symplectomorphism* via the sequence shown in Fig. 1.1 and whose maps and spaces are explained below.

map a: For a Hamiltonian function H, we define the *associated Hamiltonian vector field*, X_t^H, by the equation $\iota\left(X_t^H\right)\omega_0 = d\left(H_t\right)$. That X_t^H is well-defined depends on the non-degeneracy of ω_0. Note that if H is time-dependent, then X_t^H will be, as well. Such an X_t^H will automatically be a *symplectic vector field* (meaning simply that $\iota\left(X_t^H\right)\omega_0$ is closed). X_t^H is sometimes called the *symplectic gradient vector field* of H_t, denoted $X_t^H = sgrad\left(H_t\right)$.

map b: We will distinguish between the statement *H is compactly-supported* and *H_t is compactly supported at each $t \in [0,1]$*, which is a weaker statement, and to which we will return later in this chapter. Since H is compactly supported, X_t^H will be, also. Therefore, there exists a unique family of diffeomorphisms, h_t for $t \in [0,1]$, with velocity vector field $\dot{h}_t$, that solves the first-order initial value problem

$$\begin{cases} h_0 = id \\ \dot{h}_t = X_t^H \circ h_t. \end{cases}$$

This family is a smooth path in the compact-open topology on $Diff(\mathbb{R}^{2n})$, and is called the *Hamiltonian isotopy associated to the function H*. We say that the function H *generates* the isotopy h_t. For each $t \in [0,1]$, it turns out that $h_t \in Symp(\mathbb{R}^{2n})$ [2], so h_t is also called a *Symplectic isotopy*. Yet another name for h_t is the *symplectic gradient flow* of H.

map c: When h_t is evaluated at time $t = 1$, the result is a symplectomorphism, h_1, which is often denoted ψ^H as a reminder that it was obtained from H in the manner described above. We say that ψ^H is the *Hamiltonian symplectomorphism generated by H*, or *associated to H*. We could write ψ^C for the set of all symplectomorphisms generated by compactly supported Hamiltonian functions. A more common symbol for ψ^C is $Ham^c(\mathbb{R}^{2n})$; it is a subset of $Symp^c(\mathbb{R}^{2n})$.

An element of $Ham^c(\mathbb{R}^{2n})$ is understood to be the endpoint, the time $t = 1$ value, of a Hamiltonian isotopy, such as h_t. As mentioned above, at the intermediate times, $0 < t < 1$, h_t takes values in $Symp^c(\mathbb{R}^{2n})$. But closer inspection shows that simply by re-scaling the time variable and the height, any intermediate value, say h_λ, with $0 < \lambda < 1$, can be re-cast as the endpoint of an isotopy of its own. That is, if H_t generates isotopy h_t with endpoint h_1, then the function $t \mapsto \lambda H_{\lambda t}$ generates the isotopy $t \mapsto h_{\lambda t}$, which has $time = 1$ value h_λ. So any compactly supported Hamiltonian isotopy is actually a path in $Ham^c(\mathbb{R}^{2n})$, starting at the identity. Hence, $Ham^c(\mathbb{R}^{2n})$ is path-connected. Note that although we have seen that any compactly supported Hamiltonian function generates a path in $Ham^c(\mathbb{R}^{2n})$, it is not immediately clear that the converse is true, that any path in $Ham^c(\mathbb{R}^{2n})$ can be realized as the isotopy of some compactly supported Hamiltonian. But this is in fact the case. [2]

That $Ham^c(\mathbb{R}^{2n})$ is a normal subgroup of $Symp^c(\mathbb{R}^{2n})$ follows from the following observations: Suppose that f and g are any two elements of $Ham^c(\mathbb{R}^{2n})$, generated by Hamiltonian functions $F \in C$ and $G \in C$. Then the symplectomorphisms f^{-1} and $g \circ f$ are generated by the functions $-F_t \circ f_t$ and $G_t + F_t \circ g_t^{-1}$, both of which also have compact support. This shows that $Ham^c(\mathbb{R}^{2n})$ is a subgroup. Next, note that for any $\phi \in Symp(\mathbb{R}^{2n})$, the symplectomorphism $\phi^{-1} \circ f \circ \phi$ is generated by the function $F_t \circ \phi$, whose support is compact. This shows that the subgroup is normal.

Summarizing this section, we have $Ham^c(\mathbb{R}^{2n}) \triangleleft Symp_0^c(\mathbb{R}^{2n}) \triangleleft Symp(\mathbb{R}^{2n})$.

1.3 Hamiltonian Symplectomorphisms of Arbitrary Support

Beginning as we did above, we could omit the restriction of compact support and consider the vector space of all smooth maps $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$. (Smooth in the compact-open C^∞ topology.) But if a function $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ has non-compact support, then a solution to the initial value problem

$$\begin{cases} h_0 = id \\ \dot{h}_t = X_t^F \circ h_t \end{cases}$$

might not exist for the whole time interval $t \in [0,1]$. Example 1 in section 1.4 gives a such a function. Only if a solution to the initial value problem does exist for the whole time interval do we call the function a *Hamiltonian function.* We denote by $\mathcal{H}$ the set of Hamiltonian functions of arbitrary support. It is worth noting that, while the set C is a vector space, the set $\mathcal{H}$ is not. Example 2 in section 1.4 gives an example of two functions, $G, H \in \mathcal{H}$, such that $G + H \notin \mathcal{H}$.

For the Hamiltonian functions of arbitrary support, then, we have the same generation of vector fields, isotopies, and symplectomorphisms as in the compact case. We will have a diagram similar to the one shown in Fig. 1.1, depicting the generation of Hamiltonian symplectomorphisms from functions, but with the qualification *compact support* omitted in each box.

We denote by $\psi^{\mathcal{H}}$ the set of associated Hamiltonian symplectomorphisms, but will also use the more common symbol, $Ham(\mathbb{R}^{2n})$. It shown to be a normal subgroup of $Symp(\mathbb{R}^{2n})$ in exactly the same way that $Ham^c(\mathbb{R}^{2n})$ was shown to be, above. But in fact it is straightforward to show that $Ham(\mathbb{R}^{2n}) = Symp(\mathbb{R}^{2n})$. So all symplectomorphisms of $\mathbb{R}^{2n}$ are Hamiltonian symplectomorphisms. The proof of this claim is postponed to section 1.4. It is a fairly simple, well-known proof that uses what is sometimes referred to as the *Alexander trick*. We include only to introduce notation and the idea of the trick, in preparation for more sophisticated uses of it in chapters 2 and 4.

1.4 Normal subgroups of $Symp(\mathbb{R}^{2n})$

Much about the structure of $Symp(\mathbb{R}^{2n})$ is unknown. Here we will explore one aspect: the existence of non-trivial normal subgroups. It might be expected, since $Symp(\mathbb{R}^{2n})$ is a topological group, that we would be most interested in its *closed* normal subgroups. But, because the compact-open topology gives no control over behavior at infinity, the closure of $Ham^c(\mathbb{R}^{2n})$ in $Symp(\mathbb{R}^{2n})$ is actually all of $Symp(\mathbb{R}^{2n})$. So, we will be interested in the algebraic structure of $Symp(\mathbb{R}^{2n})$ as a discrete group.

In the case of a compact symplectic manifold (M, ω) without boundary, we know from Banyaga [1] that $Ham(M, \omega)$ is *simple*, that is, has no non-trivial normal subgroups. Less is known for non-compact spaces such as $(\mathbb{R}^{2n}, \omega_0)$. We have already seen that the subgroup $Ham^c(\mathbb{R}^{2n}, \omega_0)$ is normal, and clearly non-trivial. Again from Banyaga, we know that a simple subgroup is found within $Ham^c(\mathbb{R}^{2n})$: the kernel of the *Calabi homomorphism*. Aside from $Ham^c(\mathbb{R}^{2n})$ and its simple subgroup, however, it is not clear where one might look for non-trivial normal subgroups of $Symp(\mathbb{R}^{2n})$.

One idea suggests itself when we look again at $Ham^c(\mathbb{R}^{2n})$. That subgroup could be thought of as being obtained from the larger group $Symp(\mathbb{R}^{2n})$ by imposing a restriction on the set of allowed Hamiltonian functions, in this case the restriction being that they be compactly supported. What other restrictions, milder than requiring compact support, might we place on the set of Hamiltonian functions and obtain, as a result, a non-trivial normal subgroup of $Symp(\mathbb{R}^{2n})$ that contains $Ham^c(\mathbb{R}^{2n})$?

One milder restriction immediately comes to mind: instead of requiring that the function be compactly supported, we could require only that it be compactly supported at each time $t \in [0,1]$. We must be careful how we pose our restriction, though. The set C was defined as the vector space of all compactly supported smooth maps $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$. Any such function generates a symplectomorphism.

If we merely defined a new set of functions as the set of smooth maps $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ that are compactly supported at each time $t \in [0,1]$, then our set would be too large: it would contain functions that don't generate symplectomorphisms. These would not be called Hamiltonian functions, and are of no interest to us. In example 3 in section 1.4 of this chapter, a smooth function $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ is presented that has compact support at each time $t \in [0,1]$ and yet does not generate a symplectomorphism.

So, we want to only consider functions that are in $\mathcal{H}$ and that have compact support at each time. Let us denote by $\mathcal{H}_C$ the subset

$$\{H \in \mathcal{H} : support(H) \text{ is compact at each } t \in [0,1]\},$$

and denote by $\psi^{\mathcal{H}_C}$ the associated set of symplectomorphisms generated by these functions. One can quickly prove that $\psi^{\mathcal{H}_C}$ is a normal subgroup of $Symp(\mathbb{R}^{2n})$ by the same method that was used to show that $Ham^c(\mathbb{R}^{2n})$ is a normal subgroup. We consider two symplectomorphisms, f and g, that are generated by Hamiltonian functions $F, G \in \mathcal{H}_C$, and any $\phi \in Symp(\mathbb{R}^{2n})$. Then the symplectomorphisms f^{-1},

$g \circ f$, and $\phi^{-1} \circ f \circ \phi$ are generated by the functions $-F_t \circ f_t$, $G_t + F_t \circ g_t^{-1}$, and $F_t \circ \phi$ each of whose support is compact at each time. This shows that $\psi^{\mathcal{H}_C}$ is a normal subgroup.

Because of the inclusions $C \subset \mathcal{H}_C \subset \mathcal{H}$, we will have the following sequence of normal subgroups of symplectomorphisms: $Ham^c(\mathbb{R}^{2n}) = \psi^C \triangleleft \psi^{\mathcal{H}_C} \triangleleft \psi^{\mathcal{H}} = Ham(\mathbb{R}^{2n}) = Symp(\mathbb{R}^{2n})$

For now, we postpone the question of which of these subgroups is proper.

Other fairly obvious sub-classes of $\mathcal{H}$ that we should consider are those that are *decaying* at each time, *bounded* at each time, or *uniformly bounded.* These we will denote by $\mathcal{H}_D$, $\mathcal{H}_B$, and $\mathcal{H}_{UB}$:

$$\begin{aligned}
\mathcal{H}_D &= \left\{H \in \mathcal{H} : \forall t \in [0,1], \lim_{R\to\infty} \sup\{|H_t(x)| : \|x\| > R\} = 0\right\} \\
\mathcal{H}_B &= \left\{H \in \mathcal{H} : \forall t \in [0,1], \sup_{x\in\mathbb{R}^{2n}} \{|H_t(x)|\} < \infty\right\} \\
\mathcal{H}_{UB} &= \left\{H \in \mathcal{H} : \sup_{t\in[0,1]} \left\{\sup_{x\in\mathbb{R}^{2n}} |H_t(x)|\right\} < \infty\right\}.
\end{aligned}$$

We denote by $\psi^{\mathcal{H}_D}$, $\psi^{\mathcal{H}_B}$, and $\psi^{\mathcal{H}_{UB}}$ the associated sets of symplectomorphisms generated by these functions. One shows that these sets are in fact normal subgroups of $Symp(\mathbb{R}^{2n})$ in the same way that we showed that $Ham^c(\mathbb{R}^{2n})$ and $\psi^{\mathcal{H}_C}$ are normal subgroups. Note that because we have the sequence of inclusions of sets of functions, $C \subset \mathcal{H}_C \subset \mathcal{H}_D \subset \mathcal{H}_B \subset \mathcal{H}$, we will have this sequence of corresponding normal subgroups:

$$Ham^c\left(\mathbb{R}^{2n}\right) = \psi^C \triangleleft \psi^{\mathcal{H}_C} \triangleleft \psi^{\mathcal{H}_D} \triangleleft \psi^{\mathcal{H}_B} \triangleleft \psi^{\mathcal{H}} = Ham\left(\mathbb{R}^{2n}\right) = Symp\left(\mathbb{R}^{2n}\right).$$

We turn now to the question of which of these inclusions are proper. The requirement that the generating Hamiltonians have compact support at each time $t \in [0,1]$ seems a fairly strong restriction, and one might expect that the resulting normal subgroup, $\psi^{\mathcal{H}_C}$, would be a proper subgroup of $Symp(\mathbb{R}^{2n})$. It is rather surprising, then, that the subgroup is in fact the entire group $Symp(\mathbb{R}^{2n})$. In chapter 2, we will prove the following theorem:

Theorem A: Each f in $Symp(\mathbb{R}^{2n})$ can be generated by a Hamiltonian function H_t with the property that $support(H_t)$ is compact for each $t \in [0,1)$ and H_1 is zero.

With that, our intriguing sequence of subgroups collapses to something far less interesting:

$$Ham^c(\mathbb{R}^{2n}) = \psi^C \lhd \psi^{\mathcal{H}_C} = \psi^{\mathcal{H}_D} = \psi^{\mathcal{H}_B} = \psi^{\mathcal{H}} = Ham(\mathbb{R}^{2n}) = Symp(\mathbb{R}^{2n})$$

The sequence contains no new non-trivial normal subgroups of $Symp(\mathbb{R}^{2n})$.

We have not considered the other sequence of inclusions of sets of functions, $C \subset \mathcal{H}_{UB} \subset \mathcal{H}$. As with the other sets of symplectomorphisms, we can show that $\psi^{\mathcal{H}_{UB}}$ is a normal subgroup of $Symp(\mathbb{R}^{2n})$, so that we have the sequence $Ham^c(\mathbb{R}^{2n}) \lhd \psi^{\mathcal{H}_{UB}} \lhd Symp(\mathbb{R}^{2n})$. Clearly $Ham^c(\mathbb{R}^{2n})$ is a proper subgroup of $\psi^{\mathcal{H}_{UB}}$, but is $\psi^{\mathcal{H}_{UB}}$ a proper subgroup of $Symp(\mathbb{R}^{2n})$? The answer to this question turns out to be yes, but its proof uses notation and mathematical tools developed in Chapter 3. In that chapter, we will introduce the *extended Hofer infinity norm*, $\widehat{E_\infty}$. The chapter ends with the following *corollary*, which is what we need to prove that $\psi^{\mathcal{H}_{UB}}$ is a proper subgroup of $Symp(\mathbb{R}^{2n})$:

Corollary to the energy-capacity inequality: If a subset $\Omega \subset Symp(\mathbb{R}^{2n})$ has the property that $\widehat{E_\infty}(\psi)$ is finite for all $\psi \in \Omega$, then Ω is a proper subset of $Symp(\mathbb{R}^{2n})$, for Ω will contain no rotations.

The set of symplectomorphisms for which $\widehat{E_\infty}(\psi)$ is finite are precisely those that can be generated by uniformly-bounded Hamiltonian functions. That is to say, elements of $\psi^{\mathcal{H}_{UB}}$.

In Chapter 4, this corollary will be used in the proof that another normal subgroup of $Symp(\mathbb{R}^{2n})$ is in fact a proper normal subgroup. The description of that subgroup, which is quite a bit more complicated than those we have seen, is as follows.

> Let $U = \bigcup_{i=1}^{\infty} f_i(B^{2n}(R)) \subset \mathbb{R}^{2n}$ be any disjoint union of symplectic balls of radius $R < \frac{1}{2}$. (By symplectic ball of radius R , we mean the image, $f(B^{2n}(R))$, of a symplectic embedding, $f : B^{2n}(R) \to \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the closed ball of radius R.) Denote by $Symp_U(\mathbb{R}^{2n})$ the set of symplectomorphisms of $\mathbb{R}^{2n}$ that can be generated by Hamiltonian functions with support contained in U. Then $Symp_U(\mathbb{R}^{2n})$ is a subgroup of $Symp(\mathbb{R}^{2n})$, but it is not a normal subgroup: conjugation of an element of $Symp_U(\mathbb{R}^{2n})$ with a translation can produce an element of $Symp(\mathbb{R}^{2n})$ that is not supported in U. Define $G_U \lhd Symp(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $Symp(\mathbb{R}^{2n})$ containing $Symp_U(\mathbb{R}^{2n})$. That is, G_U contains $Symp_U(\mathbb{R}^{2n})$, is closed under conjugation by elements of $Symp(\mathbb{R}^{2n})$, and is closed under composition.

The theorem that we will prove is,

Theorem B (Chapter 4): G_U is a proper subgroup of $Symp(\mathbb{R}^{2n})$.

In the proof of that theorem, we will show that if $\psi \in G_U$, then $\widehat{E_\infty}(\psi)$ is finite. Therefore, $G_U \lhd \psi^{\mathcal{H}_{UB}}$, and hence, G_U must be a proper subgroup of $Symp(\mathbb{R}^{2n})$.

Note that it is not clear whether G_U is a proper subgroup of $\psi^{\mathcal{H}_{UB}}$ and, if it is, how it compares to $Ham^c(\mathbb{R}^{2n})$, which is also a proper normal subgroup of $\psi^{\mathcal{H}_{UB}}$. It is clear that G_U and $Ham^c(\mathbb{R}^{2n})$ have non-trivial elements in common: symplectomorphisms supported in a single one of the symplectic balls of the union U are compactly supported, so they are in both G_U and $Ham^c(\mathbb{R}^{2n})$. And, G_U contains elements of non-compact support that cannot be in $Ham^c(\mathbb{R}^{2n})$. But it is also possible that $Ham^c(\mathbb{R}^{2n})$ could contain elements that are not in G_U.

Previous examples of normal subgroups of $Symp(\mathbb{R}^{2n})$ in this chapter arose through fairly natural restrictions on the set of generating Hamiltonian functions, whereas this last example, G_U, seems rather random. But remember that $Symp(\mathbb{R}^{2n})$ is a subgroup of the group $Diff_{vol}(\mathbb{R}^{2n})$ of volume-preserving diffeomorphisms of $\mathbb{R}^{2n}$. Our subgroup G_U of $Symp(\mathbb{R}^{2n})$ becomes more interesting, when we compare it to a similarly-constructed subgroup of $Diff_{vol}(\mathbb{R}^{2n})$. That is, for the set U, described above, let $Diff_{vol}(U) \subset Diff_{vol}(\mathbb{R}^{2n})$ be the collection of volume-preserving diffeomorphisms of $\mathbb{R}^{2n}$ that are supported in U. Note that $Diff_{vol}(U)$ is a non-normal subgroup of $Diff_{vol}(\mathbb{R}^{2n})$. As above, define $G_{Diff_{vol}(U)} \triangleleft Diff_{vol}(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $Diff_{vol}(\mathbb{R}^{2n})$ containing $Diff_{vol}(U)$. McDuff [5] showed that for $n \geq 2$, $G_{Diff_{vol}(U)} = Diff_{vol}(\mathbb{R}^{2n})$, so $G_{Diff_{vol}(U)}$ is *not* a proper subgroup. So *Theorem B* distinguishes the structure of $Symp(\mathbb{R}^{2n})$ from that of $Diff_{vol}(\mathbb{R}^{2n})$.

1.5 Examples and Proofs

1.5.1 A non-Hamiltonian smooth function $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$

Our example is a time independent function, $F : \mathbb{R}^2 \to \mathbb{R}$ with these properties:

- F is supported in the set $\mathbb{R} \times (0,3)$

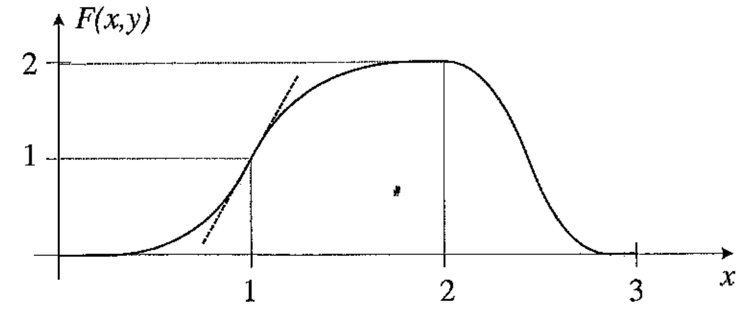

Figure 1.2: A non-Hamiltonian smooth function

- F has height 1 on the set $\mathbb{R} \times \{1\}$
- F has height 2 on the set $\mathbb{R} \times \{2\}$
- The height of F decreases monotonically as $|y-2|$ increases.
- The maximum gradient occurs at points of the form $(x, 1)$. At those points, the partial in the y direction is $\left.\frac{\partial F(x,y)}{\partial y}\right|_{(x,1)} = \frac{\pi}{2}\left(1+x^2\right).$

Figure 1.2 shows a cross section of the graph of F.

Associated to the function F is the time-independent vector field $\mathrm{X}^F = sgrad\,(F)$. Consider the solutions f_t to the initial value problem

$$\begin{cases} f_0 = id \\ \dot{f}_t = \mathrm{X}^F \circ f_t. \end{cases}$$

Let p be a point of the form $p = (a, 1)$, and let $(x_p(t), y_p(t)) = f_t(p)$ be the coordinates of the point as it moves under the influence of f_t. Note that p will move on the set $\mathbb{R} \times \{1\}$, because that is a level set of the function F, so $y_p(t)$ will be constant, with value 1. The initial value problem simplifies to a the one-variable problem

$$\begin{cases} x_p(0) = a \\ \frac{dx_p(t)}{dt} = \left.\frac{\partial F(x,y)}{\partial y}\right|_{(x_p(t),1)} = \frac{\pi}{2}\left(1 + (x_p(t))^2\right). \end{cases}$$

This has solution $x_p(t) = \tan\left(\frac{t\pi}{2} + \arctan(a)\right)$. If p is the point $(0,1)$, then $x_p(t) = \tan\left(\frac{t\pi}{2}\right)$, which is defined for $t \in [0,1)$, but not for $t = 1$. So f_t is not defined on the interval $t \in [0,1]$, and F does not generate a symplectomorphism; it is not *Hamiltonian.* Notice that in fact, f_t is not defined on *any* time interval $t \in [0,b]$, where $0 < b \leqslant 1$. To see this, let p be the point $\left(\tan\left(\frac{\pi}{2}(1-b)\right), 1\right)$, and consider its evolution, of as it moves under the influence of f_t. At time t, its x coordinate will be $x_p(t) = \tan\left(\frac{\pi}{2}(1+t-b)\right)$, which is defined for $t \in [0,b)$, but not for $t = b$.

1.5.2 Two Hamiltonian functions whose sum is not Hamiltonian

We will construct two Hamiltonian functions, G and H, whose sum is the function F from the previous example. Let $\{\chi_i\}_{i=1}^{\infty}$ be a partition of unity on $\mathbb{R}$ subordinate to the cover $\{U_i\}_{i=1}^{\infty}$, where $U_i = (i-1, i+1)$, and define $\phi_i(x,y) = \chi_i(x) \cdot F(x,y)$. Because ϕ_i is compactly supported, time-independent, and smooth, it is Hamiltonian. Notice that if $i \neq j$, then the support of ϕ_{2i} and ϕ_{2j} are disjoint. So let $G(x,y) = \sum_{i=-\infty}^{\infty} \phi_{2i}$. Then G is Hamiltonian. Similarly, let $H(x,y) = \sum_{i=-\infty}^{\infty} \phi_{2i+1}$. Then H is Hamiltonian, as well, but the sum of G and H is F, which is not Hamiltonian.

1.5.3 A smooth function $G : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$, with compact support at each time $t \in [0,1]$, that is not Hamiltonian.

In this example, let $\chi_t(x)$ be a moving cutoff function whose height, at each time, is identically 1 on the interval $x \in [\tan(t) - 1, \tan(t) + 1]$ and which is supported in the interval $x \in (\tan(t) - 2, \tan(t) + 2)$. Define $G : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ by

$$G_t(x,y) = \begin{cases} \chi_t(x) \cdot F_t(x,y) & \text{for } t \in [0,1) \\ 0 & \text{for } t = 1, \end{cases}$$

where F is the function from example 1. Then G is smooth in the compact-open C^∞ topology. Associated to the function G is the time-dependent vector field $X_t^G = sgrad(G_t)$.

For times $t \in [0,1)$, consider the solution g_t of the initial value problem

$$\begin{cases} g_0 = id \\ \dot{g}_t = X_t^G \circ g_t \end{cases}$$

Let p be the point $(0,1)$, and consider the evolution of the point p under the influence of g_t. Notice that the moving cutoff function, $\chi_t(x)$, defined above, is centered on the point $x(t) = \tan\left(\frac{t\pi}{2}\right)$. But this is exactly the x coordinate with which the same point $p = (0,1)$ moved in example 1, when under the influence of f_t. Because of the way we have constructed the time-dependent function G, at each time $t \in [0,1)$, in a neighborhood of the point $(x,y) = \left(\tan\left(\frac{t\pi}{2}\right), 1\right)$, it will have exactly the same shape as the function F. So, the point p will move in exactly the same way under the influence of g_t as it would under the influence of f_t. That is to say, its x coordinate will go to infinity, as time t approaches 1. So, there is no way that g_t can extend to an isotopy defined on the entire time interval $[0,1]$, and hence, G is not Hamiltonian.

It is worth articulating what's "wrong" with the function G, because in Chapter 2, we will construct a function H that, like G, has support that is compact at each time $t \in [0,1]$ and moves to infinity, in space, as time t approaches 1. Unlike G, however, the function H, will be *Hamiltonian.* What bad behavior of G will we avoid when constructing H?

Think again of the point $p = (0,1)$, discussed above. It starts moving, at time $t = 0$, because it sits at a place where the *gradient* of G is non-zero. One could think of the function G as a wave; the point p, a surfer. Note, however, that p moves not "down", which would be in the direction of the *negative gradient*, but rather, along the *level set* - the direction of the *symplectic gradient.* Since such a "symplectic surfer" never moves down, off the wave: the surfer will keep moving as long as the wave stays up. Only if the wave falls (or levels off) does the surfer stop moving. Since the wave - our function G -stays up and moves to infinity, in space, the surfer goes with it. We could call such a function *gnarly.* Fun as it may be for the surfer, it is of no interest to us, because it is not Hamiltonian: it does not give rise to a map that sends each point of $\mathbb{R}^2$ to some well-defined destination in $\mathbb{R}^2$.

In Chapter 2, we will be careful to build a function H that could be thought of as a succession of waves, rising in rings from the level sea, radiating outward, then falling. Successive waves will rise from spots farther and farther out from a particular spot, and they will move to infinity. But, every point - every symplectic surfer - will at some time be picked up, moved around, and let down by some wave, and will not get picked up by any subsequent rising wave. Since every point gets moved from its initial location to a well-defined final destination sometime in the interval $t \in [0, 1)$, the result, at time $t = 1$, is a well-defined symplectomorphism.

1.5.4 Proof that $Ham\left(\mathbb{R}^{2n}\right) = Symp\left(\mathbb{R}^{2n}\right)$

Let f be an arbitrary element of $Symp\left(\mathbb{R}^{2n}\right)$. With no loss of generality, we may assume that $f(0) = 0$.

We start by constructing a symplectic isotopy from a linear map to f, using the *Alexander trick.* Let $m_c : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the multiplication by the scalar c. Then for each $t \in (0, 1]$, the map $m_{\frac{1}{t}} \circ f \circ m_t$ is an element of $Symp\left(\mathbb{R}^{2n}\right)$. Note that when this symplectomorphism acts on an element $x \in \mathbb{R}^{2n}$, the result is just the difference quotient

$$m_{\frac{1}{t}} \circ f \circ m_t(x) = \frac{f(tx)}{t} = \frac{f(tx) - f(0)}{t}.$$

Because f is differentiable, the $t \to 0$ limit exists in $\mathbb{R}^{2n}$:

$$\lim_{t \to 0} m_{\frac{1}{t}} \circ \tau_{-f(0)} \circ f \circ m_t(x) = \lim_{t \to 0} \frac{f(tx) - f(0)}{t} = L_{(df)_0}(x).$$

Here, $L_{(df)_0}$ is the linear operator obtained by left multiplication by the matrix $(df)_0 \in Sp(2n)$, and is called the linearization of f at zero.

If we define a path β_t in $Symp(\mathbb{R}^{2n})$ as

$$\beta_t = \begin{cases} m_{\frac{1}{t}} \circ \tau_{-f(0)} \circ f \circ m_t & \text{if } t \in (0,1] \\ L_{(df)_0} & \text{if } t = 0 \end{cases},$$

then β_t is a continuous path in the compact-open topology on $Symp(\mathbb{R}^{2n})$, with β_0 being the linear map $L_{(df)_0}$, and $\beta_1 = f$. The map β_t is the isotopy we promised; it is the method of constructing β_t that is referred to as the Alexander trick.

Now we construct an isotopy from the identity to $L_{(df)_0}$. Because $Sp(2n)$ is path connected, there is a path $\sigma : [0,1] \to Sp(2n)$ with $\sigma_0 = I$ and $\sigma_1 = (df)_0$. Let $\alpha : [0,1] \to Symp(\mathbb{R}^{2n})$ be the map defined by $\alpha_t = L_{\sigma_t}$. Then α is an isotopy with $\alpha_0 = id$ and $\alpha_1 = L_{(df)_0}$.

Concatenating these two paths in time and smoothing, we obtain a symplectic isotopy, $\gamma = \alpha * \beta$, from the identity map to f. From the symplectic isotopy γ_t, we obtain the velocity vector field, $\dot{\gamma}_t$, which will be a time-dependent symplectic vector field. This defines a family of closed 1-forms, $\iota(\dot{\gamma}_t)\,\omega_0$. Since $H^1(\mathbb{R}^{2n}, \mathbb{R}) = 0$, there is a smooth function, $F : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ such that at each $t \in [0,1]$, $\iota(\dot{\gamma}_t)\,\omega_0 = d(F_t)$. We see that F generates f, so $f \in Ham(\mathbb{R}^{2n})$. Also note that since $Ham(\mathbb{R}^{2n})$ is contained in the path-component of the identity in $Symp(\mathbb{R}^{2n})$, which is in turn contained in the connected component of the identity, $Symp_\circ(\mathbb{R}^{2n})$, we conclude that $Ham(\mathbb{R}^{2n}) = Symp_\circ(\mathbb{R}^{2n}) = Symp(\mathbb{R}^{2n})$.

Chapter 2

A surprising fact about $Symp\left(\mathbb{R}^{2n}\right)$

2.1 Introduction and *Theorem A*

In Chapter 1, we denoted by $\mathcal{H}$ the collection of time-dependent Hamiltonian functions on $\mathbb{R}^{2n}$ with arbitrary support; we denoted by $\mathcal{H}_C$ the subset

$$\{H \in \mathcal{H} : support\,(H)\,\text{is compact at each}\; t \in [0,1]\}\,;$$

and we denoted by $\psi^{\mathcal{H}_C}$ the associated set of symplectomorphisms generated by these functions. We saw that the set was in fact a normal subgroup, $\psi^{\mathcal{H}_C} \triangleleft Symp\,(\mathbb{R}^{2n})$, but the question of whether or not this subgroup was proper remained to be answered. In this chapter, we prove that the subgroup is not proper: it is the entire group. The proof exploits the sometimes-overlooked fact that the compact-open topologies used on $Symp\,(\mathbb{R}^{2n})$ and $\mathcal{H}$ allow some unusual paths to qualify as smooth.

More specifically, in $Symp\,(\mathbb{R}^{2n})$, it is relatively easy for a continuous map $g : [0,1) \to Symp\,(\mathbb{R}^{2n})$ to be extendable to the entire time interval $t \in [0,1]$. All that is necessary is that for each $x \in \mathbb{R}^{2n}$, there is some time $t_x < 1$ such that for all times $t > t_x$, $g_t(x) = g_{t_x}(x)$. That is, the point x gets moved around during the time interval $0 \leqslant t \leqslant t_x$, and then stays put for the remainder of the time interval, $t_x \leqslant t \leqslant 1$. If that is the case, then $\lim_{t \to 1} g_t$ exists, and g_1 can be defined as this limit.

So, g_t may be defined on the entire time interval $[0,1]$,even though it may get quite wild as $t \to 1$.

Analogously, in the compact-open C^∞ topology on the function space, for a continuous function $H : \mathbb{R}^{2n} \times [0,1) \to \mathbb{R}$ to be extendable to the entire time interval $t \in [0,1]$, one need only insure that the support of the function go to infinity (in space), as $t \to 1$. If that is the case, $\lim_{t \to 1} H_t$ is the zero function, and H_1 can be defined to be this limit, thus extending H to the entire interval. So, as with g_t, H_t may be defined on the entire time interval $[0,1]$ even though it may get quite wild as $t \to 1$.When we exploit this fact in the proof, we will also be interested in insuring that the resulting function H is a Hamiltonian function, i.e. that it does generate a well-defined symplectomorphism. But this turns out to be possible in the cases of interest here. The key, as was discussed at the end of example 1.4.3, will be to make sure that the function H is constructed so that each point of $\mathbb{R}^{2n}$, as it moves with the symplectic gradient flow of H, eventually, at some time before $t = 1$, leaves the support set of H for good.

Theorem A: Each f in $Symp(\mathbb{R}^{2n})$ can be generated by a Hamiltonian function H with the property that *support* (H_t) is compact for each $t \in [0,1]$ and H_1 is zero.

Proof of Theorem A:

We will prove in *Lemma A1* that there is a sequence of Hamiltonian functions $G_{k,t}$, $k = 1, 2, \cdots$ with these properties:

1. $\forall k, \text{support}\,(G_{k,t})$ is compact for all $t \in [0,1]$.

2. $\forall k, G_{k,0} \equiv G_{k,1} \equiv 0$.

3. There is a sequence of radii, R_k, $k = 1, 2, \cdots$, with $R_k \xrightarrow{k \to \infty} \infty$, such that for all $t \in [0,1]$, $G_{k,t} | B^{2n}(R_k, f(0)) \equiv 0$.

4. $G_{k,t}$ generates a Hamiltonian isotopy $g_{k,t}$ with these properties

   (a) $g_{k,t}\left|B^{2n}\left(R_k, f\left(0\right)\right)\right. = id$ for all $t \in [0,1]$.

   (b) Let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Then $\pi_k \left|\mathrm{B}^{2n}\left(k\right)\right. = f \left|B^{2n}\left(k\right)\right.$.

Notice that property (4a) follows immediately from property (3). Also, because of property (3), the sequence $G_{k,t}$ converges in the compact-open topology to the zero function, as k approaches infinity. And, because of property (2), the functions can be concatenated in time to produce a smooth function. So we define a new function, H_t, for $t \in [0,1]$ by concatenating and re-scaling height and time in the following way. We define the sequence of *dyadic times*, $t_0 = 0$, $t_1 = \frac{1}{2}$, $t_1 = \frac{3}{4}$, $\cdots, t_k = \left(1 - \left(\frac{1}{2}\right)^k\right), \cdots$. Then the k^{th} *interval*, $t_{k-1} \leqslant t \leqslant t_k$, has length $\left(\frac{1}{2}\right)^k$. Define the function $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ piecewise in time by the formula

$$H_t = \begin{cases} 2^k G_{k,2^k(t-t_{k-1})} & \text{for } t_{k-1} \leqslant t \leqslant t_k \\ 0 & \text{for } t = 1. \end{cases}$$

Then H is continuous in the compact-open topology, and for each $t \in [0,1]$, the support of H_t is compact.

Let us now check that because of property (4), H will generate a well-defined symplectic isotopy, which we will call h_t, defined on the time interval $t \in [0,1]$. That is, we check that H is Hamiltonian. Certainly, for any number b, where $0 \leqslant b < 1$. H generates an isotopy that is well-defined on the time interval $t \in [0,b]$. So, the isotopy is well-defined for $t \in [0,1)$. Call this isotopy h_t. We show now that in fact h_t extends to the entire time interval.

Examining the value of h_t at one of the *dyadic* times, we find that because of property (4b) $h_{t_k} = \pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_1$. That is, h_{t_k} agrees with f on the set $B^{2n}\left(k\right)$. Now consider the in-between times, $t_{k-1} < t < t_k$. Because of property (4a), progress of the isotopy h_t during this time interval will not affect points in the ball $B^{2n}\left(R_k, f\left(0\right)\right)$. So every point in $\mathbb{R}^{2n}$ gets moved around by h_t during the interval

$t \in [0,1)$, but each point eventually lands in one of the balls $B^{2n}(R_k, f(0))$. where it remains for the rest of the time interval. For all time beyond t_k, all the activity caused by the isotopy h_t occurs outside of the ball $B^{2n}(R_k, f(0))$. Inside that ball, all points are in the same configuration that they would be in if they had been moved by the symplectomorphism f. Since $R_k \to \infty$. we see that as time $t \to 1$. the isotopy h_t converges in the compact-open topology to f. So in fact, H_t generates an isotopy that is well defined for $t \in [0,1]$, and the *time* $= 1$ value of that isotopy is the symplectomorphism f.

End of proof of Theorem A

2.2 *Lemma A1*

Lemma A1: For any $f \in Symp(\mathbb{R}^{2n})$, there is a sequence of Hamiltonian functions $G_{k,t}$, $k = 1, 2, \cdots$ with these properties:

1. $\forall k$, support $(G_{k,t})$ is compact for all $t \in [0,1]$.
2. $\forall k, G_{k,0} \equiv G_{k,1} \equiv 0$.
3. There is a sequence of radii, R_k, $k = 1, 2, \cdots$, with $R_k \xrightarrow{k \to \infty} \infty$, such that for all $t \in [0,1]$, $G_{k,t} \,|B^{2n}(R_k, f(0)) \equiv 0$.
4. $G_{k,t}$ generates a Hamiltonian isotopy $g_{k,t}$ with these properties
   (a) $g_{k,t} \,|B^{2n}(R_k, f(0)) = id$ for all $t \in [0,1]$.
   (b) Let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Then $\pi_k \,|B^{2n}(k) = f \,|B^{2n}(k)$.

Proof of Lemma A1

step i: Establish the sequence of radii, R_k, $k = 1, 2, \cdots$.

Let $R_k = \sup\{r : B^{2n}(r, f(0)) \subset f(B^{2n}(k))\}$. Notice that $R_k < R_{k+1}$ and that $R_k \to \infty$ as $k \to \infty$.

step ii: Introduce F_t, f_t, S_1, N_1, χ_1, $G_{1,t}$, $g_{1,t}$, and g_1.

Since $f \in Symp(\mathbb{R}^{2n})$, there is a Hamiltonian function F_t that generates f. Without loss of generality, we may assume that $F_0 \equiv 0 \equiv F_1$. Let f_t be the corresponding isotopy, so that $f = f_1$. We consider the evolution of the balls $B^{2n}(1)$ and $B^{2n}(2)$, as they move under the influence of isotopy f_t. In particular, we define the sets $S_1 = \bigcup_{t=0}^{1} f_t(B^{2n}(1))$ and $N_1 = \bigcup_{t=0}^{1} f_t(B^{2n}(2))$. We will think of S_1 as the swath of $\mathbb{R}^{2n}$ that the ball $B^{2n}(1)$ moves through, under the influence of the isotopy f_t, and we will think of N_1 as a larger neighborhood. Using these, we define a smooth cutoff function, χ_1, to be identically one on S_1 and to be zero outside N_1. We then produce a new Hamiltonian function, $G_{1,t} = F_t \cdot \chi_1$, call the corresponding isotopy $g_{1,t}$, and call its $time = 1$ value g_1.

Notice that since G_t agrees with F_t everywhere in the set S_1, the evolution of the ball $B^{2n}(1)$, under the influence of $g_{1,t}$, will be the same as its evolution would have been under the influence of f_t. So $g_1 \left| B^{2n}(1) = f \right| B^{2n}(1)$. Also, since $G_{1,t}$ is supported in N_1 at all times, we conclude that

$$g_1 = \begin{cases} f & \text{on } B^{2n}(1) \\ non - trivial & \text{on } N_1 \\ id & \text{outside } N_1 \end{cases} .$$

step iii: Introduce S_2, N_2, χ_2, $G_{2,t}$, $g_{2,t}$, and g_2

We have produced a Hamiltonian symplectomorphism, g_1,that agrees with f on a ball of radius 1. Now, we will produce an isotopy that will enlarge the region of agreement with f. It will be important that this new isotopy have no effect in the region where g_1 agrees with f, but that the new one corrects some of the disagreement between the two in outer regions. To see that disagreement more clearly, we observe the following behavior of the product $f \circ g_1^{-1}$:

$$f \circ g_1^{-1} = \begin{cases} id & \text{on } f\left(B^{2n}(1)\right) \\ non-trivial & \text{on } N_1 \\ f & \text{outside } N_1. \end{cases}$$

We can say *a priori* that there exists a Hamiltonian function, which we will call $F_{2,t}$, that generates $f \circ g_1^{-1}$. But in fact, we want to require more of $F_{2,t}$ than that it simply generate the desired symplectomorphism. We would like in addition to be able to say that for all $t \in [0, 1]$, $F_{2,t}$ is identically zero on some ball of known radius.

In *Lemma A2*, we will show that there is in fact a Hamiltonian function $F_{2,t}$ that generates $f \circ g_1^{-1}$ and such that for all $t \in [0, 1]$, $F_{2,t}$ is identically zero on the ball $B^{2n}(R_1, f(0))$. We call the corresponding Hamiltonian isotopy $f_{2,t}$, and call its corresponding $time = 1$ value f_2.

There is a problem with $F_{2,t}$, however, in that its support is not compact. So our next step is to produce a cutoff version of $F_{2,t}$, which we will call $G_{2,t}$. We will do this in a manner completely analogous to the way that we produced the Hamiltonian $G_{1,t}$ by multiplying F_t by a cutoff function. More specifically, we define the sets S_2 and N_2 as

$$S_2 = \bigcup_{t=0}^{1} f_{2,t}(N_1) = \bigcup_{t=0}^{1} f_{2,t}\left(\bigcup_{t=0}^{1} f_t\left(B^{2n}(2)\right)\right)$$
$$N_2 = \bigcup_{t=0}^{1} f_{2,t}\left(\bigcup_{t=0}^{1} f_t\left(B^{2n}(3)\right)\right)$$

So S_2 is the swath of of $\mathbb{R}^{2n}$ that the set N_1 moves through, under the influence of the isotopy $f_{2,t}$, and N_2 is a larger neighborhood. Using these, we define a smooth cutoff function, χ_2, to be identically one on S_2 and to be zero outside N_2. We then produce a new Hamiltonian function, $G_{2,t} = F_{2,t} \cdot \chi_2$, call the corresponding isotopy $g_{2,t}$, and call its $time = 1$ value, g_2.

Notice that since $G_{2,t}$ agrees with $F_{2,t}$ everywhere in the set S_2, the evolution of the set N_1, under the influence of $g_{2,t}$, will be the same as its evolution would have been under the influence of $f_{2,t}$. So $g_2\,|N_1 = f_2\,|N_1 = f \circ g_1^{-1}\,|N_1$. Considering the behavior of the composition, $g_2 \circ g_1$, we find

$$\begin{aligned} g_2 \circ g_1\,|B^{2n}(2) &= g_2|\underbrace{g_1\left(B^{2n}(2)\right)}_{\text{contained in} N_1} \circ g_1\,|B^{2n}(2) \\ &\quad \left(\text{note that } g_2\,|N_1 = \left(f \circ g_1^{-1}\right)|N_1\right) \\ &= \left(f \circ g_1^{-1}\right)|g_1\left(B^{2n}(2)\right) \circ g_1\,|B^{2n}(2) \\ &= f \circ g_1^{-1} \circ g_1\,|B^{2n}(2) \\ &= f\,|B^{2n}(2)\,. \end{aligned}$$

As for the discrepancy between $g_2 \circ g_1$ and f, we observe that

$$f \circ (g_2 \circ g_1)^{-1} = \begin{cases} id & \text{on } f\left(B^{2n}(2)\right) \\ non-trivial & \text{on } N_2 \\ f & \text{outside } N_2. \end{cases}$$

step iv: (inductive step)

Let symplectomorphisms, $g_1,\, g_2, \cdots, g_k$ be given, and let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Further suppose that a set of Hamiltonian functions $F_{1,t},\, F_{2,t}, \cdots, F_{k,t}$ has been defined, with corresponding isotopies $f_{1,t},\, f_{2,t}, \cdots, f_{k,t}$, and $time = 1$ values $f_1,\, f_2, \cdots, f_k$. Suppose that the collection of sets $S_1,\, S_2, \cdots, S_k$ and $N_1,\, N_2, \cdots, N_k$, called *swaths* and *neighborhoods*, has been defined in the following way. First, for $j = 1, \cdots, k$, recursively define the operation $sweep_j$: subsets of $\mathbb{R}^{2n} \to$ subsets of $\mathbb{R}^{2n}$ as

$$sweep_j(A) = \begin{cases} \bigcup_{t=0}^{1} f_t(A) & \text{if } j = 1 \\ \bigcup_{t=0}^{1} f_j\left(sweep_{j-1}(A)\right) & \text{if } 2 \leqslant j \leqslant \text{k}. \end{cases}$$

Then the j^{th} *swath* and j^{th}*neighborhood*, S_j and N_j, are defined by the formulas

$$\begin{aligned} S_j &= sweep_j\left(B^{2n}(j)\right) \\ N_j &= sweep_j\left(B^{2n}(j+1)\right). \end{aligned}$$

Further, suppose that the product π_k has the property

$$\pi_k = \begin{cases} f & \text{on } B^{2n}(k) \\ non-trivial & \text{on } N_k \\ id & \text{outside } N_k. \end{cases}$$

We will define F_{k+1}, $f_{k+1,t}$, f_{k+1}, $swath_{k+1}$, S_{k+1}, N_{k+1}, χ_{k+1}, $G_{k+1,t}$, $g_{k+1,t}$, and g_{k+1}.

Note that because of the behavior observed above for π_k, we know that

$$f \circ \pi_k^{-1} = \begin{cases} id & \text{on } f(B^{2n}(k)) \\ non-trivial & \text{on } N_k \\ f & \text{outside } N_k. \end{cases}$$

Using *Lemma A2*, we produce a Hamiltonian function, $F_{k+1,t}$ that generates $f \circ \pi_k^{-1}$ and that has the additional important property that for all $t \in [0,1]$, $F_{k+1,t}$ is identically zero on the ball $B^{2n}(R_k, f(0))$. The function $F_{k+1,t}$ will not have compact support, so we will produce a cutoff function that will allow us to retain the important properties of $F_{k+1,t}$ in a Hamiltonian that does have compact support.

To do that, we denote by $f_{k+1,t}$ the corresponding Hamiltonian isotopy, and denote by f_{k+1}, the *time* $= 1$ value of that isotopy. That is, $f_{k+1} = f_{k+1,1} = f \circ \pi_k^{-1}$. Now that we have f_{k+1}, we can define the map $sweep_{k+1}$: subsets of $\mathbb{R}^{2n} \to$ subsets of $\mathbb{R}^{2n}$, as well as the $(k+1)^{st}$ swath and $(k+1)^{st}$ neighborhood S_{k+1} and N_{k+1}, as

$$\begin{aligned} sweep_{k+1} &= \textstyle\bigcup_{t=0}^{1} f_{k+1}(sweep_k(A)) \\ S_{k+1} &= sweep_{k+1}(B^{2n}(k+1)) \\ N_{k+1} &= sweep_{k+1}(B^{2n}(k+2)). \end{aligned}$$

We define the cutoff function, χ_{k+1}, to be identically one on S_{k+1} and to be zero outside N_{k+1}. Finally, we define the Hamiltonian function $G_{k+1,t} = F_{k+1,t} \cdot \chi_{k+1}$. The corresponding isotopy will be called $g_{k+1,t}$, and its *time* $= 1$ value will be called g_{k+1}.

Notice that since $G_{k+1,t}$ agrees with $F_{k+1,t}$ everywhere in the set S_{k+1}, the evolution of the set N_k, under the influence of $g_{k+1,t}$, will be the same as its evolution would have been under the influence of $f_{k+1,t}$. So $g_{k+1}|N_k = f_{k+1}|N_k = f \circ \pi_k^{-1}|N_k$. Considering the behavior of the composition, $\pi_{k+1} = g_{k+1} \circ g_k \circ \cdots \circ g_2 \circ g_1$, we find

$$\begin{aligned}
\pi_{k+1}|\, B^{2n}(k+1) &= g_{k+1} \circ g_k \circ \cdots \circ g_2 \circ g_1 \\
&= g_{k+1} \circ \pi_{k-1}|\, B^{2n}(k+1) \\
&= g_{k+1}|\underbrace{\pi_{k-1}\left(B^{2n}(k+1)\right)}_{\text{contained in} N_k} \circ \pi_{k-1}|B^{2n}(k+1) \\
&\quad \left(\text{note that} g_{k+1}|N_k = \left(f \circ \pi_k^{-1}\right)|N_k\right) \\
&= \left(f \circ \pi_{k-1}^{-1}\right)|\, \pi_{k-1}\left(B^{2n}(k+1)\right) \circ \pi_{k-1}|B^{2n}(k+1) \\
&= \left(f \circ \pi_{k-1}^{-1} \circ \pi_{k-1}\right)|\, B^{2n}(k+1) \\
&= f|\, B^{2n}(k+1).
\end{aligned}$$

By induction, we can produce the promised sequence of Hamiltonian functions and isotopies.

End of proof of Lemma A1.

2.3 *Lemma A2*

Lemma A2: If $\psi \in Symp(\mathbb{R}^{2n})$ and there is some point p and radius $R > 0$ such that $\psi|B^{2n}(R,p) \equiv id$, then there is a Hamiltonian function F_t that generates ψ and which has the additional properties that $F_0 \equiv 0 \equiv F_1$ and for all $t \in [0,1]$, $F_t|B^{2n}(R,p) \equiv 0$.

Proof of Lemma A2:

We will use the Alexander trick at the point p. For $t \in (0,1]$, consider the symplectomorphism γ_t defined by $\gamma_t = \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p}$, where τ_p is the translation that sends the origin to point p, and m_t is multiplication by t. When we apply γ_t to a point x, the result is a difference quotient plus a translation: $\gamma_t(x) =$

$p + \frac{\psi(p+t(x-p))-p}{t}$ (This expressions has been simplified by the fact that $\psi(p) = p$.) At time $t = 1$, the expression simplifies to $\psi(x)$. In the limit, as t approaches zero, the expression converges to:

$$\begin{aligned} \lim_{t \to 0} \gamma_t(x) &= p + L_{(d\psi)_p}{}^{*}(x-p) \\ &= p + (x-p) \\ &= x. \end{aligned}$$

(Here, we have used the fact that because $\psi \,|B^{2n}(R, P) \equiv id$, the differential of ψ at p will be the identity matrix. So, $L_{(d\psi)_p} = id$.) Because of this convergence, we know that as $t \to 0$, γ_t converges in the compact open topology to the identity map. So we can extend the time domain of γ_t:

$$\gamma_t = \begin{cases} id & \text{for } t = 0 \\ \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p} & \text{for } 0 < t \leqslant 1. \end{cases}$$

Then γ_t is a symplectic isotopy, starting at the identity, and ending at ψ.

Moreover, we can see that for all $t \in [0,1]$, $\psi_t \,|B^{2n}(R,p) \equiv id$. To see this, let x be any point in $B^{2n}(R,p)$ and let $t \in (0,1]$. Then $\tau_p \circ m_t \circ \tau_{-p}(x)$ will also be in $B^{2n}(R,p)$. (This sentence is the reason that we needed to introduce $B^{2n}(R,p)$. We needed a star-shaped neighborhood of the point p.) Since $\psi \,|B^{2n}(R,p) \equiv id$, we see that the expression for $\psi(x)$ simplifies to

$$\begin{aligned} \gamma_t(x) &= \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p}(x) \\ &= \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \tau_p \circ m_t \circ \tau_{-p}(x) \\ &= x. \end{aligned}$$

There exists a Hamiltonian function F that generates the symplectic isotopy γ_t. Since, for each time t, γ_t restricted to $B^{2n}(R,p)$ is the identity map, at each time t, we know that F_t will be constant on that region. We may impose the normalization requirement that at each time, F_t be in fact identically zero on that region. Furthermore, we can re-scale the speed of the isotopy γ_t so that at times $t = 0$ and $t = 1$, it has zero speed. This will cause F_t to be constant (in space) at those two times. This, with the fact that at each time, F_t must be identically zero on $B^{2n}(R, p)$, tells us that $F_0 \equiv 0 \equiv F_1$.

End of proof of Lemma A2

Chapter 3

Extending Hofer's Infinity Norm

3.1 Introduction and Definitions

The group $Ham^c(\mathbb{R}^{2n})$ of Hamiltonian symplectomorphisms of compact support was introduced in Chapter 1. There, it was shown how elements of this group are generated by the family, C, of compactly supported Hamiltonian functions.

Hofer [4, 3] has introduced two norms for C that can be used to give a norm for $Ham^c(\mathbb{R}^{2n})$ $Ham^c(\mathbb{R}^{2n})$. One of these, which we will call the *infinity norm*, E_∞, will prove particularly useful in this paper, and we introduce it here.

Define the map $\| \ \|_\infty : C \to [0, \infty)$ by

$$\|H\|_\infty = \max_{t\in[0,1]} \left\{ \max_{x\in\mathbb{R}^{2n}} \{H(x,t)\} - \min_{x\in\mathbb{R}^{2n}} \{H(x,t)\} \right\}$$

Then $\| \ \|_\infty$ is a norm on C.

Define a map $E_\infty : Ham^c(\mathbb{R}^{2n}) \to [0, \infty)$ by

$$E_\infty(\psi) = \inf \{ \|H\|_\infty : H \in C \text{ and } H \text{ generates } \psi \}$$

Then it is easy to verify for that for any $\theta, \psi \in Ham^c(\mathbb{R}^{2n})$ and any $\phi \in Symp(\mathbb{R}^{2n})$,

$$E_\infty(\psi) = 0 \Leftrightarrow \psi = id \qquad \text{(non - degeneracy)}$$
$$E_\infty(\theta\psi) \leqslant E_\infty(\theta) + E_\infty(\psi) \qquad \text{(triangle inequality)}$$
$$E_\infty(\psi) = E_\infty(\psi^{-1}) = E_\infty(\phi\psi\phi^{-1}) \qquad \text{(inversion and conjugation invariance).}$$

So E_∞ is a norm on $Ham^c(\mathbb{R}^{2n})$, invariant under conjugation by elements of $Symp(\mathbb{R}^{2n})$.

It is useful to extend this norm to symplectomorphisms of arbitrary support. In Chapter 1, we denoted by $\mathcal{H}$ the set of Hamiltonian functions. These are smooth functions $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ (smooth in the compact-open C^∞ topology) that generate symplectomorphisms by the maps described in chapter 1.

We define the map $\widehat{\| \ \|_\infty} : \mathcal{H} \to [0, \infty]$ by

$$\widehat{\|H\|_\infty} = \sup_{t\in[0,1]} \left\{ \sup_{x\in\mathbb{R}^{2n}} \{H(x,t)\} - \inf_{x\in\mathbb{R}^{2n}} \{H(x,t)\} \right\}$$

Then $\widehat{\| \ \|_\infty}$ is what we might call an *extended norm* on $\mathcal{H}$. By this we mean that it retains the properties of a norm, except that it can be infinite-valued.

As discussed in Chapter 1, the set of symplectomorphisms generated by $\mathcal{H}$, which we denoted $\psi^{\mathcal{H}}$ or $Ham(\mathbb{R}^{2n})$, was actually the entire group $Symp(\mathbb{R}^{2n})$. So we can define a map $\widehat{E_\infty} : Symp(\mathbb{R}^{2n}) \to [0, \infty]$ by

$$\widehat{E_\infty}(\psi) = \inf \left\{ \widehat{\|H\|_\infty} : H \in \mathcal{H} \text{ and } H \text{ generates } \psi \right\}$$

Then it is easy to verify that $\widehat{E_\infty}$ is an extended norm on $Symp(\mathbb{R}^{2n})$, invariant under inversion and conjugation.

Notice that $\widehat{E_\infty}$ extends E_∞ in two ways. First, $\widehat{E_\infty}$ allows functions of arbitrary support in the infimum. Secondly, the domain of $\widehat{E_\infty}$ includes symplectomorphisms of arbitrary support. When restricted to symplectomorphisms ϕ of compact support, we will have the inequality $\widehat{E_\infty}(\phi) \leqslant E_\infty(\phi) = finite$, because every function allowed in the infimum on the right side is also allowed in the infimum on the left side. (One immediately wonders whether in fact $\widehat{E_\infty}(\phi) = E_\infty(\phi)$, but we will not address this question.) When applied to symplectomorphisms of non-compact support, however, $\widehat{E_\infty}$ can give either a real number or infinity as a result. Of course, E_∞ is not defined on these symplectomorphisms.

3.2 An Energy-Capacity Inequality

The question of whether or not $\widehat{E_\infty}$ is finite will be important in this paper. We will show that certain subgroups of $Symp(\mathbb{R}^{2n})$ contain only symplectomorphisms for which $\widehat{E_\infty}$ is finite, and this fact will distinguish those subgroups from the larger group. That the finiteness of $\widehat{E_\infty}$ does distinguish a subgroup from the larger group will follow from an inequality that we present in this section.

Define the Gromov width, w_G, for subsets of $\mathbb{R}^{2n}$ by

$$w_G(A) = \sup\left\{\pi \mathrm{r}^2 \middle| B^{2n}(r) \text{ embeds symplectically in } A\right\}$$

Observe that $w_G(A) \in [0, \infty]$. For symplectomorphisms $\psi \in Symp(\mathbb{R}^{2n})$, we will consider the Gromov width of compact sets A that can be moved off themselves, or displaced by ψ. That is, $\psi(A) \cap A = \phi$. We will prove the following *energy-capacity inequality*, so called because the norm of a symplectomorphism is sometimes referred to as its *energy*.

Claim (energy-capacity inequality): For any $\psi \in Symp(\mathbb{R}^{2n})$, the following holds.

$$\sup\left\{w_G(A) \text{ such that} A \subset \mathbb{R}^{2n}, \text{ compact and } \psi(A) \cap A = \phi\right\} \leqslant \widehat{E_\infty}(\psi)$$

Proof of the energy-capacity inequality:

The claim is automatically true if $\widehat{E_\infty}(\psi) = \infty$, so assume that $\widehat{E_\infty}(\psi)$ is finite, and let A be any compact set displaced by ψ. Let $\varepsilon > 0$ be any positive number. We will show that $w_G(A) \leqslant \widehat{E_\infty}(\psi) + \varepsilon$. Since ε was arbitrary, this will prove the claim. Our method will be to produce a compactly supported symplectomorphism θ such that $w_G(A) \leqslant E_\infty(\theta) \leqslant \widehat{E_\infty}(\psi) + \varepsilon$. The inequality will be obtained by the following string of inequalities, whose terminology and justifications will be given in the steps that follow:

$$\begin{aligned} w_G(A) &\leqslant c_{HZ}(A) && (Hofer - Zehnder\ capacity,\ step\ i) \\ &\leqslant e(A) && (displacement\ energy,\ step\ ii) \\ &\leqslant E_H(\theta) && (standard\ Hofer\ norm,\ step\ iii) \\ &= E_\infty(\theta) && (step\ iv) \\ &\leqslant \widehat{E_\infty}(\psi) + \varepsilon && (step\ v). \end{aligned}$$

step i: A Hamiltonian function H is called *admissible* if its associated Hamiltonian isotopy has no *non-constant periodic orbits.* That is, for all $x \in \mathbb{R}^{2n}$, if $h_{t_a}(x) = h_{t_b}(x)$ for some $0 \leqslant t_a < t_b \leqslant 1$, then in fact $h_t(x) = x$ for all $t \in [0,1]$. For a set $A \subset \mathbb{R}^{2n}$, define $H_{ad}(A, \omega_0)$ to be the set of admissible Hamiltonian functions on $\mathbb{R}^{2n}$ whose support is contained in A. The *Hofer-Zehnder capacity* of A is defined as $c_{HZ}(A) = \sup\{\|H\|_\infty : H \in H_{ad}(A, \omega_0)\}$. From [3], we have the inequality $w_g(A) \leqslant c_{HZ}(A)$.

step ii: We have seen the Hofer infinity norm, E_∞, on $Ham^c(\mathbb{R}^{2n})$. We will denote by E_H a more common norm, also introduced by Hofer [3]:

$$E_H(\phi) = \inf\left\{\int_{t=0}^{1}\left(\max_{x\in\mathbb{R}^{2n}}\{H(x,t)\} - \min_{x\in\mathbb{R}^{2n}}\{H(x,t)\}\right)dt : H \in C \text{and } \psi^H = \phi\right\}$$

We define the *displacement energy*, $e(A)$, of a compact set $A \subset \mathbb{R}^{2n}$ by

$$e(A) = \inf\left\{E_H(\xi) : \xi \in Ham^c\left(\mathbb{R}^{2n}, \omega_0\right) \text{ and } \xi \text{ displaces } A\right\}.$$

From [3], we have the inequality $c_{HZ}(A) \leqslant e(A)$

step iii: In this step, we will introduce the symplectomorphism $\theta \in Ham^c(\mathbb{R}^{2n})$ and prove the third inequality.

The Hamiltonian symplectomorphism ψ might not be compactly supported, but we are assuming that its infinity norm, $\widehat{E_\infty}(\psi)$, is finite. So, there is some Hamiltonian function H that generates ψ such that

$$\widehat{\|H\|_\infty} = \sup_{t\in[0,1]} \left\{ \sup_{x\in\mathbb{R}^{2n}} H(t,x) - \inf_{x\in\mathbb{R}^{2n}} H(t,x) \right\} \leqslant \widehat{E_\infty}(\psi) + \varepsilon,$$

where $\varepsilon > 0$ was chosen at the beginning of the proof. Denote by h_t the Hamiltonian isotopy generated by H. We can multiply H by a moving cutoff function, where the cutoff function is identically 1 on the moving image of the set A, as the set A evolves under the influence of the isotopy h_t, and is identically zero outside a compact set. The resulting function we call F; the compactly supported Hamiltonian symplectomorphism that it generates, θ. Notice that because ψ displaces A, θ also displaces A. This means that

$$e(A) = \inf \left\{ E_H(\xi) : \xi \in Ham^c\left(\mathbb{R}^{2n}, \omega_0\right) \text{ and } \xi \text{ displaces } A \right\} \leqslant E_H(\theta),$$

because θ is a particular symplectomorphism that displaces A. This proves the third inequality.

step iv: The following inequality follows immediately from the definitions of the two norms:

$$\begin{aligned} E_H(\theta) &= \inf \left\{ \int_{t=0}^{1} \left(\max_{x\in\mathbb{R}^{2n}} \{H(x,t)\} - \min_{x\in\mathbb{R}^{2n}} \{H(x,t)\} \right) dt : H \in C \text{ and } \psi^H = \theta \right\} \\ &\leqslant \inf \left\{ \max_{t\in[0,1]} \left\{ \max_{x\in\mathbb{R}^{2n}} \{H(x,t)\} - \min_{x\in\mathbb{R}^{2n}} \{H(x,t)\} \right\} : H \in C \text{ and } \psi^H = \theta \right\} \\ &= E_\infty(\theta). \end{aligned}$$

But Polterovich showed in [6] that the two norms are in fact equal.

step v: Because θ is generated by a Hamiltonian function, F, that is a cut-off version of the function that generates ψ, we have the fifth inequality:

$$\begin{aligned}
E_\infty(\theta) &= \inf\left\{\|G\|_\infty : G \in C \text{and } \psi^G = \theta\right\} \\
&\leqslant \|F\|_\infty \text{ because } F \text{ is a particular function that generates } \theta \\
&= \max_{t\in[0,1]}\left\{\max_{x\in\mathbb{R}^{2n}} F(t,x) - \min_{x\in\mathbb{R}^{2n}} F(t,x)\right\} \\
&\leqslant \sup_{t\in[0,1]}\left\{\sup_{x\in\mathbb{R}^{2n}} H(t,x) - \inf_{x\in\mathbb{R}^{2n}} H(t,x)\right\} F \text{ was obtained by cutting off } H \\
&= \widehat{\|H\|_\infty} \\
&\leqslant \widehat{E_\infty}(\psi) + \varepsilon.
\end{aligned}$$

End of proof of the energy-capacity inequality

3.3 Using the *e-c inequality* to detect proper subsets of $Symp\left(\mathbb{R}^{2n}\right)$

The energy-capacity inequality tells us that there must be elements of $Symp\left(\mathbb{R}^{2n}\right)$ whose infinity norms are not finite. For example, let $\psi \in Symp\left(\mathbb{R}^{2n}\right)$ be the counter-clockwise rotation in the $x_1 \times x_2$ plane, about the origin, through an angle of $\frac{\pi}{2}$. Then for every $M > 0$, the ball $B^{2n}(M, (2M, 0, \cdots, 0))$ is displaced by ψ. Since $w_G\left(B^{2n}(M, (2M, 0, \cdots, 0))\right) = \pi M^2$, we see that

$$\sup\left\{w_G(A) \text{ such that} A \subset \mathbb{R}^{2n}, \text{ compact and} \psi(A) \cap A = \phi\right\} = \infty$$

By the energy-capacity inequality, $\widehat{E_\infty}(\psi)$ must be infinite as well. Thus, we can state the following

Corollary of the energy-capacity inequality: If some subset $\Omega \subset Symp\left(\mathbb{R}^{2n}\right)$ has the property that $\widehat{E_\infty}(\psi)$ is finite for all $\psi \in \Omega$, then Ω is a proper subset of $Symp\left(\mathbb{R}^{2n}\right)$, for Ω will contain no rotations.

For example, in chapter 1, we discussed the normal subgroup $\psi^{\mathcal{H}_{UB}}$ in $Symp\left(\mathbb{R}^{2n}\right)$ consisting of symplectomorphisms that can be generated by Hamiltonian functions that are uniformly-bounded. We see that for any such ψ, $\widehat{E_\infty}\left(\psi\right)$ will be finite. So $\psi^{\mathcal{H}_{UB}}$ is a proper subgroup of $Symp\left(\mathbb{R}^{2n}\right)$.

Chapter 4

A new non-trivial normal subgroup of $Symp\left(\mathbb{R}^{2n}\right)$

4.1 Introduction, *Theorem B*, and *Proposition B*

In Section 1.4 of Chapter 1, the groups $Ham^c(\mathbb{R}^{2n})$ and $\psi^{\mathcal{H}_{UB}}$ were found to be proper normal subgroups of $Symp(\mathbb{R}^{2n})$. Each was obtained from the larger group by imposing restrictions on the set of generating Hamiltonian functions. In this chapter, we will describe a rather more complicated normal subgroup of $Symp(\mathbb{R}^{2n})$ and prove that it is also a proper subgroup.

The description of the group and statement of the theorem follow; the motivation for our considering the group in the first place will be in a remark following the statement of the theorem.

Theorem B:

Let $U = \bigcup_{i=1}^{\infty} f_i\left(B^{2n}(R)\right) \subset \mathbb{R}^{2n}$ be any disjoint union of symplectic balls of radius $R < \frac{1}{2}$. (By *symplectic ball of radius R*?, we mean the image, $f\left(B^{2n}(R)\right)$, of a symplectic embedding, $f : B^{2n}(R) \to \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the closed ball of radius

R.) Denote by $Symp_U(\mathbb{R}^{2n})$ the set of symplectomorphisms of $\mathbb{R}^{2n}$ that can be generated by Hamiltonian functions with support contained in U. Then $Symp_U(\mathbb{R}^{2n})$ is a subgroup of $Symp(\mathbb{R}^{2n})$, but it is not a normal subgroup: conjugation of an element of $Symp_U(\mathbb{R}^{2n})$ with a translation can produce an element of $Symp(\mathbb{R}^{2n})$ that is not supported in U. Define $G_U \triangleleft Symp(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $Symp(\mathbb{R}^{2n})$ containing $Symp_U(\mathbb{R}^{2n})$. That is, G_U contains $Symp_U(\mathbb{R}^{2n})$, is closed under conjugation by elements of $Symp(\mathbb{R}^{2n})$, and is closed under composition.

Claim (Theorem B): G_U is a *proper* subgroup of $Symp(\mathbb{R}^{2n})$

Comment: Why is this surprising? Let $Diff_{vol}(U) \subset Diff_{vol}(\mathbb{R}^n)$ be the collection of volume-preserving diffeomorphisms of $\mathbb{R}^n$ that are supported in U. Note that $Diff_{vol}(U)$ is a non-normal subgroup of $Diff_{vol}(\mathbb{R}^n)$. As above, define $G_{Diff_{vol}(U)} \triangleleft Diff_{vol}(\mathbb{R}^n)$ to be the minimal normal subgroup of $Diff_{vol}(\mathbb{R}^n)$ containing $Diff_{vol}(U)$. McDuff [5] showed that for $n \geqslant 3$, $G_{Diff_{vol}(U)} = Diff_{vol}(\mathbb{R}^n)$. So the present claim distinguishes the structure of $Symp(\mathbb{R}^{2n})$ from that of $Diff_{vol}(\mathbb{R}^{2n})$

A flow chart illustrating the structure of the proof of *Theorem B* is shown in Figure 4.1.

Proof of Theorem B:

If $g \in G_U$, then $g = g_1 g_2 \cdots g_k$, where $g_j = h_j \circ \psi_j \circ h_j^{-1}$, with $h_j \in Symp(\mathbb{R}^{2n})$ and $\psi_j \in Symp_U(\mathbb{R}^{2n})$. Note that ψ_j is supported in U, which is a disjoint union of symplectic balls. In Section 3.1, we introduced the conjugation-invariant *extended Hofer infinity norm*, $\widehat{E_\infty}$ for elements of $Symp(\mathbb{R}^{2n})$. In *Proposition B*, below, we will prove that for any element $\psi_j \in Symp_U(\mathbb{R}^{2n})$, this norm is bounded: $\widehat{E_\infty}(\psi_j) \leqslant 32$. By conjugation invariance, $\widehat{E_\infty}(g_j) \leqslant 32$, and by the triangle in-

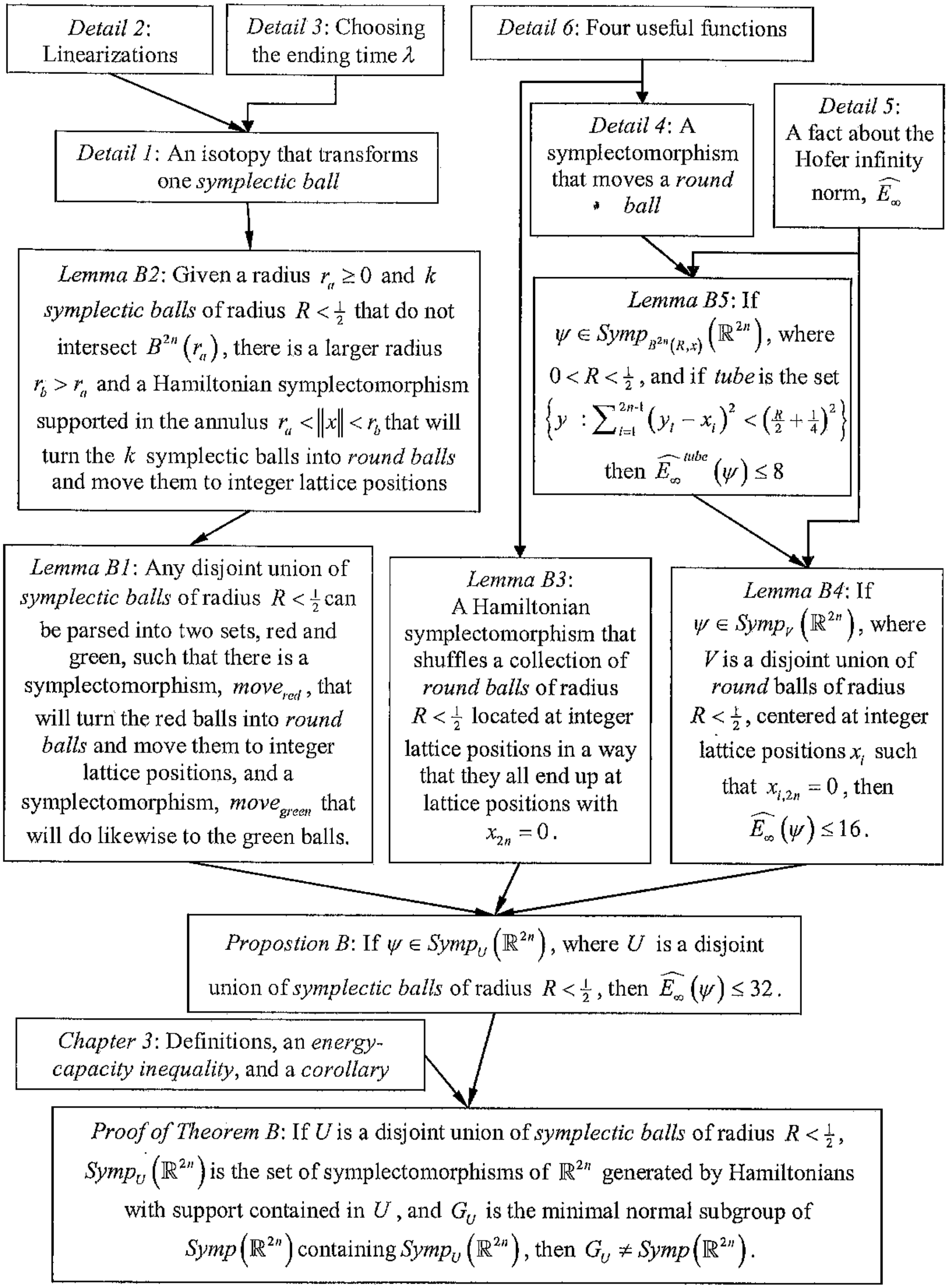

Figure 4.1: Structure of the Proof of Theorem B

equality, $\widehat{E_\infty}(g) \leqslant 32k$. So we see that for any $g \in G_U$, the norm $\widehat{E_\infty}(g)$ will be finite.

But in Section 3.3, we proved the following corollary.

Corollary of the energy-capacity inequality: If a subset $\Omega \subset Symp(\mathbb{R}^{2n})$ has the property that $\widehat{E_\infty}(\psi)$ is finite for all $\psi \in \Omega$, then Ω is a proper subset of $Symp(\mathbb{R}^{2n})$, for Ω will contain no rotations.

With that *corollary,* we see that G_U is a *proper* subgroup of $Symp(\mathbb{R}^{2n})$.

End of proof of Theorem B

Proposition B: If $\psi \in Symp(U)$, where $U = \bigcup_{i=1}^{\infty} f_i(B^{2n}(R))$ is a disjoint union of symplectic balls, then $\widehat{E_\infty}(\psi) \leqslant 32$

Proof of Proposition B:

First, recall that a *symplectic ball* is the image, $f(B^{2n}(R))$, of a symplectic embedding. We write *round ball* to denote $B^{2n}(R,x)$ or $B^{2n}(R)$.

If we were to parse the disjoint union into two sets of symplectic balls, say red ones and green ones, then we could write ψ as a composition of symplectomorphisms supported in these two sets, $\psi = \psi_r \circ \psi_g$. Then by the triangle inequality, $\widehat{E_\infty}(\psi) \leqslant \widehat{E_\infty}(\psi_r) + \widehat{E_\infty}(\psi_g)$. We will parse the union, and we will do it in a particular way, by using the result of *Lemma B1*, which we state here and prove in Section 4.2.1.

Lemma B1: Any disjoint union of symplectic balls $\bigcup_{i=1}^{\infty} f_i(B^{2n}(R))$ can be parsed into two sets, red and green, such that there are Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, $move_{red}$ and $move_{green}$,with the following properties: The symplectomorphism $move_{red}$ will turn the *red* symplectic balls into round balls and move them to integer lattice positions; $move_{green}$ will turn the *green* symplectic balls into round balls and move them to integer lattice positions.

The significance of a lattice of round balls is that they can be shuffled into a new arrangement where they are each located at lattice positions whose x_{2n} coordinate is zero. In fact, the shuffling can be accomplished with a symplectomorphism that we will construct in *Lemma B3.* We will state that lemma here and prove it in Section 4.2.3.

Lemma B3: Given a *lattice of round balls*, meaning a union $\bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$, where $0 < R < \frac{1}{2}$ and each $x_i \in \mathbb{R}^{2n}$ has integer coordinates, we claim that there is a Hamiltonian symplectomorphism, which we will call $shuffle$, that rearranges the balls so that they are again centered at integer lattice points, but now only at points such that $x_{i,2n} = 0$.

With the symplectomorphisms ψ_r, ψ_g, $move_r$, $move_g$, and $shuffle$, we can define two new symplectomorphisms, ϕ_r and ϕ_g in the following way:

$$\phi_r = shuffle \circ move_r \circ \psi_r \circ move_r^{-1} \circ shuffle^{-1}$$

$$\phi_g = shuffle \circ move_g \circ \psi_g \circ move_g^{-1} \circ shuffle^{-1}$$

Notice that both ϕ_r and ϕ_g are supported in disjoint unions of round balls centered at integer lattice positions such that $x_{2n} = 0$. So, we may apply the following *Lemma*, proven in Section 4.2.4 to give an estimate on the extended norm of each of them.

Lemma B4: If $\psi \in Symp_V(\mathbb{R}^{2n})$, where $V = \bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$ is a disjoint union of round balls with $0 < R < \frac{1}{2}$, centered at integer lattice positions x_i such that $x_{i,2n} = 0$, then its *extended Hofer infinity norm* is bounded: $\widehat{E_\infty}(\psi) \leqslant 16$

With this result, we can make the estimates $E_\infty(\phi_r) \leqslant 16$ and $E_\infty(\phi_g) \leqslant 16$. By conjugation invariance of the extended norm $\widehat{E_\infty}$, we know that $\widehat{E_\infty}(\psi_r) = \widehat{E_\infty}(\phi_r)$ and $\widehat{E_\infty}(\psi_g) = \widehat{E_\infty}(\phi_g)$. Therefore, $\widehat{E_\infty}(\psi) \leqslant \widehat{E_\infty}(\psi_r) + \widehat{E_\infty}(\psi_g) \leqslant 32$.

End of Proof of Proposition B

4.2 *Lemmas B1* through *B5*

4.2.1 *Lemma B1*

Lemma B1: Any disjoint union of symplectic balls, $\bigcup_{i=1}^{\infty} f_i(B^{2n}(R))$, can be parsed into two sets, *red* and *green*, such that there are Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$, $move_{red}$ and $move_{green}$, with the following properties. The symplectomorphism $move_{red}$ will turn the *red* symplectic balls into round balls and move them to integer lattice positions; $move_{green}$ will turn the *green* symplectic balls into round balls and move them to integer lattice positions.

Proof of Lemma B1:

The lemma will follow immediately from the following *claim.*

claim: There exists a sequence of radii, $0 = r_0 < r_1 < r_2 < \cdots$, a sequence of Hamiltonian symplectomorphisms $h_1, h_2, h_3, \cdots$, a re-numbering of the disjoint union of symplectic balls, and a sequence of important indices in that numbering $k_1 < k_2 < k_3 < \cdots$, with the following properties:

1. The open ball $\|x\| < r_1$ contains the symplectic balls numbered $1, \cdots, k_1$, which we will call the 1^{st} *batch* of balls. For $i \geqslant 2$, the i^{th} *annulus*, $r_{i-1} < \|x\| < r_{i+1}$, contains the symplectic balls numbered $k_{i-1}+1, \cdots, k_i$, which we will refer to as the i^{th} *batch*. None of the annuli are empty: For each $i \geqslant 2$, the ball $B^{2n}(r_i)$. intersects at least one symplectic ball that does not intersect $B^{2n}(r_{i-1})$, and all of those symplectic balls are completely contained in the annulus $r_{i-1} < \|x\| < r_{i+1}$.

2. The i^{th} batch of symplectic balls is red if i is odd; green, if i is even.

3. The symplectomorphism h_1 is supported in the open ball $\|x\| < r_2$. For $i \geqslant 2$, the symplectomorphism h_i is supported in the i^{th} annulus, $r_{i-1} < \|x\| < r_{i+1}$.

4. The symplectomorphism h_i turns the i^{th} batch of symplectic balls into round balls and moves them to integer lattice positions.

This claim will be proven below. First, however, we show how the lemma follows immediately from the claim.

Since the odd-numbered h_i have disjoint support, their product, $\prod_{i=1}^{\infty} h_{2i-1}$, is a well-defined Hamiltonian symplectomorphism, which we call $move_r$. This symplectomorphism moves all the red symplectic balls to integer lattice positions, while turning them into round balls. Similarly, the even-numbered h_i have disjoint support, so their product $\prod_{i=1}^{\infty} h_{2i}$ is a well-defined Hamiltonian symplectomorphism, which we call $move_g$ because it moves all the green symplectic balls to integer lattice positions, while turning them into round balls. This proves the lemma.

Proof of claim:

Denote by $sball_i$ the i^{th} symplectic ball, $f_i\left(B^{2n}(R)\right)$.

Basis step, part a: Choose r_0, r_1, r_2, h_1, k_1 and the first batch of red balls.

Let $r_0 = 0$. Choose the smallest r_1 such that $B^{2n}(r_1)$ intersects at least one of the symplectic balls from the union. This ball $B^{2n}(r_1)$ may intersect more than one symplectic ball. Renumber the union and choose an integer k_1 so that it is the first balls in the numbering of the union, $sball_1, \cdots, sball_{k_1}$, that intersect $B^{2n}(r_1)$. Color these first k_1 balls red. In section 4.2.2, we will prove the following *lemma:*

> *Lemma B2:* If $\bigcup_{i=1}^{k} f_i(B^{2n}(R))$, with $0 < R < \frac{1}{2}$, is a disjoint union of symplectic balls, then there exists $r_b > 0$ and a Hamiltonian symplectomorphism h, supported in the open ball $\|x\| < r_b$ such that $h\left(\bigcup_{i=1}^{k} f_i(B^{2n}(R))\right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i)$, a disjoint union of round balls centered on integer lattice points x_i.

(Actually, the statement of *Lemma B2* is more general; we have extracted from the general statement a simpler one that suffices for our present need.) Applying this result to our present case, we will use our k_1 for the k in the lemma, and our union, $\bigcup_{i=1}^{k_1} f_i(B^{2n}(R))$, for the union in the lemma. We will define the radius r_2 to be the radius r_b produced by the lemma, and define the symplectomorphism h_1 to be the Hamiltonian symplectomorphism h produced by the lemma. With no loss of generality, we can choose a larger r_2, if necessary, in order to insure that the ball $B^{2n}(r_2)$ intersects at least one symplectic ball that is not among the first k_1 red balls that we picked above.

Note that the re-ordering of the union of symplectic balls, along with the r_0, r_1, r_2, h_1, k_1 that we have chosen, have properties (1) - (4) listed above.

Basis step, part b: Choose r_3, h_2, k_2, and the first batch of green balls.

In the previous step, we chose r_2 large enough that the ball $B^{2n}(r_2)$ intersects at least one symplectic ball that is not red. (Keep in mind that we are still regarding the union of symplectic balls in their original state, although some have been colored red. We have proven the existence of a Hamiltonian symplectomorphism, h_1, that could manipulate these red balls - and possibly alter some other, uncolored, balls in the process - but we have not used it. So far, we have only *considered* the operation of h_1 in order to choose an appropriate radius r_2.) This ball $B^{2n}(r_2)$ may intersect more than one symplectic ball that is not red, not among the k_1 balls that we have picked so far and colored red. Renumber the union and choose an integer $k_2 > k_1$ so that it is the next balls in the numbering of the union, $sball_{k_1+1}, \cdots, sball_{k_2}$, that intersect $B^{2n}(r_2)$ and are not red. (In the renumbering, do not alter the numbering of the first k_1 balls.)

Note that there are $k_2 - k_1$ symplectic balls in this new set, none of which intersect $B^{2n}(r_1)$, and none of which have yet been colored, because in the previous step, it was those symplectic balls that *did* intersect $B^{2n}(r_1)$ that were numbered as the first k_1 balls of the union and colored red. Color green these $k_2 - k_1$ new symplectic balls that we have just chosen.

In the above section, *Basis step, part a,* we stated a simple version of the claim of *Lemma B2.* The full-strength version of the claim is applicable to our current situation, and we state it here.

Lemma B2: If $r_a \geqslant 0$ and $\bigcup_{i=1}^{k} f_i(B^{2n}(R))$, with $0 < R < \frac{1}{2}$, is a disjoint union of symplectic balls that do not intersect the closed ball $B^{2n}(r_a)$, then there exists $r_b > r_a$ and a Hamiltonian symplectomorphism h, supported in the open annulus $r_a < \|x\| < r_b$ such that $h\left(\bigcup_{i=1}^{k} f_i(B^{2n}(R))\right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i)$, a disjoint union of round balls centered on integer lattice points x_i.

We will apply this result, using our r_1 for the r_a of the lemma, our integer $k_2 - k_1$ for the integer k in the lemma, and our union, $\bigcup_{i=k_1+1}^{k_2} f_i\left(B^{2n}(R)\right)$, for the union in the lemma. The symplectomorphism h and the outer radius r_b produced by the lemma we will call h_2 and r_3. If needed, we can choose a larger r_3, large enough that the ball $B^{2n}(r_3)$ intersects at least one symplectic ball that is not among the first k_2 balls that we have picked so far.

Note that the new ordering of the union of symplectic balls, along with the r_0, r_1, r_2, r_3, h_1, h_2, k_1, and k_2 that we have chosen so far, have properties (1) - (4) listed above.

Inductive step:

Suppose one is given a list of radii, $0 = r_0 < r_1 < \cdots < r_j$; a list of Hamiltonian symplectomorphisms, $h_1, \cdots, h_{j-1}$; a numbering of the disjoint union of symplectic balls; and a list of important indices in that numbering, $k_1, \cdots, k_{j-1}$; which have properties (1) - (4) listed above.

Inductive claim: One can choose a new numbering of the union of symplectic balls, and choose r_{j+1}, h_j, k_j, and the next batch of colored balls in a way that the new numbering; the new list of radii, $0 = r_0 < r_1 < \cdots < r_{j+1}$; the new list of Hamiltonian symplectomorphisms, $h_1, \cdots, h_j$; and the new list of important indices, $k_1, \cdots, k_j$; also have properties (1) - (4). (If j is an odd number, the batch of balls that we choose in this step will be colored *red.* If j is an even number, we will be coloring balls *green.*)

Proof of inductive claim:

The radius r_j was chosen large enough that the ball $B^{2n}(r_j)$ intersects at least one symplectic ball that is not among the first k_{j-1} balls in the ordering of the union. (As before, keep in mind that we are regarding the union of symplectic balls in their original state, although some have been colored. We have proven the existence of Hamiltonian symplectomorphisms, $h_1, \cdots, h_{j-1}$, that can manipulate these balls - and possibly alter some other, uncolored, balls in the process - but we have not used them. So far, we have only *considered* the operation of the h_j in order to choose appropriate radii r_j.) This ball $B^{2n}(r_j)$ may intersect more than one symplectic ball that is not among the first k_{j-1} balls. Renumber the union and choose an integer $k_j > k_{j-1}$ so that it is the next balls in the numbering of the union, $sball_{k_{j-1}+1}, \cdots, sball_{k_j}$, that intersect $B^{2n}(r_j)$ and are not already colored red or green.

Note that there are $k_j - k_{j-1}$ symplectic balls in this new set, none of which intersect $B^{2n}(r_{j-1})$, and none of which are colored, because in the previous steps, it was those symplectic balls that *did* intersect $B^{2n}(r_{j-1})$ that were numbered as the first k_{j-1} balls of the union and colored either red or green. Color these new $k_j - k_{j-1}$ symplectic balls that we have just chosen: color them red if j is odd; green, if j is even.

We will apply *Lemma B2*, using r_j for the r_a of the lemma, our integer $k_j - k_{j-1}$ for the integer k in the lemma, and our union, $\bigcup_{i=k_{j-1}+1}^{k_j} f_i(B^{2n}(R))$, for the union in the lemma. The symplectomorphism h and the outer radius r_b produced by the lemma we will call h_j and r_{j+1}. If needed, we can choose a larger r_{j+1}, large enough that the ball $B^{2n}(r_{j+1})$ intersects at least one symplectic ball that is not among the first k_j balls that we have picked so far. Note that the new numbering; the new list of radii, $0 = r_0 < r_1 < \cdots < r_{j+1}$; the new list of Hamiltonian symplectomorphisms, $h_1, \cdots, h_j$; and the new list of important indices, $k_1, \cdots, k_j$; have properties (1) - (4) listed above.

End of proof of the inductive claim.

Conclusion: By induction, we have proved the claim.

End of Proof of Lemma B1

4.2.2 *Lemma B2*

Lemma B2: If $r_a \geqslant 0$ and $\bigcup_{i=1}^{k} f_i\left(B^{2n}(R)\right)$, with $0 < R < \frac{1}{2}$, is a disjoint union of symplectic balls that do not intersect the closed ball $B^{2n}(r_a)$, then there exists $r_b > r_a$ and a Hamiltonian symplectomorphism h, supported in the open annulus $r_a < \|x\| < r_b$, such that $h\left(\bigcup_{i=1}^{k} f_i\left(B^{2n}(R)\right)\right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i)$, a disjoint union of round balls centered on integer lattice points x_i. (In the case that $r_a = 0$, we mean simply that the k symplectic balls are disjoint, and claim that there exists some $r_b > 0$, with h supported in the open ball $\|x\| < r_b$.)

Proof of Lemma B2:

step i: Establish neighborhoods.

A *symplectic ball* is the image, $f_i\left(B^{2n}(R)\right)$, of a symplectic embedding, $f_i : B^{2n}(R) \to \mathbb{R}^{2n}$ of the closed *round ball,* $B^{2n}(R)$. By the usual definition of embedding, we know that there is a smooth map $\widehat{f}_i : U_i \to \mathbb{R}^{2n}$, where U_i is some open set, $B^{2n(R)} \subset U_i \subset \mathbb{R}^{2n}$, and $\widehat{f}_i | U_i = f_i$. From now on, we will use the symbol f_i for both f_i and $\widehat{f}_i$. We have k of these symplectic balls, which are closed, disjoint, and which do not intersect $B^{2n}(r_a)$. (Again, this last condition is omitted if $r_a = 0$.) Therefore, there is some $\varepsilon > 0$ such that $B^{2n}(R+\varepsilon) \subset U_i$ for each $i = 1, \ldots, k$, and such that the k sets, $f_i\left(B^{2n}(R+\varepsilon)\right)$, do not intersect each other or $B^{2n}(r_a)$. We will refer to the set $f_i\left(B^{2n}(R+\varepsilon)\right)$ as the i^{th} *neighborhood,* N_i. Note that the composition $\widehat{f}_i^{-1} \circ f_i$ is just the *inclusion map,* $\iota : B^{2n}(R) \to \mathbb{R}^{2n}$, which we will suppress.

step ii: Introduce $g_{i,t}$.

In *Detail 1*, found in section 4.3.1, a symplectic isotopy $g_{i,t} : [0,1] \rightarrow Symp(\mathbb{R}^{2n})$ is constructed. It has two important properties: $g_{i,0} = id$, and $g_{i,1}(f_i(B^{2n}(R))) = B^{2n}(R, x_i)$, where x_i is an integer lattice point. So the isotopy $g_{i,t}$ transforms the i^{th} *symplectic ball* into a *round ball*, centered on an integer lattice point. Also important for us will be the fact that in the process, the i^{th} ball - in fact the entire i^{th} *neighborhood* - stays outside the ball $B^{2n}(r_a)$. (If $r_a = 0$, this sentence is omitted.)

It would be convenient if we could transform the entire union, $\bigcup_{i=1}^{k} f_i(B^{2n}(R))$, of symplectic balls at once by simply acting on the union with the product of the corresponding k isotopies, $g_{k,t} \circ \cdots \circ g_{2,t} \circ g_{1,t}$, but this will not work. The supports of the various $g_{i,t}$ are not disjoint, indeed the supports are not even compact. As a result, their operations would interfere with each other (The operation of $g_{1,t}$, designed to transform the 1^{st} symplectic ball, would affect the 2^{nd} one as well, so $g_{2,t}$ would not have its intended affect on the 2^{nd} ball, etc.), and their supports would not be confined to the desired annular region. One way to get the various $g_{i,t}$ to cooperate with one another is to apply moving cutoff functions to the corresponding Hamiltonian functions, with the cutoff functions identically 1 on the images of the evolving symplectic balls, and supported inside the (slightly larger) evolving neighborhoods. This, we will do in the next step.

step iii: Introduce $G_{i,t}$, $\chi_{i,t}$.

We let $G_{i,t}$ be a Hamiltonian function that generates $g_{i,t}$. For each $i = 1, \ldots, k$, we define a cutoff function $\chi_{i,t}$ as follows. $\chi_{i,t}$ is a non-negative smooth function on $\mathbb{R}^{2n}$ that is identically 1 on the moving image $g_{i,t}(f_i(B^{2n}(R)))$ of the i^{th} symplectic ball, as it is transformed by the isotopy $g_{i,t}$, and is identically zero outside the moving image $g_{i,t}(f_i(B^{2n}(R+\varepsilon)))$ of the slightly larger i^{th} neighborhood. This insures that the support of $\chi_{i,t}$ is *compact.* (Not merely *compact at each* $t \in [0,1]$.) Moreover, if $i \neq j$, the supports of $\chi_{i,t}$ and $\chi_{j,t}$ will be disjoint at each time t, because of the way that $g_{i,t}$ is constructed. That is, in *Detail 1*, the isotopy $g_{i,t}$ is constructed in a way that insures that the evolving image of the i^{th} neighborhood remains within the confines of a linear magnification of the original i^{th} neighborhood. But the original neighborhoods are disjoint, so their linear magnifications will also be disjoint and, therefore, the evolving images of the various neighborhoods will remain disjoint. Thus, the supports of the various cutoff functions, contained in those evolving images, will remain disjoint.

Multiplying the i^{th} Hamiltonian function by this i^{th} cutoff function, we obtain a new Hamiltonian function, $\chi_{i,t}G_{i,t}$. This function agrees with the function $G_{i,t}$ on the moving image $g_{i,t}(f_i(B^{2n}(R)))$ of the i^{th} symplectic ball, as it is transformed by the isotopy $g_{i,t}$, but for each $t \in [0,1]$, the support of $\chi_{i,t}G_{i,t}$ is compact and if $i \neq j$, the supports of $\chi_{i,t}G_{i,t}$ and $\chi_{j,t}G_{j,t}$ are disjoint. Moreover, because the support of each χ_i is not merely *compact at each time* $t \in [0,1]$, but actually *compact*, the same can be said of the functions $\chi_{i,t}G_{i,t}$.

step iv: Introduce H_t, h_t, and h.

Define the Hamiltonian function H by $H_t = \sum_{i=1}^{k} \chi_{i,t}G_{i,t}$, let h_t be the symplectic isotopy it generates, and let $h = h_1$ be the *time* $= 1$ value of that isotopy. Notice that for each $i = 1, \ldots, k$, and for each $t \in [0,1]$, this function agrees with the function $G_{i,t}$ on the moving image $g_{i,t}(f_i(B^{2n}(R)))$ of the i^{th} symplectic ball, as it is transformed by the isotopy $g_{i,t}$. Therefore, the isotopy h_t will transform the i^{th} symplectic ball in precisely the same way that $g_{i,t}$ would. So, the isotopy h_t turns the i^{th} symplectic ball into a round ball and moves it to an integer lattice position, and it does this to the whole batch of balls, $i = 1, \ldots, k$, simultaneously.

step v: Describe r_b, the outer radius.

We defined the function H as the sum $H_t = \sum_{i=1}^{k} \chi_{i,t} G_{i,t}$, where the support of each $\chi_{i,t} G_{i,t}$ is compact. Therefore, the support of H is also compact. So we can choose some number $r_b > r_a$ such that the support of H is contained in the open ball $\|x\| < r_b$.

End of proof of Lemma B2

4.2.3 *Lemma B3*

Lemma B3: Given a *lattice of round balls*, meaning a union $\bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$, where $0 < R < \frac{1}{2}$ and each $x_i \in \mathbb{R}^{2n}$ has integer coordinates, we claim that there is a Hamiltonian symplectomorphism, which we will call $shuffle$, that rearranges the balls so that they are again centered at integer lattice points, but now only at points such that $x_{i,2n} = 0$.

Proof of Lemma B3:

The symplectomorphism $shuffle$ will be achieved by the composition, $shuffle = h \circ g \circ f$, of three Hamiltonian symplectomorphisms, f, g, and h, whose incremental effects on the lattice of balls are shown in Figures 4.2, 4.3, and 4.4. These symplectomorphisms effect movement only in the $x_{2n-1} \times x_{2n}$ plane, which is the plane shown. The three symplectomorphisms shown can be generated by three time-independent Hamiltonian functions, F, G, and H. We construct these Hamiltonian functions using the *wedge* and *step* functions described in *Detail 6*, found in section 4.3.6.

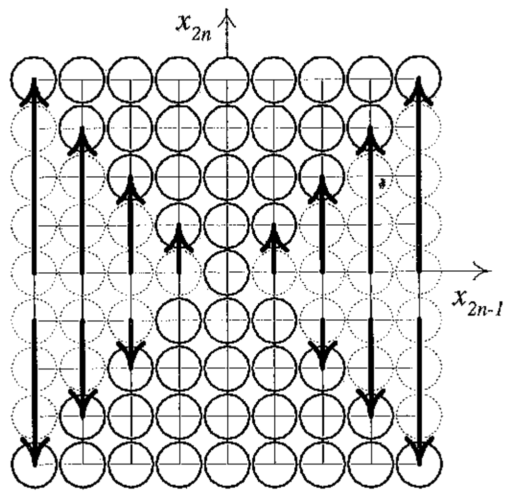

Figure 4.2: Effect of f

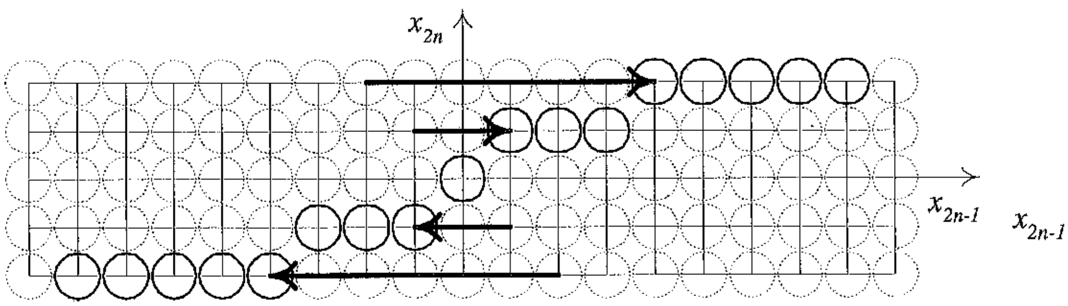

Figure 4.3: Effect of g

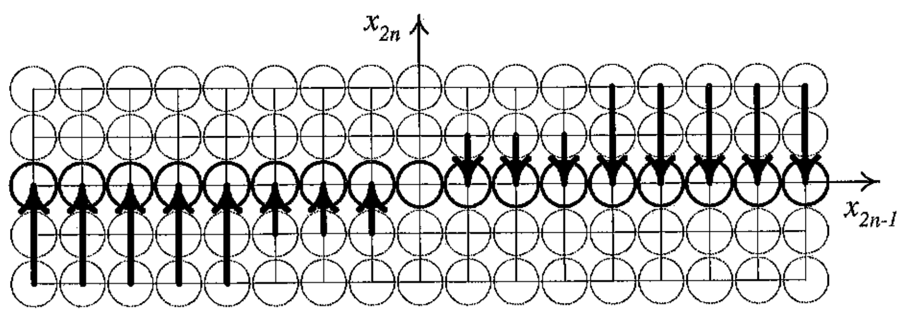

Figure 4.4: Effect of h

step i: Construction of the function F

Define the j^{th} *column* of balls to be the set of balls whose centers have $x_{2n-1} = k$. Define the *top half* of the j^{th} column to be the subset whose centers have $x_{2n} \geqslant 0$; the *bottom half*, the subset whose centers have $x_{2n} \leqslant -1$. From Fig. 4.2, showing the effect of symplectomorphism f, we see that for $j \geqslant 1$, the top half of the j^{th} and $-j^{th}$ columns must move up j units, while the bottom halves must move down $j-1$ units. (The 0^{th} column does not move.) Observe that the Hamiltonian function $wedge\,(x_{2n-1} - j)$ generates a symplectomorphism that moves the j^{th} column one unit in the x_{2n} direction. We construct the Hamiltonian function F by applying cutoff functions to this function, in order to restrict action to the top half or bottom half of the column, and multiplying it by scalars appropriate to achieve the desired displacements. (For typesetting reasons, we will write w and s for the functions *wedge* and *step* in the expression for the function F)

$$F(x) = \sum_{j=1}^{\infty} \left(\underbrace{\left(w\left(x_{2n-1} - j\right) + w\left(x_{2n-1} + j\right)\right)}_{\text{moves } j^{th} \text{ and } -j^{th} \text{ columns}} \left(\underbrace{j \cdot s\left(x_{2n}\right)}_{\text{top half up}} \underbrace{- (j-1) \cdot s\left(1 - x_{2n}\right)}_{\text{bottom half down}} \right) \right)$$

step ii: Construct the function G

Define the k^{th} *row* of balls to be the set of balls whose centers have $x_{2n} = k$. From Fig. 4.3, showing the effect of symplectomorphism g, we see that for rows $k \geqslant 1$, the initial position of the left-most ball in the k^{th} row has $x_{2n-1} = -k$. The symplectomorphism g must move that left-most ball to the right, to a final position with $x_{2n-1} = 1 + 3 + \cdots + (2k-1) = k^2$. This means that for $k \geqslant 1$, the k^{th} row must move to the right an amount $k^2 + k = k\,(k+1)$ units. Similarly, the $-k^{th}$ row must move to the left by the same amount. Accordingly, we construct the Hamiltonian function G:

$$G(x) = \sum_{k=1}^{\infty} \left(\left(\underbrace{wedge\left(x_{2n} + k\right)}_{\text{sends } -k^{th} \text{ row left}} \underbrace{- wedge\left(x_{2n} - k\right)}_{\text{sends } k^{th} \text{ row right}} \right) k\,(k+1) \right)$$

step iii: Construct the function H:

From Fig. 4.4, showing the effect of symplectomorphism h, we see that it causes columns 1 through 3 move down 1 unit, columns 5 through 9 move down 2 units, columns 10 through 16 move down 3 units, etc. (At the same time, the corresponding *negatively* numbered columns move *up* the same amounts.) Let $left_k$ denote the number of the left-most column in the collection of columns that move down k units, and let $right_k$ denote the right-most column in that collection. We will obtain explicit formulas for $left_k$ and $right_k$ by first describing them recursively.

$$left_k = \begin{cases} 1 & \text{for } k = 1 \\ left_{k-1} + (2k-1) & \text{for } k \geqslant 2 \end{cases}$$

$$right_k = \begin{cases} 3 & \text{for } k = 1 \\ right_{k-1} + (2k+1) & \text{for } k \geqslant 2 \end{cases}$$

The solutions to these recursive formulas are $left_k = k^2 - 2k + 2$ and $right_k = k^2$, for all $k \geqslant 1$. We will use these values as the lower and upper limits of a sum that defines the function H:

$$H(x) = \sum_{k=1}^{\infty} \left(k \left(\underbrace{\sum_{j=k^2-2k+2}^{k^2} \left(\underbrace{wedge\left(x_{2n-1} - j\right)}_{\text{moves } j^{th} \text{ column down}} \underbrace{- wedge\left(x_{2n-1} + j\right)}_{\text{moves } -j^{th} \text{ column up}} \right)}_{\text{the collection of columns that must be moved up or down by the amount } k} \right) \right)$$

End of proof of Lemma B3

4.2.4 *Lemma B4*

Lemma B4: If $\psi \in Symp_V\left(\mathbb{R}^{2n}\right)$, where $V = \bigcup_{i=1}^{\infty} B^{2n}\left(R, x_i\right)$ is a disjoint union of round balls with $0 < R < \frac{1}{2}$, centered at integer lattice positions x_i such that

$x_{i,2n} = 0$, then its *extended Hofer infinity norm* is bounded: $\widehat{E_\infty}(\psi) \leqslant 16$

Proof of Lemma B4:

Recall that the symbol $\psi \in Symp_V(\mathbb{R}^{2n})$ means that ψ can be generated by a Hamiltonian function supported in V. Therefore, we can write ψ as a product $\psi = \prod_{i=1}^{\infty} \psi_i$, where $support(\psi_i) \subset B^{2n}(R, x_i)$ and ψ_i can be generated by a Hamiltonian function supported in $B^{2n}(R, x_i)$. Because both the ball $B^{2n}(R, x_i)$ and the time interval $[0,1]$ are compact, such a function will be uniformly-bounded, so the norms $E_\infty(\psi_i)$ and $\widehat{E_\infty}(\psi_i)$ will be finite. But we can say more than this. We know that the norms will be finite even if we take the infimum over only those Hamiltonian functions supported in some particular set U, so long as that set U contains the ball $B^{2n}(R, x_i)$.

This notion of only taking the infimum over functions supported in a particular set will be used again, so we introduce notation for it in the case of both of the infinity norms that we have defined.

$$\begin{aligned} E_\infty^U(\psi) &= \inf\{\|H\|_\infty : H \in C,\ support(H) \subset U, \text{and } H \text{ generates } \psi\} \\ \widehat{E_\infty}^U(\psi) &= \inf\left\{\widehat{\|H\|_\infty} : H \in \mathcal{H},\ support(H) \subset U, \text{and } H \text{ generates } \psi\right\} \end{aligned}$$

With this notation, we will have $\widehat{E_\infty}^U(\psi_i) \leqslant E_\infty^U(\psi_i)$, with both being finite so long as U contains $B^{2n}(R, x_i)$.

In this first use of the notation, our choice for the set U will be a *tube* containing the i^{th} ball, pointing in the x_{2n} direction, defined as follows. First, we define $R^+ = \frac{R+\frac{1}{2}}{2} = \frac{R}{2} + \frac{1}{4}$. (We just need R^+ to be a number between R and $\frac{1}{2}$.) Then we define the i^{th} *tube* as the set

$$tube_i = \left\{B^{2n-1}(R^+, x_i) \times \mathbb{R}^1\right\} = \left\{y \in \mathbb{R}^{2n} : \sum_{k=1}^{2n-1} (y_k - x_{i,k})^2 < (R^+)^2\right\}.$$

Since the i^{th} tube will contain the i^{th} ball, in our new notation we can say that

$E_\infty^{tube_i}(\psi_i)$ is finite. But in fact, in Section 4.2.5, we will prove the following *lemma*:

Lemma B5: If ψ is a symplectomorphism generated by a Hamiltonian function supported in the ball $B^{2n}(R,x)$, $0 < R < \frac{1}{2}$, and if *tube* is the set $\left\{y : \sum_{i=1}^{2n-1}(y_i - x_i)^2 < \left(\frac{R}{2}+\frac{1}{4}\right)^2\right\}$, then $E_\infty^{tube}(\psi) \leqslant 8$.

We apply this lemma to our present situation, using our ψ_i for the symplectomorphism ψ in the lemma, and our set $tube_i$ for the set *tube* in the lemma. As a result, we can say that for each $i = 1 \cdots \infty$, $E_\infty^{tube_i}(\psi_i) \leqslant 8$. Put another way, $\max_{i=1\cdots\infty} E_\infty^{tube_i}(\psi_i) \leqslant 8$.

Recall that in *Lemma B3*, we shuffled the balls into an arrangement where the $2n^{th}$ coordinate of each ball was zero. Because of this shuffling, we know that the i^{th} tube will contain the i^{th} ball and no others. Moreover, if $i \neq j$ then $tube_i$ and $tube_j$ are disjoint. (These last two statements are the motivation for the shuffling that we did in *Lemma B*.) In *Detail 5*, found in Section 4.3.5, we prove the following fact about the extended Hofer infinity norm:

Detail 5: Let ψ_i, $i \in I$ be a finite or countable collection of Hamiltonian symplectomorphisms with support $(\psi_i) \subset U_i \subset \mathbb{R}^{2n}$, where $U_i \cap U_j = \phi$ if $i \neq j$. Further, assume that for each $i \in I$, $\widehat{E_\infty}^{U_i}(\psi_i)$ is finite and that $\max_{i\in I}\widehat{E_\infty}^{U_i}(\psi_i)$ exists. Then $\widehat{E_\infty}(\psi) \leqslant 2\max_{i\in I}\widehat{E_\infty}^{U_i}(\psi_i)$.

Applying this fact to our current situation, we will use the sets $tube_i$, for $i = 1\cdots\infty$, as the sets U_i. The result is

$$\begin{aligned}
\widehat{E_\infty}\left(\prod_{i=1}^{\infty}\psi_i\right) &\leqslant 2\max_{i=1\cdots\infty}\widehat{E_\infty}^{tube_i}(\psi_i) && \text{(by } \textit{Detail } 5\text{)}\\
&\leqslant 2\max_{i=1\cdots\infty}E_\infty^{tube_i}(\psi_i) && \text{(by definition)}\\
&\leqslant 2\,(8) && \text{(by } \textit{Lemma } B5\text{)}\\
&= 16.
\end{aligned}$$

We have shown that $\widehat{E_\infty}(\psi) \leqslant 16$.

End of Proof of Lemma B4

4.2.5 *Lemma B5*

Lemma B5: If ψ is a symplectomorphism generated by a Hamiltonian function supported in the ball $B^{2n}(R,x)$, $0 < R < \frac{1}{2}$, and if *tube* is the set

$$\left\{y : \sum_{i=1}^{2n-1}(y_i - x_i)^2 < \left(\tfrac{R}{2}+\tfrac{1}{4}\right)^2\right\},$$

then $E_\infty^{tube}(\psi) \leqslant 8$

Proof of Lemma B5:

Let $\varepsilon > 0$. We will show that $E_\infty^{tube}(\psi) \leqslant 8 + 2\varepsilon$. Since ε is arbitrary, this will prove the claim.

Without loss of generality, we may assume that the point x is the origin. To see why, suppose x is not the origin. Then the symplectomorphism $\phi = \tau_x^{-1}\circ\psi\circ\tau_x$, where τ_x is the translation that sends the origin to the point x, will be supported in the ball $\tau_{-x}(B^{2n}(R,x)) = B^{2n}(R)$, which is contained in the set $\tau_{-x}(tube)$, another tube. We could use the present claim to estimate $E_\infty^{\tau_{-x}(tube)}(\phi)$. Then, by conjugation invariance, $E_\infty^{tube}(\psi) = E_\infty^{\tau_{-x}(tube)}(\phi)$.

With that assumption, the description of our *tube* becomes simpler:

$$tube = \left\{y : \sum_{i=1}^{2n-1} y_i^2 < \left(\tfrac{R}{2}+\tfrac{1}{4}\right)^2\right\}$$

We will consider translates of the ball $B^{2n}(R)$ along the length of the tube. (That is, in the x_{2n} direction.) With that in mind, we introduce the following notation:

$ball_0 =$ the original ball $= B^{2n}(R, 0)$

$ball_k =$ ball translated k units in the x_{2n} direction $= B^{2n}(R, (0, \cdots, 0, k))$

$cell_k = \left\{ y : \sum_{i=1}^{2n-1} y_i^2 < \left(\frac{R}{2} + \frac{1}{4}\right)^2 \text{ and } k - 1 - \left(\frac{R}{2} + \frac{1}{4}\right) < y_{2n} < k + \left(\frac{R}{2} + \frac{1}{4}\right) \right\}.$

The set $cell_k$ is designed to be large enough to support a Hamiltonian function that will generate a symplectomorphism that will translate a ball of radius R centered at $(0, \cdots 0, k-1)$ to the position $(0, \cdots 0, k)$. That symplectomorphism, called σ_k, is described in *Detail 4*, below. By the construction shown there, σ_k will have three important properties:

$$\sigma_k(ball_{k-1}) = ball_k$$

$$support(\sigma_k) \subset cell_k$$

$$E_\infty^{cell_k}(\sigma_k) \leqslant 1.$$

Also note that if $|j - k| \geqslant 2$ then $cell_j$ and $cell_k$ do not intersect.

By hypothesis, ψ can be generated by a Hamiltonian function supported in $ball_0$. By compactness, such a function will be uniformly-bounded. So, ψ has finite extended Hofer infinity norm, even when we restrict the support of the functions used in the infimum. That is, $E_\infty^{tube}(\psi) \leqslant E_\infty^{cell_0}(\psi) \leqslant E_\infty^{ball_0}(\psi) < \infty$. Therefore, there is an integer $N \geqslant 1$ and a finite sequence $\psi = \psi_0, \psi_1, ..., \psi_N = id$, with each $\psi_i \in Ham(B^{2n}(\mathbb{R}^{2n}))$, such that $d_\infty(\psi_i, \psi_{i+1}) < \varepsilon$ for $i = 0, ..., N-1$, even if we take the infimum over only those Hamiltonian functions supported in $ball_0$. That is, $E_\infty^{tube}\left(\psi_i^{-1}\psi_{i+1}\right) \leqslant E_\infty^{cell_0}\left(\psi_i^{-1}\psi_{i+1}\right) \leqslant E_\infty^{ball_0}\left(\psi_i^{-1}\psi_{i+1}\right) < \varepsilon$, for $i = 0, ..., N-1$.

For $k = 0, ..., N$ we define the symplectomorphism $even_k$ by

$$even_k = \begin{cases} \psi_0 & \text{if } k = 0 \\ \sigma_{2k}\sigma_{2k-1} \cdots \sigma_2\sigma_1\psi_k\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{2k-1}^{-1}\sigma_{2k}^{-1} & \text{if } 1 \leqslant k \leqslant N. \end{cases}$$

Note that the symplectomorphism $even_k$ is supported in $ball_{2k}$, which is centered at the point $(0, \cdots, 2k)$. Also note that $even_N = id$ because $\psi_N = id$.

For $k = 0, \cdots, N-1$ define the symplectomorphism odd_k by

$$odd_k = \sigma_{2k+1}\sigma_{2k}\cdots\sigma_2\sigma_1\psi_k\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{2k}^{-1}\sigma_{2k+1}^{-1}.$$

Note that odd_k is supported in $ball_{2k+1}$, which is centered at the point $(0, \cdots, 0, 2k+1)$.

Using these terms, we rewrite ψ.

$$\psi = \psi_0 = even_0 = even_0 \left(\prod_{i=0}^{N-1} odd_i odd_i^{-1}\right)\left(\prod_{j=1}^{N-1} even_j even_j^{-1}\right) even_N^{-1}$$

In this last step, we have simply inserted a bunch of terms, along with their inverses, and one final term that equals the identity map. Notice that for any two factors in the above expression, either the supports of the two factors are disjoint, or the supports are the same and the two factors are inverses of one another. Either way, we see that the entire collection of factors commutes. This allows us to rearrange them:

$$\psi = \left(\prod_{i=0}^{N-1} even_i odd_i^{-1}\right)\left(\prod_{j=1}^{N} even_j^{-1} odd_{j-1}\right).$$

Estimating the extended infinity norm, we find

$$\begin{aligned} E_\infty^{tube}(\psi) &= E_\infty^{tube}\left(\left(\prod_{i=0}^{N-1} even_i odd_i^{-1}\right)\left(\prod_{j=1}^{N} even_j^{-1} odd_{j-1}\right)\right) \\ &\leqslant E_\infty^{tube}\left(\prod_{i=0}^{N-1} even_i odd_i^{-1}\right) + E_\infty^{tube}\left(\prod_{j=1}^{N} even_j^{-1} odd_{j-1}\right). \end{aligned} \tag{4.1}$$

Consider the expression on the left in the right side of Equation 4.1. Since $even_i$ is supported in $ball_{2i}$ and odd_i is supported in $ball_{2i+1}$, we see that $even_i odd_i$ is supported in $cell_{2i+1}$. Recall that if $|j-k| \geqslant 2$ then $cell_j$ and $cell_k$ do not intersect. Therefore, if $i \neq j$, then $cell_{2i+1}$ and $cell_{2j+1}$ do not intersect. This allows us to use the trick described in Detail 5, below to say that

$$E_\infty^{tube}\left(\prod_{i=0}^{N-1} \underbrace{even_i odd_i^{-1}}_{\text{supported in} cell_{2i+1}}\right) \leqslant 2 \max_{i=0\cdots N-1} E_\infty^{cell_{2i+1}}\left(even_i odd_i^{-1}\right). \tag{4.2}$$

For $i = 0$, we have

$$\begin{aligned}
E_\infty^{cell_1}\left(even_0 odd_0^{-1}\right) &= E_\infty^{cell_1}\left(\psi_0\sigma_1\psi_0^{-1}\sigma_1^{-1}\right) \\
&\leqslant E_\infty^{cell_1}\left(\psi_0\sigma_1\psi_0^{-1}\right) + E_\infty^{cell_1}\left(\sigma_1^{-1}\right) \\
&= E_\infty^{cell_1}\left(\sigma_1\right) + E_\infty^{cell_1}\left(\sigma_1\right) \text{ (conjugation invariance)} \\
&\leqslant 1 + 1 = 2\ .
\end{aligned} \tag{4.3}$$

For $i = 1 \cdots N-1$, we have $E_\infty^{cell_{2i+1}}\left(even_i odd_i^{-1}\right) =$

$$\begin{aligned}
&= E_\infty^{cell_{2i+1}}\Big(\overbrace{\sigma_{2i}\sigma_{2i-1}\cdots\sigma_2\sigma_1\psi_i\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{2i-1}^{-1}\sigma_{2i}^{-1}}^{even_i}\circ \\
&\qquad\circ\overbrace{\sigma_{2i+1}\sigma_{2i}\cdots\sigma_2\sigma_1\psi_i^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{2i}^{-1}\sigma_{2i+1}^{-1}}^{odd_i^{-1}}\ \Big) \\
&\leqslant E_\infty^{cell_{2i+1}}\Big(\overbrace{\underbrace{\sigma_{2i}\cdots\sigma_1\psi_i\sigma_1^{-1}\cdots\sigma_{2i}^{-1}}\underbrace{\sigma_{2i+1}}\underbrace{\sigma_{2i}\cdots\sigma_1\psi_i^{-1}\sigma_1^{-1}\cdots\sigma_{2i}^{-1}}}^{\text{We will interpret this group of terms as a conjugation.}}\Big) + \\
&\qquad + E_\infty^{cell_{2i+1}}\left(\sigma_{2i+1}^{-1}\right) \qquad \text{(triangle inequality)} \\
&= E_\infty^{cell_{2i+1}}\left(\sigma_{2i+1}\right) + E_\infty^{cell_{2i+1}}\left(\sigma_{2i+1}\right) \qquad \text{(conjugation invariance)} \\
&\leqslant 1 + 1 = 2
\end{aligned} \tag{4.4}$$

Summarizing, for the expression on the left in the right side of Equation 4.1, we have the estimate

$$E_\infty^{tube}\left(\prod_{i=0}^{N-1} even_i odd_i^{-1}\right) \leqslant 2 \max_{i=0\cdots N-1} E_\infty^{cell_{2i+1}}\left(even_i odd_i^{-1}\right) \leqslant 2\,(2) = 4. \tag{4.5}$$

Next, we estimate the expression on the right in the right side of Equation 4.1. As we did above, we will consider the supports of the factors in the product in order to be able to exploit the trick of *Detail 5.* Since $even_j$ is supported in $ball_{2j}$ and odd_{j-1} is supported in $ball_{2j-1}$, we see that $even_j odd_{j-1}$ is supported in $cell_{2j}$. As above, we know that if $i \neq j$, then $cell_{2i}$ and $cell_{2j}$ do not intersect, and that will allow us to use the trick from *Detail 5.*

$$E_\infty^{tube}\left(\prod_{j=1}^{N} \underbrace{even_j^{-1} odd_{j-1}}_{\text{supported in } cell_{2j}}\right) \leqslant 2 \max_{j=1\cdots N} E_\infty^{cell_{2j}}\left(even_j^{-1} odd_{j-1}\right)$$

For $j = 1 \cdots N$, we make the estimate $E_\infty^{cell_{2j}}\left(even_j^{-1} odd_{j-1}\right) =$

$$= E_\infty^{cell_{2j}} \left(\overbrace{\sigma_{2j} \cdots \sigma_1 \psi_j^{-1} \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{2j}^{-1}}^{even_j^{-1}} \overbrace{\sigma_{2j-1} \cdots \sigma_1 \psi_{j-1} \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1}}^{odd_{j-1}} \right)$$

$$= E_\infty^{cell_{2j}} \left(\sigma_{2j} \sigma_{2j-1} \cdots \sigma_1 \psi_j^{-1} \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1} \sigma_{2j}^{-1} \sigma_{2j-1} \cdots \sigma_1 \circ \right.$$

$$\left. \overbrace{\psi_j \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1} \sigma_{2j-1} \cdots \sigma_1 \psi_j^{-1}}^{\text{Insert group of terms that comprise the } id \text{ map.}} \quad \psi_{j-1} \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1} \right)$$

$$\leqslant E_\infty^{cell_{2j}} (\sigma_{2j}) +$$

$$+ E_\infty^{cell_{2j}} \left(\overbrace{\underbrace{\sigma_{2j-1} \cdots \sigma_1 \psi_j^{-1} \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1}} \underbrace{\sigma_{2j}^{-1}} \underbrace{\sigma_{2j-1} \cdots \sigma_1 \psi_j \sigma_1^{-1} \cdots \sigma_{2j-1}^{-1}}}^{\text{We will interpret this group of terms as a conjugation.}} \right) +$$

$$+ E_\infty^{cell_{2j}} \left(\overbrace{\underbrace{\sigma_{2j-1} \cdots \sigma_1} \underbrace{\psi_j^{-1} \psi_{j-1}} \underbrace{\sigma_1^{-1} \cdots \sigma_{2j-1}^{-1}}}^{\text{Interpret this group of terms as a conjugation.}} \right)$$

$$= E_\infty^{cell_{2j}} (\sigma_{2j}) + E_\infty^{cell_{2j}} (\sigma_{2j}) + E_\infty^{ball_0} \left(\psi_j^{-1} \psi_{j-1}\right) \text{ (conjugation inv.)}$$

$$\leqslant 1 + 1 + \varepsilon = 2 + \varepsilon.$$

Summarizing, for the expression on the right in the right side of Equation 4.1, we have the estimate

$$E_\infty^{tube} \left(\prod_{j=1}^{N} even_j^{-1} odd_{j-1} \right) \leqslant 2 \max_{j=1 \cdots N} E_\infty^{cell_{2j}} \left(even_j^{-1} odd_{j-1} \right) \leqslant 2\,(2+\varepsilon) = 4 + 2\varepsilon. \tag{4.6}$$

Plugging Equations 4.5 and 4.6 back into Equation 4.1, we have

$$\begin{aligned} E_\infty^{tube}(\psi) &\leqslant E_\infty^{tube}\left(\prod_{i=0}^{N-1} even_i odd_i^{-1}\right) + E_\infty^{tube}\left(\prod_{j=1}^{N} even_j^{-1} odd_{j-1}\right) \\ &\leqslant 4 + (4 + 2\varepsilon) = 8 + 2\varepsilon. \end{aligned}$$

End of Proof of Lemma B5

4.3 Details

4.3.1 *Detail 1:* An isotopy that transforms a symplectic ball

step i: Introduce $g_{i,t}$, $g_{i,a}$, $g_{i,b}$, $g_{i,c}$, and $g_{i,d}$.

In this section, we describe an isotopy $g_{i,t}$ that turns the i^{th} *symplectic ball* into a *round ball* centered at an integer lattice point. Each $g_{i,t}$ is obtained by a time-concatenation of four isotopies, $g_i = g_{ia} * g_{ib} * g_{ic} * g_{id}$. The operation of these isotopies is as follows: g_{ia} moves the i^{th} symplectic ball radially outward from the origin, while making it more elliptical (in the sense that will be explained below). Then, g_{ib} turns the *symplectic ball* into a *symplectic ellipse*, while holding it in place. Next, g_{ic} turns the *symplectic ellipse* into a *round ball*, while holding it in place. Finally, g_{id} translates the round ball to the nearest integer lattice position.

step ii: Describe g_{ia}.

Recall notation from *step i* of the proof of *Lemma B2*. The i^{th} *symplectic ball* is the image, $f_i\left(B^{2n}(R)\right)$, of the symplectic embedding, $f_i : B^{2n}(R) \to \mathbb{R}^{2n}$. The set $f_i\left(B^{2n}(R+\varepsilon)\right)$ is called the i^{th} *neighborhood*, with ε chosen for the whole set of k neighborhoods in order that they be disjoint and do not intersect $B^{2n}(r_a)$.

In *Detail 2*, below, the workings of a smooth family of symplectomorphisms $\gamma_{f,t} \in Symp\left(\mathbb{R}^{2n}\right)$, for $t \in [0,1)$, referred to as a *partial linearization with translation applied to a symplectic ball*, are explained. Here, we will apply such a symplectomorphism to the i^{th} ball and describe its effect at some $t \in [0,1)$.

$$\gamma_{f_i,t}\left(f_i\left(Ball^{2n}(R)\right)\right) = \underbrace{\tau_{\frac{f_i(0)}{1-t}}}_{\text{translation}} \circ \underbrace{m_{\frac{1}{1-t}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-t} \circ f_i^{-1} \circ \left(f_i\left(Ball^{2n}(R)\right)\right)}_{\text{symplectic ball becoming more elliptical; center fixed at origin}}$$

In this expression, τ_x is the translation that sends the origin to the point $x \in \mathbb{R}^{2n}$, and m_c is multiplication by the non-zero real number c. At time $t = 0$, this expression simplifies to $\gamma_{f_i,0}\left(Ball^{2n}(R)\right) = f_i\left(Ball^{2n}(R)\right)$, the original symplectic ball. As discussed in *Detail 1*, for times $0 < t < 1$, the above expression can be thought of as a difference quotient and, in the $t \to 1$ limit, it approaches the composition of a linear map with a translation.

A linear image of a ball is an ellipse; so as t approaches 1, one could say that the ball is becoming more elliptical. Furthermore, if $f_i(0) \neq 0$, the image gets translated farther and farther from the origin in the $f_i(0)$ direction. In *Detail 3* we will choose an *ending time*, $0 \leqslant \lambda < 1$, for $\gamma_{f_i,t}$. (The trick there will be to find an ending time that will work for the whole set, $i = 1, \cdots, k$, of symplectic balls.) We use this λ to re-scale the time in $\gamma_{f_i,t}$, and call the resulting map, g_{ia}:

$$g_{ia,t} = \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda t} \circ f_i^{-1} \text{for } t \in [0,1].$$

Figures 4.5 and 4.6 show the i^{th} symplectic ball evolving under the influence of g_{ia}. Also shown is the j^{th} ball evolving under the influence of g_{ja}. The symplectic balls are the darker shapes, and are described by the formulas

$$\begin{aligned} g_{ia,t}\left(f_i\left(B^{2n}(R)\right)\right) &= \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda t} \circ f_i^{-1} \circ f_i\left(B^{2n}(R)\right) \\ &= \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda t}\left(B^{2n}(R)\right). \end{aligned}$$

Surrounding the original symplectic balls are dotted regions, the i^{th} and j^{th} *neighborhoods*, $f_i\left(B^{2n}(R+\varepsilon)\right)$ and $f_j\left(B^{2n}(R+\varepsilon)\right)$, that were described in *Lemma B2*. Note that as the symplectic balls evolve, they will always be confined to regions (the larger dotted shapes in the figures below) that are simply a linear magnification of these neighborhoods. That is,

$$\begin{aligned} \underbrace{g_{ia,t}\left(f_i\left(B^{2n}(R)\right)\right)}_{\text{evolving}\, i^{th} \text{symplectic ball}} &= \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda t}\left(B^{2n}(R)\right) \\ &= m_{\frac{1}{1-\lambda t}} \circ \tau_{f_i(0)} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda t}\left(B^{2n}(R)\right) \\ &= m_{\frac{1}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t}\left(B^{2n}(R)\right) \\ &\subset m_{\frac{1}{1-\lambda t}} \circ f_i\left(B^{2n}(R)\right) \\ &\subset \underbrace{m_{\frac{1}{1-\lambda t}}}_{\text{linear magnification}} \circ \underbrace{f_i\left(B^{2n}(R+\varepsilon)\right)}_{i^{th}\text{neighborhood}}. \end{aligned}$$

What is slightly misleading about these figures is that the two isotopies do not really work simultaneously in this way. That is because the isotopy g_{ia} would also

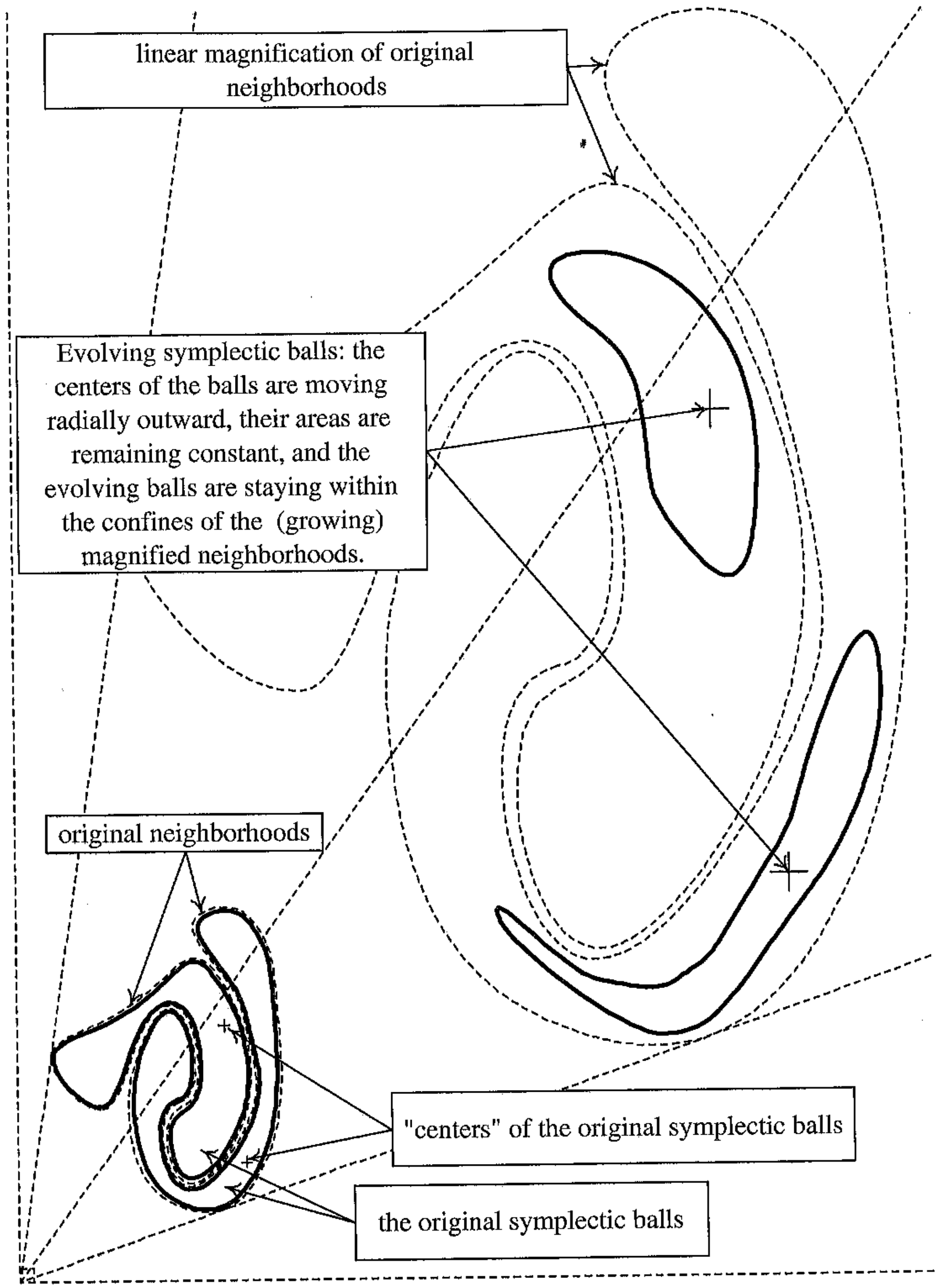

Figure 4.5: Zoomed-in view of the start of isotopies g_{ia} and g_{ja}

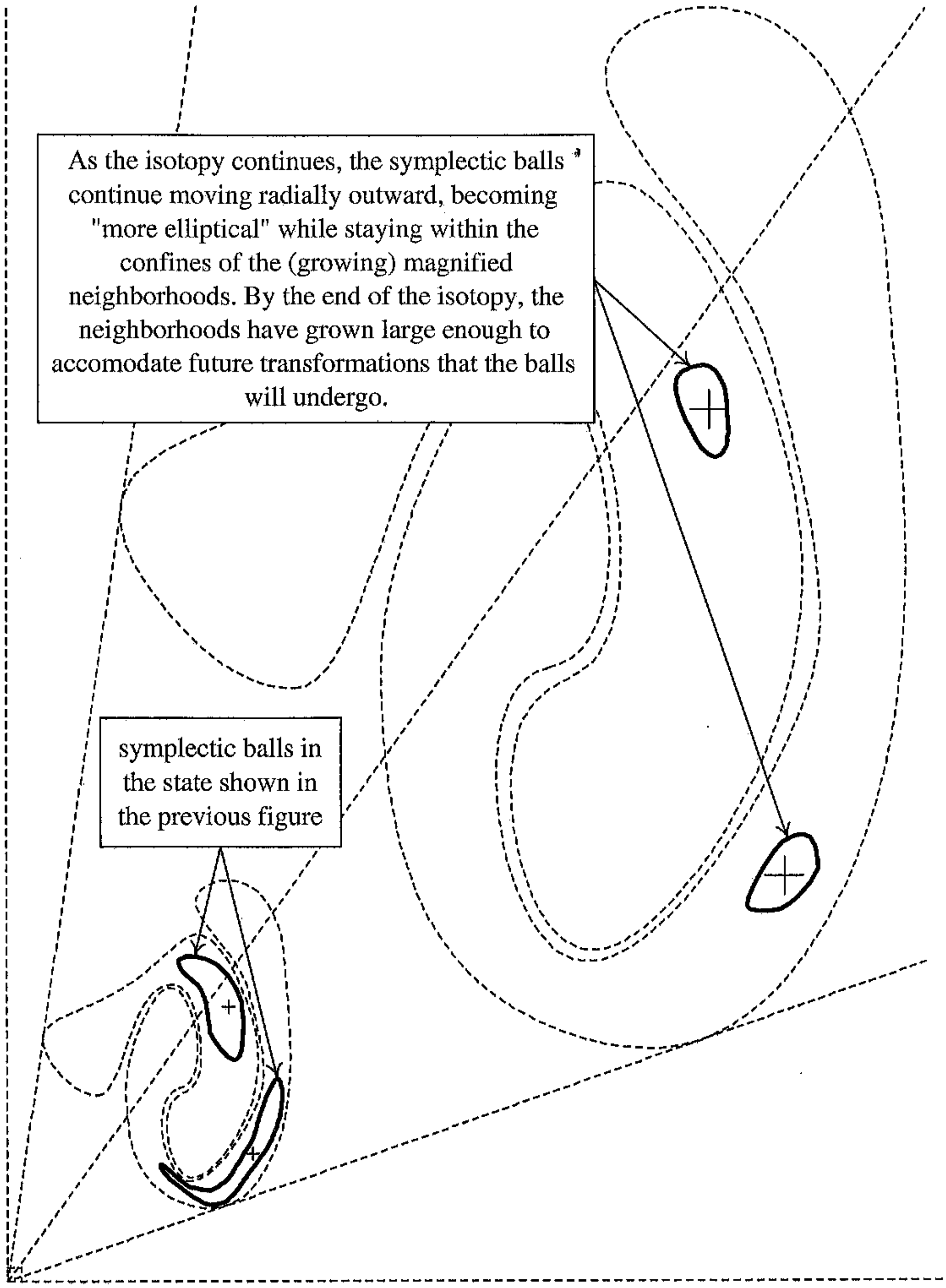

Figure 4.6: Zoomed-out view of the end of isotopies g_{ia} and g_{ja}

affect the j^{th} symplectic ball, while the isotopy g_{ja} would also affect the i^{th} symplectic ball, etc. But in *Lemma B2*, where the results of our present work are actually used, the individual isotopies g_i, with $i = 1 \cdots k$, will be incorporated into a more refined isotopy, h, that will affect the i^{th} symplectic ball in precisely the same way that g_{ia} does, while simultaneously affecting the j^{th} symplectic ball in precisely the same way that g_{ja} does, etc.

step iii: Describe $g_{ib,t}$.

At the end of the operation of the isotopy g_{ia}, the i^{th} symplectic ball has been transformed into another symplectic ball,

$$\begin{aligned} g_{ia,1}\left(f_i\left(B^{2n}\left(R\right)\right)\right) &= \tau_{\frac{f_i(0)}{1-\lambda}} \circ m_{\frac{1}{1-\lambda}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda} \circ f_i^{-1}\left(f_i\left(B^{2n}\left(R\right)\right)\right) \\ &= \tau_{\frac{f_i(0)}{1-\lambda}} \circ m_{\frac{1}{1-\lambda}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda}\left(B^{2n}\left(R\right)\right), \end{aligned}$$

that is more elliptical than the original and is centered at the point

$$g_{ia,1}\left(f_i\left(0\right)\right) = \tau_{\frac{f_i(0)}{1-\lambda}} \circ m_{\frac{1}{1-\lambda}} \circ \tau_{-f_i(0)} \circ f_i \circ m_{1-\lambda}\left(0\right) = \tfrac{1}{1-\lambda} f_i\left(0\right),$$

where λ, $0 \leqslant \lambda < 1$, is the *ending time* described in *Detail 3*. For simplicity of notation, we introduce the abbreviation θ_i for the composition $g_{ia,1} \circ f_i$. With this notation, at the end of the operation of the isotopy g_{ia}, the i^{th} symplectic ball has been transformed to the symplectic ball $\theta_i\left(B^{2n}\left(R\right)\right)$, centered at the point $\theta_i\left(0\right) = \frac{1}{1-\lambda} f_i\left(0\right)$.

In *Detail 2*, the workings of another family of symplectomorphisms, $\beta_{f,t} \in Symp\left(\mathbb{R}^{2n}\right)$, for $t \in [0,1]$, referred to as a *linearization applied to a symplectic ball*, are explained. Here, we wish apply such a symplectomorphism to the transformed i^{th} symplectic ball, $\theta_i\left(B^{2n}\left(R\right)\right)$. To accomplish that, we will substitute the new symbol θ_i for the symbol f in the expression for $\beta_{f,t}$. The result will be the isotopy $g_{ib,t}$:

$$g_{ib,t} = \beta_{\theta_i,t} = \begin{cases} \tau_{\theta_i(0)} \circ m_{\frac{1}{1-\lambda}} \circ \tau_{-\theta_i(0)} \circ \theta_i \circ m_{1-\lambda} \circ \theta_i^{-1} & \text{if } t \in [0,1) \\ \tau_{\theta_i(0)} \circ L_{(d\theta_i)_0} \circ \theta_i^{-1} & \text{if } t = 1. \end{cases}$$

In this expression, $L_{(d\theta_i)_0}$ is the linear operator obtained by left multiplication by the matrix $(d\theta_i)_0$. Because θ is a symplectomorphism, $(d\theta_i)_0$ is a symplectic matrix, and we could describe $L_{(d\theta_i)_0}(Ball^{2n}(R))$ as a *symplectic ellipse*. The results of applying the isotopy $g_{ib,t}$ to the transformed i^{th} symplectic ball $\theta_i(B^{2n}(R))$ are as follows. Because $g_{ib,0} = \beta_{\theta_i,0} = id$, at time $t = 0$ the ball is, of course, unchanged. By time $t = 1$, the ball has been transformed to

$$\begin{aligned} g_{ib,1}(\theta_i(Ball^{2n}(R))) &= \beta_{\theta_i,1}(\theta_i(Ball^{2n}(R))) \\ &= \tau_{\theta_i(0)} \circ L_{(d\theta_i)_0} \circ \theta_i^{-1}(\theta_i(Ball^{2n}(R))) \\ &= \tau_{\theta_i(0)} \circ \underbrace{L_{(d\theta_i)_0}(Ball^{2n}(R))}_{\text{symplectic ellipse}}. \end{aligned}$$

So the isotopy $g_{ib,t}$, applied to the i^{th} symplectic ball (after that ball has already been translated and deformed by isotopy $g_{ia,t}$) has the effect of turning the *symplectic ball* into a *symplectic ellipse*, while holding it in place. Figure 4.7 shows the effect of the isotopies g_{ib} and g_{jb} on the i^{th} and j^{th} symplectic balls: They have become symplectic ellipses, centered at the spots where the symplectic balls were sitting at the end of isotopies g_{ia} and g_{ja}. Also shown on the figures are the dotted regions that are the magnification of the original i^{th} and j^{th} neighborhoods. As shown in the figures, the images of the i^{th} and j^{th} symplectic balls, as they evolve under the influence of isotopies g_{ib} and g_{jb}, remain within these regions. This is not automatic. Rather, it is because, in *Detail 3*, we will be careful to choose an ending time λ sufficient to make it happen.

step iv: Describe $g_{ic,t}$.

By the end of the operation of $g_{ia} * g_{ib}$, the i^{th} *symplectic ball* has been

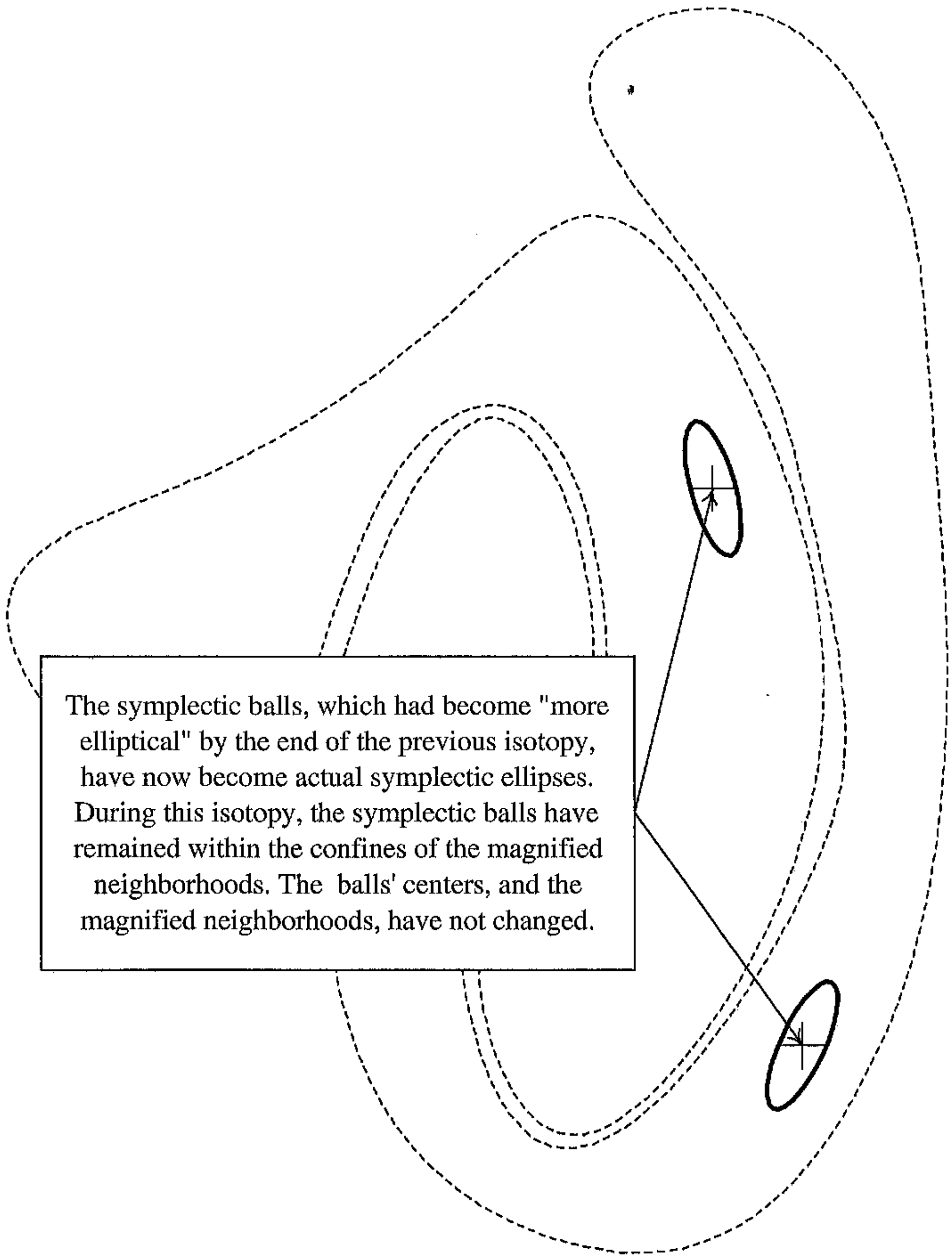

Figure 4.7: Effect of the isotopies g_{ib} and g_{jb}

turned into a *symplectic ellipse* centered at $\frac{1}{1-\lambda}f_i(0)$:

$$\begin{aligned}(g_{ia} * g_{ib})_1 \left(i^{th}\text{symplectic ball}\right) &= g_{ib,1} \circ g_{ia,1}\left(f_i\left(B^{2n}(R)\right)\right) \\ &= g_{ib,1}\left(\theta_i\left(B^{2n}(R)\right)\right) \\ &= \tau_{\theta_i(0)} \circ L_{(d\theta_i)_0}\left(B^{2n}(R)\right),\end{aligned}$$

where $\theta_i(0) = \frac{1}{1-\lambda}f_i(0)$.

Since $(d\theta_i)_0 \in Sp(\mathbb{R}^{2n})$, and $Sp(\mathbb{R}^{2n})$ is path connected, we know that there is a path $\sigma_{i,t} \in Sp(2n)$, $t \in [0,1]$ connecting the identity map to $(d\theta_i)_0^{-1}$. That is, $\sigma_{i,0} = id$ and $\sigma_{i,1} = (d\theta_i)_0^{-1}$. So we can define $g_{ic,t} = \tau_{\theta_i(0)} \circ L_{\sigma_{i,t}} \circ \tau_{-\theta_i(0)}$. At time $t = 0$, this expression reduces to $g_{ic,0} = \tau_{\theta_i(0)} \circ L_{\sigma_{i,0}} \circ \tau_{-\theta_i(0)} = id$. Considering the time $t = 1$ expression, $g_{ic,1}$, applied to the i^{th} symplectic ellipse, we find

$$\begin{aligned}g_{ic,1}\left(i^{th}\text{ symplectic ellipse}\right) &= g_{ic,1}\left((g_{ia} * g_{ib})_1\left(i^{th}\text{ symplectic ball}\right)\right) \\ &= g_{ic,1}\left(\tau_{\theta_i(0)} \circ L_{(d\theta_i)_0}\left(B^{2n}(R)\right)\right) \\ &= \tau_{\theta_i(0)} \circ L_{\sigma_{i,1}} \circ \tau_{-\theta_i(0)}\left(\tau_{\theta_i(0)} \circ L_{(d\theta_i)_0}\left(B^{2n}(R)\right)\right) \\ &= \tau_{\theta_i(0)} \circ L_{(d\theta_i)_0^{-1}} \circ \tau_{-\theta_i(0)} \circ \tau_{\theta_i(0)} \circ L_{(d\theta_i)_0}\left(B^{2n}(R)\right) \\ &= \tau_{\theta_i(0)}\left(B^{2n}(R)\right).\end{aligned}$$

So we see that $g_{ic,t}$ turns the i^{th} *symplectic ellipse* into a *round ball* while holding it in place, with its center located at $\theta_i(0) = \frac{1}{1-\lambda}f_i(0)$. Figure 4.8 shows the effect of the isotopies g_{ic} and g_{jc} on the i^{th} and j^{th} symplectic balls. They have become round balls, centered at the spots where the symplectic ellipses were sitting at the end of isotopies g_{ib} and g_{jb}. Also shown on the figures are the dotted regions that are the magnification of the original i^{th} and j^{th} neighborhoods. As shown in the figures, the images of the i^{th} and j^{th} symplectic balls, as they evolve under the influence of isotopies g_{ic} and g_{jc}, remain within these regions. Again, this is not automatic, but rather because, in *Detail 3*, we will be careful to choose an ending time λ sufficient to make it happen.

step v: Describe g_{id}.

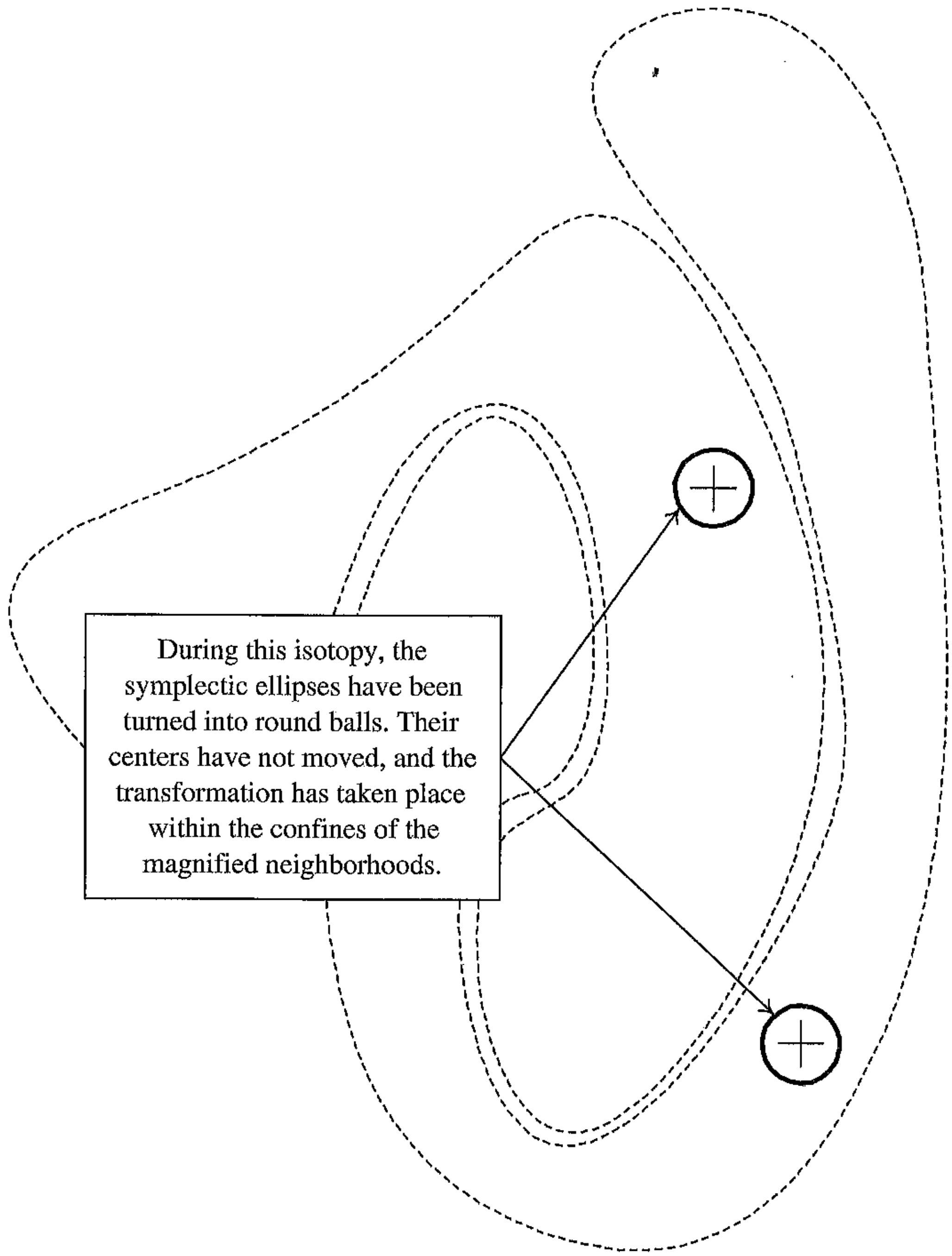

Figure 4.8: Effect of the isotopies g_{ic} and g_{jc}

At the end of the operation of $g_{ia} * g_{ib} * g_{ic}$, the i^{th} *symplectic ball* has been turned into a *round ball*, centered at $\theta_i(0) = \frac{1}{1-\lambda} f_i(0)$. Define $g_{id,t}$ to be the translation $g_{id,t} = \tau_{t\left(x_i - \frac{1}{1-\lambda} f_i(0)\right)}$, where x_i is the integer lattice point nearest $\frac{1}{1-\lambda} f_i(0)$. So $g_{id,0} = \tau_0 = id$, while $g_{id,1}$ applied to the i^{th} ball is

$$\begin{aligned} g_{id,1}\left(i^{th}\text{ball}\right) &= g_{id,1}\left(\tau_{\frac{1}{1-\lambda} f_i(0)}\left(B^{2n}(R)\right)\right) \\ &= \tau_{\left(x_i - \frac{1}{1-\lambda} f_i(0)\right)}\left(\tau_{\frac{1}{1-\lambda} f_i(0)}\left(B^{2n}(R)\right)\right) \\ &= \tau_{x_i}\left(B^{2n}(R)\right). \end{aligned}$$

We see that the isotopy $g_{id,t}$ moves the i^{th} ball to the nearest integer lattice point. Figure 4.9 shows the effect of the isotopies g_{id} and g_{jd} on the i^{th} and j^{th} balls. As above, during this isotopy, the balls remains confined to the dotted regions shown, because of our choice of the ending time, λ

End of Detail 1

4.3.2 *Detail 2:* Linearizations, moving and fixed

Part i: The *linearization at zero* (The *Alexander trick*)

Let $f \in Diff\left(\mathbb{R}^{2n}\right)$ be any diffeomorphism of $\mathbb{R}^{2n}$. Let $m_c : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the multiplication by the scalar c and let $\tau_b : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the translation by b. Then for each $t \in (0,1]$, the map $m_{\frac{1}{t}} \circ \tau_{-f(0)} \circ f \circ m_t$ is an element of $Diff\left(\mathbb{R}^{2n}\right)$. Note that when this diffeomorphism acts on an element $x \in \mathbb{R}^{2n}$, the result is just the difference quotient,

$$m_{\frac{1}{t}} \circ \tau_{-f(0)} \circ f \circ m_t(x) = \frac{f(tx) - f(0)}{t}.$$

Because f is differentiable, the $t \to 0$ limit exists in $\mathbb{R}^{2n}$:

$$\begin{aligned} \lim_{t\to 0} m_{\frac{1}{t}} \circ \tau_{-f(0)} \circ f \circ m_t(x) &= \lim_{t\to 0} \frac{f(tx) - f(0)}{t} \\ &= (df)_0\, x \\ &= L_{(df)_0}(x) \end{aligned}$$

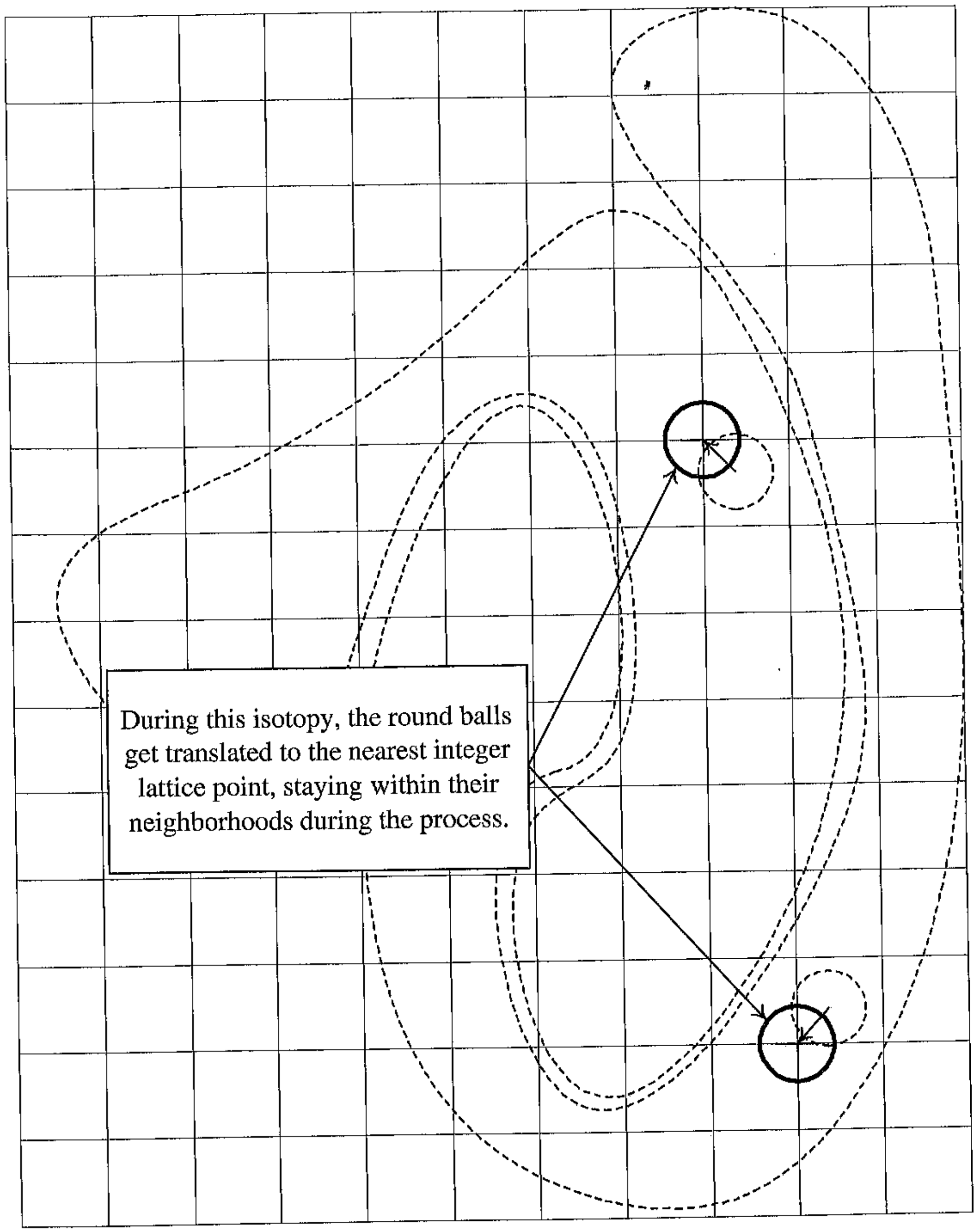

Figure 4.9: Effect of the isotopies g_{id} and g_{jd}

Here, $L_{(df)_0}$ is the linear operator obtained by left multiplication by the matrix $(df)_0$, and is called the *linearization of f at zero.* If we define a path $\alpha_{f,t}$ in $Diff(\mathbb{R}^{2n})$ as

$$\alpha_{f,t} = \begin{cases} m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} & \text{if } t \in [0,1) \\ L_{(df)_0} & \text{if } t = 1, \end{cases}$$

then $\alpha_{f,t}$ is smooth in the compact-open C^∞ topology on $Diff(\mathbb{R}^{2n})$, $\alpha_{f,0} = \tau_{-f(0)} \circ f$, and $\alpha_{f,1}$ is a linear map. We could say that α is a path from a *translation* of f to a *linearization* of f. Acting on a round ball with these maps, we have $\alpha_{f,0}(Ball^{2n}(R)) = f(Ball^{2n}(R)) - f(0)$, a diffeomorphic image of the ball, which we could call a diffeomorphic ball, and $\alpha_{f,1}(Ball^{2n}(R)) = L_{(df)_0}(Ball^{2n}(R))$, a linear image of the ball, i.e. an ellipse. Note that the center of the image, the image of $x = 0$, remains fixed at zero:

$$\alpha_{f,t}(0) = \begin{cases} f(0) - f(0) = 0 & \text{if } t \in [0,1) \\ L_{(df)_0}(0) = 0 & \text{if } t = 1 \end{cases}$$

When we actually use paths like the one above in this paper, the map f will be a symplectomorphism. In that case, we will have $\alpha_{f,t} \in Symp(\mathbb{R}^{2n})$ for all $t \in [0,1]$, and we will say *symplectic ball*, and *symplectic ellipse*, the latter because $(df)_0 \in Sp(2n)$. For the remainder of this section, we will assume that f is a symplectic map, though the techniques work for any diffeomorphism.

Part ii: The *linearization applied to a symplectic ball*

The path that we constructed above was used to transform a round ball. At time $t = 0$, the result was a symplectic ball, and at time $t = 1$, the image was a symplectic ellipse, both centered at the origin. However, our need will be to transform not a *round ball*, but rather an existing *symplectic ball*, $f(Ball^{2n}(R))$, and to do it not at the origin, but rather at the spot where the symplectic ball sits. For that reason, we compose $\alpha_{f,t}$ with a translation and with f^{-1} to achieve a new path, a path which we will call $\beta_{f,t}$:

$$\beta_{f,t} = \begin{cases} \tau_{f(0)} \circ m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1} & \text{if } t \in [0,1) \\ \tau_{f(0)} \circ L_{(df)_0} \circ f^{-1} & \text{if } t = 1 \end{cases}$$

Then $\beta_{f,t}$ is a path in $Symp\left(\mathbb{R}^{2n}\right)$, with $\beta_{f,0} = id$.

When $\beta_{f,t}$ acts on the symplectic ball $f\left(Ball^{2n}(R)\right)$, the result at time $t = 0$ is

$$\beta_{f,0}\left(f\left(Ball^{2n}(R)\right)\right) = id\left(f\left(Ball^{2n}(R)\right)\right) = f\left(Ball^{2n}(R)\right),$$

the original symplectic ball. At time $t = 1$, the result is

$$\beta_{f,1}\left(f\left(Ball^{2n}(R)\right)\right) = \tau_{f(0)} \circ L_{(df)_0} \circ f^{-1}\left(f\left(Ball^{2n}(R)\right)\right) = \tau_{f(0)} \circ L_{(df)_0}\left(Ball^{2n}(R)\right),$$

a symplectic ellipse, with center located at the same spot where the center of the original symplectic ball was located.

Part iii: Partial linearization with translation applied to a symplectic ball

In the construction of $\beta_{f,t}$, above, we inserted an additional translation into the expression for $\alpha_{f,t}$. As a result, the images of the symplectic ball, $\beta_{f,t}\left(f\left(Ball^{2n}(R)\right)\right)$, remained fixed at $f(0)$. If, instead, we *remove* the translation from the expression that describes $\alpha_{f,t}$ for $t \in [0,1)$, we create a new path in $Symp\left(\mathbb{R}^{2n}\right)$ that has the effect of moving the center of the image radially outward, in the $f(0)$ direction. We call this path $\gamma_{f,t}$:

$$\begin{aligned} \gamma_{f,t} &= m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1} \\ &= m_{\frac{1}{1-t}} \circ \underbrace{\tau_{f(0)} \circ \tau_{-f(0)}}_{id} \circ f \circ m_{1-t} \circ f^{-1} \\ &= \tau_{\frac{f(0)}{1-t}} \circ m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1}. \end{aligned}$$

This expression looks just like the one above for $\beta_{f,t}$ in the time interval $t \in [0,1)$, except that the final translation is not fixed. Rather, the amount of translation increases as time t approaches 1.

Consider the effect of the symplectomorphism $\gamma_{f,t}$ on the symplectic ball $f\left(Ball^{2n}(R)\right)$:

$$\gamma_{f,t}\left(f\left(Ball^{2n}(R)\right)\right) = \underbrace{\tau_{\frac{f(0)}{1-t}}}_{\text{translation}} \circ \underbrace{m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1}\left(f\left(Ball^{2n}(R)\right)\right)}_{\text{symplectic ball becoming more elliptical, with center fixed at origin}} .$$

At time $t = 0$, this simplifies to $\gamma_{f,0}\left(Ball^{2n}(R)\right) = f\left(Ball^{2n}(R)\right)$, the original symplectic ball. As t approaches 1, the image of the ball becomes more elliptical and, if $f(0) \neq 0$, the image gets translated farther and farther from the origin in the $f(0)$ direction, with distance from the origin going to infinity. For that reason, we cannot extend the definition of $\gamma_{f,t}$ from the time interval $t \in [0,1)$ to the entire interval $[0,1]$. But in our use of this symplectomorphism, we will choose an ending time, τ that is less than 1. The result, then, will be that as t goes from 0 to τ, the image of the ball will evolve from the original symplectic ball, $f\left(Ball^{2n}(R)\right)$, to another symplectic ball, $\gamma_{f,\tau}\left(Ball^{2n}(R)\right)$, that is *more elliptical* than the original and is translated radially outward from the origin in the $f(0)$ direction. (Of course, if $f(0)$ does equal zero, then we *could* extend the definition of $\gamma_{f,t}$ to the time interval $t \in [0,1]$, but we don't need to. For our purposes, $\gamma_{f,t}$ will be stopped at the ending time, $\tau < 1$.)

End of Detail 2

4.3.3 *Detail 3: The ending time*

Given a disjoint union of symplectic balls, $\bigcup_{i=1}^{k} f_i\left(B^{2n}(R)\right)$, we choose an *ending time*, $\lambda \geqslant 0$.

step i: Choose R_1 for one symplectic ball.

A symplectic ball is the image $f\left(B^{2n}(R)\right)$, of a symplectic embedding, $f : B^{2n}(R) \to \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the open ball of radius R. The *center* of this symplectic ball is the image of the origin, $f(0)$. Choose an R_1 such that

$f(B^{2n}(R)) \subset B^{2n}(R_1, f(0))$. Note that of course $R \leqslant R_1$, because the map f is volume-preserving.

step ii: Choose R_2 for one symplectic ball.

In *Detail 2*, we described the *linearization applied to a symplectic ball*, a technique based on the Alexander trick. Using that technique, we can construct a path in $Symp(\mathbb{R}^{2n})$ which we will call $\beta_{f,t}$, where $t \in [0,1]$. When we apply $\beta_{f,t}$ to our symplectic ball, the important results are:

$$\beta_{f,t}\left(f\left(B^{2n}(R)\right)\right) = \begin{cases} f(B^{2n}(R)), \text{the original symplectic ball,} & \text{if } t = 0 \\ \tau_{f(0)} \circ m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t}(B^{2n}(R)) & \text{if } 0 < t < 1 \\ \tau_{f(0)} \circ L_{(df)_0}(B^{2n}(R)), \text{a symplectic ellipse,} & \text{if } t = 1. \end{cases}$$

Note that in each case, the result is a symplectic image of $B^{2n}(R)$, with center located at $f_i(0)$. Choose radius R_2 such that for all $t \in [0,1]$, $\beta_{f,t}(f(B^{2n}(R))) \subset B^{2n}(R_2, f(0))$. Also note that $R_1 \leqslant R_2$.

step iii: Choose R_3 for one symplectic ball.

Because $(df)_0 \in Sp(\mathbb{R}^{2n})$, and $Sp(\mathbb{R}^{2n})$ is path connected, we know that there is a path $\sigma_t \in Sp(2n)$, $t \in [0,1]$ connecting the identity map to $(df)_0^{-1}$. That is, $\sigma_0 = id$ and $\sigma_1 = (df)_0^{-1}$. Define a map $g_t = \tau_{f(0)} \circ L_{\sigma_t} \circ \tau_{-f(0)}$. Then when g_t applied to the symplectic ellipse that we obtained at the end of *step ii*, above, the result will be

$$g_t\left(\tau_{f(0)} \circ L_{(df)_0}\left(B^{2n}(R)\right)\right) = \begin{cases} \tau_{f(0)} \circ L_{(df)_0}(B^{2n}(R)) & \text{if } t = 0 \\ \tau_{f(0)} \circ L_{\sigma_t \circ (df)_0}(B^{2n}(R)) & \text{if } 0 < t < 1 \\ \tau_{f(0)}(B^{2n}(R)) & \text{if } t = 1. \end{cases}$$

The $t = 0$ result is simply the original symplectic ellipse. The $0 < t < 1$ result is another symplectic ellipse, centered at the same location. We could think of these ellipses as becoming more spherical as time approaches 1. Finally, the $t = 1$ result is a *round ball*, centered at the same location. Choose radius R_3 such that for all $t \in [0,1]$, $g_t\left(\tau_{f(0)} \circ L_{(df)_0}\left(B^{2n}(R)\right)\right) \subset B^{2n}(R_3, f(0))$. Note that $R_2 \leqslant R_3$.

step iv: Choose R_4 for one symplectic ball.

We will be interested in determining the radius R_4 necessary to insure that a ball centered at $f(0)$ has room in its interior for all the activities of the previous three steps, and is also large enough to contain a copy of $B^{2n}(R)$ that is centered at an integer lattice point. But since $0 < R < \frac{1}{2}$, we know that a ball of radius 2 centered at any point will contain some $B^{2n}(R, x_k)$, where x_k denotes an integer lattice point. Therefore, let $R_4 = \max\{2, R_3\}$.

step v: Choose λ for one symplectic ball.

In *Detail 2*, we described the *partial linearization with translation*. It was a path $\gamma_{f,t}$ in $Symp(\mathbb{R}^{2n})$, defined by $\gamma_{f,t} = m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1}$ When we applied the symplectomorphism $\gamma_{f,t}$ to the symplectic ball $f(Ball^{2n}(R))$, the result was

$$\gamma_{f,t}\left(f\left(Ball^{2n}(R)\right)\right) = \underbrace{\tau_{\frac{f(0)}{1-t}}}_{\text{translation}} \circ \underbrace{m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1}\left(f\left(Ball^{2n}(R)\right)\right)}_{\text{symplectic ball becoming more elliptical, with center fixed at origin}} .$$

As time t approaches 1, the image becomes more elliptical, with center moving radially outward from $f(0)$.

If we examine again the defining expression for $\gamma_{f,t}$, we will notice that the evolving image of the symplectic ball remains within the confines of a simple magnification of the original neighborhood that was described in *Lemma B2*:

$$\begin{aligned}
\gamma_{f,t} f\left(B^{2n}(R)\right) &= m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1}\left(f\left(Ball^{2n}(R)\right)\right) \\
&= m_{\frac{1}{1-t}} \circ f \circ m_{1-t}\left(Ball^{2n}(R)\right) \\
&\subset m_{\frac{1}{1-t}} \circ f\left(B^{2n}(R)\right) \text{ (magnification of original symplectic ball)} \\
&\subset m_{\frac{1}{1-t}} \circ \underbrace{f\left(B^{2n}(R+\varepsilon)\right)}_{\text{original neighborhood}} \text{ (magnification of neighborhood)}
\end{aligned}$$

Also note that as it evolves under the influence of the *partial linearization with translation*, the image of the symplectic ball has precisely the same shape as it would if it were evolving under the influence of the *linearization with fixed center*, the only difference being the translation. This is easy to see if we examine the expressions for the two linearizations during the time interval $0 \leqslant t < 1$.

$$\begin{aligned}
\beta_{f,t}\left(f\left(B^{2n}(R)\right)\right) &= \underbrace{\tau_{f(0)}}_{\text{translation}} \circ \underbrace{m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t}\left(Ball^{2n}(R)\right)}_{\text{evolving shape}} \\
\gamma_{f,t}\left(f\left(Ball^{2n}(R)\right)\right) &= \underbrace{\tau_{\frac{f(0)}{1-t}}}_{\text{larger translation}} \circ \underbrace{m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t}\left(Ball^{2n}(R)\right)}_{\text{same evolving shape}}
\end{aligned}$$

In *step ii*, above, we found a radius R_2 such that during the entire evolution of the *linearization with fixed center*, the evolving image remained within the confines of a ball of radius R_2 with *fixed* center. That is, for all $t \in [0,1]$, $\beta_{f,t}\left(f\left(B^{2n}(R)\right)\right) \subset B^{2n}(R_2, f(0))$. Now we see that during the evolution of the *partial linearization with moving center*, the evolving image will remain within the confines of a *moving* ball of radius R_2,centered at $\frac{f(0)}{1-t}$. That is, for all $t \in [0,1)$, $\gamma_{f,t}\left(f\left(Ball^{2n}(R)\right)\right) \subset B^{2n}\left(R_2, \frac{f(0)}{1-t}\right)$. Since, as time t grows from 0 towards 1, the evolving symplectic ball is remaining within a moving ball of fixed radius, while the evolving neighborhood it lies in is growing without bound, we know that there is some $\lambda > 0$ such that at time $t = \lambda$, the neighborhood will be large enough so that the following will be true.

$$\gamma_{f,\lambda}\left(f\left(Ball^{2n}(R)\right)\right) \subset B^{2n}\left(R_2, \tfrac{f(0)}{1-\lambda}\right) \subset B^{2n}\left(R_4, \tfrac{f(0)}{1-\lambda}\right) \subset \underbrace{m_{\frac{1}{1-\lambda}} \circ \underbrace{f\left(B^{2n}(R+\varepsilon)\right)}_{\text{original neighborhood}}}_{\text{magnified neighborhood}}$$

This is illustrated in Fig. 4.10 and Fig. 4.11.

step vi: Choose λ for the entire collection of balls, $\bigcup_{i=1}^{k} f_i\left(B^{2n}(R)\right)$.

For $i = 1 \cdots k$, let λ_i be the number chosen by *steps (i)* through *(v)* above. Then let $\lambda = \max\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$.

Summarize our construction of λ.

Given a collection of symplectic balls, $\bigcup_{i=1}^{k} f_i\left(B^{2n}(R)\right)$, we have chosen λ in a way that, if each symplectic ball is subjected to a *partial linearization with translation* for time λ, then by the end of that time, each will have evolved to a state similar to that shown in Fig. 4.10 and Fig. 4.11.

End of Detail 3

4.3.4 *Detail 4:* A Symplectomorphism that moves a round ball

Recall the sets described in *Lemma B5.*

$$ball_0 = \text{the original ball} = B^{2n}(R, 0)$$

$$ball_k = \text{ball translated } k \text{ units} = B^{2n}(R, (0, \cdots, 0, k)), \text{ for } k \in \mathbb{Z}$$

$$cell_k = \left\{y : \sum_{i=1}^{2n-1} y_i^2 < \left(\tfrac{R}{2} + \tfrac{1}{4}\right)^2 \text{and } k - 1 - \left(\tfrac{R}{2} + \tfrac{1}{4}\right) < y_{2n} < k + \left(\tfrac{R}{2} + \tfrac{1}{4}\right)\right\}$$

$$tube = \left\{y : \sum_{i=1}^{2n-1} y_i^2 < \left(\tfrac{R}{2} + \tfrac{1}{4}\right)^2\right\}$$

Note that for each $k \in \mathbb{Z}$, we have $cell_k \subset tube$, and that if $|j - k| \geqslant 2$, then $cell_j$ and $cell_k$ do not intersect. The set $cell_k$ is designed to be large enough to support a Hamiltonian function F_k that will generate a symplectomorphism that will translate a ball centered at $(0, \cdots 0, k-1)$ to the position $(0, \cdots 0, k)$. We will describe that function now.

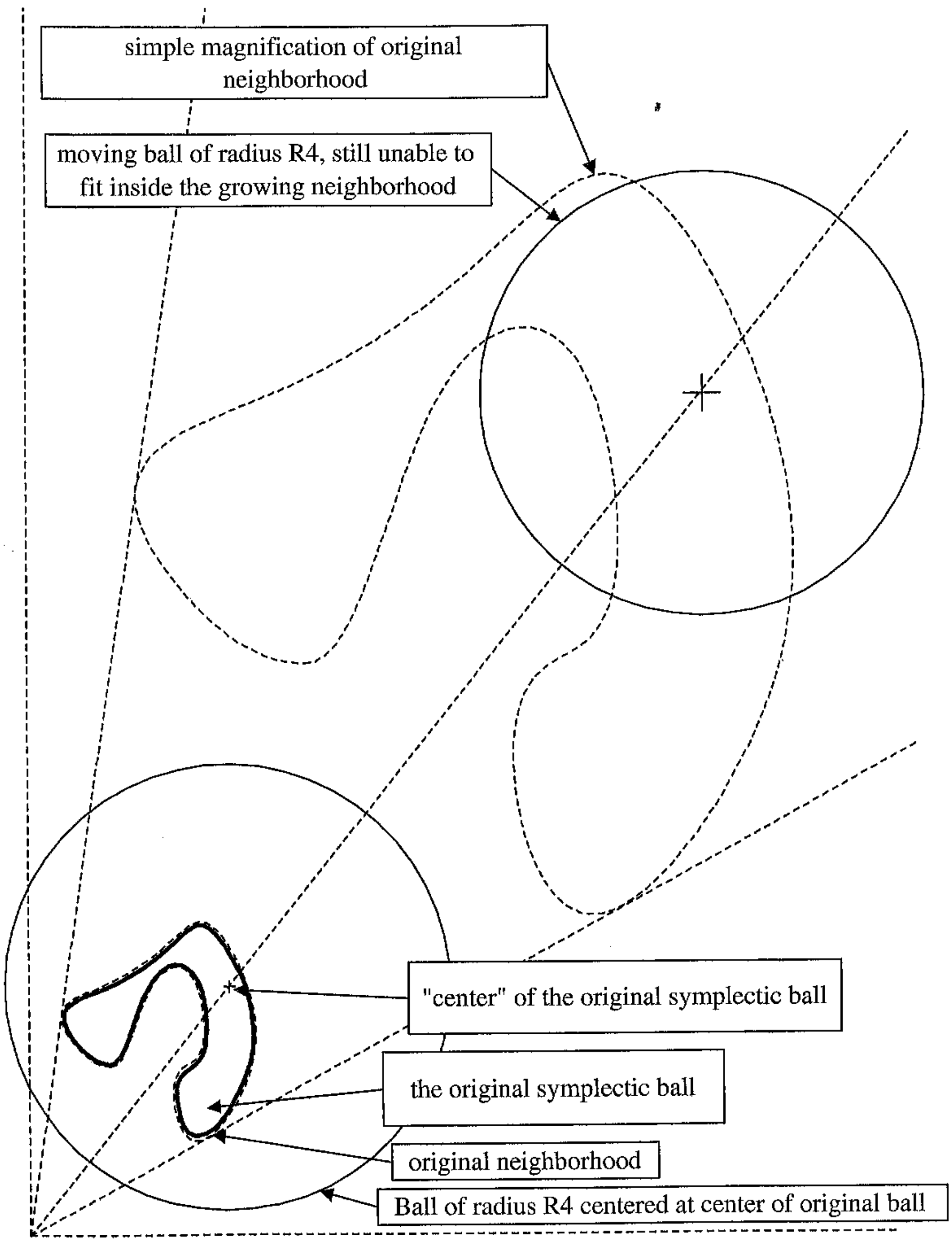

Figure 4.10: Zoomed-in view of a growing neighborhood not yet large enough

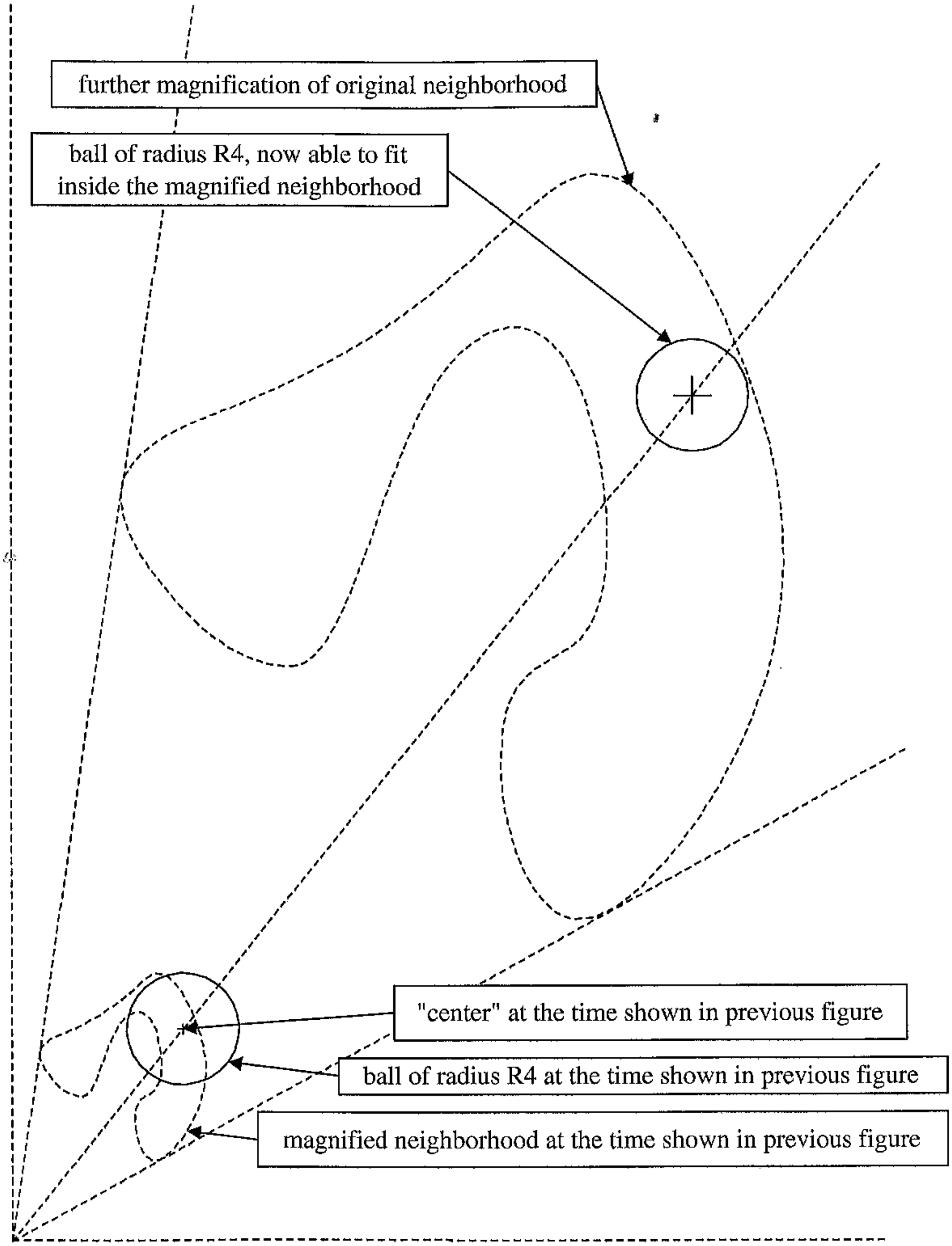

Figure 4.11: Zoomed-out view of a growing neighborhood sufficiently large

For $k \in \mathbb{Z}$, define the Hamiltonian function F_k by

$$F_k(\xi) = bump(\xi_1) \cdots bump(\xi_{2n-2})\, wedge(\xi_{2n-1})\, widebump(\xi_{2n} - k),$$

where the functions bump, wedge, and widebump are described in *Detail 6.* Notice that $support(F_k) \subset cell_k$ and $\|F_k\|_\infty \leqslant 1$. Let σ_k be the symplectomorphism generated by F_k. By the construction of F_k, we see that σ_k will have these three important properties:

$$\sigma_k(ball_{k-1}) = ball_k$$

$$support(\sigma_k) \subset cell_k \subset tube$$

$$E_\infty(\sigma_k) \leqslant E_\infty^{tube}(\sigma_k) \leqslant E_\infty^{cell_k}(\sigma_k) \leqslant \sup_{\xi \in \mathbb{R}^{2n}} |F_k(\xi)| \leqslant 1.$$

End of Detail 4

4.3.5 *Detail 5:* A fact about the *Hofer infinity norms*

Let ψ_i, $i \in I$ be a finite or countable collection of Hamiltonian symplectomorphisms with support $(\psi_i) \subset U_i \subset \mathbb{R}^{2n}$, where $U_i \cap U_j = \phi$ if $i \neq j$. Further, assume that the Hofer norm of each ψ_i is finite when the infimum in the Hofer norm is taken over only those Hamiltonian functions supported in U_i. (Hofer and Zehnder [3] have proven the same claim that we will make below, but without this additional condition. Their proof is more difficult, however. Since the weaker claim - with the additional assumption - is sufficient for us, and is easy to prove, we will state and prove it here.) Recall that this norm was introduced at the start of *Lemma B4*, where it was denoted by E_∞^U and $\widehat{E_\infty}^U$:

$$\begin{aligned} E_\infty^U(\psi) &= \inf\{\|H\|_\infty : H \in C,\ support(H) \subset U, \text{and } H \text{ generates } \psi\} \\ \widehat{E_\infty}^U(\psi) &= \inf\left\{\widehat{\|H\|_\infty} : H \in \mathcal{H},\ H \text{ generates } \psi, \text{and } support(H) \subset U\right\} \end{aligned}$$

Let $\psi = \prod_{i \in I} \psi_i$.

Claim: If $\max_{i \in I} \widehat{E_\infty}^{U_i}(\psi_i)$ exists, then $\widehat{E_\infty}(\psi) \leqslant 2 \max_{i \in I} \widehat{E_\infty}^{U_i}(\psi_i)$.

Remark: An analogous claim can be made for the norm E_∞: That is, if $\max_{i \in I} E_\infty^{U_i}(\psi_i)$ exists, then $E_\infty(\psi) \leqslant 2 \max_{i \in I} E_\infty^{U_i}(\psi_i)$. Note, however, that this norm is defined only for symplectomorphisms of compact support. Therefore, the product $\psi = \prod_{i \in I} \psi_i$ will have to be a *finite* product. We will use both inequalities in this section, but will prove the result only for the *extended norm*, $\widehat{E_\infty}$; The proof for the norm E_∞ is identical.

Proof of claim:

Let $\varepsilon > 0$. We will show that $\widehat{E_\infty}(\psi) \leqslant 2 \max_{i \in I} \widehat{E_\infty}^{U_i}(\psi_i) + 2\varepsilon$. Since ε is arbitrary, this will prove the claim.

By our assumption, for each $i \in I$, there is a Hamiltonian function $H_{i,t}$, supported in U_i, such that

$$\sup_{t \in [0,1]} \left\{ \sup_{x \in U_i} (H_{i,t}(x)) - \inf_{x \in U_i} (H_{i,t}(x)) \right\} \leqslant \widehat{E_\infty}^{U_i}(\psi_i) + \varepsilon.$$

Note that for such a function, we will have the equality

$$\begin{aligned} \widehat{\|H_i\|_\infty} &= \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^{2n}} (H_{i,t}(x)) - \inf_{x \in \mathbb{R}^{2n}} (H_{i,t}(x)) \right\} \\ &= \sup_{t \in [0,1]} \left\{ \sup_{x \in U_i} (H_{i,t}(x)) - \inf_{x \in U_i} (H_{i,t}(x)) \right\} \text{because } H_i \text{ is supported in } U_i. \end{aligned}$$

Define the Hamiltonian function $H_t = \sum_{i \in I} H_{i,t}$. Note that the sum exists, because the supports of the various $H_{i,t}$ are disjoint, and that that H_t generates ψ. That is, $\psi = \psi^H$.

Computing the *extended Hofer infinity norm* of ψ, we find

$$
\begin{aligned}
\widehat{E_\infty}(\psi) &= \widehat{E_\infty}\left(\prod_{i\in I}\psi_i\right) \\
&= \inf_{K\in\mathcal{H}}\left\{\widehat{\|K\|_\infty} : K \text{generates} \prod_{i\in I}\psi_i\right\} \\
&\leqslant \widehat{\|H\|_\infty} \text{ because } H \text{ is a particular such function} \\
&= \sup_{t\in[0,1]}\left\{\sup_{x\in\mathbb{R}^{2n}}\left(H_t(x)\right) - \inf_{x\in\mathbb{R}^{2n}}\left(H_t(x)\right)\right\} \\
&= \sup_{t\in[0,1]}\left\{\sup_{x\in\mathbb{R}^{2n}}\left(\textstyle\sum_{i\in I}H_{i,t}(x)\right) - \inf_{x\in\mathbb{R}^{2n}}\left(\textstyle\sum_{i\in I}H_{i,t}(x)\right)\right\} \text{ (definition of} H_t) \\
&= \sup_{t\in[0,1]}\left\{\sup_{i\in I}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right)\right\} - \inf_{i\in I}\left\{\inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right\} \\
&\quad \text{(because the supports of the Hamiltonian functions are disjoint.)} \\
&\leqslant \sup_{t\in[0,1]}\left\{\sup_{i\in I}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right) - \inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right. \\
&\quad \left. - \inf_{i\in I}\left\{\inf_{x\in U_i}\left(H_{i,t}(x)\right) - \sup_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right\} \\
&\quad \text{(subtracted something non-positive, added something non-negative.)} \\
&= \sup_{t\in[0,1]}\left\{\sup_{i\in I}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right) - \inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right. \\
&\quad \left. + \inf_{i\in I}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right) - \inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right\} \\
&\quad \text{(just reversed the subtraction.)} \\
&\leqslant \sup_{t\in[0,1]}\left\{2\sup_{i\in I}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right) - \inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right\} \\
&= 2\sup_{i\in I}\left\{\sup_{t\in[0,1]}\left\{\sup_{x\in U_i}\left(H_{i,t}(x)\right) - \inf_{x\in U_i}\left(H_{i,t}(x)\right)\right\}\right\} \\
&= 2\sup_{i\in I}\left\{\widehat{\|H_i\|_\infty}\right\} \\
&\leqslant 2\sup_{i\in I}\left\{\widehat{E_\infty}^{U_i}(\psi_i) + \varepsilon\right\} \\
&= 2\sup_{i\in I}\left\{\widehat{E_\infty}^{U_i}(\psi_i)\right\} + 2\varepsilon.
\end{aligned}
$$

End of proof of claim

End of Detail 5

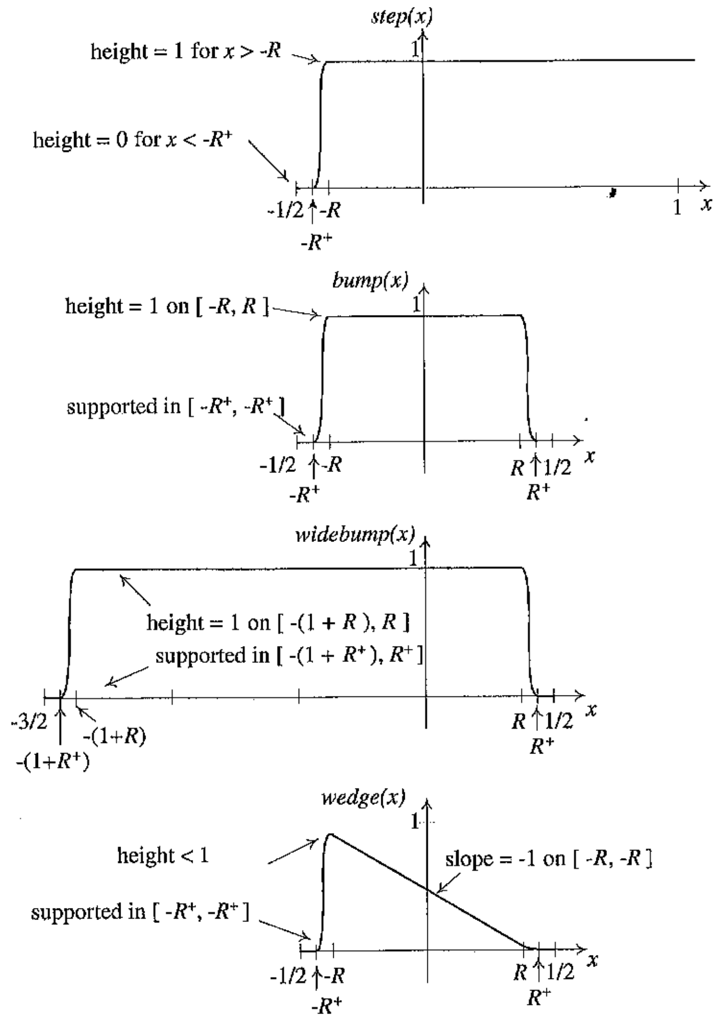

Figure 4.12: Four functions of one variable

4.3.6 *Detail 6:* Four useful functions of one variable.

For $0 < R < \frac{1}{2}$, we define $R^+ = \frac{R+\frac{1}{2}}{2} = \frac{R}{2} + \frac{1}{4}$. (We just need R^+ to be a number between R and $\frac{1}{2}$.) Figure 4.12 shows four functions of one variable that are used throughout this paper.

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