Kähler Geometry of Moduli Spaces and Universal Teichmüller Space

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Abstract of the Dissertation

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In the first part, we define Liouville action functional for compact Riemann surfaces uniformized by Kleinian groups. We discuss in detail the case where a pair of Riemann surfaces X and Y of genus g is uniformized by a quasi-Fuchsian group. The Liouville action functional is a real valued functional on the space of conformal metrics of $X \sqcup Y$ with a single critical point given by the hyperbolic metric. The critical value defines a real analytic function on the deformation space of the quasi-Fuchsian groups. We prove that the first variation of this function gives the difference between the

projective connections corresponding to the Fuchsian and quasi-Fuchsian uniformizations. The second variation of this function gives the Weil-Petersson symplectic form of the deformation space. In other words, the critical value of the Liouville action is a Kähler potential for the Weil-Petersson metric on the deformation space. We also establish a relation between the Einstein-Hilbert action for 3-dimensional gravity theory and the Liouville action, and verify the holography principle. This in turn helps to generalize our results to a large class of Kleinian groups.

In the second part, we consider natural ways to define Hermitian metrics on the universal Teichmüller curve and the universal Teichmüller space. We prove that the second variation of the spherical areas of a family of domains defines the Kirillov metric on the universal Teichmüller curve. We show that averaging Kirillov metric along the fibers gives the Weil-Petersson metric on the universal Teichmüller space. To get metrics on the finite dimensional Teichmüller spaces, we regularize the averaging procedure and obtain the Weil-Petersson metric as the result. This indicates the universal nature of the Weil-Petersson metric.

In the last part, we study variations of Laplace operators on families of Riemann surfaces of finite type.

Contents

List of Figures

To my parents.

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Chapter 1

Introduction

Riemann was one of the greatest mathematician of the 19th century. Among other things, he laid the foundation for complex function theory. The Riemann mapping theorem states that every simply connected domain that is not the whole plane is conformally equivalent to the unit disc. Although Riemann did not give a rigorous proof, his idea of using physical electrostatic theory to signify the underlying concept is very illuminating. Riemann also defined the general notion of Riemann surface and proved that every compact Riemann surface is a Riemann surface of an algebraic equation $f(x, y) = 0$. But the question whether a Riemann surface can be covered by a planar domain remained unsolved until the beginning of 20th century, when Poincaré and Koebe succeeded in proving that every Riemann surface can be uniformized by a Fuchsian group. Since then, the study of Riemann surfaces became closely connected to the study of Fuchsian groups, or more general Kleinian groups.

Recently the advent of string theory resurrected Riemann's spirit. Physical theories are used to predict mathematical results. In a lot of cases, physical intuitions also serve as guides to the proofs of the statements. However, in order not to fall into the deficiency of lack of mathematical rigor, we content ourselves with using physics as motivation. This brings us to the first part, where we study questions from Liouville theory.

Liouville theory, originated from Polyakov's approach to non-critical bosonic string theory, plays an important role in the complex geometry of the moduli spaces of Riemann surfaces. It was proved by P. Zograf and L. Takhtajan [ZT87b, ZT87c] that the critical value of the Liouville action functional is a Kähler potential of the Weil-Petersson metric on the Teichmüller space.

Specifically, Let X be a Riemann surface of genus $g > 1$, and let $\{U_\alpha\}_{\alpha\in A}$ be its open cover with charts U_α, local coordinates $z_\alpha : U_\alpha \to \mathbb{C}$, and transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$. A (holomorphic) projective connection on X is a collection $P = \{p_\alpha\}_{\alpha\in A}$, where p_α are holomorphic functions on U_α which on every $U_\alpha \cap U_\beta$ satisfy

$$p_\beta = p_\alpha \circ f_{\alpha\beta}\left(f'_{\alpha\beta}\right)^2 + \mathcal{S}(f_{\alpha\beta}),$$

where prime indicates derivative. Here $\mathcal{S}(f)$ is the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The space $\mathcal{P}(X)$ of projective connections on X is an affine space modeled on the vector space of holomorphic quadratic differentials on X.

The Schwarzian derivative satisfies the following properties.

SD1 $\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g\,(g')^2 + \mathcal{S}(g)$.

SD2 $\mathcal{S}(\gamma) = 0$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{C})$.

It follows from these properties that every planar covering of a compact Riemann surface X — a holomorphic covering $\pi : \Omega \to X$ by a domain $\Omega \subset \hat{\mathbb{C}}$, with group of deck transformations a subgroup of $\mathrm{PSL}(2, \mathbb{C})$, defines a projective connection $P_\pi = \{\mathcal{S}_{z_\alpha}(\pi^{-1})\}_{\alpha \in A}$. The Fuchsian uniformization $X \simeq \Gamma\backslash\mathbb{U}$ is the covering $\pi_F : \mathbb{U} \to X$ by the upper half-plane $\mathbb{U}$ where the group of deck transformations is a Fuchsian group Γ, and it defines Fuchsian projective connection P_F. The Schottky uniformization $X \simeq \Gamma\backslash\Omega$ is the covering $\pi_S : \Omega \to X$ by a connected domain $\Omega \subset \hat{\mathbb{C}}$ where the group of deck transformations Γ is a Schottky group — finitely-generated, strictly loxodromic, free Kleinian group. It defines Schottky projective connection P_S.

Let $\mathfrak{T}_g$ be the Teichmüller space of marked Riemann surfaces of genus $g > 1$ (with a given marked Riemann surface as the origin), defined as the space of marked normalized Fuchsian groups, and let $\mathfrak{S}_g$ be the Schottky space, defined as the space of marked normalized Schottky groups with g free generators. These spaces are complex manifolds of dimension $3g - 3$ carrying Weil-Petersson Kähler metrics, and the natural projection map $\mathfrak{T}_g \to \mathfrak{S}_g$ is a complex-analytic covering. Denote by ω_{WP} the symplectic form of the Weil-Petersson metric on spaces $\mathfrak{T}_g$ and $\mathfrak{S}_g$, and by $d = \partial + \bar{\partial}$ — the de Rham differential and its decomposition. The affine spaces $\mathcal{P}(X)$ for varying Riemann surfaces X glue together to an affine bundle $\mathfrak{P}_g \to \mathfrak{T}_g$, modeled over holomorphic cotangent bundle of $\mathfrak{T}_g$. The Fuchsian projective connection P_F is a canonical section of the affine bundle $\mathfrak{P}_g \to \mathfrak{T}_g$ [1], the Schottky projective connection is a canonical section of the affine bundle $\mathfrak{P}_g \to \mathfrak{S}_g$, and their

[1] P_F is actually a canonical section over the affine bundle $\mathfrak{P}_g \to \mathfrak{M}_g$, where $\mathfrak{M}_g$ is the moduli space of Riemann surfaces of genus g.

difference $P_F - P_S$ is a $(1,0)$-form on $\mathfrak{S}_g$. This 1-form has the following properties [ZT87c]. First, it is ∂-exact — there exists a smooth function $S : \mathfrak{S}_g \to \mathbb{R}$ such that

$$P_F - P_S = \frac{1}{2}\,\partial S. \tag{1.0.1}$$

Second, it is a $\bar{\partial}$-antiderivative, and hence a d-antiderivative by (1.0.1), of the Weil-Petersson symplectic form on $\mathfrak{S}_g$

$$\bar{\partial}(P_F - P_S) = -i\,\omega_{WP}. \tag{1.0.2}$$

It immediately follows from (1.0.1) and (1.0.2) that the function $-S$ is a Kähler potential for the Weil-Petersson metric on $\mathfrak{S}_g$, and hence on $\mathfrak{T}_g$,

$$\partial\bar{\partial} S = 2i\,\omega_{WP}. \tag{1.0.3}$$

Arguments using quantum Liouville theory (see, e.g., [Tak92] and references therein) confirm formula (1.0.1) with the function S given by the classical Liouville action, as was already proved in [ZT87c]. However, to define the Liouville action functional on a Riemann surface X is a non-trivial problem interesting in its own right (and for rigorous applications to quantum Liouville theory). Let $\mathcal{CM}(X)$ be the space (actually a cone) of smooth conformal metrics on a Riemann surface X. Every $ds^2 \in \mathcal{CM}(X)$ is a collection $\left\{e^{\phi_\alpha}|dz_\alpha|^2\right\}_{\alpha\in A}$, where functions $\phi_\alpha \in C^\infty(U_\alpha, \mathbb{R})$ satisfy

$$\phi_\alpha \circ f_{\alpha\beta} + \log|f'_{\alpha\beta}|^2 = \phi_\beta \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{1.0.4}$$

According to the uniformization theorem, X has a unique conformal metric of constant negative curvature -1, called hyperbolic, or Poincaré metric. Gaussian curvature -1 condition is equivalent to the following nonlinear PDE for functions ϕ_α on U_α,

$$\frac{\partial^2 \phi_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha} = \frac{1}{2}\, e^{\phi_\alpha}. \tag{1.0.5}$$

In string theory this PDE is called the Liouville equation. The problem is to define the Liouville action functional on Riemann surface X — a smooth functional $S : \mathcal{CM}(X) \to \mathbb{R}$ such that its Euler-Lagrange equation is the Liouville equation. At first glance it looks like an easy task. Set $U = U_\alpha$, $z = z_\alpha$ and $\phi = \phi_\alpha$, so that $ds^2 = e^\phi |dz|^2$ in U. Elementary calculus of variations shows that the Euler-Lagrange equation for the functional

$$\frac{i}{2} \iint\limits_U \left(|\phi_z|^2 + e^\phi\right) dz \wedge d\bar{z},$$

where $\phi_z = \partial\phi/\partial z$, is indeed the Liouville equation on U. Therefore it seems that the functional $\frac{i}{2}\iint_X \omega$, where ω is a 2-form on X such that

$$\omega\big|_{U_\alpha} = \omega_\alpha = \left(\left|\frac{\partial \phi_\alpha}{\partial z_\alpha}\right|^2 + e^{\phi_\alpha}\right) dz_\alpha \wedge d\bar{z}_\alpha, \tag{1.0.6}$$

does the job. However, due to the transformation law (1.0.4) the first terms in local 2-forms ω_α do not glue properly on $U_\alpha \cap U_\beta$ and a 2-form ω on X satisfying (1.0.6) does not exist!

Though the Liouville action functional can not be defined in terms of a Riemann surface X only, it can be defined in terms of planar coverings of X. Namely, let Γ be a Kleinian group with region of discontinuity Ω such that

$\Gamma\backslash\Omega \simeq X_1 \sqcup \cdots \sqcup X_n$ — a disjoint union of compact Riemann surfaces of genera > 1 including the Riemann surface X. The covering $\Omega \to X_1 \sqcup \cdots \sqcup X_n$ introduces a global "étale" coordinate, and for large variety of Kleinian groups (Class A defined in Chapter 6) it is possible, using methods in [AT97], to define a Liouville action functional $S : \mathcal{CM}(X_1 \sqcup \cdots \sqcup X_n) \to \mathbb{R}$ such that its critical value is a well-defined function on the deformation space $\mathfrak{D}(\Gamma)$. In the simplest case when X is a punctured Riemann sphere such global coordinate exists already on X, and Liouville action functional is just $\frac{i}{2}\iint_X \omega$, appropriately regularized at punctures [ZT87b]. When X is compact, one possibility is to use the "minimal" planar cover of X given by the Schottky uniformization $X \simeq \Gamma\backslash\Omega$, as in [ZT87c]. Namely, identify $\mathcal{CM}(X)$ with the affine space of smooth real-valued functions ϕ on Ω satisfying

$$\phi \circ \gamma + \log|\gamma'|^2 = \phi \quad \text{for all} \quad \gamma \in \Gamma, \tag{1.0.7}$$

and consider the 2-form $\omega[\phi] = (|\phi_z|^2 + e^{\phi})dz \wedge d\bar{z}$ on Ω. The 2-form $\omega[\phi]$ can not be pushed forward on X, so that the integral $\frac{i}{2}\iint_F \omega$ depends on the choice of a fundamental domain F for a marked Schottky group Γ. However, one can add boundary terms to this integral to ensure the independence of the choice of a fundamental domain and to guarantee that its Euler-Lagrange equation is the Liouville equation on $\Gamma\backslash\Omega$. The result is the following functional introduced in [ZT87c]

$$S[\phi] = \frac{i}{2}\iint\limits_{F} \left(|\phi_z|^2 + e^{\phi}\right) dz \wedge d\bar{z} + \frac{i}{2}\sum_{k=1}^{g}\int_{C_k}\left(\phi - \frac{1}{2}\log|\gamma_k'|^2\right)\left(\frac{\gamma_k''}{\gamma_k'}dz - \frac{\overline{\gamma_k''}}{\overline{\gamma_k'}}d\bar{z}\right) + 4\pi\sum_{k=1}^{g}\log|c(\gamma_k)|^2. \tag{1.0.8}$$

Here F is the fundamental domain of the marked Schottky group Γ with free generators $\gamma_1, \dots, \gamma_g$, bounded by $2g$ nonintersecting closed Jordan curves $C_1, \dots, C_g, C_1', \dots, C_g'$ such that $C_k' = -\gamma_k(C_k)$, $k = 1, \dots, g$, and $c(\gamma) = c$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Classical action $S : \mathfrak{S}_g \to \mathbb{R}$ that enters (1.0.1) is the critical value of this functional.

In [McM00] McMullen considered the quasi-Fuchsian projective connection P_{QF} on a Riemann surface X which is given by Bers' simultaneous uniformization of X and a fixed Riemann surface Y of the same genus and opposite orientation. Similar to formula (1.0.2), he proved

$$d(P_F - P_{QF}) = -i\,\omega_{WP}, \tag{1.0.9}$$

so that the 1-form $P_F - P_{QF}$ on $\mathfrak{T}_g$ is a d-antiderivative of the Weil-Petersson symplectic form, bounded in Teichmüller and Weil-Petersson metrics due to Kraus-Nehari inequality.

Our first result is the analog of the formula (1.0.1) for the quasi-Fuchsian case, giving the 1-form $P_F - P_{QF}$ the same treatment as to the 1-form $P_F - P_S$. Namely, let Γ be a finitely generated, purely loxodromic quasi-Fuchsian group with region of discontinuity Ω, so that $\Gamma\backslash\Omega$ is the disjoint union of two compact Riemann surfaces of the same genus $g > 1$ and opposite orientations. Denote by $\mathfrak{D}(\Gamma)$ the deformation space of Γ, and by ω_{WP} — the symplectic form of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$. To every point $\Gamma' \in \mathfrak{D}(\Gamma)$ with region of discontinuity Ω' there corresponds a pair X, Y of compact Riemann surfaces with opposite orientations simultaneously uniformized by Γ', that is, $X \sqcup Y \simeq \Gamma'\backslash\Omega'$. We will continue to denote by P_F and P_{QF} projective connections on $X \sqcup Y$ given by Fuchsian uniformizations of X and Y and Bers' simultaneous uniformization of X and Y respectively. Similarly to (1.0.1), we prove in Theorem 4.2.1 that there exists a smooth function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ such that

$$P_F - P_{QF} = \frac{1}{2}\,\partial S. \tag{1.0.10}$$

The function S is Liouville classical action for the quasi-Fuchsian group Γ — the critical value of the Liouville action functional S on $\mathcal{CM}(X \sqcup Y)$. Its construction uses double homology and cohomology complexes naturally associated with the Γ-action on Ω. Namely, the homology double complex $\mathsf{K}_{\bullet,\bullet}$ is defined as a tensor product over the integral group ring $\mathbb{Z}\Gamma$ of the standard singular chain complex of Ω and the canonical bar-resolution complex for Γ, and cohomology double complex $\mathsf{C}^{\bullet,\bullet}$ is bar-de Rham complex on Ω. The cohomology construction starts with the 2-form $\omega[\phi] \in \mathsf{C}^{2,0}$, where ϕ satisfies (1.0.7), and introduces $\theta[\phi] \in \mathsf{C}^{1,1}$ and $u \in \mathsf{C}^{1,2}$ by

$$\theta_{\gamma^{-1}}[\phi] = \left(\phi - \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right),$$

and

$$u_{\gamma_1^{-1},\gamma_2^{-1}} = -\frac{1}{2}\log|\gamma_1'|^2\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1'\,dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}\,d\bar{z}\right) + \frac{1}{2}\log|\gamma_2'\circ\gamma_1|^2\left(\frac{\gamma_1''}{\gamma_1'}dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar{z}\right).$$

Define $\Theta \in \mathsf{C}^{0,2}$ to be a group 2-cocycle satisfying $d\Theta = u$. The resulting cochain $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$ is a cocycle of degree 2 in the total complex Tot C. The corresponding homology construction starts with fundamental domain $F \in \mathsf{K}_{2,0}$ for Γ in Ω and introduces chains $L \in \mathsf{K}_{1,1}$ and $V \in \mathsf{K}_{0,2}$ such that $\Sigma = F + L - V$ is a cycle of degree 2 in the total homology complex Tot K. The Liouville action functional is given by the evaluation map,

$$S[\phi] = \frac{i}{2}\langle\Psi[\phi], \Sigma\rangle, \tag{1.0.11}$$

where $\langle\ ,\ \rangle$ is the natural pairing between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$.

In case when Γ is a Fuchsian group, the Liouville action functional on $X \simeq \Gamma\backslash\mathbb{U}$, similar to (1.0.8), can be written explicitly as follows

$$\begin{aligned} S[\phi] =& \frac{i}{2}\iint\limits_F \omega[\phi] + \frac{i}{2}\sum_{k=1}^{g}\left(\int_{a_k}\theta_{\alpha_k}[\phi] - \int_{b_k}\theta_{\beta_k}[\phi]\right) \\ &+ \frac{i}{2}\sum_{k=1}^{g}\Big(\Theta_{\alpha_k,\beta_k}(a_k(0)) - \Theta_{\beta_k,\alpha_k}(b_k(0)) + \Theta_{\gamma_k^{-1},\alpha_k\beta_k}(b_k(0))\Big) \\ &- \frac{i}{2}\sum_{k=1}^{g}\Theta_{\gamma_g^{-1}\dots\gamma_{k+1}^{-1},\gamma_k^{-1}}(b_g(0)), \end{aligned}$$

where

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + 4\pi i \varepsilon_{\gamma_1,\gamma_2}(2\log 2 + \log|c(\gamma_2)|^2),$$

$p \in \mathbb{R} \setminus \Gamma(\infty)$ and

$$\varepsilon_{\gamma_1,\gamma_2} = \begin{cases} 1 & \text{if } p < \gamma_2(\infty) < \gamma_1^{-1}p, \\ -1 & \text{if } p > \gamma_2(\infty) > \gamma_1^{-1}p, \\ 0 & \text{otherwise.} \end{cases}$$

Here a_k and b_k are edges of the fundamental domain F for Γ in $\mathbb{U}$ (see Section 2.2.1) with initial points $a_k(0)$ and $b_k(0)$, α_k and β_k are corresponding generators of Γ and $\gamma_k = \alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}$. The action functional does not depend on the choice of the fundamental domain F for Γ, nor on the choice of $p \in \mathbb{R} \setminus \Gamma(\infty)$. The Liouville action for quasi-Fuchsian group Γ is defined by a similar construction where both components of Ω are used (see Section 2.3.3).

Equation (1.0.10) is quasi-Fuchsian reciprocity. McMullen's quasi-Fuchsian reciprocity, as well as the equation $\partial(P_F - P_{QF}) = 0$, immediately follow from it. The classical action $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ is symmetric with respect to Riemann surfaces X and Y,

$$S(X, Y) = S(\bar{Y}, \bar{X}), \tag{1.0.12}$$

where $\bar{X}$ is the mirror image of X, and this property manifests the global quasi-Fuchsian reciprocity. Equation (1.0.9) now follows from (1.0.10) and (1.0.1). Its direct proof along the lines of [ZT87b, ZT87c] is given in Theorem 4.3.4. As an immediate corollary of (1.0.9) and (1.0.10), we obtain that function $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$.

Our second result is a precise relation between two and three-dimensional constructions which proves the holography principle for the quasi-Fuchsian case. Let $\mathbb{U}^3 = \{Z = (x, y, t) \in \mathbb{R}^3 \,|\, t > 0\}$ be a hyperbolic 3-space. The quasi-Fuchsian group Γ acts discontinuously on $\mathbb{U}^3 \cup \Omega$ and the quotient $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is a hyperbolic 3-manifold with boundary $\Gamma\backslash\Omega \simeq X \sqcup Y$. According to the holography principle (see, e.g., [MM02] for mathematically oriented exposition), the regularized hyperbolic volume of M — *on-shell Einstein-Hilbert action with cosmological term*, is related to the Liouville action functional $S[\phi]$.

In case when Γ is a classical Schottky group, i.e., when it has a fundamental domain bounded by Euclidean circles, holography principle was established by K. Krasnov in [Kra00]. Namely, let $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ be the corresponding hyperbolic 3-manifold (realized using the Ford fundamental region) with boundary $X \simeq \Gamma\backslash\Omega$ — a compact Riemann surface of genus $g > 1$. For every $ds^2 = e^\phi |dz|^2 \in \mathcal{CM}(X)$ consider the family H_ε of surfaces given by the equation $f(Z) = te^{\phi(z)/2} = \varepsilon > 0$ where $z = x + iy$, and let $M_\varepsilon = M \cap H_\varepsilon$. Denote by $V_\varepsilon[\phi]$ the hyperbolic volume of M_ε, by $A_\varepsilon[\phi]$ — the area of the boundary of M_ε in the metric on H_ε induced by the hyperbolic metric on $\mathbb{U}^3$, and by $A[\phi]$ — the area of X in the metric ds^2. In [Kra00] K. Krasnov obtained the following formula

$$\lim_{\varepsilon \to 0} \left(V_\varepsilon[\phi] - \frac{1}{2} A_\varepsilon[\phi] + (2g-2)\pi \log \varepsilon \right) = -\frac{1}{4} \left(S[\phi] - A[\phi] \right). \tag{1.0.13}$$

It relates three-dimensional data — the regularized volume of M, to the two-dimensional data — the Liouville action functional $S[\phi]$, thus establishing the holography principle. Note that the metric ds^2 on the boundary of M appears entirely through regularization by means of hypersurfaces H_ε, which are not Γ-invariant. As a result, arguments in [Kra00] work only for classical Schottky groups.

We extend homological algebra methods in [AT97] to the three-dimensional case when Γ is a quasi-Fuchsian group. Namely, we construct Γ-invariant cut-off function f using a partition of unity for Γ, and prove in Theorem 5.2.3 that on-shell regularized Einstein-Hilbert action functional

$$\mathcal{E}[\phi] = -4 \lim_{\varepsilon\to 0}\left(V_\epsilon[\phi] - \frac{1}{2}A_\epsilon[\phi] + 2\pi(2g-2)\log\varepsilon\right),$$

is well-defined and satisfies the quasi-Fuchsian holography principle

$$\mathcal{E}[\phi] = S[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^{\phi} d^2z - 8\pi(2g-2)\log 2.$$

Our third result is the generalization of main results for quasi-Fuchsian groups — Theorems 4.2.1, 4.3.4 and 5.2.3, to Kleinian groups. Namely, we introduce a notion of a Kleinian group of Class A for which this generalization holds. Schottky, Fuchsian, quasi-Fuchsian groups, and their free combinations are of Class A, and Class A is stable under quasiconformal deformations. We extend three-dimensional homological methods developed in Chapter 5 to the case of Kleinian group Γ of Class A acting on $\mathbb{U}^3 \cup \Omega$. In Theorem 6.2.6 we establish holography principle for Kleinian groups: we prove that the on-shell regularized Einstein-Hilbert action for the 3-manifold $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is well-defined and is related to the Liouville action functional for Γ, defined by the evaluation map (1.0.11). When Γ is a Schottky group, we get the functional

(1.0.8) introduced in [ZT87c]. Denote by $\mathfrak{D}(\Gamma)$ the deformation space of the Kleinian group Γ. To every point $\Gamma' \in \mathfrak{D}(\Gamma)$ with the region of discontinuity Ω' there corresponds a disjoint union $X_1 \sqcup \cdots \sqcup X_n \simeq \Gamma' \backslash \Omega'$ of compact Riemann surfaces simultaneously uniformized by the Kleinian group Γ'. Using the same notation, we denote by P_F projective connection on $X_1 \sqcup \cdots \sqcup X_n$ given by the Fuchsian uniformization of these Riemann surfaces and by P_K — projective connection given by their simultaneous uniformization by a Kleinian group ($P_K = P_{QF}$ for the quasi-Fuchsian case). Let $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ be the classical Liouville action. Theorem 6.3.1 states that

$$P_F - P_K = \frac{1}{2}\,\partial S,$$

which is the ultimate generalization of (1.0.1). Similarly, Theorem 6.3.3 is the statement

$$\bar{\partial}(P_F - P_K) = -i\,\omega_{WP},$$

which implies that $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$. As another immediate corollary of Theorem 6.3.1 we get McMullen's Kleinian reciprocity — Theorem 6.4.1.

After studying the Weil-Petersson geometry of deformation spaces of compact Riemann surfaces using motivation from Liouville theory, we consider Weil-Petersson geometry on family of domains bounded by Jordan curves. This brings us to the second part, where we consider a natural way to define a metric on the universal Teichmüller curve, the universal Teichmüller space and finite dimensional Teichmüller spaces.

Let $T(1)$ be the universal Teichmüller space and $\mathcal{T}(1)$ its universal curve.

In [Vel], Velling introduced a metric on $T(1)$ by using spherical areas. More precisely, consider the Bers embedding of $T(1)$ into the space

$$A_\infty(\Delta) = \left\{ \phi \text{ holomorphic on } \Delta : \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2| < \infty \right\},$$

where Δ is the unit disc. For every $Q \in A_\infty(\Delta)$, Velling considered the solution to the equation

$$\mathcal{S}(f^{tQ}) = tQ,$$

where $\mathcal{S}(f)$ is the Schwarzian derivative of the function f. This defines a family of domains $\Omega_t = f^{tQ}(\Delta)$. Velling considered the spherical area of the domain Ω_t, $A_S(f^{tQ}(\Delta))$ and proved that

$$\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))\big|_{t=0} \geq 0.$$

This gives a candidate for a metric on the tangent space at the origin of $T(1)$. Velling gave a formula for this metric in terms of some integrals. Our first result (Theorem 8.2.1) is obtained by simplifying Velling's approach to the equation $\mathcal{S}(f^{tQ}) = tQ$ and compute the metric explicitly. The result is

$$\| Q \|_S^2 = \frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))\big|_{t=0} = 2\pi \sum_{n=1}^{\infty} n|a_n|^2,$$
$$\text{where} \qquad Q = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2}.$$

As is observed by Velling, this metric is invariant under rotation, but is not invariant under the whole group of isometries of the disc, PSU(1,1). Hence it does not define a homogenuous metric on $T(1)$. We generalize Velling's approach and prove that the same method can be used to define a natural homogenuous metric on $\mathcal{T}(1)$.

Denote by $\mathrm{Diff}_+(S^1)$ the group of orientation preserving diffeomorphisms of the unit circle S^1. Let $\mathrm{M\ddot{o}b}(S^1) = \mathrm{PSU}(1,1)$ be the subgroup of Möbius transformations and S^1 the subgroup of rotations. It was known from different approaches (see, e.g., [KY87]) that all homogenuous Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$ are of the form

$$\| v \|^2 = \sum_{n=1}^{\infty} (an^3 + bn)|c_n|^2,$$

where

$$v = \sum_{n\neq 0} c_n e^{in\theta} \frac{\partial}{\partial\theta}, \qquad c_{-n} = \overline{c_n}$$

is the corresponding vector field on the unit circle. Among this family of metrics, special roles are played by the Kirillov metric

$$\| v \|^2 = \sum_{n=1}^{\infty} n|c_n|^2 \tag{1.0.14}$$

and the metric

$$\| v \|^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} (n^3 - n)|c_n|^2.$$

The latter defines the unique homogenuous Kähler metric on $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$. It was proved by Nag and Verjoysky in [NV90] that the homogenuous space $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ embeds holomorphically into $T(1)$ and the unique homogenuous Kähler metric on $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ is the pull back of the natural Weil-Petersson metric on $T(1)$. We can naturally extend this embedding to an embedding of $\text{Diff}_+(S^1)/S^1$ into $\mathcal{T}(1)$ and generalize the Kirillov metric to $\mathcal{T}(1)$. Using this embedding, we extend naturally Kirillov's ([Kir87]) identification of $\text{Diff}_+(S^1)/S^1$ with the space of smooth contours of conformal radius 1 to the identification of $\mathcal{T}(1)$ with the space of quasi-circles of conformal radius 1. Associated to a quasi-circle $\mathcal{C}$ of conformal radius 1, there is a unique holomorphic function $f : \Delta \to \Omega$, such that $f(0) = 0$ and $f'(0) = 1$, where Ω is the interior of the quasi-circle $\mathcal{C}$ containing the origin. By the definition of a quasi-circle, f can be extended to a quasiconformal map on $\hat{\mathbb{C}}$. In other words, we can identify $\mathcal{T}(1)$ with the space

$$\tilde{\mathcal{D}} = \{f : \Delta \longrightarrow \hat{\mathbb{C}} \text{ a univalent function } : f(0) = 0, f'(0) = 1,$$
$$f \text{ has a quasiconformal extension to } \hat{\mathbb{C}}\}.$$

Using Velling's approach, given a one-parameter family of holomorphic functions $f^t : \Delta \to \hat{\mathbb{C}} \in \tilde{\mathcal{D}}$, which defines a tangent vector v corresponding to $\frac{d}{dt} f^t|_{t=0}$ at the origin, we define a metric by

$$\| v \|^2 = \frac{1}{2\pi} \frac{d^2}{dt^2} A_S(f^t(\Delta))\Big|_{t=0}.$$

The proof for Theorem 8.2.1 implies that this metric coincides with the Kirillov metric (1.0.14).

In order to get a homogenuous metric on $T(1)$, we use Velling's suggestion. We average the Kirillov metric on the fiber of $\mathcal{T}(1)$ over $T(1)$ (which is the unit disc), or equivalently, we push forward the Kirillov metric from $\mathcal{T}(1)$ to $T(1)$. More precisely, the group $\text{PSU}(1, 1)$ acts transitively on the fiber of $\mathcal{T}(1)$

over $T(1)$. We identify the tangent space at the origin of $T(1)$ with $A_\infty(\Delta)$. If we choose a different base point $w \in \Delta$, we translate it to the origin by some $\gamma_w^{-1} \in \mathrm{PSU}(1,1)$. A tangent vector $Q \in A_\infty(\Delta)$ at the point w, is identified with the tangent vector $Q_w = Q \circ \gamma_w (\gamma_w')^2 \in A_\infty(\Delta)$ at the origin. We define Velling metric to be

$$\| Q \|_V^2 = \iint_\Delta \| Q_w \|_S^2 \frac{4dxdy}{(1-|w|^2)^2},$$

which is the average of $\| Q \|_S^2$ over the unit disc. Our second result is Theorem 9.1.3 and Theorem 9.1.4. We write

$$Q_w(z) = \sum_{n=2}^{\infty} (n^3 - n) a_n^w z^{n-2}.$$

We prove that whenever the vector field corresponding to Q belongs to the Sobolev class $H^{\frac{3}{2}}$, the average of the norm square of the Fourier coefficients $|a_j|^2$ of Q is given by

$$\iint_\Delta |a_j^w|^2 \frac{4dxdy}{(1-|w|^2)^2} = \frac{4\pi}{3(j^3-j)} \sum_{n=2}^{\infty} (n^3 - n)|a_n|^2.$$

This immediately imply that the Velling metric is twice the Weil-Petersson metric on the subspace of the tangent space of $T(1)$ which corresponds to $H^{\frac{3}{2}}$ vector fields.

However, the averaging procedure becomes divergent when restricted to tangent spaces of finite dimensional Teichmüller spaces. In this case, there is an infinite Fuchsian group acting on the disc and we are summing over infinitely many identical values. Instead of averaging over the disc, vaguely speaking, we should only average over a fundamental domain of the Fuchsian group. In case Γ is a cofinite Fuchsian group, we apply Velling's regularization formula to define the Velling metric on $\mathfrak{T}(\Gamma)$, the Teichmüller space of Γ,

$$\| Q \|_V^2 = \lim_{r' \to 1^-} \frac{\text{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} \| Q_w \|_S^2 \, dA_H}{\iint_{\Delta_{r'}} dA_H},$$

where $\Delta_r = \{z : |z| < r\}$, dA_H is the area form corresponding to the hyperbolic metric and $\text{Area}_H(\Gamma\backslash\Delta)$ is the hyperbolic area of the quotient Riemann surface $\Gamma\backslash\Delta$. Here $Q \in A_\infty(\Delta, \Gamma)$, where

$$A_\infty(\Delta, \Gamma) = \left\{Q \in A_\infty(\Delta) : Q \circ \gamma(\gamma')^2 = Q, \forall \gamma \in \Gamma\right\}$$

is identified with the tangent space of $\mathfrak{T}(\Gamma)$ at the origin. Our result is Theorem 9.2.2 and Theorem 9.2.3, which are the regularized versions of Theorem 9.1.3 and Theorem 9.1.4. We prove that

$$\lim_{r' \to 1^-} \frac{\text{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |a_j^w|^2 dA_H}{\iint_{\Delta_{r'}} dA_H} = \frac{8}{3(j^3 - j)} \| Q \|_{WP}^2,$$

where $\| \cdot \|_{WP}^2$ is the Weil-Petersson norm on $A_\infty(\Delta, \Gamma)$. It follows immediately that the Velling metric on $\mathfrak{T}(\Gamma)$ is twice the Weil-Petersson metric.

In Chapter 10, we consider applying Euclidean areas instead of spherical areas to define a metric on $\mathcal{T}(1)$. We define a metric by

$$\| v \|^2 = \frac{1}{2\pi} \frac{d^2}{dt^2} A_E(f^t(\Delta))\big|_{t=0},$$

where $A_E(\Omega)$ is the Euclidean area of the domain Ω, and v is the tangent vector corresponding to the one-parameter flow f^t. We prove in Theorem 10.1.1 that this metric is given by

$$\| v \|^2 = \sum_{n=1}^{\infty} (n+1)|c_n|^2.$$

In particular, we do not get a Kähler metric on $\mathcal{T}(1)$. However, after averaging and regularization, we still get the Weil-Petersson metric on $T(1)$ and the Teichmüller spaces of cofinite Fuchsian groups, which are immediate consequences of Theorem 9.1.3 and Theorem 9.2.2. In other words, Weil-Petersson metric is 'universal' in Teichmüller theory.

Finally, motivated by quantum Liouville theory, we consider variations of the Laplace operators on families of Riemann surfaces in part 3. More precisely, let $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ be the Teichmüller space of Riemann surfaces of type $(g;\nu_1,\ldots,\nu_n)$. Each point in $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ represents a normalized Fuchsian group Γ. The Selberg zeta function for the Riemann surface $X = \Gamma\backslash\mathbb{U}$ is defined for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{\gamma_0\}} \prod_{m=0}^{\infty} (1 - e^{(s+m)\log\lambda(\gamma_0)}),$$

where γ_0 runs over the set of conjugacy classes of primitive hyperbolic elements of Γ, and $0 < \lambda(\gamma) < 1$ the multiplier of γ. The function $Z(s)$ has a meromorphic continuation to the whole s–plane. The first variation of $Z(s)$, considered as defining a function on the Teichmüller space $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ depending on a parameter s is given by

$$\partial \log Z(s)|_\Gamma(z) = \frac{1}{\pi}\sum_{\gamma \text{ hyp}} \left(s\lambda(\gamma)^{s-1} + (1-s)\lambda(\gamma)^s\right)\frac{\gamma'(z)}{(z-\gamma z)^2}.$$

On the other hand, let Δ_l be the Laplace operator on l-differentials and consider the Green's function $G_s^{(l)}(z,z'), l \leq 0$, which is the kernel of the operator $(\Delta_l + \frac{1}{4}(s-2l)(s-1))^{-1}(l \leq 0, \operatorname{Re} s \geq 1)$ on the Riemann surface $X = \Gamma\backslash\mathbb{U}$. We prove the following formula by direct computation

$$\begin{aligned} &P_2\left((-\partial\rho^{q-1}\partial'(G_1^{(1-q)} - Q_1^{(1-q)}))\big|_D\right)(z) \\ &= \frac{1}{\pi}\sum_{\gamma \text{ hyp}} \left(q\lambda(\gamma)^{q-1} + (1-q)\lambda(\gamma)^q\right)\frac{\gamma'(z)}{(z-\gamma z)^2}, \qquad q \geq 2, \end{aligned}$$

where P_2 is the projection operator from the space of quadratic differentials to the subspace of holomorphic quadratic differentials. This gives us a holomorphic formula for the left hand side. An immediate consequence is

$$P_2\left((-\partial\rho^{q-1}\partial'(G_1^{(1-q)} - Q_1^{(1-q)}))\big|_D\right)(z) = \partial \log Z(q)|_\Gamma(z).$$

This formula is usually proved by using Selberg transform.

The content of this dissertation is the following. Chapters 2–6 are devoted to part 1. In Chapter 2 we give a construction of the Liouville action functional following the method in [AT97], which we review briefly in 2.1. In Section 2.2 we define and establish the main properties of the Liouville action functional in the model case when Γ is a Fuchsian group, and in Section 2.3 we consider technically more involved quasi-Fuchsian case. In Chapter 3 we recall all necessary basic facts from the deformation theory. In Chapter 4 we prove our first main result — Theorems 4.2.1 and 4.3.4. In Chapter 5 we prove the second main result — Theorem 5.2.3 on quasi-Fuchsian holography. In Chapter 6 we generalize these results for Kleinian groups of Class A: we define Liouville action functional and prove Theorems 6.2.6, 6.3.1 and 6.3.3.

Part 2 goes from Chapter 7 to Chapter 10. In Chapter 7, we review different models for the universal Teichmüller space, the universal Teichmüller curve and study their relations to the homogenuous spaces of $\mathrm{Diff}_+(S^1)$. In Chapter 8, we review Velling's approach and define a metric on the universal Teichmüller curve. We prove that it coincides with the Kirillov metric. In Chapter 9, we do the averaging and regularization on the Kirillov metric and prove that it gives the Weil-Petersson metric. In Chapter 10, we consider the Euclidean analog of Velling's approach.

Part 3 consists of Chapter 11. We prove a holomorphic formula for the variation of Laplace operators on families of Riemann surfaces.

Chapter 2

Liouville Action Functional

Let Γ be a normalized, marked, purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity Ω, so that $\Gamma\backslash\Omega \simeq X \sqcup Y$, where X and Y are compact Riemann surfaces of genus $g > 1$ with opposite orientations. Here we define Liouville action functional S_Γ for the group Γ as a functional on the space of smooth conformal metrics on $X \sqcup Y$ with the property that its Euler-Lagrange equation is the Liouville equation on $X \sqcup Y$. Its definition is based on the homological algebra methods developed in [AT97].

2.1 Homology and cohomology set-up

Let Γ be a group acting properly on a smooth manifold M. To this data one canonically associates double homology and cohomology complexes (see, e.g., [AT97] and references therein).

Let $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(M)$ be the standard singular chain complex of M with the differential ∂'. The group action on M induces a left Γ-action on $\mathsf{S}_\bullet$ by translating the chains and $\mathsf{S}_\bullet$ becomes a complex of left Γ-modules. Since the action of Γ on M is proper, $\mathsf{S}_\bullet$ is a complex of free left $\mathbb{Z}\Gamma$-modules, where $\mathbb{Z}\Gamma$ is an integral group ring of the group Γ. The complex $\mathsf{S}_\bullet$ is endowed with a right $\mathbb{Z}\Gamma$-module structure in the standard fashion: $c \cdot \gamma = \gamma^{-1}(c)$.

Let $\mathsf{B}_\bullet \equiv \mathsf{B}_\bullet(\mathbb{Z}\Gamma)$ be the canonical "bar" resolution complex for Γ with differential ∂''. Each $\mathsf{B}_n(\mathbb{Z}\Gamma)$ is a free left Γ-module on generators $[\gamma_1|\dots|\gamma_n]$, with the differential $\partial'' : \mathsf{B}_n \longrightarrow \mathsf{B}_{n-1}$ given by

$$\begin{aligned}\partial''[\gamma_1|\dots|\gamma_n] &= \gamma_1[\gamma_2|\dots|\gamma_n] + \sum_{k=1}^{n-1}(-1)^k[\gamma_1|\dots|\gamma_k\gamma_{k+1}|\dots|\gamma_n] \\ &\quad + (-1)^n[\gamma_1|\dots|\gamma_{n-1}]\,, \quad n > 1, \\ \partial''[\gamma] &= \gamma[\,] - [\,]\,, \quad n = 1,\end{aligned}$$

where $[\gamma_1|\dots|\gamma_n]$ is zero if some γ_i equals to the unit element id in Γ. Here $\mathsf{B}_0(\mathbb{Z}\Gamma)$ is a $\mathbb{Z}\Gamma$-module on one generator $[\,]$ and it can be identified with $\mathbb{Z}\Gamma$ under the isomorphism that sends $[\,]$ to 1; by definition, $\partial''[\,] = 0$.

The double homology complex $\mathsf{K}_{\bullet,\bullet}$ is defined as $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$, where the tensor product over $\mathbb{Z}\Gamma$ uses the right Γ-module structure on $\mathsf{S}_\bullet$. The associated total complex $\operatorname{Tot}\mathsf{K}$ is equipped with the total differential $\partial = \partial' + (-1)^p\partial''$ on $\mathsf{K}_{p,q}$, and the complex $\mathsf{S}_\bullet$ is identified with $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_0$ by the isomorphism $c \mapsto c \otimes [\,]$.

Corresponding double complex in cohomology is defined as follows. Denote by $\mathsf{A}^\bullet \equiv \mathsf{A}^\bullet_{\mathbb{C}}(M)$ the complexified de Rham complex on M. Each A^n is a left Γ-module with the pull-back action of Γ, i.e., $\gamma \cdot \varpi = (\gamma^{-1})^*\varpi$ for $\varpi \in \mathsf{A}^\bullet$ and $\gamma \in \Gamma$. Define the double complex $\mathsf{C}^{p,q} = \operatorname{Hom}_{\mathbb{C}}(\mathsf{B}_q, \mathsf{A}^p)$ with differentials d, the usual de Rham differential, and $\delta = (\partial'')^*$, the group coboundary. Specifically, for $\varpi \in \mathsf{C}^{p,q}$,

$$(\delta\varpi)_{\gamma_1,\cdots,\gamma_{q+1}} = \gamma_1 \cdot \varpi_{\gamma_2,\cdots,\gamma_{q+1}} + \sum_{k=1}^{q}(-1)^k \varpi_{\gamma_1,\cdots,\gamma_k\gamma_{k+1},\cdots,\gamma_{q+1}} + (-1)^{q+1}\varpi_{\gamma_1,\cdots,\gamma_q}.$$

We write the total differential on $\mathsf{C}^{p,q}$ as $D = d + (-1)^p\delta$.

There is a natural pairing between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$ which assigns to the pair $(\varpi, c \otimes [\gamma_1|\dots|\gamma_q])$ the evaluation of the p-form $\varpi_{\gamma_1,\cdots,\gamma_q}$ over the p-cycle c,

$$\langle \varpi, c \otimes [\gamma_1|\dots|\gamma_q]\rangle = \int_c \varpi_{\gamma_1,\cdots,\gamma_q}.$$

By definition,

$$\langle \delta\varpi, c\rangle = \langle \varpi, \partial'' c\rangle,$$

so that using Stokes' theorem we get

$$\langle D\varpi, c\rangle = \langle \varpi, \partial c\rangle.$$

This pairing defines a non-degenerate pairing between corresponding cohomology and homology groups $H^\bullet(\operatorname{Tot}\mathsf{C})$ and $H_\bullet(\operatorname{Tot}\mathsf{K})$, which we continue to denote by $\langle\ ,\ \rangle$. In particular, if Φ is a cocycle in $(\operatorname{Tot}\mathsf{C})^n$ and C is a cycle in $(\operatorname{Tot}\mathsf{K})_n$, then the pairing $\langle \Phi, C\rangle$ depends only on cohomology classes $[\Phi]$ and $[C]$ and not on their representatives.

It is this property that will allow us to define Liouville action functional by constructing corresponding cocycle Ψ and cycle Σ. Specifically, we consider the following two cases.

1. Γ is purely hyperbolic Fuchsian group of genus $g > 1$ and $M = \mathbb{U}$ — the upper half-plane of the complex plane $\mathbb{C}$. In this case, since $\mathbb{U}$ is acyclic, we have [AT97]

$$H_\bullet(X, \mathbb{Z}) \cong H_\bullet(\Gamma, \mathbb{Z}) \cong H_\bullet(\mathrm{Tot}\,\mathsf{K}),$$

where the three homologies are: the singular homology of $X \simeq \Gamma\backslash\mathbb{U}$, a compact Riemann surface of genus $g > 1$, the group homology of Γ, and the homology of the complex $\mathrm{Tot}\,\mathsf{K}$ with respect to the total differential ∂. Similarly, for $M = \mathbb{L}$ — the lower half-plane of the complex plane $\mathbb{C}$, we have

$$H_\bullet(\bar{X}, \mathbb{Z}) \cong H_\bullet(\Gamma, \mathbb{Z}) \cong H_\bullet(\mathrm{Tot}\,\mathsf{K}),$$

where $\bar{X} \simeq \Gamma\backslash\mathbb{L}$ is the mirror image of X — a complex-conjugate of the Riemann surface X.

2. Γ is purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity Ω consisting of two simply-connected components Ω_1 and Ω_2 separated by a quasi-circle $\mathcal{C}$. The same isomorphisms hold, where $X \simeq \Gamma\backslash\Omega_1$ and $\bar{X}$ is replaced by $Y \simeq \Gamma\backslash\Omega_2$.

2.2 The Fuchsian case

Let Γ be a marked, normalized, purely hyperbolic Fuchsian group of genus $g > 1$, let $X \simeq \Gamma\backslash\mathbb{U}$ be corresponding marked compact Riemann surface of genus g, and let $\bar{X} \simeq \Gamma\backslash\mathbb{L}$ be its mirror image. In this case it is possible to define Liouville action functionals on Riemann surfaces X and $\bar{X}$ separately. The definition will be based on the following specialization of the general construction in Section 2.1.

2.2.1 Homology computation

Here is a representation of the fundamental class $[X]$ of the Riemann surface X in $H_2(X, \mathbb{Z})$ as a cycle Σ of total degree 2 in the homology complex Tot K [AT97].

Recall that the marking of Γ is given by a system of $2g$ standard generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ satisfying the single relation

$$\gamma_1 \cdots \gamma_g = \mathrm{id},$$

where $\gamma_k = [\alpha_k, \beta_k] = \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}$. The marked group Γ is normalized, if the attracting and repelling fixed points of α_1 are, respectively, 0 and ∞, and the attracting fixed point of β_1 is 1. Every marked Fuchsian group Γ is conjugated in $\mathrm{PSL}(2, \mathbb{R})$ to a normalized marked Fuchsian group. For a given marking there is a standard choice of the fundamental domain $F \subset \mathbb{U}$ for Γ as a closed non-Euclidean polygon with $4g$ edges labeled by a_k, a_k', b_k', b_k satisfying $\alpha_k(a_k') = a_k$, $\beta_k(b_k') = b_k$, $k = 1, 2, \ldots, g$ (see Fig. 1). The orientation of the edges is chosen such that

$$\partial' F = \sum_{k=1}^{g} (a_k + b_k' - a_k' - b_k).$$

Set $\partial' a_k = a_k(1) - a_k(0)$, $\partial' b_k = b_k(1) - b_k(0)$, so that $a_k(0) = b_{k-1}(0)$. The relations between the vertices of F and the generators of Γ are the following: $\alpha_k^{-1}(a_k(0)) = b_k(1)$, $\beta_k^{-1}(b_k(0)) = a_k(1)$, $\gamma_k(b_k(0)) = b_{k-1}(0)$, where $b_0(0) = b_g(0)$.

According to the isomorphism $\mathsf{S}_\bullet \simeq \mathsf{K}_{\bullet,0}$, the fundamental domain F is identified with $F \otimes [\,] \in \mathsf{K}_{2,0}$. We have $\partial'' F = 0$ and, as it follows from the previous formula,

$$\partial' F = \sum_{k=1}^{g} \left(\beta_k^{-1}(b_k) - b_k - \alpha_k^{-1}(a_k) + a_k\right) = \partial'' L,$$

where $L \in \mathsf{K}_{1,1}$ is given by

$$L = \sum_{k=1}^{g} \left(b_k \otimes [\beta_k] - a_k \otimes [\alpha_k]\right). \tag{2.2.1}$$

There exists $V \in \mathsf{K}_{0,2}$ such that $\partial'' V = \partial' L$. A straightforward computation gives the following explicit expression

$$\begin{aligned} V = &\sum_{k=1}^{g} \left(a_k(0) \otimes [\alpha_k|\beta_k] - b_k(0) \otimes [\beta_k|\alpha_k] + b_k(0) \otimes \left[\gamma_k^{-1}|\alpha_k\beta_k\right]\right) \\ &- \sum_{k=1}^{g-1} b_g(0) \otimes \left[\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}\right]. \end{aligned} \tag{2.2.2}$$

Using $\partial'' F = 0$, $\partial' F = \partial'' L$, $\partial'' V = \partial' L$, and $\partial' V = 0$, we obtain that the element $\Sigma = F + L - V$ of total degree 2 is a cycle in $\operatorname{Tot}\mathsf{K}$, that is $\partial\Sigma = 0$. The cycle $\Sigma \in (\operatorname{Tot}\mathsf{K})_2$ represents the fundamental class $[X]$. It is proved in [AT97] that corresponding homology class $[\Sigma]$ in $H_\bullet(\operatorname{Tot}\mathsf{K})$ does not depend on the choice of the fundamental domain F for the group Γ.

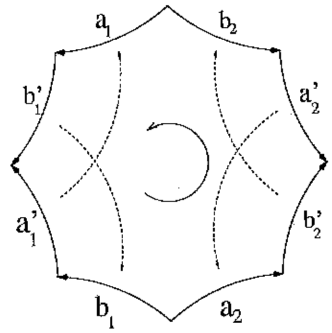

Figure 2.1: Conventions for the fundamental domain F

2.2.2 Cohomology computation

Corresponding construction in cohomology is the following. Start with the space $\mathcal{CM}(X)$ of all conformal metrics on $X \simeq \Gamma\backslash\mathbb{U}$. Every $ds^2 \in \mathcal{CM}(X)$ can be represented as $ds^2 = e^{\phi}|dz|^2$, where $\phi \in C^{\infty}(\mathbb{U}, \mathbb{R})$ satisfies

$$\phi \circ \gamma + \log|\gamma'|^2 = \phi \quad \text{for all } \gamma \in \Gamma. \tag{2.2.3}$$

In what follows we will always identify $\mathcal{CM}(X)$ with the affine subspace of $C^{\infty}(\mathbb{U}, \mathbb{R})$ defined by (2.2.3).

The "bulk" 2-form ω for the Liouville action is given by

$$\omega[\phi] = \left(|\phi_z|^2 + e^{\phi}\right) dz \wedge d\bar{z}, \tag{2.2.4}$$

where $\phi \in \mathcal{CM}(X)$. Considering it as an element in $\mathrm{C}^{2,0}$ and using (2.2.3) we get

$$\delta\omega[\phi] = d\theta[\phi],$$

where $\theta[\phi] \in \mathsf{C}^{1,1}$ is given explicitly by

$$\theta_{\gamma^{-1}}[\phi] = \left(\phi - \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right). \qquad (2.2.5)$$

Next, set

$$u = \delta\theta[\phi] \in \mathsf{C}^{1,2}.$$

From the definition of θ and $\delta^2 = 0$ it follows that the 1-form u is closed. An explicit calculation gives

$$\begin{aligned} u_{\gamma_1^{-1},\gamma_2^{-1}} = &-\frac{1}{2}\log|\gamma_1'|^2\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1'\,dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}\,d\bar{z}\right) \\ &+\frac{1}{2}\log|\gamma_2'\circ\gamma_1|^2\left(\frac{\gamma_1''}{\gamma_1'}dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar{z}\right), \end{aligned} \qquad (2.2.6)$$

and shows that u does not depend on $\phi \in \mathcal{CM}(X)$.

Remark 2.2.1. The explicit formulas above are valid in the general case, when domain $\Omega \subset \hat{\mathbb{C}}$ is invariant under the action of a Kleinian group Γ. Namely, define the 2-form ω by formula (2.2.4), where ϕ satisfies (2.2.3) in Ω. Then solution θ to the equation $\delta\omega[\phi] = d\theta[\phi]$ is given by the formula (2.2.5) and $u = \delta\theta[\phi]$ — by (2.2.6).

There exists a cochain $\Theta \in \mathsf{C}^{0,2}$ satisfying

$$d\Theta = u \text{ and } \delta\Theta = 0.$$

Indeed, since the 1-form u is closed and $\mathbb{U}$ is simply-connected, Θ can be defined as a particular antiderivative of u satisfying $\delta\Theta = 0$. This can be done as follows. Consider the hyperbolic (Poincaré) metric on $\mathbb{U}$

$$e^{\phi_{hyp}(z)}|dz|^2 = \frac{|dz|^2}{y^2}, \quad z = x + iy \in \mathbb{U}.$$

This metric is $\mathrm{PSL}(2,\mathbb{R})$-invariant and its push-forward to X is a hyperbolic metric on X. Explicit computation yields

$$\omega[\phi_{hyp}] = 2e^{\phi_{hyp}}\, dz \wedge d\bar{z},$$

so that $\delta\omega[\phi_{hyp}] = 0$. Thus the 1-form $\theta[\phi_{hyp}]$ on $\mathbb{U}$ is closed and, therefore, is exact,

$$\theta[\phi_{hyp}] = dl,$$

for some $l \in \mathsf{C}^{0,1}$. Set

$$\Theta = \delta l. \tag{2.2.7}$$

It is now immediate that $\delta\Theta = 0$ and $\delta\theta[\phi] = u = d\Theta$ for all $\phi \in \mathcal{CM}(X)$. Thus $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$ is a 2-cocycle in the cohomology complex $\mathrm{Tot}\,\mathsf{C}$, that is, $D\Psi[\phi] = 0$.

Remark 2.2.2. For every $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ define the 1-form $\theta_\gamma\,[\phi_{hyp}]$ by the same formula (2.2.5),

$$\theta_{\gamma^{-1}}[\phi_{hyp}] = -\left(2\log y + \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right). \tag{2.2.8}$$

Since for every $\gamma \in \mathrm{PSL}(2,\mathbb{R})$

$$(\delta \log y)_{\gamma^{-1}} = \log(y\circ\gamma) - \log y = \frac{1}{2}\log|\gamma'|^2,$$

the 1-form $u = \delta\theta[\phi]$ is still given by (2.2.6) and is a $\mathsf{A}^1(\mathbb{U})$-valued group 2-cocycle for $\mathrm{PSL}(2,\mathbb{R})$, that is, $(\delta u)_{\gamma_1,\gamma_2,\gamma_3} = 0$ for all $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{PSL}(2,\mathbb{R})$. Also 0-form Θ given by (2.2.7) satisfies $d\Theta = u$ and is a $\mathsf{A}^0(\mathbb{U})$-valued group 2-cocycle for $\mathrm{PSL}(2,\mathbb{R})$.

2.2.3 The action functional

The evaluation map $\langle \Psi[\phi], \Sigma\rangle$ does not depend on the choice of the fundamental domain F for Γ [AT97]. It also does not depend on a particular choice of antiderivative l, since by the Stokes' theorem

$$\langle \Theta, V\rangle = \langle \delta l, V\rangle = \langle l, \partial'' V\rangle = \langle l, \partial' L\rangle = \langle \theta[\phi_{hyp}], L\rangle. \tag{2.2.9}$$

This justifies the following definition.

Definition 2.2.3. The Liouville action functional $S[\,\cdot\,; X] : \mathcal{CM}(X) \to \mathbb{R}$ is defined by the evaluation map

$$S[\phi; X] = \frac{i}{2}\langle \Psi[\phi], \Sigma\rangle\,, \;\; \phi \in \mathcal{CM}(X).$$

For brevity, set $S[\phi] = S[\phi; X]$. The following lemma shows that the difference of any two values of the functional S is given by the bulk term only.

Lemma 2.2.4. *For all $\phi \in \mathcal{CM}(X)$ and $\sigma \in C^\infty(X, \mathbb{R})$,*

$$S[\phi + \sigma] - S[\phi] = \iint\limits_F \left(|\sigma_z|^2 + (e^\sigma + K\,\sigma - 1)\, e^\phi\right) d^2z,$$

where $d^2z = dx \wedge dy$ is the Lebesgue measure and $K = -2e^{-\phi}\phi_{z\bar z}$ is the Gaus-

sian curvature of the metric $e^{\phi}|dz|^2$.

Proof. We have

$$\omega[\phi+\sigma]-\omega[\phi]=\omega[\phi;\sigma]+d\tilde{\theta},$$

where

$$\omega[\phi;\sigma]=\left(|\sigma_z|^2+(e^{\sigma}+K\,\sigma-1)\,e^{\phi}\right)dz\wedge d\bar{z},$$

and

$$\tilde{\theta}=\sigma\left(\phi_{\bar{z}}d\bar{z}-\phi_z dz\right).$$

Since

$$\delta\tilde{\theta}_{\gamma^{-1}}=\sigma\left(\frac{\gamma''}{\gamma'}dz-\frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right)=\theta[\phi+\sigma]-\theta[\phi],$$

the assertion of the lemma follows from the Stokes' theorem. □

Corollary 2.2.5. *The Euler-Lagrange equation for the functional S is the Liouville equation, the critical point of S — the hyperbolic metric ϕ_{hyp}, is non-degenerate, and the classical action — the critical value of S, is twice the hyperbolic area of X, that is, $4\pi(2g-2)$.*

Proof. As it follows from Lemma 2.2.4,

$$\left.\frac{dS[\phi+t\sigma]}{dt}\right|_{t=0}=\iint\limits_F (K+1)\,\sigma e^{\phi}d^2z,$$

so that the Euler-Lagrange equation is the Liouville equation $K=-1$. Since

$$\left.\frac{d^2S[\phi_{hyp}+t\sigma]}{dt^2}\right|_{t=0}=\iint\limits_F\left(2|\sigma_z|^2+\sigma^2e^{\phi_{hyp}}\right)d^2z>0\quad\text{if}\quad\sigma\neq 0,$$

the critical point ϕ_{hyp} is non-degenerate. Using (2.2.9) we get

$$S[\phi_{hyp}] = \frac{i}{2}\langle \Psi[\phi_{hyp}], \Sigma\rangle = \frac{i}{2}\langle \omega[\phi_{hyp}], F\rangle = 2\iint\limits_F \frac{d^2z}{y^2} = 4\pi(2g-2).$$

□

Remark 2.2.6. Let $\Delta[\phi] = -4e^{-\phi}\partial_z\partial_{\bar{z}}$ be the Laplace operator of the metric $ds^2 = e^\phi|dz|^2$ acting on functions on X, and let $\det\Delta[\phi]$ be its zeta-function regularized determinant (see, e.g., [OPS88] for details). Denote by $A[\phi]$ the area of X with respect to the metric ds^2 and set

$$\mathcal{I}[\phi] = \log\frac{\det\Delta[\phi]}{A[\phi]}.$$

The Polyakov's "conformal anomaly" formula [Pol81] reads

$$\mathcal{I}[\phi+\sigma] - \mathcal{I}[\phi] = -\frac{1}{12\pi}\iint\limits_F \left(|\sigma_z|^2 + K\sigma\, e^\phi\right) d^2z,$$

where $\sigma \in C^\infty(X, \mathbb{R})$ (see [OPS88] for rigorous proof). Comparing it with Lemma 2.2.4 we get

$$\mathcal{I}[\phi+\sigma] + \frac{1}{12\pi}\check{S}[\phi+\sigma] = \mathcal{I}[\phi] + \frac{1}{12\pi}\check{S}[\phi],$$

where $\check{S}[\phi] = S[\phi] - A[\phi]$.

Lemma 2.2.4, Corollary 2.2.5 (without the assertion on classical action) and Remark 2.2.6 remain valid if Θ is replaced by $\Theta + c$, where c is an arbitrary group 2-cocycle with values in $\mathbb{C}$. The choice (2.2.7), or rather its analog for the quasi-Fuchsian case, will be important in Chapter 4, where we consider classical action for families of Riemann surfaces. For this purpose, we present an explicit formula for Θ as a particular antiderivative of the 1-form u.

Let $p \in \overline{\mathbb{U}}$ be an arbitrary point on the closure of $\mathbb{U}$ in $\mathbb{C}$ (nothing will depend on the choice of p). Set

$$l_\gamma(z) = \int_p^z \theta_\gamma[\phi_{hyp}] \text{ for all } \gamma \in \Gamma, \tag{2.2.10}$$

where the path of integration P connects points p and z and, possibly except p, lies entirely in $\mathbb{U}$. If $p \in \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, it is assumed that P is smooth and is not tangent to $\mathbb{R}_\infty$ at p. Such paths are called admissible. A 1-form ϑ on $\mathbb{U}$ is called integrable along admissible path P with the endpoint $p \in \mathbb{R}_\infty$, if the limit of $\int_{p'}^z \vartheta$, as $p' \to p$ along P, exists. Similarly, a path P is called Γ-closed if its endpoints are p and γp for some $\gamma \in \Gamma$, and $P \setminus \{p, \gamma p\} \subset \mathbb{U}$. A Γ-closed path P with endpoints p and γp, $p \in \mathbb{R}_\infty$, is called admissible if it is not tangent to $\mathbb{R}_\infty$ at p and there exists $p' \in P$ such that the translate by γ of the part of P between the points p' and p belongs to P. A 1-form ϑ is integrable along Γ-closed admissible path P, if the limit of $\int_{p'}^{\gamma p'} \vartheta$, as $p' \to p$ along P, exists.

Let

$$\begin{aligned} W = &\sum_{k=1}^{g} \left(P_{k-1} \otimes [\alpha_k|\beta_k] - P_k \otimes [\beta_k|\alpha_k] + P_k \otimes \left[\gamma_k^{-1}|\alpha_k\beta_k\right]\right) \\ &- \sum_{k=1}^{g-1} P_g \otimes \left[\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}\right] \in \mathsf{K}_{1,2}, \end{aligned} \tag{2.2.11}$$

where P_k is any admissible path from p to $b_k(0)$, $k = 1, \dots, g$, and $P_g = P_0$.

Since $P_k(1) = b_k(0) = a_{k+1}(0)$, we have

$$\partial' W = V - U,$$

where

$$\begin{aligned} U = \sum_{k=1}^{g} \left(p \otimes [\alpha_k | \beta_k] - p \otimes [\beta_k | \alpha_k] + p \otimes \left[\gamma_k^{-1} | \alpha_k \beta_k\right] \right) \\ - \sum_{k=1}^{g-1} p \otimes \left[\gamma_g^{-1} \dots \gamma_{k+1}^{-1} | \gamma_k^{-1}\right] \in \mathsf{K}_{1,2}. \end{aligned} \tag{2.2.12}$$

We have the following statement.

Lemma 2.2.7. *Let $\vartheta \in \mathrm{C}^{1,1}$ be a closed 1-form on $\mathbb{U}$ and $p \in \overline{\mathbb{U}}$. In case $p \in \mathbb{R}_\infty$ suppose that $\delta\vartheta$ is integrable along any admissible path with endpoints in $\Gamma \cdot p$ and ϑ is integrable along any Γ-closed admissible path with endpoints in $\Gamma \cdot p$. Then*

$$\begin{aligned} \langle \vartheta, L \rangle = \ & \langle \delta\vartheta, W \rangle \\ & + \sum_{k=1}^{g} \left(\int_p^{\alpha_k^{-1} p} \vartheta_{\beta_k} - \int_p^{\beta_k^{-1} p} \vartheta_{\alpha_k} + \int_p^{\gamma_k p} \vartheta_{\alpha_k \beta_k} - \int_p^{\gamma_{k+1} \dots \gamma_g p} \vartheta_{\gamma_k^{-1}} \right), \end{aligned}$$

where paths of integration are admissible if $p \in \mathbb{R}_\infty$.

Proof. Since ϑ_γ is closed and $\mathbb{U}$ is simply-connected, we can define a function l_γ on $\mathbb{U}$ by

$$l_\gamma(z) = \int_p^z \vartheta_\gamma,$$

where $p \in \mathbb{U}$. We have, using Stokes' theorem and $d(\delta l) = \delta(dl) = \delta\vartheta$,

$$\begin{aligned}\langle \vartheta, L\rangle &= \langle dl, L\rangle = \langle l, \partial' L\rangle = \langle l, \partial'' V\rangle = \langle \delta l, V\rangle \\ &= \langle \delta l, \partial' W\rangle + \langle \delta l, U\rangle = \langle d(\delta l), W\rangle + \langle \delta l, U\rangle \\ &= \langle \delta\vartheta, W\rangle \;\;+\;\; \langle \delta l, U\rangle.\end{aligned}$$

Since

$$(\delta l)_{\gamma_1,\gamma_2}(p) = \int_p^{\gamma_1^{-1}p} \vartheta_{\gamma_2},$$

we get the statement of the lemma if $p \in \mathbb{U}$. In case $p \in \mathbb{R}_\infty$, replace p by $p' \in \mathbb{U}$. Conditions of the lemma guarantee the convergence of integrals as $p' \to p$ along corresponding paths. □

Remark 2.2.8. Expression $\langle \delta l, U\rangle$, which appears in the statement of the lemma, does not depend on the choice of a particular antiderivative of the closed 1-form ϑ. The same statement holds if we only assume that 1-form $\delta\vartheta$ is integrable along admissible paths with endpoints in $\Gamma \cdot p$, and 1-form ϑ has an antiderivative l (not necessarily vanishing at p) such that the limit of $(\delta l)_{\gamma_1,\gamma_2}(p')$, as $p' \to p$ along admissible paths, exists.

Lemma 2.2.9. *We have*

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + \eta(p)_{\gamma_1,\gamma_2}, \tag{2.2.13}$$

where $p \in \mathbb{R} \setminus \Gamma(\infty)$ and integration goes along admissible paths. The integra-

tion constants $\eta \in C^{0,2}$ *are given by*

$$\eta(p)_{\gamma_1,\gamma_2} = 4\pi i \varepsilon(p)_{\gamma_1,\gamma_2}(2\log 2 + \log|c(\gamma_2)|^2), \tag{2.2.14}$$

and

$$\varepsilon(p)_{\gamma_1,\gamma_2} = \begin{cases} 1 & \text{if } p < \gamma_2(\infty) < \gamma_1^{-1}p, \\ -1 & \text{if } p > \gamma_2(\infty) > \gamma_1^{-1}p, \\ 0 & \text{otherwise.} \end{cases}$$

Here for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *we set* $c(\gamma) = c$.

Proof. Since

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + \int_p^{\gamma_1^{-1}p} \theta_{\gamma_2}[\phi_{hyp}],$$

it is sufficient to verify that

$$\frac{1}{2\pi i}\int_p^{\gamma_1 p} \theta_{\gamma_2^{-1}}[\phi_{hyp}] = \begin{cases} 4\log 2 + 2\log|c(\gamma_2)|^2 & \text{if } p < \gamma_2^{-1}(\infty) < \gamma_1 p, \\ -4\log 2 - 2\log|c(\gamma_2)|^2 & \text{if } p > \gamma_2^{-1}(\infty) > \gamma_1 p, \\ 0 & \text{otherwise.} \end{cases}$$

From (2.2.8) it follows that $\theta_{\gamma^{-1}}[\phi_{hyp}]$ is a closed 1-form on $\mathbb{U}$, integrable along admissible paths with $p \in \mathbb{R}\setminus\{\gamma^{-1}(\infty)\}$. Denote by $\theta^{(\varepsilon)}_{\gamma^{-1}}$ its restriction on the line $y = \varepsilon > 0$, $z = x + iy$. When $x \neq \gamma_2^{-1}(\infty)$, we obviously have

$$\lim_{\varepsilon\to 0} \theta^{(\varepsilon)}_{\gamma_2^{-1}} = 0,$$

uniformly in x on compact subsets of $\mathbb{R}\setminus\{\gamma_2^{-1}(\infty)\}$.

If $\gamma_2^{-1}(\infty)$ does not lie between points p and $\gamma_1 p$ on $\mathbb{R}$, we can approximate the path of integration by the interval on the line $y = \varepsilon$, which tends to 0 as $\varepsilon \to 0$. If $\gamma_2^{-1}(\infty)$ lies between points p and $\gamma_1 p$, we have to go around the point $\gamma_2^{-1}(\infty)$ via a small half-circle, so that

$$\int_p^{\gamma_1 p} \theta_{\gamma_2^{-1}}[\phi_{hyp}] = \lim_{r\to 0} \int_{C_r} \theta_{\gamma_2^{-1}}[\phi_{hyp}],$$

where C_r is the upper-half of the circle of radius r with center at $\gamma_2^{-1}(\infty)$, oriented clockwise if $p < \gamma_2^{-1}(\infty) < \gamma_1 p$, and counter-clockwise if $p > \gamma_2^{-1}(\infty) > \gamma_1 p$. Evaluating the limit using elementary formula

$$\int_0^{\pi} \log \sin t \, dt = -\pi \log 2,$$

and Cauchy theorem, we get the formula. □

Corollary 2.2.10. *The Liouville action functional has the following explicit representation*

$$S[\phi] = \frac{i}{2} \left(\langle \omega[\phi], F \rangle - \langle \theta[\phi], L \rangle + \langle u, W \rangle + \langle \eta, V \rangle \right).$$

Remark 2.2.11. Since $\langle \Theta, V \rangle = \langle u, W \rangle + \langle \eta, V \rangle$, it immediately follows from (2.2.9) that the Liouville action functional does not depend on the choice of the point $p \in \mathbb{R} \setminus \Gamma(\infty)$ (actually it is sufficient to assume that $p \neq \gamma_1(\infty), (\gamma_1\gamma_2)(\infty)$ for all $\gamma_1, \gamma_2 \in \Gamma$ such that $V_{\gamma_1,\gamma_2} \neq 0$). This can also be proved by direct computation using Remark 2.2.2. Namely, let $p' \in \mathbb{R}_\infty$ be another choice, $p' = \sigma^{-1} p \in \mathbb{R}_\infty$ for some $\sigma \in \mathrm{PSL}(2, \mathbb{R})$. Setting $z = p$ in the equation $(\delta\Theta)_{\sigma,\gamma_1,\gamma_2} = 0$ and using $(\delta u)_{\sigma,\gamma_1,\gamma_2} = 0$, where $\gamma_1, \gamma_2 \in \Gamma$, we get

$$\int_p^{\sigma^{-1}p} u_{\gamma_1,\gamma_2} = -(\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.2.15}$$

where all paths of integration are admissible. Using

$$\eta(p)_{\sigma\gamma_1,\gamma_2} = \eta(\sigma^{-1}p)_{\gamma_1,\gamma_2} + \eta(p)_{\sigma,\gamma_2},$$

we get from (2.2.15) that

$$\int_p^z u_{\gamma_1,\gamma_2} + \eta(p)_{\gamma_1,\gamma_2} = \int_{p'}^z u_{\gamma_1,\gamma_2} + \eta(p')_{\gamma_1,\gamma_2} + (\delta\eta_\sigma)_{\gamma_1,\gamma_2},$$

where $(\eta_\sigma)_\gamma = \eta(p)_{\sigma,\gamma}$ is constant group 1-cochain. The statement now follows from

$$\langle\delta\eta_\sigma, V\rangle = \langle\eta_\sigma, \partial'' V\rangle = \langle\eta_\sigma, \partial' L\rangle = \langle d\eta_\sigma, L\rangle = 0.$$

Another consequence of Lemmas 2.2.7 and 2.2.9 is the following.

Corollary 2.2.12. *Set*

$$\varkappa_{\gamma^{-1}} = \frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z} \in \mathsf{C}^{1,1}.$$

Then

$$\langle\varkappa, L\rangle = 4\pi i\langle\varepsilon, V\rangle = 4\pi i\,\chi(X),$$

where $\chi(X) = 2-2g$ is the Euler characteristic of Riemann surface $X \simeq \Gamma\backslash\mathbb{U}$.

Proof. Since $\delta\varkappa = 0$, the first equation immediately follows from the proofs of

Lemmas 2.2.7 and 2.2.9. To prove the second equation, observe that

$$\varkappa = \delta\varkappa_1, \quad \text{where} \quad \varkappa_1 = -\phi_z dz + \phi_{\bar z} d\bar z \quad \text{and} \quad d\varkappa_1 = 2\phi_{z\bar z}\, dz \wedge d\bar z.$$

Therefore

$$\langle \varkappa, L\rangle = \langle \delta\varkappa_1, L\rangle = \langle \varkappa_1, \partial'' L\rangle = \langle \varkappa_1, \partial' F\rangle = \langle d\varkappa_1, F\rangle.$$

The Gaussian curvature of the metric $ds^2 = e^{\phi}|dz|^2$ is $K = -2e^{-\phi}\phi_{z\bar z}$, so by Gauss-Bonnet we get

$$\langle d\varkappa_1, F\rangle = 2\iint\limits_F \phi_{z\bar z}\, dz \wedge d\bar z = 2i \iint\limits_{\Gamma\backslash\mathbb{U}} K e^{\phi} d^2 z = 4\pi i \chi(X).$$

□

Using this corollary, we can "absorb" the integration constants η by shifting $\theta[\phi] \in \mathrm{C}^{1,1}$ by a multiple of the closed 1-form $\varkappa$. Indeed, 1-form $\theta[\phi]$ satisfies the equation $\delta\omega[\phi] = d\theta[\phi]$ and is defined up to addition of a closed 1-form. Set

$$\check\theta_\gamma[\phi] = \theta_\gamma[\phi] - (2\log 2 + \log|c(\gamma)|^2)\varkappa_\gamma, \tag{2.2.16}$$

and define $\check u = \delta\check\theta[\phi]$. Explicitly,

$$\begin{aligned}\check u_{\gamma_1^{-1},\gamma_2^{-1}} = u_{\gamma_1^{-1},\gamma_2^{-1}} &- \log\frac{|c(\gamma_2)|^2}{|c(\gamma_2\gamma_1)|^2}\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1' dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'} d\bar z\right) \\ &+ \log\frac{|c(\gamma_2\gamma_1)|^2}{|c(\gamma_1)|^2}\left(\frac{\gamma_1''}{\gamma_1'}dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar z\right),\end{aligned} \tag{2.2.17}$$

where u is given by (2.2.6). As it follows from Lemma 2.2.7 and Corollary 2.2.12,

$$S[\phi] = \frac{i}{2}\left(\langle\omega[\phi], F\rangle - \langle\check{\theta}[\phi], L\rangle + \langle\check{u}, W\rangle\right). \tag{2.2.18}$$

Liouville action functional for the mirror image $\bar{X}$ is defined similarly. Namely, for every chain c in the upper half-plane $\mathbb{U}$ denote by $\bar{c}$ its mirror image in the lower half-plane $\mathbb{L}$; chain $\bar{c}$ has an opposite orientation to c. Set $\bar{\Sigma} = \bar{F} + \bar{L} - \bar{V}$, so that $\partial\bar{\Sigma} = 0$. For $\phi \in \mathcal{CM}(\bar{X})$, considered as a smooth real-valued function on $\mathbb{L}$ satisfying (2.2.3), define $\omega[\phi] \in \mathsf{C}^{2,0}$, $\theta[\phi] \in \mathsf{C}^{1,1}$ and $\Theta \in \mathsf{C}^{0,2}$ by the same formulas (2.2.4), (2.2.5) and (2.2.7). Lemma 2.2.9 has an obvious analog for the lower half-plane $\mathbb{L}$, the analog of formula (2.2.13) for $z \in \mathbb{L}$ is

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} - \eta(p)_{\gamma_1,\gamma_2}, \tag{2.2.19}$$

where the negative sign comes from the opposite orientation.

Remark 2.2.13. Similarly to (2.2.15) we get

$$\int_p^{\sigma^{-1}p} u_{\gamma_1,\gamma_2} = (\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.2.20}$$

where the path of integration, except the endpoints, lies in $\mathbb{L}$. From (2.2.15) and (2.2.20) we obtain

$$\int_C u_{\gamma_1,\gamma_2} = -2(\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.2.21}$$

where the path of integration C is a loop that starts at p, goes to $\sigma^{-1}p$ inside $\mathbb{U}$, continues inside $\mathbb{L}$ and ends at p. Note that formula (2.2.21) can also be verified directly using Stokes' theorem. Indeed, the 1-form u_{γ_1,γ_2} is closed and regular everywhere except at points $\gamma_1(\infty)$ and $\gamma_1\gamma_2(\infty)$. Integrating over small circles around these points if they lie inside C and using (2.2.14), we get the result.

Set $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$, so that $D\Psi[\phi] = 0$. The Liouville action functional for $\bar{X}$ is defined by

$$[\phi; \bar{X}] = -\frac{i}{2}\langle \Psi[\phi], \bar{\Sigma}\rangle.$$

Using an analog of Lemma 2.2.7 in the lower half-plane $\mathbb{L}$ and

$$\langle \eta, \bar{V}\rangle = \langle \eta, V\rangle,$$

we obtain

$$S[\phi; \bar{X}] = -\frac{i}{2}\left(\langle \omega[\theta], \bar{F}\rangle - \langle \theta[\phi], \bar{L}\rangle + \langle u, \bar{W}\rangle - \langle \eta, V\rangle\right).$$

Finally, we have the following definition.

Definition 2.2.14. The Liouville action functional $S_\Gamma : \mathcal{CM}(X \sqcup \bar{X}) \to \mathbb{R}$ for the Fuchsian group Γ acting on $\mathbb{U} \cup \mathbb{L}$ is defined by

$$\begin{aligned} S_\Gamma[\phi] =& S[\phi; X] + S[\phi; \bar{X}] = \frac{i}{2}\langle \Psi[\phi], \Sigma - \bar{\Sigma}\rangle \\ =& \frac{i}{2}\left(\langle \omega[\phi], F - \bar{F}\rangle - \langle \theta[\phi], L - \bar{L}\rangle + \langle u, W - \bar{W}\rangle + 2\langle \eta, V\rangle\right), \end{aligned}$$

where $\phi \in \mathcal{CM}(X \sqcup \bar{X})$.

The functional S_Γ satisfies an obvious analog of Lemma 2.2.4. Its Euler-Lagrange equation is the Liouville equation, so that its single non-degenerate critical point is the hyperbolic metric on $\mathbb{U}\cup\mathbb{L}$. Corresponding classical action is $8\pi(2g-2)$ — twice the hyperbolic area of $X \sqcup \bar{X}$. Similarly to (2.2.18) we have

$$S_\Gamma[\phi] = \frac{i}{2}\left(\langle\omega[\phi], F - \bar{F}\rangle - \langle\check{\theta}[\phi], L - \bar{L}\rangle + \langle\check{u}, W - \bar{W}\rangle\right). \qquad (2.2.22)$$

Remark 2.2.15. In the definition of S_Γ it is not necessary to choose a fundamental domain for Γ in $\mathbb{L}$ to be the mirror image of the fundamental domain in $\mathbb{U}$ since the corresponding homology class $[\Sigma - \bar{\Sigma}]$ does not depend on the choice of the fundamental domain of Γ in $\mathbb{U}\cup\mathbb{L}$.

2.3 The quasi-Fuchsian case

Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$. Its region of discontinuity Ω has two invariant components Ω_1 and Ω_2 separated by a quasi-circle $\mathcal{C}$. By definition, there exists a quasiconformal homeomorphism J_1 of $\hat{\mathbb{C}}$ with the following properties.

QF1 The mapping J_1 is holomorphic on $\mathbb{U}$ and $J_1(\mathbb{U}) = \Omega_1$, $J_1(\mathbb{L}) = \Omega_2$, and $J_1(\mathbb{R}_\infty) = \mathcal{C}$.

QF2 The mapping J_1 fixes $0, 1$ and ∞.

QF3 The group $\tilde{\Gamma} = J_1^{-1}\circ\Gamma\circ J_1$ is Fuchsian.

Due to the normalization, any two maps satisfying **QF1-QF3** agree on $\mathbb{U}$, so that the group $\tilde{\Gamma}$ is independent of the choice of the map J_1. Setting $X \simeq \tilde{\Gamma}\backslash\mathbb{U}$, we get $\tilde{\Gamma}\backslash\mathbb{U} \cup \mathbb{L} \simeq X \sqcup \bar{X}$ and $\Gamma\backslash\Omega \simeq X \sqcup Y$, where X and Y are marked compact Riemann surfaces of genus $g > 1$ with opposite orientations. Conversely, according to Bers' simultaneous uniformization theorem [Ber60], for any pair of marked compact Riemann surfaces X and Y of genus $g > 1$ with opposite orientations, there exists a unique, up to a conjugation in $\mathrm{PSL}(2, \mathbb{C})$, quasi-Fuchsian group Γ such that $\Gamma\backslash\Omega \simeq X \sqcup Y$.

Remark 2.3.1. It is customary (see, e.g., [Ahl87]) to define quasi-Fuchsian groups by requiring that the map J_1 is holomorphic in the lower half-plane $\mathbb{L}$. We will see in Chapter 4 that the above definition is somewhat more convenient.

Let μ be the Beltrami coefficient for the quasiconformal map J_1,

$$\mu = \frac{(J_1)_{\bar{z}}}{(J_1)_z},$$

that is, $J_1 = f^\mu$ — the unique, normalized solution of the Beltrami equation on $\hat{\mathbb{C}}$ with Beltrami coefficient μ. Obviously, $\mu = 0$ on $\mathbb{U}$. Define another Beltrami coefficient $\hat{\mu}$ by

$$\hat{\mu}(z) = \begin{cases} \overline{\mu(\bar{z})} & \text{if } z \in \mathbb{U}, \\ \mu(z) & \text{if } z \in \mathbb{L}. \end{cases}$$

Since $\hat{\mu}$ is symmetric, normalized solution $f^{\hat{\mu}}$ of the Beltrami equation

$$f_{\bar{z}}^{\hat{\mu}}(z) = \hat{\mu}(z) f_{z}^{\hat{\mu}}(z)$$

is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ which preserves $\mathbb{U}$ and $\mathbb{L}$. The quasiconformal map $J_2 = J_1 \circ (f^{\hat{\mu}})^{-1}$ is then conformal on the lower half-plane $\mathbb{L}$ and has properties similar to **QF1-QF3**. In particular, $J_2^{-1} \circ \Gamma \circ J_2 = \hat{\Gamma} = f^{\hat{\mu}} \circ \tilde{\Gamma} \circ (f^{\hat{\mu}})^{-1}$ is a Fuchsian group and $\hat{\Gamma} \backslash \mathbb{L} \simeq Y$. Thus for a given Γ the restriction of the map J_2 to $\mathbb{L}$ does not depend on the choice of J_2 (and hence of J_1). These properties can be summarized by the following commutative diagram

$$\begin{array}{ccc} \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} & \xrightarrow{J_1 = f^{\mu}} & \Omega_1 \cup \mathcal{C} \cup \Omega_2 \\ \Big\downarrow f^{\hat{\mu}} & & \Big\uparrow J_2 \\ \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} & \xrightarrow{=} & \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} \end{array}$$

where maps J_1, J_2 and $f^{\hat{\mu}}$ intertwine corresponding pairs of groups $\Gamma, \tilde{\Gamma}$ and $\hat{\Gamma}$.

2.3.1 Homology construction

The map J_1 induces a chain map between double complexes $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ for the pairs $\mathbb{U} \cup \mathbb{L}, \tilde{\Gamma}$ and Ω, Γ, by pushing forward chains $S_\bullet(\mathbb{U} \cup \mathbb{L}) \ni c \mapsto J_1(c) \in S_\bullet(\Omega)$ and group elements $\tilde{\Gamma} \ni \gamma \mapsto J_1 \circ \gamma \circ J_1^{-1} \in \Gamma$. We will continue to denote this chain map by J_1. Obviously, the chain map J_1 induces an isomorphism between homology groups of corresponding total complexes $\operatorname{Tot} \mathsf{K}$.

Let $\Sigma = F + L - V$ be total cycle of degree 2 representing the fundamental class of X in the total homology complex for the pair $\mathbb{U}, \tilde{\Gamma}$, constructed in the previous section, and let $\Sigma' = F' + L' - V'$ be the corresponding cycle for $\bar{X}$. The total cycle $\Sigma(\Gamma)$ of degree 2 representing fundamental class of $X \sqcup Y$ in the total complex for the pair Ω, Γ can be realized as a push-forward of the total cycle $\Sigma(\tilde{\Gamma}) = \Sigma - \Sigma'$ by J_1,

$$\Sigma(\Gamma) = J_1(\Sigma(\tilde{\Gamma})) = J_1(\Sigma) - J_1(\Sigma').$$

We will denote push-forwards by J_1 of the chains F, L, V in $\mathbb{U}$ by F_1, L_1, V_1, and push-forwards of the corresponding chains F', L', V' in $\mathbb{L}$ — by F_2, L_2, V_2, where indices 1 and 2 refer, respectively, to domains Ω_1 and Ω_2.

The definition of chains W_i is more subtle. Namely, the quasi-circle $\mathcal{C}$ is not generally smooth or even rectifiable, so that an arbitrary path from an interior point of Ω_i to $p \in \mathcal{C}$ inside Ω_i is not rectifiable either. Thus if we define W_1 as a push-forward by J_1 of W constructed using arbitrary admissible paths in $\mathbb{U}$, the paths in W_1 in general will no longer be rectifiable. The same applies to the push-forward by J_1 of the corresponding chain in $\mathbb{L}$. However, the definition of $\langle u, W_1 \rangle$ uses integration of the 1-form u_{γ_1,γ_2} along the paths in W_1, and these paths should be rectifiable in order that $\langle u, W_1 \rangle$ is well-defined. The invariant construction of such paths in Ω_i is based on the following elegant observation communicated to us by M. Lyubich.

Since the quasi-Fuchsian group Γ is normalized, it follows from **QF2** that the Fuchsian group $\tilde{\Gamma} = J_1^{-1} \circ \Gamma \circ J_1$ is also normalized and $\tilde{\alpha}_1 \in \tilde{\Gamma}$ is a dilation $\tilde{\alpha}_1 z = \tilde{\lambda} z$ with the axis $i\mathbb{R}_{\geq 0}$ and $0 < \tilde{\lambda} < 1$. Corresponding loxodromic element $\alpha_1 = J_1 \circ \tilde{\alpha}_1 \circ J_1^{-1} \in \Gamma$ is also a dilation $\alpha_1 z = \lambda z$, where $0 < |\lambda| < 1$. Choose $\tilde{z}_0 \in i\mathbb{R}_{\geq 0}$ and denote by $\tilde{I} = [\tilde{z}_0, 0]$ interval on $i\mathbb{R}_{\geq 0}$ with endpoints $\tilde{z}_0$ and 0 — the attracting fixed point of $\tilde{\alpha}_1$. Set $z_0 = J_1(\tilde{z}_0)$ and $I = J_1(\tilde{I})$. The path I connects points $z_0 \in \Omega_1$ and $0 = J_1(0) \in \mathcal{C}$ inside Ω_1, is smooth everywhere except the endpoint 0, and is rectifiable. Indeed, set $\tilde{I}_0 = [\tilde{z}_0, \tilde{\lambda}\tilde{z}_0] \subset i\mathbb{R}_{>0}$ and cover the interval $\tilde{I}$ by subintervals $\tilde{I}_n$ defined by $\tilde{I}_{n+1} = \tilde{\alpha}_1(\tilde{I}_n)$, $n = 0, 1, \ldots, \infty$. Corresponding paths $I_n = J_1(\tilde{I}_n)$ cover the path I, and due to the property $I_{n+1} = \alpha_1(I_n)$, which follows from **QF3**, we have

$$I = \bigcup_{n=0}^{\infty} \alpha_1^n(I_0).$$

Thus

$$l(I) = \sum_{n=0}^{\infty} |\lambda^n| l(I_0) = \frac{l(I_0)}{1 - |\lambda|} < \infty,$$

where $l(P)$ denotes the Euclidean length of a smooth path P.

The same construction works for every $p \in \mathcal{C} \setminus \{\infty\}$ which is a fixed point of an element in Γ, and we define Γ-contracting paths in Ω_1 at p as follows.

Definition 2.3.2. A path P connecting points $z \in \Omega_1$ and $p \in \mathcal{C} \setminus \{\infty\}$ inside Ω_1 is called Γ-contracting in Ω_1 at p, if the following conditions are satisfied.

C1 Paths P is smooth except at the point p.

C2 The point p is a fixed point for Γ.

C3 There exists $p' \in P$ and an arc P_0 on the path P such that the iterates $\gamma^n(P_0)$, $n \in \mathbb{N}$, where $\gamma \in \Gamma$ has p as the attracting fixed point, entirely cover the part of P from the point p' to the point p.

As in Section 2.2, we define Γ-closed paths and Γ-closed contracting paths in Ω_1 at p. Definition of Γ-contracting paths in Ω_2 is analogous. Finally, we define Γ-contracting paths in Ω as follows.

Definition 2.3.3. A path P is called Γ-contracting in Ω, if $P = P_1 \cup P_2$, where $P_1 \cap P_2 = p \in \mathcal{C}$, and $P_1 \setminus \{p\} \subset \Omega_1$ and $P_2 \setminus \{p\} \subset \Omega_2$ are Γ-contracting paths at p in the sense of the previous definition.

Γ-contracting paths are rectifiable.

Lemma 2.3.4. *Let Γ and Γ' be two marked normalized quasi-Fuchsian groups with regions of discontinuity Ω and Ω', and let f be normalized quasiconformal homeomorphism of $\mathbb{C}$ which intertwines Γ and Γ' and is smooth in Ω. Then the push-forward by f of a Γ-contracting path in Ω is a Γ'-contracting path in Ω'.*

Proof. Obvious: if p is the attracting fixed point for $\gamma \in \Gamma$, then $p' = f(p)$ is the attracting fixed point for $\gamma' = f \circ \gamma \circ f^{-1} \in \Gamma'$. □

Now define a chain W for the Fuchsian group $\tilde{\Gamma}$ by first connecting points $P_1(1), \dots, P_g(1)$ to some point $\tilde{z}_0 \in i\mathbb{R}_{>0}$ by smooth paths inside $\mathbb{U}$ and then connecting this point to 0 by $\tilde{I}$. The chain W' in $\mathbb{L}$ is defined similarly. Setting $W_1 = J_1(W)$ and $W_2 = J_1(W')$, we see that the chain $W_1 - W_2$ in Ω consists of Γ-contracting paths in Ω at 0. Finally, we define chain $U_1 = U_2$ as push-forward by J_1 of the corresponding chain $U = U'$ with $p = 0$.

2.3.2 Cohomology construction

Let $\mathcal{CM}(X \sqcup Y)$ be the space of all conformal metrics $ds^2 = e^{\phi}|dz|^2$ on $X \sqcup Y$, which we will always identify with the affine space of smooth real-valued functions ϕ on Ω satisfying (2.2.3). For $\phi \in \mathcal{CM}(X \sqcup Y)$ we define cochains $\omega[\phi], \theta[\phi], u, \eta$ and Θ in the total cohomology complex $\operatorname{Tot} \mathsf{C}$ for the pair Ω, Γ

by the same formulas (2.2.4), (2.2.5), (2.2.6), (2.2.14) and (2.2.13), (2.2.19) as in the Fuchsian case, where $p = 0 \in \mathcal{C}$, integration goes over Γ-contracting paths at 0, and $\tilde{\gamma} \in \tilde{\Gamma}$ are replaced by $\gamma = J_1 \circ \tilde{\gamma} \circ J_1^{-1} \in \Gamma$. The ordering of points on $\mathcal{C}$ used in the definition (2.2.14) of the constants of integration η_{γ_1,γ_2} is defined by the orientation of $\mathcal{C}$.

Remark 2.3.5. Since 1-form u is closed and regular in $\Omega_1 \cup \Omega_2$, it follows from Stokes' theorem that in the definition (2.2.13) and (2.2.19) of the cochain $\Theta \in \mathsf{C}^{0,2}$ we can use any rectifiable path from z to 0 inside Ω_1 and Ω_2 respectively.

As opposed to the Fuchsian case, we can no longer guarantee that the cochain $\omega[\phi] - \theta[\phi] - \Theta$ is a 2-cocycle in the total cohomology complex $\operatorname{Tot}\mathsf{C}$. Indeed, we have, using $\delta u = 0$,

$$(\delta\Theta)_{\gamma_1,\gamma_2,\gamma_3}(z) = \begin{cases} \int_{P_1} u_{\gamma_2,\gamma_3} + (\delta\eta)_{\gamma_1,\gamma_2,\gamma_3} = (d_1)_{\gamma_1,\gamma_2,\gamma_3} & \text{if } z \in \Omega_1, \\ \int_{P_2} u_{\gamma_2,\gamma_3} - (\delta\eta)_{\gamma_1,\gamma_2,\gamma_3} = (d_2)_{\gamma_1,\gamma_2,\gamma_3} & \text{if } z \in \Omega_2, \end{cases} \tag{2.3.1}$$

where paths of integration P_1 and P_2 are Γ-closed contracting paths connecting points 0 and $\gamma_1^{-1}(0)$ inside Ω_1 and Ω_2 respectively. Since the analog of Lemma 2.2.9 does not hold in the quasi-Fuchsian case, we can not conclude that $d_1 = d_2 = 0$. However, $d_1, d_2 \in \mathsf{C}^{0,3}$ are z-independent group 3-cocycles and

$$(d_1 - d_2)_{\gamma_1,\gamma_2,\gamma_3} = \int_C u_{\gamma_2,\gamma_3} + 2(\delta\eta)_{\gamma_1,\gamma_2,\gamma_3}, \tag{2.3.2}$$

where $C = P_1 - P_2$ is a loop that starts at 0, goes to $\gamma_1^{-1}(0)$ inside Ω_1, continues inside Ω_2 and ends at 0. In the Fuchsian case we have the equation (2.2.21), which can be derived using the Stokes' theorem (see Remark 2.2.13). The same derivation repeats verbatim for the quasi-Fuchsian case, and we get

$$\int_C u_{\gamma_2,\gamma_3} = -2(\delta\eta)_{\gamma_1,\gamma_2,\gamma_3},$$

so that $d_1 = d_2$. Since $H^3(\Gamma, \mathbb{C}) = 0$, there exists a constant 2-cochain κ such that $\delta\kappa = -d_1 = -d_2$. Then $\Theta + \kappa$ is a group 2-cocycle, that is, $\delta(\Theta + \kappa) = 0$. As the result, we obtain that

$$\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta - \kappa \in (\operatorname{Tot} \mathsf{C})^2$$

is a 2-cocycle in total cohomology complex $\operatorname{Tot} \mathsf{C}$ for the pair Ω, Γ, that is, $D\Psi[\phi] = 0$.

Remark 2.3.6. The map J_1 induces a cochain map between double cohomology complexes $\operatorname{Tot} \mathsf{C}$ for the pairs $\mathbb{U} \cup \mathbb{L}, \tilde{\Gamma}$ and Ω, Γ, by pulling back cochains and group elements,

$$(J_1 \cdot \varpi)_{\tilde{\gamma}_1,\dots,\tilde{\gamma}_q} = J_1^* \varpi_{\gamma_1,\dots,\gamma_q} \in \mathsf{C}^{p,q}(\mathbb{U} \cup \mathbb{L}),$$

where $\varpi \in \mathsf{C}^{p,q}(\Omega)$ and $\tilde{\gamma} = J_1^{-1} \circ \gamma \circ J_1$. This cochain map induces an isomorphism of the cohomology groups of corresponding total complexes $\operatorname{Tot} \mathsf{C}$. The map J_1 also induces a natural isomorphism between the affine spaces $\mathcal{CM}(X \sqcup Y)$ and $\mathcal{CM}(X \sqcup \bar{X})$,

$$J_1 \cdot \phi = \phi \circ J_1 + \log |(J_1)_z|^2 \in \mathcal{CM}(X \sqcup \bar{X}),$$

where $\phi \in \mathcal{CM}(X \sqcup Y)$. However,

$$\left|(J_1 \cdot \phi)_z\right|^2 dz \wedge d\bar{z} \neq J_1^* \left(|\phi_z|^2 dz \wedge d\bar{z}\right),$$

and cochains $\omega[\phi], \theta[\phi], u$ and Θ for the pair Ω, Γ are not pull-backs of cochains for the pair $\mathbb{U} \cup \mathbb{L}$, $\tilde{\Gamma}$ corresponding to $J_1 \cdot \phi \in \mathcal{CM}(X \sqcup \bar{X})$.

2.3.3 The Liouville action functional

Discussion in the previous section justifies the following definition.

Definition 2.3.7. The Liouville action functional $S_\Gamma : \mathcal{CM}(X \sqcup Y) \to \mathbb{R}$ for the quasi-Fuchsian group Γ is defined by

$$\begin{aligned} S_\Gamma[\phi] =& \frac{i}{2}\langle \Psi[\phi], \Sigma(\Gamma)\rangle = \frac{i}{2}\langle \Psi[\phi], \Sigma_1 - \Sigma_2\rangle \\ =& \frac{i}{2}\left(\langle \omega[\phi], F_1 - F_2\rangle - \langle \theta[\phi], L_1 - L_2\rangle + \langle \Theta + \kappa, V_1 - V_2\rangle\right), \end{aligned}$$

where $\phi \in \mathcal{CM}(X \sqcup Y)$.

Remark 2.3.8. Since $\Psi[\phi]$ is a total 2-cocycle, the Liouville action functional S_Γ does not depend on the choice of fundamental domain for Γ in Ω, i.e on the choice of fundamental domains F_1 and F_2 for Γ in Ω_1 and Ω_2. In particular, if Σ_1 and Σ_2 are push-forwards by the map J_1 of the total cycle Σ and its mirror image $\bar{\Sigma}$, then $\langle \kappa, V_1 - V_2\rangle = 0$ and we have

$$S_\Gamma[\phi] = \frac{i}{2}\left(\langle \omega[\phi], F_1 - F_2\rangle - \langle \theta[\phi], L_1 - L_2\rangle + \langle u, W_1 - W_2\rangle + 2\langle \eta, V_1\rangle\right). \tag{2.3.3}$$

In general, the constant group 2-cocycle κ drops out from the definition for any choice of fundamental domains F_1 and F_2 which is associated with the same marking of Γ, i.e. , when the same choice of standard generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ is used both in Ω_1 and in Ω_2. Indeed, in this case V_1 and V_2 have the same $\mathsf{B}_2(\mathbb{Z}\Gamma)$-structure and $\langle \kappa, V_1 - V_2 \rangle = 0$. Moreover, since 1-form u is closed and regular in $\Omega_1 \cup \Omega_2$, we can use arbitrary rectifiable paths with endpoint 0 inside Ω_1 and Ω_2 in the definition of chains W_1 and W_2 respectively.

Remark 2.3.9. We can also define chains W_1 and W_2 by using Γ-contracting paths at any Γ-fixed point $p \in \mathcal{C} \setminus \{\infty\}$. As in Remark 2.2.11 it is easy to show that

$$\langle \Theta, V_1 - V_2 \rangle = \langle u, W_1 - W_2 \rangle + 2\langle \eta, V_1 \rangle$$

does not depend on the choice of a fixed point p.

As in the Fuchsian case, the Euler-Lagrange equation for the functional S_Γ is the Liouville equation and the hyperbolic metric $e^{\phi_{hyp}}|dz|^2$ on Ω is its single non-degenerate critical point. It is explicitly given by

$$e^{\phi_{hyp}(z)} = \frac{|(J_i^{-1})'(z)|^2}{(\operatorname{Im} J_i^{-1}(z))^2} \quad \text{if} \quad z \in \Omega_i, \;\; i = 1, 2. \tag{2.3.4}$$

Remark 2.3.10. Corresponding classical action $S_\Gamma[\phi_{hyp}]$ is no longer twice the hyperbolic area of $X \sqcup Y$, as it was in the Fuchsian case, but rather non-trivially depends on Γ. This is due to the fact that in the quasi-Fuchsian case the $(1,1)$-form $\omega[\phi_{hyp}]$ on Ω is not a $(1,1)$-tensor for Γ, as it was in the Fuchsian case.

Similarly to (2.2.22) we have

$$S_{\Gamma}[\phi] = \frac{i}{2}\left(\langle\omega[\phi], F_1 - F_2\rangle - \langle\check{\theta}[\phi], L_1 - L_2\rangle + \langle\check{u}, W_1 - W_2\rangle\right), \qquad (2.3.5)$$

where F_1 and F_2 are fundamental domains for the marked group Γ in Ω_1 and Ω_2 respectively.

Chapter 3

Deformation Theory

3.1 The deformation space

Here we collect the basic facts from deformation theory of Kleinian groups (see, e.g., [Ahl87, Ber70, Ber71, Kra72b]). Let Γ be a non-elementary, finitely generated purely loxodromic Kleinian group, let Ω be its region of discontinuity, and let $\Lambda = \hat{\mathbb{C}} \setminus \Omega$ be its limit set. The deformation space $\mathfrak{D}(\Gamma)$ is defined as follows. Let $\mathcal{A}^{-1,1}(\Gamma)$ be the space of Beltrami differentials for Γ — the Banach space of $\mu \in L^\infty(\mathbb{C})$ satisfying

$$\mu(\gamma(z))\,\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \text{ for all } \gamma \in \Gamma,$$

and

$$\mu|_\Lambda = 0.$$

Denote by $\mathcal{B}^{-1,1}(\Gamma)$ the open unit ball in $\mathcal{A}^{-1,1}(\Gamma)$ with respect to $\|\cdot\|_\infty$ norm,

$$\| \mu \|_\infty = \sup_{z\in\mathbb{C}} |\mu(z)| < 1.$$

For each Beltrami coefficient $\mu \in \mathcal{B}^{-1,1}(\Gamma)$ there exists a unique homeomorphism $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfying the Beltrami equation

$$f^\mu_{\bar{z}} = \mu f^\mu_z$$

and fixing the points $0, 1$ and ∞. Set $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ and define

$$\mathfrak{D}(\Gamma) = \mathcal{B}^{-1,1}(\Gamma)/\sim,$$

where $\mu \sim \nu$ if and only if $f^\mu = f^\nu$ on Λ, which is equivalent to the condition $f^\mu \circ \gamma \circ (f^\mu)^{-1} = f^\nu \circ \gamma \circ (f^\nu)^{-1}$, for all $\gamma \in \Gamma$.

Similarly, if Δ is a union of invariant components of Γ, the deformation space $\mathfrak{D}(\Gamma, \Delta)$ is defined using Beltrami coefficients supported on Δ.

By Ahlfors finiteness theorem Ω has finitely many non-equivalent components $\Omega_1, \dots, \Omega_n$. Let Γ_i be the stabilizer subgroup of the component Ω_i, $\Gamma_i = \{\gamma \in \Gamma \mid \gamma(\Omega_i) = \Omega_i\}$ and let $X_i \simeq \Gamma_i \backslash \Omega_i$ be the corresponding compact Riemann surface of genus $g_i > 1$, $i = 1, \dots, n$. The decomposition

$$\Gamma\backslash\Omega = \Gamma_1\backslash\Omega_1 \sqcup \cdots \sqcup \Gamma_n\backslash\Omega_n$$

establishes the isomorphism [Kra72b]

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma_1, \Omega_1) \times \cdots \times \mathfrak{D}(\Gamma_n, \Omega_n).$$

Remark 3.1.1. When Γ is a purely hyperbolic Fuchsian group of genus $g > 1$, $\mathfrak{D}(\Gamma, \mathbb{U}) = \mathfrak{T}(\Gamma)$ — the Teichmüller space of Γ. Every conformal bijection $\Gamma\backslash\mathbb{U} \to X$ establishes isomorphism between $\mathfrak{T}(\Gamma)$ and $\mathfrak{T}(X)$, the Teichmüller space of marked Riemann surface X. Similarly, $\mathfrak{D}(\Gamma, \mathbb{L}) = \bar{\mathfrak{T}}(\Gamma)$, the mirror image of $\mathfrak{T}(\Gamma)$ — the complex manifold which is complex conjugate to $\mathfrak{T}(\Gamma)$. Correspondingly, $\Gamma\backslash\mathbb{L} \to \bar{X}$ establishes isomorphism $\bar{\mathfrak{T}}(\Gamma) \simeq \mathfrak{T}(\bar{X})$, so that

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{T}(X) \times \mathfrak{T}(\bar{X}).$$

The deformation space $\mathfrak{D}(\Gamma)$ is "twice larger" than the Teichmüller space $\mathfrak{T}(\Gamma)$ because its definition uses all Beltrami coefficients μ for Γ, and not only those satisfying the reflection property $\mu(\bar{z}) = \overline{\mu(z)}$, used in the definition of $\mathfrak{T}(\Gamma)$.

The deformation space $\mathfrak{D}(\Gamma)$ has a natural structure of a complex manifold, explicitly described as follows (see, e.g., [Ahl87]). Let $\mathcal{H}^{-1,1}(\Gamma)$ be the Hilbert space of Beltrami differentials for Γ with the following scalar product

$$(\mu_1, \mu_2) = \iint\limits_{\Gamma\backslash\Omega} \mu_1 \bar{\mu}_2 \rho = \iint\limits_{\Gamma\backslash\Omega} \mu_1(z)\overline{\mu_2(z)}\rho(z)\, d^2 z, \tag{3.1.1}$$

where $\mu_1, \mu_2 \in \mathcal{H}^{-1,1}(\Gamma)$ and $\rho = e^{\phi_{hyp}}$ is the density of the hyperbolic metric on $\Gamma\backslash\Omega$. Denote by $\Omega^{-1,1}(\Gamma)$ the finite-dimensional subspace of harmonic Beltrami differentials with respect to the hyperbolic metric. It consists of $\mu \in \mathcal{H}^{-1,1}(\Gamma)$ satisfying

$$\partial_z(\rho\mu) = 0.$$

The complex vector space $\Omega^{-1,1}(\Gamma)$ is identified with the holomorphic tangent space to $\mathfrak{D}(\Gamma)$ at the origin. Choose a basis $\mu_1, \dots, \mu_d$ for $\Omega^{-1,1}(\Gamma)$, let $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_d\mu_d$, and let f^μ be the normalized solution of the Beltrami equation. Then the correspondence $(\varepsilon_1, \dots, \varepsilon_d) \mapsto \Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ defines complex coordinates in a neighborhood of the origin in $\mathfrak{D}(\Gamma)$, called Bers coordinates. The holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at the origin can be naturally identified with the vector space $\Omega^{2,0}(\Gamma)$ of holomorphic quadratic differentials — holomorphic functions q on Ω satisfying

$$q(\gamma z)\gamma'(z)^2 = q(z) \text{ for all } \gamma \in \Gamma.$$

The pairing between holomorphic cotangent and tangent spaces to $\mathfrak{D}(\Gamma)$ at the origin is given by

$$q(\mu) = \iint\limits_{\Gamma\backslash\Omega} q\mu = \iint\limits_{\Gamma\backslash\Omega} q(z)\mu(z)\, d^2z.$$

There is a natural isomorphism Φ^μ between the deformation spaces $\mathfrak{D}(\Gamma)$ and $\mathfrak{D}(\Gamma^\mu)$, which maps $\Gamma^\nu \in \mathfrak{D}(\Gamma)$ to $(\Gamma^\mu)^\lambda \in \mathfrak{D}(\Gamma^\mu)$, where, in accordance with $f^\nu = f^\lambda \circ f^\mu$,

$$\lambda = \left(\frac{\nu - \mu}{1 - \nu\bar{\mu}}\frac{f_z^\mu}{\bar{f}_{\bar{z}}^\mu}\right) \circ (f^\mu)^{-1}.$$

The isomorphism Φ^μ allows us to identify the holomorphic tangent space to $\mathfrak{D}(\Gamma)$ at Γ^μ with the complex vector space $\Omega^{-1,1}(\Gamma^\mu)$, and holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at Γ^μ with the complex vector space $\Omega^{2,0}(\Gamma^\mu)$. It also allows us to introduce the Bers coordinates in the neighborhood of Γ^μ in $\mathfrak{D}(\Gamma)$, and to show directly that these coordinates transform complex-analytically. For the de Rham differential d on $\mathfrak{D}(\Gamma)$ we denote by $d = \partial + \bar{\partial}$ the decomposition into $(1,0)$ and $(0,1)$ components.

The differential of isomorphism $\Phi^\mu : \mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma^\mu)$ at $\nu = \mu$ is given by the linear map $D^\mu : \Omega^{-1,1}(\Gamma) \to \Omega^{-1,1}(\Gamma^\mu)$,

$$\nu \mapsto D^\mu \nu = P^\mu_{-1,1}\left[\left(\frac{\nu}{1-|\mu|^2}\frac{f^\mu_z}{\bar{f}^\mu_{\bar{z}}}\right)\circ (f^\mu)^{-1}\right],$$

where $P^\mu_{-1,1}$ is orthogonal projection from $\mathcal{H}^{-1,1}(\Gamma^\mu)$ to $\Omega^{-1,1}(\Gamma^\mu)$. The map D^μ allows to extend a tangent vector ν at the origin of $\mathfrak{D}(\Gamma)$ to a local vector field $\partial/\partial\varepsilon_\nu$ on the coordinate neighborhood of the origin,

$$\left.\frac{\partial}{\partial \varepsilon_\nu}\right|_{\Gamma^\mu} = D^\mu \nu \in \Omega^{-1,1}(\Gamma^\mu).$$

The scalar product (3.1.1) in $\Omega^{-1,1}(\Gamma^\mu)$ defines a Hermitian metric on the deformation space $\mathfrak{D}(\Gamma)$. This metric is called the Weil-Petersson metric and it is Kähler . We denote its symplectic form by ω_{WP},

$$\omega_{WP}\left(\frac{\partial}{\partial\varepsilon_\mu},\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right)\bigg|_{\Gamma^\lambda} = \frac{i}{2}\left(D^\lambda\mu, D^\lambda\nu\right), \quad \mu,\nu\in\Omega^{-1,1}(\Gamma).$$

3.2 Variational formulas

Here we collect necessary variational formulas. Let l and m be integers. A tensor of type (l,m) for Γ is a C^∞-function ω on Ω satisfying

$$\omega(\gamma z)\gamma'(z)^l\overline{\gamma'(z)}^m = \omega(z) \text{ for all } \gamma\in\Gamma.$$

Let ω^ε be a smooth family of tensors of type (l,m) for $\Gamma^{\varepsilon\mu}$, where $\mu \in \Omega^{-1,1}(\Gamma)$ and $\varepsilon \in \mathbb{C}$ is sufficiently small. Set

$$(f^{\varepsilon\mu})^*(\omega^\varepsilon) = \omega^\varepsilon \circ f^{\varepsilon\mu}(f_z^{\varepsilon\mu})^l(\bar{f}_{\bar{z}}^{\varepsilon\mu})^m,$$

which is a tensor of type (l,m) for Γ — a pull-back of the tensor ω^ε by $f^{\varepsilon\mu}$. The Lie derivatives of the family ω^ε along the vector fields $\partial/\partial\varepsilon_\mu$ and $\partial/\partial\bar{\varepsilon}_\mu$ are defined in the standard way,

$$L_\mu\omega = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} (f^{\varepsilon\mu})^*(\omega^\varepsilon) \text{ and } L_{\bar{\mu}}\omega = \left.\frac{\partial}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} (f^{\varepsilon\mu})^*(\omega^\varepsilon).$$

When ω is a function on $\mathfrak{D}(\Gamma)$ — a tensor of type $(0,0)$, Lie derivatives reduce to directional derivatives

$$L_\mu\omega = \partial\omega(\mu) \text{ and } L_{\bar{\mu}}\omega = \bar{\partial}\omega(\bar{\mu})$$

— the evaluation of 1-forms $\partial\omega$ and $\bar{\partial}\omega$ on tangent vectors μ and $\bar{\mu}$.

For the Lie derivatives of vector fields $\nu^{\epsilon\mu} = D^{\epsilon\mu}\nu$ we get [Wol86] that $L_\mu\nu = 0$ and $L_{\bar{\mu}}\nu$ is orthogonal to $\Omega^{-1,1}(\Gamma)$. In other words,

$$\left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu}\right] = \left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right] = 0$$

at the point Γ in $\mathfrak{D}(\Gamma)$.

For every $\Gamma^\mu \in \mathfrak{D}(\Gamma)$, the density ρ^μ of the hyperbolic metric on Ω^μ is a $(1,1)$-tensor for Γ^μ. Lie derivatives of the smooth family of $(1,1)$-tensors ρ parameterized by $\mathfrak{D}(\Gamma)$ are given by the following lemma of Ahlfors.

Lemma 3.2.1. *For every* $\mu \in \Omega^{-1,1}(\Gamma)$

$$L_\mu \rho = L_{\bar{\mu}} \rho = 0.$$

Proof. Let $\Omega_1, \ldots, \Omega_n$ be the maximal set of non-equivalent components of Ω and let $\Gamma_1, \ldots, \Gamma_n$ be the corresponding stabilizer groups,

$$\Gamma\backslash\Omega = \Gamma_1\backslash\Omega_1 \sqcup \cdots \sqcup \Gamma_n\backslash\Omega_n \simeq X_1 \sqcup \cdots \sqcup X_n.$$

For every Ω_i denote by $J_i : \mathbb{U} \to \Omega_i$ the corresponding covering map and by $\tilde{\Gamma}_i$ — the Fuchsian model of group Γ_i, characterized by the condition $\tilde{\Gamma}_i\backslash\mathbb{U} \simeq \Gamma_i\backslash\Omega_i \simeq X_i$ (see, e.g., [Kra72b]).

Let $\mu \in \Omega^{-1,1}(\Gamma)$. For every component Ω_i the quasiconformal map $f^{\varepsilon\mu}$ gives rise to the following commutative diagram

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{F^{\varepsilon\hat{\mu}_i}} & \mathbb{U} \\ \Big\downarrow{\scriptstyle J_i} & & \Big\downarrow{\scriptstyle J_i^{\varepsilon\mu}} \\ \Omega_i & \xrightarrow{f^{\varepsilon\mu}} & \Omega_i^{\varepsilon\mu} \end{array} \tag{3.2.1}$$

where $F^{\varepsilon\hat{\mu}_i}$ is the normalized quasiconformal homeomorphism of $\mathbb{U}$ with Beltrami differential $\hat{\mu}_i = J_i^*\mu$ for the Fuchsian group $\tilde{\Gamma}_i$. Let $\hat{\rho}$ be the density of the hyperbolic metric on $\mathbb{U}$; it satisfies $\hat{\rho} = J_i^*\rho$, where ρ is the density of the hyperbolic metric on Ω_i. Therefore, Beltrami differential $\hat{\mu}_i$ is harmonic with respect to the hyperbolic metric on $\mathbb{U}$. It follows from the commutativity of the diagram that

$$(f^{\varepsilon\mu})^*\rho^{\varepsilon\mu} = ((J_i^{\varepsilon\mu})^{-1} \circ f^{\varepsilon\mu})^*\hat{\rho} = (F^{\varepsilon\hat{\mu}_i} \circ J_i^{-1})^*\hat{\rho} = (J_i^{-1})^*(F^{\varepsilon\hat{\mu}_i})^*\hat{\rho}.$$

Now the assertion of the lemma reduces to

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} (F^{\varepsilon\hat{\mu}_i})^* \hat{\rho} = 0,$$

which is the classical result of Ahlfors [Ahl61]. □

Set

$$\dot{f} = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} f^{\varepsilon\mu},$$

then

$$\dot{f}(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{z(z-1)\mu(w)}{(w-z)w(w-1)}\, d^2w. \tag{3.2.2}$$

We have

$$\dot{f}_{\bar{z}} = \mu$$

and also

$$\frac{\partial}{\partial \bar{\varepsilon}}\bigg|_{\varepsilon=0} f^{\varepsilon\mu} = 0.$$

As it follows from Ahlfors lemma

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\rho^{\varepsilon\mu} \circ f^{\varepsilon\mu}\, |f_z^{\varepsilon\mu}|^2\right) = 0.$$

Using $\rho = e^{\phi_{hyp}}$ and the fact that $f^{\varepsilon\mu}$ depends holomorphically on ε, we get

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\phi_{hyp}^{\varepsilon\mu} \circ f^{\varepsilon\mu}\right) = -\dot{f}_z. \tag{3.2.3}$$

Differentiation with respect to z and $\bar{z}$ yields

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\left(\phi^{\varepsilon\mu}_{hyp}\right)_z \circ f^{\varepsilon\mu} f^{\varepsilon\mu}_z \right) = -\dot{f}_{zz}, \tag{3.2.4}$$

and

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\left(\phi^{\varepsilon\mu}_{hyp}\right)_{\bar{z}} \circ f^{\varepsilon\mu} \bar{f}^{\varepsilon\mu}_{\bar{z}} \right) = -\left((\phi_{hyp})_z \dot{f}_{\bar{z}} + \dot{f}_{z\bar{z}} \right). \tag{3.2.5}$$

For $\gamma \in \Gamma$ set $\gamma^{\varepsilon\mu} = f^{\varepsilon\mu} \circ \gamma \circ (f^{\varepsilon\mu})^{-1} \in \Gamma^{\varepsilon\mu}$. We have

$$(\gamma^{\varepsilon\mu})' \circ f^{\varepsilon\mu} f^{\varepsilon\mu}_z = f^{\varepsilon\mu}_z \circ \gamma\, \gamma',$$

and

$$\log |(\gamma^{\varepsilon\mu})' \circ f^{\varepsilon\mu}|^2 + \log |f^{\varepsilon\mu}_z|^2 = \log |f^{\varepsilon\mu}_z \circ \gamma|^2 + \log |\gamma'|^2.$$

Therefore

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\log |(\gamma^{\varepsilon\mu})' \circ f^{\varepsilon\mu}|^2 \right) = \dot{f}_z \circ \gamma - \dot{f}_z, \tag{3.2.6}$$

and, differentiating with respect to z,

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left(\frac{(\gamma^{\varepsilon\mu})''}{(\gamma^{\varepsilon\mu})'} \circ f^{\varepsilon\mu} f^{\varepsilon\mu}_z \right) = \dot{f}_{zz} \circ \gamma\, \gamma' - \dot{f}_{zz}. \tag{3.2.7}$$

Denote by

$$\mathcal{S}(h) = \left(\frac{h_{zz}}{h_z}\right)_z - \frac{1}{2}\left(\frac{h_{zz}}{h_z}\right)^2 = \frac{h_{zzz}}{h_z} - \frac{3}{2}\left(\frac{h_{zz}}{h_z}\right)^2$$

the Schwarzian derivative of the function h.

Lemma 3.2.2. *Set*

$$\dot\gamma = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \gamma^{\varepsilon\mu}, \ \gamma \in \Gamma.$$

Then for all $\gamma \in \Gamma$

$$\dot f_z \circ \gamma - \dot f_z = \frac{\dot f \circ \gamma \gamma''}{(\gamma')^2} + \left(\frac{\dot\gamma}{\gamma'}\right)', \tag{i}$$

and is well-defined on the limit set Λ. *Also we have*

$$\dot f_{z\bar z} \circ \gamma\, \overline{\gamma'} - \dot f_{z\bar z} = \frac{\gamma''}{\gamma'} \dot f_{\bar z}, \tag{ii}$$

$$\dot f_{zz} \circ \gamma\, \gamma' - \dot f_{zz} = \frac{1}{2}(\dot f_z \circ \gamma + \dot f_z)\frac{\gamma''}{\gamma'} - \frac{2\dot c}{cz+d}, \ \textit{for all } \gamma \in \Gamma. \tag{iii}$$

Proof. To prove formula (i), consider the equation

$$\dot f \circ \gamma = \dot\gamma + \gamma' \dot f, \tag{3.2.8}$$

which follows from $\gamma^{\varepsilon\mu} \circ f^{\varepsilon\mu} = f^{\varepsilon\mu} \circ \gamma$. Differentiating with respect to z gives (i). Since $\dot f$ is a homeomorphism of $\hat{\mathbb{C}}$ and $\dot\gamma/\gamma'$ is a quadratic polynomial in z, formula (i) shows that $\dot f_z \circ \gamma - \dot f_z$ is well-defined on Λ.

The formula (ii) immediately follows from $\dot f_{\bar z} = \mu$ and

$$\mu \circ \gamma \frac{\overline{\gamma'}}{\gamma'} = \mu, \ \gamma \in \Gamma.$$

To derive formula (iii), twice differentiating (3.2.8) with respect to z to obtain

$$\dot f_z \circ \gamma \gamma' = \dot\gamma' + \gamma'' \dot f + \gamma' \dot f_z,$$

and

$$\gamma'(\dot{f}_{zz} \circ \gamma\gamma' - \dot{f}_{zz}) = \dot{\gamma}'' + \gamma'''\dot{f} + 2\gamma''\dot{f}_z - \dot{f}_z \circ \gamma\gamma''.$$

Since

$$\gamma''' = \frac{3}{2}\frac{(\gamma'')^2}{\gamma'},$$

as it follows from $\mathcal{S}(\gamma) = 0$, we can eliminate $\gamma''\dot{f}$ from the two formulas above and obtain

$$\dot{f}_{zz} \circ \gamma\,\gamma' - \dot{f}_{zz} = \frac{1}{2}(\dot{f}_z \circ \gamma + \dot{f}_z)\frac{\gamma''}{\gamma'} + \frac{\dot{\gamma}''}{\gamma'} - \frac{3}{2}\frac{\gamma''\dot{\gamma}'}{(\gamma')^2}.$$

Using $2c = -\gamma''/(\gamma')^{3/2}$, we see that the last two terms in this equations are equal to $-2\dot{c}/(cz+d)$, which proves the lemma. □

Finally, we present the following formulas by Ahlfors [Ahl61]. Let $F^{\varepsilon\hat{\mu}}$ be the quasiconformal homeomorphism of $\mathbb{U}$ with Beltrami differential $\hat{\mu}$ for the Fuchsian group Γ. If $\hat{\mu}$ is harmonic on $\mathbb{U}$ with respect to the hyperbolic metric, then

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} F^{\varepsilon\hat{\mu}}_{zzz}(z) = 0, \tag{3.2.9}$$

$$\left.\frac{\partial}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} F^{\varepsilon\hat{\mu}}_{zzz}(z) = -\frac{1}{2}\hat{\rho}\overline{\hat{\mu}(z)}. \tag{3.2.10}$$

Chapter 4

Variation of the Classical Action

4.1 Classical action

Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity $\Omega = \Omega_1 \cup \Omega_2$, let $X \sqcup Y \simeq \Gamma \backslash \Omega$ be corresponding marked Riemann surfaces with opposite orientations and let

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma, \Omega_1) \times \mathfrak{D}(\Gamma, \Omega_2)$$

be the deformation space of Γ. Spaces $\mathfrak{D}(\Gamma, \Omega_1)$ and $\mathfrak{D}(\Gamma, \Omega_2)$ are isomorphic to the Teichmüller spaces $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ — they are their quasi-Fuchsian models which use Bers' simultaneous uniformization of varying Riemann surface in $\mathfrak{T}(X)$ and fixing Y and, respectively, fixing X and varying Riemann surface in $\mathfrak{T}(Y)$. Therefore,

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{T}(X) \times \mathfrak{T}(Y). \tag{4.1.1}$$

Denote by $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ corresponding affine bundle of projective connections, modeled over the holomorphic cotangent bundle of $\mathfrak{D}(\Gamma)$. We have

$$\mathfrak{P}(\Gamma) \simeq \mathfrak{P}(X) \times \mathfrak{P}(Y). \tag{4.1.2}$$

For every $\Gamma^\mu \in \mathfrak{D}(\Gamma)$ denote by $S_{\Gamma^\mu} = S_{\Gamma^\mu}[\phi_{hyp}]$ the classical Liouville action. It follows from the results in Section 2.3.3 that S_{Γ^μ} gives rise to a well-defined real-valued function S on $\mathfrak{D}(\Gamma)$. Indeed, if $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1} = f^\nu \circ \Gamma \circ (f^\nu)^{-1}$ for $\mu \sim \nu$, then corresponding total cycles $f^\mu(\Sigma(\Gamma))$ and $f^\nu(\Sigma(\Gamma))$ represent the same class in the total homology complex $\operatorname{Tot} \mathsf{K}$ for the pair Ω^μ, Γ^μ, so that

$$\langle \Psi\left[\phi_{hyp}\right], f^\mu(\Sigma(\Gamma))\rangle = \langle \Psi\left[\phi_{hyp}\right], f^\nu(\Sigma(\Gamma))\rangle.$$

Moreover, real-analytic dependence of solutions of Beltrami equation on parameters ensures that classical action S is a real-analytic function on $\mathfrak{D}(\Gamma)$.

To every $\Gamma' \in \mathfrak{D}(\Gamma)$ with the region of discontinuity Ω' there corresponds a pair of marked Riemann surfaces X' and Y' simultaneously uniformized by Γ', $X' \sqcup Y' \simeq \Gamma' \backslash \Omega'$. Set $S(X', Y') = S_{\Gamma'}$ and denote by S_Y and S_X restrictions of the function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ onto $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ respectively. Let ι be the complex conjugation and let $\bar{\Gamma} = \iota(\Gamma)$ be the quasi-Fuchsian group complex conjugated to Γ. The correspondence $\mu \mapsto \iota \circ \mu \circ \iota$ establishes complex-analytic anti-isomorphism

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\bar{\Gamma}) \simeq \mathfrak{T}(\bar{Y}) \times \mathfrak{T}(\bar{X}).$$

The classical Liouville action has the symmetry property

$$S(X', Y') = S(\bar{Y}', \bar{X}'). \tag{4.1.3}$$

For every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ set

$$\vartheta[\phi] = 2\phi_{zz} - \phi_z^2.$$

It follows from the Liouville equation that $\vartheta = \vartheta[\phi_{hyp}] \in \Omega^{2,0}(\Gamma)$, i.e., is a holomorphic quadratic differential for Γ. It follows from (2.3.4) that

$$\vartheta(z) = \begin{cases} 2\mathcal{S}\left(J_1^{-1}\right)(z) \text{ if } z \in \Omega_1, \\ 2\mathcal{S}\left(J_2^{-1}\right)(z) \text{ if } z \in \Omega_2. \end{cases} \tag{4.1.4}$$

Define a $(1,0)$-form $\boldsymbol{\vartheta}$ on the deformation space $\mathfrak{D}(\Gamma)$ by assigning to every $\Gamma' \in \mathfrak{D}(\Gamma)$ corresponding $\vartheta[\phi'_{hyp}] \in \Omega^{2,0}(\Gamma')$ — a vector in the holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at Γ'.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let P_F and P_{QF} be Fuchsian and quasi-Fuchsian projective connections on $X' \sqcup Y' \simeq \Gamma'\backslash\Omega'$, defined by the coverings $\pi_F : \mathbb{U} \cup \mathbb{L} \to X' \sqcup Y'$ and $\pi_{QF} : \Omega_1 \cup \Omega_2 \to X' \sqcup Y'$ respectively. We will continue to denote corresponding sections of the affine bundle $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ by P_F and P_{QF} respectively. The difference $P_F - P_{QF}$ is a $(1,0)$-form on $\mathfrak{D}(\Gamma)$.

Lemma 4.1.1. *On the deformation space* $\mathfrak{D}(\Gamma)$,

$$\boldsymbol{\vartheta} = 2(P_F - P_{QF}).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{U}\cup\mathbb{L} & \xrightarrow{\;J\;} & \Omega_1\cup\Omega_2 \\ \big\downarrow{\scriptstyle \pi_F} & & \big\downarrow{\scriptstyle \pi_{QF}} \\ X\sqcup Y & \xrightarrow{\;=\;} & X\sqcup Y, \end{array}$$

where the covering map J is equal to the map J_1 on component $\mathbb{U}$ and to the map J_2 on component $\mathbb{L}$. As explained in the Introduction, $P_F = \mathcal{S}(\pi_F^{-1})$ and $P_{QF} = \mathcal{S}(\pi_{QF}^{-1})$, and it follows from the property **SD1** and commutativity of the diagram that

$$\left(\mathcal{S}\left(\pi_F^{-1}\right) - \mathcal{S}\left(\pi_{QF}^{-1}\right)\right)\circ \pi_{QF}\left(\pi'_{QF}\right)^2 = \mathcal{S}\left(J^{-1}\right).$$

□

4.2 First variation

Here we compute the $(1,0)$-form ∂S on $\mathfrak{D}(\Gamma)$.

Theorem 4.2.1. *On the deformation space* $\mathfrak{D}(\Gamma)$,

$$\partial S = 2(P_F - P_{QF}).$$

Proof. It is sufficient to prove that for every $\mu \in \Omega^{-1,1}(\Gamma)$

$$L_\mu S = \boldsymbol{\vartheta}(\mu) = \iint\limits_{\Gamma\backslash\Omega} \vartheta\mu. \tag{4.2.1}$$

Indeed, using the isomorphism $\Phi^\nu : \mathfrak{D}(\Gamma) \to \mathfrak{D}(\Gamma^\nu)$, it is easy to see that variation formula (4.2.1) is valid at every point $\Gamma^{\nu} \in \mathfrak{D}(\Gamma)$ if it is valid at the origin. The actual computation of $L_{\mu}S$ is quite similar to that in [ZT87c] for the case of Schottky groups, with the clarifying role of homological algebra.

Let $\tilde{\Gamma}$ be the Fuchsian group corresponding to Γ and let $\Sigma = F + L - V$ be the corresponding total cycle of degree 2 representing the fundamental class of X in the total complex $\operatorname{Tot}\mathsf{K}$ for the pair $\mathbb{U}, \tilde{\Gamma}$. As in Section 2.3.1, set $\Sigma(\Gamma) = J_1(\Sigma - \bar{\Sigma})$. The corresponding total cycle for the pair $\Omega^{\varepsilon\mu}, \Gamma^{\varepsilon\mu} = f^{\varepsilon\mu} \circ \Gamma \circ (f^{\varepsilon\mu})^{-1}$ can be chosen as $\Sigma(\Gamma^{\varepsilon\mu}) = f^{\varepsilon\mu}(\Sigma(\Gamma))$. According to Remark 2.3.8,

$$S_{\Gamma^{\varepsilon\mu}} = \frac{i}{2} \left\langle \Psi \left[\phi^{\varepsilon\mu}_{hyp}\right], f^{\varepsilon\mu}(\Sigma(\Gamma)) \right\rangle .$$

Moreover, as it follows from Lemma 2.3.4, we can choose $\Gamma^{\varepsilon\mu}$-contracting at 0 paths of integration in the definition of $\Theta^{\varepsilon\mu}$ or, equivalently, paths in the definition of $W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu}$, to be the push-forwards by $f^{\varepsilon\mu}$ of the corresponding Γ-contracting at 0 paths. Denoting $\omega^{\varepsilon\mu} = \omega\left[\phi^{\varepsilon\mu}_{hyp}\right]$, $\check{\theta}^{\varepsilon\mu} = \check{\theta}\left[\phi^{\varepsilon\mu}_{hyp}\right]$, and using (2.3.5) we have

$$S_{\Gamma^{\varepsilon\mu}} = \frac{i}{2} \left(\langle \omega^{\varepsilon\mu}, F_1^{\varepsilon\mu} - F_2^{\varepsilon\mu} \rangle - \langle \check{\theta}^{\varepsilon\mu}, L_1^{\varepsilon\mu} - L_2^{\varepsilon\mu} \rangle + \langle \check{u}^{\varepsilon\mu}, W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu} \rangle \right).$$

Changing variables and formally differentiating under the integral sign in the term $\langle \check{u}^{\varepsilon\mu}, W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu} \rangle$, we obtain

$$\begin{aligned} L_{\mu}S &= \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} S_{\Gamma^{\varepsilon\mu}} \\ &= \frac{i}{2} \left(\langle L_{\mu}\omega, F_1 - F_2 \rangle - \langle L_{\mu}\check{\theta}, L_1 - L_2 \rangle + \langle L_{\mu}\check{u}, W_1 - W_2 \rangle \right). \end{aligned}$$

We will justify this formula at the end of the proof. Here we observe that though $\omega^{\varepsilon\mu}$, $\check{\theta}^{\varepsilon\mu}$ and $\check{u}^{\varepsilon\mu}$ are not tensors for $\Gamma^{\varepsilon\mu}$, they are differential forms on $\Omega^{\varepsilon\mu}$ so that their Lie derivatives are given by the same formulas as in Section 3.2.

Using formulas (3.2.3)-(3.2.5), we get

$$\begin{aligned} L_\mu \omega &= -\left(\left(\phi_{hyp}\right)_{\bar{z}} \dot{f}_{zz} + \left(\phi_{hyp}\right)_z \left(\left(\phi_{hyp}\right)_z \dot{f}_{\bar{z}} + \dot{f}_{z\bar{z}}\right)\right) dz \wedge d\bar{z} \\ &= \vartheta\mu \, dz \wedge d\bar{z} - d\xi, \end{aligned}$$

where

$$\xi = 2\left(\phi_{hyp}\right)_z \dot{f}_{\bar{z}} d\bar{z} - \phi_{hyp}\, d\dot{f}_z. \tag{4.2.2}$$

Since $\vartheta\mu$ is a $(1,1)$-tensor for Γ, $\delta(\vartheta\mu \, dz \wedge d\bar{z}) = 0$, so that $\delta L_\mu \omega = -\delta d\xi$. We have

$$\langle d\xi, F_1 - F_2 \rangle = \langle \xi, \partial'(F_1 - F_2) \rangle = \langle \xi, \partial''(L_1 - L_2) \rangle = \langle \delta\xi, L_1 - L_2 \rangle.$$

Set $\chi = \delta\xi + L_\mu \check{\theta}$. The 1-form χ on Ω is closed,

$$d\chi = \delta(d\xi) + L_\mu d\check{\theta} = \delta(-L_\mu \omega) + L_\mu \delta\omega = 0,$$

and satisfies

$$\delta\chi = \delta(L_\mu \check{\theta} + \delta\xi) = L_\mu \delta\check{\theta} = L_\mu \check{u}.$$

Using (3.2.3), (3.2.6), (3.2.7) and part (ii) of Lemma 3.2.2, we get

$$\begin{aligned}
L_\mu\check{\theta}_{\gamma^{-1}} =& - \dot{f}_z\frac{\gamma''}{\gamma'}\, dz + \phi_{hyp}\left(\left(\left(\dot{f}_{zz}\circ\gamma\,\gamma' - \dot{f}_{zz}\right) dz + \frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\, d\bar{z}\right) + \dot{f}_z\frac{\overline{\gamma''}}{\overline{\gamma'}}\, d\bar{z}\right.\\
&+\frac{1}{2}\left(-\left(\dot{f}_z\circ\gamma - \dot{f}_z\right)\frac{\gamma''}{\gamma'}\, dz - \log|\gamma'|^2\left(\dot{f}_{zz}\circ\gamma\,\gamma' - \dot{f}_{zz}\right) dz\right.\\
&\left.-\log|\gamma'|^2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\, d\bar{z} + \left(\dot{f}_z\circ\gamma - \dot{f}_z\right)\frac{\overline{\gamma''}}{\overline{\gamma'}}\, d\bar{z}\right)\\
&-\left(\log|c(\gamma)|^2 + 2\log 2\right) d\left(\dot{f}_z\circ\gamma - \dot{f}_z\right) - \frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right)\\
=& - \dot{f}_z\frac{\gamma''}{\gamma'}dz + \dot{f}_z\circ\gamma\frac{\overline{\gamma''}}{\overline{\gamma'}}\, d\bar{z} - d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma - \dot{f}_z\right)\right)\\
&+\phi_{hyp}\, d\left(\dot{f}_z\circ\gamma - \dot{f}_z\right) - \left(\log|c(\gamma)|^2 + 2\log 2\right) d\left(\dot{f}_z\circ\gamma - \dot{f}_z\right)\\
&-\frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right).
\end{aligned}$$

Using

$$\delta\xi_{\gamma^{-1}} = -2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\, d\bar{z} - \phi_{hyp}\, d\left(\dot{f}_z\circ\gamma - \dot{f}_z\right) + \log|\gamma'|^2 d\left(\dot{f}_z\circ\gamma\right),$$

we get

$$\begin{aligned}
\chi_{\gamma^{-1}} =& d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma + \dot{f}_z\right) - \left(\log|c(\gamma)|^2 + 2\log 2\right)\left(\dot{f}_z\circ\gamma - \dot{f}_z\right)\right)\\
&-\left(\dot{f}_z\circ\gamma + \dot{f}_z\right)\frac{\gamma''}{\gamma'}dz - 2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\, d\bar{z} - \frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right).
\end{aligned}$$

Using parts (ii) and (iii) of Lemma 3.2.2 and

$$-\frac{2\dot{c}}{cz+d} = \frac{\dot{c}}{c}\frac{\gamma''}{\gamma'}(z),$$

we finally obtain

$$\chi_{\gamma^{-1}} = d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma+\dot{f}_z+2\frac{\dot{c}(\gamma)}{c(\gamma)}\right)\right.$$
$$\left.-\left(\log|c(\gamma)|^2+2+2\log 2\right)\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\right)$$
$$= dl_{\gamma^{-1}}.$$

We have

$$\begin{aligned}\langle d\xi, F_1-F_2\rangle+\langle L_\mu\check{\theta}, L_1-L_2\rangle &= \langle\chi, L_1-L_2\rangle=\langle dl, L_1-L_2\rangle\\ &= \langle l, \partial'(L_1-L_2)\rangle=\langle l,\partial''(V_1-V_2)\rangle\\ &= \langle\delta l, V_1-V_2\rangle.\end{aligned}$$

Using $L_\mu\check{u}=d\delta l$ we get

$$\langle L_\mu\check{u}, W_1-W_2\rangle=\langle\delta l,\partial'(W_1-W_2)\rangle=\langle\delta l, V_1-V_2\rangle$$

so that

$$L_\mu S=\frac{i}{2}\langle\vartheta\mu\, dz\wedge d\bar{z}, F_1-F_2\rangle,$$

as asserted.

Finally, we justify the differentiation under the integral sign. Set

$$l_\gamma = l_\gamma^{(0)}+l_\gamma^{(1)},$$

where

$$l^{(0)}_{\gamma^{-1}} = \frac{\dot{c}(\gamma)}{c(\gamma)} \log|\gamma'|^2 - \left(\log|c(\gamma)|^2 + 2 + 2\log 2\right)(\dot{f}_z \circ \gamma - \dot{f}_z),$$
$$l^{(1)}_{\gamma^{-1}} = \frac{1}{2} \log|\gamma'|^2 \left(\dot{f}_z \circ \gamma + \dot{f}_z\right).$$

Next, we use part (i) of Lemma 3.2.2. According to it, the function $l^{(0)}_\gamma$ is continuous on $\mathcal{C} \setminus \{\gamma(\infty)\}$. Since

$$(\delta l^{(1)})_{\gamma_1^{-1},\gamma_2^{-1}} = \frac{1}{2}\left(\log|\gamma_2' \circ \gamma_1|^2(\dot{f}_z \circ \gamma_1 - \dot{f}_z) - \log|\gamma_1'|^2(\dot{f}_z \circ \gamma_2\gamma_1 - \dot{f}_z \circ \gamma_1)\right),$$

we also conclude that $(\delta l^{(1)})_{\gamma_1,\gamma_2}$, and hence the function $(\delta l)_{\gamma_1,\gamma_2}$, are continuous on $\mathcal{C} \setminus \{\gamma_1(\infty), (\gamma_1\gamma_2)(\infty)\}$. Now let $W_1^{(n)} \Subset W_1$ and $W_2^{(n)} \Subset W_2$ be a sequence of 1-chains in Ω_1 and Ω_2 obtained from W_1 and W_2 by "cutting" Γ-contracting at 0 paths at points $p_n' \in \Omega_1$ and $p_n'' \in \Omega_2$, where $p_n', p_n'' \to 0$ as $n \to \infty$. Clearly,

$$S = \lim_{n\to\infty} S_n,$$

where

$$S_n = \frac{i}{2}\left(\langle \omega, F_1 - F_2\rangle - \langle \check{\theta}, L_1 - L_2\rangle + \langle \check{u}, W_1^{(n)} - W_2^{(n)}\rangle\right).$$

Our previous arguments show that

$$L_\mu S_n = \frac{i}{2}\langle \vartheta\mu\, dz \wedge d\bar{z}, F_1 - F_2\rangle - \langle (\delta l)(p_n'), U_1\rangle + \langle (\delta l)(p_n''), U_2\rangle.$$

Since function δl is continuous at $p = 0$ and $U_1 = U_2$, we get

$$\lim_{n\to\infty} L_\mu S_n = \frac{i}{2}\,\langle \vartheta\mu, F_1 - F_2\rangle.$$

Moreover, the convergence is uniform in some neighborhood of Γ in $\mathfrak{D}(\Gamma)$, since $f^{\varepsilon\mu}$ is holomorphic at $\varepsilon = 0$. Thus

$$L_\mu S = \lim_{n\to\infty} L_\mu S_n,$$

which completes the proof. □

For fixed Riemann surface Y denote by P_F and P_{QF} sections of $\mathfrak{P}(X) \to \mathfrak{T}(X)$ corresponding to the Fuchsian uniformization of $X' \in \mathfrak{T}(X)$ and to the simultaneous uniformization of $X' \in \mathfrak{T}(X)$ and Y respectively.

Corollary 4.2.2. *On the Teichmüller space* $\mathfrak{T}(X)$,

$$P_F - P_{QF} = \frac{1}{2}\,\partial S_Y.$$

Remark 4.2.3. Conversely, Theorem 4.2.1 follows from the Corollary 4.2.2 and the symmetry property (4.1.3).

Remark 4.2.4. In the Fuchsian case the maps J_1 and J_2 are identities and similar computation shows that $\boldsymbol{\vartheta} = 0$, in accordance with $S = 8\pi(2g-2)$ being a constant function on $\mathfrak{T}(X) \times \mathfrak{T}(\bar{X})$.

4.3 Second variation

Here we compute $d\vartheta = \bar{\partial}\vartheta$. First, we have the following statement.

Lemma 4.3.1. *The quasi-Fuchsian projective connection P_{QF} is a holomorphic section of the affine bundle $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{f^{\varepsilon\mu}} & \Omega^{\varepsilon\mu} \\ \Big\downarrow \pi_{QF} & & \Big\downarrow \pi_{QF}^{\varepsilon\mu} \\ X \sqcup Y & \xrightarrow{F^{\varepsilon\mu}} & X^{\varepsilon\mu} \sqcup Y^{\varepsilon\mu} \end{array}$$

where $\mu \in \Omega^{-1,1}(\Gamma)$. We have

$$\mathcal{S}\left(\left(\pi_{QF}^{\varepsilon\mu}\right)^{-1}\right) \circ F^{\varepsilon\mu} \left(F_z^{\varepsilon\mu}\right)^2 + \mathcal{S}\left(F^{\varepsilon\mu}\right) = \mathcal{S}\left(f^{\varepsilon\mu}\right) \circ \pi_{QF}^{-1} \left(\pi_{QF}^{-1}\right)_z^2 + \mathcal{S}\left(\pi_{QF}^{-1}\right).$$

Since $f^{\varepsilon\mu}$ and, obviously $F^{\varepsilon\mu}$, are holomorphic at $\varepsilon = 0$, we get

$$\left.\frac{\partial}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} \mathcal{S}\left(\left(\pi_{QF}^{\varepsilon\mu}\right)^{-1}\right) = 0.$$

□

Using Corollary 4.2.2, Lemma 4.3 and the result [ZT87c]

$$\bar{\partial} P_F = -i\,\omega_{WP},$$

which follows from (1.0.2) since P_S is a holomorphic section of $\mathfrak{P}_g \to \mathfrak{S}_g$, we immediately get

Corollary 4.3.2. *For fixed Y*

$$\partial\bar{\partial}S_Y = -2\bar{\partial}(P_F - P_{QF}) = -2d(P_F - P_{QF}) = 2\,i\,\omega_{WP},$$

so that $-S_Y$ is a Kähler potential for the Weil-Petersson metric on $\mathfrak{T}(X)$.

Remark 4.3.3. The equation $d(P_F - P_{QF}) = -i\,\omega_{WP}$ was first proved in [McM00] and was used for the proof that moduli spaces are Kähler hyperbolic (note that symplectic form ω_{WP} used there is twice the one we are using here, and there is a missing factor $1/2$ in the computation in [McM00]). Specifically, the Kraus-Nehari inequality asserts that $P_F - P_{QF}$ is a bounded antiderivative of $-\,i\,\omega_{WP}$ with respect to Teichmüller and Weil-Petersson metrics [McM00]. In this regard, it is interesting to estimate the Kähler potential S_Y on $\mathfrak{T}(X)$. From the basic inequality of the distortion theorem (see, e.g., [Dur83])

$$\left|\frac{h''(z)}{h'(z)} - \frac{2\bar{z}}{(1-|z|^2)}\right| \leq \frac{4}{(1-|z|^2)},$$

where h is a univalent function in the unit disk, we immediately get

$$|(\phi_{hyp})_z|^2 \leq 4e^{\phi_{hyp}},$$

so that the bulk term in S_Y is bounded on $\mathfrak{T}(X)$ by $20\pi(2g-2)$. It can also be shown that other terms in S_Y have at most "linear growth" on $\mathfrak{T}(X)$, in accordance with the boundness of ∂S_Y.

The following result follows from the Corollary 4.3.2 and the symmetry property (4.1.3). For completeness, we give its proof in the form that verbatim generalizes to Kleinian groups.

Theorem 4.3.4. *The following formula holds on $\mathfrak{D}(\Gamma)$,*

$$d\boldsymbol{\vartheta} = \bar{\partial}\partial S = -2i\,\omega_{WP},$$

so that $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$.

Proof. Let $\mu, \nu \in \Omega^{-1,1}(\Gamma)$. First, using Cartan formula, we get

$$\begin{aligned} d\boldsymbol{\vartheta}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu}\right) =& L_\mu(\boldsymbol{\vartheta}(\nu)) - L_\nu(\boldsymbol{\vartheta}(\mu)) - \boldsymbol{\vartheta}\left(\left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu}\right]\right) \\ =& L_\mu(L_\nu S) - L_\nu(L_\mu S) = 0, \end{aligned}$$

which just manifests that $\partial^2 = 0$. On the other hand,

$$\begin{aligned} d\boldsymbol{\vartheta}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right) =& L_\mu(\boldsymbol{\vartheta}(\bar{\nu})) - L_{\bar{\nu}}(\boldsymbol{\vartheta}(\mu)) - \boldsymbol{\vartheta}\left(\left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right]\right) \\ =& - L_{\bar{\nu}} \iint\limits_{\Gamma\backslash\Omega} \vartheta\mu \\ =& - \iint\limits_{\Gamma\backslash\Omega} (L_{\bar{\nu}}\vartheta)\mu, \end{aligned}$$

since $\boldsymbol{\vartheta}$ is a $(1,0)$-form.

The computation of $L_{\bar{\nu}}\boldsymbol{\vartheta}$ repeats verbatim the one given in [ZT87c]. Namely, consider the commutative diagram (3.2.1) with $i = 1, 2$, and, for brevity, omit the index i. Since $(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu} = F^{\varepsilon\bar{\nu}} \circ J^{-1}$, the property **SD1** of the Schwarzian derivative (applicable when at least one of the functions is holomorphic) yields

$$\mathcal{S}(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu}\,(f_z^{\varepsilon\nu})^2 + \mathcal{S}(f^{\varepsilon\mu}) = \mathcal{S}(F^{\varepsilon\hat{\nu}}) \circ J^{-1}(J_z^{-1})^2 + \mathcal{S}(J^{-1}). \qquad (4.3.1)$$

We obtain

$$\begin{aligned}\frac{\partial}{\partial\bar{\varepsilon}_\nu}\bigg|_{\varepsilon=0} \mathcal{S}(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu}\,(f_z^{\varepsilon\nu})^2 &= \frac{\partial}{\partial\bar{\varepsilon}_\nu}\bigg|_{\varepsilon=0} \mathcal{S}(F^{\varepsilon\hat{\nu}}) \circ J^{-1}(J_z^{-1})^2 \\ &= \frac{\partial}{\partial\bar{\varepsilon}_\nu}\bigg|_{\varepsilon=0} F_{zzz}^{\varepsilon\hat{\nu}} \circ J^{-1}(J_z^{-1})^2 \\ &= -\frac{1}{2}\rho\overline{\nu(z)},\end{aligned}$$

where in the last line we have used Ahlfors formula (3.2.10). Finally,

$$d\vartheta\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right) = \iint\limits_{\Gamma\backslash\Omega} \mu\bar{\nu}\rho = -2i\,\omega_{WP}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right).$$

□

4.4 Quasi-Fuchsian reciprocity

The existence of the function S on the deformation space $\mathfrak{D}(\Gamma)$ satisfying the statement of Theorem 4.2.1 is a global form of quasi-Fuchsian reciprocity. Quasi–Fuchsian reciprocity of McMullen [McM00] follows from it as immediate corollary.

Let $\mu, \nu \in \Omega^{-1,1}(\Gamma)$ be such that μ vanishes outside Ω_1 and ν — outside Ω_2, so that Lie derivatives L_μ and L_ν stand for the variation of X for fixed Y and variation of Y for fixed X respectively.

Theorem 4.4.1. *(McMullen's quasi-Fuchsian reciprocity)*

$$\iint_X (L_\nu \mathcal{S}(J_1^{-1}))\,\mu = \iint_Y (L_\mu \mathcal{S}(J_2^{-1}))\,\nu.$$

Proof. Immediately follows from Theorem 4.2.1, since

$$L_\nu L_\mu S = 2\iint_X (L_\nu \mathcal{S}(J_1^{-1}))\,\mu,$$
$$L_\mu L_\nu S = 2\iint_Y (L_\mu \mathcal{S}(J_2^{-1}))\,\nu,$$

and $[L_\mu, L_\nu] = 0$. □

In [McM00], quasi-Fuchsian reciprocity was used to prove that $d(P_F - P_{QF}) = -i\,\omega_{WP}$. For completeness, we give here another proof of this result using earlier approach in [ZT87b], which admits generalization to other deformation spaces.

Proposition 4.4.2. *On the deformation space* $\mathfrak{D}(\Gamma)$,

$$\partial\vartheta = 0.$$

Proof. Using the same identity (4.3.1) which follows from the commutative diagram (3.2.1), we have

$$\left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} \mathcal{S}(J^{\varepsilon\nu})^{-1}\circ f^{\varepsilon\nu}\,(f_z^{\varepsilon\nu})^2 = \left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} \mathcal{S}(F^{\varepsilon\hat{\nu}})\circ J^{-1}(J_z^{-1})^2 - \left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} \mathcal{S}(f^{\varepsilon\nu})$$
$$= \left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} F_{zzz}^{\varepsilon\hat{\nu}}\circ J^{-1}(J_z^{-1})^2 - \left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} f_{zzz}^{\varepsilon\nu},$$

where we replaced μ by ν and omit index $i = 1, 2$. Differentiating (3.2.2) three times with respect to z we get

$$\left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\varepsilon=0} f_{zzz}^{\varepsilon\nu}(z) = -\frac{6}{\pi}\iint\limits_{\mathbb{C}} \frac{\nu(w)}{(z-w)^4}\, d^2w = -\frac{6}{\pi}\iint\limits_{\Gamma\backslash\Omega} K(z,w)\nu(w)d^2w, \tag{4.4.1}$$

where

$$K(z,w) = \sum_{\gamma\in\Gamma} \frac{\gamma'(w)^2}{(z-\gamma w)^4}.$$

It is well-known that for harmonic ν the integral in (4.4.1) is understood in the principal value sense (as $\lim_{\delta\to 0}$ of integral over $\mathbb{C}\setminus\{|w-z|\le\delta\}$). Therefore, using Ahlfors formula (3.2.9) we obtain

$$(L_\nu\boldsymbol{\vartheta})(z) = \frac{12}{\pi}\iint\limits_{\Gamma\backslash\Omega} K(z,w)\nu(w)d^2w,$$

and

$$\partial\boldsymbol{\vartheta}(\mu,\nu) = L_\mu\boldsymbol{\vartheta}(\nu) - L_\nu\boldsymbol{\vartheta}(\mu)$$
$$= \iint\limits_{\Gamma\backslash\Omega} (L_\mu\vartheta)(z)\,\nu(z)d^2z - \iint\limits_{\Gamma\backslash\Omega} (L_\nu\vartheta)(w)\,\mu(w)d^2w = 0,$$

since kernel $K(z,w)$ is obviously symmetric in z and w, $K(z,w) = K(w,z)$.

$\square$

Chapter 5

Holography

Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$. The group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ acts on the closure $\overline{\mathbb{U}}^3 = \mathbb{U}^3 \cup \hat{\mathbb{C}}$ of the hyperbolic 3-space $\mathbb{U}^3 = \{Z = (x, y, t) \in \mathbb{R}^3 \,|\, t > 0\}$. The action is discontinuous on $\mathbb{U}^3 \cup \Omega$ and $M = \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is a hyperbolic 3-manifold, compact in the relative topology of $\overline{\mathbb{U}}^3$, with the boundary $X \sqcup Y \simeq \Gamma\backslash\Omega$. According to the holography principle, the on-shell gravity theory on M, given by the Einstein-Hilbert action functional with the cosmological term, is equivalent to the "off-shell" gravity theory on its boundary $X \sqcup Y$, given by the Liouville action functional. Here we give a precise mathematical formulation of this principle.

5.1 Homology and cohomology set-up

We start by generalizing homological algebra methods in Chapter 2 to the three-dimensional case.

5.1.1 Homology computation

Denote by $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(\mathbb{U}^3 \cup \Omega)$ the standard singular chain complex of $\mathbb{U}^3 \cup \Omega$, and let R be a fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ such that $R \cap \Omega$ is the fundamental domain $F = F_1 - F_2$ for the group Γ in Ω (see Chapter 2). To have a better picture, consider first the case when Γ is a Fuchsian group. Then R is a region in $\overline{\mathbb{U}}^3$ bounded by the hemispheres which intersect $\hat{\mathbb{C}}$ along the circles that are orthogonal to $\mathbb{R}$ and bound the fundamental domain F (see Section 2.2.1). The fundamental region R is a three-dimensional CW-complex with a single 3-cell given by the interior of R. The 2-cells — the faces D_k, D'_k, E_k and E'_k, $k = 1, \dots, g$, are given by the parts of the boundary of R bounded by the intersections of the hemispheres and the arcs $a_k - \bar{a}_k$, $a'_k - \bar{a}'_k$, $b_k - \bar{b}_k$ and $b'_k - \bar{b}'_k$ respectively (see Fig. 1). The 1-cells — the edges, are given by the 1-cells of $F_1 - F_2$ and by $e^0_k, e^1_k, f^0_k, f^1_k$ and d_k, $k = 1, \dots, g$, defined as follows. The edges e^0_k are intersections of the faces E_{k-1} and D_k joining the vertices $\bar{a}_k(0)$ to $a_k(0)$, the edges e^1_k are intersections of the faces D_k and E'_k joining the vertices $\bar{a}_k(1)$ to $a_k(1)$; $f^0_k = e^0_{k+1}$ are intersections of E_k and D_{k+1} joining $\bar{b}_k(0)$ to $b_k(0)$, f^1_k are intersections of D'_k and E_k joining $\bar{b}_k(1)$ to $b_k(1)$, and d_k are intersections of E'_k and D'_k joining $\bar{a}'_k(1)$ to $a'_k(1)$. Finally, the 0-cells — the vertices, are given by the vertices of F. This property means that the edges of R do not intersect in $\mathbb{U}^3$. When Γ is a quasi-Fuchsian group, the fundamental region R is a topological polyhedron homeomorphic to the geodesic polyhedron for the corresponding Fuchsian group $\tilde{\Gamma}$.

As in the two-dimensional case, we construct the 3-chain representing M in the total complex $\operatorname{Tot}\mathsf{K}$ of the double homology complex $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$

as follows. First, identify R with $R \otimes [\,] \in \mathsf{K}_{3,0}$. We have $\partial'' R = 0$ and

$$\begin{aligned}\partial' R &= -F + \sum_{k=1}^{g} (D_k - D'_k - E_k + E'_k) \\ &= -F + \partial'' S,\end{aligned}$$

where $S \in \mathsf{K}_{2,1}$ is given by

$$S = \sum_{k=1}^{g} \left(E_k \otimes [\beta_k] - D_k \otimes [\alpha_k]\right).$$

Secondly,

$$\begin{aligned}\partial' S &= \sum_{k=1}^{g} \left((b_k - \bar{b}_k) \otimes [\beta_k] - (a_k - \bar{a}_k) \otimes [\alpha_k]\right) \\ &\quad - \sum_{k=1}^{g} \left(\left(f_k^1 - f_k^0\right) \otimes [\beta_k] - \left(e_k^1 - e_k^0\right) \otimes [\alpha_k]\right) \\ &= L - \partial'' E,\end{aligned}$$

where $L = L_1 - L_2$ and $E \in \mathsf{K}_{1,2}$ is given by

$$\begin{aligned}E = \sum_{k=1}^{g} &\left(e_k^0 \otimes [\alpha_k|\beta_k] - f_k^0 \otimes [\beta_k|\alpha_k] + f_k^0 \otimes [\gamma_k^{-1}|\alpha_k\beta_k]\right) \\ &- \sum_{k=1}^{g-1} f_g^0 \otimes [\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}].\end{aligned}$$

Therefore $\partial' E = V = V_1 - V_2$ and the 3-chain $R - S + E \in (\operatorname{Tot} \mathsf{K})_3$ satisfies

$$\partial(R - S + E) = -F - L + V = -\Sigma, \tag{5.1.1}$$

as asserted.

5.1.2 Cohomology computation

The PSL(2, $\mathbb{C}$)-action on $\mathbb{U}^3$ is the following. Represent $Z = (z,t) \in \mathbb{U}^3$ by a quaternion

$$\boldsymbol{Z} = x \cdot \mathbf{1} + y \cdot \boldsymbol{i} + t \cdot \boldsymbol{j} = \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix},$$

and for every $c \in \mathbb{C}$ set

$$\boldsymbol{c} = \operatorname{Re} c \cdot \mathbf{1} + \operatorname{Im} c \cdot \boldsymbol{i} = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}.$$

Then for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$ the action $Z \mapsto \gamma Z$ is given by

$$\boldsymbol{Z} \mapsto (\boldsymbol{a}\boldsymbol{Z} + \boldsymbol{b})(\boldsymbol{c}\boldsymbol{Z} + \boldsymbol{d})^{-1}.$$

Explicitly, for $Z = (z,t) \in \mathbb{U}^3$ setting $z(Z) = z$ and $t(Z) = t$ gives

$$z(\gamma Z) = \left((az+b)\overline{(cz+d)} + a\bar{c}\,t^2\right) J_\gamma(Z), \tag{5.1.2}$$

$$t(\gamma Z) = t\, J_\gamma(Z), \tag{5.1.3}$$

where

$$J_\gamma(Z) = \frac{1}{|cz+d|^2 + |ct|^2}.$$

Note that $J_\gamma^{3/2}(Z)$ is the Jacobian of the map $Z \mapsto \gamma Z$, hence it satisfies the transformation property

$$J_{\gamma_1\circ\gamma_2}(Z) = J_{\gamma_1}(\gamma_2 Z) J_{\gamma_2}(Z). \tag{5.1.4}$$

From (5.1.2) and (5.1.3) we get the following formulas for the derivatives

$$\frac{\partial z(\gamma Z)}{\partial z} = (\overline{cz+d})^2 J_\gamma^2(Z), \tag{5.1.5}$$
$$\frac{\partial z(\gamma Z)}{\partial \bar{z}} = -(\bar{c}\,t)^2 J_\gamma^2(Z), \tag{5.1.6}$$
$$\frac{\partial z(\gamma Z)}{\partial t} = 2t\bar{c}(\overline{cz+d}) J_\gamma^2(Z), \tag{5.1.7}$$

In particular,

$$\frac{\partial z(Z)}{\partial z} = \gamma'(z) + O(t^2), \quad \frac{\partial z(\gamma Z)}{\partial \bar{z}} = O(t^2), \quad \frac{\partial z(Z)}{\partial t} = O(t), \tag{5.1.8}$$

as $t \to 0$ and $z \in \hat{\mathbb{C}} \setminus \{\gamma^{-1}(\infty)\}$, where for $z \in \mathbb{C}$ we continue to use the two-dimensional notations

$$\gamma(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \gamma'(z) = \frac{1}{(cz+d)^2}, \quad \frac{\gamma''}{\gamma'}(z) = \frac{-2c}{cz+d}.$$

The hyperbolic metric on $\mathbb{U}^3$ is given by

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2},$$

and is $\mathrm{PSL}(2,\mathbb{C})$-invariant. Denote by

$$w_3 = \frac{1}{t^3}\, dx \wedge dy \wedge dt = \frac{i}{2t^3}\, dz \wedge d\bar{z} \wedge dt$$

the corresponding volume form on $\mathbb{U}^3$. The form w_3 is exact on $\mathbb{U}^3$,

$$w_3 = dw_2, \quad \text{where} \quad w_2 = -\frac{i}{4t^2}\, dz \wedge d\bar{z}. \tag{5.1.9}$$

The 2-form $w_2 \in \mathsf{C}^{2,0}$ is no longer $\mathrm{PSL}(2,\mathbb{C})$-invariant. A straightforward computation using (5.1.5)-(5.1.7) gives for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$,

$$\begin{aligned}(\delta w_2)_{\gamma^{-1}} =& \gamma^* w_2 - w_2 \\ =& \frac{i}{2}\, J_\gamma(Z) \left(|c|^2 dz \wedge d\bar{z} - \frac{c\overline{(cz+d)}}{t}\, dz \wedge dt + \frac{\bar{c}(cz+d)}{t}\, d\bar{z} \wedge dt \right).\end{aligned}$$

Since $d\delta w_2 = \delta dw_2 = \delta w_3 = 0$ and $\mathbb{U}^3$ is simply connected, this implies that there exists $w_1 \in \mathsf{C}^{1,1}$ such that $dw_1 = \delta w_2$. Explicitly,

$$(w_1)_{\gamma^{-1}} = -\frac{i}{8}\, \log\left(|ct|^2 J_\gamma(Z)\right) \left(\frac{\gamma''}{\gamma'} dz - \frac{\overline{\gamma''}}{\overline{\gamma'}} d\bar{z} \right). \tag{5.1.10}$$

Using (5.1.4) and (5.1.8) we get for $\delta w_1 \in C^{1,2}$

$$\begin{aligned}(\delta w_1)_{\gamma_1^{-1},\gamma_2^{-1}} =& -\frac{i}{8}\left(\log J_{\gamma_1}(Z) + \log\frac{|c(\gamma_2)|^2}{|c(\gamma_2\gamma_1)|^2}\right)\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\gamma_1'\,dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\overline{\gamma_1'}\,d\bar{z}\right) \\ & -\frac{i}{8}\left(\log J_{\gamma_2}(\gamma_1 Z) + \log\frac{|c(\gamma_2\gamma_1)|^2}{|c(\gamma_1)|^2}\right)\left(\frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}\,d\bar{z} - \frac{\gamma_1''}{\gamma_1'}\,dz\right) \\ & + B_{\gamma_1^{-1},\gamma_2^{-1}}(Z).\end{aligned} \tag{5.1.11}$$

Here $B_{\gamma_1^{-1},\gamma_2^{-1}}(Z) = O(t\log t)$ as $t \to 0$, uniformly on compact subsets of $\mathbb{C} \setminus \{\gamma_1^{-1}(\infty), (\gamma_2\gamma_1)^{-1}(\infty)\}$.

Clearly the 1-form δw_1 is closed,

$$d(\delta w_1) = \delta(dw_1) = \delta(\delta w_2) = 0.$$

Since $\mathbb{U}^3$ is simply connected, there exists $w_0 \in \mathrm{C}^{0,2}$ such that $w_1 = dw_0$. Moreover, using $H^3(\Gamma, \mathbb{C}) = 0$ we can always choose the antiderivative w_0 such that $\delta w_0 = 0$. Finally, set $\Phi = w_2 - w_1 - w_0 \in (\mathrm{Tot}\,\mathrm{C})^2$, so that

$$D\Phi = w_3. \tag{5.1.12}$$

5.2 Regularized Einstein-Hilbert action

In two dimensions, the critical value of the Liouville action for a Riemann surface $X \simeq \Gamma\backslash\mathbb{U}$ is proportional to the hyperbolic area of the surface (see Chapter 2). It is expected that in three dimensions the critical value of the Einstein-Hilbert action functional with cosmological term is proportional to the hyperbolic volume of the 3-manifold $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ (plus a term proportional to the induced area of the boundary). However, the hyperbolic metric diverges at the boundary of $\overline{\mathbb{U}}^3$ and for quasi-Fuchsian group Γ (as well as for general Kleinian groups [1]) the hyperbolic volume of $\Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is infinite. In [Wit98], Witten proposed a regularization of the action functional by truncating the 3-manifold M by surface $f = \varepsilon$, where the cut-off function $f \in C^\infty(\mathbb{U}^3, \mathbb{R}_{>0})$ vanishes to the first order on the boundary of $\overline{\mathbb{U}}^3$. Every

[1]Note that we are using definition of Kleinian groups as in [Mas88]. In the theory of hyperbolic 3-manifolds these groups are called Kleinian groups of the second kind.

choice of the function f defines a metric on $\mathbb{U}^3$

$$ds^2 = \frac{f^2}{t^2}(|dz|^2 + dt^2),$$

belonging to the conformal class of the hyperbolic metric. On the boundary of $\overline{\mathbb{U}}^3$ it induces the metric

$$\lim_{t \to 0} \frac{f^2(z,t)}{t^2}|dz|^2.$$

Clearly for the case of quasi-Fuchsian group Γ (or for the general Kleinian case considered in the next chapter), the cut-off function f should be Γ-automorphic. Existence of such function is guaranteed by the following result, which we formulate for the general Kleinian case.

Lemma 5.2.1. *Let Γ be non-elementary purely loxodromic Kleinian group, normalized so that $\infty \in \Lambda$, and let Δ be an invariant union of components of the region of discontinuity Ω. For every $\phi \in \mathcal{CM}(\Gamma\backslash\Delta)$ there exists Γ-automorphic function $f \in C^\infty(\mathbb{U}^3 \cup \Delta)$ which is positive on $\mathbb{U}^3$ and satisfies*

$$f(Z) = te^{\phi(z)/2} + O(t^3), \quad \textit{as } t \to 0,$$

uniformly on compact subsets of Δ.

Proof. Note that $\Gamma\backslash\Delta$ is isomorphic to at most countable disjoint union of compact Riemann surfaces. For every $\phi \in \mathcal{CM}(\Gamma\backslash\Delta)$ set $\hat{f}(z,t) = te^{\phi(z)/2}$. Clearly $\hat{f}$ is smooth, positive on $\mathbb{U}^3$, and the required asymptotic behavior is an identity. However, the function $\hat{f}$ is not Γ-automorphic. To rectify this situation we use a partition of unity. Namely, as it is proved by I. Kra in [Kra72a] (the proof generalizes verbatim for our case), there exists a function $\eta \in C^\infty(\mathbb{U}^3 \cup \Delta)$ — partition of unity for Γ on $\mathbb{U}^3 \cup \Delta$, with the following properties.

(i) $0 \leq \eta \leq 1$.

(ii) For each $Z \in \mathbb{U}^3 \cup \Delta$ there is a neighborhood U of Z and a finite subset J of Γ such that $\eta|_{\gamma(U)} = 0$ for each $\gamma \in \Gamma \setminus J$.

(iii) $\sum_{\gamma\in\Gamma} \eta(\gamma Z) = 1$ for all $Z \in \mathbb{U}^3 \cup \Delta$.

Set

$$f(Z) = \sum_{\gamma\in\Gamma} \eta(\gamma Z)\hat{f}(\gamma Z).$$

By property (ii), for every $Z \in \mathbb{U}^3 \cup \Delta$ this sum contains only finitely many non-zero terms, so that the function f is well-defined. By properties (i) and (iii) it is positive on $\mathbb{U}^3$. To prove the asymptotic behavior, we use elementary formulas

$$\begin{aligned} z(\gamma Z) &= \frac{az+b}{cz+d} + O(t^2) = \gamma(z) + O(t^2), \\ t(\gamma Z) &= \frac{t}{|cz+d|^2} + O(t^3) \quad \text{as } t \to 0, \end{aligned}$$

where $z \neq \gamma^{-1}(\infty)$. Since ϕ is smooth on Δ and

$$e^{\phi(\gamma z)/2} = e^{\phi(z)/2}|cz+d|^2,$$

we get for $z \in \Delta$

$$\begin{aligned}\hat{f}(\gamma Z) &= \left(\frac{t}{|cz+d|^2} + O(t^3)\right)\left(e^{\phi(\gamma z)/2} + O(t^2)\right) \\ &= te^{\phi(z)/2} + O(t^3),\end{aligned}$$

where the O-term depends on γ. Using properties (ii) and (iii) we finally obtain

$$\begin{aligned}f(Z) &= \sum_{\gamma\in\Gamma} \eta(\gamma Z)\left(te^{\phi(z)/2} + O(t^3)\right) \\ &= te^{\phi(z)/2} + O(t^3),\end{aligned}$$

uniformly on compact subsets of Δ. □

Returning to the case when Γ is a normalized purely loxodromic quasi-Fuchsian group, set $\Delta = \Omega$ and for every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ let f be a function given by the lemma. For $\varepsilon > 0$ let $R_\varepsilon = R \cap \{f \geq \varepsilon\}$ be the truncated fundamental region. For every chain c in $\mathbb{U}^3$ let $c_\varepsilon = c \cap \{f \geq \varepsilon\}$ be the corresponding truncated chain. Also let $F_\varepsilon = \partial' R_\epsilon \cap \{f = \varepsilon\}$ be the boundary of R_ε on the surface $f = \varepsilon$ and define chains L_ε and V_ε on $f = \varepsilon$ by the same equations $\partial' F_\varepsilon = \partial'' L_\varepsilon$ and $\partial' L_\varepsilon = \partial'' V_\varepsilon$ as chains L and V (see Sections 2.2.1 and 2.3.1). Since the truncation is Γ-invariant, for every chain $c \in \mathsf{S}_\bullet(\mathbb{U}^3)$ and $\gamma \in \Gamma$ we have

$$(\gamma c)_\varepsilon = \gamma c_\varepsilon.$$

In particular, relations between the chains, derived in Section 5.1, hold for truncated chains as well.

Let M_ε be the truncated 3-manifold with the boundary $\partial' M_\varepsilon$. For ε sufficiently small $\partial' M_\varepsilon = X_\varepsilon \sqcup Y_\varepsilon$ is diffeomorphic to $X \sqcup Y$. Denote by $V_\varepsilon[\phi]$ the hyperbolic volume of M_ε. The hyperbolic metric induces a metric on $\partial' M_\varepsilon$, and $A_\varepsilon[\phi]$ denotes the area of $\partial' M_\varepsilon$ in the induced metric.

Definition 5.2.2. The regularized on-shell Einstein-Hilbert action functional is defined by

$$\mathcal{E}_\Gamma[\phi] = -4 \lim_{\varepsilon \to 0} \left(V_\varepsilon[\phi] - \frac{1}{2} A_\varepsilon[\phi] - 2\pi\chi(X) \log \varepsilon \right),$$

where $\chi(X) = \chi(Y) = 2 - 2g$ is the Euler characteristic of X.

The main result of this chapter is the following.

Theorem 5.2.3. *(Quasi-Fuchsian holography) For every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ the regularized Einstein-Hilbert action is well-defined and*

$$\mathcal{E}_\Gamma[\phi] = \check{S}_\Gamma[\phi],$$

where $\check{S}_\Gamma[\phi]$ is the modified Liouville action functional without the area term,

$$\check{S}_\Gamma[\phi] = S_\Gamma[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^\phi d^2 z - 8\pi\,(2g-2)\log 2.$$

Proof. It is sufficient to verify the formula,

$$V_\varepsilon[\phi] - \frac{1}{2} A_\varepsilon[\phi] = 2\pi\chi(X)\log\varepsilon - \frac{1}{4}\check{S}_\Gamma[\phi] + o(1) \quad \text{as } \varepsilon \to 0, \qquad (5.2.1)$$

which is a counter-part of the formula (1.0.13) for quasi-Fuchsian groups.

The area form induced by the hyperbolic metric on the surface $f(Z) = \varepsilon$ is given by

$$\sqrt{1 + \left(\frac{f_x}{f_t}\right)^2 + \left(\frac{f_y}{f_t}\right)^2}\,\frac{dx \wedge dy}{t^2}.$$

Using

$$\frac{f_x}{f_t}(Z) = \frac{t}{2}\,\phi_x(z) + O(t^3) \quad \text{and} \quad \frac{f_y}{f_t}(Z) = \frac{t}{2}\,\phi_y(z) + O(t^3),$$

we have as $\varepsilon \to 0$

$$\begin{aligned} A_\varepsilon[\phi] &= \iint_{F_\varepsilon} \sqrt{1 + \frac{t^2}{4}(\phi_x^2 + \phi_y^2)(z) + O(t^4)}\,\frac{dx \wedge dy}{t^2} \\ &= \iint_{F_\varepsilon} \frac{dx \wedge dy}{t^2} + \frac{1}{2}\iint_F \phi_z \phi_{\bar{z}}\, dx \wedge dy + o(1) \\ &= \iint_{F_\varepsilon} \frac{dx \wedge dy}{t^2} + \frac{i}{4}\langle \breve{\omega}[\phi], F\rangle + o(1). \end{aligned}$$

Here we introduce

$$\breve{\omega}[\phi] = \omega[\phi] - e^{\phi} dz \wedge d\bar{z} = |\phi_z|^2 dz \wedge d\bar{z}, \tag{5.2.2}$$

and has used that for $Z \in F_\varepsilon$

$$t = \varepsilon e^{-\phi(z)/2} + O(\varepsilon^3), \tag{5.2.3}$$

uniformly for $Z = (z, t)$ where $z \in F$.

Next, using (5.1.1) and (5.1.12) we have,

$$\begin{aligned} V_\varepsilon[\phi] =& \ \langle w_3, R_\varepsilon\rangle \\ =& \ \langle w_3, R_\varepsilon - S_\varepsilon + E_\varepsilon\rangle \\ =& \ \langle D(w_2 - w_1 - w_0), R_\varepsilon - S_\varepsilon + E_\varepsilon\rangle \\ =& \ \langle w_2 - w_1 - w_0, \partial\left(R_\varepsilon - S_\varepsilon + E_\varepsilon\right)\rangle \\ =& -\langle w_2, F_\varepsilon\rangle + \langle w_1, L_\varepsilon\rangle - \langle w_0, V_\varepsilon\rangle. \end{aligned}$$

The terms in this formula simplify as $\varepsilon \to 0$. First of all, it follows from (5.1.9) that

$$-\langle w_2, F_\varepsilon\rangle = \frac{1}{2}\iint\limits_{F_\varepsilon} \frac{dx\wedge dy}{t^2}.$$

Secondly, using (5.2.3) and $J_\gamma(Z) = |\gamma'(z)| + O(t^2)$ as $t \to 0$, we have on L_ε

$$\begin{aligned} (w_1)_{\gamma^{-1}} &= -\frac{i}{8}\log\left(|c\varepsilon|^2 e^{-\phi}\,|\gamma'(z)|\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar z\right) + o(1) \\ &= -\frac{i}{8}\left(2\log\varepsilon - \phi + \frac{1}{2}\log|\gamma'|^2 + \log|c(\gamma)|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar z\right) + o(1). \end{aligned}$$

Therefore, as $\varepsilon \to 0$,

$$\langle w_1, L_\varepsilon\rangle = -\frac{i}{4}\langle \varkappa, L\rangle(\log\varepsilon - \log 2) + \frac{i}{8}\langle\check\theta[\phi], L\rangle + o(1),$$

where 1-forms $\varkappa_\gamma$ and $\check\theta_\gamma[\phi]$ were introduced in Corollary 2.2.12 and formula

(2.2.16) respectively. Finally,

$$\langle w_0, V_\varepsilon\rangle = \langle w_0, \partial' E_\varepsilon\rangle = \langle dw_0, E_\varepsilon\rangle = \langle \delta w_1, E_\varepsilon\rangle = \langle \delta w_1, E\rangle + o(1),$$

where we used that 1-form δw_1 is smooth on $\mathbb{U}^3$ and continuous on $\mathbb{C}\setminus\Gamma(\infty)$. Since it is closed, we can replace the 1-chain E by the 1-chain $W = W_1 - W_2$ consisting of Γ-contracting paths at 0 (see Section 2.3). It follows from (5.1.11) that $\delta w_1 = \frac{i}{8}\check{u} + o(1)$ as $t \to 0$, where the 1-form $\check{u}_{\gamma_1,\gamma_2}$ was introduced in (2.2.17), so that

$$-\langle w_0, V_\varepsilon\rangle = -\frac{i}{8}\langle \check{u}, W\rangle + o(1).$$

Putting everything together, we have as $\varepsilon \to 0$

$$V_\varepsilon[\phi] - \frac{1}{2}A_\varepsilon[\phi] = -\frac{i}{4}\langle \varkappa, L\rangle(\log\varepsilon - \log 2) - \frac{i}{8}\left(\langle\check{\omega}[\phi], F\rangle - \langle\check{\theta}[\phi], L\rangle \right.$$
$$\left. + \langle \check{u}, W\rangle\right) + o(1).$$

Using Corollary 2.2.12, trivially modified for the quasi-Fuchsian case, and (2.3.5) concludes the proof. □

A fundamental domain F for Γ in Ω is called admissible, if it is the boundary in $\mathbb{C}$ of a fundamental region R for Γ in $\mathbb{U}^3 \cup \Omega$. As an immediate consequence of the theorem we get the following.

Corollary 5.2.4. *The Liouville action functional $S_\Gamma[\phi]$ is independent of the choice of admissible fundamental domain.*

Proof. Since $V_\varepsilon[\phi]$, $A_\varepsilon[\phi]$ are intrinsically associated with the quotient manifolds $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and $X \sqcup Y \simeq \Gamma\backslash\Omega$, the statement follows from the definition of the Einstein-Hilbert action and the theorem. □

Although we proved the same result in Chapter 2 using methods of homological algebra, the above argument easily generalizes to other Kleinian groups.

Remark 5.2.5. The truncation of the 3-manifold M by the function f does depend on the choice of the realization of the fundamental group of M as a normalized discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$. Different realizations of $\pi_1(M)$ result in different choices of the function f, since f has to satisfy the asymptotic behavior in Lemma 5.2.1, where the leading term $te^{\phi(z)/2}$ is not a well-defined function on M.

Remark 5.2.6. The cochain $w_0 \in \mathsf{C}^{0,2}$ was defined as a solution of the equation $dw_0 = w_1$ satisfying $\delta w_0 = 0$. However, in the computation in Theorem 5.2.3 this condition is not needed — any choice of an antiderivative for w_1 will suffice. This is due to the fact that the chain in $(\mathrm{Tot}\,\mathsf{K})_3$ that starts with $R \in \mathsf{K}_{3,0}$ does not contain a term in $\mathsf{K}_{0,3}$, hence $\partial' E = V$. Thus we can trivially add the term $\langle \delta w_0, R_\varepsilon - S_\varepsilon + E_\varepsilon \rangle = 0$ to $V_\varepsilon[\phi]$, which through the equation $D\Phi = w_3 - \delta w_0$ still gives $\langle w_0, V \rangle = \langle dw_0, E \rangle$. Thus the absence of $\mathsf{K}_{0,3}$-components in the chain in $(\mathrm{Tot}\,\mathsf{K})_3$ implies that each term in E produces two boundary terms in V which cancel out the integration constants in definition of w_0. As a result, $S_\Gamma[\phi]$ does not depend on the choice of w_0. In the next chapter we generalize the Liouville action functional to Kleinian groups having the same property.

Chapter 6

Generalization to Kleinian Groups

6.1 Kleinian groups of Class A

Let Γ be a finitely generated Kleinian group with the region of discontinuity Ω, a maximal set of non-equivalent components $\Omega_1, \ldots, \Omega_n$ of Ω, and the limit set $\Lambda = \hat{\mathbb{C}} \setminus \Omega$. As in the quasi-Fuchsian case, a path P is called Γ-contracting in Ω, if $P = P_1 \cup P_2$, where $p \in \Lambda \setminus \{\infty\}$ is a fixed point for Γ, paths $P_1 \setminus \{p\}$ and $P_2 \setminus \{p\}$ lie entirely in distinct components of Ω and are Γ-contracting at p in the sense of Definition 2.3.2. It follows from arguments in Section 2.3.1 that Γ-contracting paths in Ω are rectifiable.

Definition 6.1.1. A Kleinian group Γ is of Class A if it satisfies the following conditions.

A1 Γ is non-elementary and purely loxodromic.

A2 Γ is geometrically finite.

A3 Γ has a fundamental region R in $\mathbb{U}^3 \cup \Omega$ which is a finite three-dimensional CW-complex with no 0-dimensional cells in $\mathbb{U}^3$ and such that $R \cap \Omega \subset$

$\Omega_1 \cup \cdots \cup \Omega_n$.

In particular, property **A1** implies that Γ is torsion-free and does not contain parabolic elements, and property **A2** asserts that Γ has a fundamental region R in $\mathbb{U}^3 \cup \Omega$ which is a finite topological polyhedron. Property **A3** means that the region R can be chosen such that the vertices of R — endpoints of edges of R, lie on $\Omega \in \hat{\mathbb{C}}$ and the boundary of R in $\hat{\mathbb{C}}$, which is a fundamental domain for Γ in Ω, is not too "exotic".

The class A is rather large: it clearly contains all purely loxodromic Schottky groups (for which the property **A3** is vacuous), Fuchsian groups, quasi-Fuchsian groups, and free combinations of these groups.

As in the previous chapter, we say that Kleinian group Γ is normalized if $\infty \in \Lambda$.

6.2 Einstein-Hilbert and Liouville functionals

For a finitely generated Kleinian group Γ let $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ be corresponding hyperbolic 3-manifold, and let $\Gamma_1, \ldots, \Gamma_n$ be the stabilizer groups of the maximal set $\Omega_1, \ldots, \Omega_n$ of non-equivalent components of Ω. We have

$$\Gamma\backslash\Omega = \Gamma_1\backslash\Omega_1 \sqcup \cdots \sqcup \Gamma_n\backslash\Omega_n \simeq X_1 \sqcup \cdots \sqcup X_n,$$

so that Riemann surfaces $X_1, \ldots, X_n$ are simultaneously uniformized by Γ. Manifold M is compact in the relative topology of $\overline{\mathbb{U}}^3$ with the disjoint union $X_1 \sqcup \cdots \sqcup X_n$ as the boundary.

6.2.1 Homology and cohomology set-up

Let $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(\mathbb{U}^3 \cup \Omega)$, $\mathsf{B}_\bullet \equiv \mathsf{B}_\bullet(\mathbb{Z}\Gamma)$ be standard singular chain and bar-resolution homology complexes and $\mathsf{K}_{\bullet,\bullet} \equiv \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ — the corresponding double complex. When Γ is a Kleinian group of Class A, we can generalize homology construction from the previous chapter and define corresponding chains R, S, E, F, L, V in total complex $\operatorname{Tot}\mathsf{K}$ as follows. Let R be a fundamental region for Γ in $\mathbb{U}^3 \sqcup \Omega$ — a closed topological polyhedron in $\overline{\mathbb{U}}^3$ satisfying property **A3**. The group Γ is generated by side pairing transformations of $R \cap \mathbb{U}^3$ and we define the chain $S \in \mathsf{K}_{2,1}$ as the sum of terms $-s \otimes \gamma^{-1}$ for each pair of sides s, s' of $R \cap \mathbb{U}^3$ identified by a transformation γ, i.e., $s' = -\gamma s$. The sides are oriented as components of the boundary and negative sign stands for the opposite orientation. We have

$$\partial' R = -F + \partial'' S, \tag{6.2.1}$$

where $F = \partial' R \cap \Omega \in \mathsf{K}_{2,0}$. Note that it is immaterial whether we choose $-s \otimes \gamma^{-1}$ or $-s' \otimes \gamma$ in the definition of S, since these terms differ by a ∂''-coboundary. Next, relations between generators of Γ determine the Γ-action on the edges of R, which, in turn, determines the chain $E \in \mathsf{K}_{1,2}$ through the equation

$$\partial' S = L - \partial'' E. \tag{6.2.2}$$

Here $L = \partial' S \cap \Omega \in \mathsf{K}_{1,1}$. Finally, property **A3** implies that

$$\partial' E = V, \tag{6.2.3}$$

where the chain $V \in \mathsf{K}_{0,2}$ lies in Ω.

Next, let the 1-chain $W \in \mathsf{K}_{1,2}$ be a "proper projection" of the 1-chain E onto Ω, i.e., W is defined by connecting every two vertices belonging to the same edge of R either by a smooth path lying entirely in one component of Ω, or by a Γ-contracting path, so that $\partial' W = V$. The existence of such 1-chain W is guaranteed by the property **A3** and the following lemma, which is of independent interest.

Lemma 6.2.1. *Let Γ be a normalized, geometrically finite, purely loxodromic Kleinian group, and let R be the fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ such that $R \cap \Omega \subset \Omega_1 \cup \cdots \cup \Omega_n$ — a union of a maximal set of non-equivalent components of Ω. If an edge e of $R \cap \mathbb{U}^3$ has endpoints v_0 and v_1 belonging to two distinct components Ω_i and Ω_j, then there exists a Γ-contracting path in Ω joining vertices v_0 and v_1. In particular, Ω_i and Ω_j has at least one common boundary point, which is a fixed point for Γ.*

Proof. There exist sides s_1 and s_2 of R such that $e \subset s_1 \cap s_2$. For each of these sides there exists a group element identifying it with another side of R. Let $\gamma \in \Gamma$ be such element for s_1. Since Γ is torsion-free and $v_0, v_1 \in \Omega$, element γ identifies the edge e with another distinct edge e' of R with endpoints $\gamma(v_0) \neq v_0$ and $\gamma(v_1) \neq v_1$. Since $R \cap \Omega \subset \Omega_1 \cup \cdots \cup \Omega_n$ — a union of a maximal set of non-equivalent components of Ω, this implies that $\gamma(v_0) \in \Omega_i$ and γ fixes Ω_i. Similarly, $\gamma(v_1) \in \Omega_j$ and γ fixes Ω_j. Now assume that attracting fixed point p of γ is not ∞ (otherwise we replace γ by γ^{-1}). Join v_0 and $\gamma(v_0)$ by a smooth path P_1^0 inside Ω_i, and let $P_1^n = \gamma^n(P_1^0)$ be its n-th γ-iterate. Since γ fixes Ω_i, the path P_1^n lies entirely inside Ω_i. Since $\lim_{n\to\infty} \gamma^n(v_0) = p$, the path $P_1 = \cup_{n=0}^{\infty} P_1^n$ joins v_0 and p, and except for the endpoint p lies entirely in Ω_i. Clearly path P_1^0 can be chosen so that the path P_1 is smooth everywhere except at p. The path P_2 joining points v_1 and p inside Ω_j is defined similarly, and the path $P = P_1 \cup P_2$ is Γ-contracting in Ω. □

Setting $\Sigma = F + L - V$ we get from (6.2.1)-(6.2.3) that

$$\partial(R - S + E) = -\Sigma.$$

Remark 6.2.2. Since $\mathbb{U}^3$ is acyclic, it follows from general arguments in [AT97] that for any geometrically finite purely loxodromic Kleinian group Γ with fundamental region R given by a closed topological polyhedron, there exist chains $S \in \mathsf{K}_{2,1}, E \in \mathsf{K}_{1,2}, T \in \mathsf{K}_{0,3}$ and chains $F \in \mathsf{K}_{2,0}, L \in \mathsf{K}_{1,1}, V \in \mathsf{K}_{0,2}$ on Ω, satisfying

$$\partial' R = -F + \partial'' S$$
$$\partial' S = L - \partial'' E$$
$$\partial' E = V + \partial'' T.$$

Property **A3** asserts that $T = 0$, and we get equations (6.2.1)-(6.2.3).

Correspondingly, let $\mathsf{A}^\bullet \equiv \mathsf{A}^\bullet_{\mathbb{C}}(\mathbb{U}^3 \cup \Omega)$ and $\mathsf{C}^{\bullet,\bullet} \equiv \mathrm{Hom}(\mathsf{B}_\bullet, \mathsf{A}^\bullet)$ be the de Rham complex on $\mathbb{U}^3 \cup \Omega$ and the bar-de Rham complex respectively. The cochains $w_3, w_2, w_1, \delta w_1, w_0$ are defined by the same formulas as in Section 5.1. For $\phi \in \mathcal{CM}(\Gamma \backslash \Omega)$ define the cochains $\omega[\phi], \theta[\phi], u$ by the same formulas (2.2.4), (2.2.5), (2.2.6), with the group elements belonging to Γ. Finally, define the cochains $\check{\theta}[\phi], \check{u}$ by (2.2.16) and (2.2.17).

6.2.2 Action functionals

Let Γ be a normalized Class A Kleinian group. For each $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ let f be the function constructed in Lemma 5.2.1. As in Section 5.2, we truncate the manifold M by the cut-off function f and define $V_\varepsilon[\phi]$, $A_\varepsilon[\phi]$.

Definition 6.2.3. The regularized on-shell Einstein-Hilbert action functional for a normalized Class A Kleinian group Γ is defined by

$$\mathcal{E}_\Gamma[\phi] = -4\lim_{\varepsilon\to 0}\left(V_\epsilon[\phi] - \frac{1}{2}A_\epsilon[\phi] - \pi(\chi(X_1) + \cdots + \chi(X_n))\log\varepsilon\right).$$

As in the quasi-Fuchsian case, a fundamental domain F for a Kleinian group Γ in Ω is called admissible, if it is the boundary in $\mathbb{C}$ of a fundamental region R for Γ in $\mathbb{U}^3$ satisfying property **A3**.

Definition 6.2.4. The Liouville action functional $S_\Gamma : \mathcal{CM}(\Gamma\backslash\Omega) \to \mathbb{R}$ for the normalized Class A Kleinian group Γ is defined by

$$S_\Gamma[\phi] = \frac{i}{2}\left(\langle\omega[\phi], F\rangle - \langle\check{\theta}[\phi], L\rangle + \langle\check{u}, W\rangle\right), \tag{6.2.4}$$

where F is an admissible fundamental domain for Γ in Ω.

Remark 6.2.5. When Γ is a purely loxodromic Schottky group (not necessarily classical Schottky group), the Liouville action functional defined above is, up to the constant term $4\pi(2g-2)\log 2$, the functional (1.0.8), introduced by P. Zograf and L. Takhtajan [ZT87c].

Using these definitions and repeating verbatim arguments in Chapter 5 we have the following result.

Theorem 6.2.6. *(Kleinian holography) For every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ the regularized Einstein-Hilbert action is well-defined and*

$$\mathcal{E}_\Gamma[\phi] = \check{S}_\Gamma[\phi] = S_\Gamma[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^\phi d^2z + 4\pi(\chi(X_1) + \cdots + \chi_(X_n)) \log 2.$$

Corollary 6.2.7. *The definition of the Liouville action functional does not depend on the choice of admissible fundamental domain F for Γ.*

As in the Fuchsian and quasi-Fuchsian cases, the Euler-Lagrange equation for the functional S_Γ is the Liouville equation, and its single critical point given by the hyperbolic metric $e^{\phi_{hyp}}|dz|^2$ on $\Gamma\backslash\Omega$ is non-degenerate. For every component Ω_i denote by $J_i : \mathbb{U} \to \Omega_i$ the corresponding covering map (unique up to a $\mathrm{PSL}(2,\mathbb{R})$-action on $\mathbb{U}$). Then the density $e^{\phi_{hyp}}$ of the hyperbolic metric is given by

$$e^{\phi_{hyp}(z)} = \frac{|(J_i^{-1})'(z)|^2}{(\mathrm{Im}\ J_i^{-1}(z))^2} \quad \text{if } z \in \Omega_i\,, \quad i = 1, \ldots, n. \tag{6.2.5}$$

Remark 6.2.8. As in Remark 2.2.6, let $\Delta[\phi] = -4e^{-\phi}\partial_z\partial_{\bar{z}}$ be the Laplace operator of the metric $ds^2 = e^\phi|dz|^2$ acting on functions on $X_1 \sqcup \cdots \sqcup X_n$, let $\det\Delta[\phi]$ be its zeta-function regularized determinant, and let

$$\mathcal{I}[\phi] = \log\frac{\det\Delta[\phi]}{A[\phi]}.$$

Polyakov's "conformal anomaly" formula and Theorem 6.2.6 give the following relation between Einstein-Hilbert action $\mathcal{E}[\phi]$ for $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and "analytic torsion" $\mathcal{I}[\phi]$ on its boundary $X_1 \sqcup \cdots \sqcup X_n \simeq \Gamma\backslash\Omega$,

$$\mathcal{I}[\phi+\sigma] + \frac{1}{12\pi}\mathcal{E}[\phi+\sigma] = \mathcal{I}[\phi] + \frac{1}{12\pi}\mathcal{E}[\phi], \quad \sigma \in C^\infty(X_1 \sqcup \cdots \sqcup X_n, \mathbb{R}).$$

6.3 Variation of the classical action

Here we generalize theorems in Chapter 4 for quasi-Fuchsian groups to Kleinian groups.

6.3.1 Classical action

Let Γ be a normalized Class A Kleinian group and let $\mathfrak{D}(\Gamma)$ be its deformation space. For every Beltrami coefficient $\mu \in \mathcal{B}^{-1,1}(\Gamma)$ the normalized quasiconformal map $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ descends to an orientation preserving homeomorphism of the quotient Riemann surfaces $\Gamma\backslash\Omega$ and $\Gamma^\mu\backslash\Omega^\mu$. This homeomorphism extends to homeomorphism of corresponding 3-manifolds $\Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and $\Gamma^\mu\backslash(\mathbb{U}^3 \cup \Omega^\mu)$, which can be lifted to orientation preserving homeomorphism of $\mathbb{U}^3$. In particular, a fundamental region of Γ is mapped to a fundamental region of Γ^μ. Hence property **A3** is stable, so that every group in $\mathfrak{D}(\Gamma)$ is of Class A. Moreover, since ∞ is a fixed point of f^μ, every group in $\mathfrak{D}(\Gamma)$ is normalized.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let $S_{\Gamma'} = S_{\Gamma'}[\phi'_{hyp}]$ be the classical Liouville action for Γ'. Since the property of the fundamental domain F being admissible is stable, Corollary 6.2.7 asserts that the classical action gives rise to a well-defined real-analytic function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$.

As in Chapter 4, let $\vartheta \in \Omega^{2,0}(\Gamma)$ be the holomorphic quadratic differential

$\Gamma\backslash\Omega$, defined by

$$\vartheta = 2(\phi_{hyp})_{zz} - (\phi_{hyp})_z^2.$$

It follows from (6.2.5) that

$$\vartheta(z) = 2\mathcal{S}(J_i^{-1})(z) \quad \text{if } z \in \Omega_i, \;\; i = 1, \ldots, n.$$

Define a $(1,0)$-form $\boldsymbol{\vartheta}$ on $\mathfrak{D}(\Gamma)$ by assigning to every $\Gamma' \in \mathfrak{D}(\Gamma)$ corresponding $\vartheta \in \Omega^{2,0}(\Gamma')$.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let P_F and P_K be Fuchsian and Kleinian projective connections on $X_1' \sqcup \cdots \sqcup X_n' \simeq \Gamma'\backslash\Omega'$, defined by the Fuchsian uniformizations of Riemann surfaces $X_1', \ldots, X_n'$ and by their simultaneous uniformization by Kleinian group Γ'. We will continue to denote corresponding sections of the affine bundle $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ by P_F and P_K respectively. The difference $P_F - P_K$ is a $(1,0)$-form on $\mathfrak{D}(\Gamma)$. As in the Section 4.1,

$$\boldsymbol{\vartheta} = 2(P_F - P_K).$$

Correspondingly, the isomorphism

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma_1, \Omega_1) \times \cdots \times \mathfrak{D}(\Gamma_n, \Omega_n)$$

defines embeddings

$$\mathfrak{D}(\Gamma_i, \Omega_i) \hookrightarrow \mathfrak{D}(\Gamma)$$

and pull-backs S_i and $(P_F - P_K)_i$ of the function S and the $(1,0)$-form $P_F - P_K$. The deformation space $\mathfrak{D}(\Gamma_i, \Omega_i)$ describes simultaneous Kleinian uniformization of Riemann surfaces $X_1, \cdots, X_n$ by varying the complex structure on X_i and keeping the complex structures on other Riemann surfaces fixed, and the $(1,0)$-form $(P_F - P_K)_i$ is the difference of corresponding projective connections.

6.3.2 First variation

Here we compute the $(1,0)$-form ∂S on $\mathfrak{D}(\Gamma)$.

Theorem 6.3.1. *On the deformation space* $\mathfrak{D}(\Gamma)$,

$$\partial S = 2(P_F - P_K).$$

Proof. Since $F^{\varepsilon\mu} = f^{\varepsilon\mu}(F)$ is an admissible fundamental domain for $\Gamma^{\varepsilon\mu}$, and, according to Lemma 2.3.4, the 1-chain $W^{\varepsilon\mu} = f^{\varepsilon\mu}(W)$ consists of $\Gamma^{\varepsilon\mu}$-contracting paths in $\Omega^{\varepsilon\mu}$, the proof repeats verbatim the proof of Theorem 4.2.1. Namely, after the change of variables we get

$$L_\mu S = \frac{i}{2}\left(\langle L_\mu \omega, F\rangle - \langle L_\mu \check{\theta}, L\rangle + \langle L_\mu, \check{u}, W\rangle\right),$$

where

$$L_\mu \omega = \vartheta \mu dz \wedge d\bar{z} - d\xi$$

and 1-form ξ is given by (4.2.2). As in the proof of Theorem 4.2.1, setting $\chi = \delta\xi + L_\mu \check{\theta}$ we get that the 1-form χ on Ω is closed,

$$d\chi = \delta(d\xi) + L_\mu d\check{\theta} = \delta(-L_\mu \omega) + L_\mu \delta\omega = 0,$$

and satisfies

$$\delta\chi = \delta(L_\mu\check{\theta} + \delta\xi) = L_\mu\delta\check{\theta} = L_\mu\check{u} = d\delta l.$$

Since the 1-chain W consists of smooth paths or Γ-contracting paths in Ω and function δl is continuous on W, the same arguments as in the proof of Theorem 4.2.1 allow to conclude that

$$L_\mu S = \frac{i}{2}\langle \vartheta\mu dz \wedge d\bar{z}, F\rangle.$$

□

Corollary 6.3.2. *Let $X_1, \dots, X_n$ be Riemann surfaces simultaneously uniformized by a Kleinian group Γ of Class A. Then on $\mathfrak{D}(\Omega_i, \Gamma_i)$*

$$(P_F - P_K)_i = \frac{1}{2}\,\partial S_i.$$

6.3.3 Second variation

Theorem 6.3.3. *On the deformation space $\mathfrak{D}(\Gamma)$,*

$$d\boldsymbol{\vartheta} = \bar{\partial}\partial S = -2i\,\omega_{WP},$$

so that $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$.

The proof is the same as the proof of Theorem 4.3.4.

6.4 Kleinian Reciprocity

Let $\mu \in \Omega^{-1,1}(\Gamma)$ be a harmonic Beltrami differential, $f^{\varepsilon\mu}$ be corresponding normalized solution of the Beltrami equation, and let $v = \dot{f}$ be corresponding vector field on $\hat{\mathbb{C}}$,

$$v(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu(w)z(z-1)}{(w-z)w(w-1)} d^2w$$

(see Section 3.2). Then

$$\varphi_\mu(z) = \frac{\partial^3}{\partial z^3} v(z) = -\frac{6}{\pi} \iint_{\mathbb{C}} \frac{\mu(w)}{(w-z)^4} d^2w$$

is a quadratic differential on $\Gamma \backslash \Omega$, holomorphic outside the support of μ.

In [McM00] McMullen proposed the following generalization of quasi-Fuchsian reciprocity.

Theorem 6.4.1. *(McMullen's Kleinian Reciprocity) Let Γ be a finitely generated Kleinian group. Then for every $\mu, \nu \in \Omega^{-1,1}(\Gamma)$*

$$\iint_{\Gamma \backslash \Omega} \varphi_\mu \nu = \iint_{\Gamma \backslash \Omega} \varphi_\nu \mu.$$

The proof in [McM00] is based on the symmetry of the kernel $K(z, w)$, defined in Section 4.2. Here we note that Theorem 6.3.1 provides a global form of Kleinian reciprocity for Class A groups from which Theorem 6.4.1 follows immediately.

Indeed, when Γ is a normalized Class A Kleinian group, Kleinian reciprocity is the statement

$$L_\mu L_\nu S = L_\nu L_\mu S,$$

since, according to (4.4.1),

$$-\frac{1}{2}L_\mu \boldsymbol{\vartheta}(z) = -\frac{6}{\pi}\iint\limits_{\mathbb{C}} \frac{\mu(w)}{(w-z)^4}d^2w = \varphi_\mu(z)$$

and

$$\iint\limits_{\Gamma\backslash\Omega} \varphi_\mu \nu = -\frac{1}{2}\iint\limits_{\Gamma\backslash\Omega} L_\mu L_\nu S.$$

Chapter 7

Universal Teichmüller Space and Universal Teichmüller Curve

7.1 Teichmüller theory

Let $T(1)$ be the universal Teichmüller space. There are two realizations of this space.

Let Δ be the open unit disc, Δ^* the exterior of the unit disc. Let $L^\infty(\Delta)$ be the complex Banach space of bounded Beltrami differentials on Δ and let $L^\infty(\Delta)_1$ be the unit ball in $L^\infty(\Delta)$. For any $\mu \in L^\infty(\Delta)_1$, we consider the following two constructions.

1. Model A: w_μ theory.

   We extend μ by reflection to Δ^*, i.e.

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \qquad z \in \Delta^*. \tag{7.1.1}$$

There is a unique quasiconformal map w_μ, fixing $-1, -i$ and 1, such that

$$(w_\mu)_{\bar z} = \mu (w_\mu)_z \,, \tag{7.1.2}$$

$$w_\mu(z) = \overline{w_\mu\left(\frac{1}{\bar z}\right)} \,. \tag{7.1.3}$$

The second identity is due to the reflection symmetry (7.1.1). As a result, w_μ fixes the unit circle S^1. We define an equivalence relation on $L^\infty(\Delta)_1$, such that $\mu \sim_1 \nu$ if and only if w_μ and w_ν induce the same map on the unit circle, or equivalently, $w_\mu^{-1} \circ w_\nu$ is the identity on the unit circle.

2. Model B: w^μ theory.

   We extend μ to be zero outside the unit disc. There is a unique quasiconformal map w^μ, fixing $0, 1$ and ∞, holomorphic outside the unit disc, such that

   $$w^\mu_{\bar z} = \mu w^\mu_z.$$

   We define an equivalence relation on $L^\infty(\Delta)_1$, such that $\mu \sim_2 \nu$ if and only if w^μ and w^ν restrict to the same map on the unit circle (and hence on Δ^*). Or equivalently, $(w^\mu)^{-1} \circ w^\nu$ is the identity on the unit circle (and hence on Δ^*).

It is well known that w_μ is the identity map on the unit circle if and only if w^μ is the identity map on the unit circle. Hence $\sim_1$ and $\sim_2$ define the same equivalence relation $\sim$ on $L^\infty(\Delta)_1$.

The universal Teichmüller space $T(1)$ is defined as a set of equivalence classes of normalized quasiconformal maps.

$$T(1) = L^\infty(\Delta)_1 / \sim .$$

The tangent space at the origin of $T(1)$ is identified with $L^\infty(\Delta)/\mathcal{N}$, where $\mathcal{N}$ is the space of infinitesimally trivial Beltrami differentials. It can be described explicitly by :

$$\mathcal{N} = \left\{ \mu \in L^\infty(\Delta) : \int_\Delta \mu\phi = 0 \ , \ \phi \in A_1(\Delta) \right\} \ , \tag{7.1.4}$$

where $A_1(\Delta)$ is the Banach space of L^1 holomorphic functions on the disc.

Using model A, we see that $T(1)$ has a group structure given by

$$\lambda * \mu = \nu \ , \qquad \text{where } w_\nu = w_\lambda \circ w_\mu \ .$$

Explicitly, it is given by

$$\nu = \frac{\mu + (\lambda \circ w_\mu)\frac{\overline{(w_\mu)_z}}{(w_\mu)_z}}{1 + \overline{\mu}(\lambda \circ w_\mu)\frac{\overline{(w_\mu)_z}}{(w_\mu)_z}} \ .$$

Using this group structure, we have a natural way to identify the tangent spaces at different points.

For convenience, we also consider a slight modification of model B.

Model B$'$: Given $\mu \in L^\infty(\Delta)_1$, we define $\tilde{\mu} \in L^\infty(\Delta^*)_1$ by

$$\tilde{\mu}(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2} \ , \qquad z \in \Delta^* \ , \tag{7.1.5}$$

and extend it to be zero outside Δ^*. We solve the Beltrami equation

$$w^{\tilde{\mu}}_{\bar{z}} = \tilde{\mu} w^{\tilde{\mu}}_{z}$$

and get a function $w^{\tilde{\mu}}$ which is holomorphic on Δ. Moreover, $w^{\tilde{\mu}}(z) = \overline{w^{\mu}(\frac{1}{\bar{z}})}$. In this model, we identify $T(1)$ with $L^\infty(\Delta^*)_1/\sim$, where $\tilde{\mu} \sim \tilde{\nu}$ if and only if $w^{\tilde{\mu}}$ and $w^{\tilde{\nu}}$ agree on the unit circle (hence on Δ). The tangent space at the origin is identified with $L^\infty(\Delta^*)/\mathcal{N}^*$, where $\mathcal{N}^*$ is the space of infinitesimally trivial Beltrami differentials on Δ^*, defined in a similar way as (7.1.4).

The universal curve $\mathcal{T}(1)$ is a fiber space over $T(1)$, with fiber over each point $[\mu]$ the unit disc Δ with complex structure given by $w^\mu(\Delta)$ (or equivalently $w_\mu(\Delta)$). In the models of $T(1)$ described above, given a Beltrami differential μ, if we only require the solution of Beltrami equation to fix the point 1, we get a family of solutions parametrized by the points in the unit disc. Hence we can identify $\mathcal{T}(1)$ as the space of the corresponding quasiconformal maps normalized such that only the point 1 is fixed. As $T(1)$, $\mathcal{T}(1)$ also has a group structure coming from the composition of maps.

7.2 Univalent functions

Let $\mathcal{D}^*$ be the space of univalent functions $g : \Delta^* \longrightarrow \hat{\mathbb{C}}$ on the exterior disc normalized such that ∞ is fixed and the Laurent expansion at ∞ is given by:

$$g(z) = z(1 + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots). \tag{7.2.1}$$

In the model B of the universal Teichmüller space given above, every point $[\mu]$ (the equivalence class defined by μ) can be represented by the holomorphic map, which is the restriction of w^μ to the exterior disc. There is a unique way to normalize w^μ by a linear fractional transformation $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ such that $\gamma \circ w^\mu$ has Laurent expansion at ∞ given by (7.2.1). Hence $T(1)$ embeds as a subspace of $\mathcal{D}^*$

Correspondingly, we consider the space $\mathcal{D}$ of univalent functions $f : \Delta \longrightarrow \hat{\mathbb{C}}$ on the unit disc, normalized such that $f(0) = 0, f'(0) = 1$ and $f''(0) = 0$, so the Taylor expansion at the origin is given by

$$f(z) = z(1 + a_2 z^2 + a_3 z^3 + \ldots). \tag{7.2.2}$$

The model B′ of the universal Teichmüller space embeds as a subspace of $\mathcal{D}$.

It is easily seen that under the transformation $g(z) \mapsto 1/g(\frac{1}{z})$ or the transformation $g(z) \mapsto \overline{1/g(\frac{1}{\bar{z}})}$ the normalization in (7.2.1) corresponds to the normalization in (7.2.2). Hence there are two natural identifications of $\mathcal{D}^*$ with $\mathcal{D}$.

We denote by $\mathcal{S}(f)$, the Schwarzian derivative of the function f, which is given by

$$\mathcal{S}(f) = \left(\frac{f_{zz}}{f_z}\right)_z - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

It satisfies the transformation property

$$\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g \; g_z^2 + \mathcal{S}(g),$$

when one of the functions f and g is holomorphic. Moreover, $\mathcal{S}(f) = 0$ if and only if f is a linear fractional transformation. Hence a normalized univalent function is uniquely determined by its Schwarzian derivative. More precisely, we have

Lemma 7.2.1 (Nehari-Kraus, Ahlfors-Weill). *Let $f : \Delta^* \longrightarrow \hat{\mathbb{C}}$ be a univalent function on Δ^*, then*

$$\sup_{z\in\Delta^*} |\mathcal{S}(f)(z)(1-|z|^2)^2| \leq 6. \tag{7.2.3}$$

Moreover if $\phi : \Delta^* \longrightarrow \hat{\mathbb{C}}$ is a holomorphic function on Δ^* such that

$$\sup_{z\in\Delta^*} |\phi(z)(1-|z|^2)^2| < 2,$$

then there is a univalent function $f : \Delta^* \longrightarrow \hat{\mathbb{C}}$, unique up to Möbius transformations, such that $\mathcal{S}(f) = \phi$ and f has a quasiconformal extension to $\mathbb{C}$.

The same lemma holds when Δ^* is replaced everywhere by Δ.

The first half of the lemma is due to Nehari and Kraus. It implies that the model B of the universal Teichmüller space embeds into a bounded subspace of the space

$$A_\infty(\Delta^*) = \left\{ \phi \text{ holomorphic on } \Delta^* : \sup_{z\in\Delta^*} |\phi(z)(1-|z|^2)^2| < \infty \right\}$$

by

$$[\mu] \mapsto \mathcal{S}(w^\mu|_{\Delta^*}).$$

This is known as Bers embedding. The second half of the lemma is due to Ahlfors and Weill. It implies that the image of $T(1)$ in $A_\infty(\Delta^*)$ contains an open ball. Under this embedding, the tangent space to $T(1)$ is identified with $A_\infty(\Delta^*)$.

Correspondingly, the model B′ of $T(1)$ embeds into

$$A_\infty(\Delta) = \left\{\phi \text{ holomorphic on } \Delta : \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2| < \infty\right\},$$

with tangent space identified with $A_\infty(\Delta)$.

Remark 7.2.2. The space of harmonic Beltrami differentials on Δ is defined as

$$\Omega^{-1,1}(\Delta) = \left\{\mu(z) = (1-|z|^2)^2\overline{\phi(z)} : \phi \in A_\infty(\Delta)\right\}.$$

It is well known that (see, e.g. [NV90, Nag88]) $\Omega^{-1,1}(\Delta)$ is a complementary subspace to $\mathcal{N}$ in $L^\infty(\Delta)$. Hence we can identify the tangent space at the origin of $T(1)$ with $\Omega^{-1,1}(\Delta)$, which is complex anti-isomorphic to $A_\infty(\Delta)$. It should not be confused with the identification above.

7.3 Homogenuous spaces of $\mathrm{Diff}_+(S^1)$

Let $\mathrm{Diff}_+(S^1)$ be the group of orientation preserving diffeomorphisms of the unit circle S^1. Let $\mathrm{Möb}(S^1)$ be the subgroup of Möbius transformations. We abuse notation and denote also by S^1 the subgroup of rotations. The Lie algebra of $\mathrm{Diff}_+(S^1)$ is the Lie algebra of C^∞ vector fields on S^1. The complexification of this Lie algebra is the Witt algebra generated by the $L_n = e^{in\theta}\frac{\partial}{\partial\theta} = iz^{n+1}\frac{\partial}{\partial z}, n \in \mathbb{Z}$. (Here $z = e^{i\theta}$). A tangent vector to $\mathrm{Diff}_+(S^1)/S^1$ at the origin is a linear combination

$$v = \sum_{n \neq 0} c_n L_n \ , \ \overline{c_n} = c_{-n},$$

where $v = \mathrm{u}(\theta)\frac{\partial}{\partial\theta}$ is the corresponding smooth real vector field on the unit circle and c_n's are the Fourier coefficients of $\mathrm{u}(\theta)$. A tangent vector to the origin of $\mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$ is of the form

$$v = \sum_{n \neq -1,0,1} c_n L_n \ , \ \overline{c_n} = c_{-n}.$$

One loses the coefficients c_{-1}, c_0, c_1 because L_{-1}, L_0, L_1 generate the $\mathrm{PSU}(1,1)$ action.

Consider the model A of the the universal Teichmüller space $T(1)$ given above. Under the equivalence relation $\sim$, the map $[\mu] \mapsto w_\mu|_{S^1}$ is well defined and one-to-one. Ahlfors-Beurling extension theorem implies that the image consists of all normalized orientation preserving quasisymmetric homeomorphisms of the unit circle (see, e.g., [Ber72]), in other words,

$$T(1) = \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{Möb}(S^1)$$

which contains normalized orientation preserving diffeomorphisms as a subgroup. Using similar reasonings, we have the identification

$$\mathcal{T}(1) = \mathrm{Homeo}_{qs}(S^1)/S^1.$$

In [Kir87], Kirillov proved that there is a natural isomorphism between the space of smooth contours with conformal radius 1 which contain 0 in their interior and the space $\mathrm{Diff}_+(S^1)/S^1$. This can be generalized to an isomorphism between $\mathcal{T}(1) = \mathrm{Homeo}_{qs}(S^1)/S^1$ and the space of all quasi-circles (image of the unit circle under a quasiconformal map) of conformal radius 1 which contain 0 in their interior. Moreover, as in [Kir87] there are two natural holomorphic functions associated to a point in $\mathcal{T}(1) = \mathrm{Homeo}_{qs}(S^1)/S^1$. For the space $T(1) = \mathrm{Homeo}_{qs}(S^1)/\mathrm{M\ddot{o}b}(S^1)$, this association is well known to Ahlfors and Bers. Since this is going to play an important role in our discussion below, we give the details for our case $\mathcal{T}(1) = \mathrm{Homeo}_{qs}(S^1)/S^1$.

Given an orientation preserving quasisymmetric homeomorphism γ of the unit circle, by Ahlfors-Beurling extension theorem, γ can be extended to be a quasiconformal map w of the plane such that it satisfies the reflection property (7.1.3). Let μ be the Beltrami differential of the map w. Up to a linear fractional transformation, w agrees with w_μ that we define in Section 7.1, i.e. $w = \sigma_1 \circ w_\mu$ for some $\sigma_1 \in \mathrm{PSU}(1,1)$. The corresponding map $w^{\tilde{\mu}}$ (Section 7.1) is holomorphic inside the unit disc Δ. Define $g = \sigma_2 \circ w^{\tilde{\mu}} \circ w^{-1}$, where $\sigma_2 \in \mathrm{PSL}(2,\mathbb{C})$ is uniquely determined by the conditions $f = \sigma_2 \circ w^{\tilde{\mu}}$ satisfies $f(0) = 0$, $f'(0) = 1$ and g satisfies $g(\infty) = \infty$. The maps f and g are holomorphic inside Δ and Δ^* respectively. They do not depend on the extension of γ and we have $\gamma = g^{-1} \circ f|_{S^1}$. The image of S^1 under f, which is the same as the image of S^1 under g, is by definition a quasi-circle $\mathcal{C}$ with conformal radius 1. By post-composing w with a rotation, the map g also satisfies $g'(\infty) > 0$.

Conversely, by definition a quasi-circle $\mathcal{C}$ with conformal radius 1 containing the origin, is the image of S^1 under a quasiconformal map $h : \mathbb{C} \to \mathbb{C}$. Let μ_1 be the Beltrami differential of $h|_\Delta$, extended to Δ^* by reflection. Let w_{μ_1}

be a solution of the corresponding Beltrami equation. Then $f = h \circ w_{\mu_1}^{-1}$ is a quasiconformal map that is holomorphic inside Δ. There is a unique way to normalize w_{μ_1} by post-composition with a PSU$(1,1)$ transformation such that $f(0) = 0$ and $f'(0) > 0$. The image of S^1 under f is the quasi-circle $\mathcal{C}$. In fact by Riemann mapping theorem, $f|_\Delta$ is uniquely determined by $\mathcal{C}$ and the normalization conditions $f(0) = 0$, $f'(0) > 0$. $\mathcal{C}$ has conformal radius 1 implies that $f'(0) = 1$. Let μ be the Beltrami differential of $f|_{\Delta^*}$, extended to Δ by reflection. Let w_μ be a solution of the corresponding Beltrami equation. Define $g = f \circ w_\mu^{-1} \circ \sigma$, where $\sigma \in$ PSU$(1,1)$ is uniquely determined so that $g(\infty) = \infty$ and $g'(\infty) > 0$. The map $\gamma = g^{-1} \circ f|_{S^1}$ is then an orientation preserving quasisymmetric homeomorphism of the unit circle.

This establishes a one-to-one correspondence between $\mathcal{T}(1)$ and the space of all quasi-circles with conformal radius 1 that contain the origin in their interior. We also establish the decomposition of an orientation preserving quasisymmetric homeomorphism of the unit circle $\gamma \mod S^1$ as $g^{-1} \circ f|_{S^1}$, where f is a holomorphic map from Δ onto the interior Ω of the quasi-circle $\mathcal{C}$ corresponding to γ and g is a holomorphic map from Δ^* onto the exterior Ω^* of $\mathcal{C}$, uniquely determined so that $f(0) = 0$, $f'(0) = 1$, $g(\infty) = \infty$ and $g'(\infty) > 0$. Using the fact that the correpondence between f and the quasi-circle $\mathcal{C}$ is one-to-one, we see that we can identify $\mathcal{T}(1)$ with the space of univalent functions

$$\tilde{\mathcal{D}} = \{f : \Delta \longrightarrow \hat{\mathbb{C}} \text{ a univalent function } : f(0) = 0, f'(0) = 1,$$
$$f \text{ has a quasiconformal extension to } \mathbb{C}\}.$$

In this picture, the tangent space to $\tilde{\mathcal{D}}$ at the origin is a subspace of

$$\hat{\mathcal{D}} = \{u : \Delta \longrightarrow \hat{\mathbb{C}} \text{ holomorphic } : u(0) = u'(0) = 0,$$
$$u \text{ has a continuous extension to } \mathbb{C}\}.$$

Remark 7.3.1. Notice that if $\gamma = w_\mu|_{S^1}$ up to post-composition with a PSU$(1,1)$ transformation, the corresponding f and g are equal to w^μ and $w^{\tilde{\mu}'}$ up to post-composition with PSL$(2,\mathbb{C})$ transformations, where $\mu' * \mu = 0$.

For more details about quasi-circles, see [Leh87, Pom92].

7.3.1 Complex structures

The almost complex structure J at the origins of $\mathcal{T}(1) \simeq \mathrm{Homeo}_{qs}(S^1)/S^1$ and $T(1) \simeq \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ is defined by:

$$Jv = \sum_n i sgn(n) c_n e^{in\theta}, \qquad \text{where } v = \sum_n c_n e^{in\theta}. \tag{7.3.1}$$

(See references in [NV90]. Notice that we differ from the definition in [NV90] by a negative sign). Hence the holomorphic tangent vectors are of the form

$$w = \frac{v - iJv}{2} = \sum_{n>0} c_n e^{in\theta}$$

and the antiholomorphic tangent vectors are of the form

$$\bar{w} = \frac{v + iJv}{2} = \sum_{n<0} c_n e^{in\theta}.$$

The corresponding holomorphic and antiholomorphic derivatives in the direction w are defined as

$$\frac{\partial}{\partial \epsilon_w} = \frac{1}{2}\left(\frac{\partial}{\partial t_v} - i\frac{\partial}{\partial t_{Jv}}\right),$$
$$\frac{\partial}{\partial \bar{\epsilon}_w} = \frac{1}{2}\left(\frac{\partial}{\partial t_v} + i\frac{\partial}{\partial t_{Jv}}\right),$$

where $\frac{\partial}{\partial t_v}$ means the Lie derivative in the direction v.

In [NV90], Nag and Verjoysky proved that the almost complex structure is integrable and corresponding complex structure coincides with the complex structure on $T(1)$, which is induced from the complex structure of $L^\infty(\Delta)$. Adapting their proof to our convention, we immediately see that the complex structure J on $\mathcal{T}(1)$ coincides with the complex structure induced from $\tilde{\mathcal{D}}$.

Remark 7.3.2. If we use model B of $T(1)$ for Bers embedding, the complex structure pull back from $A_\infty(\Delta^*)$ will be the opposite to the complex structure above. This explain our preference for model B′.

7.4 Metrics

In [Kir87] and [KY87] , Kirillov and Yuriev studied Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$. It is known that the homogeneuos Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$ must be of the form

$$\| v \|^2 = \sum_{n>0}(an^3 + bn)|c_n|^2 , \qquad v = \sum_{n\neq 0} c_n L_n. \tag{7.4.1}$$

They gave a potential to the Kähler metric $\| v \|^2 = \sum_{n>0} n|c_n|^2$. We call this particular metric Kirillov metric.

On the other hand, in order that (7.4.1) defines a metric on $\mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$, it is necessary that $an^3 + bn = 0$ for $n = -1, 0, 1$. This implies that up to a constant, there is a unique homogeneuos Kähler metric on $\mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$ given by

$$\| v \|^2 = \frac{\pi}{2} \sum_{n>0} (n^3 - n)|c_n|^2. \tag{7.4.2}$$

Let Γ be a Fuchsian group realized as a subgroup of $\mathrm{PSU}(1,1)$. Let $L^\infty(\Delta, \Gamma)$ be the space of Beltrami differentials for Γ, i.e.

$$L^\infty(\Delta, \Gamma) = \left\{ \mu \in L^\infty(\Delta) : \mu \circ \gamma \frac{\overline{\gamma'}}{\gamma'} = \mu, \forall \gamma \in \Gamma \right\} .$$

The Teichmüller space of Γ, $\mathfrak{T}(\Gamma)$ is the subspace of the universal Teichmüller space

$$\mathfrak{T}(\Gamma) = L^\infty(\Delta, \Gamma)_1 / \sim ,$$

where

$$L^\infty(\Delta, \Gamma)_1 = L^\infty(\Delta)_1 \cap L^\infty(\Delta, \Gamma) ,$$

and $\sim$ is the same equivalence relation we use to define $T(1)$. The tangent space at the origin of $\mathfrak{T}(\Gamma)$ is identified with the space of harmonic Beltrami differentials for Γ

$$\Omega^{-1,1}(\Delta,\Gamma)=\Omega^{-1,1}(\Delta)\cap L^{\infty}(\Delta,\Gamma)\ .$$

When Γ is a cofinite Fuchsian group, i.e. the quotient Riemann surface $\Gamma\backslash\Delta$ has finite hyperbolic area, it is well known that there is a canonical Hermitian metric on $\mathfrak{T}(\Gamma)$ given by

$$\langle\mu,\nu\rangle=\mathrm{Re}\iint\limits_{\Gamma\backslash\Delta}\mu\overline{\nu}\rho\ ,\qquad \mu,\nu\in\Omega^{-1,1}(\Delta,\Gamma)\ ,$$

where ρ is the area form of the hyperbolic metric on Δ. This metric is called Weil-Petersson metric. The notation $T(1)$ for the universal Teichmüller space indicates that it corresponds to the group Γ being the trivial group $\{\mathrm{id}\}$. This suggests to define the Weil-Petersson metric on $T(1)$ by

$$\langle\mu,\nu\rangle=\mathrm{Re}\iint\limits_{\Delta}\mu\overline{\nu}\rho\ ,\qquad \mu,\nu\in\Omega^{-1,1}(\Delta).$$

However, this integral does not converge for all $\mu,\nu\in\Omega^{-1,1}(\Delta)$. In particular, it diverges when both μ,ν are Beltrami differentials of a Fuchsian group that contains infinitely many elements.

Let $\mu\in L^{\infty}(\Delta)$ be a tangent vector at the origin of $T(1)$. It generates the one-parameter flow $w_{t\mu}$ and the corresponding vector field is given by $\dot{w}_{\mu}\frac{\partial}{\partial z}$,

where

$$\dot{w}_\mu(z) = \frac{1}{2\pi i} \iint\limits_{\mathbb{C}} \hat{\mu}(\zeta) R(\zeta, z) d\zeta \wedge d\bar{\zeta},$$

$$R(\zeta, z) = \frac{z(z-1)}{(\zeta - z)\zeta(\zeta - 1)}$$

and $\hat{\mu}$ is the extension of μ by reflection to $\mathbb{C}$. Restricted to S^1, we have $\dot{w}_\mu(z) = iz\mathfrak{u}(z)$, where $\mathfrak{u}(\theta)\frac{\partial}{\partial\theta}$ is the induced vector field on S^1.

In [NV90], Nag and Verjoysky proved that the pull back of the canonical Weil-Petersson metric on $T(1)$ to $\mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$ coincides with the unique homogenuous Kähler metric (7.4.2). In particular, the Weil-Petersson metric on $T(1)$ is convergent on the Sobolev class $H^{\frac{3}{2}}$ vector fields, which contains the C^2 class vector fields. Here the Sobolev space $H^s(S^1)$ is defined as

$$H^s(S^1) = \left\{ \mathfrak{u}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \mathfrak{a}_n e^{in\theta} : \sum_{n\in\mathbb{Z}} |n|^{2s} |\mathfrak{a}_n|^2 < \infty \right\}.$$

On the other hand, since $\mathcal{T}(1)$ contains $\mathrm{Diff}_+(S^1)/S^1$ as a natural subspace, we can extend Kirillov metric to $\mathcal{T}(1)$. At the origin, it is of the form

$$\| v \|^2 = \sum_{n>0} n |c_n|^2,$$

where $v = \sum_{n\neq 0} c_n e^{in\theta}\frac{\partial}{\partial\theta}$ is the corresponding vector field on the unit circle. We are going to prove that this series is always convergent. By using the group structure on $\mathcal{T}(1)$, we can transport this metric to every point on $\mathcal{T}(1)$ by right action.

Remark 7.4.1. It was proved by Reimann that the tangent space at the origin of $T(1) \simeq \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ is the Zygmund class $\Lambda(S^1)$ (see [Rei76, GS92]). However, we don't know the characterization of this class using Fourier coefficients on S^1.

7.5 Identification of tangent spaces

We have the following isomorphism which relates the two models of $\mathcal{T}(1)$.

$$\mathcal{W} : \mathrm{Homeo}_{qs}(S^1)/S^1 \longrightarrow \tilde{\mathcal{D}},$$
$$\gamma \mapsto f.$$

We want to compute the derivative of this map at the origin.

Consider the smooth one parameter flow $\gamma^t = (g^t)^{-1} \circ f^t|_{S^1}$ from the origin. From the theory of quasiconformal mappings (see, e.g., [Leh87]), we know that γ^t has a quasiconformal extension to $\mathbb{C}$, smooth on $\mathbb{C} \setminus S^1$, which depends smoothly on the parameter t. Since f^t and g^t are conformal on Δ and Δ^* respectively, this implies that f^t and g^t also have quasiconformal extensions to $\mathbb{C}$, smooth on $\mathbb{C}\setminus S^1$. In what follows, we regard γ^t, f^t and g^t as quasiconformal mappings on the plane. The corresponding vector fields

$$\frac{d}{dt}\gamma^t, \qquad \frac{d}{dt}f^t, \qquad \text{and} \qquad \frac{d}{dt}g^t$$

are continuous on $\mathbb{C}$, smooth on $\mathbb{C} \setminus S^1$.

We write

$$f^t(z) = z + tu + O(t^2) = z + tz(a_1 z + a_2 z^2 + \dots) + O(t^2),$$

for $z \in \Delta$ and

$$g^t(z) = z + tv + O(t^2) = z + tz(b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots) + O(t^2),$$

for $z \in \Delta^*$.

We denote by

$$\dot{\gamma} = \frac{d}{dt}\gamma^t\Big|_{t=0}, \qquad \dot{f} = \frac{d}{dt}f^t\Big|_{t=0} \quad \text{and} \quad \dot{g} = \frac{d}{dt}g^t\Big|_{t=0},$$

then $\dot{f}|_\Delta = u$ and $\dot{g}|_{\Delta^*} = v$.

As we mention in Section 7.2, $\mathcal{S}(f^t|_\Delta)$ belongs to a bounded subspace of $A_\infty(\Delta)$ and the corresponding tangent vector at the origin is

$$u_{zzz} = \frac{d}{dt}\mathcal{S}(f^t|_\Delta)\Big|_{t=0} \in A_\infty(\Delta).$$

Remark 7.5.1. This is not a priori clear that the embedding $T(1)$ into an open ball of the Banach space $A_\infty(\Delta)$, identifies the tangent space of $T(1) \simeq (L^\infty(\Delta^*)_1/\sim)$ with $A_\infty(\Delta)$. We sketch a proof here (see details in [Nag88]). Given a Beltrami differential $\tilde{\mu} \in L^\infty(\Delta^*)$, the curve $t \mapsto w^{t\tilde{\mu}}$ in $T(1)$ gives the curve $t \mapsto \mathcal{S}(w^{t\tilde{\mu}}|_\Delta)$ in $A_\infty(\Delta)$. We have to show that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{S}(w^{t\tilde{\mu}}|_\Delta) \in A_\infty(\Delta).$$

This is proved by using the explicit formula

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{S}(w^{t\tilde{\mu}}|_\Delta) = \dot{w}^{\tilde{\mu}}_{zzz}(z) = -\frac{6}{\pi}\iint\limits_\Delta \frac{\overline{\mu(w)}}{(1-z\bar{w})^4}\frac{|dw\wedge d\bar{w}|}{2},$$

where $\mu \in L^\infty(\Delta)$ is the reflection of $\tilde{\mu} \in L^\infty(\Delta^*)$ and the identity

$$\frac{1}{2\pi}\iint_\Delta \frac{|dw\wedge d\bar{w}|}{|1-z\bar{w}|^4} = (1-|z|^2)^{-2},$$

(see, e.g., [Kra72a]) implies that

$$\sup_{z\in\Delta} |\dot{w}^{\tilde{\mu}}_{zzz}(z)(1-|z|^2)^2| < \infty.$$

We need the following two theorems.

Theorem 7.5.2. *Let $Q \in A_\infty(\Delta)$, then the series $\sum_{n=2}^\infty n|a_n|^2$ is convergent, where $Q(z) = \sum_{n=2}^\infty (n^3-n)a_n z^{n-2}$.*

Proof. $Q \in A_\infty(\Delta) = \{\phi$ holomorphic on $\Delta : \sup_{z\in\Delta}|\phi(z)(1-|z|^2)^2| < \infty\}$ implies

$$\iint\limits_\Delta |Q(z)(1-|z|^2)^2|^2 dxdy < \infty,$$

where $z = x+iy$. But this integral is equal to

$$24\pi\sum_{n=2}^\infty \frac{n^3-n}{(n+2)(n+3)}|a_n|^2.$$

Hence

$$\sum_{n=2}^{\infty} n|a_n|^2 \leq \sum_{n=2}^{\infty} \frac{n^3-n}{(n+2)(n+3)}|a_n|^2 < \infty.$$

□

Theorem 7.5.3 ([Zyg88]). *If the function $f(z) = \mathfrak{a}_0 + \mathfrak{a}_1 z + \ldots + \mathfrak{a}_n z^n + \ldots$ is holomorphic for $|z| < 1$ and continuous for $|z| \leq 1$, and if the series $\sum_n n|\mathfrak{a}_n|^2$ is convergent, then the series*

$$\mathfrak{a}_0 + \mathfrak{a}_1 e^{i\theta} + \ldots + \mathfrak{a}_n e^{in\theta} + \ldots$$

converges uniformly to $F(e^{i\theta})$ in $0 \leq \theta \leq 2\pi$.

Since $u = \sum_{n=1}^{\infty} a_n z^{n+1}$ is holomorphic on Δ and continuous on $\mathbb{C}$, the theorems above imply that the series

$$\sum_{n=1}^{\infty} a_n e^{i(n+1)\theta}$$

converges uniformly to the continuous function $u|_{S^1}(e^{i\theta})$ on the unit circle S^1.

Similar arguments imply that the series

$$\sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}$$

converges uniformly to the continuous function $v|_{S^1}(e^{i\theta})$ on S^1.

Taking the derivative with respect to t on the relation $\gamma^t = (g^t)^{-1} \circ f^t$ and setting $t = 0$, we have

$$\dot{\gamma} = -\dot{g} + \dot{f}. \tag{7.5.1}$$

This shows that the series

$$\sum_{n=1}^{\infty} a_n e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}$$

converges uniformly to the function $(u-v)|_{S^1}$, which is $\dot{\gamma}|_{S^1}$. In particular it is the Fourier series of the function $\dot{\gamma}|_{S^1}$. Let

$$\mathrm{u}(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \frac{\partial}{\partial\theta}, \qquad c_{-n} = \overline{c_n}.$$

be the corresponding tangent vector, so that $\dot{\gamma} = iz\mathrm{u}(z)$ on S^1. Hence we have

$$i\sum_{n\in Z} c_n e^{i(n+1)\theta} = \sum_{n=1}^{\infty} a_n e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}.$$

Comparing coefficients, we have

$$a_n = ic_n\ , \qquad b_n = -ic_{-n} \qquad n \geq 1, \tag{7.5.2}$$

which implies

$$a_n = \overline{b_n}. \tag{7.5.3}$$

Theorem 7.5.4. *The derivative of $\mathcal{W}$ at the origin is*

$$D_0\mathcal{W} : T_0 Homeo_{qs}(S^1)/S^1 \to \hat{\mathcal{D}}$$
$$\sum_{n\neq 0} c_n e^{in\theta} \mapsto i\sum_{n=1}^{\infty} c_n z^{n+1}.$$

By imposing extra normalization conditions, we can pass from the models for $\mathcal{T}(1)$ to the models for $T(1)$, where the corresponding tangent vectors have no $n = \pm 1$ components.

Remark 7.5.5. In [Nag93], Nag proved a similar theorem by using explicit formulas for $\dot{\gamma}$, $\dot{f}$ and $\dot{g}$ from the theory of quasiconformal mappings. Here we use a slightly different approach, and justify that the Fourier series for $\dot{\gamma}|S^1$ indeed converges to $\dot{\gamma}|S^1$.

On the other hand, we have another isomorphism that relates the other models of $T(1)$.

$$\begin{aligned} \mathcal{B} : \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1) &\to L^\infty(\Delta)_1/\sim \to A_\infty(\Delta), \\ \gamma &\mapsto [\mu] \mapsto \mathcal{S}(w^{\tilde{\mu}}|_\Delta), \end{aligned}$$

where $\gamma = w_\mu|_{S^1}$. The map $w^{\tilde{\mu}}$ coincides with the map f up to a linear fractional transformation. However, infinitesimally composing with linear fractional transformation only affects the $n = -1, 0, 1$ components of the vector fields. Hence our argument above gives immediately

Theorem 7.5.6. *The derivative of the map $\mathcal{B}$ at the origin is*

$$\begin{aligned} D_0\mathcal{B} : T_0 Homeo_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1) &\to A_\infty(\Delta), \\ \sum_{n \neq -1,0,1} c_n e^{in\theta} &\mapsto i \sum_{n=2}^{\infty} (n^3 - n) c_n z^{n-2}. \end{aligned}$$

Using this identification, we get

Corollary 7.5.7. *The Weil-Petersson metric defined on $Q(z) = \sum_{n=2}^{\infty}(n^3 -$*

$n)a_n z^{n-2} \in A_\infty(\Delta)$ *which corresponds to* $H^{\frac{3}{2}}$ *vector fields is given by*

$$\| Q \|_{WP}^2 = \frac{\pi}{2} \sum_{n=2}^{\infty} (n^3 - n)|a_n|^2 = \frac{1}{4} \iint_{\Delta} |Q(z)|^2 (1 - |z|^2)^2 dxdy$$

Remark 7.5.8. The map $\dot{f} \mapsto \dot{f}_{zzz}$ can be viewed as sending vector fields to quadratic differentials. The theorem above implies that the Weil-Petersson metric on $A_\infty(\Delta)$ pushed forward by the embedding $T(1) \hookrightarrow A_\infty(\Delta)$ is the usual Weil-Petersson metric defined on the space of quadratic differentials. In particular, we have

$$\| Q \circ \gamma(\gamma')^2 \|_{WP}^2 = \| Q \|_{WP}^2, \qquad \text{for all } \gamma \in \mathrm{PSU}(1,1).$$

Remark 7.5.9. All the results above can be restricted to the finite dimensional Teichmüller spaces embeded in the universal Teichmüller space.

Chapter 8

Velling's Hermitian Form and Kirillov's Metric

8.1 Spherical area theorem

We denote by $A_S(\Omega)$ the spherical area of a domain Ω in $\hat{\mathbb{C}}$. It is given by

$$A_S(\Omega) = \iint_{\Omega} \frac{4dxdy}{(1+|z|^2)^2}.$$

Notice that it is invariant under rotation, i.e. $\Omega \mapsto e^{i\theta}(\Omega)$.

Following Velling [Vel], given $Q \in A_\infty(\Delta)$, consider the one parameter family of functions f^{tQ} satisfying $\mathcal{S}(f^{tQ}) = tQ$, normalized such that it belongs to $\mathcal{D}$. We consider the spherical area of the domains $\Omega_t = f^{tQ}(\Delta)$

$$\begin{aligned} A_S(\Omega_t) &= \iint_{\Omega_t} \frac{4dxdy}{(1+|z|^2)^2} \\ &= 4\iint_{\Delta} \frac{|df^{tQ}|^2}{(1+|f^{tQ}|^2)^2}. \end{aligned}$$

Velling's spherical area theorem says that

Theorem 8.1.1 (Velling[Vel]). *Let $Q \in A_\infty(\Delta)$, we have*

$$\frac{d}{dt} A_S(f^{tQ}(\Delta))|_{t=0} = 0, \tag{8.1.1}$$

$$\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))|_{t=0} \geq 0. \tag{8.1.2}$$

The first identity follows from the following theorem.

Theorem 8.1.2 (Velling[Vel]). *Let $f : \Delta \longrightarrow \hat{\mathbb{C}}$ be a univalent function (perhaps meromorphic) such that it has Taylor expansion $f(z) = z(1 + a_2 z^2 + a^3 z^3 + \ldots)$ at the origin. Then the spherical area $A_S(f(\Delta))$ satisfies*

$$A_S(f(\Delta)) \geq 2\pi,$$

with equality if and only if $f = \mathrm{id}$.

It follows from the classical area theorem.

Theorem 8.1.3. *Let $g : \Delta^* \longrightarrow \hat{\mathbb{C}}$ be a univalent function such that it has Laurent expansion at ∞ $g(z) = z + b_0 + b_1 z^{-1} + \ldots$. Then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

The second inequality in Velling's spherical area theorem implies that $\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))|_{t=0}$ is a candidate for a metric on $A_\infty(\Delta)$, which in turn induces a metric on $T(1)$. Our goal is to compute this metric explicitly and compare to the Kirillov metric and Weil-Petersson metric.

We state a lemma that is very useful for the computation:

Lemma 8.1.4 ([Zyg88]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on Δ and $\phi(r)$ an integrable function of r on $[0,1)$, then*

$$\iint\limits_{\Delta} \phi(|z|) Re\,(f(z)) dxdy = 2\pi Re\,(a_0) \int_0^1 \phi(r) dr,$$

$$\iint\limits_{\Delta} \phi(|z|) |f(z)|^2 dxdy = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 \phi(r) r^{2n+1} dr.$$

8.2 Velling's Hermitian form

Given $Q \in A_\infty(\Delta)$, we want to compute Velling's bilinear form $\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))|_{t=0}$. For t sufficiently small, let $f^{tQ} : \Delta \longrightarrow \hat{\mathbb{C}}$ be the continuous family of normalized univalent functions such that $\mathcal{S}(f^{tQ}) = tQ$. For t small, we write the perturbative expansions

$$f^{tQ}(z) = z + tu(z) + t^2 v(z) + O(t^3), \tag{8.2.1}$$

$$u(z) = z(a_2 z^2 + a_3 z^3 + \ldots) = \sum_{n=2}^{\infty} a_n z^{n+1}, \tag{8.2.2}$$

$$v(z) = z(b_2 z^2 + b_3 z^3 + \ldots) = \sum_{n=2}^{\infty} b_n z^{n+1}. \tag{8.2.3}$$

Taking t derivative and setting $t = 0$, we see that the relation between $u(z)$ and $Q(z)$ is given by

$$\frac{\partial^3}{\partial z^3} u(z) = Q(z),$$

i.e.

$$Q(z) = \sum_{n=2}^{\infty}(n^3 - n)a_n z^{n-2}.$$

We want to compute $A_S(f^{tQ}(\Delta))$ to find the term in t^2. We have

$$A_S(f^{tQ}(\Delta)) = 4\iint\limits_{\Delta} \frac{|f_z^{tQ}|^2 dxdy}{(1+|f^{tQ}|^2)^2},$$

$$\frac{|f_z^{tQ}|^2}{(1+|f^{tQ}|^2)^2} = \frac{|1+tu_z+t^2v_z|^2}{(1+|z+tu+t^2v|^2)^2} + O(t^3).$$

Hence the t^2 term is

$$\frac{v_z+\overline{v_z}+|u_z|^2}{(1+|z|^2)^2} - 2\frac{z\overline{v}+\bar{z}v+|u|^2+(z\overline{u}+\bar{z}u)(u_z+\overline{u_z})}{(1+|z|^2)^3} + 3\frac{(z\overline{u}+\bar{z}u)^2}{(1+|z|^2)^4}.$$

Using the series expansion (8.2.1) of u and v, we find that since v does not have constant term and terms in z, v will drop out from the integration. By applying Lemma 8.1.4 we get the t^2 term in $A_S(f^{tQ}(\Delta))$ is given by:

$$8\pi\sum_{n=2}^{\infty}\mathfrak{c}_n|a_n|^2$$

where

$$\mathfrak{c}_n = \int_0^1 \frac{6r^{2n+4}}{(1+r^2)^4} - \frac{(4n+6)r^{2n+2}}{(1+r^2)^3} + \frac{(n+1)^2r^{2n}}{(1+r^2)^2} rdr.$$

We compute $\mathfrak{c}_n$ by making a change of variable and repeatedly using integration by parts:

$$\mathfrak{c}_n = \frac{1}{2}\int_0^1 \frac{6r^{n+2}}{(1+r)^4} - \frac{(4n+6)r^{n+1}}{(1+r)^3} + \frac{(n+1)^2 r^n}{(1+r)^2}dr,$$

$$\int_0^1 \frac{r^{n+2}}{(1+r)^4}dr = -\frac{2n^2+7n+7}{24} + \frac{n(n+1)(n+2)}{6}\int_0^1 \frac{r^{n-1}}{1+r}dr,$$

$$\int_0^1 \frac{r^{n+1}}{(1+r)^3}dr = -\frac{2n+3}{8} + \frac{n(n+1)}{2}\int_0^1 \frac{r^{n-1}}{1+r}dr,$$

$$\int_0^1 \frac{r^n}{(1+r)^2}dr = -\frac{1}{2} + n\int_0^1 \frac{r^{n-1}}{1+r}dr.$$

Substituting into $\mathfrak{c}_n$, all the terms with integrals cancel and we are left with

$$\mathfrak{c}_n = \frac{n}{8}.$$

Therefore, we have

Theorem 8.2.1. *Let $Q \in A_\infty(\Delta)$, then*

$$\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))|_{t=0} = 2\pi\sum_{n=2}^{\infty} n|a_n|^2.$$

Theorem 7.5.2 implies that the series is convergent for all $Q \in A_\infty(\Delta)$. Hence, we can define a Hermitian bilinear form on $A_\infty(\Delta)$ by

$$\| Q \|_S^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))|_{t=0} = \sum_{n=2}^{\infty} n|a_n|^2,$$

$$\text{where} \quad Q(z) = \sum_{n=2}^{\infty}(n^3-n)a_n z^{n-2},$$

which we called Velling's Hermitian form.

8.3 Kirillov metric

We want to define a homogenuous metric on $\mathcal{T}(1)$. Using the group structure, it is sufficient to define a Hermitian metric on the tangent space at the origin. Using Velling's approach, given the tangent vector $v = \sum_{n\neq 0} c_n e^{in\theta}$ at the origin with the associated one parameter flow $\gamma^t = (g^t)^{-1} \circ f^t|_{S^1}$, we define a Hermitian form by

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}\Big|_{t=0} A_S(f^t(\Delta)).$$

The proof in Section 8.2 holds with an extra term $n = 1$ (notice that we only need the fact there are no constant terms and terms in z in the first and second order perturbations), and we get

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}\Big|_{t=0} A_S(f^t(\Delta) = \sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=1}^{\infty} n|c_n|^2$$

which is just the Kirillov metric.

As we mentioned in Section 7.4, all the homogenuous Kähler metric on $\mathrm{Diff}_+(S^1)/S^1$ must be of the form

$$\| v \|^2 = \sum_{n>0} (an^3 + bn)|c_n|^2. \qquad (8.3.1)$$

Since $A_\infty(\Delta)$ is a codimension 1 subspace of the tangent space at the origin of $\mathcal{T}(1)$, using Theorem 7.5.2, we see that the Kirillov metric converges and is well defined everywhere on $\mathcal{T}(1)$. On the other hand, up to a constant, the

Weil-Petersson metric on $A_\infty(\Delta)$ is given by

$$\| v \|^2 = \sum_{n>0} (n^3 - n)|c_n|^2.$$

We have seen that it does not converge at all elements of $A_\infty(\Delta)$. Since every metric given by (8.3.1) can be written as a linear combination of the Kirillov metric and the Weil-Petersson metric, we have

Theorem 8.3.1. *The Kirillov metric is the unique homogenuous Kähler metric on $\mathcal{T}(1)$.*

Here 'unique' means unique up to a constant.

Chapter 9

Metrics on Teichmüller Spaces

9.1 Universal Teichmüller space

Since the spherical area is only invariant under the rotation group S^1, and not the whole group of isometries of the disc–PSU$(1,1)$, we need to average the Kirillov metric over the disc to get a homogenuous metric on the universal Teichmüller space.

We identify $A_\infty(\Delta)$ as the tangent space to the universal Teichmüller space. When we choose a different base point $w \in \Delta$, we translate it to the origin by $\gamma_w^{-1}(z) = \frac{z-w}{1-\overline{w}z} \in \mathrm{PSU}(1,1)$. The tangent vector Q at the point w is identified with the tangent vector $Q_w = Q \circ \gamma_w(\gamma_w')^2 \in A_\infty(\Delta)$ at the origin, where $\gamma_w(z) = \frac{z+w}{1+\overline{w}z}$. We define the Velling metric $\| Q \|_V^2$ to be the average of $\| Q_w \|_S^2$ over the unit disc, using the invariant hyperbolic metric, i.e.,

$$\| Q \|_V^2 = \iint\limits_{\Delta} \| Q_w \|_S^2 \, \frac{4dxdy}{(1-|w|^2)^2}$$

where $w = x + iy$. To be more precise, we have a metric that is defined on the fiber space $\mathcal{T}(1)$, in order to project to an invariant metric on the quotient space $T(1)$, we use a familiar technique in compact group theory: averaging over the group action. Here we are averaging over the PSU$(1,1)$ group action, or equivalently PSU$(1,1)/S^1$ since the metric is already S^1 invariant. Notice however that, since PSU$(1,1)$ is not compact, it is not a priori clear that this averaging converge.

Remark 9.1.1. We slightly abuse terminology here. We call $\| Q \|_S^2$ Velling Hermitian form and $\| Q \|_V^2$ Velling metric although the latter is not the metric correponding to the former Hermitian form, but rather its average.

As a digression, we identify the corresponding flow for the tangent vector $Q \circ \gamma(\gamma')^2$.

Theorem 9.1.2. *Let $Q \in A_\infty(\Delta)$. Given that the normalized solution to the equation $\mathcal{S}(f^t) = tQ$ is $f^t(z) = z + tu(z)$ up to the first order in t. Then the solution to $\mathcal{S}(f^t) = t(Q \circ \gamma(\gamma')^2)$ up to the first order in t is $f^t(z) = z + t(\frac{u\circ\gamma}{\gamma'}(z) + p(z))$, where $\gamma \in$ PSU$(1,1)$ and $p(z)$ is a polynomial of degree 2 such that the solution is normalized.*

Proof. If $f^t(z) = z + tv(z) + O(t^2)$ is a solution to $\mathcal{S}(f^t) = t(Q\circ\gamma(\gamma')^2)$, taking derivative with respect to t and setting $t = 0$, we have

$$\frac{\partial^3}{\partial z^3} v(z) = \left(Q \circ \gamma(\gamma')^2\right)(z).$$

But we also have

$$\frac{\partial^3}{\partial z^3} u(z) = Q(z).$$

It is well known in the theory of Eichler integrals that this implies

$$\frac{\partial^3}{\partial z^3}\frac{u\circ\gamma}{\gamma'}(z) = \left(Q\circ\gamma(\gamma')^2\right)(z).$$

Hence the difference between $v(z)$ and $\frac{u\circ\gamma}{\gamma'}(z)$ is a polynomial $p(z)$ of degree 2, which is uniquely determined so that the coefficients of the constant term and the terms in z and z^2 of $\frac{u\circ\gamma}{\gamma'}(z) + p(z)$ vanish. □

Since the metric $\| Q \|_S^2$ is expressed in terms of the norm square of the corresponding coefficients $|a_n|^2$, it is sufficient if we can average $|a_n|^2$ for $n \geq 2$.

We denote by

$$Q_w = Q\circ\gamma_w(\gamma_w')^2 = \sum_{n=2}^{\infty}(n^3-n)a_n^w z^{n-2},$$

where

$$\gamma_w(z) = \frac{z+w}{1+\overline{w}z}.$$

Then

$$a_n^w = \frac{1}{(n^3-n)}\frac{(Q\circ\gamma_w(\gamma_w')^2)^{(n-2)}}{(n-2)!}(0), \tag{9.1.1}$$

and

$$\| Q_w \|_S^2 = \sum_{n=2}^{\infty} n|a_n^w|^2.$$

Theorem 9.1.3. *Let $Q(z) = \sum_{n=2}^{\infty}(n^3-n)a_n z^{n-2}$ be a tangent vector correponding to $H^{\frac{3}{2}}$ vector fields. For $j \geq 2$, the average of the norm square of the* *coefficients* $|a_j|^2$ *is*

$$\iint_{\Delta} |a_j^w|^2 \frac{4dxdy}{(1-|w|^2)^2} = \frac{2}{3(j^3-j)} \iint_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy$$
$$= \frac{4\pi}{3(j^3-j)} \sum_{n=2}^{\infty} (n^3-n)|a_n|^2.$$

Proof. Using (9.1.1)

$$a_j^w = \frac{1}{(j^3-j)} \frac{(Q \circ \gamma_w (\gamma_w')^2)^{(j-2)}}{(j-2)!}(0) = \frac{c_j(w)}{(j^3-j)}.$$

We write the generating function for the $c_j(w)$' s.

$$f(u,w) = \sum_{j=2}^{\infty} c_j(w) u^{j-2}$$
$$= \sum_{j=2}^{\infty} \frac{(Q \circ \gamma_w (\gamma_w')^2)^{(j-2)}}{(j-2)!}(0) u^{j-2}$$
$$= Q \circ \gamma_w(u)(\gamma_w'(u))^2.$$

Writing $u = \rho e^{i\alpha}$, we have

$$\sum_{j=2}^{\infty} |c_j(w)|^2 \rho^{2j-4} = \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\alpha}, w)|^2 d\alpha$$

and

$$\sum_{j=2}^{\infty} \iint\limits_{\Delta} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2} \rho^{2j-4} = \frac{1}{2\pi} \int_0^{2\pi} \iint\limits_{\Delta} |f(\rho e^{i\alpha}, w)|^2 \frac{dxdy}{(1-|w|^2)^2} d\alpha$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \iint\limits_{\Delta} |Q \circ \gamma_w(\rho e^{i\alpha})(\gamma_w'(\rho e^{i\alpha}))^2|^2 \frac{dxdy}{(1-|w|^2)^2} d\alpha.$$

Denote this integral by $\mathcal{I}$, substituting the series expansion of Q and using polar coordinates $w = re^{i\theta}$, we get

$$\mathcal{I} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^{2\pi} d\theta r dr d\alpha \left| \frac{(1-r^2)}{(1+r\rho e^{i(\alpha-\theta)})^4} \right|^2 \tag{9.1.2}$$
$$\sum_{n=2}^{\infty} (n^3-n) a_n \left(\frac{\rho e^{i\alpha} + re^{i\theta}}{1+r\rho e^{i(\alpha-\theta)}} \right)^{n-2} \sum_{m=2}^{\infty} (m^3-m) \overline{a_m} \left(\frac{\rho e^{-i\alpha} + re^{-i\theta}}{1+r\rho e^{-i(\alpha-\theta)}} \right)^{m-2}. \tag{9.1.3}$$

We do some juggling

$$\left(\frac{\rho e^{i\alpha} + re^{i\theta}}{1+r\rho e^{i(\alpha-\theta)}} \right)^{n-2} \left(\frac{\rho e^{-i\alpha} + re^{-i\theta}}{1+r\rho e^{-i(\alpha-\theta)}} \right)^{m-2}$$
$$= \left(\frac{\rho e^{i(\alpha-\theta)} + r}{1+r\rho e^{i(\alpha-\theta)}} \right)^{n-2} \left(\frac{\rho e^{-i(\alpha-\theta)} + r}{1+r\rho e^{-i(\alpha-\theta)}} \right)^{m-2} e^{i(n-m)\theta}$$

and make a change of variable $\alpha \mapsto (\alpha + \theta)$ to get

$$\begin{aligned}
\mathcal{I} =& \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^{2\pi} d\theta r dr d\alpha \sum_{n,m\geq 2} (n^3 - n)(m^3 - m) a_n \overline{a_m} \\
& \left(\frac{\rho e^{i\alpha} + r}{1 + r\rho e^{i\alpha}}\right)^{n-2} \left(\frac{\rho e^{-i\alpha} + r}{1 + r\rho e^{-i\alpha}}\right)^{m-2} \left|\frac{(1 - r^2)}{(1 + r\rho e^{i\alpha})^4}\right|^2 e^{i(n-m)\theta} \\
=& \int_0^{2\pi} \int_0^1 r dr d\alpha \\
& \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho e^{i\alpha} + r}{1 + r\rho e^{i\alpha}}\right)^{n-2} \left(\frac{\rho e^{-i\alpha} + r}{1 + r\rho e^{-i\alpha}}\right)^{n-2} \left|\frac{(1 - r^2)}{(1 + r\rho e^{i\alpha})^4}\right|^2 \\
=& \int_0^{2\pi} \int_0^1 r dr d\alpha \\
& \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho + r e^{i\alpha}}{1 + \rho r e^{i\alpha}}\right)^{n-2} \left(\frac{\rho + r e^{-i\alpha}}{1 + \rho r e^{-i\alpha}}\right)^{n-2} \left|\frac{(1 - r^2)}{(1 + \rho r e^{i\alpha})^4}\right|^2 \\
=& \iint_{\Delta} \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho + w}{1 + \rho w}\right)^{n-2} \left(\frac{\rho + \overline{w}}{1 + \rho \overline{w}}\right)^{n-2} \left|\frac{1 - |w|^2}{(1 + \rho w)^4}\right|^2 dx dy,
\end{aligned}$$

where we have done another juggling to get the second to the last equality. Observe that

$$\begin{aligned}
\frac{\rho + w}{1 + \rho w} &= \gamma_\rho(w), \\
\frac{1}{(1 + \rho w)^4} &= \frac{(\gamma_\rho'(w))^2}{(1 - \rho^2)^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\iint\limits_{\Delta} \left(\frac{\rho+w}{1+\rho w}\right)^{n-2} \left(\frac{\rho+\overline{w}}{1+\rho\overline{w}}\right)^{n-2} \left|\frac{1-|w|^2}{(1+\rho w)^4}\right|^2 dxdy \\
=&\iint\limits_{\Delta} \left((z^{n-2})\circ\gamma_\rho(\gamma_\rho')^2\right)(w)\overline{\left((z^{n-2})\circ\gamma_\rho(\gamma_\rho')^2\right)(w)}\frac{(1-|w|^2)^2}{(1-\rho^2)^4}dxdy \\
=&\iint\limits_{\Delta} w^{n-2}\overline{w^{n-2}}\frac{(1-|w|^2)^2}{(1-\rho^2)^4}dxdy
\end{aligned}$$

using the invariance of the Weil-Petersson metric under $\mathrm{PSU}(1,1)$ transformation. This gives us

$$\begin{aligned}
\mathcal{I}=&\iint\limits_{\Delta}\sum_{n=2}^{\infty}(n^3-n)^2|a_n|^2 w^{n-2}\overline{w^{n-2}}\frac{(1-|w|^2)^2}{(1-\rho^2)^4}dxdy \\
=&\frac{1}{(1-\rho^2)^4}\iint\limits_{\Delta}|Q(w)|^2(1-|w|^2)^2 dxdy \\
=&\sum_{j=2}^{\infty}\frac{j^3-j}{6}\rho^{2j-4}\iint\limits_{\Delta}|Q(w)|^2(1-|w|^2)^2dxdy.
\end{aligned}$$

Compare coefficients, we get

$$\begin{aligned}
\int_{\Delta}|c_j(w)|^2\frac{dxdy}{(1-|w|^2)^2} &= \frac{j^3-j}{6}\iint\limits_{\Delta}|Q(w)|^2(1-|w|^2)^2dxdy, \\
\int_{\Delta}|a_j^w|^2\frac{4dxdy}{(1-|w|^2)^2} &= \frac{2}{3(j^3-j)}\iint\limits_{\Delta}|Q(w)|^2(1-|w|^2)^2dxdy.
\end{aligned}$$

This finishes the proof. □

Theorem 9.1.4. *Velling's metric defined on the subspace corresponding to*

$H^{\frac{3}{2}}$ vector fields of the universal Teichmüller space is given by

$$\| Q \|_V^2 = \iint\limits_{\Delta} \| Q_w \|_S^2 \frac{4dxdy}{(1-|w|^2)^2} = \frac{1}{2} \iint\limits_{\Delta} |Q(w)|^2 (1-|w|^2)^2 dxdy,$$

which is twice the Weil-Petersson metric.

Proof. This is just a simple sum of the telescoping series:

$$\begin{aligned}
\iint\limits_{\Delta} \| Q_w \|_S^2 \frac{4dxdy}{(1-|w|^2)^2} &= \sum_{j=2}^{\infty} j \iint\limits_{\Delta} |a_j^w|^2 \frac{4dxdy}{(1-|w|^2)^2} \\
&= \sum_{j=2}^{\infty} \frac{2}{3(j-1)(j+1)} \iint\limits_{\Delta} |Q(w)|^2 (1-|w|^2)^2 dxdy \\
&= \frac{1}{2} \iint\limits_{\Delta} |Q(w)|^2 (1-|w|^2)^2 dxdy.
\end{aligned}$$

□

9.2 Finite dimensional Teichmüller spaces

Let Γ be a cofinite Fuchsian group. The tangent space to $\mathfrak{T}(\Gamma)$ is identified with

$$A_\infty(\Delta, \Gamma) = \{Q \in A_\infty(\Delta) : Q \circ \gamma(\gamma')^2 = Q, \forall \gamma \in \Gamma\}$$

with the Weil-Petersson metric given by

$$\| Q \|_{WP}^2 = \frac{1}{4} \iint\limits_{\Gamma\backslash\Delta} |Q(w)|^2 (1-|w|^2)^2 dxdy. \qquad (9.2.1)$$

When $Q \in A_\infty(\Delta, \Gamma)$, we cannot average $\| Q \|_S^2$ over the whole disc. Since otherwise we are summing infinitely many copies of identical integrals. Instead we use regularization procedure suggested by Velling [Vel]. We define

$$\| Q \|_V^2 = \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma \backslash \Delta) \iint_{\Delta_{r'}} \| Q_w \|_S^2 \, dA_H}{\iint_{\Delta_{r'}} dA_H},$$

where $\Delta_r = \{z : |z| < r\}$ and

$$dA_H = \frac{4dxdy}{(1 - |w|^2)^2}$$

the hyperbolic area form and $\mathrm{Area}_H(\Gamma\backslash\Delta)$ the hyperbolic area of the quotient Riemann surface $\Gamma\backslash\Delta$.

First we rewrite the Weil-Petersson metric (9.2.1) in terms of regularized integrals.

Theorem 9.2.1.

$$\| Q \|_{WP}^2 = \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma \backslash \Delta) \iint_{\Delta_{r'}} |Q(w)|^2 \frac{(1-|w|^2)^2}{4} dxdy}{\iint_{\Delta_{r'}} dA_H}.$$

Proof. We use the fact that the number of lattice points $N_{r'}$ in a disc of radius r', $\Delta_{r'}$ is given asymptotically in terms of r' by

$$N_{r'} = \frac{1}{\mathrm{Area}_H(\Gamma\backslash\Delta)} \iint\limits_{\Delta_{r'}} dA_H \Big(1 + o(1)\Big), \qquad \text{as } r' \to 1^-. \tag{9.2.2}$$

(see [Pat75]).

Let F be a fundamental domain of Γ that contains the origin. Let

$$\Gamma_{r'} = \{\gamma \in \Gamma : \gamma(0) \in \Delta_{r'}\}.$$

The number of elements in $\Gamma_{r'}$ is exactly $N_{r'}$. Using (9.2.2), we have

$$\sum_{\gamma \in \Gamma_{r'}} \iint\limits_{\gamma F} dA_H = N_{r'} \operatorname{Area}_H(\Gamma \backslash \Delta) = \iint\limits_{\Delta_{r'}} dA_H \Big(1 + o(1)\Big).$$

Since

$$\sup_{w \in \Delta} |Q(w)(1 - |w|^2)^2| < \infty,$$

we have

$$\sum_{\gamma \in \Gamma_{r'}} \iint\limits_{\gamma F} |Q(w)(1 - |w|^2)^2|^2 dA_H = \iint\limits_{\Delta_{r'}} |Q(w)(1 - |w|^2)^2|^2 dA_H \Big(1 + o(1)\Big).$$

Hence

$$\begin{aligned}
\| Q \|_{WP}^2 &= \frac{1}{4} \iint\limits_{F} |Q(w)|^2 (1 - |w|^2)^2 dxdy \\
&= \frac{1}{4N_{r'}} \sum_{\gamma \in \Gamma_{r'}} \iint\limits_{\gamma F} |Q(w)|^2 (1 - |w|^2)^2 dxdy \\
&= \frac{1}{N_{r'}} \left(\iint\limits_{\Delta_{r'}} |Q(w)|^2 \frac{(1 - |w|^2)^2}{4} dxdy \big(1 + o(1)\big) \right) \\
&= \frac{\operatorname{Area}_H(\Gamma \backslash \Delta) \iint_{\Delta_{r'}} |Q(w)|^2 \frac{(1-|w|^2)^2}{4} dxdy}{\iint_{\Delta_{r'}} dA_H} + o(1).
\end{aligned}$$

We have used (9.2.2) again to get the last equality. □

Theorem 9.2.2. *Let Γ be a cofinite Fuchsian group, $Q \in A_\infty(\Delta, \Gamma)$, then*

$$\lim_{r'\to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |a_j^w|^2 dA_H}{\iint_{\Delta_{r'}} dA_H} = \frac{8}{3(j^3-j)} \parallel Q \parallel_{WP}^2 .$$

Proof. The proof follows almost the same as in Theorem 9.1.3. We use the same notation. We have

$$\begin{aligned}\mathcal{I} &= \sum_{j=2}^{\infty} \iint_{\Delta_{r'}} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2} \rho^{2j-4} \\ &= \iint_{\Delta_{r'}} \sum_{n=2}^{\infty} (n^3-n)^2 |a_n|^2 \left(\frac{\rho+w}{1+\rho w}\right)^{n-2} \left(\frac{\rho+\overline{w}}{1+\rho\overline{w}}\right)^{n-2} \left|\frac{1-|w|^2}{(1+\rho w)^4}\right|^2 dxdy.\end{aligned}$$

Now observe that if $\gamma \in \mathrm{PSU}(1,1)$ and $Q \in A_\infty(\Delta,\Gamma)$, then $Q \circ \gamma(\gamma')^2 \in A_\infty(\Delta, \gamma^{-1}\Gamma\gamma)$, and we have

$$(\parallel Q \parallel_{WP}^2)_{\mathfrak{T}(\Gamma)} = (\parallel Q \circ \gamma(\gamma')^2 \parallel_{WP}^2)_{\mathfrak{T}(\gamma^{-1}\Gamma\gamma)}.$$

In particular, for any $u = \rho e^{i\alpha} \in \Delta$, we have

$$\parallel Q \parallel_{WP}^2 = \lim_{r'\to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |(Q \circ \gamma_u(\gamma_u')^2)(w)|^2 \frac{(1-|w|^2)^2}{4} dxdy}{\iint_{\Delta_{r'}} dA_H},$$

since $\mathrm{Area}_H(\Gamma\backslash\Delta) = \mathrm{Area}_H(\gamma_u^{-1}\Gamma\gamma_u\backslash\Delta)$. It follows that

$$\parallel Q \parallel_{WP}^2 = \lim_{r'\to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \frac{1}{2\pi} \int_0^{2\pi} \iint_{\Delta_{r'}} |(Q \circ \gamma_u(\gamma_u')^2)(w)|^2 \frac{(1-|w|^2)^2}{4} dxdyd\alpha}{\iint_{\Delta_{r'}} dA_H}.$$

But we have

$$\frac{1}{2\pi}\int_0^{2\pi}\iint\limits_{\Delta_{r'}}|(Q\circ\gamma_u(\gamma_u')^2)(w)|^2\frac{(1-|w|^2)^2}{4}dxdyd\alpha$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{r'}\int_0^{2\pi}d\theta rdrd\alpha\left|\frac{(1-r^2)(1-\rho^2)^2}{2(1+r\rho e^{i(\theta-\alpha)})^4}\right|^2$$
$$\sum_{n=2}^{\infty}(n^3-n)a_n\left(\frac{\rho e^{i\alpha}+re^{i\theta}}{1+r\rho e^{i(\theta-\alpha)}}\right)^{n-2}\sum_{m=2}^{\infty}(m^3-m)\overline{a_m}\left(\frac{\rho e^{-i\alpha}+re^{-i\theta}}{1+r\rho e^{-i(\theta-\alpha)}}\right)^{m-2}.$$

This is similar to the integral (9.1.2) with the role of θ and α interchanged, so it equals to

$$\frac{(1-\rho^2)^4}{4}\iint\limits_{\Delta_{r'}}\sum_{n=2}^{\infty}(n^3-n)^2|a_n|^2\left(\frac{\rho+w}{1+\rho w}\right)^{n-2}\left(\frac{\rho+\overline{w}}{1+\rho\overline{w}}\right)^{n-2}\left|\frac{1-|w|^2}{(1+\rho w)^4}\right|^2dxdy$$
$$=\frac{(1-\rho^2)^4}{4}\mathcal{I}.$$

Hence

$$\sum_{j=2}^{\infty}\lim_{r'\to 1^-}\frac{\text{Area}_H(\Gamma\backslash\Delta)\iint_{\Delta_{r'}}|c_j(w)|^2\frac{dxdy}{(1-|w|^2)^2}}{\iint_{\Delta_{r'}}dA_H}\rho^{2j-4}$$
$$=\frac{4}{(1-\rho^2)^4}\parallel Q\parallel_{WP}^2.$$

Compare coefficients, we have

$$\lim_{r'\to 1^-}\frac{\text{Area}_H(\Gamma\backslash\Delta)\iint_{\Delta_{r'}}|c_j(w)|^2\frac{dxdy}{(1-|w|^2)^2}}{\iint_{\Delta_{r'}}dA_H}=\frac{2(j^3-j)}{3}\parallel Q\parallel_{WP}^2$$

and

$$\lim_{r'\to 1^-}\frac{\mathrm{Area}_H(\Gamma\backslash\Delta)\iint_{\Delta_{r'}}|a_j^w|^2 dA_H}{\iint_{\Delta_{r'}} dA_H} = \frac{8}{3(j^3-j)}\parallel Q\parallel_{WP}^2 .$$

□

Theorem 9.2.3. *Let* Γ *be a cofinite Fuchsian group. The Velling metric on the Teichmüller space* $\mathfrak{T}(\Gamma)$ *is twice the Weil-Petersson metric. i.e.*

$$\parallel Q\parallel_V^2 = 2\parallel Q\parallel_{WP}^2 .$$

Proof. The same as Theorem 9.1.4. □

For a general Fuchsian group Γ and $Q\in A_\infty(\Delta,\Gamma)$, we can define

$$\parallel Q\parallel_V^2 = \lim_{r'\to 1^-}\frac{\iint_{\Delta_{r'}\cap F(\Gamma)} dA_H \iint_{\Delta_{r'}}\parallel Q_w\parallel_S^2\, dA_H}{\iint_{\Delta_{r'}} dA_H},$$

where $F(\Gamma)$ is a fundamental domain of Γ acting on Δ. When Γ is the trivial group, it reduces to integrating over the whole disc, which coincides with our original definition.

Chapter 10

Euclidean Area and Kirillov-Yuriev Potential

10.1 Euclidean area

Here we use Velling's approach, but instead of spherical area we consider the Euclidean area. We find the corresponding Hermitian form at the origin of the universal Teichmüller curve.

Given a tangent vector v with the associated one parameter flow $\gamma^t = (g^t)^{-1} \circ f^t|_{S^1}$, and the family of domains Ω_t bounded by the quasi-circles $\mathcal{C}_t$, we consider the Euclidean area of the domains Ω_t:

$$A_E(f^t(\Delta)) = \iint_{\Omega_t} dxdy = -\frac{i}{2} \int_{\mathcal{C}_t} \overline{z} dz.$$

We denote by

$$f^t(z) = \sum_{n=0}^{\infty} a_n(t) z^{n+1}\,, \qquad z \in \Delta, \qquad a_0(t) \equiv 1\,,$$

$$g^t(z) = \sum_{n=0}^{\infty} b_n(t) z^{1-n}, \qquad z \in \Delta^*.$$

Using the fact that f^t and g^t extend to quasiconformal mappings on the whole plane and are absolutely continuous on S^1, with $f^t(S^1) = g^t(S^1) = \mathcal{C}_t$ we can evaluate the Euclidean area in two different ways:

$$A_E(f^t(\Delta)) = -\frac{i}{2}\int_{S^1} \overline{f^t} f^t_z dz = \pi \sum_{n=0}^{\infty}(n+1)|a_n(t)|^2,$$
$$A_E(f^t(\Delta)) = -\frac{i}{2}\int_{S^1} \overline{g^t} g^t_z dz = \pi \sum_{n=0}^{\infty}(1-n)|b_n(t)|^2.$$

This gives the equality

$$\sum_{n=0}^{\infty}(n+1)|a_n(t)|^2 = \sum_{n=0}^{\infty}(1-n)|b_n(t)|^2. \tag{10.1.1}$$

The analog of the spherical area theorem is

Theorem 10.1.1.

$$\frac{d}{dt}A_E(f^t(\Delta))\big|_{t=0} = 0,$$
$$\frac{d^2}{dt^2}A_E(f^t(\Delta))\big|_{t=0} = 2\pi \sum_{n=1}^{\infty}(n+1)\dot{a}_n(0)\overline{\dot{a}_n(0)} \geq 0.$$

Proof. We compute directly

$$\frac{d}{dt}A_E(f^t(\Delta)) = \pi \sum_{n=0}^{\infty}(n+1)(\dot{a}_n(t)\overline{a_n(t)} + a_n(t)\overline{\dot{a}_n(t)}).$$

But since

$$\dot{a}_0(0) = 0 \qquad \textit{and} \qquad a_n(0) = 0\ , \quad \forall n \geq 1,$$

it follows immediately that

$$\frac{d}{dt}A_E(f^t(\Delta))\big|_{t=0} = 0$$

and

$$\frac{d^2}{dt^2}A_E(f^t(\Delta))\big|_{t=0} = 2\pi\sum_{n=1}^{\infty}(n+1)\dot{a_n}(0)\overline{\dot{a_n}(0)} \geq 0.$$

□

As in Section 8.2, we write $a_n = \dot{a_n}(0)$, and define a new Hermitian metric on $\mathcal{T}(1)$ by

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}A_E(f^t(\Delta))\big|_{t=0} = \sum_{n=1}^{\infty}(n+1)|a_n|^2$$

at the origin and extend it to other points by the right group action. Notice that this does not belong to the family of Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$. Hence it is also not Kähler on $\mathcal{T}(1)$. However, if we do the averaging procedure to project it as a homogenuous metric on $T(1)$, we still get a multiple of the Weil-Petersson metric defined on vectors corresponding to $H^{\frac{3}{2}}$ vector fields by Theorem 9.1.3. After regularization of the averaging procedure, we also get a multiple of the Weil-Petersson metric on Teichmüller spaces of cofinite Fuchsian groups, which follows from Theorem 9.2.2.

10.2 Kirillov-Yuriev Potential

We use the equality (10.1.1) to get information about the variation of b_0. First we take derivative with respect to t on both sides,

$$\begin{aligned}
&\frac{d}{dt}b_0(t)\overline{b_0(t)} + b_0(t)\frac{d}{dt}\overline{b_0(t)} \\
&= \sum_{n=1}^{\infty}(n+1)\left(\frac{d}{dt}a_n(t)\overline{a_n(t)} + a_n(t)\frac{d}{dt}\overline{a_n(t)}\right) \\
&\quad + \sum_{n=1}^{\infty}(n-1)\left(\frac{d}{dt}b_n(t)\overline{b_n(t)} + b_n(t)\frac{d}{dt}\overline{b_n(t)}\right).
\end{aligned}$$

Setting $t = 0$, and using the fact that $b_0(0) = 1$, we have

$$\left(\frac{d}{dt}b_0(t) + \frac{d}{dt}\overline{b_0(t)}\right)\Big|_{t=0} = 0. \tag{10.2.1}$$

We write

$$a_n = \frac{d}{dt}a_n(t)\Big|_{t=0} \qquad \text{and} \qquad b_n = \frac{d}{dt}b_n(t)\Big|_{t=0}.$$

We take the second derivative, set $t = 0$ and use (7.5.3), which gives

$$\begin{aligned}
\left(\frac{d^2}{dt^2}b_0(t) + \frac{d^2}{dt^2}\overline{b_0(t)} + 2\left|\frac{d}{dt}b_0(t)\right|^2\right)\Big|_{t=0} &= 2\sum_{n=1}^{\infty}(n+1)|a_n|^2 + 2\sum_{n=1}^{\infty}(n-1)|b_n|^2 \\
&= 4\sum_{n=1}^{\infty}n|a_n|^2.
\end{aligned}$$

Notice that $|b_0| = |g'(\infty)|$. Using (10.2.1), we have

$$\frac{d^2}{dt^2}\log|g'(\infty)|\Big|_{t=0} = \frac{1}{2}\Big(\frac{d^2}{dt^2}b_0(t) + \frac{d^2}{dt^2}\overline{b_0(t)} + 2\Big|\frac{d}{dt}b_0(t)\Big|^2\Big)\Big|_{t=0}$$
$$= 2\sum_{n=1}^{\infty} n|a_n|^2,$$

Given a tangent vector $v = \sum c_n e^{in\theta}$, let $w = \frac{1}{2}(v - iJv)$ and $\frac{\partial}{\partial \epsilon_w}$, $\frac{\partial}{\partial \bar{\epsilon}_w}$ the corresponding holomorphic and antiholomorphic derivatives (see Section 7.3.1).

From the identity (7.5.2) and the definition of J, we have

$$|a_n(v)| = \left|\frac{\partial}{\partial t_v} a_n(t_v)\Big|_{t=0}\right| = \left|\frac{\partial}{\partial t_{Jv}} a_n(t_{Jv})\Big|_{t=0}\right| = |a_n(Jv)|,$$

which immediately gives

$$\frac{\partial^2}{\partial \epsilon_w \partial \bar{\epsilon}_w}\log|g'(\infty)| = \frac{1}{4}\left(\frac{\partial^2}{\partial t_v^2} + \frac{\partial^2}{\partial t_{Jv}^2}\right)\log|g'(\infty)|$$
$$= \frac{1}{2}\left(\sum_n n|a_n(v)|^2 + \sum_n n|a_n(Jv)|^2\right)$$
$$= \sum_n n|a_n(v)|^2.$$

In [KY87], Kirillov and Yuriev proved that $\log|g'(\infty)|$ is a potential for a Kähler metric on $\mathrm{Diff}_+(S^1)/S^1$ and stated that the Kähler metric is the Kirillov metric. This directly follows from our computation above.

Chapter 11

Variations of Laplace Operators and Selberg Zeta Function

11.1 Mathematical set-up

Let $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ be the Teichmüller space of Riemann surfaces of type $(g;\nu_1,\cdots,\nu_n)$, $g+n/2>1$ and $2\le\nu_1\le\ldots\le\nu_n\le\infty$. A point on $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ corresponds to a normalized Fuchsian group Γ that is generated by the $2g$ hyperbolic elements $\alpha_1,\cdots,\alpha_g;\beta_1,\cdots,\beta_g$ and the n elements $\kappa_1,\cdots,\kappa_n$ such that κ_j is elliptic of order ν_j if $\nu_j<\infty$ and κ_j is parabolic if $\nu_j=\infty$. Moreover, they satisfy the following relation:

$$[\alpha_1,\beta_1]\ldots[\alpha_g,\beta_g]\kappa_1\ldots\kappa_n=id,$$

where $[\alpha_k,\beta_k]=\alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}$. Γ is normalized such that the attracting and repelling fixed points of α_1 are 0 and ∞ respectively and the attracting fixed point of β_1 is 1.

Let $\mathcal{T}_{(g;\nu_1,\cdots,\nu_n)}$ be the universal Teichmüller curve. The fiber over a point Γ

is the Riemann surface $X = \Gamma\backslash\mathbb{U}$, where $\mathbb{U}$ is the upper half plane. Denote by $\mathcal{H}^{k,l}$ the space of automorphic forms of weight $(2k, 2l)$ of Γ, which corresponds to the space of (k,l)–tensors on $X = \Gamma\backslash\mathbb{U}$. For k an integer, denote by $\overline{\partial}_k$ the $\overline{\partial}$–operators acting on k–differentials ($(k,0)$–tensors) on X. $\overline{\partial}_k^*$ its adjoint operators and $\Delta_k = \overline{\partial}_k^*\overline{\partial}_k$ the k–Laplacian. Denote by $\Omega^{k,0}$ the subspace $\operatorname{Ker}\Delta_k = \operatorname{Ker}\overline{\partial}_k$ in $\mathcal{H}^{k,0}$, consisting of holomorphic k–differentials. Denote by P_k the orthogonal projection (with respect to the Hodge metric corresponding to the hyperbolic metric) of $\mathcal{H}^{k,0}$ to $\Omega^{k,0}$.

11.2 Selberg zeta function

The Selberg Zeta function $Z(s)$ of a Riemann surface $X = \Gamma\backslash\mathbb{U}$ is defined for $\operatorname{Re} s > 1$ by the absolutely convergent product

$$Z(s) = \prod_{\{\gamma_0\}}\prod_{m=0}^{\infty}(1 - e^{(s+m)\log\lambda(\gamma_0)}),$$

where γ_0 runs over the set of conjugacy classes of primitive hyperbolic elements of Γ, and $0 < \lambda(\gamma) < 1$ the multiplier of γ. The function $Z(s)$ has a meromorphic continuation to the whole s–plane.

Under a deformation of Γ (given by a quasiconformal map) $f^\mu : \Gamma\backslash\mathbb{U} \longrightarrow \Gamma^\mu\backslash\mathbb{U}$, let $\gamma^\mu = f^\mu \circ \gamma \circ (f^\mu)^{-1}$. Then $\log\lambda(\gamma^\mu)$ defines a function on the Teichmüller space $\mathfrak{T}_{(g,\nu_1,\cdots,\nu_n)}$, the geodesic length function. The following formula is well known (see, e.g., [IT92]).

$$\partial\log\lambda(\gamma)(z) = -\frac{1}{\pi}\sum_{\sigma\in\langle\gamma\rangle\backslash\Gamma}\frac{\sigma'(z)^2(p_2-p_1)^2}{(\sigma z - p_1)^2(\sigma z - p_2)^2}, \qquad (11.2.1)$$

where p_1, p_2 are fixed points of γ. Here ∂ is the $(1,0)$ component of the de Rham differential on the Teichmüller space and we identify the cotangent bundle of the Teichmüller space at the point Γ with the space of holomorphic quadratic differentials on $\Gamma\backslash\mathbb{U}$. The series on the right hand side of this formula is known as the relative Poincare series of the element γ. It can be rewritten as

$$\sum_{\sigma\sim\gamma}(\lambda(\sigma)^{\frac{1}{2}} - \lambda(\sigma)^{-\frac{1}{2}})^2 \frac{\sigma'(z)}{(z-\sigma z)^2}, \tag{11.2.2}$$

where we sum over all the elements σ that are conjugate to γ.

We consider $Z(s)_\Gamma$ as defining a function on the Teichmüller space $\mathfrak{T}_{(g;\nu_1,\cdots,\nu_n)}$ (which depends on the parameter s). Using the formula (11.2.1), we will get

Theorem 11.2.1. *For Re $s > 1$,*

$$\partial \log Z(s)|_\Gamma(z) = \frac{1}{\pi}\sum_{\gamma\ hyp}\left(s\lambda(\gamma)^{s-1} + (1-s)\lambda(\gamma)^s\right)\frac{\gamma'(z)}{(z-\gamma z)^2}, \tag{11.2.3}$$

where the sum runs over all the hyperbolic elements in Γ.

Proof. Applying the formula (11.2.1), we have

$$\partial \log Z(s)|_\Gamma(z) = \frac{1}{\pi}\sum_{\{\gamma_0\}}\sum_{m=0}^{\infty}\frac{(s+m)\lambda(\gamma_0)^{s+m}}{1-\lambda(\gamma_0)^{s+m}}\sum_{\sigma\in\langle\gamma_0\rangle\backslash\Gamma}\frac{\sigma'(z)^2(p_2-p_1)^2}{(\sigma z-p_1)^2(\sigma z-p_2)^2}. \tag{11.2.4}$$

We do some manipulation on the sum over m term:

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(s+m)\lambda^{s+m}}{1-\lambda^{s+m}} &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(s+m)\lambda^{k(s+m)} \\ &= \sum_{k=1}^{\infty}\lambda^{ks}\sum_{m=0}^{\infty}(s\lambda^{mk}+m\lambda^{mk}) \\ &= \sum_{k=1}^{\infty}\lambda^{ks}\left(\frac{s}{1-\lambda^k}+\frac{\lambda^k}{(1-\lambda^k)^2}\right) \\ &= \sum_{k=1}^{\infty}\frac{s\lambda^{k(s-1)}+(1-s)\lambda^{ks}}{(\lambda^{\frac{k}{2}}-\lambda^{-\frac{k}{2}})^2}.\end{aligned}$$

Inserting into the right hand side of (11.2.4), we have

$$\partial \log Z(s) = \frac{1}{\pi}\sum_{\{\gamma_0\}}\sum_{k=1}^{\infty}\frac{s\lambda(\gamma_0)^{k(s-1)}+(1-s)\lambda(\gamma_0)^{ks}}{(\lambda(\gamma_0)^{\frac{k}{2}}-\lambda(\gamma_0)^{-\frac{k}{2}})^2}\sum_{\sigma\in\langle\gamma_0\rangle\backslash\Gamma}\frac{\sigma'(z)^2(p_2-p_1)^2}{(\sigma z-p_1)^2(\sigma z-p_2)^2}. \tag{11.2.5}$$

Recall that we can write every group element $\gamma\in\Gamma$ as γ_0^k, where k is a positive integer. The multipliers satisfy the relation $\lambda(\gamma)=\lambda(\gamma_0)^k$. We can rewrite the above sum as

$$\sum_{\{\gamma\}}\frac{s\lambda(\gamma)^{s-1}+(1-s)\lambda(\gamma)^{s}}{(\lambda(\gamma)^{\frac{1}{2}}-\lambda(\gamma)^{-\frac{1}{2}})^2}\sum_{\sigma\in\langle\gamma_0\rangle\backslash\Gamma}\frac{\sigma'(z)^2(p_2-p_1)^2}{(\sigma z-p_1)^2(\sigma z-p_2)^2},$$

where now the sum runs over conjugacy classes of hyperbolic elements. Since $\gamma=\gamma_0^k$ and γ_0 have the same fixed points and centralizer, analogous to (11.2.2), we can rewrite the inner sum as

$$\sum_{\sigma\sim\gamma}(\lambda(\gamma)^{\frac{1}{2}}-\lambda(\gamma)^{-\frac{1}{2}})^2\frac{\sigma'(z)}{(z-\sigma z)^2}.$$

We use the fact that $\lambda(\sigma) = \lambda(\gamma)$ when σ is conjugate to γ. Summing over conjugacy classes of hyperbolic γ and then summing over all the elements conjugate to γ, we end up with summing over all hyperbolic elements. So we finally have (11.2.3). □

11.3 Variations of Laplace operators

For l a non–positive integer, denote by $Q_s^{(l)}(z, z')$ the resolvent kernel of the Laplace operator Δ_l on $\mathbb{U}$, i.e. $Q_s^{(l)}(z, z')$ is the kernel of the operator $(\Delta_l + \frac{1}{4}(s-2l)(s-1))^{-1}(l \leq 0, \operatorname{Re} s \geq 1)$. Denote by $G_s^{(l)}(z, z'), l \leq 0$ the kernel of the operator $(\Delta_l + \frac{1}{4}(s-2l)(s-1))^{-1}(l \leq 0, \operatorname{Re} s \geq 1)$ on the Riemann surface $X = \Gamma \backslash \mathbb{U}$. It is given by the absolutely convergent series

$$G_s^{(l)}(z, z') = \sum_{\gamma \in \Gamma} Q_s^{(l)}(z, \gamma z'), \tag{11.3.1}$$

which admit term by term differentiation with respect to z and z'.

We have the following result.

Theorem 11.3.1. *Let $q \geq 2$ be an integer. We have the following formula:*

$$P_2\left((-\partial \rho^{q-1} \partial' (G_1^{(1-q)} - Q_1^{(1-q)}))\big|_D\right)(z) \tag{11.3.2}$$
$$= \frac{1}{\pi} \sum_{\gamma\ hyp} \left(q\lambda(\gamma)^{q-1} + (1-q)\lambda(\gamma)^q\right) \frac{\gamma'(z)}{(z-\gamma z)^2}, \tag{11.3.3}$$

where ρ is the hyperbolic metric on $\mathbb{U}$ and $\big|_D$ is a restriction of the kernel to the diagonal $z = z'$.

Proof. We use the formula (see [ZT87a]):

$$(\rho')^{(q-1)}\partial' Q_1^{(1-q)}(z,z') = \frac{1}{\pi}\frac{1}{z-z'}(\frac{\bar{z}-z}{\bar{z}-z'})^{2q-1}$$

and (11.3.1) to get

$$-\partial\rho^{q-1}\partial'(G_1^{(1-q)} - Q_1^{(1-q)})|_D = \frac{1}{\pi}\sum_{\gamma\neq id}\frac{\gamma'(z)\overline{\gamma'(z)}^{q-1}}{(\gamma z - z)^2}\left(\frac{z-\bar{z}}{z-\overline{\gamma z}}\right)^{2q-2}.$$

It is well known (see [Ahl87]) that the kernel $K(z,w)$ for the projection P_2 is given by

$$-\frac{3}{\pi}\sum_{\gamma\in\Gamma}\frac{(w-\overline{w})^2}{(\gamma z-\overline{w})^4}\gamma'(z)^2.$$

We denote by $\mathcal{I}$ the left hand side of the equation (11.3.2), we have

$$\mathcal{I} = -\frac{3}{\pi^2}\iint\limits_{\Gamma\backslash\mathbb{U}} K(z,w)\sum_{\gamma\neq id}\frac{\gamma'(w)\overline{\gamma'(w)}^{q-1}}{(\gamma w - w)^2}\left(\frac{w-\overline{w}}{w-\overline{\gamma w}}\right)^{2q-2}\left|\frac{dw\wedge d\overline{w}}{2}\right|.$$

We unravel the sum over γ in the term $K(z,w)$ and get

$$\mathcal{I} = -\frac{3}{\pi^2}\iint\limits_{\mathbb{U}}\frac{(w-\overline{w})^2}{(z-\overline{w})^4}\sum_{\gamma\neq id}\frac{\gamma'(w)\overline{\gamma'(w)}^{q-1}}{(\gamma w - w)^2}\left(\frac{w-\overline{w}}{w-\overline{\gamma w}}\right)^{2q-2}\left|\frac{dw\wedge d\overline{w}}{2}\right|$$

Interchanging the integration and summation, we have

$$\mathcal{I} = -\frac{3}{\pi^2} \sum_{\gamma \neq id} \iint_{\mathbb{U}} \frac{(w-\overline{w})^2}{(z-\overline{w})^4} \frac{\gamma'(w)\overline{\gamma'(w)}^{q-1}}{(\gamma w - w)^2} \left(\frac{w-\overline{w}}{w-\overline{\gamma w}} \right)^{2q-2} \left| \frac{dw \wedge d\overline{w}}{2} \right| \tag{11.3.4}$$

$$= -\frac{3}{\pi^2} \sum_{\gamma \neq id} \mathcal{I}_\gamma \tag{11.3.5}$$

Divide it into 3 cases:

(I) $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is hyperbolic.

Let p_1, p_2 be the attracting and repelling fixed points and $0 < \lambda < 1$ the multiplier of γ. We diagonalize γ as $\sigma \circ D \circ \sigma^{-1}$, where

$$\sigma^{-1}(z) = \frac{z-p_1}{z-p_2}, \tag{11.3.6}$$

$$D(z) = \lambda z. \tag{11.3.7}$$

Then we make the substitution $w \to \sigma w$ in the integral $\mathcal{I}_\gamma$ and get

$$\mathcal{I}_\gamma = \frac{\lambda^{q-1}}{(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^2} \int_{\mathbb{U}} \frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z - \overline{w})^4} \frac{(w-\overline{w})^{2q}}{w^2(w-\lambda\overline{w})^{2q-2}} \left| \frac{dw \wedge d\overline{w}}{2} \right|$$

we want to write the expression

$$\frac{(w-\overline{w})^{2q}}{w^2(w-\lambda\overline{w})^{2q-2}}$$

as $\partial_w I$. Using elementary mathematics, by treating $\overline{w}$ as a constant, we can write the partial fraction expansion

$$\frac{(w-\overline{w})^{2q}}{w^2(w-\lambda\overline{w})^{2q-2}} = 1+\sum_{j=1}^{2}\frac{A_j}{w^j}+\sum_{j=1}^{2q-2}\frac{B_j}{(w-\lambda\overline{w})^j}, \tag{11.3.8}$$

where A_j, B_j are functions of $\overline{w}$. Hence I is given by

$$\begin{aligned} I =& A_1\log w + B_1\log(w-\lambda\overline{w}) + (w-\overline{w}) \\ &+ A_2(-\frac{1}{w}+\frac{1}{\overline{w}}) + \sum_{j=2}^{2q-2}\frac{B_j}{j-1}(-\frac{1}{(w-\lambda\overline{w})^{j-1}}+\frac{1}{(\overline{w}-\lambda w)^{j-1}}) \\ =& I_1 + I_2, \end{aligned}$$

where

$$I_2 = (w-\overline{w}) + A_2(\overline{w})(-\frac{1}{w}+\frac{1}{\overline{w}}) + \sum_{j=2}^{2q-2}\frac{B_j(\overline{w})}{j-1}\left(-\frac{1}{(w-\lambda\overline{w})^{j-1}}+\frac{1}{(\overline{w}-\lambda w)^{j-1}}\right)$$

vanishes on the real axis $w=\overline{w}$. Now we have

$$\begin{aligned} \mathcal{I}_\gamma &= \frac{\lambda^{q-1}}{(\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}})^2}\iint_{\mathbb{U}}\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-\overline{w})^4}\,\partial_w(I_1+I_2)\,\left|\frac{dw\wedge d\overline{w}}{2}\right| \\ &= \frac{i}{2}\frac{\lambda^{q-1}}{(\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}})^2}\int_{\mathbb{R}}\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-\overline{w})^4}\,I_1\,d\overline{w} \end{aligned}$$

by Stokes' theorem. In the term I_1, we take the principal values on the logarithm, i.e. $\log w = \log|w| + i\arg w$, where $0\le \arg w < 2\pi$. We have

$$\mathcal{I}_\gamma = \frac{i}{2}\frac{\lambda^{q-1}}{(\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}})^2}\int_{\mathbb{R}}\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-x)^4}(A_1(x)\log x + B_1(x)\log(x-\lambda x))\,dx. \tag{11.3.9}$$

We want to compare it to the integral

$$\lim_{R\to\infty}\int_{C_R}\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-w)^4}\left(A_1(w)\log w+B_1(w)(\log(1-\lambda)+\log w)\right)\,dw,$$

where C_R is the union of the line segment from the point $(-R,0)$ to $(R,0)$ and the semicircle $\{w:|w|=R,\operatorname{Im}w\le 0\}$ going from $(R,0)$ to $(-R,0)$. Here the branch of the logarithm is taken such that $0<\arg w\le 2\pi$. Notice that since $\sigma^{-1}z$ is on the upper half-plane, the integrand is holomorphic in the region bounded by C_R. By Cauchy's theorem, the integral is identically zero. On the other hand, the integral along the semicircle $\{w:|w|=R,\operatorname{Im}w\le 0\}$ tends to zero as $R\to 0$. So we have

$$0=\int_{\mathbb{R}}\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-x)^4}\left(A_1(x)\log x+B_1(x)(\log(1-\lambda)+\log x)\right)\,dx. \quad (11.3.10)$$

We compare the integrals (11.3.9) and (11.3.10) by looking at the branched values of the logarithms carefully. We get

$$\mathcal{I}_\gamma=\frac{\pi\lambda^{q-1}}{(\lambda^{\frac{1}{2}}-\lambda^{-\frac{1}{2}})^2}\int_0^\infty\frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z-x)^4}(A_1(x)+B_1(x))\,dx. \quad (11.3.11)$$

We find the A_1, B_1 as functions of $\overline{w}$ from the expression (11.3.8) by treating

$\overline{w}$ as a constant. We have

$$(A_1 + B_1)(\overline{w}) = \lim_{x\to\infty} x\left(\frac{(x-\overline{w})^{2q}}{x^2(x-\lambda\overline{w})^{2q-2}} - 1\right)$$
$$= -2(q + (1-q)\lambda)\overline{w}.$$

Substituting into (11.3.11), we have

$$\begin{aligned}\mathcal{I}_\gamma &= -\,2\pi\frac{q\lambda^{q-1} + (1-q)\lambda^q}{(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^2}\int_0^\infty \frac{x}{(\sigma^{-1}z - x)^4}((\sigma^{-1})'(z))^2\,dx \\ &= -\,\frac{\pi}{3}\frac{q\lambda^{q-1} + (1-q)\lambda^q}{(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^2}\left(\frac{(\sigma^{-1})'(z)}{\sigma^{-1}z}\right)^2 \\ &= -\,\frac{\pi}{3}(q\lambda^{q-1} + (1-q)\lambda^q)\frac{\gamma'(z)}{(z-\gamma z)^2},\end{aligned}$$

where we have used the formulas (11.3.6) and the formulas

$$c(p_1 - p_2) = \lambda^{-\frac{1}{2}} - \lambda^{\frac{1}{2}},$$
$$\gamma w - w = \frac{c(w-p_1)(w-p_2)}{cw+d}.$$

(II) $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is parabolic.

Let p be the fixed point. We diagonalize γ as $\sigma \circ T \circ \sigma^{-1}$, where

$$\sigma^{-1}(z) = \frac{1}{z-p},$$
$$T(z) = z + b\,,\ b \neq 0.$$

Then we make the substitution $w \to \sigma w$ in the integral $\mathcal{I}_\gamma$ and get

$$\mathcal{I}_\gamma = \frac{1}{b^2} \iint\limits_{\mathbb{U}} \frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z - \overline{w})^4} \frac{(w-\overline{w})^{2q}}{(w-\overline{w}-b)^{2q-2}} \left| \frac{dw \wedge d\overline{w}}{2} \right|$$

We expand the term

$$\frac{(w-\overline{w})^{2q}}{(w-\overline{w}-b)^{2q-2}}$$

as

$$\sum_{j=0}^{2q} \binom{2q}{j} (w-\overline{w}-b)^{2-j} b^j,$$

so that we can write it as $\partial_w I$, where

$$I = \binom{2q}{3} b^3 \log(w-\overline{w}-b) + \sum_{\substack{0 \le j \le 2q \\ j \neq 3}} \binom{2q}{j} b^j \frac{(w-\overline{w}-b)^{3-j} - (\overline{w}-w-b)^{3-j}}{3-j}$$

$$= I_1 + I_2,$$

and

$$I_2 = \sum_{\substack{0 \le j \le 2q \\ j \neq 3}} \binom{2q}{j} b^j \frac{(w-\overline{w}-b)^{3-j} - (\overline{w}-w-b)^{3-j}}{3-j}$$

vanishes on the real axis $w = \overline{w}$. So we have

$$\begin{aligned} \mathcal{I}_\gamma &= \frac{i}{2b^2} \int_{\mathbb{R}} \frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z - \overline{w})^4} I_1 \, d\overline{w} \\ &= \frac{i}{2b^2} \int_{\mathbb{R}} \frac{((\sigma^{-1})'(z))^2}{(\sigma^{-1}z - x)^4} \binom{2q}{3} b^3 \log(-b) \, dx. \end{aligned}$$

Since

$$\frac{1}{(\sigma^{-1}z - x)^4}$$

can be extended as a holomorphic function

$$\frac{1}{(\sigma^{-1}z - w)^4}$$

in the lower half plane and it behaves as R^{-4} as $|w| = R$ tends to ∞. By Cauchy's theorem, the integral is identically zero.

(III) $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is elliptic.

Let p be the fixed point in $\mathbb{U}$ and $\overline{p}$ – the complex conjugate of p, the other fixed point. We diagonalize γ as $\sigma \circ D \circ \sigma^{-1}$, where

$$\sigma^{-1}(z) = \frac{z-p}{z-\overline{p}},$$
$$D(z) = e^{i\alpha}z.$$

We make the substitution $w \to \sigma w$ in the integral $\mathcal{I}_\gamma$. Notice that σ^{-1} maps the upper half plane onto the unit disc $\Delta = \{z : |z| < 1\}$. After the substitution, we get an integral over Δ

$$\mathcal{I}_\gamma = \frac{e^{i(1-q)\alpha}}{(e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}})^2} \iint\limits_{\Delta} \frac{((\sigma^{-1})'(z))^2}{(1-(\sigma^{-1}z)\overline{w})^4} \frac{(1-w\overline{w})^{2q}}{w^2(1-e^{-i\alpha}w\overline{w})^{2q-2}} \left|\frac{dw \wedge d\overline{w}}{2}\right|.$$

Using polar coordinates, we have

$$\mathcal{I}_\gamma = \frac{e^{i(1-q)\alpha}}{(e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}})^2} \int_0^1 \int_0^{2\pi} \frac{((\sigma^{-1})'(z))^2}{(1-r(\sigma^{-1}z)e^{-i\theta})^4} \frac{(1-r^2)^{2q}}{r^2 e^{2i\theta}(1-e^{-i\alpha}r^2)^{2q-2}}\, r d\theta dr.$$

Using standard technique to evaluate the integration over θ term

$$\int_0^{2\pi} \frac{d\theta}{e^{2i\theta}(1-r(\sigma^{-1}z)e^{-i\theta})^4} = \oint_{|u|=1} \frac{1}{u^2(1-r(\sigma^{-1}z)u^{-1})^4} \frac{du}{iu}$$
$$= \oint_{|u|=1} \frac{udu}{i(u-r(\sigma^{-1}z))^4}$$
$$= 0.$$

So $\mathcal{I}_\gamma$ is identically zero.

We see that (11.3.4) only have contributions from the hyperbolic elements. Summing the terms together, we prove (11.3.2). □

Combining Theorems 11.2.1 and 11.3.1, we immediately have

Corollary 11.3.2. *Let $q \geq 2$ be an integer, we have*

$$\partial \log Z(q)|_\Gamma(z) = P_2\left((-\partial\rho^{q-1}\partial'(G_1^{(1-q)} - Q_1^{(1-q)}))|_D\right)(z) \qquad (11.3.12)$$

Remark 11.3.3. The formula (11.3.12) is usually proved by means of Selberg transform. For the proof, see [TZ91].

Remark 11.3.4. When $X = \Gamma\backslash\mathbb{U}$ is a compact Riemann surface, we can give a separate proof that

$$\partial \log \det \Delta_q|_\Gamma = P_2\left((-\partial\rho^{q-1}\partial'(G_1^{(1-q)} - Q_1^{(1-q)}))|_D\right)(z)$$

(see, e.g., [ZT87a]). We immediately get

$$\partial \log Z(q)|_\Gamma(z) = \partial \log \det \Delta_q|_\Gamma,$$

which can be integrated on the Teichmüller space to prove that

$$Z(q) = c(q) \det \Delta_q, \qquad q \geq 2,$$

where $c(q)$ is an integration constant that does not depend on the moduli. This formula is usually proved by using Selberg Trace Formula (see [Sar87], [DP86]).

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