A SYMPLECTIC ALEXANDER TRICK AND SPACES OF SYMPLECTIC SECTIONS

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Abstract. This thesis consists of two sections, connected by a thread. The symplectomorphism group of a 2-dimensional surface S is homotopy equivalent to the orbit of a filling system of curves on S. In the first section, we give a generalization of this statement to dimension 4. The filling system of curves is replaced by a decomposition of M into a disjoint union of an isotropic 2-complex L and a disc bundle over a symplectic surface Σ. This decomposition is due to Paul Biran. We show that one can recover the homotopy type of the symplectomorphism group of M from the orbit of the pair (L, Σ). This allows us to compute the homotopy type of certain spaces of Lagrangian submanifolds, for example the space of Lagrangian $RP^2 \subset CP^2$ isotopic to the standard one.

In the second section, we consider the product of two n-manifolds: $\Sigma\times\Gamma$, each equipped with a volume form σ_Σ and σ_Γ. We show that there is a homotopy equivalence between Σ_S- the space of sections S of this product fibration such that the product form $\pi_\Sigma^*\sigma_\Sigma + \pi_\Gamma^*\sigma_\Gamma|_S$ is a volume form – and NS_a^{2K+a} – spaces of maps $\Sigma \to \Gamma$ with constrained numbers of pre-images. This allows us to compute various identities between the spaces Σ_S for different volume forms. In the case that $n = 2$, these sections are the symplectic sections of the product fibration. Finally we compute NS_a^{2K+a} for certain cases.

1. Motivation

From one point of view, the geometry and topology of a $2-$dimensional surface S is dominated by the study of simple closed curves on S. If S is equipped with a symplectic structure ω, these curves are Lagrangian submanifolds of S.

It is becoming increasingly clear that the study of higher dimensional symplectic manifolds is also dominated by the structure of their Lagrangian, and more generally isotropic, submanifolds. In the first part of this thesis, we will explain a generalization of one facet of simple closed curves on a surface to the theory of isotropic submanifolds of a $4-$dimensional symplectic manifold.

We say that a system $\{\gamma_i\}$ of simple closed curves on S fills S if $S\backslash\{\gamma_i\}$ consists solely of discs. In this case, the symplectmorphism group of S, $Symp(S)$, is homotopy equivalent to the orbit of $\{\gamma_i\}$ under the action of $Symp(S)$. For if we examine the stabilizer of $\{\gamma_i\}$ in $Symp(S)$ we find that it consists of the symplectomorphisms of a disjoint union of discs, fixing their boundaries. This is a contractible set. One proves this by applying the well-known Alexander trick.

Paul Biran [1]recently showed that every Kahler manifold M whose symplectic form lies in a rational cohomology class admits a decomposition

$$M = L \coprod E$$

where L is an embedded, isotropic cell complex and E is a symplectic disc bundle over a hypersurface Σ. We will argue that a Biran decomposition of a symplectic $4-$manifold should be regarded as the $4-$dimensional analogue of a filling system of curves. In the case that M is a surface: L is a filling system of curves on the surface, Σ is a union of points -one in each disc inside $M\backslash L$ and E is the union of discs.

We will, at least in dimension 4, provide the necessary "symplectic Alexander trick". We will reduce the homotopy type of $Symp(M)$ to the orbit of pairs (L, Σ) under that group. All this is explained in greater details in the next section. Our argument will rely heavily on the relatively mature field of J-holomorphic spheres in sphere bundles over surfaces. Through this theory, which we owe to Gromov, Lalonde and McDuff [2, 3, **?**], we will reduce the $4-$dimensional theory to a parametric Alexander trick.

In the second part of this thesis, we consider $n-$manifolds Γ and Σ endowed with volume forms σ_Γ and σ_Σ. We then consider the homotopy type $Maps(\Sigma, \Gamma)$ such that the restriction of $\pi_\Sigma^*\sigma_\Sigma + \pi_\Gamma^*\sigma_\Gamma$ to the graph of the map f is again a volume form.

One can combine the arguments of the first section with those of the second to show that, even for symplectic fibrations of $S^2 \times S^2$ by spheres, the homotopy type of the spaces of symplectic sections must sometimes change (at least for certain homology classes) as one deforms the fibration. However this matierial is not included here.

Part 1. A Symplectic Alexander Trick

2. Summary of Results

Definition 2.1. A smoothly embedded cell complex is

(1) An abstract smooth cell complex C – the interior of each cell is endowed with a smooth structure.
(2) A continuous map

$$i : C \hookrightarrow M$$

which is a smooth embedding when restricted to the interior of each cell in C.

We say that a smoothly embedded cell complex is isotropic with respect to a symplectic structure ω, if $i^*(\omega) = 0$ on the interior of each cell

Definition 2.2. Let (M, ω) be a symplectic manifold. Let J be an almost complex structure compatible with ω. Let Σ_λ a symplectic hypersurface of M Poincare dual to $\lambda\omega$, and such that:

(1) There is a smoothly embedded, isotropic cell complex L_λ disjoint from Σ_λ. In what follows we will call this cell complex an **Isotropic Spine** of M.
(2) $M - L_\lambda$ is a symplectic disc bundle E over Σ_λ, such that the fibers have area $\frac{1}{\lambda}$ with respect to ω. This bundle is symplectomorphic to the unit disc bundle in the normal bundle to $\Sigma_\lambda$with symplectic form:

$$\pi^*\omega|_\Sigma + \frac{1}{\lambda}d(r^2\alpha)$$

where r is the radial coordinate in the fiber, and α is the connection 1 form coming from the hermitian metric $\omega(\cdot, J, \cdot)$ on the normal bundle. α is normalized so that its total integral around the boundary of a fiber is $\frac{1}{\lambda}$.

We call such a configuration $(L_\lambda, E \to \Sigma_\lambda)$ a **decomposition** of M.

Theorem 2.3. *(Biran* [1]*) Let M be a Kahler manifold with a symplectic, holomorphic hypersurface Σ_λ Poincare dual to $\lambda\omega$. Then there is a decomposition $(L_\lambda, E \to \Sigma_\lambda)$.*

When combined with the following theorem of Kodaira and Donaldson one sees that every Kahler manifold whose symplectic form has a rational cohomology class admits a decomposition.

Theorem 2.4. *(Kodaira/Donaldson* [1]*) Let M be a symplectic manifold whose symplectic form ω has an integral cohomology class. Then* *there is a* $\lambda_0 \in Z^+$*such that for all* $\lambda > \lambda_0$ *in there is a symplectic hypersurface* Σ_λ *Poincare dual to* $\lambda\omega$*. If* M *is Kahler with integral compatible complex structure* J *this surface can be made* J*- holomorphic.*

While Theoreom 2.3requires that M be Kahler, and the surface Σ_λ be holomorophic, Biran states that his proofs can probably be generalized to all symplectic manifolds, with $\epsilon-$holomorphic hypersurface Σ_λ. He makes his assumptions only for technical facility, and the confidence of familiar surroundings. It is probably safe to expect that every symplectic manifold whose form has a rational cohomology class admits a decomposition.

In this paper we will work with spaces of germs of mappings. A germ does not have a specified domain, and as a result most natural topologies on spaces of germs have unwanted pathologies. To avoid these, we will work at times in the category of Kan complexes, simplicial sets that satisfy the extension condition. [5]

We will call the germ of a neighborhood of a set a **framing**, and we will call a set along with its framing a **framed** set. If a set is denoted X, X with its framing will be denoted X^F.

Let $(L, E \to \Sigma)$ be a decomposition of M.

Definition 2.5. Let X be a topological space. By $\Delta(X)$ we denote the Kan complex of continuous maps

$$\Delta^n \to X$$

If ϕ is a continuous map, we denote the corresponding map of Kan complexes by ϕ^Δ. If Y is a Kan complex, $T(Y)$ denotes its geometric realization. If ϕ is a map of Kan complexes, we denote the corresponding continuous map by ϕ^T.

Definition 2.6. Denote the space of unparamaterized symplectic surfaces abstractly symplectomorphic to Σ and disjoint from a set $X \subset M$ by Σ_X.

Definition 2.7. Denote by $\mathcal{L}$ the Kan complex of embeddings

$$\phi : L^F \to M$$

such that $\phi^*\omega$ vanishes on L. We now describe the simplices in $\mathcal{L}$. An n-simplex in $\mathcal{L}$ consists of the following data:

(1) A neighborhood U of L in M.
(2) A continuous map $\phi : \Delta^n :\to Symp(U, M)$, where $Symp(U, M)$ denotes a symplectic embedding of U into M, which admits an extension to a symplectomorphism of all of M.

Two such pairs (U_1, ϕ_1) and (U_2, ϕ_2) are equivalent if there exists a neighborhood U_3 of L such that $U_3 \subset U_1$, $U_3 \subset U_2$ and

$$\phi_1|_{U_3} = \phi_2|_{U_3}$$

The degeneration maps to the faces are given by restricting ϕ to the faces of Δ^n.

Definition 2.8. Denote by $\mathcal{L}_\Sigma$ the following Kan complex :
an n-simplex consists of a triple (U, ϕ, ψ) where:

(1) U is a neighborhood of L in M.
(2) A continuous map $\phi : \Delta^n :\to Symp(U, M)$
Two such pairs (U_1, ϕ_1) and (U_2, ϕ_2) are equivalent if there exists a neighborhood U_3 of L such that $U_3 \subset U_1$, $U_3 \subset U_2$ and

$$\phi_1|_{U_3} = \phi_2|_{U_3}$$

The degeneration maps to the faces are given by restricting ϕ to the faces of Δ^n.
(3) A continous map $\psi : \Delta^n \to \Sigma$ such that for all $x \in \Delta^n$, $\psi(x) \in \Sigma_{\phi(x)(L)}$

Note that $Symp(M)$ acts on $\mathcal{L}_\Sigma$. For any symplectomorphism will carry L^F to another such spine, and will preserves the the homology class of Σ as it is Poincare dual to $\lambda[\omega]$.

This paper is devoted to the proof and application of the following theorem:

Theorem 2.9. *Let (M, ω)be a Symplectic $4-$manifold with decomposition $(L_\lambda, E \to \Sigma_\lambda)$ such that $[\Sigma_\lambda] \cdot [\Sigma_\lambda]$is even. Then $\Delta(Symp(M))$is weakly homotopy equivalent to $\mathcal{L}_\Sigma$.*

We do not require that M be Kahler, only that it has decomposition. However *we do restrict ourselves to dimension 4, and to decompositions where the self intersection of Σ is even.*

The assumption on the self intersection of Σ is technical, made mostly for the sake of clarity, and we hope to remove it shortly. It is made only to ensure that when the disc bundle E is compactified fiberwise, the resulting sphere bundle is trivial.

One should note however that every Kahler manifold with rational symplectic form admits a decomposition where the self intersection of Σ is even. For as

$$\Sigma_{2\lambda_0} \cdot \Sigma_{2\lambda_0} = 4\Sigma_{\lambda_0} \cdot \Sigma_{\lambda_0}$$

one can always, by theorem , find a hypersurface with even self intersection. Biran's theorem then provides a decomposition.

That we require M to be dimension 4 may well prove more resistant. Some aspects of the theory of J-holomorphic curves still apply; however some do not, and the proof will require some reorginization. I have not done a serious analysis of this case.

Proposition 2.10. *The map $\pi : \mathcal{L}_\Sigma \to \mathcal{L}$ which forgets the hypersurface is a Kan fibration. The fiber of π is $\Delta(\Sigma_L)$.*

One can compute this fiber by passing to a compactification of the disk bundle E. The details of this are somewhat technical (see Section 3), however it allows us to separate the problem of understanding the topology of the symplectomorphism group into two parts: embeddings of Lagrangian spines up to symplectic equivalence, and a "universal" problem about symplectic embeddings in $S^2 \times \Sigma$. If Σ is a sphere this problem admits a complete solution - the fiber of π is contractible. In these cases the symplectomorphism group also admits a complete computation. This allows us to compute spaces of symplectically equivalent Lagrangian spines in these cases:

Theorem 2.11. *The space of Lagrangian embeddings of $RP^2 \hookrightarrow CP^2$ isotopic to the standard one is homotopy equivalent to $Symp(CP^2)$.*

By work of Lalonde and McDuff, symplectic structures on rational surfaces are classified by their cohomology class. We establish the following convention:

Definition 2.12. By $S^2 \times \Sigma_{a,b}$we will denote $S^2 \times \Sigma$ with the following symplectic structure: Let $\tau_{sph}$and $\tau_\Sigma$denote fixed volume forms on S^2 and Σ respectively such that each form has total integral 1. Then endow $S^2 \times \Sigma$ with the symplectic structure $a\pi^*_{sph}\tau_{sph} + b\pi^*_\Sigma\tau_\Sigma$, where $\pi_{sph}$and π_Σ denote the projection onto the respective manifold.

Theorem 2.13. *The space of Lagrangian embeddings $S^2 \hookrightarrow S^2 \times S^2_{1,1}$ isotopic to the standard embedding of the diagonal is homotopy equivalent to the identity component of $Symp(S^2 \times S^2_{1,1}) \simeq SO(3) \times SO(3)$.*

3. Scaffolding of Proof

3.1. **Statement and introduction to proof.** We henceforth consider a symplectic 4-manifold (M, ω) with Biran decomposition $(L, E \to \Sigma)$, such that Σ has even self intersection k. We embark on the proof of:

Theorem. 2.9 *Let (M, ω) be a Symplectic $4-$manifold with decomposition $(L_\lambda, E \to \Sigma_\lambda)$ such that $k = [\Sigma_\lambda]\cdot[\Sigma_\lambda]$ is even. Then $Symp(M)$is weakly homotopy equivalent to $\mathcal{L}_\Sigma$.*

Note that both $Symp(M,\omega)$ and $\mathcal{L}_\Sigma$ are invariant under scaling the symplectic structure by a constant factor. Thus we safely replace ω by $\lambda\omega$ and reduce to the case that the class of Σ and the symplectic form are Poincare dual, and E's fibers have area 1.

One should not confuse this "scaffolding" for a sketch: the core geometric ideas of the proof of Theorem 2.9 lie in Section 4. *This Section will seek only to perform a series of reductions, transforming Theorem 2.9 into a pair of Propositions (3.12 and 3.13)* about the action of the symplectomorphism group of a rational surface on symplectic curves. This rational surface appears as the fiberwise compactification of the disc bundle $M - L^F$ into a sphere bundle over Σ. .The proofs of Propositions 3.12 and 3.13 proceed by applying the ample resources of the theory of J-holmorphic spheres in rational surfaces to this compactification. They are contained in Section 4.

3.1.1. *Conventions:* Throughout this paper we will be computing and comparing the stabilizers of the action of various groups on various geometric objects. To keep our heads straight it will be helpful to adopt a few notational conventions.

- $Symp(M,\omega)$ denotes the diffeomorphisms of M which preserve ω. If either M or ω is clear they will be omitted.
- G_S denotes the elements in the group G which preserve the set S. i.e :$\{g \in G : g(S) \subset S\}$.
- $G_{P(S)}$ denotes the elements in the group G which preserve the set S, and a parametrization $P(S)$. These are $\{g \in G : g(s) = s, \forall s \in S\}$.
- $G_{P(S^F)}$denotes the elements in the group G which preserve the set S, and a parametrization of a framing of that S,$P(S^F)$. These are $\{g \in G : \exists \text{ neighborhood } N_S \supset S : g(s) = s, \forall s \in N_S\}$. We endow $G_{P(S^F)}$with the direct limit topology.
- If we fix or preserve more than one set we will denote this by seperating the two with a comma. Eg: $G_{X,P(Y)}$ denotes the elements in G which preserve X and fix Y.

These are only notational guidelines, we will at each turn define each object considered. As such we will not treat them as sacrosanct, we will often use this notation with spaces "G" which are not groups, but which act like them for the purposes of our paper, in the hopes that this will cause more suggestion than confusion. Moreover not every use of subscript denotes a group preserving something. For instance $\mathcal{L}_\Sigma$ defined above, is not a group at all. However as the "groups" considered will always have either $Symp$ or $Diff$ embedded in their notation this practice should not cause confusion.

3.2. **Restatement of Theorem 2.9 in terms of $Symp_{P(L^F)}$.** In this section we will consider the action of $Symp$ on various geometric objects. In each case, we denote the orbit map:

$$\zeta \in Symp \to \zeta(x)$$

by $\phi_{(x)}$.

Proposition 3.1. *$\phi_{(\Sigma,L)} : \Delta(Symp(M)) \to \mathcal{L}_\Sigma$ is a homotopy equivalence if $\phi_{(\Sigma)} : Symp_{P(L^F)} \to \Sigma_L$ is a homotopy equivalence.*

Proof. The action of $Symp(M)$ on $\mathcal{L}$ results in the Kan fibration:

$$\Delta(Symp_{P(L^F)}) \to \Delta(Symp) \to \mathcal{L}$$

Forgetting the hypersurface Σ results in another Kan fibration:

$$\Delta(\Sigma_L) \to \mathcal{L}_\Sigma \to \mathcal{L}$$

The action of $Symp$ on L_Σ gives a morphism of these two fibrations:

$$\begin{array}{ccccc} \Delta(Symp_{P(L^F)}) & \to & \Delta(Symp) & \to & \mathcal{L} \\ \downarrow \phi^\Delta_{(\Sigma)} & & \downarrow \phi_{(L,\Sigma)} & & \downarrow (id) \\ \Delta(\Sigma_L) & \to & \mathcal{L}_\Sigma & \to & \mathcal{L} \end{array}$$

This yields a morphism of the associated exact sequences of Kan homotopy groups.[] Thus, by the 5-Lemma (Lemma 10.12), to show that $\phi_{(L,\Sigma)}$ is a homotopy equivalence (Theorem 2.9) it is sufficient to show that

$$\phi^\Delta_{(\Sigma)} : \Delta(Symp_{P(L^F)}) \to \Delta(\Sigma_L)$$

is a (weak) homotopy equivalence. By theorem 16.6 in [5], it is sufficient to show that the map on the underlying spaces

$$\phi_{(\Sigma)} : Symp_{P(L^F)} \to \Sigma_L$$

is a homotopy equivalence. □

Unfortunately it is difficult to translate the reduction of Proposition 3.1 to an amenable statement on a compactification of $M - L = E$ (see Remark 3.5). Instead we must consider a distinguished system of neighborhoods $L^\epsilon \supset L$ and compactifications of their complements.

E is the unit disc bundle in the normal bundle to Σ, N_Σ. Denote by $E_{1-\epsilon} \subset E$ the $\{x \in E : ||x|| \leq 1-\epsilon\}$, and denote its complement $M\backslash E^\epsilon$ by L^ϵ. We will end this section by reducing "$\phi_{(\Sigma)} : Symp_{P(L^F)} \to \Sigma_L$ is a homotopy equivalence" to a statement which admits a ready translation to a compactification of $E_{1-\epsilon}$.

Definition 3.2. Denote by Σ_{L^ϵ} the space of unparameterized, embedded symplectic surfaces S in $M\backslash L^\epsilon$ which are abstractly symplectomorphic to Σ.

Proposition 3.3. *$\phi_{(\Sigma)} : Symp_{P(L^F)} \to \Sigma_L$ is a homotopy equivalence if $\phi_{(\Sigma)} : Symp_{P(L^{\epsilon F})} \to \Sigma_{L^\epsilon}$ is a homotopy equivalence for every $0 < \epsilon < 1$.*

Proof. Every $\phi \in Symp_{P(L^F)}$ fixes some neighborhood N_ϕ of L. $N_\phi \supset L^\epsilon$ for some $\epsilon > 0$. Thus $\phi \in Symp_{P(L^{\epsilon F})}$. Thus the direct system:

$$Symp_{P(L^{\epsilon_1^F})} \hookrightarrow Symp_{P(L^{\epsilon_2^F})} \hookrightarrow ...$$

has limit $Symp_{P(L^F)}$. Similarily every embedding of $\eta : \Sigma \to M$ which misses L, also misses L^ϵ for some $\epsilon > 0$. Thus the direct system:

$$\Sigma_{L^{\epsilon_1}} \hookrightarrow \Sigma_{L^{\epsilon_2}} \hookrightarrow ...$$

has limit Σ_L.

We consider the action of each $Symp_{P(L^\epsilon)}$ on Σ_{L^ϵ}. The resulting orbit maps ψ_i yield a morphism of direct systems

$$\begin{array}{ccccccc} Symp_{P(L^{\epsilon_1^F})} & \hookrightarrow & Symp_{P(L^{\epsilon_2^F})} & \hookrightarrow & ... & \hookrightarrow & Symp_{P(L^F)} \\ \downarrow \psi_1 & & \downarrow \psi_2 & & ... & & \downarrow \\ \Sigma_{L^{\epsilon_1}} & \hookrightarrow & \Sigma_{L^{\epsilon_2}} & \hookrightarrow & ... & \hookrightarrow & \Sigma_{L^F} \end{array}$$

Finally note that any compact family $N_\phi \subset Symp_{P(L^F)}$ lies in $Symp_{P(L^\epsilon)}$ for some ϵ, and any compact family $N_\eta \subset \Sigma_L$ lies some Σ_{L^ϵ}. Thus if each ψ_i is a (weak) homotopy equivalence, ψ must also be a (weak) homotopy equivalence. □

We now begin with a discussion of our compactification:

3.3. **Compactification of $E_{1-\epsilon}$ via Symplectic Cutting (a la Lerman).** We apply the Lerman's Symplectic Cutting [4] to achieve our compactification.

Consider $E \subset M$ as the unit disc bundle in the normal bundle to Σ, N_Σ. Denote by $E_{1-\epsilon} \subset E$ the $\{x \in E : ||x|| \leq 1 - \epsilon\}$.

Lemma 3.4. *There is a surjective C^∞ map $\Psi : E_{1-\epsilon} \to \hat{E_{1-\epsilon}}$ where $\hat{E_{1-\epsilon}}$ is a symplectic sphere bundle over Σ. Topologically ψ is given by the collapse of the boundary circle in each fiber of the disc bundle. ψ:*

1. *is a symplectomorphism on the interior of $E_{1-\epsilon}$,*
2. *maps the boundary of $E_{1-\epsilon}$ to a symplectic section $Z_\infty$of this bundle whose self intersection is $-k$,*

(3) *maps the zero section of* $E_{1-\epsilon}$ *to a symplectic section* Z_0 *whose self intersection is* k.
(4) *The symplectic form* ω *on* $\hat{E_{1-\epsilon}}$ *has cohomology class* $(1-\epsilon)PD([Z_0])+\epsilon kPD([F])$ *where* $[F]$*denotes the class of the fiber of* $\hat{E_{1-\epsilon}}$.

Proof. The bundle $E \to \Sigma$ is given as the unit disc bundle in N_Σ in the hermitian metric induced from that on M. Place the following coordinates on the fiber of $E = D^2$: r, is a radial coordinate $r = |w|$, and the angular coordinate t lies in $[0,1]$, (i.e. $t = \frac{\theta}{2\pi}$) Then the symplectic structure on E is given by:

$$\pi^*\omega|_\Sigma + d(|w|^2\alpha)$$

where $\alpha = dt$. This structure is invariant under the circle action $S(t)$ given by the Hamiltonian function $\mu = |w|^2$.

We now consider the S^1 action $P(t)$ on the product:

$$(E \times C, \omega \oplus \tau)$$

where C denotes the complex numbers and τ denotes their standard complex structure, scaled by the constant factor $\frac{1}{\pi}$. The action is given by:

$$P(t)(m,z) = (S(t)m, e^{2\pi it}z)$$

$P(t)$ is Hamiltonian with function:

$$\zeta = \mu + ||z||^2$$

Let $\hat{E_{1-\epsilon}}$be the symplectic reduction of $((E \times C, \omega \oplus \tau), P(t))$ along the level set $\zeta = 1-\epsilon$. The level set

$$\zeta_{1-\epsilon} := \{(m,z) : \zeta(m,z) = 1-\epsilon\}$$

has the following structure:

$$\zeta_{1-\epsilon} = \left\{(m,z) : \mu(m) < 1-\epsilon \text{ and } z = e^{2\pi it}\sqrt{\mu(m)-(1-\epsilon)}\right\} \coprod \{(m,0) : \mu(m) = 1-\epsilon\}$$

Where both members of the disjoint union are invariant under the S^1 action. The map $i : E_{1-\epsilon} \to E \times C$ given by:

$$i : (m) = (m, \sqrt{(1-\epsilon)-\mu(m)}$$

is a symplectic embedding, whose image is contained in the level set $\zeta_{1-\epsilon}$. I claim that the composition of i with the quotient of $\zeta_{1-\epsilon}$ by $P(t)$:

$$\pi_Q : \zeta_{1-\epsilon} \to \zeta_{1-\epsilon}/S^1 = \hat{E_{1-\epsilon}}$$

gives a map:

$$\psi = \pi_Q i : E_{1-\epsilon} \to \hat{E_{1-\epsilon}}$$

with the properties above.

Symplectomorphism on $int(E_{1-\epsilon})$**:** $i(int(E_{1-\epsilon}))$ is transverse to the S^1 action $P(t)$ on $\zeta_{1-\epsilon}$. Thus composition with the quotient by this action:

$$\pi_Q : \zeta_{1-\epsilon} \to \zeta_{1-\epsilon}/S^1 = \hat{E_{1-\epsilon}}$$

yields a symplectic embedding into the symplectic reduction:

$$\pi_Q \circ i : int(E_{1-\epsilon}) \to \hat{E_{1-\epsilon}}$$

Maps boundary to symplectic section Z_∞; $\hat{E_{1-\epsilon}}$ **is a sphere bundle**: We now examine the restriction of $\pi_Q i$ to the boundary $(\delta E_{1-\epsilon})$.

$$\delta E_{1-\epsilon} = \{m : \mu(m) = 1 - \epsilon\}$$

Thus $\delta E_{1-\epsilon}$is the level set $\mu_{1-\epsilon}$. T

$$i(\delta E_{1-\epsilon}) = \{(m, 0) : \mu(m) = 1 - \epsilon\}$$

$P(t)$ then preserves $i(\delta E_{1-\epsilon})$, and its action there is that induced by μ. Thus

$$\pi_Q \circ i : \delta E_{1-\epsilon} \to \hat{E_{1-\epsilon}}$$

maps $\delta E_{1-\epsilon}$ to an embedded copy of the symplectic reduction of $\mu_{1-\epsilon}$ within $\hat{E_{1-\epsilon}}$. This then is a symplectic submanifold. The action induce by μ is $S(t)$, given by the rotation in each fiber of E_ϵ. $S(t)$ acts transiatively on each disc's boundary, and thus $\pi_Q \circ i$ collapses each of these.

Z_0 **and** Z_∞ **have correct self intersection**: As ψ is a symplectomorphism near Σ:

$$\begin{aligned} Z_0 \cdot Z_0 &= \Sigma \cdot \Sigma \\ &= k \end{aligned}$$

Denote by $[F]$ the homology class of the fiber of $\hat{E_{1-\epsilon}}$. As $\hat{E_{1-\epsilon}}$ is a sphere bundle over Σ, $H_2(\hat{E_{1-\epsilon}})$ is generated by $[F]$ and $[Z_0]$. Thus:

$$Z_\infty = aZ_0 + bF$$

As $[Z_\infty] \cdot F = 1$, $a = 1$. As $[Z_\infty] \cdot [Z_0] = 0$, $b = -k$. Thus

$$Z_\infty \cdot Z_\infty = (Z_0 - kF) \cdot (Z_0 - kF) = k - sk = -k$$

$[\omega]$ **has cohomology class** $(1-\epsilon)PD([Z_0]) + \epsilon k PD([F])$**:** The classes $PD([Z_0])$ and $PD([F])$ span $H^2(\hat{E_{1-\epsilon}})$. Thus:

$$[\omega] = aPD([Z_0]) + bPD([F])$$

for some a and b.

$$\omega([F]) = 1 - \epsilon$$

and thus $a = 1 - \epsilon$.

$$\omega([Z_0]) = [Z_0] \cdot [Z_0] = k$$

as the symplectic form on M was Poincare dual to Σ, and $\psi = i \circ \pi_Q$ is a symplectomorphism near Σ. As

$$\begin{aligned} \omega([Z_0]) &= (1-\epsilon)k + b \\ b &= \epsilon k \end{aligned}$$

End Lemma 3.4 □

Remark 3.5. *Why ϵ can't be 0:* Note that for cohomological reasons any compactification $p : E \to \hat{E}$ of the entire disc bundle E cannot map δE to a *symplectic* section. To understand the symplectomorphisms of M fixing L we must understand $Symp(\hat{E})_{sp(\delta E)}$, the symplectomorphisms of the compactification which fix $p(\delta E)$. This is possible only if $p(\delta E)$ is adapted to the symplectic form in some way. Moreover some adaptations are superior: the condition "$p(\delta E)$ symplectic" is much more pliable than "$p(\delta E)$ Lagrangian", at least for the arguments we will propose.

This compactification $\psi : E_{1-\epsilon} \to \hat{E_{1-\epsilon}}$ thus serves two roles: it allows us to play in the more comfortable compact terrain, and *it converts the problem of computing the stabilizer of an isotropic object to that of a symplectic object.* For this dual service we pay a price: we cannot compactify all of $M - L$, and must be satisifed with compactifying the complement of a neighborhood $L^\epsilon = M - int(E_{1-\epsilon})$ of L.

3.3.1. *$\hat{E_{1-\epsilon}}$is a trivial bundle over Σ.* In this subsection we show that if $k = \Sigma \cdot \Sigma$ is even, $\hat{E_{1-\epsilon}}$is a trivial bundle over Σ:

Lemma 3.6. *There are exactly 2 topological S^2 bundles over any surface Σ. $\hat{E_{1-\epsilon}}$ is the trivial bundle if k is even and nontrivial if k is odd.*

Proof. Bundles over Σ are in bijection with $\pi_0(Maps(\Sigma, BSO(3))$. Let γ be a one skelton of Σ, which gives Σ a cell decomposition with only one 2-cell. Then there is a natural fibration:)

Which induces the following maps on π_0:

$$... \to \pi_0(Maps(S^2, BSO(3))) \to \pi_0(Maps(\Sigma, BSO(3))) \to \pi_0(Maps(\gamma, BSO(3))$$

We first show that $\pi_0(Maps(\gamma, BSO(3))$ is trivial. $Maps(\gamma, BSO(3))$ fibers over $BSO(3)$ with fiber the based maps $Maps(\gamma, BSO(3))_*$. As $BSO(3)$ is connected it is sufficient to show that these based maps are connected. γ is a bouquet of circles. Thus $Maps(\gamma, BSO(3))_* \cong \prod Maps(S^1, BSO(3))_*$. As $\pi_1(BSO(3)) \cong \pi_0(S0(3))$ is the trivial group, this last space is the product of connected spaces, and thus connected.

We next show that $Maps(S^2, BSO(3))$ has 2 components. Again it fibers over $S0(3)$ with fiber the based maps $Maps(S^2, BSO(3))_*$. This fibration induces the following exact sequence of homotopy groups.

$$...\pi_1(BSO(3)) \to \pi_0(Maps(S^2, BSO(3))_*) \to \pi_0(Maps(S^2, BSO(3))) \to \pi_0(BSO(3))$$

Again since $BSO(3)$ is both connected and simply connected, the connected components of based and unbased maps coincide.

$$\pi_0(Maps(S^2, BSO(3))) \cong \pi_0(Maps(S^2, BSO(3)))_* \cong \pi_2(BSO(3)) \cong \pi_1(SO(3)) \cong Z_2$$

Thus we have:

$$... \to Z_2 \to \pi_0(Maps(\Sigma, BSO(3))) \to pt$$

S and thus there are 2 S^2 bundles over Σ.

I claim that these two bundles are distinguished by the parity of the self intersection of their sections. That is

(1) If S_1 and S_2 are two sections of a sphere bundle P over Σ then
$$S_1 \cdot S_1 = S_2 \cdot S_2 (\text{mod}2)$$
(2) This parity is 0 for the trivial bundle, and 1 for the non trivial bundle.

For if we denote the homology class of the fiber of the bundle by $[F]$ we have:

$$[S_2] = [S_1] + k[F]$$

Thus

$$[S_2] \cdot [S_2] = [S_1] \cdot [S_1] + 2k[S_1] \cdot [F] + k^2[F] \cdot [F] = [S_1] \cdot [S_1] + 2k$$

To see the second claim it is enough to note that the trivial bundle admits a section with self intersection 0. And that one can construct Hirzebruch surfaces of any genus with sections whose self intersection is odd.

□

Proposition 3.7. *$\hat{E_{1-\epsilon}}$is symplectomorphic to $S^2 \times \Sigma_{1-\epsilon, \frac{k}{2}(1+\epsilon)}$.*

Proof. By Lemma 3.6, it is diffeomorphic to $S^2 \times \Sigma$. Symplectic structures ω on $S^2 \times \Sigma$ are classified by their cohomology class. The proposition thus follows from condition 4 in Lemma 3.4. □

3.4. **Translation of Theorem 2.9 to Compactification.** In subsection 3.2 we reduced Theorem 2.9 to the following Proposition:

Proposition 3.8. *For each* $0 < \epsilon < 1$, $Symp(M)_{P(L^{\epsilon F})}$ *is homotopy equivalent to* Σ_{L^ϵ}.

In this subsection we translate both sides of Proposition 3.3 to statements within the compactification $\hat{E_{1-\epsilon}}$. Denote the elements of $Symp(\hat{E_{1-\epsilon}})$ which fix Z^F_∞ by $Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty)}$.

Lemma 3.9. *For each* $0 < \epsilon < 1$, $Symp(M)_{P(L^{\epsilon F})}$ *is homeomorphic to* $Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty)}$.

Proof. Restricting to $E_{1-\epsilon}$ gives a homeomorphism from $Symp(M)_{P(L^{\epsilon F})}$ to $Symp(E_{1-\epsilon})_{\delta E^F}$ -the symplectomorphisms of $E_{1-\epsilon}$which fix both the boundary and a framing of that boundary. I claim the compactification $\Psi : E_{1-\epsilon} \to \hat{E_{1-\epsilon}}$ described in Lemma yields a homeomorphism $\Psi_* : Symp(E_{1-\epsilon})_{P(\delta E^F)} \to Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty)}$.

Let $\eta \in Symp(E_{1-\epsilon})_{P(\delta E^F)}$. Then define $\Psi_*(\eta) = \Psi\eta\Psi^{-1}$ on $\hat{E_{1-\epsilon}} \backslash Z_\infty$, and extend $\Psi_*(\eta)$ to be the identity on Z_∞. As

$$\Psi|_{int(E_{1-\epsilon})} \to \hat{E_{1-\epsilon}} \backslash Z_\infty$$

is a symplectomorphism and as η preserves the interior of $E_{1-\epsilon}$, $\Psi\eta\Psi^{-1}|_{int(E_{1-\epsilon})}$ is a well defined map in $Symp(\hat{E_{1-\epsilon}} \backslash Z_\infty)$. As η is the identity near $\delta(E_{1-\epsilon})$, $\Psi\eta\Psi^{-1}$ is the identity near Z_∞ and this extension is smooth and in $Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty)}$. The inverse map is defined in the same way: $\Psi_*^{-1}(\eta) = \Psi^{-1}\eta\Psi$ on $int(E_{1-\epsilon})$, and extend $\Psi_*^{-1}(\eta)$ to be the identity on $\delta E_{1-\epsilon}$. As η is the identity near Z_∞, $\Psi\eta\Psi^{-1}$ is the identity near $\delta E_{1-\epsilon}$ and thus this extension is smooth and in $Symp(E_{1-\epsilon})_{P(\delta E^F)}$.
End Lemma 3.9

□

Definition 3.10. Denote by $\Sigma^\epsilon_{Z_\infty}$ the pairs of disjoint symplectic curves (Z_∞, Z) in $\hat{E_{1-\epsilon}}$ where Z is a curve abstractly symplectomorphic to Z_0, and Z_∞ denotes the fixed curve at infinity.

Lemma 3.11. $\Sigma \to (Z_{\infty,} \Psi(\Sigma))$ *gives a homeomorphism* $\Psi_* : \Sigma_{L^\epsilon} \to \Sigma^\epsilon_{Z_\infty}$

Proof. $\Psi|_{M-L^\epsilon} \to (\hat{E_{1-\epsilon}} \backslash Z_\infty)$ is a symplectomorphism. □

Finally Proposition 3.3 is equivalent to the combination of the following two propositions about the action of *Symp* on symplectic curves in $E_{1-\epsilon}$:

Proposition 3.12. *For each* $0 < \epsilon < 1 Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty)}$ *acts transitively on* $\Sigma^\epsilon_{Z_\infty}$.

Proposition 3.13. *For each* $0 < \epsilon < 1$ $Symp(\hat{E_{1-\epsilon}})_{P(Z^F_\infty),Z_0}$ *is contractible.*

For armed with these we have a fibration:

$$Symp_{P(Z^F_\infty),Z_0} \rightarrow Symp_{P(Z^F_\infty)} \rightarrow \Sigma^\epsilon_{Z_\infty}$$

with contractible fiber, and thus the following chain of homotopy equivalences.

$$Symp_{P(L^{\epsilon F})} \simeq_{(3.9)} Symp_{P(Z^F_\infty)} \simeq_{(\text{fibration})} \Sigma^\epsilon_{Z_\infty} \simeq_{(3.11)} \Sigma_{L^\epsilon}$$

The proofs of Propositions 3.12 and 3.13 will occur in 4.4 and 4.5 respectively.

4. Curves and Fibrations in $\hat{E_{1-\epsilon}}$

4.1. J-holomorphic curves and rational surfaces.

In this subsection we supply the necessary background from the theory of J-holomorphic curves on symplectic sphere bundles over surfaces. The main geometric ingredient in our proof is the following Proposition:

Proposition 4.1. *(Gromov-Mcduff[?]) Consider* $\Sigma \times S^2$ *with a symplectic form* ω. *Then if either:*

1. Σ *is not a sphere.*
2. $\omega([\Sigma \times pt]) \geq \omega([pt \times S^2])$

then for every almost complex structure J *tamed by* ω, *there is a* J*-holomorphic fibration by spheres in class* $[pt \times S^2]$.

We do not reproduce the proof of this Proposition here.

Lemma 4.2. *Consider* $\Sigma \times S^2$ *with a symplectic form* ω, *satisfying the hypotheses of Proposition 4.1. Let* $\{S_i\}$*be a collection of symplectic curves such that* $[S_i] \cdot [S^2 \times pt] = 1$, J *a tamed almost complex structure which preserves each curve. Then there is a* J*- holomorphic fibration by 2 spheres* F *in the class of* $[S^2 \times pt]$ *which is transverse to each curve, and such that each fiber meets each curve in exactly one point.*

Proof. By Proposition 4.1 there is a unique J-holomorphic fibration F by 2 spheres in class $[S^2 \times pt]$. As J is tamed this fibration is symplectic. By positivity of intersection each fiber must meet each curve transversely, and precisely once. **End Lemma 4.2** □

Remark 4.3. The curves $\{S_i\}$ are then symplectic sections of F.

4.2. **A softening of the symplectomorphism group.** In this subsection we will construct a large open neighborhood of $Symp(\hat{E_{1-\epsilon}})$ within its diffeomorphism group. This neighborhood will have the same homotopy type as $Symp(\hat{E_{1-\epsilon}})$, but it will be far easier to work with. In particular, it will be much easier to understand the "action" of this neighborhood on various objects.

Our compactification $\hat{E_{1-\epsilon}}$ comes equipped with a symplectic fibration by 2-spheres, we denote this fibration by F and we consider the triple (F_Z, Z_0, Z_∞) where Z_0 and Z_∞ are the symplectic sections discussed in Lemma 3.4.

Definition 4.4. Denote the diffeomorphisms of $\hat{E_{1-\epsilon}}$ which fix H_2 by $Diff^2$.

Definition 4.5. Denote the orbit of (F_Z, Z_0, Z_∞) under $Diff^2$ by $\mathcal{F}_0^\infty$. Denote the triples such that each member is symplectic by $S\mathcal{F}_0^\infty$. The orbit map fibers $Diff^2$ over $\mathcal{F}_0^\infty$. Consider the restriction of this fibration to $S\mathcal{F}_0^\infty$and denote the induced total space by $Diff(S\mathcal{F}_0^\infty)$. These are the elements which take (F_Z, Z_0, Z_∞) to another triple in $S\mathcal{F}_0^\infty$.

Lemma 4.6. *$Symp(\hat{E_{1-\epsilon}}) \subset Diff(S\mathcal{F}_0^\infty)$*

Proof. If $\gamma \in Symp(\hat{E_{1-\epsilon}})$ it is clear that each member of the triple $(\gamma(F_Z), \gamma(Z_0,)\gamma(Z_\infty))$ is symplectic.What is required then is to show that $Symp(\hat{E_{1-\epsilon}}) \subset Diff^2$. Each $\gamma \in Symp(\hat{E_{1-\epsilon}})$ preserves ω, and thus also the cohomology class

$$[\omega] = (1-\epsilon)PD([Z_0]) + \epsilon k PD([F])$$

Thus, as $[\omega]$ is Poincare dual to

$$(1-\epsilon)([Z_0]) + \epsilon k([F])$$

γ preserves this homology class as well. As $[Z_0]$ and $[F_Z]$ together span $H^2(\hat{E_{1-\epsilon}})$, it is enough for us to show that γ preserves $[F_Z]$. By Proposition 3.7 $\hat{E_{1-\epsilon}}$ is symplectomorphic to $\Sigma \times S^2_{1-\epsilon, \frac{k}{2}(1+\epsilon)}$, thus $[F_Z]$ is characterized by the following properties:

(1) $[F_Z]$ is spherical.
(2) $[F_Z] \cdot [F_Z] = 0$.
(3) $\omega([F_Z]) < \omega([pt \times S^2])$

Thus γ must fix $[F_Z]$.

□

Definition 4.7. Denote the space of closed 2-forms which restrict to symplectic forms which agree with the orientation induced by ω on each member of the triple (F_Z, Z_0, Z_∞) by P. Denote the symplectic members of P which induce the same orientation as ω by P_{Sym}. Finally, denote those forms in P with cohomology class $[\omega]$ by P^ω, and let $P^\omega_{Sym} \subset P^\omega$ be the symplectic members of P^ω.

Definition 4.8. Denote by $\pi_F : \hat{E_{1-\epsilon}} \to \Sigma$ the map induced by the leaves of F_Z .

Proposition 4.9. *P^ω_{Sym} is weakly contractible.*

Proof. This argument is more or less the same as one of the arguments that Lalonde-McDuff use to classify symplectic structures on ruled surfaces[3]. We simply do it with more parameters.

We remind the reader that

$$[\omega] = (1-\epsilon)PD([Z_0]) + \epsilon k PD([F_Z])$$

To lighten our notation in the calculations ahead, we let

$$\begin{aligned} a &:= 1-\epsilon \\ b &:= \epsilon k \end{aligned}$$

Then,

$$[\omega] = aPD([Z_0]) + bPD([F_Z])$$

Note that every form in P_{Sym} induces the same orientation on the ambient manifold, as well as the fibers of F_Z.Thus they also induce the same orientation on N_F the normal bundle to the fibers of F_Z. Choose a volume form σ_Σ on Σ such orientation induced by $\pi_F^*(\sigma_\Sigma)$ on N_F agrees with that of P_{Sym}.

To construct our contraction of spheres in P^ω_{Sym} we will require the following properties of the affine flow given by $\Theta_\kappa : \alpha \to \alpha + \kappa\pi_F^\star(\sigma_\Sigma)$.

Lemma 4.10. *The affine flow on 2-forms, given by $\Theta_\kappa : \alpha \to \alpha + \kappa\pi_F^\star(\sigma_\Sigma)$, satisfies the following conditions:*

1. *For any compact family $\Gamma \subset P$ there is a $\kappa > 0$ such that the entire family $\Theta_\kappa(\Gamma) \subset P_{Sym}$*
2. *Θ preserves P_{Sym}: If $\Gamma \subset P_{Sym}$ then $\Theta_\kappa(\Gamma) \subset P_{Sym}$*
3. *Θ preserves P: If $\Gamma \subset P$ then $\Theta_\kappa(\Gamma) \subset P$*
4. *If we denote the convex hull of a set X by $Conv(X)$: $Conv(\Theta(\Gamma)) = \Theta(Conv(\Gamma))$.*

Proof.

$$\begin{aligned}\Theta(\kappa)(\alpha)\wedge\Theta(\kappa)(\alpha) &= (\alpha+\kappa\pi_F^\star(\sigma_\Sigma))\wedge(\alpha+\kappa\pi_F^\star(\sigma_\Sigma))\\ &= \alpha\wedge\alpha+2\kappa\alpha\wedge\pi_F^\star(\sigma_\Sigma)+\kappa^2\pi_F^\star(\sigma_\Sigma)\wedge\pi_F^\star(\sigma_\Sigma)\\ &= \alpha\wedge\alpha+2\kappa\alpha\wedge\pi_F^\star(\sigma_\Sigma)\end{aligned}$$

Let $\alpha \in P$. Let $\{v_1, v_2, h_1, h_2\}$ be an oriented basis of the tangent space to a point such that $\{v_1, v_2\}$ span the tangent space to F_Z. If η is closed two form, $\eta\wedge\eta(v_1, v_2, h_1, h_2)$ is positive if and only if η is symplectic, and induces the same orientation as ω. Thus $\alpha\wedge\alpha(v_1, v_2, h_1, h_2) > 0$ if and only if $\alpha \in P_{Sym}$. I claim that $\alpha \wedge \pi_F^\star(\sigma_\Sigma)(v_1, v_2, h_1, h_2) > 0$. As this second term dominates for large κ this will show both claim 1 and claim 2.

$$\alpha\wedge\pi_F^\star(\sigma_\Sigma)(v_1,v_2,h_1,h_2) = \Sigma_{\mu\in S_4}\alpha(\mu v_1,\mu v_2)\pi_F^\star(\sigma_\Sigma)(\mu h_1,\mu h_2)sign(\mu)$$

The only non-vanishing pairings in this sum are those of the form:

$$\alpha(v_1,v_2)\pi_F^\star(\sigma_\Sigma)(h_1,h_2)$$

for $\pi_F^\star(\sigma_\Sigma)$ vanishes on any pair of vectors which contains a vertical vector v_i. Terms of this form are strictly positive due to our choice of sign of σ_Σ, and α's positivity on F_Z. As $\pi_F^\star(\sigma_\Sigma)$ is positive on both Z_0 and Z_∞,Θ preserves P (claim 3). Θ preserves convex hulls (claim 4) as it is affine. **End Lemma 4.10** □

Let $\phi : S^n \to P^\omega_{Sym}$ be a sphere of symplectic forms based at ω. (For the sake of this argument we define the 0-sphere to be the boundary of the 1-disk, that is the union of 2 points, one specified as the basepoint). Let κ be such that $\Theta_\kappa Conv(\phi(S^n)) \subset P_{Sym}$. Begin by homotoping ϕ so that it is constant in a neighborhood U_b of the basepoint b. Let

$$\chi : S^n \to [0,1]$$

be a continous function on the sphere such that $\alpha(b) = 0$, and $\alpha = 1$ outside U_b. We introduce χ to insure that we preserve the basepoint of ϕ throughout the homotopy. It plays no essential role in the construction.

We follow with the homotopy given by $\Psi_\Theta(x,t) = \Theta_{\chi(x)t}(\phi(x))$ as t travels from 0 to κ. The image of Ψ_Θ in H^2 is a line of classes $[\omega] + t\kappa PD([F])$. Compose this homotopy with the contraction of $\Theta_\kappa(\phi(S^n))$ within $Conv(\phi(\Theta_\kappa S^n)$. As $Conv(\phi(\Theta_\kappa S^n)) = \Theta_\kappa Conv(\phi(S^n)) \subset P_{Sym}$ this is also a contraction within P_{Sym}. Denote the resulting homotopy by ϕ_t.

The cohomology classes of the forms $\phi_t(x)$ lie on the line $[\omega]+t\kappa[F]$., where $PD(\cdot)$ denotes Poincare duality. We will now alter the homotopy

ϕ_t by adding a sufficient multiples of a Thom class of the section for each value of t so that the form $\phi_t(S^n)$ is Poincare Dual to $[\omega] = aPD([Z_0]) + bPD([F_Z])$. This process is called inflation. Its earliest appearance came in the papers of Lalonde-McDuff on the classification of symplectic structures on ruled surfaces. The version we will use is more refined:

Lemma 4.11. *(McDuff) Let (M, ω) be a symplectic 4-manifold with a compact family of tamed almost complex structures $\zeta_J : \Gamma \to \mathcal{J}$, which make a symplectic curve C with $C \cdot C \geq 0$ holomorphic. Then for each $\beta > 0$ there is a compact family of closed 2 forms $\zeta_\kappa : \Gamma \to \Omega^2$, supported in an arbitrarily small neighborhood of C, and such that the form $\omega + \zeta_\kappa(\gamma)$ is symplectic, tames $\zeta_J(\gamma)$ and has cohomology class $[\omega] + \beta PD([C])$*[1]

The proof of this Lemma is in []. We content ourselves with its application: Consider the normal bundle to Σ given by N_Σ with symplectic structure σ. By Weinstein's symplectic neighborhood theorem we can find a family of embeddings

$$\psi_{x \in S^n, t \in I} : N_\Sigma \to M$$

of the normal bundle to Σ, which map the zero section S_0 of N_Σ to Z_0 and such that

$$\psi^*_{x,t}(\phi_t(x)) = \sigma \tag{4.1}$$

on some neighborhood $U_{x,t}$ of S_0. As the family $\psi_{x,t}$ is compact, we can find a single neighborhood U_ψ of the zero section such that Equation 4.1 holds restricted to U_ψ for all x and t. $\psi^{-1}_{x,t}(F, Z_0)$ is then a family of fibrations on U_ψ. By Proposition 10.9 we can find a family of

$$\zeta_J : S^n \times I \to \mathcal{J}(U_\psi.\sigma)$$

of almost complex structures on U_ψ which are tamed by σ and such that $\psi^{-1}_{x,t}(F, Z_0)$ is $\zeta_J(x,t)$ holomorphic.

We then apply McDuff's Lemma above with:

(1) $M = (U_\psi, \sigma)$
(2) $C = S_0$
(3) $\beta = \frac{a\kappa}{b}$
(4) $\zeta_J = \zeta_J$.

[1]This is actually a good bit more refined than what we require. Mcduff achieves positivity on all holomorphic planes, while we require only positivity on fibration and section. Still it suffices, and there doesn't seem to be a need to populate the literature with weaker inflation lemmas.

This provides us a family of forms $\zeta_\tau : S^n \times I \to \Omega^2(U_\psi)$, such that $\sigma + \zeta_\tau(x,t)$ tames $\zeta_J(x,t)$. The set of forms taming a given almost complex structure is convex. Thus as σ and $\sigma + \zeta_\tau(x,t)$ both tame $\zeta_J(x,t)$ so does $\sigma + t\zeta_\tau(x,t)$ for $t \in [0,1]$. We obtain a set of forms $\sigma + t\zeta_\tau(x,t)$ such that:

(1) $\sigma + t\zeta_\tau(x,t) = \sigma$ outside the neighborhood of S_0.
(2) $\psi_{x,t}^{-1}(F, Z_0)$ is symplectic with respect to the form $\sigma + \zeta_\tau(x,t)$.

Transporting back to N we gain a family of forms $(\psi_{x,t}^{-1})^*(t\zeta_\tau(x,t))$ such that the form

$$\phi_t(x) + (\psi_{x,t}^{-1})^*(t\zeta_\tau(x,t))$$

is symplectic and positive on the triple (F_Z, Z_0, Z_∞).

Thus the homotopy ϕ_t^1:

$$\phi_t^1(x) \quad = \quad \phi_t(x) + \chi(x)t(\psi_{x,t}^{-1})^*(\zeta_\tau(x,t))$$

lies in P_{Sym}. Moreover:

$$\begin{aligned}
[\phi_t^1(x)] &= [\phi_t(x)] + \chi(x)t[(\psi_{x,t}^{-1})^*(\zeta_\tau(x,t))] \\
&= aPD([Z_0]) + bPD([F_Z]) + \chi(x)t\kappa PD([F_Z]) + \chi(x)t\frac{a\kappa}{b}PD([Z_0]) \\
&= (1 + \frac{\kappa\chi(x)t}{b})(aPD([Z_0]) + bPD([F_Z])) \\
&= (1 + \frac{\kappa\chi(x)t}{b})[\omega]
\end{aligned}$$

Thus we have moved our homotopy to one which takes place only in classes which are multiples of $[\omega]$. One can then rescale each part of the homotopy by the appropriate constant factor to obtain a homotopy of our sphere within the original cohomology class:

$$[\phi_t^2(x)] \quad = \frac{1}{(1+\frac{\kappa\chi(x)t}{b})} \quad (\phi_t(x) + \chi(x)t(\psi_{x,t}^{-1})^*(\zeta_\tau(x,t)))$$

$\phi_1^2(x)$ is then constant outside U_b, and maps U_b to the line of forms:

$$\frac{1}{(1+\frac{\kappa\chi(x)t}{b})}(\omega + \chi(x)t\kappa\pi_F^*(\sigma_\Sigma) + (\psi_{x,t}^{-1})^*\zeta_\tau(\chi(x)t))$$

and one can complete the homotopy by retracting this line down to its base point. **End Proposition 4.9** $\square$

Proposition 4.12. *The inclusion $Symp(\hat{E_{1-\epsilon}}) \to Diff(S\mathcal{F}_0^\infty)$ is a weak deformation retract.*

Proof. Let $\psi : D^n \to Diff(S\mathcal{F}_0^\infty)$ such that $\psi(\delta D^n) \subset Symp(\hat{E_{1-\epsilon}})$. We will produce a retraction of ψ to a disc of symplectomorphisms, while fixing its boundary.

Consider the disc of symplectic forms $\psi^\star(\omega) = \bigcup_{d \in D^n} \psi^*(d)(\omega)$. As $\psi(d)$ is a symplectomorphism for $d \in \delta D^n$, this disc is a sphere based at ω once we quotient out its boundary. As

$$\psi(d)((F_Z, Z_0, Z_\infty)) \in S\mathcal{F}_0^\infty$$

for each $d \in S^n = D^n/\delta$, each form in $\psi^\star(\omega)$ makes each member of the triple (F_Z, Z_0, Z_∞) symplectic. As $Diff(S\mathcal{F}_0^\infty) \subset Diff^2$ each form in $\psi^\star(\omega)$ has cohomology class $[\omega]$. Thus $\psi^\star(\omega) \subset P^\omega_{Sym}$. By Lemma 4.9 P^ω_{Sym} is weakly contractible, and thus that we can find a contraction of $\psi^\star_d(\sigma)$ to the constant sphere. Moser's Lemma then yields:

$$M_{\psi,t} : D^n \times I \to Diff$$

such that: □

(1) $M_{\psi,1}(d)^*(\omega) = \psi^\star_d(\omega)$
(2) $M_{\psi,0}(d) = id$
(3) $M_{\psi,t}(\delta D) = id$

$M^{-1}_{\psi,t}(d)\psi(d) : D^n \times I \to Diff(S\mathcal{F}_0^\infty)$ then yields a retraction of ψ into $Symp$ as t travels from 0 to 1.**End Proposition 4.12**

4.3. Application to the action of *Symp* on geometric objects.

We now characterize $\mathcal{SF}_0^\infty$:

Lemma 4.13. *$\mathcal{SF}_0^\infty = \mathcal{Z}_{0,\infty,F}$, the space of all triples (F_S, S_0, S_∞) where $(S_0, S_\infty) \in \mathcal{Z}_{0,\infty}$and F_S is a symplectic fibration by two spheres in class $[F]$ which makes each of the symplectic curves S_0 and S_∞ into symplectic sections of F_S.*

Proof. Each element in $Diff^2$ preserves homology class of each member of the triple. Thus $\mathcal{SF}_0^\infty \subseteq \mathcal{Z}_{0,\infty,F}$. To show the reverse inclusion we must construct a diffeomorphism ζ carrying the triple (F_Z, Z_0, Z_∞) to any other triple $(F_S, S_0, S_\infty) \in \mathcal{Z}_{0,\infty,F}$. By Proposition (??) in [Mcduff] there is a diffeomorphism taking any symplectic fibrations by 2-spheres with fiber in class $[F]$ to any other. Let $\Phi_F \in Diff$ such that $\Phi_F(F_Z) = F_S$ As the fibrations we consider are all product fibrations it is sufficient to show that:

Lemma 4.14. *Let F be the product fibration on $\Sigma \times S^2$. There is a fiber preserving diffeomorphism η which transforms any pair of sections $(\Sigma^1_k, \Sigma^1_{-k}))$ to any other $(\Sigma^2_k, \Sigma^2_{-k})$.*

Proof. Sections in class $[\Sigma_k]$ are given by graphs of deg k maps from Σ to S^2.As all such maps are homotopic we can find an isotopy between Σ^1_k and Σ^2_k. We can then find a fibration preserving path of diffeomorphisms which induces this isotopy. Denote the end of this path by η_0. Then $\eta_0(\Sigma^1_k, \Sigma^1_{-k}) = (\Sigma^2_k, \eta_0(\Sigma^1_{-k}))$. Denote the sections of F which miss Σ^2_k by $\mathcal{S}_2$. These are sections of the disc bundle $F - \Sigma^2_k$. Thus $\mathcal{S}_2$ is contractible, and we may can find an isotopy of sections from $\eta_0(\Sigma^1_{-k})$ to Σ^2_{-k} lying in $\mathcal{S}_2$. This isotopy may then be induced by a path of diffeomorphims which preserve both F and Σ^2_k. Call the end of this path of diffeomorphims η^∞_0. Then $\eta^\infty_0\eta_0(\Sigma^1_k, \Sigma^1_{-k}) = (\Sigma^2_k, \Sigma^2_{-k})$, so we can take $\eta = \eta^\infty_0\eta_0$. **End Lemma 4.14** □

$\eta\Phi_F$is then a diffeomorphism carrying (F_Z, Z_0, Z_∞) into (F_S, S_0, S_∞). **End Lemma 4.13**

□

Denote by $\mathcal{Z}_{0,\infty}$ the space of pairs (S_0, S_∞) of symplectic curves in $\hat{E}_{1-\epsilon}$ such that $[S_0] = [Z_0]$ and $[S_\infty] = [Z_\infty]$. Then the pair $(Z_0, Z_\infty) \in \mathcal{Z}_{0,\infty}$.
We remind the reader that $\Sigma^\epsilon_{Z_\infty}$ is defined as the pairs of disjoint symplectic curves (Z_∞, Z) such that Z is abstractly symplectomorphic to Z_0.

Lemma 4.15. $\Sigma^\epsilon_{Z_\infty} \subset \mathcal{Z}_{0,\infty}$

Proof. We must show that if Z is a symplectic curve in $\hat{E}_{1-\epsilon}\backslash Z_\infty$, abstractly symplectomorphic to Z_0 then $[Z] = [Z_0]$. As $[Z_0]$ and $[F]$ span $H_2(\hat{E}_{1-\epsilon})$

$$[Z] = a[Z_0] + b[F]$$

I claim that $b = 0$. For as Z misses Z_∞:

$$\begin{aligned} 0 &= [Z] \cdot [Z_\infty] \\ &= (a[Z_0] + b[F]) \cdot [Z_\infty] \\ &= a[Z_0] \cdot [Z_\infty] + b[F] \cdot [Z_\infty] \\ &= 0 + b \end{aligned}$$

Moreover as the Z and Z_0 are abstractly symplectomorphic

$$\begin{aligned} \omega[Z] &= \omega[Z_0] \\ a\omega[Z_0] &= \omega[Z_0] \end{aligned}$$

and thus

$$a = 1$$

thus $[Z] = [Z_0]$, and $\Sigma^{\epsilon}_{Z_\infty} \subset Z_{0,\infty}$. □

Proposition 4.16. *The forgetful map $\pi : S\mathcal{F}_0^\infty \to \mathcal{Z}_{0,\infty}$ is a fibration with contractible fiber.*

Proof. The proof of this Proposition will rely heavily on the results in the Appendix on almost complex structures.

We begin by showing that π is a fibration. π is surjective, for as the curves in (Z_0, Z_∞) are disjoint from one another we can find a tamed almost complex structure J which makes each curve holomorphic. Lemma 4.2 then provides a fibration F so that $(F, Z_{0,\infty}) \in S\mathcal{F}_0^\infty$.

I claim that π has path lifting: Let B be a polyhedron. We consider $\Phi : B \times I \to \mathcal{Z}_{0,\infty}$, along with a lifting $\Phi_{lift} : B \times 0 \to S\mathcal{F}_0^\infty$. We aim to extend Φ_{lift} to all of $B \times I$. By Proposition 10.11 there is a $\Phi^J : B \times I \to \mathcal{J}$, , such that $\Phi(b,t)$ is $\Phi^J(b,t)$ holomorphic, and $\Phi_{lift}(b,0)$ is $\Phi^J(b,0)$ holomorphic. Applying Lemma 4.2 we gain a family of fibrations, $\Phi_{lift}(b,t)$ extending our original lifting on $B \times 0$.

Finally we show that π has contractible fiber. Denote by $\mathcal{J}_{0,\infty}$ the tamed almost complex structures which make both Z_0 and Z_∞ holomorphic. It is enough to show that the map

$$\begin{aligned} \rho : \mathcal{J}_{0,\infty} &\to \pi^{-1}(Z_0, Z_\infty) \\ \rho(J) &= (F, Z_0, Z_\infty) \end{aligned}$$

where F is the unique J-holomorphic fibration determined by Lemma is a fibration with contractible fiber. For then ρ will be a weak homotopy equivalence, and as $\mathcal{J}_{0,\infty}$ is also contractible by Proposition 10.8, so must $\pi^{-1}(Z_0, Z_\infty)$ be contractible. We commence with this task.

We first show that ρ is a fibration on its image, i.e. that it has path lifting: Let B be a polyhedron. Consider $\Phi : B \times I \to \pi^{-1}(Z_0, Z_\infty)$, along with a lifting $\Phi_{lift} : B \times 0 \to \mathcal{J}_{0,\infty}$ such that $\Phi(b,0)$ is $\Phi_{lift}(b,0)$ holomorphic. Then Proposition 10.10 allows us to extend Φ_{lift} to all of $B \times I$.

Let $(F, Z_0, Z_\infty) \in \pi^{-1}(Z_0, Z)$. Then $\rho^{-1}(F, Z_0, Z_\infty) = \mathcal{J}_{F_0^\infty}$ the space of almost complex structures making each member of the triple holomorphic. I claim that $\mathcal{J}_{F_0^\infty}$ is nonempty and contractible. Thus ρ will be surjective with contractible fiber.

That $\mathcal{J}_{F_0^\infty}$ is nonempty is immediate from Proposition 10.10. To see that it is also (weakly) contractible it is enough to show that any map $\Phi^J : S^n \to \mathcal{J}_{F_0^\infty}$ admits an extension to the $n+1$ ball B^{n+1}. We apply Proposition10.10 with

(1) $B = B^{n+1}$ the $n+1$ ball.
(2) $\Phi : B \to S\mathcal{F}_0^\infty$ the constant map $\Phi(b,t) = (F, Q_0, Q_\infty)$.
(3) $Q = S^n$

□

Proposition 4.17. *Symp acts transitively on* $\mathcal{Z}_{0,\infty}$.

Proof. It is enough to show that there is a symplectomorphism carrying (Z_0, Z_∞) to any other pair (Z_0^1, Z_∞^1) in $\mathcal{Z}_{0,\infty}$. Let J be an almost complex structure leaving both Z_0 and Z_∞ invariant. Apply Lemma 4.2 and denote the resulting fibration by F. Then by Lemma 4.13 there is a $\alpha_1 \in Diff(S\mathcal{F}_0^\infty)$ which carries (F, Z_0, Z_∞) into (F^1, Z_0^1, Z_∞^1). Since $Symp \hookrightarrow Diff(S\mathcal{F}_0^\infty)$ is a deformation retract by Proposition 4.12, there is an isotopy α_t through $Diff(S\mathcal{F}_0^\infty)$ to a symplectomorphism α_0. Applying this isotopy to (Z_0, Z_∞) yields a path of pairs of curves $\alpha_t(Z_0, Z_\infty)$ which begins at $\alpha_1(Z_0, Z_\infty) = (Z_0^1, Z_\infty^1)$ and ends at $\alpha_0(Z_0, Z_\infty)$ within the orbit of (Z_0, Z_∞) under *Symp*. One can then induce this path $\alpha_t(Z_0, Z_\infty)$ by a path of symplectomorphisms Ψ_t.constructed by an easy application of Moser's Lemma. Then $\Psi_1\alpha_0(Z_0, Z_\infty) = (Z_0^1, Z_\infty^1)$. □

4.4. **Proof of Proposition 3.12.** We combine the background from subsection 4 to prove Proposition 3.12:

Proposition. 3.12*For every* $0 < \epsilon < 1$, $Symp(\hat{E_{1-\epsilon}})_{P(Z_\infty^F)}$ *acts transitively on* $\Sigma^\epsilon_{Z_\infty}$.

Proof. By Proposition 4.17 *Symp* acts transitively on $\mathcal{Z}_{0,\infty}$. By Lemma 4.15, $\Sigma^\epsilon_{Z_\infty} \subset \mathcal{Z}_{0,\infty}$. $Symp(\hat{E_{1-\epsilon}})_{P(Z_\infty^F)}$ are then precisely the symplectomorphisms which preserve $\Sigma^\epsilon_{Z_\infty}$, and act transitively on this space. □

4.5. **Proof of Proposition 3.13.** In this subsection we will leverage the background developed in 4 to complete the proof of Proposition 3.13:

Proposition. 3.13*For every* $0 < \epsilon < 1$, $Symp(\hat{E_{1-\epsilon}})_{P(Z_\infty^F), Z_0}$ *is contractible.*

Henceforth we will suppress the $\hat{E_{1-\epsilon}}$ from our notation of symplectomorphism groups.

Denote by $Symp_{Z_\infty, Z_0}$ the symplectomorphisms that preserve both Z_∞ and Z_0. Denote by $Diff(S\mathcal{F}_0^\infty)_{Z_\infty, Z_0}$ the diffeomorphisms in $Diff(S\mathcal{F}_0^\infty)$ which do the same.

Proposition 4.18. $Symp_{Z_\infty, Z_0} \hookrightarrow Diff(S\mathcal{F}_0^\infty)_{Z_\infty, Z_0}$ *is a homotopy equivalence.*

Symp acts transitively on $\mathcal{Z}_{0,\infty}$ by Corollary 4.17. Thus the orbit map $\phi : Symp \to \mathcal{Z}_{0,\infty}$ is a fibration.

Consider

$$\eta : Diff(S\mathcal{F}_0^\infty) \to SF_0^\infty \to \mathcal{Z}_{0,\infty}$$

The first map $Diff(S\mathcal{F}_0^\infty) \to SF_0^\infty$ is a fibration by Definition 4.5. The second $SF_0^\infty \to \mathcal{Z}_{0,\infty}$ is a fibration by Proposition 4.16 . Thus so is η -the composition of the two. The fiber of η is $Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0}$.

The inclusion $Symp_{Z_\infty} \hookrightarrow Diff(S\mathcal{F}_0^\infty)_{Z_\infty}$ yields a morphism of fibrations:

$$\begin{array}{ccc} Symp_{Z_\infty,Z_0} & \overset{i_1}{\hookrightarrow} & Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0} \\ \downarrow & & \downarrow \\ Symp & \overset{i_2}{\hookrightarrow} & Diff(S\mathcal{F}_0^\infty) \\ \downarrow \phi & & \downarrow \eta \\ \mathcal{Z}_{0,\infty} & \overset{(id)}{\to} & \mathcal{Z}_{0,\infty} \end{array}$$

i_2 and (id) are homotopy equivalences, thus so is i_1 by the 5-Lemma (Lemma 10.12).

Proposition 4.19. *$Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0}$ is homotopy equivalent to $Diff_{F_0^\infty}$, the diffeomorphisms which preserve the fibration F_Z and both sections Z_0 and Z_∞*

Proof. Denote the subset of $S\mathcal{F}_0^\infty$ given by triples (F, S_0, S_∞) where $S_0 = Z_0$ and $S_\infty = Z_\infty$ by $\mathcal{F}(Z_0, Z_\infty)$. Restricting the fibration of $Diff(S\mathcal{F}_0^\infty) \to S\mathcal{F}_0^\infty$ to $\mathcal{F}(Z_0, Z_\infty)$ yields a fibration:

$$Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0} \to \mathcal{F}(Z_0, Z_\infty)$$

with fiber $Diff_{F_0^\infty}$.

As $\mathcal{F}(Z_0, Z_\infty)$ is the fiber of the forgetful fibration $\pi : S\mathcal{F}_0^\infty \to \mathcal{Z}_{0,\infty}$ it is contractible by Lemma 4.16. □

Next we aim to calculate the symplectomorphisms which preserve Z_0 and *fix* Z_∞. We compare these to $Diff_{F_0^{P(\infty)}}$ - the diffeomorphisms preserving Z_0, *fixing* Z_∞, and *preserving* F_Z - a space that admits ready computation. Again we proceed by constructing a morphism of fibrations. However it is not easy to do this directly, and we will find it easier to reintroduce the $Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0}$ and their newest incarnation: $Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),Z_0}$ the diffeomorphisms in $Diff(S\mathcal{F}_0^\infty)$ which *fix* Z_∞ and preserve Z_0.

Proposition 4.20. *$Symp_{P(Z_\infty),Z_0} \hookrightarrow Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),Z_0} \hookleftarrow Diff_{F_0^{P(\infty)}}$ are each homotopy equivalences*

Proof.

$$\begin{array}{ccccc} Symp_{P(Z_\infty),Z_0} & \to & Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),Z_0} & \longleftrightarrow & Diff_{F_0^{P(\infty)}} \\ \downarrow & & \downarrow & & \downarrow \\ Symp_{Z_\infty,Z_0} & \hookrightarrow & Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0} & \longleftrightarrow & Diff_{F_0^\infty} \\ \downarrow \phi & & \downarrow \eta & & \downarrow \pi \\ Symp(Z_\infty) & \hookrightarrow & Diff(Z_\infty) & \leftarrow (id) & Diff(Z_\infty) \end{array}$$

π is a group homomorphism, and thus a fibration on its image. Moreover it is surjective- given a diffeomorphism of Z_∞ we lift it to $Diff_{F_0^\infty}$ in the following way: First use the product structure to lift $\delta \in Diff(Z_\infty)$ to a diffeomorphism μ which preserves F_Z (but may not preserves the sections). Then compose μ_δ with a diffeomorphism ζ which preserves the fibration and takes $\mu_\delta(Z_\infty)$ to Z_∞ and $\mu_\delta(Z_0)$ to Z_0. As ζ preserves the fibration, the composition $\zeta \cdot \mu_\delta$ still induces δ on Z_∞.

η is surjective as π is its restriction to $Diff_{F_0^\infty}$. If $\psi : P \times I \to Diff(Z_\infty)$ is a family of paths with an initial lifting $\psi_{lift} : P \times 0 \to Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0}$ we can extend this lifting by endowing the original triple (F_Z, Z_0, Z_∞) with a connection for which both $Z_\infty$and Z_0 are parallel. Then we use the diffeomorphisms given by ψ_{lift} to induce a connection on each fibration $\psi_{lift}(p)(F_Z)$. Finally we use these connections to lift each path of diffeomorphisms $\psi(p,t)$ to those preserving the fibration $\psi_{lift}(p)(F_Z)$.

To see that ϕ is surjective on π_0 note that both $Symp(Z_\infty) \hookrightarrow Diff(Z_\infty)$ and $Symp_{Z_\infty,Z_0} \hookrightarrow Diff(S\mathcal{F}_0^\infty)_{Z_\infty,Z_0}$ are homotopy equivalences, and thus isomorphisms on π_0. Thus, as the above diagram commutes, and η is surjective on π_0, so η must also be surjective on connected components. To see that it has path lifting: If $\psi : P \times I \to Symp(Z_\infty)$ is a family of paths with an initial lifting $\psi_{lift} : P \times 0 \to Symp_{Z_\infty,Z_0}$, first extend ψ_{lift} to a symplectic automorphics of the normal bundle to Z_∞, and then to a diffeomorphism in a neighborhood of Z_∞. Near Z_∞ the forms $\psi(p,t)^*\omega$ remain in a convex neighborhood of ω, one can thus use Moser to adjust $\psi(p,t)$ to be a family of symplectomorphisms near Z_∞. Then, as the embedding of $Z_\infty \hookrightarrow M$ is injective on H_1, $H_2(M) \to H_2(M, Z_\infty)$ is surjective. Thus we may apply the symplectic isotopy theorem to extend the lifting to the rest of the manifold. As $Z_\infty$and Z_0 are disjoint one can arrange this cut off so the resulting lifting preserves Z_0.

Finally we note that all of the maps between total spaces and base spaces are homotopy equivalences thus, by the five lemma both of the outer fibers must be homotopy equivalent to the inner one, and thus to each other.

□

Finally we aim to calculate the symplectomorphisms which preserve Z_0, and fix both Z_∞ *and its normal bundle.* These we denote by $Symp_{P(Z_\infty)^F, Z_0}$. Denote the fiber preserving diffeomorphisms which preserve Z_0, and fix both $Z_\infty$and its normal bundle by $Diff_{F_0^{P(\infty)^F}}$.

Proposition 4.21. *$Symp_{P(Z_\infty)^F, Z_0}$ is homotopy equivalent to $Diff_{F_0^{P(\infty)^F}}$.*

Proof. First we fiber each space $Symp_{P(Z_\infty),Z_0} \hookrightarrow Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),Z_0} \hookleftarrow Diff_{F_0^{P(\infty)}}$ over the automorphisms of the normal bundle to Z_∞. They fit together in the following morphism of fibrations:

$$\begin{array}{ccccc}
Symp_{P(Z_\infty),P(N_{Z_\infty}),Z_0} & \to & Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),P(N_{Z_\infty}),Z_0} & \hookleftarrow & Diff_{F_0^{P(\infty),N(\infty)}} \\
\downarrow & & \downarrow & & \downarrow \\
Symp_{P(Z_\infty),Z_0} & \hookrightarrow & Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),Z_0} & \hookleftarrow & Diff_{F_0^{P(\infty)}} \\
\downarrow \phi & & \downarrow \eta & & \downarrow \pi \\
Sp(N_{Z_\infty}) & \hookrightarrow & GL^+(N_{Z_\infty}) & \leftarrow (id) & GL^+(N_{Z_\infty})
\end{array}$$

where $Sp(N_{Z_\infty})$ consists of the symplectic automorphisms of N_{Z_∞} and $GL^+(N_{Z_\infty})$ consists of the orientation preserving automorphisms. Each map is a group homomorphism, thus it is enough to show that each is surjective. This follows by standard arguments for the right two maps – one can use the exponential of a metric for which F is totally geodesic lift an path of automorphisms of the normal bundle to a path of diffeomorphisms ψ_t in a neighborhood of Z_∞, which preserve the fibration F. One can then use a bump function χ, supported in a neighborhood of Z_∞ to obtain a path of diffeomorphisms $\psi_{\chi(t)t}$ supported near Z_∞, which preserve F. This shows that π and thus η is surjective. To show that ϕ is surjective we apply the same Moser argument from the previous lemma.

Finally we note the commutative diagram:

$$\begin{array}{ccccc}
Symp_{P(Z_\infty),P(N_{Z_\infty}),Z_0} & \to & Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty),P(N_{Z_\infty}),Z_0} & \hookleftarrow & Diff_{F_0^{P(\infty),N(\infty)}} \\
\uparrow & & \uparrow & & \uparrow \\
Symp_{P(Z_\infty^F),Z_0} & \hookrightarrow & Diff(S\mathcal{F}_0^\infty)_{P(Z_\infty^F),Z_0} & \hookleftarrow & Diff_{F_0^{P(\infty^F)}}
\end{array}$$

Again the right two vertical inclusions are homotopy equivalences by standard arguments, and the left inclusion follows by an application of Moser. ∎

Proposition. *3.13 $Symp_{P(Z_\infty^F),Z_0}$is contractible.*

Now follows from combining the above Proposition 4.21 with the following "parametric Alexander trick":

Lemma 4.22. *$Diff_{F_0^{P(\infty)F}}$ is contractible.*

Proof. As the elements in $Diff_{F_0^{P(\infty)F}}$ fix the section Z_∞, they must carry each fiber of F_Z into itself. Thus $Diff_{F_0^{P(\infty)F}}$ consists of

$$Maps(\Sigma, Diff(S^2)_{0,\infty^F})$$

where $Diff(S^2)_{0,\infty^F}$ consists of the diffeomorphisms of S^2 which fix a point 0 and the neighborhood of another point ∞. This is a contractible set, thus $Maps(\Sigma, Diff(S^2)_{0,\infty^F})$ is also a contractible set. □

5. Computation of $\Sigma^\epsilon_{Z_\infty}$

Proposition 5.1. *If Σ is a sphere $\Sigma^\epsilon_{Z_\infty}$ is contractible.*

Proof. Denote by $\mathcal{J}_\infty$the set of tamed almost complex structures on $\hat{E_{1-\epsilon}}$which make Z_∞holomorphic. Denote by $\mathcal{J}^S_\infty$ the space of pairs(J,S) where $J \in \mathcal{J}_\infty$ and $S = (Z_\infty, S_0) \in \Sigma^\epsilon_{Z_\infty}$ such that both curves are J-holomorphic.

In the following two lemmas 5.2 and 5.3 we will show that both $\mathcal{J}_\infty$ and $\Sigma^\epsilon_{Z_\infty}$ are homotopy equivalent to $\mathcal{J}^S_\infty$. Thus as $\mathcal{J}_\infty$ is contractible, this will show that $\Sigma^\epsilon_{Z_\infty}$ must also be contractible. □

Lemma 5.2. *The projection $\pi_\Sigma : \mathcal{J}^S_\infty \to \Sigma^\epsilon_{Z_\infty}$is a fibration with contractible fiber, and thus a homotopy equivalence.*

Proof. The fiber of π_Σ is the set of tamed complex structures which make both $Z_\infty$and S_0 holomorphic. As $Z_\infty$and S_0 form a disjoint pair of symplectic curves this is a contractible set. □

Denote the 2 disc by D^2.

Lemma 5.3. *The projection $\pi_J : \mathcal{J}^S_\infty \to \mathcal{J}_\infty$ is a fibration with contractible fiber and thus a homotopy equivalence.*

Proof. Let $J \in \mathcal{J}_\infty$. Fix $k+1$ distinct points x_i on Z_∞. By Proposition4.1 there is a unique J-holomorphic curve F_i in class $[F]$ which passes through x_i.

As both S_0 and the F_i are J-holomorphic they must intersect positively. Thus S_0 meets each F_i in precisely one point σ_i. As S_0 misses Z_∞, $\sigma_i \in F_i - x_i \simeq D^2$. Lemma 5.4 below shows that for any $k+1$-tuple in $\prod_{i=1..k+1} F_i - x_i$ there is a unique such curve S_0: □

We remind the reader that by 4.15 $[S_0] = [Z_0]$.

Lemma 5.4. *Let $J \in \mathcal{J}_\infty$. If $\Sigma = S^2$, then there is a unique, smooth J-holomorphic curve in Z_0 through any $k+1$ points in $\hat{E_{1-\epsilon}} \backslash Z_\infty$.*

Proof. For a generic J the moduli space of J holomorphic curves through q points has dimension:

$$4 + 2c_1(T(\hat{E_{1-\epsilon}}))([Z_0]) + 2q - 6 - nq$$

$$2c_1(T(\hat{E_{1-\epsilon}}))([Z_0]) = 2(\chi(Z_0) + [Z_0] \cdot [Z_0]) = 4 + 2k$$

so for the dimension to be 0 we need:

$$q = k + 1$$

The Gromov Witten Invariant for this class is 1. Thus there is a J-holomorphic curve Θ through any $k+1$ points. I claim that this curve is unique. Let $\Theta_1$and Θ_2 be two curves through these $k+1$ points. Then these two curves must coincide by positivity of intersection as $[Z_0] \cdot [Z_0] = k$.

I claim that Θ is always smooth and irreducible. For as the set of generic almost complex structures is dense one can always approximate J by a sequence of complex structures J_n so that the J_n holomorphic curve through these q points Θ_n is smooth. The sequence of curves Θ_n then converges to Θ, and Θ is thus controlled by Gromov compactness. It consists of a union of J-holomorphic spheres, which meet in points. In Lemma 5.5 below, we will now show that the need to:

(1) Intersect the curves in class $[F]$ positively. (Curves in class $[F]$ exist for every J tamed by ω by Proposition 4.1.)
(2) Intersect Z_∞ positively. ($J \in J_\infty$ and thus Z_∞ is a J_∞ holomorphic curve.)

eliminate all such nodal curves, save those of the form:

$$Z_\infty \bigcup_{i=1}^{k} F_i$$

where the F_i are (possibly repeated) spheres in class F. However curves of this last form are eliminated as well. They have only k fiber curves F_i, they cannot pass through all $k+1$ points. For I remind you that each point lies off Z_∞ and in a distinct J-holomorphic fiber.

□

Lemma 5.5. *Every nodal curve Θ consist of:*

$$Z_\infty \bigcup_{i=1}^{k} F_i$$

where the F_i are (possibly repeated) spheres in class F.

Proof. We begin by proving a weaker statement, namely that Θ must consist of:

(1) A curve in class $[Z_0] - l[F]$ for $l \in \mathcal{Z}, l > 0$.
(2) A collection of curves which are fibers for the J- holomorphic fibration by 2-spheres.

The second homology of $\hat{E_{1-\epsilon}}$ is spanned by $[Z_0]$ and the fiber class $[F]$. The class of each irreducible component Θ_i of a curve may thus be written $a_i[Z_0] + b_i[F]$. Each $a_i > 0$ as $a_i = [\Sigma_i] \cdot [F]$ and each of the classes are represented by a holomorphic curve.

The union of these components lies in class $[Z_0]$ thus:

$$\Sigma_i(a_i[Z_0] + b_i[F]) = [Z_0]$$

As all the a_i are positive integers, the only possibility which remains is that one $a_i = 1$ and the rest vanish. Moreover for all i such that $a_i = 0$, b_i must be positive, as ω evaluated on each component must be positive. Thus we have reduced ourselves to:

(1) A curve in class $[Z_0] - l[F]$ for $l \in \mathbb{Z}, l > 0$.
(2) A collection of curves in class $b_i[F]$ $b_i > 0$ such that $\Sigma_i b_i = l$.

Since there is a unique curve through each point in class $[F]$ these curves of "type 2" must be unions of fibers in F. we will now show that the only J-holomorphic curve in class $[Z_0] - l[F]$ $(l > 0)$ is Z_∞, with $l = k$.

Denote another such J-holomorphic curve by Z_{other}. Distinct J-holomorphic curves must intersect each other positively. Since $J \in Z_\infty$there is a J-holomorphic curve in class $[Z_\infty]$. But:

$$[Z_\infty] \cdot ([Z_{other}]) = ([Z_0] - l[F]) \cdot ([Z_0] - k[F]) = [Z_0]^2 - k - l = 0 - l$$

which is negative. Thus $Z_{other} = Z_\infty$S.

□

Corollary 5.6. *If Σ is a sphere, $Symp_{P(L^F)}$ is contractible.*

Proof. By Propositions 3.3 and 3.8, $\phi_{(\Sigma)} : Symp_{P(L^F)} \to \Sigma_L$ is a homotopy equivalence. Σ_{L^F} is give by the direct limit:

$$\Sigma_{L^{\epsilon_1}} \hookrightarrow \Sigma_{L^{\epsilon_2}} \hookrightarrow ... \hookrightarrow \Sigma_L$$

as each $\Sigma_{L^{\epsilon_i}}$ is contractible by Proposition 5.1, so must Σ_L be contractible. □

6. Applications to spaces of Lagrangian Embeddings

6.1. **Getting rid of framings.**

Definition 6.1. Let (M, ω) be a symplectic 4-manifold with the decomposition $(L, E \to \Sigma)$ such that $\phi : L \hookrightarrow M$ is a smooth submanifold of M. Then we denote by $\mathcal{L}_\Sigma^-$ the space of pairs (ψ, S) where

(1) $\psi : L \hookrightarrow M$ is a Lagrangian embedding of L symplectically equivalent to ϕ.
(2) S is a symplectic embedded unparamaterized surface which is abstractly symplectomorphic to Σ and disjoint from $\psi(L)$.

If the spine of the decomposition L is a smooth submanifold, and L satisfies suitable cohomological assumptions, one does not have to introduce Kan complexes. The situation is much simpler. In this section, we will show that in this case $Symp(M)$ is homotopy equivalent to $\mathcal{L}_\Sigma^-$

Proposition 6.2. *Let $L \hookrightarrow M$ be a Lagrangian submanifold. Suppose that $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective. Then $Symp_{P(L^F)} \hookrightarrow Symp_{P(L)}$ is a homotopy equivalence.*

Denote by $Symp_{P(L),P(N_L)}$ the symplectomorphisms which fix both L and act trivially on its normal bundle.

Lemma 6.3. *Suppose that $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective, then $Symp_{P(L^F)} \hookrightarrow Symp_{P(L),P(N_L)}$ is a deformation retract.*

Proof. We will deform any family in $Symp_{P(L),P(N_L)}$ into $Symp_{P(L^F)}$. As a neighborhood of L is symplectomorphic to T^*L and we will perform our deformation there. If $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective we can then apply the isotopy extension theorem [**?**] to extend it to all of M.

Denote by λ multiplication by λ in the fibers of T^*L. This multiplication scales the symplectic structure by λ. Thus conjugation by λ takes symplectomorphisms into themselves. We consider the conjugation of elements in $Symp_{P(L^F)}$ by $\lambda : \psi \to \frac{1}{\lambda}\psi\lambda$, as λ tends from 1 to ∞.

Near the zero section we can write any symplectomorphism in $Symp_{P(L),P(N_L)}$ as a Taylor series whose linear term is the identity map:

$$Id + quadratic + cubic...$$

The quadratic, cubic and higher order terms tend to zero under this conjugation. More specifically for the nth degree term:

$$X^n_{\lambda^{-1}\psi\lambda} = \frac{1}{\lambda^{n-1}} X^n_\psi$$

Where X^n_ψ denotes the nth degree term of $\psi's$ taylor series, and $X^n_{\lambda^{-1}\psi\lambda}$denotes the nth degree term of the conjugated symplectomorphism. Thus in the limit $\lambda \to \infty$ these higher order terms tend to zero and $\lambda^{-1}\psi\lambda \in Symp_{P(L^F)}$. □

We complete the proof of Proposition with the following Lemma:

Lemma 6.4. *Every symplectomorphism in $Symp_L$ also fixes N_L*

This follows from the corresponding linear statement:

Lemma 6.5. *Let (V, ω) be a symplectic vector space. Let $L \hookrightarrow V$ be a Lagrangian subspace. Then the only sympletic linear map which fixes L is the identity map.*

Proof. We first note that $\psi : v \to \omega(v, \cdot)$ gives an identification of V/L with the linear functions on L. ψ is well defined on the quotient V/L as L is Lagrangian, and thus for any $l \in L$ $\omega(v+l, \cdot) = \omega(v, \cdot)$. Further, the image of ψ seperates vectors in L. For if $l \in L$, there is a $v \in V$ such that $\omega(v, l)$ is non zero. As L is lagrangian, this v cannot lie in L. Thus if we denote its image in V/L by $\overline{v}$ the functional $\omega(\overline{v}, \cdot)$ is also nonzero on l. Thus $\psi(V/L)$ seperates vectors and so ψ is surjective. As the dimensions of V/L and L^* coincide the map is an isomorphism. Further any linear map η which preserves both L and the symplectic form must also preserve this identification- the induced map on V/L is the adjoint of $\eta|_L$. Thus if $\eta|_L$ is the identity, so is the induced map on V/L. □

Theorem 6.6. *Let (M, ω) be a symplectic 4-manifold with the decomposition $(L, E \to \Sigma)$ such that $\phi : L \hookrightarrow M$ is a smooth submanifold of M, and such that $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective. Then $Symp(M)$ is homotopic equivalent to $\mathcal{L}^-_\Sigma$.*

Proof. We apply our machinery to this decomposition. We consider the fibration:

$$\begin{array}{ccccc} Symp_{P(L)} & \to & Symp & \to & \mathcal{L}^- \\ \downarrow \phi_{(\Sigma)} & & \downarrow \phi_{(L,\Sigma)} & & \downarrow (id) \\ \Sigma_L & \to & \mathcal{L}^-_\Sigma & \to & \mathcal{L}^- \end{array}$$

where $Symp_L$ denotes the stabilizer of L. I claim that $\phi_{(\Sigma)}$ is a homotopy equivalence. The theorem will then follow from the 5-Lemma, Lemma10.12.

Consider the following commutative diagram:

$$\begin{array}{ccc} Symp_{P(L^F)} & \overset{i}{\hookrightarrow} & Symp_{P(L)} \\ \downarrow \phi'_{(\Sigma)} & & \downarrow \phi_{(\Sigma)} \\ \Sigma_L & \to & \Sigma_L \end{array}$$

By Proposition 6.2, the inclusion i is a homotopy equivalence. By Propositions 3.3 and 3.8 $\phi'_{(\Sigma)}$ is a homotopy equivalence. Thus by the commutivity of the diagram, ϕ_Σ is also a homotopy equivalence. □

Corollary 6.7. *Let (M, ω) be a symplectic 4-manifold with the decomposition $(L, E \to \Sigma)$ such that $\phi : L \hookrightarrow M$ is a smooth submanifold of M, $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective, and Σ is a sphere. Then the space $\mathcal{L}_\phi$ of Lagrangian embeddings isotopic to ϕ is homotopy equivalent to the identity component of $Symp(M)$.*

Proof. We re-examine the fibration:

$$Symp_{P(L)} \to Symp \to \mathcal{L}^-$$

As Σ is a sphere, Corollary 5.6implies that $Symp_{P(L^F)}$ is contractible. Thus $Symp_{P(L)}$ is also contractible, and $Symp(M)$ is homotopy equivalent to L^-. Moreover if $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective one can induce any isotopy of the embedding $\phi : L \hookrightarrow M$ by a path of symplectomorphisms. □

6.2. **Applications to spaces of Embeddings.** We now apply Corollary 6.7 to compute spaces of Lagrangian submanifolds in cases where we know the homotopy type of the symplectomorphism group of the ambient manifold M.

Corollary 6.8. *$\mathcal{L}_{RP^2}$, the space of Lagrangian embeddings of $RP^2 \hookrightarrow CP^2$ isotopic to the standard one is homotopy equivalent to $Symp(CP^2)$.*

Proof. We note the following proposition

Proposition 6.9. *(Biran [1]) There is a decomposition of CP^2 with :*

1. *Σ a quadric (and thus a sphere).*
2. *L the standard $RP^2 \hookrightarrow CP^2$.*

$H_1(RP^2)$ is torsion, thus $H_2(M) \otimes \mathbb{R} \to H_2(M, L) \otimes \mathbb{R}$ is surjective, and we can apply Corollary 6.7

□

Corollary 6.10. *$\mathcal{L}_{S^2}$the space of Lagrangian embeddings $S^2 \hookrightarrow S^2 \times S^2_{1,1}$ isotopic to the standard embedding of the diagonal is homotopy equivalent to the identity component of $Symp(S^2 \times S^2_{1,1}) \simeq SO(3) \times SO(3)$.*

Proof. We note the following proposition

Proposition 6.11. *(Biran[1]) There is a decomposition of $S^2 \times S^2_{1,1}$ with :*

1. *Σ the diagonal*

(2) *L the antidiagonal*

$H_1(S^2)$ vanishes, thus $H_2(M)\otimes\mathbb{R} \to H_2(M,L)\otimes\mathbb{R}$ is surjective, and we can apply Corollary 6.7

□

Part 2. Positive Sections

7. Introduction and summary of results

We consider the product of two n-manifolds $\Sigma \times \Gamma$. We endow each factor with a volume form given by σ_Σ and σ_Γ respectively. These volume forms induce a product n-form on $\Sigma \times \Gamma$:

$$\sigma = \pi_\Sigma^\star \sigma_\Sigma + \pi_\Gamma^\star \sigma_\Gamma$$

Denote the space of C^1 sections of π_Σ in class $[\Sigma \times pt] + a[pt \times \Gamma]$ by Σ_{F_a}.

Definition 7.1. We will call the sections $S \in \Sigma_{F_a}$ such that $\sigma|_S$ is a volume form the **positive sections**.

Products of volume forms such as σ are determined by their volume on each factor, up to diffeomorphism which preserve the product structure, and thus the fibration π_Σ. This is an immediate consequence of Moser's Lemma. The space of positive sections of π_Σ depends only on the ratio $\frac{vol(\Sigma)}{vol(\Gamma)}$, as it is invariant under scaling the form $\sigma$by a constant factor. We denote this ratio by K, and we denote the positive sections in Σ_{F_a} by $P\Sigma_a^K$. When $n = 2$ the positive sections will be the symplectic sections with respect to σ.

Denote the degree a, C^1 maps from Σ to Γ by $C_a^1(\Sigma, \Gamma)$. There is a canonical homeomorpism $\Phi : \Sigma_{F_a} \to C_a^1(\Sigma, \Gamma)$ given by considering the section $S \in \Sigma_{F_a}$as the graph of a map $\Phi(S) \in C_a^1(\Sigma, \Gamma)$. We will reserve Φ throughout this paper to denote this identification.

Definition 7.2. A C^1 map $f : \Sigma \to \Gamma$ is called **non $Q-$Surjective** if there is an open ball in $U \subset \Gamma$ such that every $x \in U$ has less than Q pre-images. If Q is 1 this is the space of non-surjective maps.

Denote the space of C^1 smooth deg a, non $Q-$surjective maps $\Sigma \to \Gamma$ by NS_a^Q Here $a \in \mathbb{Z}$, and $Q \in \mathbb{R}$. However, $NS_a^Q = \emptyset$ if $Q < |a|$. Moreover NS_a^Q changes as a function of Q only at discrete intervals:

Lemma 7.3. $NS_a^{Q_1} = NS_a^{Q_2}$ *if* $\lfloor \frac{Q_1 - a}{2} \rfloor = \lfloor \frac{Q_2 - a}{2} \rfloor$.

Proof. Let $f \in NS_a^Q$. Let $U \subset \Gamma$ be an open ball such that every $x \in U$ has less than Q pre-images. Let $x \in U$ be a regular value of f. As Γ and Σ are both compact, the set of regular values of f is open. Let U_{reg} be a neighborhood of x consisting of regular values of f. Then for every y in U, $f^{-1}(y)$ has the same cardinality.

$$Card(f^{-1}(y)) = a + 2l < Q$$

for l a positive integer. The claim follows. □

The main Theorem of this section is the following:

Theorem 7.4. *$\Phi(P\Sigma_a^K) \subset NS_a^{2K+a}$, and the inclusion $\Phi : P\Sigma_a^K \hookrightarrow NS_a^{2K+a}$ is a deformation retract.*

Note that neither $P\Sigma_a^K$ nor NS_a^{2K+a} is changed if we scale σ by a constant factor so that $vol(\Gamma) = 1$. For simplicity of exposition we do so. K henceforth denotes $vol(\Sigma)$.

Theorem 7.4 has the following corollaries:

Corollary 7.5. *$P\Sigma_a^{K_1} = P\Sigma_a^{K_2}$ if $\lfloor K_1 \rfloor = \lfloor K_2 \rfloor$*

Proof. This follows immediately by combining Theorem 7.4 with Lemma 7.3 □

Corollary 7.6. *Suppose that Γ admits a degree -1 diffeomorphism ϕ, then $P\Sigma_a^K$ is homotopy equivalent to $P\Sigma_{-a}^{K+a}$.*

Proof. ϕ determines a homeorphism:

$$\begin{aligned} \phi_* : NS_a^{2K+a} &\rightarrow NS_{-a}^{2K+a} \\ \alpha &\rightarrow \phi \circ \alpha \end{aligned}$$

By Theorem 7.4 we have homotopy equivalences:

$$\begin{aligned} P\Sigma_a^K &\simeq NS_a^{2K+a} \\ P\Sigma_{-a}^{K+a} &\simeq NS_{-a}^{2K+a} \end{aligned}$$

combining these with the homeomorphism ϕ yields the Corollary. □

Remark 7.7. In a later paper, we will combine this result with identities in the spaces of symplectic embeddings in $S^2 \times S^2$ to show that the homotopy type of the space of sections of a fibration must change (in certain classes) as the fibration moves in the space of symplectic fibrations.

We now commence with the proof of Theorem 7.4. It will carry us through the next two sections.

8. Proof of Theorem 7.4

Definition 8.1. Let α be an n-form on an oriented n-manifold Σ. Denote by $Dgn(\alpha) \subset \Sigma$ the $x \in \Sigma$ such that $\alpha(x) \leq 0$, where this sign is determined by the orientation of M. Denote by $Neg(\alpha) \supset Dgn(\alpha)$ the $x \in \Sigma$ such that $\alpha(x) < 0$.

If $f : \Sigma \to \Gamma$ and σ_Γ is a volume form on on Γ we will sometimes denote $Dgn(f^*\sigma_\Gamma)$ and $Neg(f^*\sigma_\Gamma)$ by $Dgn(f)$ and $Neg(f)$ respectively, for these sets depend only on f, and not on the choice of volume form σ_Γ.

8.1. Definition and Basic Properties of Negative Area.

Definition 8.2. Define the **negative area** of an n-form α denoted $NA(\alpha)$ to be $-\int_{Neg(\alpha)} \alpha$.

If $f : \Sigma \to \Gamma$ we will denote $NA(f^*\sigma_\Gamma)$ by $NA(f)$.

For regular values of $f : x \in reg(f) \subset \Gamma$ denote by $\mu_f(x)$ the cardinality of $f^{-1}(x) \cap Neg(f)$ Then:

Lemma 8.3. *Let η be a volume form on Γ. Then $NA(f^*\eta) = \int_{reg(X)} \mu_f(x)\eta$*

Proof. $f|_{Neg(f)}$ is a covering map over each connected component of $reg(x)$, the regular values of f in X. This may be the empty cover over certain components - some regular x may have no negative preimages.Thus

$$f|_{Neg(f) \cap f^{-1}(X_i)}$$

is a cover. $\mu_f(x)$ is constant for $x \in X_i$, and gives the number of sheets in this cover. Thus

$$\int_{Neg(f) \cap f^{-1}(X_i)} f^*\eta = \mu_f(x) \int_{X_i} \mu$$

We gain the Lemma by integrating over each X_i. □

Lemma 8.4. *Let $\gamma \in Diff(\Sigma)$, α be an n-form on Σ. Then $NA(\gamma^*\alpha) = NA(\alpha)$*

Proof. $Neg(\gamma^*\alpha) = \gamma^{-1}(Neg(\alpha))$. Then

$$\int_{Neg(\gamma^*\alpha)} \gamma^*\alpha = \int_{\gamma^{-1}Neg(\alpha)} \gamma^*\alpha = \int_{Neg(\alpha)} \alpha$$

End Lemma 8.4 □

Lemma 8.5. *$NA : \Omega^n(\Sigma) \to \mathbb{R}$ is continuous in the C^0 topology on forms.*

Proof. Let $\alpha \in \Omega^n(\Sigma)$. Let β be C^0close to α. Then $\int_\Sigma |\alpha - \beta| < \delta$, where we can take δ to be as small as we like by moving β closer to α. Then

$$\int_{Neg(\alpha)\cup Neg(\beta)} \alpha - \int_{Neg(\alpha)} \alpha < \int_{Neg(\alpha)\cup Neg(\beta)\setminus Neg(\alpha)} \alpha - \beta < \int_\Sigma |\alpha - \beta| < \delta$$

Similarily:

$$\int_{Neg(\alpha)\cup Neg(\beta)} \beta - \int_{Neg(\beta)} \beta < \int_{Neg(\alpha)\cup Neg(\beta)\setminus Neg(\alpha)} \beta - \alpha < \int_\Sigma |\alpha - \beta| < \delta$$

Finally

$$|\int_{Neg(\alpha)\cup Neg(\beta)} \alpha - \int_{Neg(\alpha)\cup Neg(\beta)} \beta| < \int_{Neg(\alpha)\cup Neg(\beta)} |\alpha - \beta| < \delta$$

Thus

$$\left| \int_{Neg(\alpha)} \alpha - \int_{Neg(\beta)} \beta \right| < 3\delta$$

End Lemma 8.5 □

Definition 8.6. Denote by $NA_a^K$the space of degree a maps $f \in C^1(\Sigma, \Gamma)$ such that $NA(f^*\sigma_\Gamma) < K$.

8.2. **Symplectic Sections are a Deformation Retract of Maps with Bounded Negative Area.** This section is devoted to the proof of the following proposition:

Proposition 8.7. *$\Phi(P\Sigma_a^K) \subset NA_a^K$ and the inclusion $\Phi : P\Sigma_a^K \hookrightarrow NA_a^K$is a deformation retract.*

We first note that $\Phi : P\Sigma_a^K \subset NA_a^K$. For if $S \in P\Sigma_a^K$ the following equation holds for any domain U in S :

$$\int_U \sigma = \int_U \pi_\Sigma^*(\sigma_\Sigma) + \int_U \pi_\Gamma^*(\sigma_\Gamma) > 0$$

holds for integration over any subset of the section. If we take this domain to be the subset $Dgn(\Phi(S))$

$$\int_{Dgn(\Phi(S))} \pi_\Sigma^*(\sigma_\Sigma) - NA(\Phi(S)) > 0$$

and thus

$$NA(\Phi(S)) < \int_{Dgn(\Phi(S))} \pi_\Sigma^*(\sigma_\Sigma) < [\sigma](\Sigma) = K$$

Now consider a disc ρ of non-surjective maps with boundary in $\Phi(P\Sigma_a^K)$:

$$\rho : (D^n, \delta D^n) \to (NA_a^K, \Phi(P\Sigma_a^K))$$

We will construct a retraction of this disc into the positive sections $\Phi(P\Sigma_a^K)$. i.e. we will construct a homotopy of pairs

$$\rho_t : (D^n \times I, \delta D^n \times I) \to (NA_a^K, \Phi(P\Sigma_a^K))$$

such that:

(1) $\rho_0(d) = \rho$
(2) $\rho_1(D^n) \subset \Phi(P\Sigma_a^K)$
(3) $\rho_t|_{\delta D^n} = \rho|_{\delta D^n}$ for all t

Denote the space of volume forms on Σ by $Vol(\Sigma)$. Denote those in class $[\sigma_\Sigma]$ by $Vol(\Sigma)_{[\sigma]}$. We will construct ρ_t by constructing a family

$$\zeta_\rho : (D^m, \delta D^m) \to (Vol(|\Sigma)_{[\sigma]}, \sigma_\Sigma)$$

Then using Moser's Lemma we will provide a family of diffeomorphisms ϕ_t of Σ. $\phi_t \times id$ will then be a family of diffeomorphisms of $\Sigma \times \Gamma$ which will induce ρ_t.

Lemma 8.8. *Let $\rho : D^m \to NA_a^K$ be a disc of maps, such that $\rho(\delta D^m) \subset \Phi(P\Sigma_a^K)$. Then there is a continuous map $\zeta_\rho : D^m \to Vol(\Sigma)$ such that:*

a *With respect to the form $\pi_\Gamma^* \sigma_\Gamma + \pi_\Sigma^* \zeta_\rho(d)$ the section $\Phi^{-1}(\rho(d))$ is positive.*
b *For all d in the family $[\zeta_\rho(d)] = [\sigma_\Sigma]$*
c $\zeta_\rho(\delta D^m) = \sigma_\Sigma$

Proof. We first note that if one restricts $\pi^* \sigma_\Gamma$ to the section $\Phi^{-1}(\rho(d))$, and then uses π_Σ to identify $\Phi^{-1}(\rho(d))$ with Σ, the resulting form is $\rho(d)^*(\sigma_\Gamma)$. Thus Condition (a) can be rephrased as: "the form

$$\rho(d)^*(\sigma_\Gamma) + \zeta_\rho(d)$$

is a volume form on Σ."

Denote $\rho(d)^*(\sigma_\Gamma)$ by $\sigma_\Gamma(d)$ to lighten our notation. We begin by adding a positive form to σ_Σ where $\sigma_\Gamma(d) + \sigma_\Sigma$ is (nearly) degenerate. Let $\delta \in \mathcal{V}ol(\Sigma)$. We will think of δ as small, and we will specify how small shortly. Let

$$U_\delta = \left(\bigcup_{d \in \Sigma} Neg((\sigma_\Gamma(d) + \sigma_\Sigma - \delta) \times d \right)$$

Claim 8.9. U_δ is open, and $\delta - \sigma_\Gamma(d) > 0|_{U_\delta}$.

Proof. Let $(x,d) \in U_\delta$. Trivialize the tangent bundles of Σ and Γ in neighborhoods U_x of x and $U_{\rho(d)(x)}$ of $\rho(d)(x)$ respectively. Use these trivializations to identify T_{x_i} with T_x and $T_{\rho(d_i)x_i}$ with $T_{\rho(d)x}$ for $(x,d) \in U_x \times U_{\rho(d)(x)}$. Then, for (x_0,d_0) near (x,d) the anti-linear $n-$forms $\sigma_\Gamma(d_0) = \rho(d_0)^*(\sigma_\Gamma)_0$ are near the antilinear $n-$form $\rho(d)^*(\sigma_\Gamma)_x = \sigma_\Gamma(d)$. δ_x is also near δ_{x_0}. Thus as $\sigma_\Gamma(d)_x < \delta_x$ so must $\sigma_\Gamma(d_0)_{x_0} < \delta_{x_0}$. The second statement is immediate from the definition. ☐

Let $\phi_1 : \Sigma \to \mathbb{R}$ be a function such that:

(1) $\phi_1(x) > 0$ for $x \in U_\delta$
(2) $\phi_1(x) = 0$ for $x \in \Sigma \backslash U_\delta$

The map:

$$\zeta^1_\rho(d) = \phi_1(\delta - (\sigma_\Gamma(d) + \sigma_\Sigma)) + \sigma_\Sigma$$

then achieves condition (a), but may well fail the rest. In particular its volume is probably not $\int_\Sigma \sigma_\Sigma$. In fact:

$$\int_\Sigma \zeta^1_\rho(d) \geq \int_\Sigma \sigma_\Sigma \tag{8.1}$$

We now aim to modify ζ^1_ρ so that $\int_\Sigma \zeta^1_\rho(d) = \int_\Sigma \sigma_\Sigma$, and thus achieve condition (b).

Let f be a smooth function on $D^m \times \Sigma$ such that:

i $f(x) = 1$ for $x \in \bar{U}_\delta$ (the closure of U_δ)
ii $0 < f(x) < 1$ elsewhere

We now consider the map:

$$\Psi : (d,k) \quad \to \quad (d, \int_\Sigma f^k_d \zeta^1_\rho(d))$$

As k approaches infinity the function f_d approaches the characteristic function of $\bar{U}_\delta$, and thus the integral $\int_\Sigma f^k_d \zeta^1_\rho(d)$ approaches $\int_{U_\delta(d)} \zeta^1_\rho(d)$. Moreover $\Psi(\cdot, d)$ is montone decreasing in k. Thus the map gives a diffeomorphism:

$$D^m \times [0,\infty) \quad \to D^m \times \quad (\int_{U_\delta(d)} \zeta^1_\rho(d), \int_\Sigma \zeta^1_\rho(d)]$$

Note that :

$$\begin{aligned}\int_{U_\delta(d)} \zeta_\rho^1(d) &= (NA(\rho(d)) + \int \delta_{U_\delta(d)} + \int_{(U_\delta(d)-Deg(\sigma_\Gamma(d)))} \zeta_\rho^1(d) \\ &= (NA(\rho(d)) + \epsilon_d)\end{aligned}$$

where we can make ϵ_d as small as we like by making both δ and our neighborhoods U_δ small. Thus, as we assume that $NA(\rho(d)) < \int \sigma_\Sigma$, we may force ϵ_d to be small enough so that

$$\int_{U_\delta(d)} \zeta_\rho^1(d) = (NA(\rho(d)) + \epsilon_d) < \int \sigma_\Sigma$$

Combining this with 8.1 we see that the interval $(\int_{U_\delta(d)} \zeta_\rho^1(d), \int_\Sigma \zeta_\rho^1(d)]$ must contain $\int \sigma_\Sigma$.

Choose $k(d)$ such that $\Psi(d, k(d)) = (d, \int_\Sigma \sigma_\Sigma)$. Then

$$\int_\Sigma f_d^{k(d)} \zeta_\rho^1(d) = \int_\Sigma \sigma_\Sigma$$

and

$$\zeta_\rho^2 = f^{k(d)} \zeta_\rho^1$$

will thus satisfy (b). Moreover ζ_ρ^2 will still satisfy condition (a) as $f^{k(d)} = 1$ on U_δ, and thus $\zeta_\rho^2 = \zeta_\rho^1$ there.

The only condition on ζ_ρ which remains is (c). By assumption $\rho(\delta D)$ is symplectic with respect to our original form $\sigma = \pi_\Gamma^\star \sigma_\Gamma + \pi_\Sigma^* \sigma_\Sigma$. The condition of positivity is open, thus there is some neighborhood $U_{\delta D}$ of δD^m such that $\pi_\Gamma^\star \sigma_\Gamma + \pi_\Sigma^* \sigma_\Sigma$ makes each map $\rho(U_{\delta D})$ positive. Let $\phi_{\delta D}$, ϕ_{int} be a partition of unity subordinate to the cover given by $U_{\delta D}$ and a slightly smaller interior disc $D_{int} \subset D^m$. Then, as both conditions (a) and (b) are convex. :

$$\zeta_\rho = \phi_{\delta D} \sigma_\Sigma + \phi_{int} \zeta_\rho^2$$

provides our ζ_ρ satisfying each condition. **End Lemma 8.8**

□

8.3. **ζ_ρ to a retraction of ρ via Moser's lemma.** We now use this family of forms ζ_ρ to construct our isotopy of sections. Consider the homotopy:

$$\zeta_t = t\sigma_\Sigma + (1-t)\zeta_\rho$$

where σ_Σ denotes the constant map $D^m \to \sigma_\Sigma$. ζ_t is a homotopy of ζ_ρ to σ_Σ which fixes δD^m throughout. Moreover

$$[\zeta_t(d)] = t[\sigma_\Sigma] + (1-t)[\zeta_\rho(d)] = [\sigma_\Sigma]$$

as $[\zeta_\rho(d)] = [\sigma_\Sigma]$ by Lemma 8.8 condition 2.Thus Moser's Lemma applies and so if we denote the diffeomorphisms of Σ by $Diff(\Sigma)$ we obtain:

$$M_\zeta : (D^m \times I, \delta D \times I) \to (Diff(\Sigma), Id)$$

such that $M_\zeta(d,1)^*(\sigma_\Sigma) = \zeta_\rho(d)$.

Let

$$\rho_t = \rho(d) M_\zeta^{-1}(d,t)$$

By Lemma 8.4:

$$NA(\rho(d) M_\zeta^{-1}(d,t)) = NA(\rho(d)) < K$$

Thus $\rho_t(d) \in NA_a^K$. I claim that $\Phi^{-1}(\rho_1(d))$is symplectic with respect to our original form $\sigma = \pi_\Sigma^\star \sigma_\Sigma + \pi_\Gamma^\star \sigma_\Gamma$. For $\Phi^{-1}(\rho_1(d))$ is given by the graph of $\rho_1(d)$:

$$(M_\zeta(d,1)x, \rho(d)(x)) \in \Sigma \times \Gamma$$

This section is the same as that obtained by applying $(M_\zeta(d,1) \times Id)$ to $\Phi^{-1}(\rho(d))$. That is:

$$(M_\zeta(d,1) \times Id)(\Phi^{-1}(\rho(d)) \quad = \quad (M_\zeta(d,1)x, \rho(d)(x))$$

Thus

$$\begin{aligned} \sigma|_{\Phi^{-1}(\rho_1(d))} &= (M_\zeta(d,1) \times Id)^* \sigma|_{\Phi^{-1}(\rho(d))} \\ &= (\sigma_\Gamma + \zeta_\rho(d))|_{\Phi^{-1}(\rho(d))} \end{aligned}$$

which is everywhere nondegenerate by condition (b) of Lemma 8.8 .
End Proposition 8.7

8.4. **Maps of bounded negative area are a weak deformation retract of non Q-surjective maps.** For $f \in C^1(\Sigma, \Gamma)$ denote by $\mu_f(x)$ the cardinality of $f^{-1}(x) \cap Neg(f)$.

Lemma 8.10. *Let f be a degree a map in $C^1(\Sigma,\Gamma)\backslash NS_a^{2K+a}$ then $\mu_f(x) \geq K$ for all regular points $x \in X$.*

Proof. If $a \geq 0$: Then x has (at least) $2K$ "excess" pre-images. Half of these must be negative.

If $a < 0$: Then x has (at least) $2K - 2|a|$ "excess" pre-images. Again half of these must be negative. We also have $|a|$ negative pre-images coming from the degree of the map. This again yields K in total. □

Proposition 8.11. *NA_a^K is contained in NS_a^{2K+a}, and the inclusion $i : NA_a^K \hookrightarrow NS_a^{2K+a}$ is a weak deformation retract.*

Proof. We first show that $NA_a^K \subset NS_a^{2K+a}$. Let f be a degree a map in $C^1(\Sigma, \Gamma) - NS_a^{2K+a}$. I claim that f is not in NA_a^K. For by Lemma 8.10 f must cover a dense set $X \subset \Gamma$ at least $(2K + a)$ times.

Claim 8.12. By Lemma 8.3,

$$\begin{aligned} NA(f) &= \int_{reg(f)} \mu_f \sigma \\ &= \int_{reg(f)\cap X} \mu_f \sigma \\ &\geq K \int_{reg(f)\cap X} \sigma \\ &= K \end{aligned}$$

Thus f is not in NA_a^K, and $NA_a^K \subset NS_a^{2K+a}$. We will show that this inclusion is a weak deformation retract. Consider then a map of pairs:

$$\phi : (D^n, \delta D) \to (NS_a^{2K+a}, NA_a^K)$$

Our strategy is the same as before. Denote the volume forms on Γ by $\mathcal{V}ol(\Gamma)$. We will construct a map η_ϕ

$$\eta_\phi : (D^n, \delta D) \to (\mathcal{V}ol(\Gamma)_{[\sigma]}, \sigma_\Gamma)$$

such that

$$NA(\phi(d), \eta_\phi(d)) < K$$

We will then contract η_ϕ to the constant map $D^n \to \sigma_\Gamma$. Moser will then yield a family of diffeomorphisms. Post composition with these diffeomorphisms will contract our disc of maps to those with negative area $< K$, while fixing the boundary.

Lemma 8.13. *Let $\phi : (D^n, \delta D) \to (NS_a^{2K+a}, NA_a^K)$ There is a continuous function $\eta_\phi : (D^n, \delta D) \to (\mathcal{V}ol(\Gamma)_{[\sigma]}, \sigma_\Gamma)$ such that*

$$NA(\phi(d), \eta_\phi(d)) < K$$

Proof. First we construct a form η_ϵ such that for a fixed d the map $\phi(d)$ has $NA(\phi(d), \eta_\epsilon) < K$:

Partition the sphere into a set $X_<$ with less than $2K + a$ pre images, and its complement $X_\geq$. As $\phi : (D^n) \subset NS_a^{2K+a}$, $X_<$ has nonempty interior. We may thus find a volume form η_ϵ such that makes $X_<$ very large:

$$\int_{X_<} \eta_\epsilon = 1 - \epsilon$$

and $X_\geq$very small:

$$\int_{X_\geq} \eta_\epsilon = \epsilon$$

We may further require that

$$\frac{\eta_\epsilon(x)}{\sigma_\Gamma(x)} < C(\epsilon)$$

for all $x \in X_\geq$where $C(\epsilon) > 0$ is a constant which can be made as small as we like for small ϵ.

Now $x \in X_<$ has $< 2K + a$ pre images under a map and thus:

$$\mu_{\phi(d)}(x) < K - \delta$$

for $x \in X$, and some definite $\delta > 0$, given by the difference between K and the next lowest integer. By Lemma 8.10:

$$\begin{aligned} NA(\phi(d)|_{\phi(d)^{-1}(X_<)}, \eta_\epsilon) &= \int_{X_<} \mu_{\phi(d)}(x)\eta_\epsilon \\ &< (K-\delta)\int_{X_<} \eta_\epsilon \\ &< (K-\delta)(vol_{X_<}). \end{aligned}$$

On $X_\geq$ we have the following bound:

$$NA(\phi(d)^*\eta_\epsilon|_{\phi(d)^{-1}(X_\geq)}) < C(\epsilon)\left(NA(\phi(d)^*\sigma_\Gamma|_{\phi(d)^{-1}(X_\geq)})\right)$$

As:

$$NA(\phi(d)^*\eta_\epsilon) = NA(\phi(d)^*\eta_\epsilon|_{\phi(d)^{-1}(X_<)}) + NA(\phi(d)^*\eta_\epsilon|_{\phi(d)^{-1}(X_\geq)})$$

We have:

$$NA(\phi(d)^*\eta_\epsilon) < (K-\delta)(1-\epsilon) + C(\epsilon)\left(NA(\phi(d)^*\sigma_\Gamma|_{\phi(d)^{-1}(X_\geq)})\right)$$

which approaches $K - \delta$ as $\epsilon \to 0$. It is thus less than K for small ϵ.

By Lemma 8.5, $NA(\cdot, \eta_\epsilon) : C^1(\Sigma, \Gamma) \to R$ is continuous, thus there is some neighborhood of d denoted U_d such that for $d_1 \in U_d$ $NA(d_1, \eta_\epsilon) < K$. Let

$$\bigcup_{d_i} U_{d_i}$$

be a covering of $int(D^n)$ by open sets U_{d_i} with forms η^i_ϵ such that for $d \in U_{d_i}$ $NA(\phi(d)^*\eta^i_\epsilon) < K$. Let $U_{\delta D}$ be a neighborhood of the boundary such that $NA(\phi(d)^*\eta_\phi) < K$ for $d \in U_{\delta D}$. Finally let $\{\gamma_i, \gamma_\delta\}$ be a partition of unity subordinate to the covering of D^n given by the U_{d_i} and $U_{\delta D}$. Then, as both $NA(f, \cdot) < K$ and having cohomology class $[\sigma_\Gamma]$ are convex conditions:

$$\eta_\phi = \gamma_\delta \sigma_\Gamma + \Sigma_i \gamma_i \eta^i_\epsilon$$

satisfies each condition. **End Lemma 8.13** □

8.5. η_ϕ**to a retraction of** ϕ **via Moser's lemma.** We now use this family of forms η_ϕ to construct our isotopy of sections. Consider the homotopy:

$$\eta_t = t\sigma_\Sigma + (1-t)\eta_\phi$$

where σ_Σ denotes the constant map $D^n \to \sigma_\Sigma$. ζ_t is a homotopy of η_ϕ to σ_Σ which fixes δD^n throughout. Moreover

$$[\eta_t(d)] = t[\sigma_\Sigma] + (1-t)[\eta_\phi(d)] = [\sigma_\Sigma]$$

as $[\eta_\phi(d)] = [\sigma_\Sigma]$ by Lemma 8.8 condition (b) Moser's Lemma applies and so if we denote the diffeomorphisms of Σ by $Diff(\Sigma)$ we obtain:

$$M_\eta : (D^n \times I, \delta D \times I) \to (Diff(\Sigma), Id)$$

such that $M_\eta(d,1)^*(\sigma_\Sigma) = \eta_\phi(d)$. Let

$$\phi_t(d) = M_\eta(d,t) \circ \phi(d)$$

Clearly ρ_t remains in NS^K_a:postcomposing a map with a diffeomorphism doesn't change its Q-surjectivity.

I claim that

$$NA(\phi_1(d)^*\sigma_\Gamma) < K$$

For $\phi_1(d) = M_\eta(d,1) \circ \phi(d)$. Thus $\phi_1(d)^*\sigma_\Gamma = \phi(d)^* M_\eta(d,1)^*\sigma_\Gamma$. So

$$\begin{aligned} NA(\phi_1(d)^*\sigma_\Gamma) &= NA(\phi(d)^* M_\eta(d,1)^*\sigma_\Gamma) \\ &= NA(\phi(d)^*\eta_\phi(d)) < K \end{aligned}$$

End Proposition 8.11

□

Combining Propositions 8.7 and 8.11 we achieve Theorem 7.4
End Theorem 7.4

9. $S^2 \to S^2$

In this section we examine the case where $\Sigma = \Gamma = S^2$. Then $\sigma = \pi_\Sigma^\star \sigma_\Sigma + \pi_\Gamma^\star \sigma_\Gamma$ is a symplectic form, and the positive sections $P\Sigma_a^K$ are symplectic sections for the product fibration.

Our goal will be the proofs of the following two theorems:

Theorem 9.1. *$P\Sigma_0^K$ is homology equivalent to S^2 for $K \in (0,1)$*

Theorem 9.2. *$P\Sigma_a^K$ is homology equivalent to $SO(3)$ for :*

(1) $K \in [0,1)$, $a = 1$
(2) $K \in [1,2)$, $a = -1$

9.1. Simplicial Approximation.

Let T_0 be a triangulation of Γ. We consider the system of triangulations T_i of Γ, where T_{i+1} denotes the barycentric subdivision of T_i.

Convention: In the arguments that follow, it will be convenient for our triangles to have smooth boundary. We replace each triangle Δ in T_i by a smooth, closed neighborhood $\widetilde{\Delta} \supset \Delta$ which "rounds off" the corners of Δ. We will abuse notation throughout; when we refer to a triangle Δ in a triangulation T_i we will mean this smooth neighborhood $\widetilde{\Delta}$.

Definition 9.3. .Denote by (NS_a^{a+1}, T_i) the degree a maps $f \in C^1(\Sigma, \Gamma)$ such that there is a triangle $\Delta \in T_i$ such that $f^{-1}(\Delta)$ consists of a disjoint discs D_i, and $f : D_i \to \Delta$ is a diffeomorphism, and f maps $\Sigma \backslash \bigcup D_i \to \Gamma \backslash \Delta$.

If $a = 0$ these are the degree 0 maps which miss Δ.

This definition is stable under refinement: If $\Delta_1 \subset \Delta$ then $f^{-1}(\Delta_1)$ consists of a disjoint discs $D_i^1 \subset D_i$ and f restricted to these is a diffeomorphism, and it maps their complement to $\Gamma \backslash \Delta_1$. Thus as every triangle $\Delta \in T_i$ contains several triangles in T_{i+1} $(NS_a^Q, T_i) \hookrightarrow (NS_a^k, T_{i+1})$. Moreover we have:

Lemma 9.4. *For $0 < Q \leq 2$, $NS_a^{|a|+Q}$ is the direct limit of the system*
.

$$(NS_a^{a+1}, T_0) \hookrightarrow (NS_a^{a+1}, T_1) \hookrightarrow \cdots (NS_a^{a+1}, T_i) \hookrightarrow (NS_a^{a+1}, T_{i+1}) \hookrightarrow \cdots$$

By Lemma 7.3 it is enough to consider the case $Q = 1$. Moreover, it is enough to show that $f \in C^1(\Sigma, \Gamma)$ is in NS_a^{a+1} if and only if there exists a disc $D \subset \Sigma$ such that $f^{-1}(D)$ consists of a disjoint discs D_i, and $f : D_i \to \Delta$ is a diffeomorphism, for every such D will contain a triangle $\Delta \in T^i$ for $i >> 0$. It is clear that such f lie in $NS_a^{|a|+Q}$. We commence with the converse:

If $f \in NS_a^{|a|+Q}$, then there is some disc $D_1 \subset \Gamma$ such that every point in D_1 has a pre-images. Let $x \in D_1$ be a regular value of f. Then f is a diffeomorphism in a neighborhood U_{y_i} of each $y_i \in f^{-1}(x)$. We may then choose our discs D_i' to be disjoint, lying in U_{y_i}, and such that $f(D_i) \subset D_2$. Denote by D_i the connected components of $D \subset f^{-1} \bigcap f(D_i')$. Then $f : D_i \to D$ is a diffeomorphism, and f maps the complement of $\bigcup_i D_i$ to the complement of D.

9.2. Coverings of (NS_a^K, T_i).

Definition 9.5. Let T be a triangulation of Γ, Denote by $\{\Delta_k, k \in S\}$ the closed n-simplices in Γ. Let $U_k \subset (NS_a^{a+Q}, T)$ be the maps such that $f^{-1}(\Delta_k)$ consists of a disjoint discs D_i, and $f : D_i \to \Delta_k$ is a diffeomorphism, and f maps $\Sigma \backslash \bigcup D_i \to \Gamma \backslash \Delta_k$. Then $\bigcup_{k \in S} U_k = (NS_a^{a+Q}, T)$.

These coverings will be used to construct homology equivalences $S^2 \to (NS_0^2, T_i)$, and $Diff(S^2) \to (NS_1^3, T_i)$ via the following proposition.

Proposition 9.6. *Let $f : X \to Y$ be a continuous map. Let U_Y^i be a finite covering of Y by m open sets, and denote $f^{-1}(U_Y^i)$ by U_X^i. Suppose that the map f is a homology equivalence on each U_X^i and each of their mutual intersections, then f is a homology equivalence.*

Proof. **Idea:** Given a covering $\bigcup_{i \le m} U_X^i$ of a space X, one can compute the homology of X from the the homology of the U_x^i and their mutual intersections via an inductive application of Mayer-Vietoris. This proposition follows from noting that f provides a isomorphism of the "inductive processes" resulting from the covers $\bigcup_{i \le m} U_X^i$ and $\bigcup_{i \le m} U_Y^i$ of X and Y respectively.

Formal proof: The trick in carrying out this idea is to find an inductive hypothesis of the proper strength. The following suffices:

Suppose:

$$f|(\bigcup_{i \le k} U_X^i) \cap (\bigcap_{k < i \le l} U_X^i)$$

is a homology equivalence, for all $k < k_0$ then

$$f|(\bigcup_{i \le k_0} U_X^i) \cap (\bigcap_{k_0 < i \le l} U_X^i)$$

is a homology equivalence.

The conclusion of the proposition, that $f : X \to Y$ is a homology equivalence, is the special case $k_0 = l = m$. Note moreover that the initial case $k_0 = 0$ follows by our assumption that:

$$f : (\bigcap_{i \in S} U_X^i) \to (\bigcap_{i \in S} U_Y^i)$$

is a homology equivalence for each subset $S \subset \{1,..k\}$.

We thus proceed with the proof of the inductive hypothesis. So that our notation does not overwhelm our argument we introduce the following abbreviations:

$$\begin{aligned} X_{i_0..i_j} &:= \bigcup_{i_0 \le i \le i_j} U_X^i \\ X^{i_0..i_j} &:= \bigcap_{i_0 \le i \le i_j} U_X^i \\ X_{i_0..i_j}^{l_0..l_q} &:= X_{i_0..i_j} \cap X^{l_0..l_q} \end{aligned}$$

and similarly for $Y_{i_0..i_j}$ and $Y^{i_0..i_j}$. Note that $X_{i_0..i_0} = X^{i_0..i_0} = U_{i_0}$. Then:

$$\begin{aligned} X_{1..k}^{k+1..l} &= (\bigcup_{i \le k} U_X^i) \cap (\bigcap_{k < i \le l} U_X^i) \\ &= \left((\bigcup_{i \le k-1} U_X^i) \cap (\bigcap_{k < i \le l} U_X^i) \right) \cup (\bigcap_{k-1 < i \le l} U_X^i) \\ &= (X_{1..k-1}^{k+1..l}) \cup (X^{k..l}) \end{aligned}$$

We then apply Mayer-Vietoris to the covers given by $(X_{1..k-1}^{k+1..l}) \cup (X^{k..l})$ and $(Y_{1..k-1}^{k+1..l}) \cup (Y^{k+1..l})$ of X and Y respectively. The result is the following morphism of exact sequences:

$$\begin{array}{ccccc} H_*\left((X_{1..k-1}^{k+1..l}) \cap (X^{k..l})\right) & \to & H_*\left((X_{1..k-1}^{k+1..l}) \oplus (X^{k..l})\right) & \to & H_*\left((X_{1..k-1}^{k+1..l}) \cup (X^{k..l})\right) \\ \downarrow f_*(left) & & \downarrow f_* \oplus f_*(middle) & & \downarrow f_*(right) \\ H_*\left((Y_{1..k-1}^{k+1..l}) \cap (Y^{k..l})\right) & \to & H_*\left((X_{1..k-1}^{k+1..l}) \oplus (X^{k..l})\right) & \to & H_*\left((Y_{1..k-1}^{k+1..l}) \cup (Y^{k..l})\right) \end{array}$$

We wish to show that the right morphism:

$$\begin{array}{c} H_*\left((X_{1..k-1}^{k+1..l}) \cup (X^{k..l})\right) \\ \downarrow f_*(right) \\ H_*\left((Y_{1..k-1}^{k+1..l}) \cup (Y^{k..l})\right) \end{array}$$

is an isomorphism. By the 5-lemma it is sufficient to show that the other two maps induced by f are isomorphisms. The inductive hypothesis implies that each factor of the middle map:

$$H_*\left((X^{k+1..l}_{1..k-1})\oplus(X^{k..l})\right)$$
$$\downarrow f_*\oplus f_*(middle)$$
$$H_*\left((X^{k+1..l}_{1..k-1})\oplus(X^{k..l})\right)$$

is an isomorphism, and thus so is their sum. For the left map:

$$H_*\left((X^{k+1..l}_{1..k-1})\cap(X^{k..l})\right)$$
$$\downarrow f_*(left)$$
$$H_*\left((Y^{k+1..l}_{1..k-1})\cap(Y^{k..l})\right)$$

we note that:

$$\begin{aligned}(X_{1..k-1}\cap X^{k+1..l})\cap(X^{k..l}) &= (\bigcup_{i\le k-1} U^i_X)\cap(\bigcap_{k<i\le l} U^i_X)\cap(\bigcap_{k-1<i\le l} U^i_X)\\ &= (\bigcup_{i\le k-1} U^i_X)\cap(\bigcap_{k-<i\le l} U^i_X)\\ &= (X_{1..k-1}\cap X^{k..l})\end{aligned}$$

and analogously

$$(Y_{1..k-1}\cap Y^{k+1..l})\cap(Y^{k..l}) \quad = \quad (Y_{1..k-1}Y^{k..l})$$

thus our left map

$$H_*\left((X^{k+1..l}_{1..k-1})\cap(X^{k..l})\right)$$
$$\downarrow f_*(left)$$
$$H_*\left((Y^{k+1..l}_{1..k-1})\cap(Y^{k..l})\right)$$

is also an isomorphism by the inductive hypothesis.Homology equivalence $S^2 \to (NS^2_0)$ □

We now restrict ourselves to the case $\Sigma = \Gamma = S^2$. Denote the constant maps from $S^2 \to S^2$ by CM. Then CM is homeomorphic to S^2, and$CM \subset (NS^2_0 T_i)$.

Theorem 9.7. *$\eta : CM \hookrightarrow NS^2_0$ is a homology equivalence.*

If $i < j$, η factors as:

$$\eta : CM \hookrightarrow (NS^2_0, T_i) \hookrightarrow (NS^2_0, T_j) \hookrightarrow NS^2_0$$

Moreover as,

$$\bigcup_{i=0..\infty} (NS^2_0, T_i) = NS^2_0$$

and as homology commutes with direct limits, it is sufficient to show that:

Lemma 9.8. *$f : CM \hookrightarrow (NS^2_0, T_i)$ is a homology equivalence.*

Proof. We consider the cover U_{Δ_j} of $(NS^2_0 T_i)$ provided by Definition . The sets $V_{\Delta_j} := f^{-1}(U_{\Delta_j})$ then cover CM. I claim that

$$f : \bigcap_{j\in S} V_{\Delta_j} \to \bigcap_{j\in S} U_{\Delta_j}$$

is a homology equivalence for all indexing sets $S \subset 1..m$. When combined with Proposition 9.6 this will show that f must be a homology equivalence.

$\bigcap_{j\in S} U_{\Delta_j}$ consists of the maps $S^2 \to S^2$ which miss $\bigcup_{j\in S} \Delta_j$. $\bigcap_{j\in S} V_{\Delta_j}$ consists of the constant maps $S^2 \to S^2$ which miss $\bigcup_{j\in S} \Delta_j$. Denote $S^2\backslash \bigcup_{j\in S} \Delta_j$ by S^2_Δ. It may have many connected components. We denote its kth component by $(S^2_\Delta)_k$.

Fix a point x in the domain. Then over each $(S^2_\Delta)_k$ we have a fibration, induced by evaluating each map at x:

$$\begin{array}{c} Maps_*(S^2, (S^2_\Delta)_k) \\ \downarrow \\ \bigcap_{j\in S} U_{\Delta_j} \\ \downarrow \pi_x \\ (S^2_\Delta)_k \end{array}$$

where $Maps_*(S^2, (S^2_\Delta)_k)$ denotes the based maps from $S^2$to $(S^2_\Delta)_k$, and $\pi(\gamma) = \gamma(x)$. Evaluation at x also induces a fibration of $\bigcap_{j\in S} V_{\Delta_j}$over $\bigcap_{j\in S} V_{\Delta_j}$:

$$\begin{array}{c} pt \\ \downarrow \\ \bigcap_{j\in S} V_{\Delta_j} \\ \downarrow \pi_x \\ (S^2_\Delta)_k \end{array}$$

the inclusion f then induces a morphism of these fibrations:

$$\begin{array}{ccc} f : pt & \hookrightarrow & Maps_*(S^2, (S^2_\Delta)_k) \\ \downarrow & & \downarrow \\ f : \bigcap_{j\in S} V_{\Delta_j} & \hookrightarrow & \bigcap_{j\in S} U_{\Delta_j} \\ \downarrow \rho_x & & \downarrow \pi_x \\ (S^2_\Delta)_k & id & (S^2_\Delta)_k \end{array}$$

$(S^2_\Delta)_k$ is either a disc, or a bouquet of circles. In either case it is a $K(\pi, 1)$. For any space X,

$$\pi_l(Maps_*(S^2, X)) \cong \pi_{l+2}(X)$$

Thus if X is a $K(\pi, 1)$, $Maps_*(S^2, X)$ is weakly contractible.

So both ρ_x and π_x are (weak) homotopy equivalences. Thus

$$f : \bigcap_{j \in S} V_{\Delta_j} \hookrightarrow \bigcap_{j \in S} U_{\Delta_j}$$

is also a homotopy equivalence for every indexing set S. $f : CM \to (NS^2_0, T_i)$ itself is thus a homology equivalence by Proposition 9.6 . □

9.2.1. *Homology equivalence $Diff(S^2) \to (NS^3_1, T_i)$.*

Theorem 9.9. *$\eta : Diff(S^2) \hookrightarrow NS^3_1$ is a homology equivalence.*

If $i < j$, η factors as:

$$\eta : Diff(S^2) \hookrightarrow (NS^3_1, T^i_\Gamma) \hookrightarrow (NS^3_1, T^j_\Gamma) \hookrightarrow NS^3_1$$

Moreover as,

$$\bigcup_{i=0..\infty} (NS^3_1, T^i_\Gamma) = NS^3_1$$

and as homology commutes with direct limits, it is sufficient to show that:

Lemma 9.10. *$\eta : Diff(S^2) \hookrightarrow (NS^3_1, T^i_\Gamma)$ is a homology equivalence.*

Proof. Denote the covering described in Definition by U_j. We aim to prove this Lemma (and thus our theorem) by applying Proposition 9.6 to the η and the cover given by U_j. Thus we must show that

$$\eta| \bigcap_{j \in S} V_j \to \bigcap_{j \in S} U_j$$

is a homology equivalence, where $\{V_j\}$ denotes the cover of $Diff(S^2)$ given by $\{\eta^{-1}(U_j)\}$. As $U_j \supset Diff(S^2)$, this cover is trivial: each V_j, and thus all of their mutual intersections, consists of all of $Diff(S^2)$.

Denote $\bigcup_{j \in S} \Delta_j$ by Δ. $\bigcap_{j \in S} U_j$ consists of the maps $S^2 \to S^2$ which are diffeomorphisms on $f^{-1}(\Delta)$, and which map $S^2 \backslash f^{-1}(\Delta)$ to $S^2 \backslash \Delta$. To remind us of its content we will denote $\bigcap_{j \in S} U_j$ by NS_Δ. Denote $f^{-1}(\Delta)$ by θ_f.

Denote the connected components of the boundary of Δ by K_h for $h \in H$. Denote the connected components of $S^2 \backslash \bigcup_j \Delta_j$ by A_l where

$$A_l = D^2 - m_l \text{ discs}$$

Note that one may have $m_l = m_k$ though $k \neq l$.

The connected components of Δ have the same form. We denote these by

$$B_q = D^2 - m_q \text{ discs}$$

Definition 9.11. Denote by $\mathcal{C}$ the space of orientation preserving embeddings $g : \Delta \to S^2$ such that if

$$\bigcup_{j \in J \subset H} K_j$$

bound some A_l then $\bigcup_{j \in J \subset H} g(K_j)$ also bound a connected set in $S^2 \backslash g(\Delta)$

Lemma 9.12. *There is a fibration:*

$$\begin{aligned} \pi_1 : NS_\Delta &\to \mathcal{C} \\ \pi_2 : Diff(S^2) &\to \mathcal{C} \end{aligned}$$

where $\pi_i(f)$ is the embedding

$$f^{-1} : \Delta \to \theta_f$$

Proof. We begin by showing that $im(\pi_i) \subset \mathcal{C}$. As $im(\pi_i) \subset im(\pi_1)$ it is enough to show this for $i = 1$. Let $f \in NS_\Delta$ and suppose that

$$K_J = \bigcup_{j \in J \subset I} K_j$$

bound some connected component A_l of $S^2 \backslash \Delta$.

Both $S^2 \backslash \theta_f$ and $S^2 \backslash \Delta$ are complements of embeddings of $\Delta = \bigcup_{i=1}^k B_q$ into S^2. If φ is any embedding

$$\varphi(\bigcup_{q=1}^k (B_q = D^2 \backslash m_q \text{discs}) \to S^2$$

then

$$S^2 \backslash \varphi(\bigcup_{i=1}^k B_q)$$

has Σm_q components. Thus both $S^2 \backslash \theta_f$ and $S^2 \backslash \Delta$ have the same number of components. f maps $S^2 \backslash \Delta$ to $S^2 \backslash \theta_f$. As f is degree 1 this map is surjective. Thus f induces a bijection on connected components between those two spaces. $f^{-1}(B_l)$ is then a connected component of $S^2 \backslash \theta_f$. Denote this component by C_{m_l}. As

$$f : C_l \to B_l$$

is surjective

$$f(\delta C_l) \subset \delta B_l = K_J$$

thus $f^{-1}(K_J)$ bounds C_m.

Next we show that π_i is surjective. It is enough to show this for π_1. Let $\gamma \in \mathcal{C}$. We seek to construct an f inducing γ. It is more natural to construct f^{-1} and this is the task we take up.

Let

$$f^{-1}|_\Delta = \gamma$$

We now aim to extend f^{-1} over each connected component A_{m_l} of $S^2\backslash\Delta$. $f^{-1}(\delta A_{m_l})$ bound a component C_l of $S^2\backslash\theta_f$. C_l like A_l is a genus 0 surface. Both have the same number of boundary components. Thus C_l is abstractly diffeomorphic to A_l and moreover we can construct a diffeomorphism

$$C_l \to A_l$$

which extends f^{-1} on δA_l.

Finally, we show that each π_i has path lifting. Here we will explain the proof for π_1, the proof for $\pi_2$is identical except by obvious substitutions. Let P be a polyhedron and let

$$\Phi : P \times I \to \mathcal{C}$$

be a family of embeddings. Let

$$\Phi_{lift} : P \times \{0\} \to NS_\Delta$$

be an initial lifting of Φ. We seek to extend Φ_{lift} to a lifting over all $P \times I$. Let $\phi(p,t)$ be a family of diffeomorphims in $Diff(S^2)$ such that

$$\phi(p,t) \circ (\Phi(p,0)) = \Phi(p,t)$$

Then let

$$\Phi_{lift}(p,t) = \Phi_{lift}(p,0) \circ \phi^{-1}(p,t)$$

This is the required lifting. To construct the analogous proof for π_2 replace NS_Δ everywhere by $Diff(S^2)$.

□

□

Corollary 9.13. *$\mathcal{C}$ is connected.*

Proof. This follows immediately from the surjectivity of π_2. □

We denote the degree 1 self maps of A_l which are the identity near the boundary by $(A_l, A_l)_1^\circ$. Then the fiber of π_1 is homeomorphic to

$$\coprod_i (A_l, A_l)_1^\circ$$

Summarizing we have:

$$\begin{array}{c} \coprod_i (A_l, A_{l_i})_1^\circ \\ \downarrow \\ NS_\Delta \\ \downarrow \pi_1 \\ \mathcal{C} \end{array}$$

We have a similar fibration of $Diff(S^2)$:

$$\begin{array}{c} \coprod_i Diff(A_l)^\circ \\ \downarrow \\ Diff(S^2) \\ \downarrow \pi_2 \\ \mathcal{C} \end{array}$$

Where $Diff(A_l)^\circ$ denotes the diffeomorphisms of A_l which are the identity near the boundary. The inclusion η induces a morphism of these fibrations:

$$\begin{array}{ccc} \eta : \coprod_i Diff(A_l)^\circ & \hookrightarrow & \coprod_i (A_l, A_l)_1^\circ \\ \downarrow & & \downarrow \\ \eta : Diff(S^2) & \hookrightarrow & NS_\Delta \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{C} & id & \mathcal{C} \end{array}$$

Moreover the induced map

$$\eta : Diff(A_l)^\circ \hookrightarrow (A_l, A_l)_1^\circ$$

is a homotopy equivalence. Thus

$$\eta : Diff(S^2) \hookrightarrow NS_\Delta$$

is also a homotopy equivalence by the 5-lemma. Our Lemma, and thus Theorem 9.9, then follow from applying Proposition 9.6 to η and the cover given by the U_j.

10. Appendix

10.1. **Tamed almost complex structures preserving sub-bundles.** In this subsection we collect the results we require about tamed almost complex structures preserving sub-bundles. They are listed below in order of their difficulty. The first two are classical, the last less so, and we provide a proof of it here.

Definition 10.1. If $\pi : V \to B$ is a symplectic vector bundle, Let $\pi_J : \mathcal{J}(V) \to B$ be the bundle such that $\pi_J^{-1}(b)$ are the tamed almost complex structures on $\pi^{-1}(b)$. If $\eta_i \subset V$ are symplectic sub bundles, let $\pi_J : \mathcal{J}(V, \eta_1, \eta_2, ...) \to B$ be the (possibly locally non trivial) bundle where $\pi_J^{-1}(b)$ are the tamed almost complex structures on $\pi^{-1}(b)$, which preserve each η_i.

The first result goes back at least to Gromov's seminal paper.

Lemma 10.2. *Let Let $(V, \omega) \to B$ be a symplectic vector bundle over a polyhedron B. Then $\rho : \mathcal{J}(V) \to B$ is a bundle with contractible fibers.*

Next we consider almost complex structures preserving a given plane field. This is also a classical result:

Lemma 10.3. *Let $(V, \omega) \to B$ be a 4 dimensional symplectic vector bundle over a polyhedron B. Let $\vartheta \subset V$ be a 2 dimensional symplectic sub-bundle of V. Then $\rho : \mathcal{J}(V, \vartheta) \to B$ is a bundle with contractible fibers.*

Let $Q \subset B$ be a sub-polyhedron of B. Then given a section ϕ_Q of $\rho : \mathcal{J}(V, \vartheta) \to Q$, Lemma allows to construct a section ϕ of $\rho : \mathcal{J}(V, \vartheta) \to B$ extending ϕ_Q.

Finally we will need to consider the tamed almost complex structures preserving 2 plane fields. Preserving 2 planes requires a good deal more work than preserving one, as pairs of symplectic planes have moduli. This, while probably not new, is much less well known and we include a proof of it here.

Proposition 10.4. *Let $(V, \omega) \to B$ be a 4 dimensional symplectic vector bundle over a polyhedron B. Let $\vartheta_1, \vartheta_2 \subset V$ be 2 dimensional symplectic sub-bundles of V, such that ϑ_1, ϑ_2 intersect transversely in each fiber, and the the symplectic orthogonal projection $\pi_{12}^{\perp} : \vartheta_1 \to \vartheta_2$ is orientation preserving. Let $Q \subset B$ be a sub-polyhedron, and let ϕ_Q be a section of $\rho : \mathcal{J}(V, \vartheta_1, \vartheta_2) \to B$, defined over Q.*

Lemma 10.5. *Then: There is a section ϕ of ρ which extends ϕ_Q.*

Proof. Constructing ϕ is equivalent to constructing a section of $\phi^1 \oplus \phi^2$ of $J(\vartheta_1) \oplus J(\vartheta_2)$ such that the resulting almost complex structure is tamed by ω. Denote by ϕ_Q^i the sections such that

$$\phi_Q = \phi_Q^1 \oplus \phi_Q^2$$

Constructing ϕ_1 alone is fairly simple, for by Lemma:

$$J(\vartheta_1) \to B$$

is a bundle with contractible fibers. Thus this bundle admits a section ϕ^1 extending ϕ_Q^1. We now proceed with the problem of constructing ϕ^2 extending ϕ_Q^2.

Our main tool in will be the following Lemma in Linear Algebra, which provides the almost complex structures satisfying our conditions with (something resembling a) convex structure. This will allow to use partitions of unity to construct ϕ^2.

Lemma 10.6. *Let V, P be 2 symplectic planes in R^4 with symplectic structure ω. Let $\pi_\perp : P \to V^\perp$ denote the symplectic orthogonal projection. Suppose that $\pi_\perp$ is orientation preserving. Fix an $\omega-$tame complex structure J_P on P, and $s_1 \in V, s_1 \neq 0$. Denote the space of almost complex structures J_V on V such that $J = (J_P \oplus J_V)$ is ω-tame by $\mathcal{J}_V$.*

Then: $\Phi_{s_1} : \mathcal{J}_V \to V$ given by $J_V \to J_V(s_1)$ gives a homeomorphism of $\mathcal{J}_V$ onto a convex set.

The proof of this Lemma is involved, and we defer it to the end of this subsection. For now we concentrate on its application to our argument. It has the following immediate consequence

Lemma 10.7. *Let s be a section of ϑ_2 ,non vanishing over a set $X_s \subset B$. Then there is a section $\phi^2 \in \mathcal{J}(\vartheta_2)$ such that ϕ^2 extends ϕ^2_Q and $\phi^1 \oplus \phi^2$ is tamed by ω over X_s.*

Proof. As s is non-vanishing, α is determined by its action on s. Lemma 10.6 tells us that the set of allowable choices for $\alpha(s)$ form an open, convex set. As X_s is paracompact, so we can use a partition of unity to construct a section s_α of ϑ_2 over X_s, so that for

$$\begin{aligned} \phi^2 : s &\to s_\alpha \\ \phi^2 : s_\alpha &\to -s \end{aligned}$$

the almost complex structure $\phi^1 \oplus \phi^2$ is tamed by ω. □

Proposition 10.4 then follows by applying a partition of unity to a covering X_{s_i} coming from a finite set of sections $\{s_i\}$ of $\Phi^{1*}_Z(\eta)$ such that $\bigcup_i X_{s_i} = \Sigma \times P$.

□

10.1.1. *Proof of Linear Algebra Lemma.* In this subsection we provide the proof of the promised linear algebra lemma.

Lemma. 10.6*Let V, P be 2 symplectic planes in R^4 with symplectic structure ω. Let $\pi_\perp : P \to V^\perp$ denote the symplectic orthogonal projection. Suppose that $\pi_\perp$ is orientation preserving. Fix an $\omega-$tame complex structure J_P on P, and $s_1 \in V, s_1 \neq 0$. Denote the space of almost complex structures J_V on V such that $J = (J_P \oplus J_V)$ is ω-tame by $\mathcal{J}_V$.*

Then: $\Phi_{s_1} : \mathcal{J}_V \to V$ given by $J_V \to J_V(s_1)$ gives a homeomorphism of $\mathcal{J}_V$ onto a convex set.

Proof. We begin by establishing some useful coordinates. Let $\pi : P \to V$ denote the symplectic orthogonal projection.

Let

$$p \in \pi^{-1}(s_1) \subset P$$

Then

$$p = w_1 + s_1$$

where $w_1 \in V^{\perp}$.

$$J_P p = w_2 + s_2$$

where $w_2 \in V^{\perp}, s_2 \in V^{\perp}$.

Then

$$Jw_1 = w_2 + s_2 - J_V s_1$$

Applying J to both sides of this equation we compute Jw_2:

$$Jw_2 = -w_1 - J_V s_2 - s_1$$

Throughout this lemma we will suppress ω and just denote the pairing of two vectors p and q by (p, q).

$\pi_{\perp}$ is orientation preserving. Thus as $(\pi_{\perp}(s), J\pi_{\perp}(s)) > 0$ their projections to $V^{\perp}$ must pair with the same sign, and so $(w_1, w_2) > 0$ as well. To lessen our burden of constants: scale s_1 so that

$$(w_1, w_2) = 1$$

This scaling in turn dilates the image of Φ_{s_1}, and thus does not affect its convexity.

As P is symplectic $(w_1, w_2) + (s_1, s_2) > 0$ and thus

$$(s_1, s_2) > -1 \tag{10.1}$$

We now commence in earnest. Let $w + v \in V^{\perp} \oplus V = R^4$. What must we require of J_V so that $(w + v, J(w + v)) > 0$ for all such pairs w and v ?

$$(w + v, J(w + v)) \quad = \quad (w, Jw) + (w, Jv) + (v, Jw) + (v, Jv)$$

$(w, Jv) = 0$ as J must preserve V. And we have:

$$(w, Jw) + (v, Jw) + (v, Jv)$$

$(v, Jw) = (v, q)$ where q is the projection of Jw to V. This term may well be negative. We seek to bound its absolute value in terms of the other 2 (positive) terms. We replace v by $-Jv$ throughout the equation. As we seek a bound for all pairs w,v this has no effect on our task. Moreover as $(v, Jv) = (-JJv, -Jv)$ this has no effect on the third term. As J_V has determinate 1, it preserves $\omega|_V$, thus $(\cdot, J_V \cdot)$ is a (symmetric) inner product on V. The second term (v, q) becomes $(-J_V v, q) = (v, J_V q)$.

We seek to show that:

$$|(v, J_V q)| \leq (w, Jw) + (v, Jv)$$

Note that the right-hand side of the inequality depends only on the magnitude of v and not its direction. Cauchy-Schwartz then implies that:

$$|(v, J_V q)| \leq (v, J_V q)^{\frac{1}{2}} (q, J_V q)^{\frac{1}{2}}$$

If $v = kq$ this bound is achieved. Thus *the tamed J_V are precisely those such that:*

$$(v, J_V v)^{\frac{1}{2}} (q, J_V q)^{\dot{\frac{1}{2}}} < (w, Jw) + (v, Jv) \tag{10.2}$$

We now unpack this inequality. Write w as $aw_1 + bw_2$. Then

$$\begin{aligned} (w, Jw) &= (aw_1 + bw_2, J(aw_1 + bw_2)) \\ &= (aw_1 + bw_2, a(w_2 + s_2 - J_V s_1) + b(-w_1 - J_V s_2 - s_1)) \\ &= a^2(w_1, w_2) - b^2(w_2, w_1) \\ &= a^2 + b^2 \end{aligned}$$

$$(q, J_V q) = (as_2 - aJ_V s_1 - bJ_V s_2 - bs_1, aJ_V s_2 + as_1 + bs_2 - bJ_V s_1)$$

Expanding the right hand side creates 16 pairings, however some of them are 0, and the 4 "ab" terms all cancel. Upon summing we are left with:

$$(q, J_V q) = \lambda(s_1, J_V s_1) + \lambda(s_2, J_V s_2) - 2\lambda(s_1, s_2)$$

where we denote $(a^2 + b^2)$ by λ.

Our inequality 10.2 then reads:

$$(v, J_V v)^{\frac{1}{2}} (\lambda(s_1, J_V s_1) + \lambda(s_2, J_V s_2) - 2\lambda(s_1, s_2))^{\frac{1}{2}} < \lambda + (v, J_V v)$$

If $v = 0$ the inequality places no restriction on J_V. Thus we may assume that v is not zero. Since the condition $(w+v, J(w+v)) > 0$ is invariant under scaling by a positive constant, we may scale the vector $v + w$ so that $(v, J_V v) = 1$, if we assume that J_V tames ω on V. We do so, are left with one free parameter $\lambda > 0$.

$$(\lambda(s_1, J_V s_1) + \lambda(s_2, J_V s_2) - 2\lambda(s_1, s_2))^{\frac{1}{2}} < \lambda + 1$$

squaring both sides yields:

$$\lambda|(s_1, J_V s_1) + (s_2, J_V s_2) - 2(s_1, s_2)| \quad < \quad \lambda^2 + 2\lambda + 1$$

This may be achieved, for all λ if and only if:

$$|(s_1, J_V s_1) + (s_2, J_V s_2) - 2(s_1, s_2)| \quad < \quad 4 \tag{10.3}$$

At this point our proof bifurcates into two cases:

Case 1. $rk(\pi = 2))$

$$\begin{aligned} \Phi_s(J_V) &= J_V s_1 \\ J_V s_1 &= c s_1 + d s_2 \end{aligned} \tag{10.4}$$

where $c, d \in R$.

We now describe the constraints that 10.3 places on c and d. We can compute $J_V s_2$ by applying J_V to both sides of equation 10.6. We get:

$$J_V s_2 = -\frac{1}{d}(c^2 + 1)s_1 - c s_2$$

Substituting into 10.3 we get:

$$|(d + \frac{c^2 + 1}{d} - 2(s_1, s_2)| < 4$$

J_V's tameness restricted to V [2] translates to d having the same sign as (s_1, s_2). Thus $(d + \frac{c^2+1}{d})$ and $-2(s_1, s_2)$ have opposite sign, and our inequality is equivalent to:

$$|(d + \frac{c^2 + 1}{d})| < 4 + 2(s_1, s_2)$$

If we denote $4 + 2(s_1, s_2)$ by γ, the set of whose solutions of this inequality form disc, centered at

$$(c, d) = (0, \frac{\gamma}{2})$$

with radius $(\frac{\gamma^2}{4} - 1)^{\frac{1}{2}}$. Since $(s_1, s_2) > -1$ by 10.1 , $\gamma > 2$, and this disc is nonempty.

Case 2. $rk(\pi) = 1$

Write s as

$$s \quad = \quad \alpha s_1 + \beta s_3$$

where $\alpha, \beta \in \mathbb{R}$, and $s_3 \in V$, such that $(s_1, s_3) = 1$. We introduce s_3 because s_1 and s_2 are linearly dependent.

[2] we assumed this when we scaled v so that $(v, Jv) = 1$

Then if

$$J_V s_1 = c s_1 + d s_3$$

we have that

$$J_V s_3 = -\frac{1}{d}(c^2+1)s_1 - c s_3$$

$$\Phi_s(J_V) = J_V s$$

$$J_V s = \alpha J_V s_1 + \beta J_V s_3 \tag{10.5}$$

$$= (\alpha c + -\frac{\beta}{d}(c^2+1))s_1 + (\alpha d - \beta c s_1) \tag{10.6}$$

As $s_2 = k s_1$, the constraints that 10.3 places on c and d are much weaker. Substituting into 10.3 we get:

$$|d| < k_0 = \frac{4}{1+k^2}$$

and no condition on c. J_V's tameness restricted to V [3] translates to d having the same sign as (s_1, s_3), and thus the image of Φ_s are the vectors

$$(\alpha c + -\frac{\beta}{d}(c^2+1))s_1 + (\alpha d - \beta c s_1)$$

where c is free, α and β are fixed constants one of which must be non-zero, and

$$0 \leq d < \frac{4}{1+k^2}$$

These vectors form a convex set. In fact the map

$$(c,d) \rightarrow (\alpha c + -\frac{\beta}{d}(c^2+1), \alpha d - \beta c)$$

gives a diffeomorphism of the strip,

$$\{(c,d) : 0 < d < k_0\}$$

onto the region in the plane to the convex side of the parabola:

$$(c,d) : (\alpha c + -\frac{\beta}{k_0}(c^2+1), \alpha k_0 - \beta c)$$

The case $\beta = 0$ is degenerate and yields the band

$$(c,d) : 0 < d < 4$$

Case 3. $rk(\pi) = 0$

[3] we assumed this when we scaled v so that $(v, Jv) = 1$

This case actual requires no proof at all, for here the two planes are orthogonal. Thus for any J_V tamed by $\omega|_V$, $J_V \oplus J_P$ is tamed by ω, and $im(\Phi_s)$ is a half plane.

□

10.2. **Applications.** In this subsection we collect some immediate consequences of the Lemmas in the previous subsection. Each follows quickly from the lemmas in the previous section.

Let $\{S_i\}$ be a collection of disjoint symplectic curves. Denote the tamed almost complex structures preserving the tangent space to each curve by $J_\star$.

Proposition 10.8. *$J_\star$ is contractible.*

Proposition 10.9. *Let N be a symplectic 4-manifold, let P be a polyhedron, and let $\Phi : P \to S\mathcal{F}^*$ be a family of symplectic fibrations with symplectic section. Then there is a family $\Phi^J : P \to \mathcal{J}$ of tamed almost complex structures such that $\Phi(p)$ is $\Phi^J(p)$ holomorphic. Moreover if Q is a sub polyhedron of P and $\Phi^J|_Q$makes $\Phi|_Q$ holomorphic, we can extend $\Phi^J|_Q$ to such map on all P.*

Proposition 10.10. *Let P be a polyhedron. Let $\Phi : P \to S\mathcal{F}_0^\infty$. There is a map $\Phi^J : P \to \mathcal{J}$ such that $\Phi(p)$ is $\Phi^J(p)$ holomorphic. Moreover if Q is a sub polyhedron of P and $\Phi^J|_Q$makes $\Phi|_Q$ holomorphic, we can extend $\Phi^J|_Q$ to such map on all P.*

Proposition 10.11. *Let $\Phi : B \times I \to \mathcal{Z}_{0,\infty}$, along with a lifting $\Phi_{lift} : B \times 0 \to S\mathcal{F}_0^\infty$. There is a $\Phi^J : B \times I \to \mathcal{J}$, such that $\Phi(b,t)$ is $\Phi^J(b,t)$ holomorphic, and $\Phi_{lift}(b,0)$ is $\Phi^J(b,0)$ holomorphic.*

10.3. **A non-Traditional 5 Lemma.** The 5-Lemma is usually presented in the context of chain complexes, and as such it is usually stated as a Lemma about abelian groups. However its usual proof actually applies in much more generality. As we will require it we present the more general statement here. The proof is the standard one, which we reproduce from Spanier, with cosmetic changes due to the different language.

Lemma 10.12. *Let*

$$\begin{array}{ccccccccc} G_5 & \stackrel{\alpha_5}{\to} & G_4 & \stackrel{\alpha_4}{\to} & G_3 & \stackrel{\alpha_3}{\to} & G_2 & \stackrel{\alpha_2}{\to} & G_1 \\ \gamma_5 \downarrow & & \gamma_4 \downarrow & & \gamma_3 \downarrow & & \gamma_2 \downarrow & & \gamma_1 \downarrow \\ H_5 & \stackrel{\beta_5}{\to} & H_4 & \stackrel{\beta_4}{\to} & H_3 & \stackrel{\beta_3}{\to} & H_2 & \stackrel{\beta_2}{\to} & H_1 \end{array}$$

be a diagram of pointed sets, with each row exact. Suppose that G_3 and G_2 are groups, γ_i makes H_i a G_i-set, and the morphisms α_3 and

β_3 respect this structure. Suppose further that γ_1, γ_2, γ_4 and γ_5 are bijections then γ_3 is a bijection.

Proof. Denote the base point of each set by e. For G_3 this is also the identity element. We first show that γ_3 is an injection. It is enough to show that $\gamma_3^{-1}(e) = e$, as γ_3 is a morphism of G_3 sets. Suppose $\gamma_3(g_3) = e$. Then $\gamma_2\alpha_3(g_3) = \beta_3\gamma_3(g_3) = e$. Thus $\alpha_3(g_3) = e$, as γ_2 is injective. Thus, by exactness, there is a $g_4 \in G_4$ such that $\alpha_4(g_4) = g_3$. Then $\beta_4\gamma_4(g_4) = \gamma_3(g_3) = e$. Thus there is an $h_5 \in H_5$ such that $\beta_5(h_5) = \gamma_4(g_4)$, and so $g_4 = \alpha_5(g_5)$. Therefore $g_3 = \alpha_4\alpha_5(g_5) = e$.

Next we show that γ_3 is surjective. Let $h_3 \in H_3$. There is a $g_2 \in G_2$such that $\gamma_2(g_2) = \beta_3(h_3)$. Then $\gamma_1\alpha_2(g_2) = \beta_2\beta_3(h_3) = e$. Therefore $\alpha_2(g_2) = e$, and there is $g_3 \in G_3$ such that $\alpha_3(g_3) = g_2$. Then

$$\begin{aligned} \beta_3(h_3 \cdot \gamma_3(g_3^{-1})) &= \beta_3(h_3) \cdot \gamma_2(\alpha_3(g_3)^{-1}) \\ &= \beta_3(h_3) \cdot \gamma_2(g_2^{-1})) \\ &= e \end{aligned}$$

and thus there is an $h_4 \in H_4$ such that $\beta_4(h_4) = h_3 \cdot \gamma_3(g_3^{-1})$. Let $g_4 \in G_4$ be such that $\gamma_4(g_4) = h_4$. Then $\alpha_4(g_4)g_3 \in G_3$ and

$$\begin{aligned} \gamma_3(\alpha_4(g_4)g_3) &= \beta_4(h_4)\gamma_3(g_3) \\ &= h_3 \cdot \gamma_3(g_3^{-1})\gamma_3(g_3) \\ &= h_3 \cdot \gamma_3(g_3)^{-1} \cdot \gamma_3(g_3) \\ &= h_3 \end{aligned}$$

□

Remark 10.13. To prove that γ_3 is injective we needed only that γ_2 and γ_4 are injective, and $\gamma_5$was surjective. We also needed only that G_3 was a group and that H_3 was a G_3 set. We did not need the structures on G_4 and H_4. To prove that γ_3 was surjective we needed that $\gamma_2$and γ_4 were surjective, and that γ_1 was injective. However the above statement is general enough for our purposes.

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