On Generalizations of the Scalar Curvature

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Abstract of the Dissertation
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We study the scalar curvature and its generalizations in various contexts. First, We prove the existence of infinite dimensional families of non-Kähler almost-Kähler metrics with constant scalar curvature on a compact Kähler manifold of $\dim_{\mathbb{C}} \geq 2$ with constant scalar curvature and non-degenerate Futaki invariant. By the same method we also obtain infinite dimensional families of non-Kähler almost-Kähler metrics with $s + s^*$ constant, where s is scalar curvature and s^* is $*$-scalar curvature. Explicit examples are presented to show that there exist compact non-Kähler almost-Kähler 4-manifolds with the property that $s + s^*$ is a negative constant and the almost-Kähler form belongs to the lowest eigenspace of self-dual Weyl curvature at each point.

Secondly we consider $aK+s$, where a is a nonnegative constant, K is the sectional curvature and s is the scalar curvature on a Riemannian manifold X. It is shown that if X admits a metric with $aK + s > 0$, then so does any manifold obtained from X by surgeries of codimension ≥ 3. This implies the existence of such metrics on certain compact simply connected manifolds of dimension ≥ 5 by using the cobordism argument. We also study the corresponding minimal volume problem. As a corollary, we derive that every compact simply connected manifold of dimension ≥ 5 and every compact complex surface of Kodaira dimension ≤ 1 whose minimal model is not of Class VII collapse with $aK + s$ bounded below.

Dedicated to my parents.

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Chapter 1

Introduction

It has been one of main interests in Riemannian geometry to study the relations between curvature and topology. One of the great successes in this field is that of scalar curvature. On an n-dimensional Riemannian manifold M, the scalar curvature is a smooth function $s : M \to \mathbb{R}$ defined as

$$s(p) = \sum_{i \neq j} K(e_i, e_j),$$

where K denotes the sectional curvature and $(e_1, \cdots, e_n)$ is an orthonormal basis for the tangent space at $p \in M$. From an another view point, the scalar curvature at a point p is a measure of how fast the volume of the ball of radius r around p is growing with r in compared to that of the euclidean metric ball. More precisely,

$$\frac{\mathrm{Vol} B_r(M, p)}{\mathrm{Vol} B_r(\mathbb{R}^n, 0)} = 1 - \frac{s(p)}{6(n+2)} r^2 + O(r^3),$$

where $B_r(\cdot,\cdot)$ denotes the ball of radius r around the given point. Since the scalar curvature is an average of the sectional curvatures at each point, it has more flexibility than the sectional curvature and Ricci curvature so that it has many implications.

The first remarkable fact about the scalar curvature is that on a compact manifold of dimension ≥ 3 each conformal class of metrics has a metric with constant scalar curvature. These metrics minimize the normalized total scalar curvature $\int_M s d\mu / \mathrm{Vol}^{\frac{n-2}{n}}$ in their conformal classes. The metric with constant scalar curvature as a canonical metric on a manifold with a given structure appears in other contexts.

The classical uniformization theorem of Riemann surface says that on a compact complex 1-dimensional manifold, each Kähler class can be represented by a metric of constant curvature. In higher dimensions, Calabi [7, 8] sought to find extremal Kähler metrics by minimizing the L^2-norm of scalar curvature over a given Kähler class. Any Kähler metric of constant scalar curvature is extremal, but he showed that the converse is generally false. The celebrated theorem of Aubin [1] and Yau [45] guarantees the existence of Kähler-Einstein metric in the Kähler class of the first chern class c_1 on any compact complex manifold with $c_1 < 0$. This metric is unique up to homothety. Yau [46] also showed that in case of $c_1 = 0$, each Kähler class can be represented as a unique Ricci-flat Kähler metric. In case $c_1 > 0$, by Tian [43] a compact complex surface admits Kähler-Einstein metric iff the Lie algebra of biholomorphism group is reductive. Recently Chen [9] showed the uniqueness of Kähler metric of constant scalar curvature in each Kähler class on any complex surface with $c_1 < 0$ or $c_1 = 0$.

On a smooth compact oriented 4-manifolds, one may instead fix a *polarization* meaning a maximal linear subspace H^+ of $H^2(M, \mathbb{R})$ for which the intersection form is positive definite. A Riemannian metric is called H^+-adapted if the space of self-dual harmonic 2-forms coincides with H^+. Thus one can try to find extremal metrics minimizing the L^2-norm of scalar curvature among all H^+-adapted metrics. For example, on a compact complex surface of Kähler type, fixing a Kähler class $[\omega]$ gives rise to a polarization by $H^+ = \mathbb{R}[\omega] \oplus ReH^{2,0} \oplus ImH^{2,0}$ where $ReH^{2,0}$ and $ImH^{2,0}$ denote the de Rham classes represented by real and imaginary parts of holomorphic 2-forms respectively. LeBrun [20] has shown that on a polarized 4-manifold with a nontrivial Seiberg-Witten invariant for a spin^c structure whose first chern class has nonzero projection to H^+, these extremal metrics minimizing the L^2-norm of scalar curvature are exactly Kähler metrics of negative constant scalar curvature.

Later LeBrun [23, 24] proved similar results involving Weyl curvature. Recall that on a oriented Riemannian 4-manifold, $\Lambda^2 T^*M$ has a direct sum decomposition $\Lambda^+ \oplus \Lambda^-$ of self-dual 2-forms and anti-self-dual 2-forms, and in these terms the curvature tensor R as a section of $End(\Lambda^2 T^* M)$ has a decomposition

$$R = \left(\begin{array}{c|c} W_+ + \frac{s}{12} Id_{\Lambda^+} & B \\ \hline B^t & W_- + \frac{s}{12} Id_{\Lambda^-} \end{array} \right).$$

into irreducible pieces. Here s denotes the scalar curvature, and the self-dual and anti-self-dual Weyl curvatures $W_\pm$ are the trace-free pieces of the appropriate blocks. B can be identified with the trace-free part $r - \frac{s}{4}g$ of the Ricci curvature. Let w be the lowest eigenvalue of W_+ at each point. Then

Theorem 1.0.1 (LeBrun) *Let (M, H^+, g) be a polarized smooth compact oriented 4-manifold with a H^+-adapted Riemannian metric g. Suppose M has a spin^c structure c so that Seiberg-Witten equations have a solution for every H^+-adapted metric. Let c_1 be the first chern class of c with nonzero orthogonal projection c_1^+ to H^+. Then for any constant $\delta \in [0, \frac{1}{3}]$,*

$$\int_M ((1-\delta)s + \delta \cdot 6w)^2 d\mu \geq 32\pi^2 (c_1^+)^2. \qquad (\S1.0.1)$$

When $\delta \in [0, \frac{1}{3})$, the equality holds if and only if g is a Kähler metric of negative constant scalar curvature and $(c_1^+)^2 = \left(c_1(M)^+\right)^2$ where $c_1(M)$ denotes the first chern class of the almost complex structure.

When $\delta = \frac{1}{3}$, the equality is equivalent to that g is an almost-Kähler metric and $(c_1^+)^2 = \left(c_1(M)^+\right)^2$ such that the almost-Kähler form belongs to the lowest eigenspace of W_+ at each point, and the sum $s + s^*$ of scalar curvature and $*$-scalar curvature is a negative constant.

Here, an almost-Kähler manifold means a symplectic manifold endowed with the metric defined by

$$g(J\cdot, \cdot) = \omega(\cdot, \cdot)$$

for the symplectic form ω and a compatible almost complex structure J. Such a manifold is Kähler iff the almost complex structure is integrable. The $*$-scalar curvature of an almost-Kähler manifold is a natural partner for the scalar curvature, which is defined as the function $s^* = s + \frac{1}{2}|\nabla J|^2$. Note that s^* exactly coincides with s on a Kähler manifold. The Blair's formula [4]

$$\int_M \frac{1}{2}(s + s^*)\frac{\omega^n}{n!} = 4\pi c_1(M) \cdot [\omega]^{n-1} \qquad (\S1.0.2)$$

for $2n$-dimensional almost-Kähler manifold M indicates that $\frac{1}{2}(s+s^*)$ is a more canonical quantity than s alone on an almost-Kähler manifold.

Any Kähler metric of negative constant scalar curvature clearly attains the equality in §1.0.1 when $\delta = \frac{1}{3}$. The question thus arises whether such a almost-Kähler metric saturating the inequality in case of $\delta = \frac{1}{3}$ can be strictly almost-Kähler in the sense that the almost complex structure is not integrable. Another natural question is whether the estimate of these can be extended to $\delta > \frac{1}{3}$. We will present an example that answers both of the questions. Moreover we will show that on a compact Kähler manifold of $\dim_{\mathbb{C}} \geq 2$ with constant scalar curvature and non-degenerate linearized Futaki invariant has an infinite dimensional family modulo diffeomorphisms of strictly almost-Kähler metrics with constant $s+s^*$. In the same way we can also obtain an infinite dimensional family modulo diffeomorphisms of strictly almost-Kähler metrics with constant scalar curvature.

The next topic we are going to discuss originates from the question "Which manifolds admit metrics of positive scalar curvature?" First of all it is easy to see that every compact manifold of dimension ≥ 3 admits a metric with negative scalar curvature. On a compact orientable 2-dimensional manifold M, the Gauss-Bonnet theorem

$$\int_M s d\mu = 4\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M gives a complete answer to the question. In higher dimensions, α-genus turned out to be the relevant topological quantity. It was Lichnerowicz [28] who first realized that a compact spin $4n$-manifold with non-vanishing $\hat{A}$-genus cannot admit a metric of positive scalar curvature. Noting that $\hat{A}$-genus can be extended to the ring homomorphism α from the spin cobordism ring Ω_*^{Spin} onto $KO_*(\mathrm{pt})$, Hitchin [14] proved that any compact spin manifold M admitting a metric with positive scalar curvature has $\alpha(M) = 0$. The converse is also true if we further assume X is simply connected and of dimension ≥ 5 by the works of Gromov and Lawson [13], Rosenberg [36], and finally Stolz [38]. In case of non-spin manifolds, Gromov and Lawson [13] proved that any compact simply connected non-spin manifold of dimension ≥ 5 admits a metric of positive scalar curvature. All these existence results of positive scalar curvature are based on the following main lemma of Gromov and Lawson [13] which was also proved independently by Schoen and Yau [37] : *If a manifold M admits a metric of positive scalar curvature, then any manifold obtained from M by surgeries of codimension ≥ 3 also admits a metric of positive scalar curvature.*

Of course this lemma is generally false for the sectional curvature in which case much is still unknown. As an interpolating case between scalar curvature s and sectional curvature K, we can consider $\frac{1}{2}(\underline{K} + \frac{s}{n(n-1)})$ where the *bottom sectional curvature* $\underline{K}$ is defined as the minimum of sectional curvatures over all 2-planes at each point and n is the dimension of the manifold, or more generally $\lambda\underline{K} + (1-\lambda)\frac{s}{n(n-1)}$ where $\lambda \in [0, 1)$ is a constant. Actually it turns out that $\lambda \underline{K} + (1-\lambda)\frac{s}{n(n-1)}$ with $\lambda \in [0,1)$ behaves in the same way as s in many respects. Indeed we will prove that if M admits a metric with $\lambda \underline{K} + (1-\lambda)\frac{s}{n(n-1)} > 0$, then so does any manifold obtained from M by surgeries of codimension ≥ 3. This implies that any compact simply connected non-spin manifold of dimension ≥ 5 and any compact simply connected spin manifold M of dimension ≥ 5 with $\alpha(M) = 0$ carry a metric with $\lambda \underline{K} + (1-\lambda)\frac{s}{n(n-1)} > 0$.

Our main surgery lemma enables us to compute the minimal volumes for many classes of manifolds. Consider minimal volume invariants of a compact manifold as a way of quantifying the degree to which some negative curvature might be an inevitable feature of all possible geometry on the manifold. Following LeBrun [24] we define *Gromov minimal volume* as

$$\mathrm{Vol}_K(M) := \inf_g \; \{ \mathrm{Vol}(M,g) \mid K_g \geq -1 \},$$

and *Yamabe minimal volume* as

$$\mathrm{Vol}_s(M) := \inf_g \left\{ \mathrm{Vol}(M,g) \;\middle|\; \frac{s_g}{n(n-1)} \geq -1 \right\}.$$

where n is the dimension of the manifold. In contrast to Gromov minimal volume which is difficult to compute in general, Yamabe minimal volume is much simple. Indeed Petean [34] proved that any compact simply connected manifold M of dimension ≥ 5 has $\mathrm{Vol}_s(M) = 0$, in other words it collapses with scalar curvature bounded below. LeBrun [21] has computed it on 4-manifolds with nontrivial Seiberg-Witten invariant and it turned out to be highly nontrivial. We expect similar results in case of the λ-*mixed minimal* *volume* for $0 \leq \lambda < 1$ which is defined [24] as

$$\mathrm{Vol}_{\lambda,K,s}(M) := \inf_g \left\{ \mathrm{Vol}(M,g) \;\middle|\; \lambda K_g + (1-\lambda)\frac{s_g}{n(n-1)} \geq -1 \right\}.$$

Indeed we will show that the surgery in codimension ≥ 3 does not increase the mixed minimal volume. This will lead to the conclusion that every compact simply connected manifold of dimension ≥ 5 and every compact simply connected complex surface of Kodaira dimension ≤ 1 whose minimal model is not of Class VII collapse with $\lambda \underline{K} + (1-\lambda)\frac{s}{n(n-1)}$ bounded below.

Chapter 2

Preliminaries

2.1 Notations and conventions

Let E be a smooth real vector bundle endowed with a metric on a compact real m-dimensional Riemannian manifold M. Then $L^p_k(E)$ denotes the real Banach space of sections of E whose first k derivatives have bounded L^p-norms. The Sobolev imbedding theorem states that if $k > \frac{m}{p}+l$, then $L^p_k(E) \subset C^l(E)$, the space of continuous sections whose derivatives of order up to l, are continuous. When E is a trivial line bundle, it is usually denoted as $L^p_k(M)$ or more briefly L^p_k, which is a Banach algebra for $k > \frac{m}{p}$. Throughout this paper, $k > \frac{m}{p}$ will be assumed. Finally by $L^p_k(M)/\mathbb{R}$, we mean the quotient Banach space of $L^p_k(M)$ by the closed subspace of constant functions. By abuse of notation, any element of $L^p_k(M)/\mathbb{R}$ will be denoted with or without $[\cdot]$, e.g. $[\varphi]$ or φ.

Chapter 3

Almost-Kähler metric with constant scalar curvature

3.1 Seiberg-Witten theory

Let M be a smooth compact oriented Riemannian 4-manifold and $P_{SO(4)}$ be the principal $SO(4)$ bundle of orthonormal frames. The group $Spin^c(4)$ is $(SU(2) \times SU(2) \times U(1))/\{\pm 1\}$, where $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$. A spin^c structure on M is by definition an equivalence class of lifts of $P_{SO(4)}$ to a principal $Spin^c(4)$ bundle $P \to M$. P can be viewed as a double cover of $P_{SO(4)} \times P_{U(1)}$. Thus the Levi-Civita connection on M and a connection A on $P_{U(1)}$ induces a connection ∇_A on P.

A spin^c lift has two canonical associated $\mathbb{C}^2$ bundles $V_\pm$ on M coming from the two homomorphisms of $Spin^c(4)$ to $U(2) = SU(2) \times U(1)/\{\pm 1\}$. $V_\pm$ are distinguished by the fact that the projective bundles $\mathbb{P}(V_\pm)$ are the unit sphere bundles $S(\Lambda^\pm)$ respectively, and their determinant line bundles are both equal to the associated line bundle L of $P_{U(1)}$.

The Seiberg-Witten equations [44]

$$D_A \Phi = 0$$

$$F_A^+ = (\Phi \otimes \Phi^*)_0$$

are equations for an unknown Hermitian connection A on L and an unknown section Φ of V_+. Here D_A is the Dirac operator coupled to A which is defined as the composition of Clifford multiplication with the covariant differentiation by ∇^A. F_A^+ denotes the self-dual part of the curvature 2-form of A and $(\cdot)_0$ denotes the trace-free part.

Now suppose that M has an almost complex structure on TM under which the metric is invariant. Then there is a canonical spin^c structure coming from the almost complex structure so that $V_+ = \underline{\mathbb{C}} \oplus K^{-1}$ and $V_- = (T^*M)^{0,1}$, where $\underline{\mathbb{C}}$ is the trivial complex line bundle and $K^{-1} = \Lambda^2(T^*M)^{0,1}$. Consider 1 as a section of $\underline{\mathbb{C}}$ with a unit norm and let A_0 be a unique connection on K^{-1} such that $\nabla^{A_0} 1 \in T^*M \otimes K^{-1}$. On an almost-Kähler manifold the Seiberg-Witten equations for the canonical spin^c structure can be written more concretely.

$$\overline{\partial}_a \alpha + \overline{\partial}_a^* \beta = 0$$

$$F_A^+ = i\frac{|\alpha|^2 - |\beta|^2}{4}\omega + \frac{\overline{\alpha}\beta - \alpha\overline{\beta}}{2},$$

where (α, β) is a section of V_+, and $\overline{\partial}_a$ denotes the anti-holomorphic part of $d + \frac{A-A_0}{2}$.

Here we present two theorems which will be used later.

Theorem 3.1.1 *Suppose M is an almost complex 4-manifold with a hermitian metric. Let $e_1, e_2 = J(e_1), e_3, e_4 = J(e_3)$ be a local orthonormal frame of TM and (ϑ_{ji}) be the connection 1-form of the Levi-Civita connection on TM. Then $A_0 = i(\vartheta_{21} + \vartheta_{43})$.*

Proof. Let $(f^1, \cdots, f^4)$ be the dual coframe field. The connection form (ω_{ji}) is given by Cartan structure equations

$$df^j = -\vartheta_{ji} \wedge f^i,$$

and ∇_{A_0} is given by

$$d + \frac{A_0}{2} + \sum_{i<j} \frac{\vartheta_{ji}}{2} cl(f^i \wedge f^j).$$

Here $cl(\cdot)$ denotes Clifford multiplication defined by

$$cl(v) \cdot \gamma = \sqrt{2}(v^{0,1} \wedge \gamma - i(\overline{v^{1,0}})\gamma)$$

for $v \in T^*M \otimes \mathbb{C}$ and $\gamma \in V_+ \oplus V_-$, where $v^{p,q}$ denotes the (p,q)-component of v, and $\imath$ is the interior product. Let $\pi_1 : V_+ = \underline{\mathbb{C}} \oplus K^{-1} \to \underline{\mathbb{C}}$ be the projection map. Then

$$\begin{aligned} 0 = \pi_1(\nabla^{A_0} 1) &= \frac{A_0}{2} + \frac{\vartheta_{21}}{2} cl(f^1 \wedge f^2) \cdot 1 + \frac{\vartheta_{43}}{2} cl(f^3 \wedge f^4) \cdot 1 \\ &= \frac{A_0}{2} + (\frac{\vartheta_{21}}{2} + \frac{\vartheta_{43}}{2})(-i). \end{aligned}$$

□

Theorem 3.1.2 *Let M be an almost complex 4-manifold with a hermitian metric. Suppose A and $\Phi \in C^\infty(M)$ be a solution of Seiberg-Witten equations for the canonical spin^c structure such that $|\Phi|$ is a nonzero constant and $\nabla^A \Phi \in T^*M \otimes K^{-1}$. Then*

$$\nabla^* \nabla F_A^+ = \frac{4|\nabla^A \Phi|^2}{|\Phi|^2} F_A^+,$$

where ∇ denotes the Levi-Civita connection.

Proof. From the Weitzenböck formula

$$0 = 4 D_A D_A \Phi = 4(\nabla^A)^* \nabla^A \Phi + s\Phi + |\Phi|^2 \Phi,$$

where s denotes the scalar curvature. Taking the inner product with Φ gives

$$0 = 2\Delta|\Phi|^2 + 4|\nabla^A \Phi|^2 + s|\Phi|^2 + |\Phi|^4 = 4|\nabla^A \Phi|^2 + s|\Phi|^2 + |\Phi|^4.$$

Fix $x \in M$ and choose a local orthonormal tangent frame field $(e_1, \cdots, e_4)$ with the property that $(\nabla e_j)_x = 0$ for all j. Then we have at the point x that

$$\begin{aligned}
\nabla^* \nabla F_A^+ &= (\nabla^A)^* \nabla^A (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} Id) \\
&= \sum_k -\nabla^A_{e_k} (\nabla^A_{e_k} \Phi \otimes \Phi^* + \Phi \otimes \nabla^A_{e_k} \Phi^*) \\
&= (\nabla^A)^* \nabla^A \Phi \otimes \Phi^* - 2 \sum_k \nabla^A_{e_k} \Phi \otimes (\nabla^A_{e_k} \Phi)^* + \Phi \otimes ((\nabla^A)^* \nabla^A \Phi)^* \\
&= -\frac{s + |\Phi|^2}{2} \Phi \otimes \Phi^* - 2 \sum_k \nabla^A_{e_k} \Phi \otimes (\nabla^{A_0}_{e_k} \Phi)^* \\
&= -\frac{s + |\Phi|^2}{2} (F_A^+ + \frac{|\Phi|^2}{2} Id) - 2 |\nabla^A \Phi|^2 (Id - \frac{\Phi \otimes \Phi^*}{|\Phi|^2}) \\
&= -\frac{s + |\Phi|^2}{2} F_A^+ - (-|\nabla^A \Phi|^2 + 2 |\nabla^A \Phi|^2) Id + 2 |\nabla^A \Phi|^2 \frac{\Phi \otimes \Phi^*}{|\Phi|^2} \\
&= -\frac{s + |\Phi|^2}{2} F_A^+ + \frac{2 |\nabla^A \Phi|^2}{|\Phi|^2} (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} Id) \\
&= \frac{-(s + |\Phi|^2)|\Phi|^2 + 4 |\nabla^A \Phi|^2}{2 |\Phi|^2} F_A^+ \\
&= \frac{4 |\nabla^A \Phi|^2}{|\Phi|^2} F_A^+.
\end{aligned}$$

□

3.2 Extremal almost-Kähler 4-manifold

Let M be a smooth 4-manifold $\Sigma \times T^2$ where T^2 denotes a torus and Σ is any compact Riemann surface of genus bigger than 1. We consider it as a torus bundle over Σ. Define a metric $g_0 = g_\Sigma \times g_{T^2}$ on M, where g_{T^2} is the flat metric and g_Σ is a metric of constant curvature. With the compatible complex structure M is a Kähler manifold of negative constant scalar curvature.

To get our desired metric, we want to perturb the metric on each fiber by rescaling the longitude and the meridian keeping the volume unchanged. In a local coordinate, the perturbed metric looks like

$$g_h = \frac{dx^2 + dy^2}{c^2 y^2} + \frac{1}{h^2} dz^2 + h^2 du^2 \qquad (\S 3.2.1)$$

where $-c^2$ is the Gaussian curvature of g_Σ, and $\{dz, du\}$ forms a global parallel frame for g_{T^2}, and our perturbing function h is any non-constant smooth function on Σ with C^2-norm $||h - 1||_{C^2}$ sufficiently small. By adjusting the almost complex structure on each fiber, g_h can be also made to be almost-Kähler with the same almost-Kähler form

$$\omega = \frac{1}{c^2 y^2} dx \wedge dy + dz \wedge du.$$

The almost complex structure J has to be as follows:

$$J(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \quad J(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x},$$

$$J(h\frac{\partial}{\partial z}) = \frac{1}{h}\frac{\partial}{\partial u}, \quad J(\frac{1}{h}\frac{\partial}{\partial u}) = -h\frac{\partial}{\partial z}.$$

Then the first chern class $c_1(M)$ of the almost complex structure and its self-dual harmonic projection $c_1(M)^+$ does not change by the above perturbation and is nonzero.

Note $b_2^+(M) > 1$. By Witten's computation [44] on Kähler surfaces or more generally Taubes' result [41] on symplectic 4-manifolds, M has the nonzero Seiberg-Witten invariant for the canonical spin^c structure. So M with the canonical spin^c structure has a solution of Seiberg-Witten equations for any metric. But it's noteworthy that (M, g_h) actually has a solution which is the same as that of (M, g_0). Let (e_1, e_2, e_3, e_4) denote a local orthonormal frame field $(|cy|\frac{\partial}{\partial x}, |cy|\frac{\partial}{\partial y}, h\frac{\partial}{\partial z}, \frac{1}{h}\frac{\partial}{\partial u})$, and (ϑ_{ji}) be the corresponding connection 1-form of the Levi-Civita connection on TM.

Theorem 3.2.1 *Let a_{21} be the connection 1-form of the Levi-Civita connection on (Σ, g_Σ). Then for (M, g_h) above,*

$$A = ia_{21}, \quad (\alpha, \beta) = (\sqrt{2c^2}, 0)$$

is a solution of Seiberg-Witten equations for the canonical spin^c structure.

Proof. The direct computation using Cartan structure equations shows that $\vartheta_{21} = a_{21}$ and $\vartheta_{43} = 0$. By theorem 3.1.1, $A = ia_{21}$ is the connection on K^{-1} which makes $\nabla^A 1$ belong to K^{-1} and hence $D_A(\sqrt{2c^2}) = 0$. Let ω_Σ be the

Kähler form of (Σ, g_Σ).

$$\begin{aligned} F_A^+ = (ida_{21})^+ &= (ic^2\omega_\Sigma)^+ \\ &= ic^2\frac{\omega}{2} \\ &= i\frac{|\alpha|^2 - |\beta|^2}{4}\omega + \frac{\overline{\alpha}\beta - \alpha\overline{\beta}}{2}. \end{aligned}$$

□

Now we prove our main theorems.

Theorem 3.2.2 *Let $M = \Sigma \times T^2$ with the genus of $\Sigma > 1$. Then M admits a non-Kähler almost-Kähler metric such that*

$$\int_M (\frac{2}{3}s + 2w)^2 d\mu = 32\pi^2(c_1(M)^+)^2,$$

where $c_1(M)^+$ is the self-dual harmonic part of the first chern class $c_1(M)$ of the almost complex structure. Moreover, the moduli space of such metrics on M modulo diffeomorphisms is infinite dimensional.

Proof. The ansatz for the metric is g_h defined above. We will show that the almost-Kähler form ω belongs to the lowest eigenspace of W_+ at each point, and $s + s^*$ is a negative constant.

On a local coordinate the curvature tensor R is computed by using the formula

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm},$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Using a coordinate expression (§3.2.1), all nonzero Christoffel symbols are computed as follows:

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \Gamma^x_{zz} = \frac{c^2y^2h_x}{h^3}, \quad \Gamma^x_{uu} = -c^2y^2hh_x,$$

$$\Gamma^y_{xx} = -\Gamma^y_{yy} = \frac{1}{y}, \quad \Gamma^y_{zz} = \frac{c^2y^2h_y}{h^3}, \quad \Gamma^y_{uu} = -c^2y^2hh_y,$$

$$\Gamma^u_{xu} = \Gamma^u_{ux} = -\Gamma^z_{xz} = -\Gamma^z_{zx} = \frac{h_x}{h},$$

$$\Gamma^u_{yu} = \Gamma^u_{uy} = -\Gamma^z_{yz} = -\Gamma^z_{zy} = \frac{h_y}{h}.$$

With respect to the orthonormal frame (e_1, e_2, e_3, e_4) previously defined, all the nonzero components of R are as follows:

$$R_{1212} = c^2, \quad R_{1313} = c^2y^2(\frac{-h_{xx}h + 2h_x^2}{h^2} + \frac{h_y}{yh}),$$

$$R_{1414} = \frac{c^2y^2}{h}(h_{xx} - \frac{h_y}{y}), \quad R_{1323} = c^2y^2(\frac{-h_{xy}h + 2h_xh_y}{h^2} - \frac{h_x}{yh}),$$

$$R_{1424} = \frac{c^2y^2}{h}(h_{xy} + \frac{h_x}{y}), \quad R_{2323} = c^2y^2(\frac{-h_{yy}h + 2h_y^2}{h^2} - \frac{h_y}{yh}),$$

$$R_{2424} = \frac{c^2y^2}{h}(h_{yy} + \frac{h_y}{y}), \quad R_{3434} = -\frac{c^2y^2}{h^2}(h_x^2 + h_y^2),$$

where h_x denotes the partial derivative of h with respect to x etc.

Let K_{ij} denote the sectional curvature of the plane generated by e_i and e_j. Then

$$
\begin{aligned}
s = 2\sum_{i<j} K_{ij} &= 2(R_{1221} + R_{1331} + R_{1441} + R_{2332} + R_{2442} + R_{3443}) \\
&= -2c^2 - 2\frac{|dh|^2}{h^2},
\end{aligned}
$$

and

$$
\begin{aligned}
s^* = 2R(\omega, \omega) &= 2R(e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_2 + e_3 \wedge e_4) \\
&= 2(R_{1221} + R_{1243} + R_{3421} + R_{3443}) \\
&= -2c^2 + 2\frac{|dh|^2}{h^2}.
\end{aligned}
$$

So $s + s^* = -4c^2$ is a negative constant. And since

$$
|\nabla J|^2 = 2(s^* - s) = 8\frac{|dh|^2}{h^2}
$$

is not identically zero, J is not integrable.

From

$$
R(\omega, e_1 \wedge e_3 - e_2 \wedge e_4) = R_{1231} - R_{1242} + R_{3431} - R_{3442} = 0
$$

and

$$
R(\omega, e_1 \wedge e_4 + e_2 \wedge e_3) = R_{1241} + R_{1232} + R_{3441} + R_{3432} = 0,
$$

we deduce that ω belongs to the eigenspace of W_+, and for a sufficiently small perturbation the eigenvalues have to be still lowest at each point by Lemma 3.2.1 below.

Recall that the space of metrics $\mathcal{M}$ on M is a Hilbert manifold. At any $g \in \mathcal{M}$, the tangent space $T_g\mathcal{M}$ is the space of symmetric bilinear forms on TM, and is endowed with the inner product induced from g so that it has an orthogonal splitting $Im\delta_g^* \oplus Ker\delta_g$ where δ_g means the divergence and δ_g^* is its formal adjoint. Note that $Im\delta_g^*$ represents exactly the direction for the orbit of g by the diffeomorphisms on M. Consider the smooth 1-parameter family of our perturbations

$$g_{e^{tf}} = g_0 + t(-2f dz \otimes dz + 2f du \otimes du) + O(t^2).$$

Since f is constant on each fiber,

$$\delta_{g_0}(-2f dz \otimes dz + 2f du \otimes du) = -(-2\frac{\partial f}{\partial z} dz + 2\frac{\partial f}{\partial u} du) = 0.$$

So the tangent space at g_0 to the space of our metrics modulo diffeomorphisms is isomorphic to $C^\infty(\Sigma)/\mathbb{R}$ which is infinite dimensional. □

Lemma 3.2.1 *Let (M, J, g) be a Kähler manifold of real dimension four, oriented so that the Kähler form ω is self-dual. Then*

$$W_+(\omega) = \frac{s}{6}\omega, \quad \textit{and} \quad W_+(\eta) = -\frac{s}{12}\eta$$

for any self-dual 2-form η orthogonal to ω.

Proof. See [3]. □

Remark. (Alternative Proof of Theorem 3.2.2) From theorem 3.2.1, we know $\omega = -\frac{2i}{c^2}F_A^+$. So ω belongs to the eigenspace of $\nabla^*\nabla$ by theorem 3.1.2. Now recall the Weitzenböck formula [5]

$$\Delta\xi = \nabla^*\nabla\xi - 2W_+(\xi,\cdot) + \frac{s}{3}\xi,$$

for any self-dual 2-form ξ. Then it follows that ω belongs to the eigenspace of W_+. From theorem 3.2.1 and theorem 3.1.2, we also have

$$\begin{aligned}
s + s^* &= s + 2R(\omega,\omega) = s + 2W_+(\omega,\omega) + \frac{s}{3} \\
&= \frac{4}{3}s + (-\Delta\omega + \nabla^*\nabla\omega + \frac{s}{3}\omega, \omega) \\
&= \frac{4}{3}s + 0 + \frac{4|\nabla^A\Phi|^2}{|\Phi|^2}\cdot 2 + \frac{s}{3}\cdot 2 \\
&= \frac{2s|\Phi|^2 + 8|\nabla^A\Phi|^2}{|\Phi|^2} \\
&= \frac{-2|\Phi|^4}{|\Phi|^2} \\
&= -4c^2.
\end{aligned}$$

Remark. It is also noteworthy that the projection map $\pi : (M, g_h) \to (\Sigma, g_\Sigma)$ is a Riemannian submersion with the vanishing O'Neill tensor, i.e. the horizontal distribution is integrable. Moreover each fiber is a minimal submanifold, i.e the trace of its second fundamental form is identically zero, because $h\frac{\partial}{\partial z}$ and $\frac{1}{h}\frac{\partial}{\partial u}$ are killing fields. In this case the scalar curvature can also be computed using the formula 9.37 in [3]. Let $\check{s}$ be the scalar curvature of (Σ, g_Σ), and $\hat{s}$ be the scalar curvature of the fiber with the induced metric. Denote the vertical component of a vector $v \in TM$ by $\hat{v}$. Then

$$
\begin{aligned}
s &= \breve{s} \circ \pi + \hat{s} - (|\widehat{\nabla_{e_3} e_1}|^2 + |\widehat{\nabla_{e_4} e_1}|^2 + |\widehat{\nabla_{e_3} e_2}|^2 + |\widehat{\nabla_{e_4} e_2}|^2) \\
&= -2c^2 + 0 - (c^2 y^2 \frac{h_x^2}{h^2} + c^2 y^2 \frac{h_x^2}{h^2} + c^2 y^2 \frac{h_y^2}{h^2} + c^2 y^2 \frac{h_y^2}{h^2}) \\
&= -2c^2 - 2\frac{|dh|^2}{h^2}.
\end{aligned}
$$

Theorem 3.2.3 *Let (M, g_h) be as above. Then there exists a constant $\delta' > \frac{1}{3}$ such that*

$$
\int_M ((1-\delta)s + \delta \cdot 6w)^2 d\mu < 32\pi^2 (c_1(M)^+)^2
$$

for $\delta \in (\frac{1}{3}, \delta')$.

Proof. Since we already know that ω belongs to the lowest eigenspace of W_+,

$$
w = \frac{1}{2} W_+(\omega, \omega) = \frac{s^*}{4} - \frac{s}{12}.
$$

Hence

$$
\begin{aligned}
\int_M ((1-\delta)s + \delta \cdot 6w)^2 d\mu &= \int_M ((1 - \frac{3\delta}{2})s + \frac{3\delta}{2} s^*)^2 d\mu \\
&= \int_M (-2c^2 + 2(3\delta - 1)\frac{|dh|^2}{h^2})^2 d\mu \\
&< \int_M (-2c^2)^2 d\mu
\end{aligned}
$$

for a sufficiently small $3\delta - 1 > 0$. Considering the case of g_0 which is a Kähler metric of negative constant scalar curvature $-2c^2$, we have

$$\int_M (-2c^2)^2 d\mu = 32\pi^2 (c_1(M)^+)^2,$$

thus proving the theorem. □

For any metric $6w \geq -\sqrt{24}|W_+|$, simply because W_+ is trace-free. So the estimate (§1.0.1) leads to

$$(1-\delta)\|s\| + \delta\|\sqrt{24}W_+\| \geq 4\sqrt{2}\pi|c_1^+| \qquad (\S 3.2.2)$$

for any constant $\delta \in [0, \frac{1}{3}]$, where $\|\cdot\|$ denotes L^2-norm. Finally let's check the optimality of this estimate for our (M, g_h).

Theorem 3.2.4 *Let (M, g_h) be as before. Then the estimate in (§3.2.2) for $c_1 = c_1(M)$ is true for any $\delta \in [0, 1]$.*

Proof. Since

$$\|s\| > \|s_{g_0}\| = \|\sqrt{24}W_{+,g_0}\| = 4\sqrt{2}\pi|c_1(M)^+|,$$

where s_{g_0} and W_{+,g_0} are the scalar curvature and the self-dual Weyl curvature of the metric g_0 respectively, it is enough to show that $\|W_+\| > \|W_{+,g_0}\|$.

With respect to the orthonormal frame (e_1, e_2, e_3, e_4), the Ricci curvature is as follows:

$$r_{11} = -c^2 - \frac{2c^2y^2{h_x}^2}{h^2}, \quad r_{22} = -c^2 - \frac{2c^2y^2{h_y}^2}{h^2},$$

$$r_{33} = -r_{44} = \frac{\Delta h}{h} - \frac{|dh|^2}{h^2}, \quad r_{12} = r_{21} = -\frac{2c^2y^2h_xh_y}{h^2},$$

and all others are zero.

Let $h = 1 + tf$. Then by the generalized Gauss-Bonnet formula

$$\begin{aligned}
\int_M |W_+|^2 d\mu &= 4\pi^2(2\chi + 3\tau) - \int_M \frac{s^2}{24} d\mu + \int_M \frac{|r - \frac{s}{4}g_h|^2}{2} d\mu \\
&= 4\pi^2(2\chi + 3\tau) - \int_M \frac{4c^4 + t^2(8c^2|df|^2) + O(t^3)}{24} d\mu \\
&\quad + \int_M \frac{c^4 + t^2(2c^2|df|^2 + 2|\Delta f|^2) + O(t^3)}{2} d\mu \\
&= \int_M |W_{+,g_0}|^2 d\mu + t^2 \int_M (\frac{2c^2|df|^2}{3} + |\Delta f|^2) d\mu + O(t^3).
\end{aligned}$$

Since f is not a constant function, for a sufficiently small $t \neq 0$

$$\int_M |W_+|^2 d\mu > \int_M |W_{+,g_0}|^2 d\mu,$$

completing the proof. $\square$

3.3 Existence of almost-Kähler metrics with constant scalar curvature

We will now consider the deformation problem of constant scalar curvature almost-Kähler metrics on compact Kähler manifolds. Let (M, g_0, ω_0, J_0) be a compact Kähler manifold of complex dimension $\dim_{\mathbb{C}}(M) = n \geq 2$ with constant scalar curvature. We will vary the metric g_0 in a family of almost-Kähler metrics $g_{\varphi,\alpha,t}$ in 3 parameters, (φ, α, t) to be explained below, such that $g_0 = g_{0,0,0}$. We shall explain how to vary g_0 in three steps.

Step I : Before we actually vary the metrics, here we pre-fix some data, which consist of $p, U, f^1, f^2, \cdots, f^{2n}, a_1, a_2, \cdots, a_n$. We shall explain these now.

Choose any point $p \in M$. Take any trivializing neighborhood $U \ni p$ of the tangent bundle TM and a local orthonormal frame $\{f^1, f^2, \cdots, f^{2n}\}$ for (TM, g_0) on U such that $J_0 f^{2i-1} = f^{2i}$ for $i = 1, \cdots, n$. Then take any smooth functions $a_1, a_2, \cdots, a_n$ on M compactly supported in U satisfying a generic condition which involves only g_0, p, U, f^i. Indeed the condition is exactly that the quantity of (§3.3.3), appearing below in the proof of Theorem 3.3.1, is nonzero. (For instance one may choose a_i's to satisfy that $a_1(p) = a_2(p) = 0$ and $f^3(a_1)(p) \neq 0$.)

Step II : We fix J_0 and deform the metric and the Kähler form. We denote the space of real ω_0-harmonic $(1,1)$-forms by $\mathcal{H}^{1,1}$. For $[\varphi] \in L^2_{k+4}/\mathbb{R}$ and $\alpha \in \mathcal{H}^{1,1}$ with sufficiently small norms, $\omega_{\varphi,\alpha} = \omega_0 + \alpha + i\partial\bar{\partial}\varphi$ is also a Kähler form with respect to J_0. Let $g_{\varphi,\alpha}$ be the corresponding metric.

Step III : We fix the form $\omega_{\varphi,\alpha}$ and deform the almost complex structure and the metric to get $g_{\varphi,\alpha,t}$.

Now to define $g_{\varphi,\alpha,t}$, we need an orthonormal frame $\{f^1_{\varphi,\alpha}, \cdots, f^{2n}_{\varphi,\alpha}\}$ for $g_{\varphi,\alpha}$ on U compatible with J_0 i.e. $J_0 f^{2i-1}_{\varphi,\alpha} = f^{2i}_{\varphi,\alpha}$. Take $f^1_{\varphi,\alpha} = \frac{f^1}{\|f^1\|_{g_{\varphi,\alpha}}}$ and $f^2_{\varphi,\alpha} = J_0 f^1_{\varphi,\alpha}$. Let $\tilde{f}^3$ be the orthogonal projection of f^3 to the $(n-2)$-plane orthogonal to $f^i_{\varphi,\alpha}$'s for $i < 3$, i.e. $\tilde{f}^3 = f^3 - \langle f^3, f^1_{\varphi,\alpha}\rangle_{g_{\varphi,\alpha}} f^1_{\varphi,\alpha} - \langle f^3, f^2_{\varphi,\alpha}\rangle_{g_{\varphi,\alpha}} f^2_{\varphi,\alpha}$. Now define, $f^3_{\varphi,\alpha} = \frac{\tilde{f}^3}{\|\tilde{f}^3\|_{g_{\varphi,\alpha}}}$ and $f^4_{\varphi,\alpha} = J_0 f^3_{\varphi,\alpha}$. Other $f^i_{\varphi,\alpha}$'s are defined inductively. For a real number t define

$$g_{\varphi,\alpha,t} = \begin{cases} \sum_{i=1}^n e^{ta_i} (f^{2i-1}_{\varphi,\alpha})^* \otimes (f^{2i-1}_{\varphi,\alpha})^* + e^{-ta_i} (f^{2i}_{\varphi,\alpha})^* \otimes (f^{2i}_{\varphi,\alpha})^* & \text{on } U \\ g_{\varphi,\alpha} & \text{elsewhere,} \end{cases}$$

where $(f^i_{\varphi,\alpha})^*$ denotes the dual of $f^i_{\varphi,\alpha}$. Since a_i's are compactly supported in U, $g_{\varphi,\alpha,t}$ is a L^2_{k+2}-metric on M. An orthonormal frame field for $g_{\varphi,\alpha,t}$ on U is $f^{2i-1}_{\varphi,\alpha,t} = e^{-\frac{ta_i}{2}} f^{2i-1}_{\varphi,\alpha}$ and $f^{2i}_{\varphi,\alpha,t} = e^{+\frac{ta_i}{2}} f^{2i}_{\varphi,\alpha}$, $i = 1, 2, \cdots, n$. Thus $g_{\varphi,\alpha,t}$ is almost-Kähler with almost-Kähler form $\omega_{\varphi,\alpha}$ and the almost complex structure

J_t defined by

$$J_t(f^{2i-1}_{\varphi,\alpha,t}) = f^{2i}_{\varphi,\alpha,t}, \quad J_t(f^{2i}_{\varphi,\alpha,t}) = -f^{2i-1}_{\varphi,\alpha,t} \quad \text{on } U$$

$$J_t = J_0 \qquad\qquad\qquad\qquad \text{elsewhere,}$$

for $i = 1, \cdots, n$. This finishes the step III.

We claim that a deformation of this type produces strictly almost-Kähler metrics of constant scalar curvature.

Theorem 3.3.1 *Let (M, g_0, ω_0, J_0) be a compact Kähler manifold of* $\dim_{\mathbb{C}} \geq 2$ *with constant scalar curvature. Suppose that the linearized Futaki invariant is non-degenerate at the cohomology class $[\omega_0]$.*

Then near (g_0, ω_0, J_0) there exist strictly almost Kähler metrics of constant scalar curvature, which are Kähler away from any small open subset of M. They form an infinite dimensional family of metrics modulo diffeomorphisms.

Proof. Let $(\vartheta^{ij}_{\varphi,\alpha,t})$ be the Riemannian connection form of the metric $g_{\varphi,\alpha,t}$ with respect to the frame $f^1_{\varphi,\alpha,t}, \cdots, f^{2n}_{\varphi,\alpha,t}$ defined above. Consider the map between Banach spaces given by

$$\mathcal{F} : L^2_{k+4}/\mathbb{R} \times \mathcal{H}^{1,1} \times \mathbb{R} \supset \mathcal{V} \longrightarrow L^2_k/\mathbb{R} \times \mathbb{R}$$

$$([\varphi],\alpha,t) \longmapsto (S(\varphi,\alpha,t), w(\varphi,\alpha,t)),$$

where $\mathcal{V}$ is a sufficiently small neighborhood of $([0],0,0)$, $S(\varphi,\alpha,t)$ is the scalar curvature mod constants of $g_{\varphi,\alpha,t}$, and $w(\varphi,\alpha,t)$ is defined as $[(\vartheta^{13}_{\varphi,\alpha,t} - \vartheta^{24}_{\varphi,\alpha,t})(f^1)]_{x=p}$, where $[\cdot]_{x=p}$ means the evaluation at $x=p$. This $w(\varphi,\alpha,t)$ map is designed to detect the non-integrabilty of the corresponding almost complex structure J_t of $g_{\varphi,\alpha,t}$. In fact, for any Kähler metric (g,ω,J) with the Riemannian connection ∇, (ϑ^{ij}) and a J-compatible frame $\{e^1,\cdots,e^{2n}\}$,

$$\begin{aligned}
\vartheta^{13}-\vartheta^{24} &= g(\nabla e^3,e^1) - g(\nabla e^4,e^2)\\
&= g(\nabla e^3,e^1) - g(\nabla Je^3,Je^1)\\
&= g(\nabla e^3,e^1) - g(J\nabla e^3,Je^1)\\
&= 0.
\end{aligned}$$

We now show that $\mathcal{F}$ is a smooth map between Banach spaces. For (φ,α,t) in $\mathcal{V}$, $S(\varphi,\alpha,t)$ can be written as $H(x,\partial^\beta\varphi,\alpha,t), x\in M, |\beta|\le 4$, where H is a smooth function defined on the range of $(x,\partial^\beta\varphi,\alpha,t)$. Note that here α can be considered as a real parameter in $\mathbb{R}^{h^{1,1}}\simeq\mathcal{H}^{1,1}$. Clearly the linear map $f\mapsto\partial f$ is a smooth map from $L^2_{k+1}/\mathbb{R}$ to L^2_k. By [31] it follows that S is a smooth map. The other map $w(\varphi,\alpha,t)$ is written as the composition of the evaluation map $\mathbf{eval}_p : L^2_k\to\mathbb{R}$ and $\tilde H(x,\partial^\beta\varphi,\alpha,t), x\in M, |\beta|\le 3$, where $\tilde H$ is also a smooth function defined on the range of $(x,\partial^\beta,\varphi,\alpha,t)$ with α considered again as a real parameter in $\mathbb{R}^{h^{1,1}}\simeq\mathcal{H}^{1,1}$. The evaluation map $f\mapsto f(p)$ is bounded linear and hence a smooth map from L^2_k to $\mathbb{R}$. Thus w is also smooth.

Let's compute the Fréchet derivative $D\mathcal{F}$ of $\mathcal{F}$ at $([0],0,0)$. First, $DS|_{([0],0,0)}$ was already computed by LeBrun and Simanca in [26]. Let $\mathcal{L} : L^2_{k+4}/\mathbb{R} \to L^2_k/\mathbb{R}$ be the differential operator $-\frac{1}{2}(\Delta^2_{g_0} + ric_{g_0} \cdot \nabla_{g_0}\nabla_{g_0})$ where Δ_{g_0}, ric_{g_0}, and ∇_{g_0} are the Laplace-Beltrami operator, the Ricci tensor, and the Riemannian connection of g_0 respectively. Then $DS|_{([0],0,0)}$ may be expressed as a 1-by-3 matrix:

$$DS|_{([0],0,0)} = \begin{pmatrix} \mathcal{L} & -2\langle \rho, \cdot \rangle_{g_0} & * \end{pmatrix},$$

where ρ is the Ricci form of g_0. They showed [25] that $DS|_{([0],0,0)}$ is surjective when the linearized Futaki invariant is non-degenerate at $[\omega_0]$.

For $Dw|_{([0],0,0)}$, we claim that

$$Dw|_{([0],0,0)} = \begin{pmatrix} 0 & 0 & b \end{pmatrix},$$

where b is a real number which is nonzero for our generic choice of a_i's (in step I). The first two entries are obviously zero because $g_{\varphi,\alpha}$ is still Kähler with the complex structure J_0. Let ∇^t be the Riemannian connection of $g_{0,0,t}$, and define

$$h := \frac{dg_{0,0,t}}{dt}|_0 = \sum_{i=1}^{n} a_i (f^{2i-1})^* \otimes (f^{2i-1})^* - a_i (f^{2i})^* \otimes (f^{2i})^*.$$

Note that

$$\begin{aligned} \omega^{13}_{0,0,t}(f^1) = g_{0,0,t}(\nabla^t_{f^1} f^3_{0,0,t}, f^1_{0,0,t}) &= e^{ta_1} g_0(\nabla^t_{f^1}(e^{-\frac{ta_2}{2}} f^3), e^{-\frac{ta_1}{2}} f^1) \\ &= e^{\frac{t(a_1-a_2)}{2}} g_0(\nabla^t_{f^1} f^3, f^1), \end{aligned}$$

$$\omega^{24}_{0,0,t}(f^1) = g_{0,0,t}(\nabla^t_{f^1} f^4_{0,0,t}, f^2_{0,0,t}) \;=\; e^{-ta_1} g_0(\nabla^t_{f^1}(e^{\frac{ta_2}{2}} f^4), e^{\frac{ta_1}{2}} f^2)$$
$$= e^{-\frac{t(a_1-a_2)}{2}} g_0(\nabla^t_{f^1} f^4, f^2).$$

Recall the formula in [3]

$$g_0((\nabla^t_X Y)'_{t=0}, Z) = \frac{1}{2}\{(\nabla^0_X h)(Y,Z) + (\nabla^0_Y h)(X,Z) - (\nabla^0_Z h)(X,Y)\}.$$

For clarity's sake, let (ϑ^{ij}) be the Riemannian connection form of g_0. Then

$$\begin{aligned}
\frac{\partial w}{\partial t}|_{([0],0,0)} &= [\frac{d}{dt}|_0(\vartheta^{13}_{0,0,t} - \vartheta^{24}_{0,0,t})(f^1)]_{x=p} \\
&= [\frac{a_1-a_2}{2} g_0(\nabla^0_{f^1} f^3, f^1) \\
&\quad + \frac{1}{2}\{(\nabla^0_{f^1} h)(f^3, f^1) + (\nabla^0_{f^3} h)(f^1, f^1) - (\nabla^0_{f^1} h)(f^1, f^3)\} \\
&\quad -(-\frac{a_1-a_2}{2}) g_0(\nabla^0_{f^1} f^4, f^2) \\
&\quad -\frac{1}{2}\{(\nabla^0_{f^1} h)(f^4, f^2) + (\nabla^0_{f^4} h)(f^1, f^2) - (\nabla^0_{f^2} h)(f^1, f^4)\}]_{x=p},
\end{aligned}$$

which is computed to

$$\begin{aligned}
&[\frac{a_1-a_2}{2}\vartheta^{13}(f^1) + \frac{1}{2}\{f^3(h(f^1,f^1)) - h(\nabla^0_{f^3} f^1, f^1) - h(f^1, \nabla^0_{f^3} f^1)\} \\
&\quad + \frac{a_1-a_2}{2}\vartheta^{24}(f^1) - \frac{1}{2}\{-h(\nabla^0_{f^1} f^4, f^2) - h(f^4, \nabla^0_{f^1} f^2) \\
&\quad - h(\nabla^0_{f^4} f^1, f^2) - h(f^1, \nabla^0_{f^4} f^2) + h(\nabla^0_{f^2} f^1, f^4) + h(f^1, \nabla^0_{f^2} f^4)\}]_{x=p} \\
&= [\frac{a_1-a_2}{2}\vartheta^{13}(f^1) + \frac{1}{2} f^3(a_1) + \frac{1}{2}\{2a_1\vartheta^{12}(f^4) - (a_1+a_2)\vartheta^{14}(f^2)\}]_{x=p},
\end{aligned}$$

(§3.3.3)

where $f^3(\cdot)$ denotes the directional derivative in the direction of f^3. Thus $\frac{\partial w}{\partial t}|_{([0],0,0)}$ is nonzero for generic a_1 and a_2, (already chosen so in step I).

Summarizing the above, we now have

$$DF|_{([0],0,0)} = \begin{pmatrix} \mathcal{L} & -2\langle \rho, \cdot \rangle_{g_0} & * \\ & & \\ 0 & 0 & b \end{pmatrix},$$

which is surjective. From the choice of our a_i functions it is easy to see that it has $h^{1,1}$-dimensional kernel. Applying the Implicit Function Theorem, for a sufficiently small neighborhood $\mathcal{V}$ of 0, $\mathcal{F}^{-1}(\{[0]\} \times \mathbb{R} - \{([0], 0)\}) \bigcap \mathcal{V}$ gives a $(h^{1,1} + 1)$-dimensional submanifold consisting of strictly almost-Kähler L^2_{k+4}-metrics of constant scalar curvature.

To show the smoothness of these metrics, we resort to the elliptic regularity. Consider the smooth 4-th order non-linear differential operator $\tilde{S}$ smoothly parameterized by $r = (r_1, \cdots, r_{h^{1,1}}) \in \mathbb{R}^{h^{1,1}}$ and $t \in \mathbb{R}$

$$\tilde{S} : L^2_{k+4} \supset \tilde{\mathcal{V}} \longrightarrow L^2_k$$

$$\varphi \longmapsto \text{the scalar curvature of } g_{\varphi, \sum_i r_i \alpha_i, t}$$

where $\tilde{\mathcal{V}}$ is a sufficiently small neighborhood of 0 in L^2_{k+4}, and $\{\alpha_1, \cdots, \alpha_{h^{1,1}}\}$ is a basis for $\mathcal{H}^{1,1}$. The derivative at 0 when $r = t = 0$ is $-\frac{1}{2}(\Delta^2_{g_0} + ric_{g_0} \cdot \nabla_{g_0} \nabla_{g_0})$ which is elliptic. Observe that $\tilde{S} = \tilde{S}(x, \partial^\beta \varphi, r, t), |\beta| \leq 4$ is smooth in all its arguments. Since $\tilde{S}$ is elliptic at $0 \in L^2_{k+4} \subset C^4$, $\tilde{S}$ is also elliptic at φ when $\|\varphi\|_{L^2_{k+4}} + \| \ r\| + |t|$ is sufficiently small by Lemma 3.3.1 below. Then the elliptic regularity theorem 6.8.1 in [30] gives the smoothness of our solution satisfying $\tilde{S}(\varphi) = \text{constant}$.

Now let us show that these metrics form an infinite dimensional family modulo diffeomorphisms. First, denoting the space of Riemannian metrics on M by $\mathcal{M}$, we recall from [3], Chapter 4 that at any $g \in \mathcal{M}$ the tangent space $T_g\mathcal{M}$ is the space of symmetric 2-tensor fields and is endowed with the metric induced from g so that it has an orthogonal splitting $Im\delta_g^* \oplus Ker\delta_g$ where δ_g means the divergence and δ_g^* is its formal adjoint. Note that the image of δ_g^*, $Im\delta_g^*$, represents exactly the directions for the orbit of g by the diffeomorphisms on M.

Let $c_a(t) = (\varphi_a(t), \alpha_a(t), t) \in L^2_{k+4}/\mathbb{R} \times \mathcal{H}^{1,1} \times \mathbb{R}$ be any smooth curve in $\mathcal{F}^{-1}(\{[0]\} \times \mathbb{R}) \bigcap \mathcal{V}$ with the property that $c_a(0) = g_0$ and $c_a(t) \notin \mathcal{F}^{-1}\{([0], 0)\}$ for $t \neq 0$. Here the subscript a represents the n-tuple $a_1, a_2, \cdots, a_n$. Let h_a be the tangent vector at g_0 to the image of $c_a(t)$ under the obvious smooth map $L^2_{k+4}/\mathbb{R} \times \mathcal{H}^{1,1} \times \mathbb{R} \longrightarrow \mathcal{M}$. Then

$$h_a = \frac{d}{dt}|_0 (i\partial\overline{\partial}\varphi_a(t) + \alpha_a(t))(\cdot, J_0\cdot) + \sum_{i=1}^{n} a_i((f^{2i-1})^* \otimes (f^{2i-1})^* - (f^{2i})^* \otimes (f^{2i})^*).$$

We are going to show that by varying a we can produce infinitely many linearly independent h_a's modulo $Im\delta_{g_0}^*$. For a natural number m, let $h_{a^j}, j = 1, \cdots, m$ be the tangent vectors at g_0 corresponding to $a^j = (a_1^j, a_2^j, \cdots, a_n^j)$, $j = 1, \cdots, m$ respectively. For simplicity we let $a_i^j \equiv 0$ for $j = 1, \cdots, m$ and $i = 2, \cdots, n$ (Then e.g. $a_1^j(p) = 0$ and $f^3(a_1^j)|_{x=p} \neq 0$ will be enough to ensure our restriction on a^j for $j = 1, \cdots, m$). Consider the linear combination $h := \sum_{j=1}^m c_j h_{a^j}$, $c_j \in \mathbb{R}$.

By definition, we have that $h \notin Im\delta^*_{g_0}$ iff $\langle h, \tilde{h}\rangle_{L^2} \neq 0$ for some symmetric 2-tensor field $\tilde{h}$ with the property that $\delta_{g_0}\tilde{h} = 0$. We choose $\tilde{h}$ to be a nonzero J_0-anti-invariant symmetric 2-tensor field on M such that $\delta_{g_0}\tilde{h} = 0$. Its existence comes from the under-determined ellipticity of δ_{g_0} and the result of [6]. Note that $(f^1)^* \otimes (f^1)^* - (f^2)^* \otimes (f^2)^*$ is a J_0-anti-invariant symmetric 2-tensor field on U. By redefining p, U, f^i if necessary, it is not hard to see that we may assume the function $\langle (f^1)^* \otimes (f^1)^* - (f^2)^* \otimes (f^2)^*, \tilde{h}\rangle_{g_0}$ to be never-zero on U. Since $\frac{d}{dt}|_0(i\partial\overline{\partial}\varphi_a(t) + \alpha_a(t))(\cdot, J_0\cdot)$ is J_0-invariant for any a,

$$\langle \sum_{j=1}^{m} c_j h_{a^j}, \tilde{h}\rangle_{L^2} = \langle \sum_{j=1}^{m} c_j a_1^j ((f^1)^* \otimes (f^1)^* - (f^2)^* \otimes (f^2)^*), \tilde{h}\rangle_{L^2}.$$

We can choose a_1^j's so that the above is nonzero for any nontrivial linear combination. This implies that $h_{a^1}, \cdots, h_{a^m}$ are linearly independent modulo $Im\delta^*_{g_0}$. Since m is arbitrary, we proved the infinite dimensionality.

The last statement is obvious, since $J_t = J_0$ is integrable outside U. This finishes the proof of Theorem 3.3.1. □

In the argument above to prove the infinite dimensionality one can vary a effectively in *uncountably* many ways.

Lemma 3.3.1 *Let E and F be smooth vector bundles of the same dimension on a compact manifold M. Let $G(x, \partial^\beta\varphi, t), |\beta| \leq k, t \in \mathbb{R}^m$ be a smooth* *map of all of its arguments so that* $G : C^\infty(E) \to C^\infty(F)$ *defines a smooth m-parameter family of smooth differential operator of order k. Suppose* $G(x, \partial^\beta \varphi, \tilde{t})$ *is elliptic at* $\tilde{\varphi} \in C^k$. *Then for* $\varphi \in C^k$ *and* $t \in \mathbb{R}^m$ *with* $\|\varphi - \tilde{\varphi}\|_{C^k} + \|t - \tilde{t}\|$ *sufficiently small,* $G(x, \partial^\beta \varphi, t)$ *is also elliptic at* φ.

Proof. The linearization at φ is the linear operator $P(\psi) = \frac{d}{ds}|_0 G(x, \partial^\beta(\varphi + s\psi), t)$. To compute the symbol $\sigma_\xi(P; x) : E_x \to F_x$ at $x \in M$ and for $\xi \in T_x^* M$, choose a smooth function g satisfying that g is zero at x and $dg(x) = \xi$. Then for $\psi \in C^\infty(E)$ with $\psi(x) = v \neq 0$,

$$\sigma_\xi(P; x)v = \frac{i^k}{k!} P(g^k \psi)|_x = \frac{i^k}{k!} \frac{d}{ds}|_0 G(x, \partial^\beta(\varphi + s g^k \psi), t)|_x,$$

which is also nonzero when $\|\varphi - \tilde{\varphi}\|_{C^k} + \|t - \tilde{t}\|$ is sufficiently small. Since M is compact, the symbol is invertible for all $x \in M$. □

If a stronger condition is assumed, we can get almost-Kähler metrics of constant scalar curvature without changing the given symplectic form:

Corollary 3.3.1 *Let* (M, g_0, ω_0, J_0) *be a compact Kähler manifold of complex dimension* $\dim_{\mathbb{C}}(M) \geq 2$ *with constant scalar curvature, where every global holomorphic vector field is parallel. Then near* (g_0, ω_0, J_0) *there exists an infinite dimensional family, modulo diffeomorphisms, of strictly almost-Kähler metrics of constant scalar curvature with the same symplectic form* ω_0.

Proof. First, we are going to find such metrics with the symplectic form in the same cohomology class $[\omega_0]$. Consider the deformation without $\mathcal{H}^{1,1}$ and the corresponding map denoted by the same symbol $\mathcal{F} : L^2_{k+4}/\mathbb{R} \times \mathbb{R} \supset \mathcal{V} \longrightarrow L^2_k/\mathbb{R} \times \mathbb{R}$. Then

$$D\mathcal{F}|_{([0],0)} = \begin{pmatrix} \mathcal{L} & * \\ & \\ 0 & b \end{pmatrix}.$$

By [26], $\mathcal{L}$ is an isomorphism when every global holomorphic vector is parallel. So $D\mathcal{F}|_{([0],0)}$ is an isomorphism and the Inverse Function Theorem says that $(\mathcal{F}^{-1}(\{[0]\} \times \mathbb{R}) \bigcap \mathcal{V}) - \{([0], 0)\}$ is a smooth 1-dimensional submanifold consisting of strictly almost Kähler metrics with the symplectic form in the class $[\omega_0]$. Of course they are smooth metrics by the elliptic regularity. Let this family be $(g_t, \omega_0 + i\partial\overline{\partial}\varphi_t, J_t)$, $t \in \mathbb{R} - \{0\}$. Note that $\omega_0 + i\partial\overline{\partial}\varphi_t = \omega_0 + d(\frac{\partial\varphi_t - \overline{\partial}\varphi_t}{2i})$. Now by Moser's argument in [29], there exists a family of diffeomorphisms Ψ_t such that $\Psi_t{}^*(\omega_0 + d(\frac{\partial\varphi_t - \overline{\partial}\varphi_t}{2i})) = \omega_0$. Thus $(\Psi_t{}^* g_t, \omega_0, \Psi_t{}^* J_t)$ is our desired metrics. The infinite dimensionality of the metrics modulo diffeomorphisms follows from the proof of Theorem 3.3.1 □

By refining the map $\mathcal{F}$, we show the existence of 'prescribed' constant-scalar-curvature almost-Kähler metrics in the next Corollary to Theorem 3.3.1. This is useful in particular to get zero scalar curvature metrics, see Example 2 below.

Corollary 3.3.2 *Let (M, g_0, ω_0, J_0) be a compact Kähler manifold of complex dimension $\dim_{\mathbb{C}}(M) = n \geq 2$ with constant scalar curvature c, where every global holomorphic vector field is parallel and the first Chern class c_1 is not a constant multiple of $[\omega_0]$. Then there exists an infinite dimensional family, modulo diffeomorphisms, of strictly almost-Kähler metrics g of constant scalar curvature near (g_0, ω_0, J_0), with the property that*

$$\frac{\int_M s_g d\mu_g}{(Vol_g)^{\frac{n-1}{n}}} = \frac{\int_M c d\mu_{g_0}}{(Vol_{g_0})^{\frac{n-1}{n}}},$$

where s_g is the scalar curvature and Vol_g denotes the volume.

Proof. Let η be the orthogonal projection of c_1 to the orthogonal complement of $\mathbb{R} \cdot \omega_0$ in $\mathcal{H}^{1,1}$. Then consider the deformation $g_{\varphi, r\eta, t}$, $r \in \mathbb{R}$ and the corresponding map $\mathcal{F}(\varphi, r, t) = (S, f, w)$

$$L^2_{k+4}/\mathbb{R} \times \mathbb{R} \times \mathbb{R} \supset \mathcal{V} \longrightarrow L^2_k/\mathbb{R} \times \mathbb{R} \times \mathbb{R},$$

where $f(\varphi, r, t) = \frac{\int_M S(\varphi,r,t) d\mu_{g_{\varphi,r\eta,t}}}{(Vol_{g_{\varphi,r\eta,t}})^{\frac{n-1}{n}}} - \frac{\int_M c d\mu_{g_0}}{(Vol_{g_0})^{\frac{n-1}{n}}}$. Note that f is also a smooth map. Since the deformation by φ alone produces Kähler metrics with the same volumes, by applying Blair's formula (§1.0.2)

$$\frac{\partial f}{\partial \varphi}|_{([0],0,0)} = 0.$$

On the other hand,

$$\begin{aligned}
\frac{\partial f}{\partial r}|_{([0],0,0)} &= \frac{1}{(Vol_{g_0})^{\frac{2(n-1)}{n}}}\{(\int_M -2\langle \rho, \eta\rangle_{g_0} d\mu_{g_0} + \int_M c\frac{{\omega_0}^{n-1}}{(n-1)!}\wedge\eta)(Vol_{g_0})^{\frac{n-1}{n}} \\
&\quad - \int_M c d\mu_{g_0}\frac{n-1}{n}(Vol_{g_0})^{\frac{-1}{n}}\int_M \frac{{\omega_0}^{n-1}}{(n-1)!}\wedge\eta\} \\
&= \frac{\int_M -2\langle 2\pi c_1, \eta\rangle_{g_0} d\mu_{g_0}(Vol_{g_0})^{\frac{n-1}{n}}}{(Vol_{g_0})^{\frac{2(n-1)}{n}}}
\end{aligned}$$

is nonzero and let it be a. Noting that $\frac{dw}{dr}|_{([0],0,0)} = 0$,

$$D\mathcal{F}|_{([0],0,0)} = \begin{pmatrix} \mathcal{L} & * & * \\ 0 & a & * \\ 0 & 0 & b \end{pmatrix}$$

is an isomorphism. Again the elements in $(\mathcal{F}^{-1}(\{[0]\}\times\{0\}\times\mathbb{R})\bigcap\mathcal{V}) - \{([0],0,0)\}$ are our desired metrics. The smoothness of these metrics and the infinite dimensionality follow in the same way as in the proof of Theorem 3.3.1. □

We prove the following $(s+s^*)$-versions of Theorem 3.3.1 and Corollary 3.3.1.

Theorem 3.3.2 *Let (M, g_0, ω_0, J_0) be as in Theorem 3.3.1. Then there exists an infinite dimensional family, modulo diffeomorphisms, of strictly almost-Kähler metrics with constant $s+s^*$ near (g_0, ω_0, J_0).*

Proof. We only describe the map to be considered and its derivative. The rest of the proof will be similar to that of Theorem 3.3.1. To ensure constant $s+s^*$ instead of s in this case, we define $\tilde{\mathcal{F}} = (S, w) : L^2_{k+4}/\mathbb{R} \times \mathcal{H}^{1,1} \times \mathbb{R} \supset \mathcal{V} \longrightarrow L^2_k/\mathbb{R} \times \mathbb{R}$, where the w map is same as before. $S(\varphi, \alpha, t)$ is $\frac{1}{2}$ times the sum of the scalar curvature and the $*$-scalar curvature of $g_{\varphi,\alpha,t}$ mod constants. Note that S is a smooth map. Since s and s^* coincide in a Kähler manifold, $D\tilde{\mathcal{F}}|_{([0],0,0)}$ is of the same form, i.e.

$$D\tilde{\mathcal{F}}|_{([0],0,0)} = \begin{pmatrix} \mathcal{L} & -2\langle \rho, \cdot \rangle_{g_0} & * \\ & & \\ 0 & 0 & b \end{pmatrix}.$$

□

Corollary 3.3.3 *Let (M, g_0, ω_0, J_0) be as in Corollary 3.3.1. Then near (g_0, ω_0, J_0) there exists an infinite dimensional family, modulo diffeomorphisms, of strictly almost-Kähler metrics of constant $s + s^*$ with the same symplectic form ω_0.*

Proof. With the modified map $\tilde{\mathcal{F}}$, the proof will be similar to that in Corollary 3.3.1. □

Remark. By applying the same methods as all above, we can also prove analogues of Theorem 3.3.1 and Corollary 3.3.1 for the $*$-scalar curvature s^*.

Remark. The linearized Futaki invariant is non-degenerate if for example, every global holomorphic vector field is parallel, which is not quite a strong restriction, because the existence of a nonzero parallel holomorphic vector field would not only force the center of the fundamental group to contain $\mathbb{Z} \oplus \mathbb{Z}$, but also dictate that the universal cover of the manifold be biholomorphic to $\mathbb{C}$ cross a complex manifold of lower dimension [26].

Example. As the primary application of Theorem 3.3.1 or Corollary 3.3.1, we get infinitely many strictly almost-Kähler metrics with constant scalar curvature near most of the Kähler Einstein metrics whose existence were generally shown by Yau [46] and near most of constant-scalar-curvature Kähler metrics in [17, 25, 27]. In particular such metrics with negative constant scalar curvature exist on complex tori or $K3$ surfaces. It is interesting to find the metrics of Theorem 3.3.1 or Corollary 3.3.1 explicitly. See the following example for some explicit metrics on complex tori.

Example. In [15], Jelonek constructed explicitly an infinite dimensional family of strictly almost-Kähler metrics with non-positive constant scalar curvature on the real $2n$-dimensional torus T^{2n} for $n \geq 3$. Let $u, v \in C^\infty(\mathbb{R})$ be any smooth real-valued functions satisfying $u(1+x) = u(x), v(1+x) = v(x)$, and $(u')^2 + (v')^2 = 1$. Think of T^{2n} as $T^2 \times T^{2n-2}$. On T^2, define two functions $f(x,y) = e^{tu(lx+my)}, h(x,y) = e^{tv(lx+my)}$ where $t \in \mathbb{R}$ is a nonzero constant, and $l, m \in \mathbb{Z}$ are integer constants satisfying $l^2 + m^2 > 0$. Let's denote the flat metric on T^2 by g_{T^2} and the standard metric on $S^1 = \mathbb{R}/\mathbb{Z}$ by g_{S^1}. Then for example on T^6,

$$g_{f,h} = g_{T^2} + f g_{S^1} + f^{-1} g_{S^1} + h g_{S^1} + h^{-1} g_{S^1}$$

gives strictly almost-Kähler metric of constant scalar curvature $-2t^2(l^2 + m^2)$ with the same symplectic form as that of $g_{1,1}$. This may be viewed as a special case of deformations of Theorem 3.3.1: with the notation in section 3, $n = 3$, $U = M$, $a_1 = 1$, $\phi = 0$.

Example. Kähler metrics with zero scalar curvature on compact complex surfaces M are absolute minima of the squared L^2-norm functional of the Riemannian curvature tensor over the space of smooth Riemannian metrics on M, [27]. Our Corollary 3.3.2 can produce infinitely many strictly almost-Kähler metrics with zero scalar curvature near most of Kähler metrics with zero scalar curvature.

Indeed, consider the Kähler metrics with zero scalar curvature which were found in [17, 27]. They exist on some blow-ups of minimal ruled surfaces over any compact smooth Riemann surface and most of these ruled surfaces have no nontrivial holomorphic vector fields. It is easy to see that c_1 is not a constant multiple of $[\omega_0]$ on these. So by Corollary 3.3.2, there exists an infinite dimensional family of strictly almost-Kähler metrics with zero scalar curvature on these manifolds. Similar existence should generally hold in higher dimensions.

Chapter 4

Manifolds of positive scalar curvature

4.1 Surgery and curvature

Just for notational conveniences, let's consider $a\underline{K} + s$ where a is a non-negative constant.

Theorem 4.1.1 *Let X be a Riemannian manifold with $a\underline{K} + s > c$ where c is a constant. Then any manifold obtained from X by performing surgeries in codimension ≥ 3 also carries a metric with $a\underline{K} + s > c$.*

The same is true of the "connected sum" along embedded spheres with *trivial normal bundles in codimension* ≥ 3.

Proof. Our proof of theorem 4.1.1 is almost parallel to that of Gromov-Lawson [13]. First, let's consider the surgery case. Let S^p be an embedded sphere with trivial normal bundle N of dimension ≥ 3. The idea of proof is to deform a metric on $N - S^p$ keeping $a\underline{K} + s > c$ to make a cylindrical end isometric to the riemannian product $S^p \times S^{q-1} \times [0,1]$ with the standard metric, so that we can glue the riemannian product $D^{p+1} \times S^{q-1}$ with the standard metric. Take a global orthonormal frame of N to get a global trivialization $S^p \times \mathbb{R}^q$ of N. Define a function $r : S^p \times \mathbb{R}^q \to \mathbb{R}$ by $r(y,x) = \|x\|$. Set $N(\rho) \equiv S^p \times D^q(\rho) := \{(y,x) \in N : r(y,x) \leq \rho\}$ and $S^p \times S^{q-1}(\rho)$ to be its boundary. Choose $\bar{r} > 0$ so that the exponential map $\exp : N \to X$ is an embedding on $N(\bar{r})$. Pullback the metric of X to $S^p \times D^q(\bar{r})$ by the exponential map. Recall the following simple observation in [13].

Lemma 4.1.1 (Gromov-Lawson) *Let D be a geodesic normal coordinate on a Riemannian manifold of dimension $n \geq 2$. Let $S^{n-1}(\varepsilon)$ be the hypersurface $\{x \in D : \|x\| = \varepsilon\}$ with the induced metric g_ε and let $g_{0,\varepsilon}$ be the standard metric on the sphere of radius ε in $\mathbb{R}^n$. Then the principal curvature of $S^{n-1}(\varepsilon)$ in D are of the form $-\frac{1}{\varepsilon} + O(\varepsilon)$ for ε small, and $\frac{1}{\varepsilon^2} g_\varepsilon \to \frac{1}{\varepsilon^2} g_{0,\varepsilon} = g_{0,1}$ in the C^∞ topology as $\varepsilon \to 0$.*

Proof. See [13] □

Consider the hypersurface $S^p \times S^{q-1}(\varepsilon)$ in $S^p \times D^q(\bar{r})$. Note that $\frac{\partial}{\partial r}$ is the outward normal frame field on $S^p \times S^{q-1}(\varepsilon)$. Let $\widehat{II}$ be the second fundamental form of $S^p \times S^{q-1}(\varepsilon)$, considered as a real-valued symmetric bilinear 2-form by taking the inner product with the normal. We estimate it as follows.

Lemma 4.1.2 *Let $(TS^{q-1}(\varepsilon))^{\perp}$ be the orthogonal complement of $TS^{q-1}(\varepsilon) \subset T(S^p \times S^{q-1}(\varepsilon))$. Then for ε sufficiently small, $TS^{q-1}(\varepsilon)$ is the span of the $q-1$ principal vectors with principal curvatures of the form $-\frac{1}{\varepsilon} + O(\varepsilon)$, and $(TS^{q-1}(\varepsilon))^{\perp}$ is the span of the p principal vectors with principal curvatures of the form $O(1)$.*

Proof. Let $(y, x) \in S^p \times S^{q-1}(\varepsilon)$. Take a geodesic normal coordinate $D^n(\varepsilon)$ of $S^p \times S^{q-1}(\varepsilon)$ at $(y, 0)$, consisting of points with the distance from the origin less than or equal to ε and let $S^n(\varepsilon)$ be its boundary. Note that $\frac{\partial}{\partial r}\big|_{(y,x)}$ is also a unit normal vector to $S^n(\varepsilon)$, and $S^p \times S^{q-1}(\varepsilon)$ and $S^n(\varepsilon)$ are tangent to each other intersecting at $\{y\} \times S^{q-1}(\varepsilon)$. Denote the second fundamental form of $S^n(\varepsilon)$ by $\widetilde{II}$.

Now for a unit tangent vector v of $S^p \times S^{q-1}(\varepsilon)$ at (y, x), write it as $v = a_1 v_1 + a_2 v_2$ for some constants a_1 and a_2, where $v_1 \in T(\{y\} \times S^{q-1}(\varepsilon))$

and $v_2 \in (T(\{y\} \times S^{q-1}(\varepsilon)))^{\perp}$ are of unit norm. By the abuse of notation, v_1 and v_2 will also denote local extensions keeping $v_1 \in T(\{y\} \times S^{q-1}(\varepsilon))$. Then at (y, x),

$$\begin{aligned}
\widehat{II}(v_1, v_2) &= \langle \nabla_{v_2} v_1, \frac{\partial}{\partial r} \rangle \\
&= \widetilde{II}(v_1, v_2) \\
&= \frac{1}{2}(\widetilde{II}(v, v) - a_1^2 \widetilde{II}(v_1, v_1) - a_2^2 \widetilde{II}(v_2, v_2)) \\
&= \frac{1}{2}((-\frac{1}{\varepsilon} + O(\varepsilon)) - a_1^2(-\frac{1}{\varepsilon} + O(\varepsilon)) - a_2^2(-\frac{1}{\varepsilon} + O(\varepsilon))) \\
&= O(\varepsilon),
\end{aligned}$$

using $a_1^2 + a_2^2 = 1$ at the last step. So

$$\begin{aligned}
\widehat{II}(v, v) &= a_1^2 \widehat{II}(v_1, v_1) + a_2^2 \widehat{II}(v_2, v_2) + 2a_1 a_2 \widehat{II}(v_1, v_2) \\
&= a_1^2(-\frac{1}{\varepsilon} + O(\varepsilon)) + a_2^2 \langle \nabla_{v_2} v_2, \frac{\partial}{\partial r} \rangle + 2a_1 a_2 O(\varepsilon) \\
&= a_1^2(-\frac{1}{\varepsilon} + O(\varepsilon)) + a_2^2 O(1) + O(\varepsilon) \\
&= -\frac{a_1^2}{\varepsilon} + O(1).
\end{aligned}$$

Thus if v is a principal vector, then $|a_1| = 0$ or $|a_1| = 1$. This completes the proof. □

From now on we assume that $\bar{r}$ also satisfies the condition of lemma 4.1.2. Being prepared with this, let's get down to making a cylindrical end diffeomorphic to $S^p \times S^{q-1} \times \mathbb{R}$ having a metric with $a\underline{K} + s > c$. Following Gromov-Lawson [13], we construct it as a smooth hypersurface M of the riemannian product $N(\bar{r}) \times \mathbb{R}$ by the relation

$$M = \{(y, x, t) \in S^p \times D^q(\bar{r}) \times \mathbb{R} : (t, \|x\|) \in \gamma\},$$

where γ is a smooth curve in the (t, r)-plane as pictured below. The curve γ starts along the positive r-axis and ends as a straight line parallel to the t-axis.

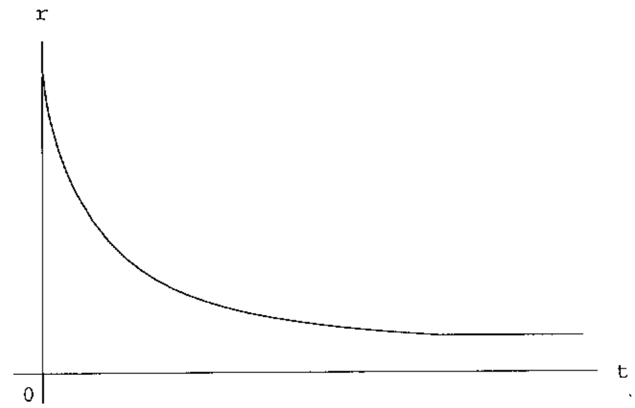

Figure 4.1: Curve γ

Our main goal is to show that we can choose γ so that the metric induced on M has $a\underline{K} + s > c$. Let $\theta(t, r)$ be the acute angle between the normal to M and the t-axis. Then we see that $\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t}$ is a unit normal. Let $(y, x) \in S^p \times D^q(\bar{r})$ and l be a geodesic ray in $S^p \times D^q(\bar{r})$ from $(y, 0)$ to (y, x). Since $l \times \mathbb{R}$ is totally geodesic in $S^p \times D^q(\bar{r}) \times \mathbb{R}$, $\gamma_l := M \cap (l \times \mathbb{R})$ is a principal curve on M. Let $k \geq 0$ be its curvature function. Let's determine other principal directions. They all belong to $T(S^p \times S^{q-1}(r(y, x))) \subset TM$, because they are orthogonal to γ_l. Note that $\nabla_v(\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t}) = \sin\theta\nabla_v\frac{\partial}{\partial r}$ for $v \in T(S^p \times S^{q-1}(r))$. Thus by lemma 4.1.2, $TS^{q-1}(r)$ is the span of the $q-1$

principal vectors with principal curvatures of the form $(-\frac{1}{r}+O(r))\sin\theta$, and $(TS^{q-1}(r))^{\perp}$ is the span of the p principal vectors with principal curvatures of the form $O(1)\sin\theta$. We choose $e_1,\cdots,e_n$ to be an orthonormal basis of $T_{(y,x)}M$ such that e_1 is tangent to γ_l and $e_2,\cdots,e_q$ are principal vectors belonging to $TS^{q-1}(r)$ and $e_{q+1},\cdots,e_n$ are principal vectors belonging to $(TS^{q-1}(r))^{\perp}$.

The Gauss curvature equation states that the sectional curvature $K(u_1,u_2)$ of M, corresponding to the plane $u_1\wedge u_2$ for orthonormal $u_1,u_2\in TM$ is given by

$$K(u_1,u_2)=\widetilde{K}(u_1,u_2)+\det\begin{pmatrix} II(u_1,u_1) & II(u_1,u_2) \\ II(u_2,u_1) & II(u_2,u_2) \end{pmatrix}, \qquad (\S 4.1.1)$$

where $\widetilde{K}$ is the sectional curvature of $N(\bar{r})\times\mathbb{R}$ and II is the second fundamental form of M in $N(\bar{r})\times\mathbb{R}$. To estimate $\widetilde{K}$ we need to estimate the angle made by t-axis and the tangent vector of M at each point.

Lemma 4.1.3 *Any unit tangent vector u to M at the point where the acute angle between the normal and t-axis is θ, can be written as $u=(\cos\alpha)H+\sin\alpha\frac{\partial}{\partial t}$ for unit $H\in TN(\bar{r})$ and $-\theta\leq\alpha\leq\theta$.*

Proof. We may assume $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. Suppose $\alpha > \theta$. Then

$$\begin{aligned} 0 &= \langle u, \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t}\rangle = \cos\alpha\sin\theta\langle H, \frac{\partial}{\partial r}\rangle + \sin\alpha\cos\theta \\ &= \sin(\alpha-\theta) + \cos\alpha\sin\theta(\langle H, \frac{\partial}{\partial r}\rangle + 1) > 0, \end{aligned}$$

yielding a contradiction.

Now Suppose $\alpha < -\theta$. Then

$$\langle u, \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial t}\rangle = \sin(\alpha+\theta) + \cos\alpha\sin\theta(\langle H, \frac{\partial}{\partial r}\rangle - 1) < 0,$$

yielding a contradiction too. □

Now let $u_1 = (\cos\alpha_1)H_1 + \sin\alpha_1\frac{\partial}{\partial t}$ and $u_2 = (\cos\alpha_2)H_2 + \sin\alpha_2\frac{\partial}{\partial t}$ for $H_1, H_2 \in TN(\bar{r})$ and $-\theta \leq \alpha_1, \alpha_2 \leq \theta$ be orthonormal vectors in TM. We will denote the sectional curvature, Ricci curvature, and scalar curvature of $N(\bar{r})$ by K^N, Ric^N, and s^N respectively. And let $\widetilde{R}$ be the Riemann curvature tensor of $N(\bar{r}) \times \mathbb{R}$. Since $\frac{\partial}{\partial t}$ is parallel, $\widetilde{R}$ vanishes on it. Thus

$$\begin{aligned} \widetilde{K}(u_1, u_2) &\equiv \widetilde{R}(u_1, u_2, u_2, u_1) \\ &= \cos^2\alpha_1\cos^2\alpha_2\ \widetilde{R}(H_1, H_2, H_2, H_1) \\ &= \cos^2\alpha_1\cos^2\alpha_2\ K^N(H_1, H_2) \\ &\geq \min(\underline{K}^N, \cos^4\theta\ \underline{K}^N). \end{aligned}$$

With respect to the orthonormal basis $e_1, \cdots, e_n$ previously defined,

$$\widetilde{K}(e_1, e_j) = \cos^2\theta\ K^N(\frac{\partial}{\partial r}, e_j), \quad \widetilde{K}(e_i, e_j) = K^N(e_i, e_j)$$

for $i, j = 2, \cdots, n$.

Now we want to estimate the second term $\det(II(\cdot,\cdot))$ in the Gauss equation (§4.1.1). With respect to e_i's which are eigenvectors of II, we just have

$$\det \begin{pmatrix} II(e_i, e_i) & II(e_i, e_j) \\ II(e_j, e_i) & II(e_j, e_j) \end{pmatrix} = \lambda_i \lambda_j,$$

where λ_i is the principal curvature corresponding to e_i. In general we have the following.

Lemma 4.1.4 $\det \begin{pmatrix} II(u_1, u_1) & II(u_1, u_2) \\ II(u_2, u_1) & II(u_2, u_2) \end{pmatrix} \geq (-\frac{4}{r} + O(1))k \sin\theta + O(1)\frac{\sin^2\theta}{r}.$

Proof. Let $u_1 = \sum_{i=1}^n b_i e_i$ and $u_2 = \sum_{i=1}^n c_i e_i$ for some constants b_i and c_i satisfying $\sum_{i=1}^n b_i^2 = \sum_{i=1}^n c_i^2 = 1$. Then

$$II(u_1, u_1) = kb_1^2 + \sum_{i=2}^{q} b_i^2(-\frac{1}{r} + O(r))\sin\theta + \sum_{j=q+1}^{n} b_j^2 O(1)\sin\theta$$

$$II(u_2, u_2) = kc_1^2 + \sum_{i=2}^{q} c_i^2(-\frac{1}{r} + O(r))\sin\theta + \sum_{j=q+1}^{n} c_j^2 O(1)\sin\theta$$

$$II(u_1, u_2) = II(u_2, u_1) = kb_1c_1 + \sum_{i=2}^{q} b_i c_i(-\frac{1}{r} + O(r))\sin\theta + \sum_{j=q+1}^{n} b_j c_j O(1)\sin\theta,$$

and hence

$$\begin{aligned}
\det(II) &= -\sum_{i=1}^{q}(b_1c_i - c_1b_i)^2\frac{k\sin\theta}{r} + (\sum_{i=1}^{q} b_i^2\sum_{l=1}^{q} c_l^2 - \sum_{i=1}^{q} b_ic_i\sum_{l=1}^{q} b_lc_l)\frac{\sin^2\theta}{r^2} \\
&\quad + O(1)k\sin\theta + O(1)\frac{\sin^2\theta}{r} \\
&\geq -\frac{4k\sin\theta}{r} + (\sum_{i=1}^{q} b_i^2\sum_{l=1}^{q} c_l^2 - (\sum_{i=1}^{q} b_i^2)^2(\sum_{l=1}^{q} c_l^2)^2)\frac{\sin^2\theta}{r^2} \\
&\quad + O(1)k\sin\theta + O(1)\frac{\sin^2\theta}{r} \\
&\geq -\frac{4k\sin\theta}{r} + O(1)k\sin\theta + O(1)\frac{\sin^2\theta}{r}.
\end{aligned}$$

□

Summarizing all the above, we estimate $a\underline{K} + s$ of M as follows.

$$\begin{aligned}
s &= \sum_{i\neq j} K(e_i, e_j) \\
&= 2\sum_{j=2}^{n}\cos^2\theta\ K^N(\frac{\partial}{\partial r}, e_j) + 2(q-1)(-\frac{1}{r} + O(r))k\sin\theta + 2pO(1)k\sin\theta \\
&\quad + \sum_{i,j\geq 2} K^N(e_i, e_j) + (q-1)(q-2)(-\frac{1}{r} + O(r))^2\sin^2\theta \\
&\quad + 2p(q-1)O(1)(-\frac{1}{r} + O(r))\sin^2\theta + p(p-1)O(1)^2\sin^2\theta \\
&= s^N - 2Ric^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})\sin^2\theta + (-\frac{2(q-1)}{r} + O(1))k\sin\theta \\
&\quad + (q-1)(q-2)\frac{\sin^2\theta}{r^2} + O(1)\frac{\sin^2\theta}{r},
\end{aligned}$$

and

$$\underline{K} \;\geq\; \min(\underline{K}^N, \cos^4\theta\ \underline{K}^N) + (-\frac{4}{r} + O(1))k\sin\theta + O(1)\frac{\sin^2\theta}{r}.$$

Thus

$$\begin{aligned} a\underline{K} + s \;\geq\;& a\cdot\min(\underline{K}^N, \cos^4\theta\ \underline{K}^N) + s^N \\ & +(q-1)(q-2)(\frac{1}{r^2} + \frac{O(1)}{r})\sin^2\theta \\ & -2(2a+q-1)(\frac{1}{r} + O(1))k\sin\theta. \end{aligned} \qquad (\S 4.1.2)$$

We now construct γ so that $a\underline{K} + s > c$ in 5 steps.

STEP 0: Let $r_0 > 0$ be a small number satisfying $|O(1)| < \frac{1}{2r_0} < \frac{1}{2r_0^2}$ for the two $O(1)$'s in (§4.1.2). Then $(0, r_0)$ is going to be the starting point of γ in (t,r)-plane. So (§4.1.2) can be rewritten as

$$\begin{aligned} a\underline{K} + s \;\geq\;& a\cdot\min(\underline{K}^N, \cos^4\theta\ \underline{K}^N) + s^N \\ & +\frac{(q-1)(q-2)}{2}\frac{1}{r^2}\sin^2\theta - 3(2a+q-1)\frac{k}{r}\sin\theta. \end{aligned} \qquad (\S 4.1.3)$$

STEP 1: By continuity, make a small bend of γ to an angle $\theta = \theta_0 > 0$ keeping the right hand side of (§4.1.3) greater than c and satisfying

$$\inf_{(y,x)\in N(r_0)} (a\cos^4\theta_0\ \underline{K}^N + s^N) > c. \qquad (\S 4.1.4)$$

Let's call the resulting radius r_1 for a later purpose.

STEP 2: Then owing to (§4.1.4), we can let γ proceed down as long as we want as a straight line which keeps the angle θ_0 unchanged and the right hand side of (§4.1.3) greater than c.(Note k=0.) Let's go down to $r = r_2$ such that

$$\inf_{\substack{0 \le \theta \le \frac{\pi}{2} \\ (y,x) \in N(r_0)}} (a \cos^4 \theta \, \underline{K}^N + s^N) + \frac{(q-1)(q-2)}{4r_2^2} \sin^2 \theta_0 > c. \qquad (\S 4.1.5)$$

STEP 3: Make a bend of γ after the following prescription of the curvature function $k(L)$ parameterized by the arc length L. Here, k_0,the maximum

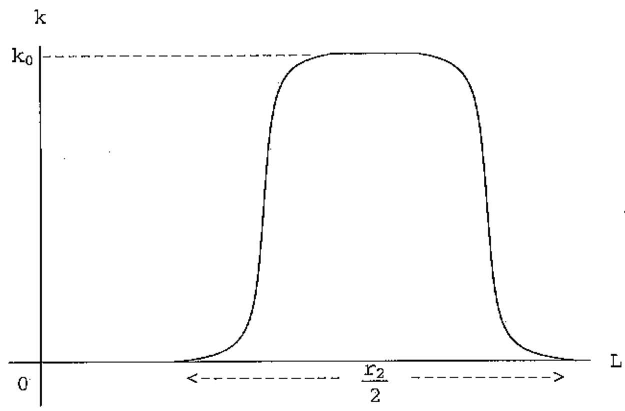

Figure 4.2: Curvature fucntion $k(L)$

of k, is defined as $\frac{(q-1)(q-2)\sin\theta_0}{12r_2(q-1+2a)}$ so that

$$\frac{(q-1)(q-2)}{4r^2} \sin^2 \theta - 3(2a + q - 1)\frac{k}{r} \sin \theta \ge 0$$

is ensured during this process. Hence, combined with (§4.1.5) it guarantees $a\underline{K} + s > c$. Note that γ does not cross the line $r = \frac{r_2}{2}$ because the length of the bend is $r = \frac{r_2}{2}$ and it has begun at the radius $r = r_2$. The amount of the bend $\Delta\theta$ is

$$\Delta\theta = \int k \, dL \approx k_0 \cdot \frac{r_2}{2} = \frac{(q-1)(q-2)\sin\theta_0}{24(q-1+2a)},$$

which is independent of the starting radius.

STEP 4: Repeat the step 3 with the curvature prescription completely determined only by the ending radius of the previous process until we achieve a total bend of $\frac{\pi}{2}$. $(([\frac{\pi}{2}/\Delta\theta]+1)$-times is enough.)

Let τ be the final radius and let ds_τ^2 be the metric on the boundary $\partial M = S^p \times S^{q-1}(\tau)$. As $\tau \to 0$, ds_τ^2 converges in C^∞ to the metric induced on the τ-sphere bundle of N from the natural metric on N using the normal connection, which is a riemannian submersion with totally geodesic fibers with the standard euclidean metric of curvature $\frac{1}{\tau^2}$. Let's call this metric h_τ. Consider a smooth homotopy $H_\tau^1(x,t) = \varphi(t)ds_\tau^2 + (1-\varphi(t))h_\tau$ for $(x,t) \in \partial M \times [0,1]$, where $\varphi : [0,1] \to [0,1]$ is a smooth decreasing function which is 1 near 0 and 0 near 1. As $\tau \to 0$, the metric $H_\tau^1(x,t) + dt^2$ on $\partial M \times [0,1]$ converges in C^∞ to $h_\tau + dt^2$, whose $a\underline{K} + s$ tends to infinity as $\tau \to 0$ by O'Neill's formula in [3]. We can also homotope h_τ through riemannian submersions with totally geodesic fibers to one where the metric on S^p is standard and then deform the horizontal planes to those of the product metric. Let these two homotopies be $H_\tau^2(x,t), x \in \partial M, t \in [0,1]$. Note that $\partial M \times [0,1]$ with the metric $H_\tau^2(x,t) + dt^2$ is also a riemannian submersion with totally geodesic fibers S^{q-1}. Thus again by O'Neill's formula, it has $a\underline{K} + s > c$ when τ is sufficiently small.

Finally we have a cylindrical end isometric to the product $S^p \times S^{q-1}(\tau) \times [0,1]$ with the standard metric. Now we can glue the riemannian product $D^{p+1} \times S^{q-1}$ with the standard metric to get a smooth Riemannian manifold with $a\underline{K} + s > c$. The case of the "connected sum" along S^p is done in the same way. This completes the proof. □

Remark. Note that in the above we didn't change the original metric outside of a arbitrarily small neighborhood $N(r_0)$ of S^p where the surgery or the "connected sum" is performed. This fact will be used in the proof of theorem 4.2.1.

Corollary 4.1.1 *Any compact simply connected non-spin manifold X of dimension ≥ 5 carries a metric with $a\underline{K} + s > 0$.*

Proof. In [13], Gromov-Lawson showed that if X is oriented cobordant to another manifold X', then X can be obtained from X' by surgeries in codimension ≥ 3. So it is enough to show that there exists a set of generators for the oriented cobordism ring Ω_*^{SO}, each of which carries a metric with $a\underline{K} + s > 0$. As in [13], let's consider $\mathbb{C}P^n$, Milnor manifolds, Dold manifolds, and manifolds of type V as generators. We know Fubini-Study metric on $\mathbb{C}P^n$ has positive sectional curvature. Milnor manifolds, the hypersurfaces of degree $(1,1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$, are projective space bundles over a projective space, which is a riemannian submersion with totally geodesic fibers. By shrinking the metric uniformly in the fibers, we get $a\underline{K} + s > 0$. Dold manifolds, $(S^n \times \mathbb{C}P^m)/Z_2$ where Z_2 acts by -1 on the left and conjugation the right, has a metric of positive sectional curvature. Manifolds of type V carry a metric which is locally a riemannian product of Dold manifolds and flat $\mathbb{R}^k$. This completes the proof. □

Corollary 4.1.2 *Any compact simply connected spin manifold X of dimension ≥ 5 with $\alpha(X) = 0$ carries a metric with $a\underline{K} + s > 0$.*

Proof. Again by Gromov and Lawson [13], if X is spin cobordant to another manifold X', then X can be obtained from X' by surgeries in codimension ≥ 3. In [38] Stolz actually showed that any compact spin manifold X with $\alpha(X) = 0$ is spin cobordant to the total space of a fiber bundle with the fiber $\mathbb{H}P^2$ and the structure group $PSp(3)$. Note that $\mathbb{H}P^2$ admits a metric of positive sectional curvature in which $PSp(3)$ acts by isometries. So $\mathbb{H}P^2$ bundle is a riemannian submersion with totally geodesic fibers and $a\underline{K}+s > 0$ is achieved by shrinking the fibers sufficiently small. This concludes the proof.

□

In particular, every compact simply connected manifold of dimension 5, 6, or 7 admits a metric with $a\underline{K} + s > 0$, because the spin cobordism group in dimension 5, 6, or 7 is trivial.

4.2 Minimal volume

Theorem 4.2.1 *Let X_1 and X_2 be compact n-manifolds for $n \geq 3$. Then*

$$Vol_{\lambda,K,s}(X_1 \# X_2) \leq Vol_{\lambda,K,s}(X_1) + Vol_{\lambda,K,s}(X_1).$$

for any $\lambda \in [0,1)$. The same is true of the "connected sum" along embedded spheres with trivial normal bundles in codimension ≥ 3.

Proof. Let ε be any positive constant and take a metric g_i on X_i such that $\mathrm{Vol}(X_i, g_i) \leq \mathrm{Vol}_{\lambda,K,s}(X_i) + \frac{\varepsilon}{6}$ for $i = 1, 2$. We make cylindrical ends of X_1 and X_2 with $\lambda \underline{K} + (1-\lambda)\frac{s}{n(n-1)} > -1$ following the procedure in the proof of theorem 4.1.1. We claim that the volume of transitional region M can be made arbitrarily small by taking r_0 very small. First we can assume the length of γ_l in step 1 is less than any constant, say 1. Furthermore we can take γ_l in step 1 to satisfy $2(r_0 - r) < \frac{\tan\theta}{2} < \sin\theta$ at each point so that we have $r_0 < 2(r_0 - r_1) < \sin\theta_0$ in particular. Thus we have

$$\begin{aligned} \text{Total length of } \gamma_l &\leq 1 + \frac{r_0}{2}([\frac{\pi}{2}/\Delta\theta] + 1) \\ &= 1 + \frac{r_0}{2}([\frac{12\pi(q-1+2n^2-2n)}{(q-1)(q-2)\sin\theta_0}] + 1) \\ &\leq 1 + \frac{6\pi(q-1+2n^2-2n)}{(q-1)(q-2)} + \frac{r_0}{2}, \end{aligned}$$

and hence

$$\text{volume of } M \leq (\text{lengh of } \gamma_l) \cdot \sup_{\tau \leq r \leq r_0} \mathrm{Vol}(S^p \times S^{q-1}(r), g_i)$$

can be made smaller than $\frac{\varepsilon}{6}$ by taking r_0 sufficiently small. Obviously the volume of the homotopy region can also be made smaller than $\frac{\varepsilon}{6}$ when τ is sufficiently small. Therefore the resulting glued manifold has a volume smaller than

$$\mathrm{Vol}_{\lambda,K,s}(X_1) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \mathrm{Vol}_{\lambda,K,s}(X_2) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6}$$
$$= \mathrm{Vol}_{\lambda,K,s}(X_1) + \mathrm{Vol}_{\lambda,K,s}(X_1) + \varepsilon,$$

completing the proof. □

Since S^n for $n \geq 2$ admits a metric of positive sectional curvature, we immediately get

Corollary 4.2.1 *Let X' be a compact manifold obtained from X by surgeries in codimension ≥ 3. Then*

$$Vol_{\lambda,K,s}(X') \leq Vol_{\lambda,K,s}(X)$$

for any $\lambda \in [0, 1)$.

Following Petean [33], by performing "inverse surgeries" on X' we have

Theorem 4.2.2 *Let X' be a n-dimensional manifold obtained from a compact manifold X by surgeries on spheres of dimension $\neq 1, n-1, n-2$ for $n \geq 4$. Then*

$$Vol_{\lambda,K,s}(X') = Vol_{\lambda,K,s}(X)$$

for any $\lambda \in [0,1)$.

Proof. If $n-p-1 \leq n-3$ i.e. $p \geq 2$, then we can perform a surgery along ∂D^{n-p} again to get back the original manifold X. In case $p=0$, note that doing 0-dimensional surgery is the same as taking the connected sum with $S^1 \times S^{n-1}$. So doing a surgery on the S^1 in X' gives us X back. Now applying corollary 4.2.1, we get

$$\mathrm{Vol}_{\lambda,K,s}(X) \leq \mathrm{Vol}_{\lambda,K,s}(X') \leq \mathrm{Vol}_{\lambda,K,s}(X).$$

□

In [24] and [21] LeBrun proved that any compact complex surface of general type X satisfies

$$\mathrm{Vol}_{\frac{1}{2},K,s}(X) \geq \frac{9}{4}\mathrm{Vol}_s(X) = \frac{\pi^2}{2}c_1^2(M),$$

where M is the minimal model of X and every compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$ saturates the equality. It is an interesting question what else can satisfy the equality. Our previous results can produce another simple examples. Since a blow-up of a complex surface is diffeomorphic to the connected sum with $\overline{\mathbb{C}P^2}$ whose minimal volumes are all zero, a blown-up manifold has the same $\mathrm{Vol}_{\frac{1}{2},K,s}(X)$ and $\mathrm{Vol}_s(X)$. Thus we have

Corollary 4.2.2 *Suppose X is birational to a compact complex-hyperbolic 4-manifold. Then*

$$Vol_{\frac{1}{2},K,s}(X) = \frac{9}{4}Vol_s(X) = \frac{\pi^2}{2}c_1^2(M)$$

where M is the minimal model of X.

As the case of Yamabe minimal volume in Petean [34] and Paternain and Petean [32], we also obtain similar results.

Theorem 4.2.3 *Let X be a compact simply connected manifold of dimension ≥ 5. Then $Vol_{\lambda,K,s}(X) = 0$ for any $\lambda \in [0, 1)$.*

Proof. When X is non-spin, $\mathrm{Vol}_{\lambda,K,s}(X) = 0$ by corollary 4.1.1. When X is spin, the following proposition gives the proof. □

Proposition 4.2.1 *Every element in the spin cobordism group Ω_n^{Spin} for $n \geq 1$ can be represented by a connected spin manifold with $Vol_{\lambda,K,s} = 0$.*

Proof. Overall the proof is parallel to that of Petean [34]. When $n = 1$, we have $\Omega_1^{Spin} \cong \mathbb{Z}_2$, and its nontrivial element is S^1 with the "bad" spin structure, i.e. the disconnected 2-fold covering of S^1, and its trivial element is S^1 with the spin structure coming from the boundary of the 2-disk. When $n = 2$, we also have $\Omega_2^{Spin} \cong \mathbb{Z}_2$, and its nontrivial element can be represented by the torus $S^1 \times S^1$ for the nontrivial element $S^1 \in \Omega_1^{Spin}$, and its trivial element can be represented by S^2 with its unique spin structure. When $n = 3, 5, 6$, or 7, Ω_n^{Spin} is trivial and its only element can be represented by S^n with its unique spin structure. In all these cases, obviously $\mathrm{Vol}_{\lambda,K,s} = 0$.

For $n = 4$, $\Omega_4^{Spin} \cong \mathbb{Z}$ is generated by K3 surface and its trivial element can be represented by S^4.

Lemma 4.2.1 *$Vol_{\lambda,K,s}$(K3 surface) $= 0$.*

Proof. For convenience set $a = n(n-1)\frac{\lambda}{1-\lambda}$. Let Γ be the involution of a complex torus T^4 given by $z \to -z$ and $\widehat{T^4}$ be a blow up of T^4 at 16 fixed points. Note $\widehat{T^4}$ is diffeomorphic to the connected sum $T^4 \# 16\overline{\mathbb{C}P^2}$. The involution Γ extends to an involution on $\widehat{T^4}$ with 16 exceptional 2-spheres as fixed points. Put the flat metric on T^4. Let X_1 be a cylindrical ended manifold with $a\underline{K} + s > -\frac{n(n-1)}{1-\lambda}$ made from $T^4 - \{16 \text{ fixed points}\}$ following the procedure in the proof of theorem 4.1.1. Let r_0 be the starting radius and τ be the final radius. Note that Γ extends to an isometry on X_1 and $\partial X_1/\Gamma$

is isometric to $\mathbb{R}P^3$ with the standard metric of curvature $\frac{1}{\tau^2}$.

Now note that $\overline{\mathbb{C}P^2}$ with a closed ball deleted is diffeomorphic to the cotangent bundle T^*S^2 of S^2. We are going to show that T^*S^2 admits a metric with $a\underline{K}+s>0$ and a cylindrical end isometric to the quotient of a Berger sphere by the involution $x \mapsto -x$. Consider S^3 with the standard metric of curvature 1, which is bi-invariant. Let $\{\sigma^1, \sigma^2, \sigma^3\}$ be an orthonormal coframing where σ^1 corresponds to the Hopf fibers. Consider a riemannian submersion

$$((0,1] \times S^1 \times S^3, dr^2 + \tan^2(2r)d\theta^2 + [(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2])$$

$$\downarrow$$

$$((0,1] \times \mathbb{R}P^3, dr^2 + [\sin^2(2r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2]),$$

which is just a quotient by the isometric circle action i.e. the complex scalar multiplication on both S^1 and S^3. Note that these two metrics extend smoothly up to $r=0$ so that they give a riemannian submersion from $\mathbb{R}^2 \times S^3$ onto T^*S^2.(See lemma 4.1 in [35].) Using the formulae for the warped product in [35], we can easily show that the sectional curvature of the plane tangent to $\mathbb{R}^2$ of the riemannian product $\mathbb{R}^2 \times S^3$ is $-\frac{(\tan(2r))''}{\tan(2r)} = -8\sec^2(2r)$, and the base manifold T^*S^2 has constant scalar curvature 12. Since sectional curvatures do not decrease under the riemannian submersion, $a\underline{K}+s \geq 0+12$ at the zero section of T^*S^2, and hence we can take δ sufficiently small so that $a\underline{K}+s>4$ and $s>4$ for $(r,x) \in [0,\delta] \times \mathbb{R}P^3$.

To make a cylindrical end we again use a similar method to that of theorem 4.1.1. We will adopt the same notations. First note that the induced metric g_r on $X_r = \{r\} \times \mathbb{R}P^3$ for $r > 0$ gives a Berger sphere metric whose sectional curvatures lie in the interval $[\sin^2(2r), 4 - 3\sin^2(2r)]$. Let $e_1 = \frac{\partial}{\partial r}, e_2, e_3, e_4$ be an orthonormal frame dual to $dr, \sin(2r)\sigma^1, \sigma^2, \sigma^3$ respectively. Then the second fundamental form $\widehat{II}$ of X_r with respect to a unit normal $\frac{\partial}{\partial r}$ is computed as follows:

$$\widehat{II}(e_2) = -\frac{(\sin(2r))'}{\sin(2r)} e_2 = -2\cot(2r) e_2$$

$$\widehat{II}(e_i) = -\frac{(1)'}{1} e_i = 0, \ i = 3, 4.$$

We construct the hypersurface M in the riemannian product $T^*S^2 \times [0,1]$ as

$$M = \{(r, x, t) \in [0, \delta] \times \mathbb{R}P^3 \times [0,1] : (r, t) \in \gamma\},$$

where γ is pictured below. As before γ finishes as a straight line parallel to the t-axis but $\sin\theta \frac{\partial}{\partial r} - \cos\theta \frac{\partial}{\partial t}$ is a unit normal where θ is the acute angle between the normal and the t-axis. The important point is that this time the curvature k of γ is non-positive!

Since the integral curve of $\frac{\partial}{\partial r}$ denoted by l is a geodesic on T^*S^2 this time also, $\gamma_l = M \cap (l \times [0,1])$ is a principal curve with a curvature k. The other principal directions are tangent to X_r with principal curvatures $-2\cot(2r)\sin\theta, 0$

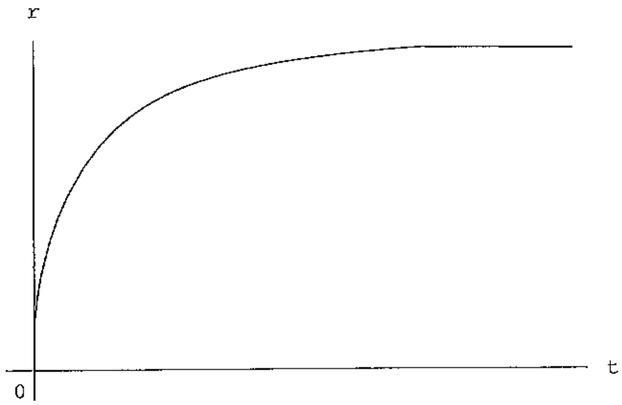

Figure 4.3: Curve γ

and 0. Using the same notations as in the proof of theorem 4.1.1, we get

$$\begin{aligned}\underline{K} &\geq \min(\underline{K}^N, \cos^4\theta\underline{K}^N) + \det(II) \\ &= \min(\underline{K}^N, \cos^4\theta\underline{K}^N) - 2k\cot(2r)\sin\theta(b_1c_2 - b_2c_1)^2 \\ &\geq \min(\underline{K}^N, \cos^4\theta\underline{K}^N),\end{aligned}$$

and

$$\begin{aligned}s &= s^N - Ric^N(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})\sin^2\theta + 4k(-\cot(2r)\sin\theta) \\ &= s^N - (-\frac{(\sin(2r))''}{\sin(2r)} - 2\frac{(1)''}{1})\sin^2\theta + 4k(-\cot(2r)\sin\theta) \\ &\geq s^N - 4.\end{aligned}$$

Thus M has

$$a\underline{K} + s \geq a\min(\underline{K}^N, \cos^4\theta\underline{K}^N) + s^N - 4 > 0$$

for any choice of such γ.

Let δ_1 be the final radius. The boundary ∂M is the $\mathbb{Z}_2$-quotient of the Berger sphere with the Hopf fibers shrunk by $\sin(2\delta_1)$. To glue it onto $\partial X_1/\Gamma$, first we homotope it to the standard metric of curvature 1 keeping $a\underline{K}+s>0$. Consider a smooth homotopy

$$H(x,t)=(\varphi(t)\sin^2(2\delta_1)+1-\varphi(t))(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2$$

for $(x,t)\in\mathbb{R}P^3\times[0,1]$, where $\varphi:[0,1]\to[0,1]$ is a smooth decreasing function which is 1 near 0 and 0 near 1. By the lemma below, there exists a $T>0$ such that the metric $H(x,t/T)+dt^2$ on $\mathbb{R}P^3\times[0,1]$ has $a\underline{K}+s>0$. Next, we rescale the whole thing by multiplying τ. Now we can glue 16 copies of them to $\partial X_1/\Gamma$ to get our desired K3 surface.

By taking r_0 sufficiently small, we achieve $\mathrm{Vol}_{\lambda,K,s}=0$. □

Lemma 4.2.2 *Let g_t, $0\le t\le 1$ be a family of metrics on a compact n-manifold X, which vary smoothly with respect to t. If g_t has $a\underline{K}+s>c$ for all t where c is a nonnegative constant, then there exists a constant $t_0>0$ such that for all $T\ge t_0$, the metric $g_{t/T}+dt^2$ on $X\times[0,T]$ has $a\underline{K}+s>c$.*

Proof. Equivalently we show that there exists ε_0 such that for all $0<\varepsilon\le\varepsilon_0$, the metric $d\sigma_\varepsilon^2=\varepsilon^2g_t+dt^2$ on $X\times[0,1]$ has $a\underline{K}+s>c$. Let $(x_1,\cdots,x_n)$ be a local coordinate on X. For notational convenience we set $t=x_{n+1}$, and write $d\sigma_\varepsilon^2=\sum\gamma_{ij}^\varepsilon dx^idx^j$. Fix $\bar{x}=(\bar{x}_1,\cdots,\bar{x}_{n+1})$ and assume $\gamma_{ij}^1(\bar{x})=\delta_{ij}$.

Recall the Christoffel symbols are given by

$$(\Gamma^\varepsilon)^k_{ij} = \frac{1}{2}\sum_l (\gamma^\varepsilon)^{kl}\{\frac{\partial \gamma^\varepsilon_{il}}{\partial x_j} + \frac{\partial \gamma^\varepsilon_{jl}}{\partial x_i} - \frac{\partial \gamma^\varepsilon_{ij}}{\partial x_l}\},$$

and the Riemann curvature tensor is given by

$$(R^\varepsilon)^l_{ijk} = \partial_i(\Gamma^\varepsilon)^l_{jk} - \partial_j(\Gamma^\varepsilon)^l_{ik} + (\Gamma^\varepsilon)^m_{jk}(\Gamma^\varepsilon)^l_{im} - (\Gamma^\varepsilon)^m_{ik}(\Gamma^\varepsilon)^l_{jm}.$$

So at $\bar{x}$ we have

$$(\Gamma^\varepsilon)^{n+1}_{(n+1)(n+1)} = (\Gamma^\varepsilon)^{n+1}_{i(n+1)} = 0 \quad (\Gamma^\varepsilon)^{n+1}_{ij} = \varepsilon^2 O(1) \quad (\Gamma^\varepsilon)^k_{i(n+1)} = O(1)$$

$$(\Gamma^\varepsilon)^k_{ij} = O(1) \quad (R^\varepsilon)_{ijk(n+1)} = \varepsilon^2 O(1) \quad (R^\varepsilon)_{i(n+1)k(n+1)} = \varepsilon^2 O(1) \tag{§4.2.6}$$

for all $i, j, k = 1, \cdots, n$.

Let $\overline{X}$ be the hypersurface defined by $x_{n+1} = \bar{x}_{n+1}$. Then the second fundamental form of $\overline{X}$ is given by $(\Gamma^\varepsilon)^{n+1}_{ij} = \varepsilon^2 O(1)$ for $i, j = 1, \cdots, n$, and hence its norm is of the form $O(1)$. Denote the sectional curvature and the scalar curvature of the hypersurface $\overline{X}$ with respect to the metric $g_{\bar{x}_{n+1}}$ by $K_{\overline{X}}$ and $s_{\overline{X}}$ respectively. It follows from the Gauss curvature equation that for a plane P belonging to $T_{\bar{x}}\overline{X}$ the sectional curvature K^ε of $d\sigma^2_\varepsilon$ is given by

$$K^\varepsilon(P) = \frac{1}{\varepsilon^2}K_{\overline{X}}(P) + O(1). \tag{§4.2.7}$$

So the scalar curvature s^ε of $d\sigma^2_\varepsilon$ is

$$\begin{aligned} s^\varepsilon &= \frac{1}{\varepsilon^2}s_{\overline{X}} + O(1) + 2\sum_{i=1}^n \frac{1}{\varepsilon^2}(R^\varepsilon)_{i(n+1)(n+1)i} \\ &= \frac{1}{\varepsilon^2}s_{\overline{X}} + O(1) \end{aligned}$$

at $\bar{x}$. Let $u_1 = (\cos\alpha_1)H_1 + \sin\alpha_1 \frac{\partial}{\partial t}$ and $u_2 = (\cos\alpha_2)H_2 + \sin\alpha_2 \frac{\partial}{\partial t}$ for $H_1, H_2 \in T_{\bar{x}}\overline{X}$ be orthonormal vectors in $T_{\bar{x}}(X \times [0,1])$. Let P be the plane generated by H_1 and H_2. Then by using (§4.2.6) and (§4.2.7)

$$\begin{aligned} K^{\varepsilon}(u_1, u_2) &= R^{\varepsilon}(u_1, u_2, u_2, u_1) \\ &= \cos^2\alpha_1 \cos^2\alpha_2 \; R^{\varepsilon}(H_1, H_2, H_2, H_1) + \frac{O(1)}{\varepsilon} + O(1) \\ &= \cos^2\alpha_1 \cos^2\alpha_2 \; (\frac{1}{\varepsilon^2} K_{\overline{X}}(P) + O(1)) + \frac{O(1)}{\varepsilon} + O(1) \\ &\geq \frac{1}{\varepsilon^2} \min(\underline{K}_{\overline{X}}, 0) + \frac{O(1)}{\varepsilon}. \end{aligned}$$

Therefore at $\bar{x}$ we have

$$a\underline{K}^{\varepsilon} + s^{\varepsilon} \geq \frac{1}{\varepsilon^2} \min(a\underline{K}_{\overline{X}} + s_{\overline{X}}, s_{\overline{X}}) + \frac{O(1)}{\varepsilon}.$$

Let $A > 0$ be the infimum of $\min(a\underline{K}_{\overline{X}} + s_{\overline{X}}, s_{\overline{X}})$ over the whole $X \times [0,1]$. Since $X \times [0,1]$ is compact, we can take a constant B such that $O(1) \geq B$ at any point $x \in X \times [0,1]$. Thus at any point $x \in X \times [0,1]$

$$a\underline{K}^{\varepsilon} + s^{\varepsilon} \geq \frac{A}{\varepsilon^2} + \frac{B}{\varepsilon}.$$

By taking ε very small, we achieve $a\underline{K}^{\varepsilon} + s^{\varepsilon} > c$ as claimed. □

Thus any element in Ω_4^{Spin}, which is spin cobordant to a connected sum of a finite number of K3 surface with S^4 has $\mathrm{Vol}_{\lambda,K,s} = 0$ by theorem 4.2.1.

For $n \geq 8$, we need to look at compact 8-dimensional manifolds with holonomy Spin(7) constructed by Joyce [16]. These are spin and have $\hat{A}$-genus

1, i.e. under α they all get mapped to a generator of $KO_8(pt)$. As one of them, let's consider J_8 defined as follows. Take the standard 8-torus $T^8 = \mathbb{R}^8/\mathbb{Z}^8$ and consider involutions $\alpha, \beta, \gamma, \delta$ defined as

$$\alpha(x) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),$$

$$\beta(x) = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),$$

$$\gamma(x) = (\frac{1}{2} - x_1, \frac{1}{2} - x_2, x_3, x_4, \frac{1}{2} - x_5, \frac{1}{2} - x_6, x_7, x_8),$$

$$\delta(x) = (-x_1, x_2, \frac{1}{2} - x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7, x_8)$$

for $x = (x_1, \cdots, x_8) \in T^8$. The corresponding fixed point sets are as follows:

$$S_\alpha = \{(p_1, p_2, p_3, p_4, x_5, x_6, x_7, x_8) : p_i = 0 \text{ or } \frac{1}{2},\ x_i \in [0, 1)\},$$

$$S_\beta = \{(x_1, x_2, x_3, x_4, p_5, p_6, p_7, p_8) : p_i = 0 \text{ or } \frac{1}{2},\ x_i \in [0, 1)\},$$

$$S_\gamma = \{(q_1, q_2, x_3, x_4, q_5, q_6, x_7, x_8) : q_i = \frac{1}{4} \text{ or } \frac{3}{4},\ x_i \in [0, 1)\},$$

$$S_\delta = \{(p_1, x_2, q_3, x_4, q_5, x_6, q_7, x_8) : p_1 = 0 \text{ or } \frac{1}{2},\ q_i = \frac{1}{4} \text{ or } \frac{3}{4},\ x_i \in [0, 1)\}.$$

They are all disjoint except $S_\alpha \cap S_\beta$ which is the set of 256 points. Let $\Gamma \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the group generated by $\alpha, \beta, \gamma, \delta$, and $\pi : T^8 \to T^8/\Gamma$ be the quotient map.

Following Joyce [16], there are 3 types of singularities in T^8/Γ. Note that β fixes S_α which is 16 T^4, but $\langle\gamma, \delta\rangle \cong (\mathbb{Z}_2)^2$ acts freely upon it. On each T^4, β has the standard action of -1. Therefore $\pi(S_\alpha)$ becomes 4-copies of $T^4/\{\pm 1\}$ whose small neighborhood in T^8/Γ is of the type $B^4/\{\pm 1\} \times T^4/\{\pm 1\}$ where

B^4 is a 4-dimensional ball around the origin. The same is true for $\pi(S_\beta)$. By the free action of $\langle\gamma,\delta\rangle$, $\pi(S_\alpha \cap S_\beta)$ is 64 points. A small neighborhood of each point is of the type $B^4/\{\pm 1\} \times B^4/\{\pm 1\}$. On S_γ, $\langle\alpha,\beta,\delta\rangle \cong (\mathbb{Z}_2)^3$ identifies 16 copies of T^4 into 2 copies, and so a neighborhood of $\pi(S_\gamma)$ is 2 copies of the form $T^4 \times B^4/\{\pm 1\}$. The same thing holds for $\pi(S_\delta)$. We want to resolve these singularities as we did in the construction of K3 surface. For the 8 singularities of the type $B^4/\{\pm 1\} \times T^4/\{\pm 1\}$, we replace each of them with $T^*S^2 \times T^4/\{\pm 1\}$, and for the 4 singularities of the form $T^4 \times B^4/\{\pm 1\}$, we replace each with $T^4 \times T^*S^2$. Of course, by this process each of 64 singularities of the type $B^4/\{\pm 1\} \times B^4/\{\pm 1\}$ is resolved into $T^*S^2 \times T^*S^2$. Thus we get our desired smooth manifold J_8.

The important thing is $\mathrm{Vol}_{\lambda,K,s}(J_8) = 0$. Start with the flat orbifold metric on T^8/Γ coming from the flat metric on T^8, and then the same method of construction as that of K3 surface in lemma 4.2.1 gives this result.

Being prepared with J_8, let's consider any class $[X] \in \Omega_n^{Spin}$ for $n \geq 8$. Since α is surjective, we can find a compact spin manifold P of dimension $n-8$ such that $\alpha[P]$ is a generator of $KO_{n-8}(pt)$. By Bott periodicity, $\alpha[P \times J_8]$ is a generator of $KO_n(pt)$. So there exists a $k \in \mathbb{Z}$ such that $\alpha[X] = k \cdot \alpha[P \times J_8]$. By Stolz [38], $[X] - k[P \times J_8]$ is spin cobordant to the total space of a $\mathbb{H}P^2$ fiber bundle, and hence $[X] - k[P \times J_8]$ has $\mathrm{Vol}_{\lambda,K,s} = 0$. Therefore $[X]$ which is spin cobordant to the connected sum of $[X] - k[P \times J_8]$ and $k[P \times J_8]$ also has $\mathrm{Vol}_{\lambda,K,s} = 0$. This completes the proof.

□

Theorem 4.2.4 *Let X be a compact complex surface of Kodaira dimension ≤ 1, which is not of Class VII. Then $Vol_{\lambda,K,s}(X) = 0$ for any $\lambda \in [0, 1)$.*

Proof. Since the blow-up does not increase $\mathrm{Vol}_{\lambda,K,s}$ by theorem 4.2.1, it is enough to consider the minimal ones. The Kodaira-Enriques classification [2] tells us that a minimal complex surface of Kodaira dimension ≤ 1 is either $\mathbb{C}P^2$ or a geometrically ruled surface or of Class VII or deformation equivalent to an elliptic surface. Any minimal complex surface of Kodaira dimension 0 or 1 is deformation equivalent to an elliptic surface. A geometrically ruled surface is a $\mathbb{C}P^1$ bundle with structure group $PGL(2, \mathbb{C})$ over a Riemann surface and hence it admits a riemannian submersion with totally geodesic fibers isometric to Fubini-Study metric. So $a\underline{K} + s > 0$ is achieved by shrinking the fibers, and hence $\mathrm{Vol}_{\lambda,K,s} = 0$.

To show that every elliptic surface has $\mathrm{Vol}_{\lambda,K,s} = 0$, we consider special diffeomorphic models of minimal elliptic surfaces following LeBrun [22]. Any minimal elliptic surface is diffeomorphic to a fiber sum of an elliptic surface with Euler characteristic zero and copies of the rational elliptic surface. Any elliptic surface M with Euler characteristic zero is an orbi-bundle over the 2-orbifold Σ with structure group equal to the isometry group of a flat 2-torus. For any orbifold metric h on Σ, we can construct a smooth metric g on M by gluing together local product metrics using a partition of unity. Here h is taken to be flat in the neighborhoods of fiber sum constructions. This is a riemannian submersion with totally geodesic fibers. We can arrange that $a\underline{K}+s > -1$ by rescaling and the volume is arbitrarily small by shrinking the fibers.

For a smooth model of the rational elliptic surface $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$, consider the 4-orbifold V obtained from $\mathbb{R} \times T^3$ by dividing by the involution induced by $-1 : \mathbb{R}^4 \to \mathbb{R}^4$, and $B = (\mathbb{R} \times S^1)/\mathbb{Z}_2$ defined analogously. Let $\pi : V \to B$ be the map induced by projection $\mathbb{R} \times T^3 \to \mathbb{R} \times S^1$ to the first two coordinates. Then our smooth model $\hat{V}$ is obtained from $\pi : V \to B$ by replacing each of the 8 singular points of V with a 2-sphere of self-intersection -2, and adding a smooth fiber at infinity. To define a metric, start with a flat orbifold metric on V rescaled so that the neighborhood of fiber sum construction is isometric to that of the above M. So the volume of V is also made small. And in the same way as we did in the case of the $K3$ surface in lemma 4.2.1, we remove a small ball of each of 8 singular points, and glue a neighborhood of the zero section of the cotangent bundle T^*S^2 of S^2. Thus we get the fiber sum having a metric with $a\underline{K} + s > -1$ and an arbitrarily small volume.

□

Remark. A surface of Class VII is by definition a complex surface with Kodaira dimension $-\infty$ and the first betti number equal to 1. The complete classification of these surfaces is still lacking. One of well-known examples is a Hopf surface. It is diffeomorphic to $S^1 \times S^3$ and hence has $\text{Vol}_K = \text{Vol}_{\lambda,K,s} = 0$.

Remark. Paternain and Petean [32] proved that every elliptic surface has $\text{Vol}_K = \text{Vol}_{\lambda,K,s} = 0$.

Finally we remark that these minimal volumes crucially depend on the smooth structure of a manifold. For example, let X be a simply connected complex surface of general type and Y be a simply connected complex surface of Kodaira dimension ≤ 1 with the same geometric genus as X.(Such Y always exists.) Let k be any positive integer such that $c_1^2(X) - c_1^2(Y) + k > 0$. Then the 4-manifolds $X_k \equiv X\#(c_1^2(X) - c_1^2(Y) + k)\overline{\mathbb{C}P^2}$ and $Y_k \equiv Y\#k\overline{\mathbb{C}P^2}$ are homeomorphic according to Freedman's classification [11]. But their minimal volumes are different. Indeed

$$\text{Vol}_{\frac{1}{2},K,s}(X_k) \geq \frac{\pi^2}{2} c_1^2(M) > 0,$$

where M is the minimal model of X and $\text{Vol}_{\lambda,K,s}(Y_k) = 0$ by theorem 4.2.4. The following theorem hints that the mixed minimal volume invariant is significant as a smooth invariant rather than a purely topological invariant.

Theorem 4.2.5 *Assume the* 11/8*-conjecture is true and let* $\lambda \in [0, 1)$. *Then every smooth compact simply connected* 4*-manifold is homeomorphic to one which has* $Vol_{\lambda,K,s} = 0$

Proof. As suggested in [32], we have to rely on Freedman's [11] and Donaldson's [10] well-known results on the classification of smooth compact simply connected 4-manifolds. Connected sums of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ realize all odd intersection forms and all definite intersection forms. The 11/8-conjecture implies every even indefinite intersection form can be realized by taking connected sums of K3 surfaces and $S^2 \times S^2$. All these building blocks have been proved to have $\mathrm{Vol}_{\lambda,K,s} = 0$, hence the theorem follows. □

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