The Dimension of Escaping Geodesics

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Abstract of the Dissertation

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Suppose M is a hyperbolic manifold. This may be described as a quotient $\mathbb{D}/G$, where G is a Fuchsian group acting on the hyperbolic disc $\mathbb{D}$. Consider the set of geodesic rays originating at a fixed point p of M. Some of these geodesics will return to a compact set infinitely often; correspond to the so called conical limit points on the limit set of G. Others will go to infinity, i.e. $\text{dist}(\gamma(t), p) \to \infty$; these are called the escaping geodesics.

We will consider those escaping geodesics which escape at the fastest possible rate, and find the Hausdorff dimension of the corresponding terminal points on the boundary of $\mathbb{D}$. In dimension 2, for a geometrically infinite Fuchsian group, if the injectivity radius of $M = \mathbb{D}/G$ is bounded above and away from zero, then these points have full dimension. In dimension 3, when G is a geometrically infinite and topologically tame Kleinian group, if the injectivity radius of $M = \mathbb{B}/G$ is bounded away from zero, the dimension of these points is 2, which is again maximal.

We also obtain a result concerning the quasi-conformal self-maps of jungle gyms. If the dilatation is compactly supported, then the induced map on the boundary of the covering disc $\mathbb{D}$ is differentiable with non-zero derivative on a set of Hausdorff dimension 1.

To my Family

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1 Dimension of Deep points

1.1 Introduction

Consider G, a discrete, torsion free subgroup of isometries of the hyperbolic metric on the hyperbolic three ball $\mathbb{B}$; i.e. a Kleinian group. Passing to the quotient $\mathbb{B}/G$ by identification of the G-equivalent points we obtain the quotient space M, which is a manifold. Suppose this group G is non-elementary and denote its limit set by Λ. In the limit set we have conical and non-conical points.

A point x on the boundary of the ball is a non-conical point if there is a geodesic ray ending at x so that the projection of this geodesic will eventually leave any compact set and tend to the ideal boundary. Among these points there is a subset, which escapes to the ideal boundary at the fastest possible rate. These are called deep points. The original definition of deep points is due to McMullen [18]. A point is a *deep point* if there is a geodesic ray $\gamma : [0, \infty) \to C(\Lambda)$ in the convex hull parameterized by arclength and terminating at x, so that for some $\delta > 0$

$$\frac{\operatorname{dist}\big(\gamma(t), \partial C(\Lambda)\big)}{t} \geq \delta$$

for all t, i.e. the depth of γ inside the convex hull of the limit set Λ increases linearly with the hyperbolic length.

We can generalize this notation by taking any Lipschitz function $\phi(t)$: $[1,\infty) \to [1,\infty)$ with the property of $\lim_{t\to\infty} \phi(t) = \infty$. Fixing a point $z_0 \in M = \mathbb{B}/G$, we consider the set of geodesic rays starting at z_0 and parameterized by the hyperbolic arclength. Define the set of geodesics in the convex core which escape at a rate ϕ as

$$\Gamma_\phi^C = \Big\{\gamma : \frac{1}{C} \le \frac{\operatorname{dist}\big(\gamma(t), z_0\big)}{\phi(t)} \le C\Big\}.$$

Let Λ_ϕ^C denote the terminating points of the geodesics in Γ_ϕ^C, and let $\Lambda_\phi = \cup_C \Lambda_\phi^C$.

The main theorem of this paper is the following:

Theorem 1.1.1. *Suppose G is a geometrically infinite, topologically tame Kleinian group, $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and there is a Green's function on M. Let $\phi(t) : [1,\infty) \to [1,\infty)$ be a Lipschitz function satisfying* $\lim_{t\to\infty} \phi(t) = \infty$, *then* $\dim_{\mathrm{H}}(\Lambda_\phi) = 2$.

The definitions will be given later in Sections 1.2, 1.3 and 1.4. The idea of the proof: we can find a positive harmonic function u on the manifold M (Lemma 1.4.1). Then lift this u to the covering space $\mathbb{B}$, so we have a hyperbolic harmonic function U on the ball $\mathbb{B}$. This hyperbolic harmonic function is a Poisson integral of some positive measure μ, which is supported on the limit set. Using this measure construct a Bloch martingale $\{f_n\}$ on the dyadic squares Q of length 2^{-n} by defining f_n as

$$f_n(x) = \frac{\mu(Q(x))}{m(Q(x))}.$$

With the help of a technical lemma (Lemma 1.4.4) we can find a Cantor set, which has Hausdorff dimension two (Lemma 1.3.5), on which the martingale grows approximately at the same rate as the given Lipschitz function ϕ, i.e.

$$\frac{1}{C} \leq \left| \frac{f_n(Q)}{\phi(n)} \right| \leq C.$$

On Q the martingale $f_n(Q)$ has bounded distance from the harmonic function U on the top of the Carleson square drawn over Q ([6], Lemma 1.4.2); therefore

$$\frac{1}{C} \leq \left| \frac{U(z_Q)}{\phi(n)} \right| \leq C.$$

Finally, $U(z)$ approximately gives the distance from $\gamma(t)$ to the base point, which gives estimation for $\text{dist}(\gamma(t), \partial C(\Lambda))$ on manifolds specified in the main theorem.

An analogous theorem can also be given for Fuchsian groups:

Theorem 1.1.2. *Suppose G is a geometrically infinite Fuchsian group, $M = \mathbb{B}/G$ has injectivity radius bounded and bounded away from zero, and there is a Green's function on M. Let $\phi(t) : [1, \infty) \to [1, \infty)$ be a Lipschitz function satisfying $\lim_{t \to \infty} \phi(t) = \infty$, then $\dim_{\text{H}}(\Lambda_\phi) = 1$.*

Taking $\phi(t) = t$ we can get the dimension of deep points in such sets using the theorem above.

Corollary 1.1.3. *If G is a geometrically infinite, topologically tame Kleinian group and $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and has a Green's function, then the deep points have dimension 2.*

1.2 Definitions and notations

A *similarity* of $\mathbb{R}^d$ is a map $f(x) = Ax + b$, where $b \in \mathbb{R}^d$ and A is a conformal matrix, i.e. a positive scalar multiple of an orthogonal matrix. The *reflection* in the unit sphere is given by $J(x) = \frac{x}{|x|^2}$. The *full* or *general Möbius group* $GM(\bar{\mathbb{R}}^d)$ acting in $\bar{\mathbb{R}}^d$ is defined as the group generated by the similarities and by the reflection J ([4], [19]). Let $GM(\mathbb{B})$ denote the subgroup of this full Möbius group, which leaves the unit ball $\mathbb{B} \in \mathbb{R}^d$ invariant. This group can also be characterized as the group of isometries of the hyperbolic metric on the hyperbolic ball $\mathbb{B}$. The action of a Möbius transformation extends from $\bar{\mathbb{R}}^d$ to the $(d+1)$-dimensional hyperbolic space $\mathbb{H}^{d+1}$ by its Poincaré-extension. There are isomorphisms showing that $GM(\bar{\mathbb{R}}^{d-1}) \cong GM(\mathbb{B}^d) \cong GM(\mathbb{H}^d)$.

The *Möbius group* $M(\bar{\mathbb{R}}^d)$ acting in $\bar{\mathbb{R}}^d$ is the subgroup of the general Möbius group consisting of all the orientation preserving elements. The subgroup of $M(\bar{\mathbb{R}}^d)$ which preserves the upper-half plane $\mathbb{H} = \{x \in \bar{\mathbb{R}}^d : x_d > 0\}$ or the unit ball $\mathbb{B} = \{x \in \bar{\mathbb{R}}^d : |x| < 1\}$ will be denoted by $M(\mathbb{H})$ and $M(\mathbb{B})$, respectively. A discrete group G of $M(\mathbb{B})$ in dimension 3 is called *Kleinian group*. A *Fuchsian group* is a Kleinian group which stabilizes a round disc on $\partial\mathbb{B}$, the sphere at infinity. In this paper we consider only *non-elementary* groups, that is, G has no finite orbit in $\bar{\mathbb{H}}^3$.

If G is a discrete subgroup of $M(\mathbb{B})$, the orbit $G(a)$ of any point $a \in \mathbb{B}$ can accumulate only on the boundary of $\mathbb{B}$. So we call a point $x \in S = \partial\mathbb{B}$ a *limit point*, if there is an orbit $G(a)$ accumulating at x. The *limit set* is the set of limit points and denoted by $\Lambda(G)$ or simply by Λ. For non-elementary groups the limit set is the accumulation points of a single orbit. It can be shown, that this definition is independent of the particular orbit and the limit set is a closed subset of S ([4]). The complementary set $S\backslash\Lambda$ of Λ is called the *ordinary set*, and denoted by Ω.

Let G be a Kleinian group, then the quotient space Ω/G, which is obtained from the ordinary set of G by identifying equivalent points under the mappings of G, is a marked (possibly disconnected) Riemann surface ([17]). If Ω/G is a finite marked Riemann surface (i.e. a finite union of compact surfaces, each with at most a finite number of punctures), then we call G *analytically finite.* Ahlfors finiteness theorem shows that G is analytically finite if it is finitely generated.

A *convex polyhedron* of $\mathbb{B}$ (or $\mathbb{H}$) is the intersection of countably many open half-spaces, where only finitely many of the hyperplanes, defining these half-planes, meet any compact subset of $\mathbb{B}$ (or $\mathbb{H}$) (see [17]). A polyhedron D is a *fundamental polyhedron* for the discrete group G if

(i) for every $g \in G\backslash\{\mathrm{id}\}$, $g(D) \cap D = \emptyset$,

(ii) for every $x \in \mathbb{B}$, there is a $g \in G$, with $g(x) \in \bar{D}$,

(iii) the sides of D are paired by elements of G, and

(iv) any compact set meets only finitely many G-translates of D.

Let $a \in \mathbb{B}$ be a point not fixed by any non-trivial element of the discrete group G, then the set

$$D_a = \text{interior}\{x \in \mathbb{B} : \rho(x, a) \leq \rho(x, g(a)), g \in G\}$$

is called the *Dirichlet region* centered at a. The Dirichlet region is a fundamental polyhedron ([4], [17]).

A Möbius group G is called *geometrically finite* if some convex fundamental polyhedron has finitely many faces. In dimensions 2 and 3 the standard definition of geometric finiteness is that the Dirichlet region must have finitely many faces. It is known, that this criterion implies that every Dirichlet region and every convex fundamental polyhedron has finitely many faces ([4], [19]). Moreover, geometric finiteness implies that the group is finitely generated, and therefore analytically finite.

The *convex hull* of $\Lambda \subset S$, denoted by $C(\Lambda)$, is the smallest convex subset of $\mathbb{B}$ containing all geodesics with both endpoints in Λ. The *convex core* of a hyperbolic manifold $M = \mathbb{B}/G$ is given by the quotient $C(\Lambda)/G$ and denoted by $C(M)$. For $x \in M$ the *injectivity radius*, $\text{inj}(x)$, is half the distance between the two closest distinct lifts of x to $\mathbb{B}$. In the theorem we assume that the injectivity radius is bounded away from zero uniformly on M, which in dimensions 2 and 3 implies that G has no parabolic elements.

A Kleinian group is called *topologically tame*, if the corresponding quotient manifold $M = \mathbb{B}/G$ is homeomorphic to the interior of a compact 3-

manifold with boundary. This implies that the convex core $C(M)$ consists of a compact piece and a finite number of ends E_j, which are topologically equivalent to $S \times \mathbb{R}_+$ for some compact surface S. We note here, that Canary showed in [8] that topological tameness is equivalent to analytical tameness in dimension 3. Moreover, if G is topological tame, then there is an upper bound for the injectivity radius inside the convex core.

In the introduction we already gave the definition of a deep point defined by a geodesic ray in the convex hull of Λ. An equivalent definition can also be given on the quotient manifold, as in [6]. A point $x \in \Lambda$ is deep, if the geodesic ray γ ending at x satisfies

$$\frac{\operatorname{dist}\big(\tilde{\gamma}(t), M \backslash C(M)\big)}{t} \geq \delta > 0$$

for all $t \geq t_0$, where $\tilde{\gamma}$ denotes the corresponding curve on the quotient space to γ. We also note here, that Fernández and Melián in [10] studied the size of the set of escaping geodesics starting at a point of the hyperbolic surface.

1.3 Dyadic martingale and Hausdorff measure

Definitions: An *nth generation dyadic cube* in $\mathbb{R}^d$ is

$$Q_n = \big\{x = (x_1, x_2, \ldots, x_d) : a_i \leq x_i < a_i + 2^{-n}, 1 \leq i \leq d\big\}$$

where $a = (a_1, a_2, \ldots, a_d)$ is the corner of the cube, and each coordinate of a is of the form $a_i = \frac{m_i}{2^n}$ with an integer m_i. The collection of these dyadic cubes is denoted by $\mathcal{D}_n$. For any given point $x \in \mathbb{R}^d$, let $Q_n(x)$ denote the unique nth generation dyadic cube which contains the point x. The mth generation descendants of Q_n are the dyadic sub-cubes of Q_n with sidelength of $2^{-m}|Q_n|$. There are 2^{md} of them.

Suppose Q_0 is a unit cube in $\mathbb{R}^d$. Then a sequence of functions $\{f_n\}_{n=0}^{\infty}$ is said to be a *dyadic martingale* on Q_0 if

1. f_n is measurable on each $Q_n \in \mathcal{D}$,
2. $\frac{1}{|Q_n|} \int_{Q_n} f_n < \infty$,
3. $\frac{1}{|Q_m|} \int_{Q_m} f_n = f_m$ for all $m < n$.

In addition to this usual definition, we will also require that f_n must be constant on the nth generation dyadic cubes. Since in this paper we will use only dyadic martingales, so we will often omit the "dyadic" attribute.

If a finite measure μ is given on Q_0, then the functions

$$f_n(x) = \frac{\mu(Q_n(x))}{|Q_n(x)|_d}$$

define a dyadic martingale, where $|Q_n(x)|_d$ (or just $|Q_n(x)|$ if the notation is clear from the text) denotes the d-dimensional Lebesgue measure of $Q_n(x)$. We define the *martingale differences* as $\Delta f_n(x) = f_{n+1}(x) - f_n(x)$, and the *martingale square function* as

$$S_f(x) = \Big(\sum_{n=1}^{\infty} \|\chi_{Q_n(x)} \Delta f_n\|_\infty^2\Big)^{1/2}.$$

Although this sum is not necessarily convergent, we will have a nice result if $\|S_f(x)\|_\infty < \infty$. A martingale is called *Bloch* if $\sup_n \|\Delta f_n\|_\infty < \infty$. If $\{f_n\}$ is an L^1-bounded martingale, then f_n converges a.e. to a function f with $\|f\|_1 < \infty$. For more results on the convergence of martingales, you may see [11]. We will use two estimates for dyadic martingale that we shall prove first.

Lemma 1.3.1. *Let f_n be a dyadic martingale on $Q_0 \subset \mathbb{R}^d$ with limit function f. Suppose $\|S_f\|_\infty < \infty$. Then for $\lambda \geq 0$,*

$$|x \in Q_0 : f(x) - f_0(x) \geq \lambda| \leq \exp\Big(-\frac{\lambda^2}{2\|S_f\|_\infty^2}\Big).$$

The following proof is due to Herman Rubin and quoted from the paper of Chang-Wilson-Wolff [9, Theorem 3.1]. First we introduce a generally used notation. Let $\mathcal{G}_n$ be the σ-algebra generated by the 2^{nd} dyadic subcubes on Q_0 of sidelength 2^{-n} and let $E(f, \mathcal{G}_n)$ denote the conditional expectation of f on $\mathcal{G}_n$, that is

$$E(f, \mathcal{G}_n)(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} f, \quad \text{where} \quad x \in Q_n(x).$$

Notice, that using this notation a sequence of functions $\{f_n\}$ is dyadic martingale if $E(f_{n+1}, \mathcal{G}_n) = f_n$.

Proof. We may assume $f_0 = 0$. Fix $t > 0$, and define $q_n : Q_0 \to \mathbb{R}$, $n \geq 1$ by

$$q_n(x) = e^{tf_n(x)} \left(\prod_{j=0}^{n-1} E(e^{t\Delta f_j(x)}, \mathcal{G}_j) \right)^{-1}.$$

These q_n form a martingale:

$$\begin{aligned} E(q_{n+1}, \mathcal{G}_n) &= E\left(e^{tf_{n+1}} \left(\prod_{j=0}^{n} E(e^{t\Delta f_j}, \mathcal{G}_j) \right)^{-1}, \mathcal{G}_n \right) \\ &= E\left(e^{t\Delta f_n} e^{tf_n} \left(\prod_{j=0}^{n} E(e^{t\Delta f_j}, \mathcal{G}_j) \right)^{-1}, \mathcal{G}_n \right) \\ &= E\Big(e^{t\Delta f_n} \big(E(e^{t\Delta f_n}, \mathcal{G}_n) \big)^{-1} q_n, \mathcal{G}_n \Big) \\ &= E(e^{t\Delta f_n}, \mathcal{G}_n) \big(E(e^{t\Delta f_n}, \mathcal{G}_n) \big)^{-1} q_n \\ &= q_n. \end{aligned}$$

It follows that $\int_{Q_0} q_n = 1$ for all n. Using the elementary inequalities, which we will show below,

$$\int e^{\phi} d\mu \leq \cosh\big(\|\phi\|_\infty\big) \leq \exp\Big(\frac{1}{2}\|\phi\|_\infty^2\Big)$$

for a probability measure μ with the property of $\int \phi d\mu = 0$, we find that

$$E(e^{t\Delta f_j}, \mathcal{G}_j)(x) \leq \exp\Big(\frac{1}{2} t^2 \|\chi_{Q_j(x)} \Delta f_j\|_\infty^2\Big).$$

So for all n and x

$$\prod_{j=1}^{n-1} E(e^{t\Delta f_j}, \mathcal{G}_j)(x) \leq \exp\left(\frac{t^2}{2}\|Sf\|_\infty^2\right),$$

and by the equality $\int_{Q_0} q_n = 1$ we get that

$$\int_{Q_0} e^{tf_n} \leq \exp\left(\frac{t^2}{2}\|Sf\|_\infty^2\right)$$

for all n. Letting n go to infinity gives $\int_{Q_0} e^{tf} \leq \exp(\frac{t^2}{2}\|Sf\|_\infty^2)$. Now take $t = \frac{\lambda}{\|Sf\|_\infty^2}$ and apply Tsebyshev's inequality to get the proof of the lemma,

$$\begin{aligned}
\big|\{x \in Q_0 : f(x) \geq \lambda\}\big| &= \left|\left\{x \in Q_0 : \exp\left(\frac{\lambda}{\|Sf\|_\infty^2} f\right) \geq \exp\left(\frac{\lambda^2}{\|Sf\|_\infty^2}\right)\right\}\right| \\
&\leq \frac{\int \exp\left(\frac{\lambda}{\|Sf\|_\infty^2} f\right)}{\exp\left(\frac{\lambda^2}{\|Sf\|_\infty^2}\right)} \\
&\leq \exp\left(\frac{\lambda^2}{2\|Sf\|_\infty^2} - \frac{\lambda^2}{\|Sf\|_\infty^2}\right) \\
&= \exp\left(-\frac{1}{2}\frac{\lambda^2}{\|Sf\|_\infty^2}\right).
\end{aligned}$$

□

In the lemma above, we used two elementary inequalities:

$$\int e^{\phi} d\mu \leq \cosh(\|\phi\|_\infty) \leq \exp\left(\frac{1}{2}\|\phi\|_\infty^2\right)$$

which are valid when μ is a probability measure and $\int \phi d\mu = 0$.

The second inequality comes easily from the Taylor expansion of the two sides, because

$$\frac{x^{2n}}{(2n)!} \leq \frac{x^{2n}}{2^n n!} = \frac{\left(\frac{1}{2}x^2\right)^n}{n!}.$$

The first inequality can be shown by the usual argument in analysis: proving it first for step functions by induction, then using Lebesgue dominated convergence theorem it can be proven for any other function.

Suppose first that ϕ is a step function with only two values, i.e. $\phi(x) = a\chi_A + b\chi_B$, where $\mu(A) + \mu(B) = C < \infty$ and $\int \phi d\mu = a\mu(A) + b\mu(B) = 0$. Using these assumptions and elementary calculus we get that

$$\begin{aligned}\int e^{\phi(x)} d\mu &= e^a\mu(A) + e^b\mu(B) \\ &= e^a\frac{-bC}{a-b} + e^b\frac{aC}{a-b} \\ &= C\frac{ae^b - be^a}{a-b} \\ &= Cf_a(b).\end{aligned}$$

Suppose that $\|\phi\|_\infty = a$, and consider the function $f_a(b) = \frac{ae^b - be^a}{a-b}$ as a function of b. Differentiating $f_a(b)$ with respect to b we get that

$$\begin{aligned}\frac{\partial}{\partial b} f_a(b) &= \frac{(ae^b - e^a)(a-b) + ae^b - be^a}{(a-b)^2} \\ &= \frac{a}{(a-b)^2}(ae^b - be^b + e^b - e^a) \\ &= \frac{a}{(a-b)^2} g(b).\end{aligned}$$

By differentiating $g(b)$ with respect to b we see that the function $g(b)$ is an increasing function on the interval $[-a, a]$ and its value at a is 0. Therefore

$g(b) \leq 0$ for all $|b| \leq a$ and so is $\frac{\partial}{\partial b} f_a(b)$. Hence $f_a(b)$ is a non-increasing function on the interval $[-a, a]$, and

$$\int e^{\phi(x)} d\mu = C f_a(b) \leq C f_a(-a) = C \frac{ae^{-a} + ae^{a}}{2a} = C \cosh a.$$

If $\|\phi\|_\infty = -a > 0$, we can use the same calculations. In this case $g(b)$ is a decreasing functions on the interval $[a, -a]$ with initial value $g(a) = 0$. Therefore $g(b) < 0$ on the interval $(a, -a)$ and so $\frac{\partial}{\partial b} f_a(b) > 0$ for all $|b| < -a$. This means the function $f_a(b)$ is increasing on the interval $[a, -a]$, and

$$\int e^{\phi(x)} d\mu \leq C f_a(b) \leq C f_a(-a) = C \frac{ae^{-a} + ae^{a}}{2a} = C \cosh a.$$

We can use induction to show the inequality for arbitrary step functions.

Let

$$\phi(x) = \sum_{i=1}^{n+1} c_i \chi_{C_i}$$

so that $\int \phi d\mu = 0$ and $\sum \mu(C_i) = C < \infty$ and suppose that

$$|c_1|\mu(C_1) = \min_{1 \leq i \leq n+1} \{|c_i|\mu(C_i)\}.$$

We are able to write ϕ as a sum of two step functions each having at most n function-values, therefore we can use the induction step for each of them. So write

$$\phi(x) = f(x)\chi_A + g(x)\chi_B,$$

where

$$\begin{aligned} f &= c_1\chi_{C_1} + c_2\chi_{C_2'}, \\ g &= c_2\chi_{C_2''} + \sum_{i=3}^{n+1} c_i\chi_{C_i}, \\ A &= C_1 \cup C_2' \\ B &= C_2'' \cup \left(\bigcup_{i=3}^{n+1} C_i\right). \end{aligned}$$

Choosing the decomposition of the set $C_2 = C_2' \cup C_2''$ carefully we may assume that both $\int_A f d\mu = 0$ and $\int_B g d\mu = 0$. Therefore

$$\begin{aligned} \int e^{\phi} d\mu &= \int_A e^f + \int_B e^g \\ &\leq \mu(A)\cosh\|f\|_\infty + \mu(B)\cosh\|g\|_\infty \\ &\leq C\cosh\|\phi\|_\infty. \end{aligned}$$

If $\phi(x)$ is not a step function, we can approximate it by step functions and use the Lebesgue dominated convergence theorem to prove the inequality.

Lemma 1.3.2. *Suppose μ is a probability measure on X. Suppose F is a measurable, real valued function on X so that $\int_X F d\mu = 0$ and $\|F\|_4 \leq B\|F\|_2$. Then*

$$\mu\left(\left\{x : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\right\}\right) \geq \frac{1}{64B^{12}}.$$

The following proof was given in [6].

Proof. Without loss of generality we may assume $\|F\|_2 = 1$. Hölder's inequality implies that

$$1 = \|F\|_2 = \left(\int F^{2/3}F^{4/3}\right)^{1/2} \leq \|F\|_1^{1/3}\|F\|_4^{2/3} \leq \|F\|_1^{1/3}B^{2/3},$$

which implies $\|F\|_1 \geq B^{-2}$. Write $F = F_+ - F_-$ as the difference of its positive and negative parts. Since F has mean value zero we must have $\|F_-\|_1 \geq \frac{1}{2B^2}$, and hence $F(x) \leq -\frac{1}{2B^2}$ at some point.

To show F is negative on a set of large measure, let

$$E_1 = \left\{x \in Q \,:\, F_- \leq \frac{1}{\sqrt{8}B^2}\right\}, \quad E_2 = \left\{x \in Q \,:\, F_- \geq \frac{1}{\sqrt{8}B^2}\right\}.$$

Since

$$\|F_-\|_2^2 \geq \|F_-\|_1^2 \geq \frac{1}{4B^4} \quad \text{and} \quad \int_{E_1} (F_-)^2 d\mu \leq \frac{1}{8B^4},$$

we deduce $\int_{E_2}(F_-)^2 d\mu \geq \frac{1}{8B^4}$. Thus if $\mu(E_2) < \frac{1}{64B^{12}}$, we would get

$$\frac{1}{8B^4} \leq \int_{E_2} F_-^2 d\mu \leq \left(\int F_-^4 d\mu\right)^{1/2} \mu(E_2)^{1/2} < B^2\frac{1}{8B^6} = \frac{1}{8B^4},$$

which is contradiction, and this proves the lemma. □

To prove our main theorem we will need the following two lemmas for dyadic martingales.

Lemma 1.3.3. *Suppose μ is a positive measure on the cube $[0,1]^d$, $d \geq 1$, so that the corresponding dyadic martingale defined by $f_n(x) = \frac{\mu\left(Q_n(x)\right)}{|Q_n(x)|}$ is Bloch and $\frac{1}{|Q_n|}\|\Delta f_n\|_2^2 \geq \delta > 0$ whenever $f_n(x) \geq 1$ on $Q_n(x)$. There is an $\epsilon > 0$ and $M < \infty$ so that given any sufficiently large n, there is a constant C so that the following holds. Let Q be any dyadic cube, and let f_Q denote that function in the martingale, which is defined by Q, i.e. $f_Q = \frac{\mu(Q)}{|Q|}$ on Q. Suppose $f_Q \geq C$. Then among the 2^{dn} nth generation descendants of Q, at least $\epsilon 2^{dn}$ satisfy $Mn \geq f_{Q'} - f_Q \geq 1$, and at least $\epsilon 2^{dn}$ satisfy $-Mn \leq f_{Q'} - f_Q \leq -1$.*

Proof. Suppose $\sup_n\|\Delta f_n\|_\infty = L < \infty$ and $\frac{1}{|Q_n|}\|\Delta f_n\|_2^2 \geq \delta > 0$ whenever $f_n(x) \geq 1$ and fix an ϵ with $0 < \epsilon \leq \min\left\{\frac{\delta^6}{2^{16}L^{12}}, 1\right\}$. By an appropriate scaling we may assume that $|Q| = 1$. Then the martingale square function for the sequence $\{f_0, f_1, ..., f_n\}$ is

$$S_f(x) = \left(\sum_{j=1}^{n}\|\chi_{Q_j(x)}\Delta f_j\|_\infty^2\right)^{1/2} \leq \sqrt{n}L.$$

Let $F = \Delta f_1 + ... + \Delta f_n$, and suppose that $n > \frac{64L^4}{\delta^3}$ and that $f_Q \geq 1 + nL = C$. Then $f_j \geq 1$ for all $1 \leq j \leq n$, so $\|\Delta f_j\|_2^2 \geq \delta$ for all $1 \leq j \leq n$. The system $\{\Delta f_j\}_{j=1}^n$ is orthogonal, therefore

$$nL^2 \geq \|F\|_2^2 = \sum_{j=1}^{n}\|\Delta f_j\|_2^2 \geq n\delta.$$

Let $\lambda(t) = \left|\left\{x : |F(x)| > t\right\}\right|$ define the distribution function of $|F|$. Then

$$\int |F|^p = p\int_0^\infty t^{p-1}\lambda(t)dt$$

and by Lemma 1.3.1, $\lambda(t) \leq e^{-\frac{t^2}{2\|S_F\|_\infty^2}} \leq e^{-\frac{t^2}{2L^2 n}}$. Therefore

$$\begin{aligned}\|F\|_4^4 = \int |F|^4 &= 4\int_0^\infty t^3 \lambda(t)dt \\ &\leq 4\int_0^\infty t^3 e^{-\frac{t^2}{2L^2n}} dt \\ &= 8L^4n^2 \int_0^\infty y e^{-y} dy \\ &= 8L^4 n^2.\end{aligned}$$

Hence $\|F\|_4 \leq \sqrt[4]{8} L\sqrt{n} = B\delta\sqrt{n} \leq B\|F\|_2$ with the constant $B = \frac{\sqrt[4]{8}L}{\sqrt{\delta}}$. Now we can apply Lemma 1.3.2 so

$$\mu\Big(\Big\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\Big\}\Big) \geq \frac{1}{64B^{12}}.$$

Using that $\|F\|_2 \geq \sqrt{n\delta}$ and the assumptions that $\sqrt{n} > \frac{8L^2}{\sqrt{\delta^3}} = \frac{\sqrt{8}}{\sqrt{\delta}}B^2$ we get that:

$$\begin{aligned}\mu\big(\{x \in Q : F(x) \leq -1\}\big) &\geq \mu\Big(\Big\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\sqrt{n\delta}\Big\}\Big) \\ &\geq \mu\Big(\Big\{x \in Q : F(x) \leq -\frac{1}{\sqrt{8}B^2}\|F\|_2\Big\}\Big) \\ &\geq \frac{1}{64B^{12}} = \Big(\frac{\delta}{L}\Big)^{12}\frac{1}{8^5} \\ &\geq 2\epsilon.\end{aligned}$$

Switching F with $-F$, with the same assumptions, we get

$$\mu\big(\{x \in Q : F(x) \geq 1\}\big) \geq 2\epsilon.$$

Next, consider the following subsequence $\{f_0, ..., f_n\}$. By Lemma 1.3.1, for a positive constant $M \geq L\sqrt{-2\ln\epsilon}$

$$\begin{aligned} \left|\{x \in Q : f_n(x) - f_0(x) \geq nM\}\right| &\leq \exp\left(\frac{-n^2M^2}{2\|S_f\|_\infty^2}\right) \\ &\leq \exp\left(\frac{-nM^2}{2L^2}\right) \\ &\leq \left(\exp\left(-\frac{M^2}{2L^2}\right)\right)^n \\ &\leq \epsilon^n \\ &\leq \epsilon, \end{aligned}$$

for every $n \geq 1$ and if $\epsilon < 1$. Repeating this argument with $\{-f_0, ..., -f_n\}$ we get that

$$\left|\{x \in Q : f_n(x) - f_0(x) \leq -nM\}\right| \leq \epsilon.$$

Therefore, for every sufficiently large n there is a constant $C = 1 + nL$ so that if $f_Q \geq C$ then

$$\mu\big(\{x \in Q : 1 \leq F(x) = f_n(x) - f_0(x) \leq nM\}\big) \geq \epsilon$$

and

$$\mu\big(\{x \in Q : -1 \geq F(x) = f_n(x) - f_0(x) \geq -nM\}\big) \geq \epsilon.$$

In other words, among all of the 2^{dn} nth generation descendants of Q at least $\epsilon 2^{dn}$ satisfy the inequality $Mn \geq f_{Q'} - f_Q \geq 1$, and at least $\epsilon 2^{dn}$ of them satisfy $-Mn \leq f_{Q'} - f_Q \leq -1$. $\square$

Definitions: Suppose ϕ is an increasing, continuous function from $[0,\infty)$ to itself such that $\phi(0)=0$. For a given set E we define the *Hausdorff content* as

$$\mathcal{H}^{\phi}_{\infty}(E)=\inf\Big\{\sum\phi(r_j): E\subset\cup_j D(x_j,r_j)\Big\}.$$

Specially, if $\phi(t)=t^{\alpha}$ we denote $\mathcal{H}^{\phi}_{\infty}$ by $\mathcal{H}^{\alpha}_{\infty}$. The *Hausdorff dimension* of this set E is

$$\dim_{\mathrm{H}}(E)=\inf\big\{\alpha:\mathcal{H}^{\alpha}_{\infty}(E)=0\big\}.$$

Lemma 1.3.4 (Mass Distribution Principle). *If E supports a strictly positive measure μ which satisfies*

$$\mu\big(B(x,r)\big)\le Cr^{d}$$

for every ball $B(x,r)$, then $\mathcal{H}^{d}_{\infty}\ge\mu(E)/C$ and therefore $\dim_{\mathrm{H}}(E)\ge d$.

For more details on Hausdorff dimension, you may see [7].

Lemma 1.3.5. *Suppose E_n is a union of closed dyadic cubes of generation k_n so that $E_0\supset E_1\supset E_2\supset\ldots$ and there are constants N and ϵ with*

1. *$|k_n-k_{n+1}|=N$ for all n,*
2. *If $Q\in E_n$ is generation k_n, then $|E_{n+1}\cap Q|_d\ge\epsilon|Q|_d$.*

If $E=\cap_n E_n$, then $\dim_{\mathrm{H}}(E)\ge d-C(N,\epsilon)$ where $C(N,\epsilon)\to 0$, whenever $\epsilon>0$ is fixed and $N\to\infty$.

Proof. Fix $1 > \epsilon > 0$ and $N > 0$, and consider the probability measure μ defined on the Cantor set E, so that for a given $Q \in E_n$ each $Q' \in Q \cap E_{n+1}$ has the same mass. To prove the lemma, we will use the Mass Distribution Principle 1.3.4. From the first hypothesis we can deduce that $k_n = k_0 + nN$. Using the second hypothesis we can find an upper estimation for the measure of each dyadic cube. For each $Q \in E_n$

$$\begin{aligned}\epsilon \frac{1}{2^{dk_n}} &= \epsilon\,|Q|_d \\ &\leq |E_{n+1} \cap Q|_d \\ &= \big(\text{the number of dyadic cubes in } E_{n+1} \cap Q\big)\frac{1}{2^{dk_{n+1}}} \\ &= \big(\text{the number of dyadic cubes in } E_{n+1} \cap Q\big)\frac{1}{2^{dk_n}}\,\frac{1}{2^{dN}}.\end{aligned}$$

Therefore

$$\big(\text{the number of dyadic cubes in } E_{n+1} \cap Q\big) \geq \epsilon\, 2^{dN}.$$

Suppose that in E_0 there are B dyadic cubes, then

$$\big(\text{the number of dyadic cubes in } E_n\big) \geq B\,(\epsilon\, 2^{dN})^n.$$

Our measure defined on the Cantor set E gives equal mass to each cube in E_n, therefore

$$\mu(Q : Q \in E_n) \leq \frac{1}{B\,\epsilon^n\, 2^{dNn}}.$$

Let $B(x,r)$ be a ball of radius r and choose n so large that $\frac{1}{2^{k_{n+1}}} \leq r \leq \frac{1}{2^{k_n}}$, i.e. $\frac{1}{2^{k_0+(n+1)N}} \leq r \leq \frac{1}{2^{k_0+nN}}$. Then $B(x,r)$ can hit at most 2^d of the k_nth generation cubes so

$$\begin{aligned}
\mu\big(B(x,r)\cap E_n\big) &\leq 2^d \mu(Q : Q \in E_n) \\
&\leq \frac{2^d}{B\epsilon^n 2^{Nnd}} \\
&= \frac{2^d}{B}\frac{1}{2^{n\log_2\epsilon}}\frac{2^{Nd+k_0d}}{2^{(k_0+(n+1)N)d}} \\
&\leq \frac{2^{(1+k_0)d}}{B} r^{\frac{n\log_2\epsilon}{k_0+(n+1)N}}\left(\frac{1}{r}\right)^{\frac{Nd}{k_0+nN}} r^d \\
&= \frac{2^{(1+k_0)d}}{B} r^{d-\left(\frac{Nd}{k_0+nN}-\frac{n\log_2\epsilon}{k_0+(n+1)N}\right)} \\
&\leq \frac{2^{(1+k_0)d}}{B} r^{d-\left(\frac{Nd}{k_0+N^2}-\frac{\log_2\epsilon}{N}\right)}
\end{aligned}$$

if $n \geq N$ and $r < 1$, because then $\frac{Nd}{k_0+nN} \leq \frac{Nd}{k_0+N^2}$ and $\frac{n\log_2\epsilon}{k_0+(n+1)N} \geq \frac{\log_2\epsilon}{N}$.

By the Mass Distribution Principle 1.3.4, for any fixed $\epsilon > 0$ and $N > 0$ the Hausdorff dimension $\dim_{\mathrm{H}}(E) \geq d - C(N,\epsilon)$ where $C(N,\epsilon) = \frac{Nd}{k_0+N^2} - \frac{\log_2\epsilon}{N}$. Moreover $C(N,\epsilon) \to 0$ as $N \to \infty$ and $\epsilon > 0$ is fixed. □

1.4 The hyperbolic space and harmonic functions

Definitions: The unit ball $\mathbb{B}$ in $\mathbb{R}^n$ is the disc model for the n-dimensional hyperbolic space equipped with the hyperbolic metric

$$d\rho = \frac{2|dx|}{1-|x|^2}.$$

An alternative model of the hyperbolic n-space is the upper half plane $\mathbb{H} = \{x = (x_1, x_2, \ldots, x_n) : x_n > 0\} \subset \mathbb{R}^n$ equipped with the metric

$$d\rho = \frac{|dx|}{x_n}.$$

Using the hyperbolic metric defined in $\mathbb{B}$ we may construct the hyperbolic volume element:

$$dV_H = \frac{2^n dx_1 dx_2 \cdots dx_n}{(1 - |x|^2)^n}.$$

On the upper half plane model the volume element is:

$$dV_H = \frac{dx_1 dx_2 \cdots dx_n}{x_n^n}.$$

The hyperbolic Laplace-Beltrami operator for the unit ball $\mathbb{B} \subset \mathbb{R}^n$ is given by

$$\Delta_H = \frac{(1-r^2)^2}{4}\left[\Delta + \frac{2(n-2)r}{1-r^2}\frac{\partial}{\partial r}\right],$$

where $r = |x|$. On the upper half plane

$$\Delta_H = x_n^2\left[\Delta - \frac{n-2}{x_n}\frac{\partial}{\partial x_n}\right].$$

A function f is called hyperbolically harmonic if it satisfies the hyperbolic Laplace equation, $\Delta_H f = 0$.

We define the *Green's function* on a quotient manifold M as follows. F

is a Green's function on M with a pole at the projection of a point a, if there exists a function $f : \mathbb{B}\backslash\{G(a)\} \to \mathbb{R}$ such that the projection of f is F and the followings are true for f:

- f is a hyperbolic harmonic function on $\mathbb{B}\backslash\{G(a)\}$,
- $f \circ g = f$ for all $g \in G$,
- $\lim_{z \to a}\left(f(z) - \frac{1}{z-a}\right)$ =exists, i.e. f has singularity $\frac{1}{z-a}$ at the point a,
- f is the smallest positive function with these properties.

The hyperbolic version of Green's formula:

$$\int_D (u\Delta_H v - v\Delta_H u) dV_H = \int_{\partial D}\left(u\frac{\partial v}{\partial n_H} - v\frac{\partial u}{\partial n_H}\right) d\sigma_H,$$

where in $\mathbb{B} = \left\{x \in \mathbb{R}^n : |x| < 1\right\}$:

$$\begin{aligned} dV_H &= \frac{2^n dx_1 dx_2 \cdots dx_n}{(1-|x|^2)^n} \\ d\sigma_H &= \frac{2^{n-1} d\sigma}{(1-|x|^2)^{n-1}} \\ \frac{\partial v}{\partial n_H} &= \frac{1-|x|^2}{2}\frac{\partial v}{\partial n} \\ \nabla_H u &= \frac{1-|x|^2}{2}\nabla u \end{aligned}$$

in $\mathbb{H} = \{x = (x_1, \dots x_n) \in \mathbb{R}^n, x_n > 0\}$:

$$dV_H = \frac{dx_1 \cdots dx_n}{x_n^n}$$
$$d\sigma_H = \frac{d\sigma}{x_n^{n-1}}$$
$$\frac{\partial v}{\partial n_H} = x_n \frac{\partial v}{\partial n}$$
$$\nabla_H u = x_n \nabla u$$

are the hyperbolic counterparts of the volume element, area element, normal derivative and gradient in the hyperbolic ball. More detailed description can be found in [2] and [19].

Lemma 1.4.1. *Suppose G is topologically tame, geometrically infinite, $M = \mathbb{B}/G$ has injectivity radius bounded below by $\epsilon > 0$ and that Green's function $G(w, z)$ exists on M. Then there exists a positive harmonic function U on M such that*

$$\sup_{z \in M} |\nabla U(z)| \le 1$$

and U tends to zero in the geometrically finite ends of M. If, in addition, G is topologically tame then for any $a_0 > 0$ there are constants a_1 and a_2 so that

$$\int_{B(z,a_1)} |\nabla U|^2 dV \ge a_2,$$

for every z such that $\operatorname{dist}(z, C(M)) \le a_0$. Moreover, $U(z)$ tends to $+\infty$ in the geometrically infinite ends as $\operatorname{dist}(z, \partial C(M)) \to \infty$.

The proof of this lemma can be found in [6].

Lemma 1.4.2. *Suppose U on $\mathbb{R}^{n+1}_+$ is the hyperbolic Poisson integral of the positive measure μ and satisfies $|\nabla_H U(z)| \leq 1$. For a square $Q \in \mathbb{R}^n$, let $Q_t = \{(\underline{x}, t) : \underline{x} \in Q\}$. Then there is an $A < \infty$ so that*

$$|U(z_Q) - \frac{1}{|Q|}\int_Q U(\underline{x}, t)d\underline{x}| \leq A,$$

for any $0 < t \leq \ell(Q)$, where $\ell(Q)$ denotes the side-length of Q, and

$$|U(z_Q) - \frac{1}{|Q|}\int_Q d\mu| \leq A.$$

The proof of this lemma in dimension $n = 2$ was given in [6].

Proof. If Q is a cube in $\mathbb{R}^n$, then $\hat{Q} = Q \times [0, \ell(Q)]$ is called the Carleson cube in $\mathbb{R}^{n+1}_+$ with base Q, and let z_Q denote the center of $\hat{Q}$. By rescaling we may assume $z_Q = (\underline{0}, 1)$ and by replacing U by $U - U(z_Q)$ we may assume $U(z_Q) = 0$. Let $\psi \in C_\infty(\mathbb{R}^{n+1}_+)$ be chosen with $\text{supp}(\psi) \subset 3Q$. Our first goal is to prove

$$\int_Q \psi(\underline{x}, t)U(\underline{x}, t)d\underline{x} = O(1),$$

with bounds depending only on the C^2-norm of ψ (in particular, independent of t).

We begin by assuming U is smooth up to the boundary (later we will apply these estimates to functions of the form $U(\underline{x}, y + t)$). Let $W(z)$ be the hyperbolic Poisson extension of the characteristic function of Q to $\mathbb{R}^{n+1}_+$ and let

$F(z) = y^n\psi(z)W(z)$. We now apply Green's theorem (in hyperbolic setting) to $U(z)$ and $F(z)$ to get

$$\int_{\mathbb{R}^{n+1}_+} U\Delta_H F - F\Delta_H U dV_H = \int_{\mathbb{R}^n} U\frac{\partial F}{\partial n_H} - F\frac{\partial U}{\partial n_H} d\sigma_H.$$

Since $\Delta_H U = 0$ this becomes

$$\int_{\mathbb{R}^{n+1}_+} U\Delta_H F dV_H = \int_{\mathbb{R}^n} U\frac{\partial F}{\partial n_H} - F\frac{\partial U}{\partial n_H} d\sigma_H.$$

First we estimate the right hand side. Since $|\nabla_H U| = O(1)$ and $F = O(y^n)$ we get,

$$\int_{\mathbb{R}^n} F\frac{\partial U}{\partial n_H} d\sigma_H = O(1).$$

To estimate the first term on the right hand side, note that

$$\begin{aligned}\frac{\partial F}{\partial n_H}(z) &= \frac{\partial \psi(z)}{\partial n_H} y^n W(z) + \psi(z)\frac{\partial y^n}{\partial n_H} W(z) + \psi(z)y^n\frac{\partial W(z)}{\partial n_H}\\ &= O(y^{n+1}) + \psi(z)y^n W(z) + O\Big(y^n \min\big(W(z), 1 - W(z)\big)\Big).\end{aligned}$$

Now break the integral $\int_{\mathbb{R}^n} U\frac{\partial F}{\partial n_H} d\sigma_H$ into three terms according to this equation. As $y \to 0$ the integral of the first term over the boundary tends to zero. The integral of the second term gives $\int_{\mathbb{R}^n} U\psi y^n W d\sigma_H = \int_Q U(\underline{x})\psi(\underline{x})d\underline{x}$. Since $\min\big(W(z), 1 - W(z)\big) \to 0$ almost everywhere as $y \to 0$, the Lebesgue dominated convergence theorem implies the integral of the third term also tends to zero. Thus we deduce

$$\int_{\mathbb{R}^n} U\frac{\partial F}{\partial n_H} - F\frac{\partial U}{\partial n_H} d\sigma_H = \int_Q U(\underline{x})\psi(\underline{x})d\underline{x} + O(1).$$

Now we estimate the left side of the equation above. Since ψ is C^2 up to the boundary, we have $\nabla_H \psi = O(y)$ and $\Delta_H \psi = O(y^2)$. Thus

$$\begin{aligned}
\Delta_H F &= \Delta_H(\psi y^n W) \\
&= \Delta_H \psi y^n W + \psi \Delta_H y^n W + \psi y^n \Delta_H W \\
&\quad + \nabla_H \psi \nabla_H y^n W + \nabla_H \psi y^n \nabla_H W + \psi \nabla_H y^n \nabla_H W \\
&= O(y^{n+2}) + O(y^n) + 0 + O(y^{n+1}) + O(y^{n+1}) \\
&\quad + O(y^n \nabla_H W).
\end{aligned}$$

The integral $\int_{\mathbb{R}^{n+1}_+} U\Delta_H F dV_H$ breaks into six terms, the first five of which are either zero or are obviously bounded (since $U(z) = O(\log \frac{1}{y})$). To bound the last term, we use the fact that $|\nabla_H W| = O(y|\nabla W|)$ and the simple estimate

$$\int_{3Q} |\nabla W(\underline{x}, y) d\underline{x}| = O(1),$$

independent of y. This gives of order

$$\begin{aligned}
\int_{|z|<3} y^n |\nabla_H W| \log \frac{1}{y} dV_H &= \int_{0<y<3} \int_{|x_i|<3} |\nabla W| \log \frac{1}{y} d\underline{x} dy \\
&= \int_{0<y<3} \log \frac{1}{y} dy \\
&= O(1).
\end{aligned}$$

Thus we have now proven that if U is hyperbolically harmonic, has bounded hyperbolic gradient and $U(z_Q) = 0$ then $\int_Q U\psi d\underline{x} = O(1)$, with bounds depending only on the C^2 norm of ψ. To deduce the first part of the lemma, set $F(z) = U\big(z + (\underline{0}, t)\big) - U(\underline{0}, 1-t)$ and apply the preceding case to F. Choose ψ so that $\psi \equiv 1$ on Q. Then

$$\int_Q U(\underline{x}, t)d\underline{x} = \int_Q F d\underline{x} + U(\underline{0}, 1-t) = U(z_Q) + O(1),$$

as desired. To prove the second part of the lemma we simply wish to note that

$$\lim_{t\to 0} \frac{1}{|Q|_n}\int_Q U(\underline{x}, t)d\underline{x} = \frac{1}{|Q|_n}\int_Q d\mu.$$

To prove this, we first observe that $\mu(\partial Q) = 0$. This is because we can cover ∂Q by $2n2^{(n-1)m}$ squares of size 2^{-m}, and U is bounded by Cm in the tops of the corresponding Carleson boxes. Hence the μ measure of each square is at most $Cm2^{-nm}$ and the total μ measure of ∂Q is at most $Cm2^{-nm}2n2^{-(n-1)m} = 2Cmn2^{-m} \to 0$. Next note that $\int_Q P_H(\underline{x}, t)d\underline{x}$ converges to 1 inside Q and to 0 outside Q, and hence to χ_Q μ-almost everywhere. Now use the fact that U is the Poisson integral of μ, Fubini's theorem and the Lebesgue dominated convergence theorem to get

$$\begin{aligned}\frac{1}{|Q|_n}\int_Q U(\underline{x}, t)d\underline{x} &= \frac{1}{|Q|_n}\int_Q \left[\int_{\mathbb{R}^n} P_H(\underline{x}, t)d\mu\right] d\underline{x}\\ &= \int_{\mathbb{R}^n}\left[\frac{1}{|Q|_n}\int_Q P_H(\underline{x}, t)d\underline{x}\right] d\mu\\ &\to \frac{1}{|Q|_n}\int_{\mathbb{R}^n} \chi_Q d\mu.\end{aligned}$$

This completes the proof of the lemma. □

Remark 1.4.3. Define a martingale on the dyadic cubes using this positive measure μ by

$$f_Q(x) = \frac{\mu(Q(x))}{|Q(x)|}.$$

Then by the lemma above, there exists a constant A so that $|U(z_Q) - f_Q| \leq A$.

Lemma 1.4.4. *Suppose G is a topologically tame and geometrically infinite Kleinian group, so that $M = \mathbb{B}/G$ has injectivity radius bounded below by some positive epsilon and there exists a Green's function on M. Let U ba a positive harmonic function on M for which $\sup_{z\in M}|\nabla_H U(z)| \leq 1$, and let $\{f_m\}$ denote the corresponding martingale as defined in the remark above. Then this martingale $\{f_m\}$ has bounded differences away from zero in the L_2 norm, whenever its value larger than some fixed constant C.*

Proof. Suppose $f_m > C$ on the dyadic cube Q_m. From Lemma 1.4.2 we know that $|U(z_Q) - f_m| \leq A$, where z_Q denotes the center of the Carleson square in $\mathbb{R}^{n+1}_+$ with base Q_m. Since G is topologically tame and $\text{inj}(z) \geq \epsilon > 0$, the convex core $C(M)$ can be written as a compact part and a finite number of ends E_j, each of which is topologically equivalent to $S \times \mathbb{R}^+$ with some compact surface S ([8]). We may suppose that we are already in such and end.

First, we will show that for the given constant A, there exists a constant L, so that for all $v \in C(M)$ with $U(v) \geq C$ we can find another point w with $\rho(v, w) \leq L$ and $|U(v) - U(w)| \geq 6A$. Lemma 1.4.1 says, there exist r and a

so that

$$\int_{B(z,r)} |\nabla U|^2 dV \geq a$$

for every $z \in C(M)$. Consider a geodesic ray on M originating at the point v and going to infinity in the convex core. We may put disjoint discs of radius r along this geodesic, say N discs, and denote w the endpoint, so $\rho(v, w) = 2rN$. Cut E_j at v and at w, and call these surfaces Σ_1 and Σ_2, respectively, and let T denote the part of E_j between these cuts. Moreover, we may also assume that $U(v) = 0$. Green's Theorem says that

$$\int_T f\Delta g - g\Delta f dV_H = \int_{\Sigma_1 \cup \Sigma_2} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} d\sigma.$$

Let $f = 1$ and $g = U^2$, then

$$\int_T \Delta(U^2) dV = \int_{\Sigma_1 \cup \Sigma_2} \frac{\partial (U^2)}{\partial n} d\sigma.$$

By elementary calculations we will get from this that

$$\int_T |\nabla U|^2 dV = \int_{\Sigma_1 \cup \Sigma_2} U\frac{\partial U}{\partial n} d\sigma.$$

Using Lemma 1.4.1, we can estimate the left-hand side by

$$\int_T |\nabla U|^2 \geq Na.$$

For the estimation of the right-hand side of the equality we can use that

$|\nabla U| \le 1$, so

$$\int_{\Sigma_1} U\frac{\partial U}{\partial n}d\sigma \le \mathrm{diam}(\Sigma_1)\,\mathrm{area}(\Sigma_1),$$
$$\int_{\Sigma_2} U\frac{\partial U}{\partial n}d\sigma \le \big(U(w) + \mathrm{diam}(\Sigma_2)\big)\,\mathrm{area}(\Sigma_2).$$

Since $E_j = S \times \mathbb{R}^+$, $\mathrm{diam}(\Sigma_j) \le D$ and $\mathrm{area}(\Sigma_j) \le S$ along the entire end E_j. Using these estimations in the equality above reduced from Green's Theorem, we get that

$$Na \le DS + \big(U(w) + D\big)S,$$

and so

$$\frac{Na}{S} - 2D \le U(w).$$

Therefore, we can choose a uniform N large enough so that $|U(v)-U(w)| \ge 6A$ and let $L = 2rN$.

Next, we will show that there is a point w such that $\rho(z_Q, w) \le 3L$, but $|U(z_Q) - U(w)| \ge 3A$. Start at the point z_Q on the Carleson square and go straight down toward the boundary by hyperbolic distance $2L$, call this point v. As we just showed above, there exists a w such that $\rho(v,w) \le L$ and $|U(v) - U(w)| \ge 6A$, which means that either $|U(z_Q) - U(v)| \ge 3A$ or $|U(z_Q) - U(w)| \ge 3A$. Assume the latter is true.

Finally, we show that there is a subfamily $\{f_{m_i}\}$ in the original martingale sequence with bounded differences away from zero in L_2-norm, and $m_{i+1}-m_i \le$

$3L$ for all i. From Lemma 1.4.2, $|U(z_Q) - f_m| \leq A$ and we can find a w such that $|U(z_Q) - U(w)| \geq 3A$, but $\rho(z_Q, w) \leq 3L$. We may also assume that w is in the middle of a Carleson square, since $|\nabla U| \leq 1$. This Carleson square is different from the original Q_m, call it $Q_{m'}$, and let $f_{m'}$ be the martingale function determined by the size of this square. Then $|U(w) - f_{m'}| \leq A$, $|U(z_Q) - f_m| \leq A$ and $|U(z_Q) - U(w)| \geq 3A$, so $|f_m - f_{m'}| \geq A$ on $Q_{m'}$ while $|m - m'| \leq 3L$. Therefore

$$\frac{1}{|Q_m|} \int_{Q_m} |f_m - f_{m'}|^2 dx \geq \frac{1}{|Q_m|} A^2 \frac{|Q_m|}{2^{3L}} = \delta > 0.$$

□

1.5 The proof of the theorem

Suppose G is a topologically tame, geometrically infinite Kleinian group and the quotient manifold $M = \mathbb{B}/G$ has injectivity radius bounded away from zero. This implies that G has no parabolic elements. Suppose $\phi(t) : [1, \infty) \to [1, \infty)$ is Lipschitz, i.e.

$$|\phi(t) - \phi(s)| \leq B|s - t|$$

for some $B < \infty$ and satisfies $\lim_{t \to \infty} \phi(t) = \infty$. Fix a point $z_0 \in M$ and consider the set of geodesic rays starting at z_0, parameterized by hyperbolic arclength. Define the set of geodesics in the convex core which escape at rate

ϕ as

$$\Gamma_\phi^C = \left\{\gamma : C^{-1} \le \frac{\operatorname{dist}(\gamma(t), z_0)}{\phi(t)} \le C\right\}.$$

Let Λ_ϕ^C denote the terminal points of these geodesics, and let $\Lambda_\phi = \cup_C \Lambda_\phi^C$.

Theorem 1.5.1. *Suppose G is a geometrically infinite, topologically tame Kleinian group and $M = \mathbb{B}/G$ has injectivity radius bounded away from zero and there is a Green's Function on M. Let $\phi(t) : [1,\infty) \to [1,\infty)$ be a Lipschitz function satisfying* $\lim_{t\to\infty} \phi(t) = \infty$, *then* $\dim_{\mathrm{H}}(\Lambda_\phi) = 2$.

The analogous theorem for Fuchsian groups:

Theorem 1.5.2. *Suppose G is a geometrically infinite Fuchsian group, $M = \mathbb{B}/G$ has bounded injectivity radius which is also bounded away from zero and there is a Green's function on M. Let $\phi(t) : [1,\infty) \to [1,\infty)$ be a Lipschitz function satisfying* $\lim_{t\to\infty} \phi(t) = \infty$, *then* $\dim_{\mathrm{H}}(\Lambda_\phi) = 1$.

Proof of Theorem 1.5.1. By Lemma 1.4.1 there exists a positive harmonic function U on M with $\sup_{z\in M}|\nabla U(z)| \le 1$ and U tends to zero in the geometrically finite ends of M. This U lifts to a hyperbolic harmonic function (which we will also call U) on $\mathbb{B}$, and this function is a Poisson integral of some positive measure μ supported on the limit set. Consider the corresponding dyadic Bloch martingale $f_Q(x) = \frac{\mu(Q(x))}{|Q(x)|}$.

Using Lemma 1.4.4 we may pass into a subsequence of $\{f_Q\}$ for which the martingale differences are bounded away from zero whenever the value of the martingale is not less than a constant C. Notice that even if we work with this subsequence in the future we can still use the previous lemmas because there are upper and lower bounds for the number of generations we skipped. To simplify our indexes we will suppose that $\{f_Q\}$ is a Bloch martingale and it has differences bounded away from zero whenever the value is not less than C.

Using Lemma 1.3.3 we can create a Cantor set $\{E_l\}$ of nested dyadic cubes, so that the dyadic martingale and the function ϕ are comparable there. As in Lemma 1.3.3 find appropriate $\epsilon > 0$, $M < \infty$, fix a sufficiently large N, and also fix the corresponding constant C. Since U tends to infinity on the geometrically infinity ends we may suppose that $f_Q \geq C$, except for finite many generations of cubes.

First notice that we may suppose that the function ϕ is Lipschitz with a Lipschitz constant $\frac{1}{N}$, i.e. $|\phi(x) - \phi(y)| \leq \frac{1}{N}|x - y|$. This implies that

$$\phi(k_n) - 1 \leq \phi(k_{n+1}) \leq \phi(k_n) + 1$$

for all $n \in \mathbb{N}$. In case $|\phi(x) - \phi(y)| \leq B|x - y|$ with a bigger constant B than $\frac{1}{N}$, we can rescale our function by choosing $\Phi(x) = \frac{1}{BN}\phi(x)$. Then $|\Phi(x) - \Phi(y)| \leq \frac{1}{N}|x - y|$, and if we prove that $\frac{1}{D} \leq \left|\frac{f_n(Q)}{\Phi(n)}\right| \leq D$ on a set Q, then

$$\frac{1}{BND} \leq \left|\frac{f_n(Q)}{\phi(n)}\right| \leq \frac{D}{BN}$$

will also be true. The second thing we should notice is that if the difference $|f_{k_{n+1}} - \phi(k_{n+1})|$ is bounded, that also means the quotient $\left|\frac{f_{k_{n+1}}}{\phi(k_{n+1})}\right|$ is bounded, since $\lim_{t\to\infty} \phi(t) = \infty$. We will show that the difference is bounded.

Define E_0 as the collection of those largest cubes Q where $f_Q \geq C$ and let k_0 be the number which denotes the generation of these cubes. Then there is a positive constant D so that $|f_{k_0}(Q) - \phi(k_0)| \leq D$ on all $Q \in E_0$. We may also assume that $D \geq MN$. We define the sets $\{E_l\}$ inductively. Suppose we already have the set E_n defined, and the quotient or the difference of f_{k_l} and $\phi(k_l)$ is bounded on all the previous sets, say $\frac{1}{D} \leq |f_{k_l} - \phi(k_l)| \leq D$ for all $l \leq n$. Let $k_{n+1} = k_n + N$ and for each $Q \in E_n$ compare $f_{k_n}(Q)$ to $\phi(k_{n+1})$:

If $f_{k_n}(Q) < \phi(k_{n+1})$ then choose those Nth generation descendants Q' of Q for which $MN \geq f_{k_{n+1}} - f_{k_n} \geq 1$. According to Lemma 1.3.3 there are at least $\epsilon 2^{dN}$ of them, and then $f_{k_{n+1}}(Q') = f_{k_n} + a$ where $a \in [1, MN]$. Therefore, $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$ because

$$f_{k_{n+1}} - \phi(k_{n+1}) = f_{k_n} + a - \phi(k_{n+1}) < a \leq MN \leq D$$

and

$$\begin{aligned}\phi(k_{n+1}) - f_{k_{n+1}} &= \phi(k_{n+1}) - f_{k_n} - a \\ &\leq \phi(k_n) - f_{k_n} + 1 - a \\ &\leq D + 1 - a \\ &\leq D.\end{aligned}$$

If $f_{k_n}(Q) \geq \phi(k_{n+1})$ then choose those Nth generation descendants Q' of Q for which $-MN \leq f_{k_{n+1}} - f_{k_n} \leq -1$. From Lemma 1.3.3 we know that there are at least $\epsilon 2^{dN}$ of them, and then $f_{k_n}(Q') = f_{k_n} - a$ where $a \in [1, MN]$.

Therefore $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$, because

$$f_{k_{n+1}} - \phi(k_{n+1}) = f_{k_n} - a - \phi(k_{n+1}) > -a \geq -MN \geq -D,$$

and

$$\begin{aligned} \phi(k_{n+1}) - f_{k_{n+1}} &= \phi(k_{n+1}) - f_{k_n} + a \\ &\geq \phi(k_n) - f_{k_n} - 1 + a \\ &\geq -D + a - 1 \\ &\geq -D. \end{aligned}$$

Define

$$E_{n+1} = \cup_{Q \in E_n} \{\text{all the chosen descendants of } Q\}.$$

Then $E_{n+1} \subset E_n$ and for all $Q' \in E_{n+1}$ we have $|f_{k_{n+1}}(Q') - \phi(k_{n+1})| \leq D$, moreover

$$|E_{n+1} \cap Q|_d \geq \epsilon 2^{dN} \frac{|Q|_d}{2^{dN}} = \epsilon |Q|_d$$

for all $Q \in E_n$. Since $\lim_{t \to \infty} \phi(t) = \infty$, the inequality $|f_n(Q) - \phi(n)| \leq D$ implies that the quotient $\left| \frac{f_n(Q)}{\phi(n)} \right|$ is also bounded above. Moreover, it is bounded away from zero for sufficiently large values of n. Therefore, for all

$Q \subset E_n$:

$$\frac{1}{D} \leq \left| \frac{f_{k_n}(Q)}{\phi(k_n)} \right| \leq D.$$

The nested sets $\{E_l\}$ defined this way will satisfy the requirements of Lemma 1.3.5, so the Hausdorff dimension of the set $E = \cap_l E_l$ is

$$\dim_{\mathrm{H}}(E) \geq d - C(N, \epsilon)$$

with $\lim_{N\to\infty} C(N, \epsilon) = 0$. According to Theorem 3 in Sullivan's paper [21]

$$\frac{1}{C} \leq \left| \frac{U(z)}{\mathrm{dist}(z, z_0)} \right| \leq C.$$

Since $|U(z) - f_{Q_z}| \leq A$ and $\frac{1}{D} \leq \left| \frac{f_n(Q)}{\phi(n)} \right| \leq D$, so

$$\frac{1}{C} \leq \left| \frac{\mathrm{dist}\big(\gamma(n), z_0\big)}{\phi(n)} \right| \leq C.$$

Therefore $\dim_{\mathrm{H}}(\Lambda_\phi) = d$. $\square$

Fix a point $z_0 \in M$ and consider the set of geodesic rays starting at z_0 parameterized by hyperbolic arclength. A point $x \in \Lambda$ is a *deep point* if there is a geodesic ray $\gamma : [0, \infty) \to C(\Lambda)$ parameterized by arclength and terminating at x, such that for some $\delta > 0$

$$\frac{\mathrm{dist}\big(\gamma(t), \partial C(\Lambda)\big)}{t} \geq \delta$$

for all $t \geq t_0$.

Choosing $\phi(t) = t$ the set Λ_ϕ determines the deep points and using the theorem above we get the following Corollary:

Corollary 1.5.3. *If G is non-compact, topologically tame Kleinian group and $M = \mathbb{B}/G$ has injectivity radius bounded away from zero, then the deep points have dimension 2. For Fuchsian group the dimension of the deep points is one.*

2 Differentiability of Quasiconformal maps on the Jungle gym

2.1 Introduction

Suppose G is a Fuchsian group covering of the '1-dimensional jungle gym', pictured in Figure 2.1. Consider a quasiconformal deformation f of this surface by using a dilatation in a ball U with compact closure on the 'jungle gym', i.e. on the quotient Riemann surface $\mathbb{D}/G$. We can lift this map f to the universal covering space, to the hyperbolic disc $\mathbb{D}$. The lifted map $F : \mathbb{D} \to \mathbb{D}$ is a quasiconformal self-map of the hyperbolic disc, it has the same complex dilatation as f, and the dilatation is supported in the lifts of the ball U, i.e. in a union of hyperbolic balls in $\mathbb{D}$. Moreover, any quasiconformal self-mapping

Figure 2.1: 1-dimensional jungle gym

of a disc, in particular this map F, admits a homeomorphic extension to the boundary. Outside the unit ball F is defined by the reflection across the boundary of the unit ball. We will show that this extended map, call it also F, is differentiable with a non-zero derivative on a set of Hausdorff dimension of 1.

Theorem 2.1.1. *Let f be a quasiconformal self-map of the jungle gym so that the dilatation of f is compactly supported, and F be its lifted map to the hyperbolic disc extended to the boundary of the disc. Then F is differentiable with non-zero derivative at the deep points, and the Hausdorff dimension of these points is* 1.

To show that the function F is differentiable with non-zero derivative at the deep points, we are going to use a Theorem by O. Lehto from [13]. For the second part, we would like to use Theorem 1.5.1 to show that the set of deep points has full dimension. Although that proof relied on the existence of a Green's function on the quotient manifold (Lemma 1.4.1), in the special case of the jungle gym we can still make the theorem work, as follows. Sullivan in [21] showed that on manifolds like this one, there are no positive non-constant superharmonic function, so there is no positive non-constant harmonic function either. But we can construct a harmonic function on each half of the jungle gym separately. Cut the jungle gym into two quasi-cylinders with a curve through the point z_0. Then on each quasi-cylinder, M_i, we are able to construct a positive harmonic function, h_i, so that

$$\frac{1}{c}\,\mathrm{dist}(x, \partial M_i) \le h_i(x) \le c\,\mathrm{dist}(x, \partial M_i)$$

with some constant $c < \infty$. For these harmonic functions we can apply Theorem 1.5.1, which shows that the dimension of deep points has full dimension.

2.2 Definitions and notations

2.2.1 Quasiconformal mappings

There are several ways to define quasiconformal mappings. The definition I will give here is called the analytic definition. Another equivalent definition, the geometric, can be found in several books, for example in [15] or [16].

Let f be a diffeomorphism between domains A and B of the extended complex plane, i.e. homeomorphism which with its inverse is continuously differentiable. We also assume that f is a sense-preserving map, i.e.

$$J(f) = |\partial f|^2 - |\bar{\partial} f|^2 > 0,$$

where

$$\partial f = \frac{1}{2}(f_x - i f_y) \text{ and } \bar{\partial} f = \frac{1}{2}(f_x + i f_y).$$

The *dilatation quotient* is defined as

$$D_f = \frac{\max_\alpha |\partial_\alpha f|}{\min_\alpha |\partial_\alpha f|} = \frac{|\partial f| + |\bar{\partial} f|}{|\partial f| - |\bar{\partial} f|},$$

where

$$\partial_\alpha f = \lim_{r\to 0} \frac{f(z+e^{i\alpha}) - f(z)}{re^{i\alpha}} = \partial f + \bar{\partial} f e^{-2i\alpha}$$

the derivative of f in the direction α. Note that the mapping f is conformal if and only if $\bar{\partial} f \equiv 0$, i.e. $D_f \equiv 1$. We call a sense preserving diffeomorphism f *K-quasiconformal* if $D_f(z) \le K$ holds everywhere.

The function

$$\mu = \frac{\bar{\partial} f}{\partial f}$$

is called the *complex dilatation* of f. This is a continuous function and $|\mu(z)| < 1$. A sense preserving diffeomorphism $f : A \to B$ is *quasiconformal* if $\sup_{z\in A}|\mu(z)| < 1$. In particular, f is *K-quasiconformal* $(K \ge 1)$ if

$$|\mu(z)| \le \frac{K-1}{K+1} < 1$$

for all $z \in A$.

A homeomorphism $f : S_1 \to S_2$ between the Riemann surfaces S_1 (with atlas H_1) and S_2 (with atlas H_2) is called *K-quasiconformal* if $h_2 \circ f \circ h_1^{-1}$ is K-quasiconformal on its domain for all $h_i \in H_i$. This is an invariant definition, because the change of local coordinates does not change the maximal dilatation, since the mappings $h_i \circ k_i^{-1}$ are conformal by definition. For every point $p \in S_1$ choose $h_i \in H_i$ defined in a neighbourhood of p and $f(p)$ respectively. Let $w = h_2 \circ f \circ h_1^{-1}$, then $\mu = \frac{\bar{\partial} w}{\partial w}$ exists almost everywhere in the domain of

w. We call it the *complex dilatation of* f.

Let (D, Π_i) be the universal covering space of S_i. Then a K-quasiconformal mapping f from S_1 to S_2 induces a map $F : D \to D$, which is also quasiconformal with the same dilatation, because $\Pi_2 \circ F = f \circ \Pi_1$.

A quasiconformal mapping f of a Jordan domain A onto another Jordan domain B can always be extended to a homeomorphism from the closure of A to the closure of B ([16]). In particular, a quasiconformal self-map in a disc can be extended to the boundary of the disc homeomorphically. Let F be a quasiconformal mapping from a domain A to a domain B, and let γ, γ' be free boundary arcs or curves of these domains respectively. If γ' corresponds to γ under the mapping F and they are quasiconformal, then F can be continued to a quasiconformal mapping of a domain containing $A \cup \gamma$. So does our map $F : S^1 \to S^1$ ([16]). If A and B are two n-tuply connected domains whose boundary curves are quasiconformal, then every quasiconformal mapping $F : A \to B$ can be extended to a quasiconformal mapping of the whole plane. The following weaker theorem is still true in higher dimensions. If D and D' are quasiconformally equivalent to a ball, then every quasiconformal mapping $f : D \to D'$ can be extended to a homeomorphism $F : D \to D'$. For a detailed description on quasiconformal mappings in the n-dimensional space you may see [27].

The generalization of the complex dilatation and the dilatation quotient for $n \geq 2$ is the matrix dilatation ([1], [23]). Let $f : U \to V$ be a differentiable map at $x \in U$, where U and V are domains of $\mathbb{R}^n$. The *dilatation matrix* of f

at x is

$$\mu(x) = |\det f'(x)|^{-\frac{2}{n}} f'(x)^T f'(x),$$

where $f'(x)$ denotes the differential at x and $f'(x)^T$ is its transpose. If $n = 2$ the dilatation matrix gives the direction of the principal axes of the dilatation ellipsoid and their ratios, D_f.

Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a homeomorphic map. This map is called *K-quasiconformal* if

$$\|f'(x)\|^n \leq K J_f(x) \; a.e.$$

where $J_f(x)$ denotes the Jacobian determinant of $f'(x)$ and $\|f'(x)\|$ is the matrix norm, i.e. $\|f'(x)\| = \sup_h |f'(x)h|$, where h is a unit vector of $\mathbb{R}^n$ ([5] and [27]).

The *outer*, *inner* and *linear dilatation* of f at x respectively:

$$K_f(x) = \frac{\|f'(x)\|^n}{J_f(x)}, \quad L_f(x) = \frac{J_f(x)}{\ell(f'(x))^n}, \quad D_f(x) = \frac{\|f'(x)\|}{\ell(f'(x))},$$

where $\ell(f'(x)) = \inf_h |f'(x)h|$. These dilatation coefficients are well-defined at regular points of f and we let $K_f(x) = L_f(x) = D_f(x) = 1$ at the non-regular points and for constant mapping. Usually theorems are shown only for one of these coefficients, since the following inequalities are true for them for every

$n \geq 2$

$$L_f(x) \leq K_f^{n-1}(x), \quad K_f(x) \leq L_f^{n-1}(x), \quad D_f^n(x) = K_f(x)L_f(x).$$

2.2.2 Deformation of Riemann surfaces

Let S_i be a Riemann surface and (D, Π_i) a universal covering space of S_i, and G_i the cover transformation group, i.e. D/G_i is equivalent to S_i. Suppose f is a continuous map from S_1 into S_2 (see Figure 2.2). Then f induces a continuous mapping $F : D \to D$ so that

$$f \circ \Pi_1 = \Pi_2 \circ F.$$

If f is a homeomorphism, then so is the lifted map F, and if f is conformal, then F is conformal ([14]).

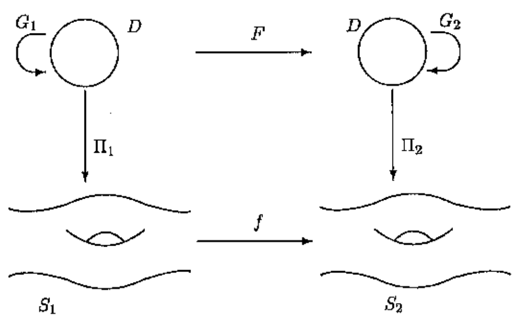

Figure 2.2: Deformation of Riemann Surfaces

The map f also induces a mapping $\Gamma : G_1 \to G_2$. Take $g_1 \in G_1$, then

$\Pi_2 \circ F \circ g_1 = f \circ \Pi_1 \circ g_1 = f \circ \Pi_1 = \Pi_2 \circ F$, therefore for every $z \in D$ there is a transformation $g_2 \in G_2$ such that $(F \circ g_1)(z) = (g_2 \circ F)(z)$. Define $\Gamma(g_1) = g_2$. If f is continuous, then this map Γ is well defined homomorphism and bijective if F is bijective.

Conversely, a continuous mapping $F : D \to D$ can be projected to a mapping $f : S_1 \to S_2$ if and only if $G_2F = FG_1$. We have seen that it is a necessary condition. It is also also sufficient, and we can construct f as follows. Let $p \in S_1$, choose $z \in \Pi_1^{-1}(p)$ and set $f(p) = \Pi_2(F(z))$. Then f is well defined, because if $z' \in \Pi_1^{-1}(p)$ then there exits $g_1 \in G_1$ such that $g_1(z) = z'$ and $\Pi_2 \circ F(z') = \Pi_2 \circ F \circ g_1(z) = \Pi_2 \circ g_2 \circ F(z) = \Pi_2 \circ F(z)$. The construction shows that the induced mappings F and Γ are not uniquely determined by f. We may replace F by $h_2 \circ F \circ h_1$ where $h_i \in G_i$, and $\Gamma(g_1)$ by $h_2 \circ \Gamma(h_1 \circ g_1 \circ h_1^{-1}) \circ h_2^{-1}$. We call such groups homomorphism equivalent.

2.3 Rigidity of Möbius groups, overview of results

2.3.1 Möbius groups

Let G_1 and G_2 be two groups of Möbius transformations of $\bar{\mathbb{R}}^n$, and let A_i be a G_i-invariant subset of $\bar{\mathbb{R}}^n$, i.e. $g(A_i) = A_i$ for all $g \in G_i$. We say that a map $f : A_1 \to A_2$ is G_1-*compatible* if there is a homomorphism $\phi : G_1 \to G_2$

such that

$$f \circ g(x) = \phi(g) \circ f(x)$$

for all $g \in G_1$ and $x \in A_1$. In this case we call ϕ the *induced homomorphism* by f.

A map $f : U \to \bar{\mathbb{R}}^n$, $U \subset \bar{\mathbb{R}}^n$, is *differentiable* at a point $x \in \mathbb{R}^n$, if there is an affine map α of $\mathbb{R}^n$ so that

$$\frac{|f(y) - \alpha(y)|}{|y - x|} \to 0 \quad \text{as } y \to x \text{ in } U.$$

If the map α can be chosen to be an affine homeomorphism, then f is differentiable with a non-vanishing Jacobian at x.

A point $x \in \bar{\mathbb{R}}^n$ is called *radial point* of the group G (*point of approximation* or *conical point*) if there exists a sequence of different $g_i \in G$ so that for a given $z \in \mathbb{H}^{n+1}$ and for a hyperbolic line L terminating at the point x, we can find K such that the hyperbolic distances $\text{dist}(g_i(z), L)$ are all bounded by K, and $\lim_{i \to \infty} g_i(z) = x$ in $\bar{\mathbb{H}}^{n+1}$. On the quotient manifold this means that there is a geodesic ray returning to a compact set infinitely often.

Tukia showed ([23] Theorem A) that if $G = G_1 = G_2$ is a group of Möbius transformations of $\bar{\mathbb{R}}^n$ and $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ is a G-compatible map which is differentiable with a non-vanishing Jacobian at a radial point of G, then it is a Möbius transformation unless there is a point $z \in \bar{\mathbb{R}}^n$ fixed by every element $g \in G$. If there is such a point z fixed by every element of G, then there are two Möbius transformations h and h' so that $h' f h|_{\mathbb{R}^n}$ is an affine homeomorphism.

In the same paper, Tukia also showed ([23] Theorem D) a similar result for a map defined only on smaller G-compatible set, for example if the function is defined on the limit set of the group. The theorem says that if $A \subseteq \bar{\mathbb{R}}^n$ is a G-invariant set containing at least three points and if $f : A \to \bar{\mathbb{R}}^n$ is a G-compatible map of $\bar{\mathbb{R}}^n$ which is differentiable with a non-vanishing Jacobian at a radial points x of G, then f is an affine map of $\bar{\mathbb{R}}^n$, up to composition with Möbius transformations. Moreover, if A is a k-sphere for some $k \leq n$ (i.e. A is the image of $\bar{\mathbb{R}}^k$ under a Möbius transformation of $\bar{\mathbb{R}}^n$) and there is no point fixed by every element g of G, then f is a Möbius transformation. In the theorem it is essential that x is a radial point ([23] D2). In the second part it is necessary, that there is no common fixed point in G, as we can see it in the next example.

Let G be the group of orientation preserving similarity maps of $\mathbb{R}^n$, i.e. $G = \{g \mid g(z) = \lambda z + b$, where $\lambda > 0$ and $b \in \mathbb{R}^n\}$. Every element in this group fixes ∞, and every point in $\mathbb{R}^n$ is a radial point of G. So take $A = \mathbb{R}^n$, and choose an affine homeomorphism α for f. Since $\alpha g \alpha^{-1}$ is an orientation preserving similarity for every $g \in G$, the map $f = \alpha$ induces a group isomorphism $\phi : G \to G$, $g \mapsto fgf^{-1}$. By definition the map f is differentiable with non-vanishing Jacobian, but this is not a Möbius transformation unless α is a similarity.

If $\phi : G_1 \to G_2$ is an isomorphism induced by a map f, then this map f is not always homeomorphism. How can a non-homeomorphic map f look? In [26] Tukia gives an example for this situation. The groups G_1 and G_2 are two finitely generated Fuchsian groups of the first kind containing parabolic elements, and $\phi : G_1 \to G_2$ is an isomorphism between them. Then the map f_ϕ (which induces ϕ) will be continuous outside the parabolic fixed points of G_1 and not continuous on a dense set, on the set of the parabolic fixed points. This map f_ϕ is injective outside a countable set, which consists of the fixed points of certain hyperbolic elements, and maps a set of full measure onto a null-set.

2.3.2 Geometrically finite and convex co-compact groups

A Möbius group G of $\bar{\mathbb{R}}^n$ is called *geometrically finite*, if its action on $\mathbb{H}^{n+1}$ has a finite sided fundamental polyhedron. In the previous section we saw that not every isomorphism $\phi : G_1 \to G_2$ is induced by a homeomorphism, however if the groups are geometrically finite, then there is always a Borel map $f : \Lambda(G_1) \to \Lambda(G_2)$ inducing ϕ ([25], [26]). This function can be constructed so that it is continuous and injective outside the parabolic fixed points of G_1. If a rank-one parabolic fixed point of G_1 is mapped onto a loxodromic fixed point of G_2, then f is not continuous at this parabolic point. Moreover, the preimage of a rank-one parabolic fixed point of G_2 may consist of two points. Therefore, if G_1 and G_2 are two non-elementary, geometrically finite Möbius groups of $\bar{\mathbb{R}}^n$ and if $\phi : G_1 \to G_2$ is an isomorphism which carries parabolic elements bijectively onto parabolic elements, then there is a homeomorphism $f : \Lambda(G_1) \to \Lambda(G_2)$ inducing ϕ.

A group G is called *convex co-compact*, if H_G/G is compact, where H_G

denotes the smallest hyperbolically convex and closed subset of $\mathbb{H}^{n+1}$ such that $\Lambda(G) \subset \bar{H}_G$. (If $n = 1$ then H_G is the usual Nielsen region.) Another characterization of the convex co-compact groups is that G is convex co-compact iff $M_G = \big(\bar{\mathbb{H}}^{n+1} \setminus \Lambda(G)\big)/G$ is compact. Moreover, a geometrically finite group is convex co-compact iff it contains no parabolic element. If the groups G_1 and G_2 are convex co-compact then f is a homeomorphism, since they do not contain parabolic elements ([24]).

Tukia showed ([23]) that if G_1 is a non-elementary Möbius group, and if f is differentiable with a non-vanishing Jacobian at a radial point of G_1, then $f|_{\Lambda(G_1)}$ is a Möbius transformation. For a geometrically finite Möbius group (or if $\mathbb{H}^{n+1}/G$ has finite hyperbolic volume) a limit point is a radial point unless it is fixed by some parabolic element. Therefore, if G_1 and G_2 are two non-elementary, geometrically finite Möbius groups, and $f : \Lambda(G_1) \to \Lambda(G_2)$ is not a Möbius transformation, then the set of points where f is differentiable with a non-vanishing Jacobian is contained in the set of parabolic fixed points, so it is at most a countable set. If G_1 has no parabolic element, then this set is empty.

There is an application of this for Fuchsian groups. Let G_i be Fuchsian groups of the first kind acting in $\mathbb{H}^2$ such that $\mathbb{H}^2/G_1$ is compact, so a limit point is a radial point unless it is fixed by a parabolic element. Let $f : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ be the map on the limit sets which fixes ∞, then f can have at no $x \in \mathbb{R}$ a finite, non-zero derivative unless f is a Möbius transformation.

2.3.3 Divergence type groups

A discrete group G fixing the n-dimensional unit ball $\mathbb{B}$ is called *divergence type* if the series

$$\sum_{g\in G}\big(1-|g(0)|\big)^{n-1}$$

is divergent. The convergence of the series above is necessary and sufficient for the existence of a Green's function on the quotient manifold $\mathbb{B}/G$. A discrete group G has *finite volume*, if it has a fundamental region in $\mathbb{B}$ of finite hyperbolic volume. If the group G preserves $\mathbb{B}$ and has finite volume, then G is of divergence type ([19]).

Let G_1 and G_2 be two discrete Möbius groups acting on $\bar{\mathbb{R}}^n$, $n \geq 2$, such that $\mathbb{H}^{n+1}/G_i$ has finite hyperbolic volume, and let $\phi : G_1 \to G_2$ be an isomorphism. Then there is a quasiconformal map $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ inducing ϕ, and this map f is a Möbius transformation. More generally, if G is a discrete Möbius group of divergence type, and $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$, $n \geq 2$ is a quasiconformal and G-compatible map, then f is a Möbius transformation ([23]). Agard's result for quasiconformal deformations: let G be a discrete groups of Möbius transformations in $\bar{\mathbb{R}}^n$ and suppose it is of divergence type. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiconformal map such that fgf^{-1} is a Möbius transformation for all $g \in G$. Then if $n \geq 2$, f is a Möbius transformation; and if $n = 1$ then f is either a Möbius transformation or it is singular ([1]). In dimension one, instead of quasiconformal we mean quasi-symmetric, and the map $f : \bar{\mathbb{R}}^1 \to \bar{\mathbb{R}}^1$ can be very irregular. These results are often quoted as follows. Let G be a discrete group of Möbius transformations on $\mathbb{H}^{n+1}$ (or on $\mathbb{B}^{n+1}$), $n \geq 1$. We say that G has the *Mostow rigidity property* if for each homeomorphism $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ (or S^n) with $f \circ G \circ f^{-1}$ a Möbius group, it holds that either f is a Möbius transformation itself or completely singular. Therefore the divergence groups have the Mostow rigidity property for all $n \geq 1$. A stronger result for Fuchsian groups is due to Astala and Zinsmeister ([3]): a Fuchsian group has the Mostow rigidity property if and only if it is of divergence type.

2.3.4 Fuchsian groups

Kuusalo ([12]) has a more extensive study on Fuchsian groups acting in the unit disc D (or in the upper half plane $\mathbb{H}^2$). Consider an isomorphism $\phi : G_1 \to G_2$ of two Fuchsian groups and suppose there exists a homeomorphism $f : D \to D$ inducing ϕ. He calls the action of ϕ *geometric*. If G_1 and G_2 are Fuchsian groups of the first kind then f has a unique homeomorphic extension $\bar{f} : \bar{D} \to \bar{D}$, and the boundary map $F = f|_{\partial D}$ is uniquely determined by ϕ ([20], [22]).

If F is a boundary map of two Fuchsian groups of the first kind as above, and if one of the quotient Riemann surfaces D/G_i is of class O_{HB} (i.e. has no non-constant bounded hyperbolic harmonic function on it), then F is either absolutely continuous or completely singular. Moreover, if D/G_i is of class O_G (i.e. has no Green's function), then F is either linear fractional or singular.

A consequence of this dichotomy is that if the groups G_i are finitely generated Fuchsian groups of the first kind acting in $\mathbb{H}^2$, so that $F : \mathbb{R} \to \mathbb{R}$ is increasing, then F is either affine map or a completely singular quasi-symmetric function ([12]).

Here we should mention again a result about some special boundary maps of Fuchsian groups we showed earlier. If G_i are Fuchsian groups such that $\mathbb{H}^2/G_1$ is compact, and if $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ fixes ∞, then F can have at no $x \in \mathbb{R}$ a finite, non-zero derivative unless f is a Möbius transformation ([23]).

2.4 Differentiable points on the Jungle gym

Now consider the original assumptions, that G is a Fuchsian group covering of the '1-dimensional jungle gym' and let f be a quasiconformal deformation f of this surface by using a dilatation in a ball U with compact closure on the 'jungle gym'. Lift this map f to the universal covering space (see Figure 2.2), to the hyperbolic disc $\mathbb{D}$. The lifted map $F : \mathbb{D} \to \mathbb{D}$ is quasiconformal self-map of the hyperbolic disc, it has the same complex dilatation as f, and the dilatation is supported in the lifts of the ball U, i.e. in a union of hyperbolic balls in $\mathbb{D}$. As we showed in Section 2.2.1, this quasiconformal self-mapping F of a disc admits a homeomorphic extension to the boundary. We would like to show that this extended map, that we also name F, is differentiable with a non-zero derivative on a set of Hausdorff dimension of 1.

We will use Lehto's theorem on differentiability of quasiconformal mappings to show that the corresponding boundary map on the circle is differentiable with non-zero derivative at the deep points.

Theorem 2.4.1. (O. Lehto [13]) *In a domain D, let $\mu(z)$ be measurable with $\sup|\mu(z)| = k < 1$, and let $w = w(z)$ be a quasiconformal mapping whose complex dilatation is equal to $\mu(z)$ a.e. If*

$$I(z_0) = \iint_D \frac{|\mu(z) - \mu(z_0)|}{|z - z_0|^2} d\sigma < \infty, \quad z_0 \in D,$$

then at $z = z_0$, $w(z)$ is totally differentiable, $J(z) > 0$, and $w(z)$ has the complex dilatation $\mu(z_0)$.

First, we will show that the integral in Lehto's theorem is finite over a small Euclidean ball around every deep point x_0. Fix a point z_0 inside U, where U denotes the compact support of the dilatation, and consider a geodesic ray $\gamma(t)$ corresponding to this deep point and starting at the point z_0. By the definition of deep points this means

$$1 \geq \frac{\operatorname{dist}\big(\gamma(t), z_0\big)}{t} \geq \delta > 0$$

for all t, i.e. $\operatorname{dist}\big(\gamma(t), U\big) > \delta t - \operatorname{diam}(U) = \delta t - D$ for all $t > \frac{D}{\delta}$. Because of this inequality there is a region around this geodesic that none of the lifted preimages of U can hit. Call this set H_1 and its symmetric image over the boundary H_2. We know that $\mu = 0$ in the two regions H_1 and H_2 touching the ideal boundary at the deep limit point x_0. Therefore it will be enough to show that the integral $I(x_0)$ in Lehto's theorem is finite in a neighborhood of x_0 outside these regions H_1 and H_2.

Let A_n denote the annulus with radii e^{-n+1} and e^{-n} about the point x_0. We will use the upper half plane model as the universal covering space for the surface. We may suppose that $x_0 = 0$ and the initial point of the geodesic ray is $z_0 = i$. We shall show that the integral in Lehto's theorem over the set $\cup_n(A_n \backslash H_j)$ is finite, $j = 1, 2$.

Lemma 2.4.2. $\iint\limits_{A_n \backslash H_1} d\sigma \leq ce^{-2n}e^{-\delta n}$.

Proof. Since $\text{dist}(\gamma(t), U) > \delta t - D$ for all $t > t_0$, in particular for $t = n$, the set U cannot intersect with the hyperbolic ball of radius $\delta n - D$ around the point ie^{-n}. Estimate the area of $A_n \backslash H_1$ by the area of the wedge shown in Figure 2.3. For this estimation we need to find the angle $\alpha = AOA'\angle$ of the wedge. Since $0 \leq \alpha \leq \frac{\pi}{2}$, we know that $\frac{2}{\pi}\alpha \leq \sin\alpha$, so enough to estimate $\sin\alpha = \frac{y}{e^{-n+1}}$.

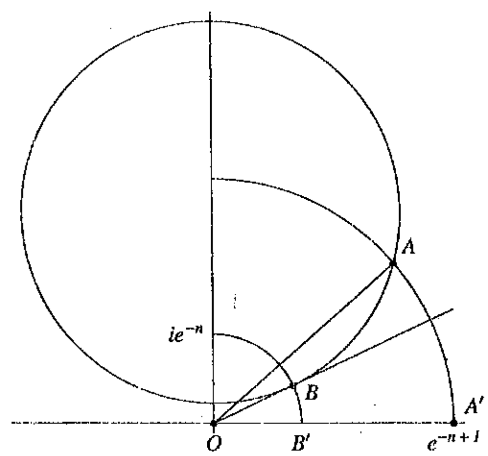

Figure 2.3: Estimation for the integral

The point $A = x + iy$ is the intersection of the hyperbolic ball with the

Euclidean ball, so

$$x^2 + \left(y - e^{-n}\cosh(\delta n - D)\right)^2 = \left(e^{-n}\sinh(\delta n - D)\right)^2$$
$$x^2 + y^2 = e^{-2n+2}.$$

Eliminating x^2, we get that

$$-2ye^{-n}\cosh(\delta n - D) + e^{-2n}\cosh^2(\delta n - D) = e^{-2n}\sinh^2(\delta n - D) - e^{-2n+2},$$

and rearranging this we have the following

$$e^{-n}(1+e^2) = 2y\cosh(\delta n - D).$$

Therefore

$$\begin{aligned}\sin\alpha &= \frac{y}{e^{-n+1}} \\ &= \frac{1+e^2}{2e\cosh(\delta n - D)} \\ &\leq \frac{1+e^2}{2e}\frac{1}{e^{\delta n - D}} \\ &= \frac{(e^2+1)e^{D-1}}{2}e^{-\delta n}.\end{aligned}$$

We should also justify that the picture above is correct, i.e. the wedge does cover the whole set $A_n \backslash H_1$. For this we have to show that $AOA'\angle$ is bigger than $BOB'\angle$, or equivalently $\sin\alpha = \sin(AOA'\angle) > \sin(BOB'\angle) = \sin\beta$. We can calculate $\sin\beta$ similarly as we did for $\sin\alpha$.

Point B is the intersection of the same hyperbolic ball and the Euclidean ball of radius e^{-n}, so

$$x^2 + \left(y - e^{-n}\cosh(\delta n - D)\right)^2 = \left(e^{-n}\sinh(\delta n - D)\right)^2$$
$$x^2 + y^2 = e^{-2n}.$$

So solve this system for $\sin\beta = \frac{y}{e^{-n}}$

$$\begin{aligned} -2ye^{-n}\cosh(\delta n - D) + e^{-2n}\cosh^2(\delta n - D) &= e^{-2n}\left(\sinh^2(\delta n - D) - 1\right) \\ e^{-n}\left(\cosh^2(\delta n - D) - \sinh^1(\delta n - D) + 1\right) &= 2y\cosh(\delta n - D) \\ e^{-n} &= y\cosh(\delta n - D) \\ \frac{y}{e^{-n}} &= \frac{1}{\cosh(\delta n - D)}. \end{aligned}$$

Therefore

$$\sin\beta = \frac{y}{e^{-n}} = \frac{1}{\cosh(\delta n - D)} < \frac{1+e^2}{2e}\frac{1}{\cosh(\delta n - D)} = \sin\alpha,$$

This means, Figure 2.3 above is correct and we can use angle α to give upper bound for the area $A_n \backslash H_1$. Using this estimation for the area of $A_n \backslash (H_1 \cup H_2)$, where the dilatation may differ from zero is:

$$
\begin{aligned}
\iint\limits_{A_n \backslash (H_1 \cup H_2)} d\sigma &\leq \left[(e^{-n+1})^2 - (e^{-n})^2\right] \pi \frac{2\alpha}{2\pi} \\
&= e^{-2n}(e^2 - 1)\alpha \\
&\leq e^{-2n}(e^2 - 1)\frac{\pi}{2} \sin \alpha \\
&\leq e^{-2n}(e^2 - 1)\frac{\pi}{2} \frac{(e^2 + 1)e^{D-1}}{2} e^{-\delta n} \\
&= ce^{-2n}e^{-\delta n}.
\end{aligned}
$$

□

Remark 2.4.3. In general, if $\operatorname{dist}\big(\gamma(t), z_0\big) \geq \phi(t)$ for all t, then with the same calculation we can show that

$$\iint\limits_{A_n \backslash H_1} d\sigma \leq ce^{-2n}e^{\phi(n)}.$$

Lemma 2.4.4. *If x_0 is a deep point then*

$$I(x_0) = \iint\limits_{D} \frac{|\mu(x) - \mu(x_0)|}{|x - x_0|^2} d\sigma < \infty.$$

Proof. Using Lemma 2.4.2 the integral $I(x_0)$ over the euclidean ball of radius

e^{-N_0}, where N_0 is an integer bigger than $\frac{1+D}{\delta}$ is:

$$\begin{aligned} I(x_0) &= \iint \frac{|\mu(x) - \mu(x_0)|}{|x - x_0|^2} d\sigma \\ &\leq \sum_{n=N_0}^{\infty} \iint_{A_n \setminus (H_1 \cup H_2)} \frac{|\mu(x) - \mu(x_0)|}{|x - x_0|^2} d\sigma \\ &\leq \sum_{n=N_0}^{\infty} \iint_{A_n \setminus (H_1 \cup H_2)} \frac{2}{e^{-2n}} d\sigma \\ &\leq \sum_{n=N_0}^{\infty} \frac{2}{e^{-2n}} e^{-2n} (e^2 - 1) \frac{\pi}{2} \frac{(e^2+1)e^{D-1}}{2} e^{-\delta n} \\ &= \frac{(e^4 - 1)e^{D-1}\pi}{2} \sum_{n=N_0}^{\infty} e^{-\delta n} < \infty. \end{aligned}$$

□

Therefore we have shown that F is differentiable with non-zero derivative at the deep points.

Remark 2.4.5. In the general case, when $\operatorname{dist}(\gamma(t), z_0) \geq \phi(t)$ for all t, then

$$I(x_0) \leq c \sum_{n=N_0}^{\infty} e^{-\phi(n)}.$$

In particular, if $\phi(t) \geq (1 + \epsilon) \ln t$ with some positive ϵ, then

$$I(x_0) \leq c \sum e^{-\phi(n)} \leq c \sum \frac{1}{n^{1+\epsilon}} < \infty.$$

2.5 Harmonic functions

In this section we collect a few facts about harmonic functions and special properties of harmonic functions living on surfaces like the half of the jungle gym.

Lemma 2.5.1 (Harnack's Inequality). *For $0 \le t < 1$, define the functions $\alpha(t) = \frac{1-t}{(1+t)^{d-1}}$ and $\beta(t) = \frac{1+t}{(1-t)^{d-1}}$, where d denotes the dimension. If u is a positive harmonic function on $B(a, R)$ and $|x - a| \le r < R$, then*

$$\alpha\Big(\frac{r}{R}\Big)u(a) \le u(x) \le \beta\Big(\frac{r}{R}\Big)u(a).$$

Proof. Recall, that the Poisson formula for the unit ball $B(0, 1)$ is

$$v(x) = \frac{1}{\sigma(S)} \int\limits_S \frac{1 - |x|^2}{|x - \xi|^d} v(\xi) d\sigma(\xi)$$

in dimension d. If u is a positive harmonic function on the closed call $\bar{B}(0, R)$, then using the Poisson formula for $u(Rx) = v(x)$ we get

$$
\begin{aligned}
u(x) &= \frac{1}{\sigma(S)} \int\limits_S \frac{1-|\frac{x}{R}|^2}{|\frac{x}{R}-\xi|^d} u(R\xi)d\sigma(\xi) \\
&\leq \frac{1-|\frac{x}{R}|^2}{\left(1-|\frac{x}{R}|\right)^d} \frac{1}{\sigma(S)} \int\limits_S u(R\xi)d\sigma(\xi) \\
&= \frac{1-|\frac{x}{R}|^2}{\left(1-|\frac{x}{R}|\right)^d} u(0) \\
&= \frac{1+|\frac{x}{R}|}{\left(1-|\frac{x}{R}|\right)^{d-1}} u(0) \\
&= \beta\left(\left|\frac{x}{R}\right|\right) u(0) \\
&\leq \beta\left(\frac{r}{R}\right) u(0)
\end{aligned}
$$

since the function $\beta(t)$ is increasing in the interval $(0,1)$.

Similarly,

$$
\begin{aligned}
u(x) &= \frac{1}{\sigma(\partial B(0,R))} \int\limits_{\partial B(0,R)} \frac{1-|\frac{x}{R}|^2}{|\frac{x}{R}-\xi|^d} u(\xi)d\sigma(\xi) \\
&\geq \frac{1-|\frac{x}{R}|^2}{\left(1+|\frac{x}{R}|\right)^d} u(0) \\
&= \frac{1-|\frac{x}{R}|}{\left(1+|\frac{x}{R}|\right)^{d-1}} u(0) \\
&= \alpha\left(\left|\frac{x}{R}\right|\right) u(0) \\
&\geq \alpha\left(\frac{r}{R}\right) u(0)
\end{aligned}
$$

since the function $\alpha(t)$ is decreasing for $0 < t < 1$. □

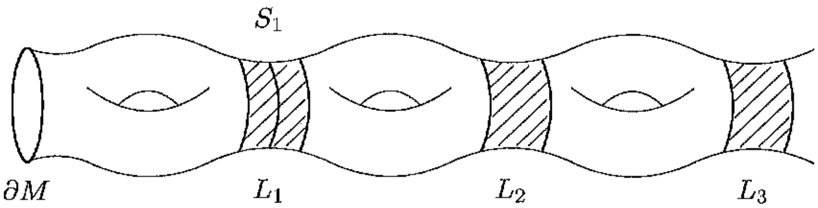

Figure 2.4: Quasi-cylinder

Let M be a quasi-cylinder as demonstrated in Figure 2.4. That is, a complete Riemann manifold with compact boundary, and with bounded geometry locally, i.e. each point of M away from the boundary has a ball neighbourhood of radius 1 which is a geometrically bounded distortion of a unit ball in Euclidean space. Furthermore, we suppose that M has the following additional properties: let S_n denote the hypersurface at distance n from the boundary of M and suppose S_n is homologous to ∂M and $\text{diam}(S_n) \leq D$. Suppose there are closed bands $L_1, L_2, \ldots$ on M, so that each band L_i contains a unit neighbourhood of S_i with $\text{vol}(L_i) \leq V$.

Lemma 2.5.2. *Let L_i denote the closed band on M, which is a unit neighbourhood of S_i and* $\text{vol}(L_i) \leq V$. *Then there exist a constant c so that*

$$\max_{x \in L_i} u(x) \leq c \min_{x \in L_i} u(x).$$

Proof. Fix a point x_i in each L_i where u takes its maximum on L_i. Lift these points up to the universal covering space $\mathbb{B}$ so that x_i lifts to the origin. Denote the lifted image of x_i by $\hat{x}_i$, similarly for other lifted points and sets. Take a ball $B(\hat{x}_i, R_i) = B(0, R_i)$ around the origin so that $L_i \subset B(0, R_i)/G \subset M \backslash \partial M$.

Since $\text{vol}(L_i)$ are bounded there exists R so that

$$\hat{L}_i \subset B(0,R) \subset B(0,2R) \subset (M \backslash \partial M)\check{}.$$

Since u is a non-negative harmonic function on $B(0,2R)$ by the Harnack's Inequality 2.5.1

$$\frac{2R-r}{2R+r}u(0) \leq u(re^{i\theta}) \leq \frac{2R+r}{2R-r}u(0)$$

for all $0 \leq r < 2R$ and all θ. So for all $z \in \hat{L}_i$, and for all i

$$\frac{u(0)}{4} \leq u(z) \leq 4u(0).$$

This implies that

$$\max_{x \in L_i} u(x) \leq 4 \min_{x \in L_i} u(x).$$

□

Corollary 2.5.3. *If u is a positive harmonic function on M then either u is bounded or $u(x) \to \infty$ as $x \to \infty$ in M.*

Proof. If all the $\max_{x \in L_i} u(x)$ are bounded then by the maximum principle h is bounded. Otherwise, there exists a subsequence of $\left\{\max_{x \in L_i} u(x)\right\}_i$ which tends to infinity. The inequality above shows that $\min_{x \in L_i} u(x)$ also tends to infinity. By the maximum principle, the minimum between the bands L_i and

L_{i+1} is attained on the boundary which is greater or equal than

$$\min\{\min_{x\in L_i} u(x), \min_{x\in L_{i+1}} u(x)\}.$$

Therefore $u(x) \to \infty$ as $x \to \infty$. □

Theorem 2.5.4. *Let M be a complete oriented Riemann manifold and h be a harmonic function on it. Suppose $M_+ = h^{-1}[0,\infty)$ has a compact boundary and that $h^{-1}[a,b]$ is compact for all $0 \leq a < b < \infty$. Assume that each point of M_+ (away from the boundary) has a ball neighbourhood of radius one which is a geometrically bounded distortion of a unit ball in euclidean space. Then the gradient of h is uniformly bounded on M_+.*

The proof of this theorem can be found in [21].

Lemma 2.5.5. *Let M be a quasi-cylinder, and let S_n denote the hypersurface at distance n from the boundary of M and suppose S_n is homologous to ∂M with* $\mathrm{diam}(S_n) \leq D$. *Then M admits a non-constant positive harmonic function h so that*

$$\frac{1}{c}\,\mathrm{dist}(x,\partial M) \leq h(x) \leq c\,\mathrm{dist}(x,\partial M).$$

Theorem 2.5.4 and Lemma 2.5.5 are due to D. Sullivan and were published in [21].

Proof. For each n construct a harmonic function h_n which is 0 on ∂M and n on S_n. Then this function is non-negative by the minimum principle. Fix a point $p \in M$, then by multiplying each function by a positive constant we may suppose that $h_n(p) = 1$. Let K be a compact neighbourhood of this point p. By Harnack's inequality 2.5.1 $\alpha = \alpha h_n(p) \le h_n(z) \le \beta h_n(p) = \beta$ on this set K, where α and β depend on the sets K and $M \backslash \partial M$ only. This shows that the family $\{h_n\}$ is pointwise bounded on the compact set K.

The Harnack's inequality 2.5.1 gives the following estimation for the gradient of any harmonic function u on $B(a, R)$:

$$\begin{aligned}
|\partial_v u(a)| &= \left| \lim_{\epsilon \to 0} \frac{u(a + v\epsilon) - u(a)}{\epsilon} \right| \\
&= \lim_{\epsilon \to 0} \left| \frac{\frac{u(a+v\epsilon)}{u(a)} - 1}{\epsilon} \right| u(a) \\
&\le u(a) \lim_{\epsilon \to 0} \frac{\max\{\beta(\frac{\epsilon}{R}) - 1, 1 - \alpha(\frac{\epsilon}{R})\}}{|\epsilon|} \\
&= u(a) \frac{d}{R}.
\end{aligned}$$

The family of harmonic functions $\{h_n\}$ on the set K has therefore bounded gradient, since $\mathrm{dist}(K, \partial M)$ is bounded away from zero. So $\{h_n\}$ is equicontinuous and pointwise collection of complex functions, so by the Arzela-Ascoli theorem it has a subsequence that converges uniformly on compact subsets of K. Take a larger compact neighbourhood and apply the Arzela-Ascoli theorem again, we gain a subsequence of this which converges uniformly on the compact subsets of this bigger set. Continue this process, then take the diagonal subsequence. That converges uniformly on each set K and $h_n(p) = 1$ for all n. So $\{h_n\}$ converges uniformly on each compact subsets of M to a harmonic function h. Applying Theorem 2.5.4 for h, we can even see that the gradient of h is uniformly bounded by a constant, so the right-hand side of

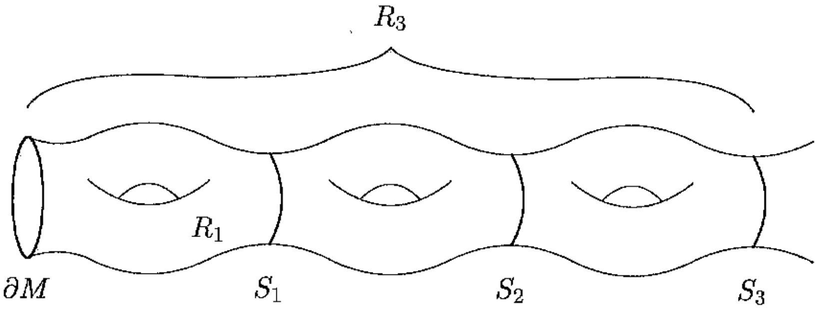

Figure 2.5: Quasi-cylinder

the inequality of Lemma 2.5.5 is proven.

From the construction the harmonic function h is not constant and we have to show that there is some constant c so that $\frac{1}{c}\operatorname{dist}(x, \partial M) \leq h(x)$. We will use the notations illustrated on Figure 2.5.

The following argument is true for any non-constant harmonic function on the surface. By Green's identity $\int_{\partial\Omega} \nabla h ds = 0$ whenever h is a harmonic function on Ω. Apply this equality for that part of the surface M which is bounded by two hypersurfaces S_m and S_n to get

$$\int_{S_n} \nabla h ds = \int_{S_m} \nabla h ds = \int_{\partial M} \nabla h ds > 0.$$

Since the length of S_n is bounded and the integral of the gradient of h is bounded away from zero over S_n, there must be a point $p_n \in S_n$ so that $\nabla h(p_n) \geq c > 0$. Let R_n denote those points on the surface which has distance less than n, i.e. that part which is bounded by ∂M and S_n.

On each hypersurface S_n we have a point p_n where the gradient has some definite value, say $\nabla h(p_n) > c$. The function h is harmonic so $h = \Re\{f(z)\}$

real part of a complex analytic function, at least locally since M has bounded geometry locally. Then we know that $|\nabla h|^2 = \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 = |f'(z)|^2$ and if we take the unit disc D around the point p_n then

$$\frac{1}{|D|}\int_D |f'| \geq \left|\frac{1}{|D|}\int_D f'\right| = |f'(p_n)| > c,$$

which means $|\nabla h| = |f'(z)| > c$ for all points on D. Therefore

$$\int_{R_n} |\nabla h|^2 dV \geq c_1 n.$$

We can use Green's formula I:

$$\int_\Omega \nabla u \nabla v dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n}ds - \int_\Omega u\Delta v dxdy$$

with $u = v = h$ to rewrite the integral:

$$c_1 n \leq \int_{R_n} (\nabla h)^2 dV = \int_{\partial R_n} h\frac{\partial h}{\partial n}ds - \int_{R_n} h\Delta h ds = \int_{\partial M} h\frac{\partial h}{\partial n}ds + \int_{S_n} h\frac{\partial h}{\partial n}ds.$$

Since the gradient is bounded by some constant K, $\left|\frac{\partial h}{\partial n}\right| \leq K$ and therefore

$$c_1 n - \int_{\partial M} h\frac{\partial h}{\partial n} \leq \int_{S_n} h\frac{\partial h}{\partial n}ds \leq \int_{S_n} Khds.$$

Using that $\int_{\partial M} h\frac{\partial h}{\partial n}$ is a constant, we can deduce that $\int_{S_n} hds \geq c_2 n$ with some constant c_2. Moreover

$$\int_{S_n} hds \leq \int_0^{\text{length}(S_n)} h(x_0) + Kxdx \leq \text{length}(S_n)h(x_0) + \left(\text{length}(S_n)\right)^2 \frac{K}{2}$$

for any $x_0 \in S_n$. Combining this with the inequality $\int_{S_n} hds \geq nc_2$ and using that each S_n has bounded length, we get that

$$h(x_0) \geq nc_3 = \text{dist}(x_0, \partial M)c_3.$$

for any $x_0 \in S_n$. We can apply the minimum principle for any point on M between S_n and S_{n+1} to get the $\frac{1}{c}\,\text{dist}(x, \partial M) \leq h(x) \leq c\,\text{dist}(x, \partial M)$ everywhere on M. □

Remark 2.5.6. The proof above is true for any non-constant harmonic function, therefore for any non-constant positive harmonic function on the quasi-cylinder there exists some constant $c < \infty$, such that

$$\frac{1}{c}\,\text{dist}(x, \partial M) \leq h(x) \leq c\,\text{dist}(x, \partial M).$$

2.6 Proof of the theorem and its corollaries

Now we can prove the main theorem of this chapter:

Theorem 2.6.1. *Let f be a quasiconformal self-map of the jungle gym so* *that the dilatation of f is compactly supported. Let F be its lifted map to the hyperbolic disc extended to the boundary of the disc. Then F is differentiable with non-zero derivative at the deep points, and the Hausdorff dimension of these points is* 1.

Proof. In Lemma 2.4.4 we showed that $I(x_0)$ is finite at every deep point, and so by Lehto's Theorem 2.4.1 we can conclude that the map F is totally differentiable there with nonzero derivative. Now cut the jungle gym in half. On each half there is a positive harmonic function which grows linearly as x goes to infinity (Lemma 2.5.5). Even though there is no Green's function on this manifold we may still use Theorem 1.5.2, because the harmonic functions on each half of the quasi-cylinder have the properties described in Lemma 1.4.1. Therefore the Hausdorff dimension of deep points is 1. By Theorem 2.4.1 and Lemma 2.4.4 the map F is differentiable with non-zero derivative at the deep points, so we can conclude that the Hausdorff dimension of those points where F is differentiable with non-zero derivative is 1. □

Remark 2.6.2. Let M be a hyperbolic manifold, so that one of its ends is a quasi-cylinder, i.e. this end has bounded geometry locally and if S_n denotes the hypersurface at distance n from the beginning of this end, then S_n is homologous to S^1 with $\operatorname{diam}(S_n) \leq D$. Consider a quasiconformal deformation of this surface M by dilatation with compact support. Lift this deformation to the universal covering space $\mathbb{B}$, extend to the boundary of $\mathbb{B}$ and call it F. Then F is differentiable at the deep points, which have full dimension, therefore the Hausdorff dimension of the points where F is differentiable with non-zero derivative is also full.

Corollary 2.6.3. *Consider a quasi-symmetric homeomorphism f of the unit circle which conjugates two divergence type groups with quasi-cylindrical end, like the covering group of the jungle gym. Suppose that the dilatation has compact support. Then f is differentiable with non-zero derivative on a set of dimension one and the image of this set also has dimension one. We claim that there is no subset E of the circle so that E and $f(E^c)$ both have dimension less than* 1.

Proof. In Theorem 2.6.1 we have shown that the set

$$F = \{x : f'(x) \text{ exists and non-zero}\}$$

has dimension one. Define the set

$$F_M = \{x : \frac{1}{M} < |f'(x)| < M\}.$$

By the definition, $F_{M_1} \subset F_{M_2}$ whenever $M_1 < M_2$, and $\cup_M F_M = F$. Therefore for all $\epsilon > 0$ there exists M so that $\dim_{\mathrm{H}}(F_M) \geq 1 - \epsilon$.

First we will show that if $x \in F_M$ there exists a neighbourhood B_x of x so that $\frac{1}{2M}|B_x| \leq |f(B_x)| \leq 2M|B_x|$. Since $x \in F_M$ so

$$\frac{1}{M} \leq \lim_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \leq M.$$

Therefore if $|y-x|$ is small enough then $\frac{1}{2M}|y-x| < |f(y)-f(x)| < 2M|y-x|$.

This means we can choose a neighbourhood B_x around the point x so that

$$\frac{1}{2M}|B_x| < |f(B_x)| < 2M|B_x|.$$

Notice that the diameter of this neighbourhood depends on M and on the point x. To get rid of the dependence on x, define the set

$$F_{M,\delta} = \left\{x \in F_M : \forall B_x, |B_x| \leq \delta, \frac{|B_x|}{2M} \leq |f(B_x)| \leq 2M|B_x|\right\}.$$

Then $F_M = \cup_\delta F_{M,\delta}$ and $F_{M,\delta_1} \subset F_{M,\delta_2}$ if $\delta_1 > \delta_2$. Therefore, for any small positive epsilon we can find M and δ so that $\dim_{\mathrm{H}}(F_{M,\delta}) > 1 - \epsilon$.

Now we will show that $\dim_{\mathrm{H}}\big(f(F_{M,\delta})\big) = \dim_{\mathrm{H}}(F_{M,\delta})$. On the one hand, if we take any covering of $F_{M,\delta}$ with balls $\{U_i\}$ of diameter less than δ, then

$$\frac{|U_i|}{2M} \leq |f(U_i)| \leq 2M|U_i|$$

and $\{f(U_i)\}$ gives a cover for $f(F_{M,\delta})$. This gives an estimation for the Hausdorff content of $f(F_{M,\delta})$:

$$\mathcal{H}^\alpha\big(f(F_{M,\delta})\big) \leq \sum |f(U_i)|^\alpha \leq (2M)^\alpha \sum |U_i|^\alpha$$

for any covering $\{U_i\}$ of the set $F_{M,\delta}$, so

$$\mathcal{H}^\alpha\big(f(F_{M,\delta})\big) \leq \mathcal{H}^\alpha(F_{M,\delta})$$

and therefore

$$\dim_{\mathrm{H}}\big(f(F_{M,\delta})\big) \leq \dim_{\mathrm{H}}(F_{M,\delta}).$$

On the other hand, take a covering $\{V_i\}$ of the set $f(F_{M,\delta})$ so that $|V_i| \leq \frac{\delta}{4M}$. We may assume that $V_i \cap f(F_{M,\delta})$ is not empty for any i, so there exists a point $x \in f^{-1}(V_i) \cap F_{M,\delta}$. From the definition of the set $F_{M,\delta}$, x has a δ-neighbourhood B_x, for which

$$\frac{|B_x|}{2M} \leq |f(B_x)| \leq 2M|B_x|.$$

Since $|V_i| \leq \frac{\delta}{4M}$, the set V_i lies inside the image set of B_x, or equivalently $f^{-1}(V_i) \subset B_x$, and so by the definition $F_{M,\delta}$

$$\frac{|f^{-1}(V_i)|}{2M} \leq |V_i| \leq 2M|f^{-1}(V_i)|.$$

The set $F_{M,\delta}$ is covered by $\{f^{-1}(V_i)\}$, so

$$\mathcal{H}^\alpha(F_{M,\delta}) \leq \sum |f^{-1}(V_i)|^\alpha \leq (2M)^\alpha \sum |V_i|^\alpha,$$

and hence

$$\mathcal{H}^\alpha(F_{M,\delta}) \leq \mathcal{H}^\alpha\big(f(F_{M,\delta})\big),$$
$$\dim_{\mathrm{H}}(F_{M,\delta}) \leq \dim_{\mathrm{H}}\big(f(F_{M,\delta})\big).$$

Therefore we proved the equality of the Hausdorff dimension of $F_{M,\delta}$ and its image under the map f. Since the dimension is preserved for every δ and M, $\dim_{\mathrm{H}}(F) = \dim_{\mathrm{H}}\big(f(F)\big)$.

For the last statement of the corollary assume that $\dim_{\mathrm{H}}(E) = 1 - \epsilon < 1$, then we have to prove that $\dim_{\mathrm{H}}\big(f(E^c)\big) = 1$. Since the dimension of E is strictly less than 1 but $\dim_{\mathrm{H}}(F) = 1 > \dim_{\mathrm{H}}(E)$, so

$$\dim_{\mathrm{H}}(F \cap E) \leq \dim_{\mathrm{H}}(E) = 1 - \epsilon.$$

We may write F as the disjoint union of the two sets, $F \backslash E$ and $F \cap E$. Then $\dim_{\mathrm{H}}(F) = \max\big\{\dim_{\mathrm{H}}(F\backslash E), \dim_{\mathrm{H}}(F \cap E)\big\}$, and since $\dim_{\mathrm{H}}(F) = 1$ but $\dim_{\mathrm{H}}(F \cap E) = 1 - \epsilon$ the dimension of $F\backslash E$ must be one. The map f preserves the Hausdorff dimension of F, as we have shown in the first part of this proof, therefore $\dim_{\mathrm{H}}(F\backslash E) = 1$ and so does the dimension of $f(E^c)$. □

Corollary 2.6.4. *There is a quasiconformal deformation of a divergence type group with quasi-cylindrical end on the quotient manifold, so that the limit set might not be a circle, but does have tangents on a set of dimension one.*

Proof. As previously, let f be a quasiconformal deformation of the quotient manifold, whose dilatation is supported in a compact set on the manifold. Lift this deformation up to $\mathbb{D}$, and now define the dilatation on $\mathbb{R}^2\backslash\mathbb{D}$ to be zero. This defines a quasiconformal deformation of the group which sends S^1 to a quasicircle. The same argument as in Theorem 2.6.1 shows that the lifted map F is differentiable with non-zero derivative at the deep points. Therefore the quasicircle has tangents on a set of Hausdorff dimension one. □

Bibliography

[1] S. Agard. A geometric proof of Mostow's rigidity theorem for goups of divergence type. *Acta Math.*, 151:231–252., 1983.

[2] L. V. Ahlfors. *Möbius transformations in several dimensions.* Lecture Notes, School of Mathematics, University of Minnesota, Minneapolis, 1981.

[3] K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fichsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212., 1994.

[4] A. F. Beardon. *The Geometry of Discrete Groups.* Springer-Verlag, New York, 1983.

[5] C. Bishop, V. Y. Gutlyanskiĭ, O. Martio, and M. Vuorinen. Conformal Distortion in Space. to appear.

[6] C. J. Bishop and P. W. Jones. The law of the iterated logarithm for Kleinian groups. *Contemporary Mathematics*, 211:17–50., 1997.

[7] C. J. Bishop and Y. Peres. Hausdorff dimension and Fractal Sets. to appear.

[8] R. D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6:1–35., 1993.

[9] S.-Y. A. Chang, J. M. Wilson, and T. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.*, 60:217–246, 1985.

[10] J. L. Fernández and M. V. Melián. Escaping geodesics of Riemann surfaces. to appear.

[11] P. Hall and C. C. Heyde. *Martingale Limit Theory and Its Application.* Academic Press, New York, 1980.

[12] T. Kuusalo. Boundary mappings of geometric isomorphisms of Fuchsian groups. *Ann. Acad. Sci. Fenn. Ser. A I*, 545:1–7., 1973.

[13] O. Lehto. On the differentiability of Quasiconformal Mapping with Prescribed Complex Dilatation. *Ann. Acad. Sci. Fenn.*, 275, 1960.

[14] O. Lehto. *Conformal Mappings and Teichmüller Spaces.* Israel Institute of Technology, Haifa, Israel, 1973.

[15] O. Lehto. *Univalent Functions and Teichmüller Spaces.* Springer-Verlag, New York, 1987.

[16] O. Lehto and K. I. Virtanen. *Quasiconformal Mappings in the Plane.* Springer-Verlag, New York, 1973.

[17] B. Maskit. *Kleinian Groups.* Springer-Verlag, New York, 1988.

[18] C. T. McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle*, volume 142 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, 1996.

[19] P. J. Nicholls. *The Ergodic Theory of Discrete Groups.* Cambridge University Press, Cambridge, 1989.

[20] T. Sorvali. The boundary mapping induced by an isomorphism of covering groups. *Ann. Acad. Sci. Fenn. A I.*, 526:1–31, 1972.

[21] D. Sullivan. Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension two. *Geometry Symposium*, 504:127–144, 1980.

[22] P. Tukia. On discrete groups of the unit disk and their isomorphisms. *Ann. Acad. Sci. Fenn. A I.*, 504:1–45, 1972.

[23] P. Tukia. Differentiability and rigidity of Möbius groups. *Invent. Math.*, 82:557–578., 1985.

[24] P. Tukia. On isomorphisms of geometrically finite Möbius groups. *Publ. Math. Inst. Hautes Étud. Sci.*, 61:171–214., 1985.

[25] P. Tukia. *A remark on a paper by Floyd, In: Holomorphic functions and moduli*, volume 11, pages 165–172. Springer-Verlag, New York, 1988.

[26] P. Tukia. A rigidity theorem for Möbius groups. *Invent. Math.*, 97:405–431., 1989.

[27] J. Väisälä. *Lectures on n-dimansional Quasiconformal Mappings.* Springer-Verlag, New York, 1971.