Invariants of Real k-planes in $\mathbf{R}^{2n}$ under the Standard Action of $\mathbf{U(n)}$

A Dissertation, Presented

by

Stewart Mandell

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

May 2001

State University of New York

at Stony Brook

The Graduate School

Stewart Mandell

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Detlef Gromoll
Professor of Mathematics
Dissertation Director

Lowell Jones

Lowell Jones
Professor of Mathematics
Chairman of Dissertation

Alexander Kirillov
Assistant Professor of Mathematics

Martin Rocek
Department of Physics
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation,

Invariants of Real k-planes in $\mathbf{R}^{2n}$ under the Standard Action of $\mathbf{U}(\mathbf{n})$

by

Stewart Mandell

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

2001

In our main result, we describe an explicit set of generators for invariants of real k-planes in R^{2n} under the action of the unitary group $U(n)$.

We build on the seminal work of Herman Weyl where an invariant theory is developed for the classical groups, $SL(n)$, $O(n)$, $SP(2n)$, see [Weyl46]. For each irreducible submodule in the $GL(n)$ module, $S^k(S^2(R^n))$, there is exactly one $SO(n)$ invariant which we describe explicitly. We call these "primitive" $SO(n)$ invariants of $S^k(S^2)$. They are generated by Gram determinants.

Similarly, for each irreducible of the $GL(2n)$ module $S^k(\Lambda^2(R^{2n}))$, we describe its one $SP(2n)$ invariant. These are generated by Pfaffian analogs of the Gram determinant.

Next we translate what we mean by polynomial invariants of k-planes into the language of representations on vector spaces. They correspond to homogeneous polynomials associated to certain irreducibles of rectangular Young diagrams with k rows, q columns.

The standard action of $U(n)$ on R^{2n} is the sames as $SO(2n) \cap SP(2n)$. Using Capelli's identity and a geometric argument, we prove that all invariants for this action are generated by the orthogonal and symplectic products, $< \, , \, >$ and $[\, , \,]$. The polynomial algebra of invariants is thus $S^*(S^2) \otimes S^*(\Lambda^2)$.

Using the Littlewood–Richardson Rule, we prove that given any subdiagram S_λ of the rectangular Young diagram S_{q^k} on k rows, q columns, then S_{q^k} appears as an irreducible of multiplicity 1, in the module $S_\lambda \otimes S_\mu$, where S_μ is the complement of S_λ in S_{q^k} rotated 180 degrees.

This leads to an explicit set of generators for the standard action of $U(n)$ on real k-planes.

Contents

Chapter 1

Introduction

We give an overview of the main facts from invariant theory referring the reader to [FH91] for details. Our field is the real numbers R although the following holds for any field of characteristic 0.

Irreducible representations of the symmetric group, S_d, are in 1-1 correspondence with partitions of the number d. We represent partitions of d by sequences $\lambda = (\lambda_1, \cdots, \lambda_k)$ with $d = \lambda_1 + \cdots + \lambda_k$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. k is called the length of the partition. To each partition, we have its associated Young diagram, with λ_i boxes in the ith row, counting from top down, rows lined up on the left. We denote the partition with rectangular diagram consisting of k rows and q columns by (q^k). A Young tableaux is a diagram filled with the numbers $1 \cdots d$ in some order. A standard tableaux is one in which the entries in each row and column form a strictly increasing sequence. A Young symmetrizer c_λ is the element of the group algebra $R[S_d]$ obtained by symmetrizing the rows and then alternating the columns of a Young tableaux. Each $R[S_d]$ module $R[S_d] \cdot c_\lambda$ appears as an irreducible representation of S_d in its group algebra. The c_λ are the projectors onto irreducible components. Rearrangements of elements in the tableaux give isomorphic represention which we denote generically as S_λ. Both the multiplicity and the dimension of the irreducible component S_λ in $R[S_d]$ are equal to the number of standard tableaux of shape λ.

If S_λ and S_μ are irreducible representations of S_{d_1} and S_{d_2} respectively, we have a representation of $S_{d_1+d_2}$ which we denote, $S_\lambda \otimes S_\mu$, induced from the external tensor product representation $S_\lambda \boxtimes S_\mu$ of $S_{d_1} \times S_{d_2}$. $S_\lambda \otimes S_\mu$ decomposes as $S_\lambda \otimes S_\mu = \sum_\nu N^\nu_{\lambda\mu} S_\nu$ where the irreducible component S_ν of S_{m+n} appears in $S_\lambda \otimes S_\mu$ with multiplicity $N^\nu_{\lambda\mu}$. $N^\nu_{\lambda\mu}$ can be computed explicitly by the Littlewood–Richardson rule.

Let V be any $GL(n)$ module. $GL(n)$ acts on the left on $V^{\otimes d}$ by $g \cdot (v_1 \otimes \cdots \otimes v_d) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_d$, while S_d acts on right by $(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}$ for $\sigma \in S_d$, and these actions clearly commute. In particular we have a $GL(n)$ module $V^{\otimes d} \cdot c_\lambda$ called the Weyl module V_λ.

When $V = R^n$, there is a 1-1 correspondence $S_\lambda \Leftrightarrow V_\lambda$, between irreducible components of the $GL(n)$ module $V^{\otimes d}$ and those irreducible components of S_d corresponding to partitions of d with length $\leq n$. V_λ appears in $V^{\otimes d}$ with multiplicity equal to the number of standard tableaux of shape λ. The dimension of V_λ is the number of semi-standard tableaux of shape λ, e.g. all possible fillings of its diagram with entries from the set $\{1, \cdots, n\}$ with rows weakly and columns strictly increasing.

Important examples are $V_{(d)} = S^d(V)$, the symmetric product, and $V_{(1^d)} = \Lambda^d(V)$, the wedge product.

More formally $V^{\otimes d} = \sum_\lambda V_\lambda \otimes S_\lambda$ for all partitions λ of d where V_λ, S_λ are irreducible components of $GL(n)$, S_d respectively.

All irreducible $GL(n)$ modules are of the form $\det^a V_{(\lambda_1, \cdots, \lambda_n)}$, for a an integer and all sequences $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Each irreducible $GL(n)$ module is generated by a unique highest weight vector, e.g. an eigenvector for diagonal $GL(n)$ elements, which is annihilated by upper triangular

$gl(n)$ elements. Here $gl(n)$ acts on $V^{\otimes d}$ via the Leibnitz rule.

$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_d) = g \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_d + v_1 \otimes g \cdot v_2 \otimes \cdots \otimes v_d + \cdots v_1 \otimes v_2 \otimes \cdots \otimes g \cdot v_d$

For $V = R^n$ and λ, μ partitions of d_1 and d_2 respectively, of length $\leq n$, the $GL(n)$ module $V_\lambda \otimes V_\mu \subset V^{\otimes(d_1+d_2)}$ decomposes as $V_\lambda \otimes V_\mu = \sum_\nu N^\nu_{\lambda\mu} V_\nu$ where ν are partitions of d_1+d_2 of length $\leq n$ and $N^\nu_{\lambda\mu}$ are the Littlewood–Richardson coefficients.

There is another important operation on $GL(n)$ Weyl modules called plethysm, $V_\lambda \circ V_\mu$. Since V_μ is a $GL(n)$ module we can define the $GL(n)$ Weyl module $V_\lambda(V_\mu) \subset V_\mu^{\otimes d_1}$.

When $V = R^n$, V_μ is a $GL(n)$ submodule of $V^{\otimes d_2}$, and $V_\lambda(V_\mu)$ becomes a $GL(n)$ submodule of $V^{\otimes d_1 d_2}$. This will decompose into irreducible components $V_\lambda \circ V_\mu = \sum_\nu M^\nu_{\lambda\mu} V_\nu$ where ν are partitions of $d_1 d_2$ of length $\leq n$. Unfortunately, there is no general method for computing the coefficients, $M^\nu_{\lambda\mu}$, and the plethysm problem is solved for only special cases. We will be interested in the examples $V_{(k)} \circ V_{(2)} = S^k(S^2)$ and $V_{(k)} \circ V_{(1,1)} = S^k(\Lambda^2)$.

For H a group acting on V, a polynomial invariant of k vectors $v_1, \cdots, v_k$ in V is a polynomial $P(v_1, \cdots, v_k)$ invariant under the action of H: $P(h \cdot v_1, \cdots, h \cdot v_k) = P(v_1, \cdots, v_k) \;\; \forall h \in H$. The H invariant polynomials on V form an algebra. For finite, compact Lie, and the classical groups, the algebra of invariants is finitely generated, see [Weyl46]. Describing an explicit set of generators is called a First Fundamental Theorem (FFT). In [Weyl46], an FFT is given for the standard representation for all the classical groups, but in general, this is a difficult problem whose solution is known in some special cases.

We note that the algebra of polynomial invariants of a compact Lie group acting on R^n will distinguish orbits, see [Schwarz75]. An analogous result will hold for Grassmannians, where polynomial invariants are homogeneous coordinates.

Chapter 2

$\mathbb{SO}(n)$ invariants of $\mathbb{S}^k(\mathbb{S}^2)$

We show that for each irreducible component of the $GL(n)$ module $S^k(S^2R^n)$, there is exactly one $SO(n)$ invariant, which we describe explicitly in terms of Gram determinants.

2.1 Decomposition of $\mathbb{S}^k(\mathbb{S}^2)$ via $\mathbb{GL}(n)$

According to [Howe89], the decomposition of $S^k(S^2)$ as a $GL(n)$ module, consists of irreducible components, each occuring with multiplicity 1, corresponding to Young diagrams of size 2k with rows of even length. Highest weight vectors for the irreducible components corresponding to Young diagrams with 2 elements in each row, are given by

$$\delta_1 \equiv x_{11}\,, \quad \delta_2 \equiv det\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\,, \quad \delta_3 \equiv det\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}\,, \quad \cdots, \quad \delta_n$$

where $x_1, \cdots, x_n$ is the canonical basis for R^n, x_{ij} is $x_i \cdot x_j$, an element of S^2, and $\delta_k \in S^k(S^2)$ is the expansion of the determinant with symmetrization replacing multiplication of elements x_{ij}.

These correspond to Young diagrams:

$$S_{(2)} = \bullet\bullet, \quad S_{(2,2)} = \begin{array}{c}\bullet\bullet\\ \bullet\bullet\end{array}, \quad S_{(2,2,2)} = \begin{array}{c}\bullet\bullet\\ \bullet\bullet\\ \bullet\bullet\end{array}, \text{ etc.}$$

Monomials in the δ_i give highest weight vectors for all irreducible components of $S^k(S^2)$ corresponding to Young diagrams of size $2k$ with rows of even length. If $\delta_\lambda = \delta_1^{n_1}\delta_2^{n_2}\cdots\delta_s^{n_s}$ then δ_λ is a highest weight vector for $V_\lambda \subset S^k(S^2)$, where $k = n_1 + 2n_2 + \cdots + sn_s$ and λ is the partition $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ of $2k$.

$$\begin{aligned}
&\lambda_1 = 2(n_1 + n_2 + \cdots + n_s)\\
&\lambda_2 = 2(n_2 + \cdots + n_s)\\
&\cdots\\
&\lambda_s = 2n_s
\end{aligned}$$

The monomials in δ_λ consist of fully symmetrized monomials in the $x_{ij} \in S^2$, e.g. $\delta_1\delta_2 = x_{11} \cdot x_{11} \cdot x_{22} - x_{11} \cdot x_{12} \cdot x_{12}$ From the above we clearly have

Theorem 2.1.1 *The irreducible component of $S^k(S^2)$ corresponding to highest weight δ_k is generated by elements of the form*

$$\delta_k(v_1, v_1, v_2, v_2, \cdots, v_k, v_k) \equiv det \begin{vmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_k \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \cdots & v_2 \cdot v_k \\ \cdots & \cdots & \cdots & \cdots \\ v_k \cdot v_1 & v_k \cdot v_2 & \cdots & v_k \cdot v_k \end{vmatrix}$$

where $v_1, \cdots, v_k$ are vectors in R^n.

2.2 Primitive $\mathbb{SO}(n)$ invariants of $\mathbb{S}^k(\mathbb{S}^2)$

By the First Fundamental Theorem for the standard action of $SO(n)$ on R^n, see [Weyl46], invariants are polynomials in the orthogonal product, $\langle , \rangle$, e.g. elements in $S^*(S^2)$. Let Γ_{2k} be the element of $S^k(S^2)^*$ defined by

$$\Gamma_{2k}(v_1, v_2, v_3, v_4, \cdots, v_{2k-1}, v_{2k}) \;=\; \langle v_1, v_2\rangle \langle v_3, v_4\rangle \cdots \langle v_{2k-1}, v_{2k}\rangle$$

S_{2k} acts on Γ_{2k} by

$$\sigma \cdot \Gamma_{2k}(v_1, v_2, v_3, v_4, \cdots, v_{2k-1}, v_{2k}) = \langle v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}\rangle \cdots \langle v_{\sigma^{-1}(2k-1)}, v_{\sigma^{-1}(2k)}\rangle$$

and all $SO(n)$ invariants of $(R^n)^{\otimes 2k}$ must be linear combinations of the $\sigma \cdot \Gamma_{2k}$.

For (α, β) the natural pairing of $GL(n)$ modules $S^k(S^2)^*$, $S^k(S^2)$ we have

$$(\Gamma_{2k}, \delta_k) = \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_k\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_k\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle x_k, x_1\rangle & \langle x_k, x_2\rangle & \cdots & \langle x_k, x_k\rangle \end{vmatrix} = 1$$

and $(\sigma \cdot \Gamma_{2k}, \delta_k)$ for any $\sigma \in S_{2k}$ gives either the same result, up to sign, or 0.

It follows that there is only one $SO(n)$ invariant for the $GL(n)$ module generated by δ_k. Multiplying invariants we see that there is one $SO(n)$ invariant for each irreducible component of $S^k(S^2)$.

Definition 2.2.1 *The unique $SO(n)$ invariants corresponding to irreducible components of $S^k(S^2)$ are called* **primitive SO(n) invariants of $\mathbf{S^k(S^2)}$**.

Theorem 2.2.2 *The primitive $SO(n)$ invariant for the irreducible component of $S^k(S^2)$ with highest weight δ_k, is the Gram determinant. Its value on the element*

$\delta_k(v_1, v_1, v_2, v_2, \cdots, v_k, v_k)$ *is*

$$\gamma_k(v_1, v_1, v_2, v_2, \cdots, v_k, v_k) = \begin{vmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_k\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \cdots & \langle v_2, v_k\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle v_k, v_1\rangle & \langle v_k, v_2\rangle & \cdots & \langle v_k, v_k\rangle \end{vmatrix}$$

Proof: This is just Γ_{2k} acting on $\delta_k(v_1, v_1, v_2, v_2, \cdots, v_k, v_k)$

We use the notation, $\langle v_1, v_1, \cdots, v_k, v_k\rangle \equiv \gamma_k(v_1, v_1, \cdots v_k, v_k)$, for the Gram determinant.

Corollary 2.2.3 *The primitive $SO(n)$ invariant for the irreducible component of the $GL(n)$ module $S^k(S^2)$, corresponding to the highest weight vector $\delta_\lambda = \delta_1^{n_1} \cdots \delta_s^{n_s}$ is $\gamma_\lambda = \gamma_1^{n_1} \cdots \gamma_s^{n_s}$*

Chapter 3

$\mathbb{SP}(2n)$ invariants of $\mathbb{S}^k(\Lambda^2)$

We show that for each irreducible component of $S^k(\Lambda^2 R^{2n})$ as a $GL(2n)$ module, there is exactly one $SP(2n)$ invariant, which we describe explicitly in terms of Pfaffians.

3.1 Decomposition of $\mathbb{S}^k(\Lambda^2)$ via $\mathbb{GL}(2n)$

The decomposition of $S^k(\Lambda^2)$ as a $GL(2n)$ module, consists of irreducible components, each occuring with multiplicity 1, corresponding to Young diagrams of size $2k$ with columns of even length, see [Howe89].

The following theorem, whose proof is similar to the symmetric case, is well known.

Theorem 3.1.1 *Highest weight vectors for the irreducible components corresponding to Young diagrams with 1 column of even length are given by*

$$\sigma_1 \equiv \text{pfaff}\begin{vmatrix} 0 & y_{12} \\ y_{21} & 0 \end{vmatrix}, \quad \sigma_2 \equiv \text{pfaff}\begin{vmatrix} 0 & y_{12} & y_{13} & y_{14} \\ y_{21} & 0 & y_{23} & y_{24} \\ y_{31} & y_{32} & 0 & y_{34} \\ y_{41} & y_{42} & y_{43} & 0 \end{vmatrix}, \quad \cdots, \quad \sigma_n$$

where $x_1, \cdots, x_{2n}$ is the canonical basis for R^{2n}, y_{ij} is $x_i \wedge x_j$, an element of Λ^2, and pfaff is the analog for tensors in $S^k(\Lambda^2)$ of the usual Pfaffian on skew symmetric matrices. Thus $\sigma_1 = y_{12}$, $\sigma_2 = y_{12} \cdot y_{34} - y_{13} \cdot y_{24} + y_{14} \cdot y_{23}$, etc.

These correspond to Young diagrams

$$S_{(1,1)} = \begin{array}{c}\circ\\ \circ\end{array}, \quad S_{(1,1,1,1)} = \begin{array}{c}\circ\\ \circ\\ \circ\\ \circ\end{array}, \text{ etc.}$$

Monomials in the σ_i give highest weight vectors for all irreducible components of $S^k(\Lambda^2)$ corresponding to Young diagrams on 2k with columns of even length. If

$$\sigma_\lambda = \sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_s^{n_s}$$

then σ_λ is a highest weight vector for $S^k(\Lambda^2)$, where $k = n_1 + 2n_2 + \cdots + sn_s$ and λ is the partition $(\lambda_1, \lambda_2, \cdots, \lambda_{2s-1}, \lambda_{2s})$ of $2k$.

$$\begin{aligned}
&\lambda_1,\ \lambda_2 = n_1 + n_2 + \cdots + n_s \\
&\lambda_3,\ \lambda_4 = n_2 + \cdots + n_s \\
&\cdots \\
&\lambda_{2s-1},\ \lambda_{2s} = n_s
\end{aligned}$$

The monomials in σ_λ consist of fully symmetrized monomials in the $y_{ij} \in \Lambda^2$, e.g.

$$\sigma_1\sigma_2 = y_{12} \cdot y_{12} \cdot y_{34} - y_{12} \cdot y_{13} \cdot y_{24} + y_{12} \cdot y_{14} \cdot y_{23}$$

It follows immediately

Theorem 3.1.2 *The irreducible component of $S^k(\Lambda^2)$ corresponding to highest weight*

σ_k is generated by elements of the form

$$\sigma_k(v_1, v_2, \cdots, v_{2k_1}, v_{2k}) \equiv \mathrm{pfaff} \begin{vmatrix} 0 & v_1 \wedge v_2 & \cdots & v_1 \wedge v_{2k} \\ v_2 \wedge v_1 & 0 & \cdots & v_2 \wedge v_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ v_{2k} \wedge v_1 & v_{2k} \wedge v_2 & \cdots & 0 \end{vmatrix}$$

where $v_1, \cdots, v_{2k}$ are vectors in R^{2n}.

3.2 Primitive $\mathbb{SP}(2n)$ invariants of $\mathbb{S}^k(\Lambda^2)$

By the First Fundamental Theorem for the standard action of $SP(2n)$ on R^{2n}, see [Weyl46], invariants are polynomials in the symplectic product, [,], hence are elements in $S^*(\Lambda^2)$. Let P_{2k} be the element of $(S^k(\Lambda^2))^*$ defined by

$$P_{2k}(v_1, v_2, v_3, v_4, \cdots, v_{2k-1}, v_{2k}) = [v_1, v_2]\,[v_3, v_4] \cdots [v_{2k-1}, v_{2k}]$$

Let S_{2k} acts on P_{2k} by

$$\sigma \cdot P_{2k}(v_1, v_2, v_3, v_4, \cdots, v_{2k-1}, v_{2k}) = [v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}] \cdots [v_{\sigma^{-1}(2k-1)}, v_{\sigma^{-1}(2k)}]$$

and all $SP(2n)$ invariants of $(R^{2n})^{\otimes 2k}$ must be linear combinations of the $\sigma \cdot P_{2k}$.

For (α, β) the natural pairing of $GL(2n)$ modules $S^k(\Lambda^2)^*$, $S^k(\Lambda^2)$ we have

$$(P_{2k}, \sigma_k) = \begin{vmatrix} 0 & [x_1, x_2] & \cdots & [x_1, x_{2k}] \\ [x_2, x_1] & 0 & \cdots & [x_2, x_{2k}] \\ \cdots & \cdots & \cdots & \cdots \\ [x_{2k}, x_1] & [x_{2k}, x_2] & \cdots & 0 \end{vmatrix} = 1$$

and $(\sigma \cdot P_{2k}\,,\ \sigma_k)$ for any $\sigma \in S_{2k}$ gives either the same result, up to sign, or 0.

It follows that there is only one $SP(2n)$ invariant for the $GL(2n)$ module generated by σ_k. Multiplying invariants we see that there is one $SP(2n)$ invariant for each irreducible component of $S^k(\Lambda^2)$.

Definition 3.2.1 *The unique $SP(2n)$ invariants corresponding to irreducible components of $S^k(\Lambda^2)$ are called* **primitive SP(2n) invariants of $\mathbf{S^k(\Lambda^2)}$**.

Theorem 3.2.2 *The primitive $SP(2n)$ invariant for the irreducible component of of the $GL(2n)$ module, $S^k(\Lambda^2)$ with highest weight σ_k, is the Pfaffian invariant. Its value on the element $\sigma_k(v_1, v_2, \cdots, v_{2k})$ is*

$$\psi_k(v_1, v_2, , \cdots, v_{2k}) = \text{pfaff} \begin{vmatrix} 0 & [v_1, v_2] & \cdots & [v_1, v_{2k}] \\ [v_2, v_1] & 0 & \cdots & [v_2, v_{2k}] \\ \cdots & \cdots & \cdots & \cdots \\ [v_{2k}, v_1] & [v_{2k}, v_2] & \cdots & 0 \end{vmatrix}$$

Proof: This is just P_{2k} acting on $\sigma_k(v_1, v_2, \cdots, v_{2k})$.

We use the notation, $[v_1, v_2, \cdots, v_{2k}] \equiv \psi_k(v_1, v_2, \cdots, v_{2k})$, for the Pfaffian invariant.

Corollary 3.2.3 *The primitive $SP(2n)$ invariant for the irreducible component of the $GL(n)$ module $S^k(\Lambda^2)$ corresponding to the highest weight vector $\sigma_\lambda = \sigma_1^{n_1} \cdots \sigma_s^{n_s}$ is $\psi_\lambda = \psi_1^{n_1} \cdots \psi_s^{n_s}$*

Chapter 4

k-planes in $\mathbb{R}^n$

Let $G_{k,n}$ be the Grassmannian of k-planes in R^n. $G_{k,n}$ is the orbit space of the action of $GL(k)$ on k-tuples of linearly independent vectors in R^n.

$$(v_1, \cdots, v_k) \equiv g \cdot (v_1, \cdots, v_k) = (w_1, \cdots, w_k)$$

with $w_i = \sum_j g_{ij} v_j, \;\; i, j = 1 \cdots k$

Equivalently, k-planes are non-zero, decomposable elements of $\Lambda^k(R^n)$, $v_1 \wedge \cdots \wedge v_k$, equivalent up to constant, which we will denote by $G(v_1 \wedge \cdots \wedge v_k)$. We have $g \cdot (v_1 \wedge \cdots \wedge v_k) = w_1 \wedge \cdots \wedge w_k = det(g) \cdot (v_1 \wedge \cdots \wedge v_k)$.

4.1 H-Invariant Polynomials of k-planes

Let H be a group acting on R^n. The action of H on k-tuples of vectors in R^n, $h \cdot (v_1, \cdots, v_k) = (h \cdot v_1, \cdots, h \cdot v_k)$, clearly commutes with the action of $GL(k)$, and thus gives an action of H on $G_{k,n}$. Similarly $h \cdot (v_1 \wedge \cdots \wedge v_k) = (h \cdot v_1 \wedge \cdots \wedge h \cdot v_k)$, hence $h \cdot G(v_1 \wedge \cdots \wedge v_k) = G(h \cdot v_1 \wedge \cdots h \cdot \wedge v_k)$.

Our goal is to describe H-invariant polynomials on $G_{n,k}$. However, it is well known that $G_{n,k}$ is a projective variety, and thus there can be no polynomial functions in the usual sense of this word on $G_{n,k}$. Instead, we should consider functions on the set of k-tuples of vectors $v_1, \ldots, v_k$ which are invariant up to a constant under the action of $GL(k)$ More precisely, we consider functions $m(v_1, \ldots, v_k)$ such that $m(g(v_1, \ldots, v_k)) = (\det g)^q m(v_1, \ldots, v_k)$; we will call such functions "homogeneous functions of degree q". They can be viewed as sections of L^q where L is a canonical line bundle on $G_{k,n}$. This leads to the following definition.

Definition 4.1.1 Multilinear H-Invariants of q k-planes, *are H-invariant, multilinear functions of kq vectors in R^n, $(v_1^1, \cdots, v_k^1), \cdots, (v_1^q, \cdots, v_k^q)$*

$$m(h \cdot v_1^1, \cdots, h \cdot v_k^1, \cdots, h \cdot v_1^q \cdots h \cdot v_k^q) \; = \; m(v_1^1, \cdots, v_k^1, \cdots, v_1^q, \cdots, v_k^q)$$

compatable with the $GL(k)$ action on k-tuples of vectors.

$$m(g_1 \cdot (v_1^1, \cdots, v_k^1), \cdots, g_q \cdot (v_1^q \cdots v_k^q)) \; = \; det(g_1) \cdots det(g_q) \; m(v_1^1, \cdots, v_k^1, \cdots, v_1^q \cdots v_k^q)$$

for $g_1, \cdots, g_q$ in $GL(k)$.

It is clear that up to a constant, these invariants are functions on the q k-planes

$$G(v_1^1 \wedge \cdots \wedge v_k^1), \cdots, G(v_1^q \wedge \cdots \wedge v_k^q)$$

Since the transpose of 2 vectors within a k-tuple is an element of $GL(k)$ of determinant -1, we have

$$m(v_1^1, \cdots, v_k^1, \cdots, v_1^q \cdots v_k^q) \; = \; f(v_1^1 \wedge \cdots \wedge v_k^1 \otimes \cdots \otimes v_1^q \wedge \cdots \wedge v_k^q)$$

with $f \in ((\Lambda^k R^n)^{\otimes q})^*$,

$$f(h \cdot v_1^1 \wedge \cdots \wedge h \cdot v_k^1 \otimes \cdots \otimes h \cdot v_1^q \wedge \cdots \wedge h \cdot v_k^q) \;=\; f(v_1^1 \wedge \cdots \wedge v_k^1 \otimes \cdots \otimes v_1^q \wedge \cdots \wedge v_k^q)$$

and

$f(g_1(v_1^1 \wedge \cdots \wedge v_k^1) \otimes \cdots \otimes g_q(v_1^q \wedge \cdots \wedge v_k^q)) \;=$
$det(g_1) \cdots det(g_q)\; f(v_1^1 \wedge \cdots \wedge v_k^1 \otimes \cdots \otimes v_1^q \wedge \cdots \wedge v_k^q)$

We are interested in the case where we have q copies of 1 k-plane, $G(v_1 \wedge \cdots \wedge v_k)$. From the familiar processes of restitution and polarization, see [Weyl46, Processi76], the following lemma follows.

Lemma 4.1.2 *There is a 1-1 correspondence between H-invariant multilinear forms in kq vectors in R^n, $m(v_1^1, \cdots, v_k^1, v_1^2, \cdots, v_k^2, \cdots, v_1^q, \cdots, v_k^q)$, symmetric in each set of the q vectors $\{v_1^1, v_1^2, \cdots, v_1^q\}, \cdots, \{v_k^1, v_k^2, \cdots, v_k^q\}$ and H-invariant polynomials of 1 k-plane of degree q in k vectors,*

$$p(v_1, \cdots v_k) \;=\; m(\underbrace{v_1, \cdots, v_k, \cdots, v_1, \cdots, v_k}_{q\ times})$$

with

$$p(h \cdot v_1, \cdots, h \cdot v_k) \;=\; p(v_1, \cdots, v_k)$$

and

$$p(g \cdot (v_1, \cdots, \cdot v_k)) \;=\; det(g)^q p(v_1, \cdots, v_k)$$

From the above considerations we have

Theorem 4.1.3 *H-invariant polynomials of 1 k-plane of degree q constitute the H-invariant vector subspace of the Weyl submodule of $((R^n)^{\otimes kq})^*$ corresponding to the* *rectangular Young diagram $S_{(q^k)}$, with k rows and q columns.*

$$W = (V_{(q^k)}(R^n)^*)^H \subset (((R^n)^{\otimes kq})^*)^H$$

The number of linearly independent H-invariant polynomials of 1 k-plane of degree q is the dimension of W.

4.2 Examples

The following examples are well known to experts but difficult to find in the literature.

4.2.1 $H = Id$

The following result is classical.

Theorem 4.2.1 *Id-invariant polynomials of 1 k-plane in R^n are generated by the $\binom{n}{k}$ Plücker coordinates of the k-plane $G(v_1 \wedge \cdots \wedge v_k)$, e.g. the $k \times k$ minors of the matrix*

$$\begin{matrix} v_1^1 & \cdots & v_1^n \\ \cdots & \cdots & \cdots \\ v_k^1 & \cdots & v_k^n \end{matrix}$$

These are polynomial invariants of 1 k-plane of degree 1 corresponding to the Young diagram with k rows and 1 column. They give homogeneous coordinates that distinguish k-planes in R^n.

4.2.2 $H = \mathbb{O}(n)$

Theorem 4.2.2 *$O(n)$-invariant polynomials of 1 k-plane in R^n are generated by the Gram determinant, $\gamma_k(v_1, v_1, \cdots, v_k, v_k)$, corresponding to the Young diagram with k* *rows and 2 columns. This is a polynomial invariant of 1 k-plane of degree 2.*

Proof: Follows from Theorem 2.2.2. In particular q must be even.

The above result is in agreement with the obvious geometric fact that the orbit space of k-planes in R^n, under $O(n)$ is a point.

4.2.3 $H = \mathbb{SP}(2n)$

Theorem 4.2.3 *$SP(2n)$-invariant polynomials of $2k$-planes in R^{2n} are generated by the Pfaffian invariant, $\psi_k(v_1, v_2, \cdots, v_{2k})$, corresponding to the Young diagram with 2k rows and 1 column. This is a polynomial invariant of 1 2k-plane of degree 1.*

Proof: Follows from Theorem 3.2.2.

$SP(2n)$ is not compact and polynomial invariants of k-planes will not distinguish orbits. Even on $2k$ planes, the above invariant will be 0 if the restriction of the symplectic form has nullity.

Chapter 5

$\mathbb{U}(n)$ invariants for $\mathbb{R}^{2n}$

The standard action of $U(n)$ is on C^n. We are interested in the standard action of $U(n)$ on R^{2n}. We develop the invariant theory for this case, along the lines of [Weyl46].

5.1 First Fundamental Theorem for $\mathbb{U}(n)$ on $\mathbb{R}^{2n}$

Lemma 5.1.1 *$U(n) = O(2n, R) \cap SP(2n, R)$ acting on R^{2n}.*

Proof: Let $C = A + iB$ in $U(n)$. On R^{2n} this is the matrix $C_R = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$. Then $CC^* = I \rightarrow C_R$ is in $O(2n) \cap SP(2n)$. Similarly, D in $O(2n) \cap SP(2n) \rightarrow D = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ with $A + iB$ in $U(n)$.

Theorem 5.1.2 (First Fundamental Theorem for U(n) on $\mathbf{R^{2n}}$) *All polynomial invariants of $O(2n) \cap SP(2n)$ depending on an arbitrary number of vectors in R^{2n} are expressable in terms of the orthogonal and symplectic product*

$$\langle u, v \rangle, \ [u, v]$$

Proof: This follows from Weyl's, [Weyl46], formal algebraic argument via Capelli's identity plus the following geometric lemma. Capelli's identity for the case at hand says that if $\langle u, v\rangle$, $[u, v]$ generate a complete set of $O(2n) \cap SP(2n)$ polynomial invariants for all $2n$ vectors $(v_1, \cdots, v_{2n})$ in R^{2n}, then they will generate a complete set of polynomial invariants for all m vectors $(v_1, \cdots, v_m)$ in R^{2n}, for all $m > 2n$.

Lemma 5.1.3 *Given 2n-1 vectors $v_1, \cdots, v_{2n-1}$ in R^{2n}, one can find a coordinate system in R^{2n} both symplectic and orthogonal, such that the first coordinate of each vector $v_1, \cdots v_{2n-1}$ vanishes.*

Proof: The proof is exactly the same as that for $U(2n) \cap SP(2n, C)$ acting on C^{2n}, see [Weyl46, page 172]. As in Weyl's proof, we can assume that our 2n-1 vectors span a 2n-1 dimensional hyperplane P, in R^{2n}. We remind the reader.

Define the linear operator $\tilde{}$ by $[x, y] = \langle \tilde{x}, y\rangle$. Explicitly, if $x = x_1, x_{1'}, \cdots, x_n, x_{n'}$, then $\tilde{x} = -x_{1'}, x_1, \cdots, -x_{n'}, x_n$. The operation $\tilde{}$ has the following properties:

A. $\langle \tilde{x}, \tilde{x}\rangle = \langle x, x\rangle$

B. $\tilde{\tilde{x}} = -x$

C. $\langle \tilde{x}, y\rangle = -\langle x, \tilde{y}\rangle$

Let e_1 be a unit vector perpendicular to P. Let $e_{1'} = \tilde{e_1}$. Then

$\langle e_{1'}, e_{1'}\rangle = \langle e_1, e_1\rangle = 1$ by A.

$\langle e_1, e_{1'}\rangle = [e_1, e_1] = 0$ by definition, hence $e_{1'}$ is in P.

$[e_1, e_{1'}] = -\langle \tilde{\tilde{e_1}}, e_1\rangle = \langle e_1, e_1\rangle = 1$ by B.

Let P_1 be the subspace of P where $[u, e_1] = 0$, $[u, e_{1'}] = 0$. This is equivalent to $\langle u, e_{1'}\rangle = 0$, $\langle u, e_1\rangle = 0$. Hence by C., P_1 is closed under $\tilde{}$ and we can continue our inductive construction of a basis both orthogonal and symplectic.

From the results of sections 2.2 and 3.3 it follows that

$\Gamma_{2i}(v_1, v_2, \cdots, v_{2i})\, P_{2j}(w_1, w_2, \cdots, w_{2j}) =$

$\langle v_1, v_2\rangle \cdots \langle v_{2i-1}, v_{2i}\rangle\, [w_1, w_2] \cdots [w_{2j-1}, w_{2j}]$

is a $U(n)$ invariant of $(R^{2n})^{\otimes 2k}$ for $2i + 2j = 2k$, with $\Gamma_{2i}\, P_{2j} \in (S^i(S^2))^* \otimes (S^j(\Lambda^2))^*$. Linear combinations of the action of S_{2k} on $\Gamma_{2i}\, P_{2j}$ generate all such invariants.

5.2 First Fundamental Theorem for real k-planes under $\mathbb{U}(n)$

We will start with examples which are clearly $U(n)$ polynomial invariants of k-planes in R^n. We will soon see that these examples give a complete set of generators.

5.2.1 Examples

We use the notation $\mathcal{A}(\cdots)$ for alternation along the columns of Young diagrams with entries labeled by vectors $v_1, \cdots, v_k$.

$$\begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{matrix} \quad \text{or} \quad \begin{matrix} v_1 & v_1 \\ v_2 & v_2 \\ \vdots & \vdots \\ v_k & v_k \end{matrix}$$

In the examples, solid dots in Young diagrams correspond to Gram determinants, while open dots correspond to Pfaffians.

$k = 4$

$[v_1, v_2, v_3, v_4]$

$\mathcal{A}(< v_1, v_1, v_2, v_2 > [v_3, v_4]^2)$

$< v_1, v_1, v_2, v_2, v_3, v_3, v_4, v_4 >$

corresponding to diagrams

○	●●	●●
○	●●	●●
○	○○	●●
○	○○	●●

$k = 5$

$\mathcal{A}(< v_1, v_1 > [v_2, v_3, v_4, v_5]^2)$

$\mathcal{A}(< v_1, v_1, v_2, v_2, v_3, v_3 > [v_4, v_5]^2)$

$< v_1, v_1, v_2, v_2, v_3, v_3, v_4, v_4, v_5, v_5 >$

corresponding to diagrams

●●	●●	●●
○○	●●	●●
○○	●●	●●
○○	○○	●●
○○	○○	●●

5.2.2 First Fundamental Theorem (FFT)

Theorem 5.2.1 (FFT for real k-planes under U(n)) *$U(n)$ invariant polynomials of k-planes in R^{2n} are generated by*

k even

$[v_1, \cdots, v_k]$

$\mathcal{A}(\langle v_1, v_1, v_2, v_2 \rangle [v_3, \cdots, v_k]^2)$

$\mathcal{A}(\langle v_1, v_1, v_2, v_2, v_3, v_3, v_4, v_4 \rangle [v_5, \cdots, v_k]^2)$

. . . .

$\langle v_1, v_1, v_2, \cdots, v_k, v_k \rangle$

k odd

$\mathcal{A}(\langle v_1, v_1\rangle [v_2, \cdots, v_k]^2)$

$\mathcal{A}(\langle v_1, v_1, v_2, v_2, v_3, v_3\rangle [v_4, \cdots, v_k]^2)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\langle v_1, v_1, v_2, \cdots, v_k, v_k\rangle$

These are primitive $U(n)$ invariants of k-planes corresponding to irreducible components of the $GL(2n)$ module, $S^*(S^2) \otimes S^*(\Lambda^2)$, generated by

k even

$\sigma_k(v_1, \cdots, v_k)$

$\mathcal{A}\left(\delta_2(v_1, v_1, v_2, v_2)\sigma_{k-2}^2(v_3, \cdots, v_k)\right)$

$\mathcal{A}\left(\delta_4(v_1, v_1, v_2, v_2, v_3, v_3, v_4, v_4)\sigma_{k-4}^2(v_5, \cdots, v_k)\right)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\delta_k(v_1, v_1, v_2, \cdots, v_k, v_k)$

k odd

$\mathcal{A}\left(\delta_1(v_1, v_1)\sigma_{k-2}^2(v_2, \cdots, v_k)\right)$

$\mathcal{A}\left(\delta_3(v_1, v_1, v_2, v_2, v_3, v_3)\sigma_{k-3}^2(v_4, \cdots, v_k)\right)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\delta_k(v_1, v_1, v_2, \cdots, v_k, v_k)$

Proof: See section 5.4.

5.3 A Rectangular Lemma

The proof of the FFT for real k-planes under $U(n)$ will be based on the following lemma.

Lemma 5.3.1 (Rectangular Lemma) *Given a subdiagram S_λ of the rectangular Young diagram S_{q^k} with k rows, q columns, there is only one μ such that $S_\lambda \otimes S_\mu$ contains S_{q^k}, and for this μ, S_{q^k} appears in $S_\lambda \otimes S_\mu$ with multiplicity* 1. *The diagram S_μ is obtained by rotating the skew diagram $S_{(q^k)} - S_\lambda$ 180 degrees.*

Proof: See section 5.5.

5.4 Proof of FFT for Real k-planes in $\mathbb{U}(n)$

From Theorems 4.1.3 and 5.1.1 $U(n)$-invariant polynomials of 1 k-plane in R^{2n} of degree q constitute

$$V_{(q^k)}((R^{2n})^*)^{U(n)} \subset (((R^{2n})^{\otimes kq})^*)^{U(n)} = \sum_{2i+2j=kq} S^i(S^2) \otimes S^j(\Lambda^2)$$

Let V_{q^k} be an irreducible component of $S^i(S^2) \otimes S^j(\Lambda^2)$, $2i + 2j = kq$. $S^i(S^2) \otimes S^j(\Lambda^2) = \sum_{\lambda,\mu} V_\lambda \otimes V_\mu$ where V_λ is an irreducible of $S^i(S^2)$ corresponding to a diagram on $2i$ with rows of even length, and V_μ is an irreducible component of $S^j(\Lambda^2)$ corresponding to a Young diagram of size $2j$ with columns of even length.

V_{q^k} is thus an irreducible component of some $V_\lambda \otimes V_\mu$ and hence S_{q^k} is an irreducible component of $S_\lambda \otimes S_\mu$. S_{q^k} is obtained by appending elements of S_μ to S_λ by the Littlewood–Richardson rule, see [FH91].

By the Lemma, this can be done in only one way, and S_μ is $S_{(q^k)} - S_\lambda$ rotated 180 degrees.

We will explicitly exhibit heighest weight vectors for the irreducible component, V_{q^k} of $V_\lambda \otimes V_\mu$. One should refer to the examples below in what follows. Since S_λ has rows of even length, its entries in column 1 and column 2, if they exist, must be the same length l, reading from top down. Entries of S_μ in columns 1 and 2 of S_{q^k}, reading from bottom up, are of even length $k-l$, which may be zero. $\mathcal{A}(\delta_l\,\sigma_{k-l}^2)$ is thus a highest weight vector for the first two columns corresponding to the invariant $\mathcal{A}\left(\langle v_1, v_1, \cdots, v_l, v_l\rangle\,[v_{l+1}, \cdots, v_k]^2\right)$. This is a primitive $U(n)$-invariant of 1 k-planes of degree 2. We continue column by column in this manner. If entries of S_λ appear, we get highest weight vectors and invariants, similar to the above, on two columns. If, however, no entries of S_λ appear, all remaining columns will have highest weight vector σ_k, corresponding to the invariant $[v_1, \cdots, v_k]$, a primitive $U(n)$-invariant of 1 k-planes of degree 1. Multiplying these highest weight vectors, and primitive invariants, will give us a highest weight vector and a corresponding primitive $U(n)$-invariant of 1 k-plane of degree q.

Examples:

$k=4,\ q=5,\ \lambda=(4,4,2,2),\ \mu=(3,3,1,1)$

●●●●○
●●●●○
●●○○○
●●○○○

$k=5,\ q=4,\ \lambda=(4,2,2),\ \mu=(4,4,2,2)$

●●●●
●●○○
●●○○
○○○○
○○○○

5.5 Proof of the Rectangular Lemma

To decompose, $S^i(S^2)\otimes S^j(\Lambda^2)$ one must decompose the "outer" product of their irreducible components, see [FH91]. This is done via the Littlewood–Richard rule. If

λ is a partition of m and μ a partition of n, then

$$S_\lambda \otimes S_\mu = \sum_\nu N^\nu_{\lambda\mu} S_\nu$$

where ν is a partition of $m+n$ and the integer $N^\nu_{\lambda\mu}$ is the number of ways to construct the diagram of S_ν from the diagrams of S_λ and S_μ via the Littlewood–Richardson rules which we recapitulate.

Fill in the diagram of S_μ with 1's in the first row, 2's in the second row, etc.

We enumerate all possible ways of appending all the 1's to the diagram of S_λ, next all the 2's, etc. such that the following rules are obeyed.

LR 1 At each step we maintain the shape of a young diagram (weakly decreasing rows).

LR 2 No two 1's, 2's, etc can appear in the same column.

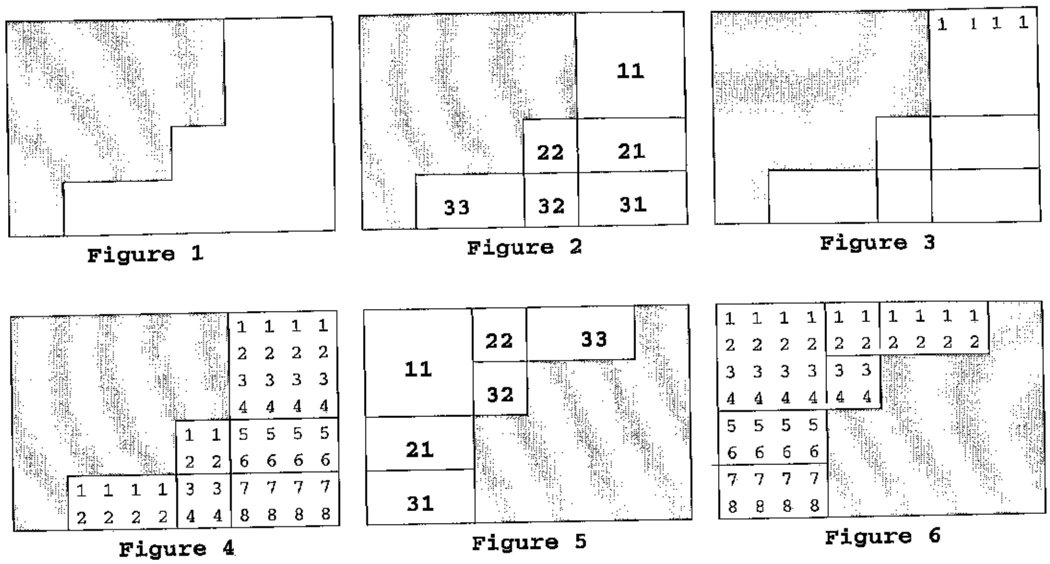
LR 3 For each diagram constructed via [LR 1, LR 2] construct the sequence obtained by reading the appended numbers from right to left, up to down.

LR 4 We keep only those diagrams that form a strict μ expansion. This means the first element must be a 1, the second element a 1 or 2. At each step as we march across the sequence the number of 1's encountered must be $\geq$ the number of 2's encountered must be $\geq$ the number of 3's encountered, etc. Example: 123112344

$N^\nu_{\lambda\mu}$ is the number of Young diagrams of shape ν, constructed in this manner. It is the multiplicity in which the irreducible component S_ν appears in the representation of $S_\lambda \otimes S_\mu$.

Given a subdiagram S_λ of the rectangular young diagram S_{q^k}, we prove there is only one diagram S_μ, which can be appended to S_λ in only one way, via the

Littlewood–Richardson rule, to make S_{q^k}. This will be the skew diagram $S_{q^k} - S_\lambda$ rotated 180 degrees.



Rectangular Lemma Illustration

We refer to the illustration.

1. Figure 1: We have $S(q^k)$ with k = 8, q = 12, and S_λ is the shaded young diagram $S_{(8^4,6^2,2^2)}$.

2. Figure 2: By extending the edges of S_λ , we partition the skew diagram $S_{(q^k)} - S_\lambda$ into blocks which we enumerate top to bottom, right to left. By [LR 4] the top right corner of $B_{1,1}$ must be a 1 and by [LR 1, LR 4], a 1 must have been placed in the top left corner of $B_{1,1}$, and filled out to complete $B'_{1,1}s$ entire top row. See Figure 3. By [LR 2], any remaining 1's of S_μ must appear in blocks to the left of $B_{1,1}$.

3. By [LR 4] the right edge of the second row of $B_{1,1}$ must be a 2 and again by [LR 1, LR 4] 2's must fill out the entire second row of $B_{1,1}$. By [LR 2], remaining 2's of S_μ appear in blocks to the left of $B_{1,1}$

4. Continuing, we fill out the rows of $B_{1,1}$ with 1's, 2's, etc., up to n_{11} (4 in Figure 4). All additional 1's, 2's, n_{11}'s, of S_μ appear in blocks to the left of $B_{1,1}$. See Figure 4.

5. By [LR 4] the number on the right edge of the first row of $B_{2,1}$ must be $n_{11}+1$ (5 in Figure 4). By [LR 1] we must have appended this number at some starting point on this row, and continued consecutively. This starting point can only be the left edge of $B_{2,1}$, for if it began right of the edge, the edge to this position would contain numbers in the set $1, 2, \cdots, n_{11}$ by [LR 1], contradicting [LR 2]. If it began left of the edge, there would appear a string of $n_{11}+1$ greater than the row length of B_{11} contradicting [LR 4]. In this manner we continue filling the rows of the first column of blocks, $B_{1,1}$, $B_{2,1}$, etc. with consecutive integers. See Figure 4.

6. We turn our attention to the second column of blocks. By [LR 4] the top row, right edge of $B_{2,2}$ must be a 1. By [LR 1] and since all 1's are appended first, we must have started with a 1 at the left edge of $B'_{2,2}s$ top row and continued consecutively. Figure 4.

7. We continue this procedure until each column of blocks consists of rows filled with 1's, 2's, etc.

8. In Figure 5 we rearrange the blocks so they're enumerated from top to bottom, left to right, and anchored at the top left corner of S_{q^k}. Figure 6 shows that S_μ is indeed $S_{(q^k)} - S_\lambda$ rotated 180 degrees.

Bibliography

[Weyl46] H Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton, N.J.(1946)

[Schwarz75] G.W. Schwarz, *Smooth functions invariant under the action of a compact Lie Group*, Topology **14** (1975) 63-68

[Processi76] C. Procesi, *The Invariant Theory of* $n \times n$ *Matrices*, Advances In Mathematics **19** (1976) 306-381

[wF97] William Fulton, *Young Tableaux*, Cambridge University Press (1997)

[FH91] William Fulton, Joe Harris, *Representation Theory*, Graduate Texts in Mathematics, Springer Verlag **129** (1991)

[Howe89] Roger Howe *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989)

[BD85] Theodor Brocker, Tammo tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, Springer Verlag **98** (1985)