Weierstrass Representations of Minimal Real Kähler Submanifolds

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Abstract of the Dissertation

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Since the nineteenth century, Weierstrass representations have been used to investigate minimal surfaces in Euclidean 3-space. In the last two decades, it emerged that minimal Kähler submanifolds of Euclidean spaces share many of the features of minimal surfaces. In this dissertation, we try to find similar representations for these minimal real Kähler submanifolds.

First, we modify a method developed by M. Dajczer and D. Gromoll to give a simple way of describing minimal real Kähler hypersurfaces. As an application, we are able to give local examples of superminimal surfaces in the 4-sphere.

Then, based on the formulae for the classical Weierstrass representation, we find a coordinate system for the homogeneous space of all isotropic complex planes in arbitrary complex vector spaces of dimension at least 5. We utilize this coordinate system to give a local characterization of minimal real Kähler surfaces (of real dimension 4) in Euclidean spaces.

Finally, using this characterization, we are able to give a complete local classification and construction methods for all minimal real Kähler surfaces in Euclidean 6-space, at least away from certain isolated singularities. Employing these construction methods, we also give some explicit new examples for such submanifolds.

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1 Introduction

The study of minimal surfaces is one of the classical areas of geometry, both for its beauty and its applications to other areas of Mathematics and Physics. Minimal surfaces are those surfaces in Euclidean 3–space that minimize the area compared to all other surfaces with the same boundary curve.

Already since the nineteenth century, it is known that there is an intimate relationship between the structure of these surfaces and complex analysis. Namely, let $f = (f_1, f_2, f_3) : U \to \mathbf{R}^3$ be a parametrization of a minimal surface by so–called *isothermal coordinates* $(x, y) \in U$, defined in some region U in the plane; this means that the partial derivative vectors of f with respect to these coordinates x and y have the same length and are orthogonal at every point of the surface. (For the existence of such coordinates and proofs of the following results, see e.g. [S], p. 387–397, or any classical treatment of minimal surfaces, as in [O]). Then the component functions f_j are *harmonic functions* with respect to these coordinates, i.e. they satisfy the Laplace equation

$$0 = \triangle f_j = \frac{\partial^2 f_j}{\partial x^2} + \frac{\partial^2 f_j}{\partial y^2}\,, \quad \text{for } j = 1\,, 2\,, 3\,.$$

But by complex analysis, this means that the f_j are locally the real parts of *holomorphic* functions $F_j : V \to \mathbf{C}$, where V is some subset of U, viewed as a region in the complex plane $\mathbf{C}$ with its complex coordinate $z = x + iy$. Thus, for every minimal surface in Euclidean 3–space, we can (locally) find a holomorphic map $F : V \to \mathbf{C}^3$, the so–called *holomorphic representative* of f in the given isothermal coordinates, such that

$$f = \sqrt{2}\,\mathrm{Re}(F)\,.$$

Here, the factor $\sqrt{2}$ is commonly introduced to make F isometric to f, if both are regarded as immersions from V (with the by f induced metric) into $\mathbf{C}^3 \cong \mathbf{R}^6$, for $\mathbf{R}^3 \cong \mathbf{R}^3 \times \{0\} \subset \mathbf{R}^6$ (compare also page 3).

But we have even more structure. If we take the *complex* derivative of F with respect to z, we can observe that $\varphi := \frac{\partial F}{\partial z} \in \mathbf{C}^3$ is a so–called *isotropic vector* of $\mathbf{C}^3$, which means the following. Taking the standard *symmetric* inner product in $\mathbf{C}^3$, i.e.

$$v \cdot w := \sum_{j=1}^{3} v_j\, w_j \quad \text{for } v\,, w \in \mathbf{C}^3$$

(**NO** complex conjugates!), and writing $v^2 := v \cdot v$, we find that

$$\varphi^2 = \frac{\partial F}{\partial z} \cdot \frac{\partial F}{\partial z} = 0 \, .$$

But it is fairly easy to describe (almost) all vectors $X \in \mathbf{C}^3$ that are isotropic in this sense. Namely, if λ is any non–zero complex number, and ξ any other complex number, it is easy to check that

$$\varphi = \lambda \left(\frac{1-\xi^2}{2} \, , \, i \, \frac{1+\xi^2}{2} \, , \, \xi \right) \tag{1}$$

is isotropic. On the other hand, if $\varphi = (\varphi_1 \, , \varphi_2 \, , \varphi_3)$ is isotropic, and if for the first two coordinates we have $\varphi_1 \neq i \, \varphi_2$, then we can always find a $\lambda \neq 0$ and a ξ as above such that φ has the form of (1); namely, simply take $\lambda := \varphi_1 - i \, \varphi_2$ and $\xi := \frac{\varphi_3}{\lambda}$. The map φ as in (1) is called the (local) **complex Gauss map** of our minimal surface with respect to the given isothermal coordinates.

We will show later (Lemma 1.3) that, by slightly rotating the minimal surface, we can always achieve $\varphi_1 \neq i \, \varphi_2$, at least locally. Summarizing, we thus have the following result:

Locally and up to isometry, every minimal surface can be parametrized with respect to isothermal coordinates $(x \, , y)$ in the following way:

$$\begin{aligned} f_1(x \, , y) &= \sqrt{2} \ \mathrm{Re} \int \lambda \, \frac{1-\xi^2}{2} \, dz \\ f_2(x \, , y) &= \sqrt{2} \ \mathrm{Re} \int i \, \lambda \, \frac{1+\xi^2}{2} \, dz \\ f_3(x \, , y) &= \sqrt{2} \ \mathrm{Re} \int \lambda \, \xi \, dz \end{aligned}$$

where $z = x + i \, y$, and λ and ξ are certain holomorphic functions in z.

Conversely, it is not hard to show that, for any two holomorphic functions ξ and λ (the latter nowhere zero), the above formulas give a minimal surface in Euclidean 3–space.

This parametrization is called a **Weierstrass representation** of the minimal surface. In fact, one does not have to rotate the surface to avoid points z

where we might have $\lambda(z) = 0$, if we allow ξ to be *meromorphic*, with poles precisely at the points where λ has zeroes, and their order being exactly half of the order of the zeroes of λ (see [S], page 395). However, we will later avoid this more general description.

Another phenomenon of minimal surfaces in Euclidean 3–space is that they always allow so–called *associated families*, which are also called "isometric deformations". These are one–parameter families of minimal isometric immersions from some two–dimensional parameter manifold into $\mathbf{R}^3$ that are not congruent to each other, but all have the same (real) Gauss map. Using Weierstrass representations, they are extremely easy to describe. We have that if F is a holomorphic representative of a minimal isometric immersion f with respect to an isothermal coordinate system (as desribed above), then the associated family $\{f_\theta \,|\, \theta \in \mathbf{R}\}$ to $f = f_0$ is given by

$$f_\theta = \sqrt{2}\,\mathrm{Re}(e^{i\theta} F)\,.$$

The classical example of such an associated family is the isometric deformation of the helicoid into the catenoid. (For a nice picture of this deformation see the June/July 1999 issue of the Notices of the American Mathematical Society, Volume 46, Number 6, page 649.)

The associated family $\{f_\theta \,|\, \theta \in \mathbf{R}\}$ of a minimal isometric immersion $f : U \to \mathbf{R}^3$ also gives a very simple way to express a holomorphic representative F of f without reference to an isothermal coordinate system; namely, we can write

$$F = \frac{1}{\sqrt{2}}\,(f \oplus f_{-\pi/2}) \cong \frac{1}{\sqrt{2}}\,(f + i\, f_{-\pi/2})\,,$$

where we identify $\mathbf{R}^3 \oplus \mathbf{R}^3 \cong \mathbf{C}^3$, with respect to the standard complex structure $J(u\,,\,v) = (-v\,,\,u)$ on $\mathbf{R}^3 \oplus \mathbf{R}^3$; see e.g. [L], page 143[1].

As mentioned above, associated families of minimal surfaces are the classical counterexamples to the fact that the Gauss map of an isometric immersion $f : M \to \mathbf{R}^N$ (i.e. the map that assigns to each point $p \in M$ the

[1]There, F is defined to be $\frac{1}{\sqrt{2}}\,(f_{\pi/2} \oplus f) \cong \frac{1}{\sqrt{2}}\,(f_{\pi/2} + i\, f)$, which means that we would have $f = \sqrt{2}\,\mathrm{Im}(F)$. This differs from our holomorphic representative simply by a multiplicative factor of i.

image of its tangent space f_*T_pM in Euclidean space) does, in general, *not* determine its image $f(M)$ up to congruence. In 1985, M. Dajczer and D. Gromoll asked the question if there are other examples of this kind. In [D-G$_1$], they prove that there is, in fact, a wider class of isometric immersions which display this behavior, namely **circular Kähler manifolds**, i.e. isometric immersions $f : M \to Q^N$ from a Kähler manifold M into a space of constant curvature whose second fundamental form α satisfies

$$\alpha(JX\, ,\, Y) = \alpha(X\, ,\, JY)$$

for all vector fields X and Y on M, J being the complex structure on M. It is easy to see that "f circular" always implies "f minimal", which in general means that the second fundamental form α of f has vanishing trace, i.e.

$$\mathrm{tr}(\alpha) := \sum_{j=1}^{n} \alpha(X_j\, ,\, X_j) = 0$$

for every orthonormal basis frame $X_1, \ldots, X_n$ on M. Dajczer and Gromoll show that circular immersions always allow associated families, defined at least on any simply connected open subset of M. More explicitly, we have that for any fixed point $p_0 \in M$, f_θ is given by the line integral

$$f_\theta(p) = \int_{p_0}^{p} f_* \circ J_\theta\ ,$$

where $J_\theta := \cos\theta\ I + \sin\theta\ J$ (I being the identity tensor on TM; see [D-G$_1$], formula (1.15) on page 17). Note that, in particular, we have that

$$(f_{-\pi/2})_* = -f_* \circ J\ . \qquad (2)$$

Then Dajczer and Gromoll are able to prove that essentially all local examples of non–congruent isometric immersions with the same Gauss map are of this kind.

In this dissertation, we will focus on the Euclidean case, i.e. on so–called **minimal real Kähler submanifolds**. These are minimal isometric immersions $f : M \to \mathbf{R}^N$ from a Kähler manifold into an Euclidean space. As M. Dajczer and L. Rodríguez show in [D-R$_1$], for these immersions "circular" and "minimal" mean exactly the same thing, whereas for immersions into spaces of constant, non–zero curvature, "circular" is far more restrictive than "minimal"; namely, M has to be a *surface* for f to be circular in this case (see Proposition 1.8 on page 16 in [D-G$_1$]).

In even codimension, it is particularly easy to find examples for minimal real Kähler submanifolds. As shown in [D_2], page 139, every *holomorphic* isometric immersion $f : M^{2n} \to \mathbf{C}^N$ from a Kähler manifold M (of *complex* dimension n) into a complex vector space $\mathbf{C}^N$ will become a minimal real Kähler submanifold, if we view $\mathbf{C}^N \cong \mathbf{R}^{2N}$ as Euclidean $2N$–space. This means, of course, that f will have even *real* codimension $2(N-n)$. But one can also find minimal real Kähler manifolds in odd codimensions; in fact, in the same article [D-G_1] mentioned above, Dajczer and Gromoll classify all real Kähler *hypersurfaces*, i.e. in *real* codimension one, minimal and non–minimal. For more on real Kähler hypersurfaces, see Chapter 2.

Allowing associated families is not the only phenomenon that minimal real Kähler submanifolds and minimal surfaces in Euclidean 3–space have in common. As with minimal surfaces, minimal real Kähler immersions $f : M \to \mathbf{R}^N$ always have (local) holomorphic representatives $F : U \to \mathbf{C}^N$, where U is a suitable, usually simply connected open subset of M. They can be defined using the associated families $\{f_\theta : U \to \mathbf{R}^N \,|\, \theta \in \mathbf{R}\}$ of f discussed above, namely:

$$F = \frac{1}{\sqrt{2}}\,(f \oplus f_{-\pi/2}) \cong \frac{1}{\sqrt{2}}\,(f + i\, f_{-\pi/2}) \qquad (3)$$

(see e.g. [D-G_1], formula (1.17)[2]). Again we identify $\mathbf{R}^N \oplus \mathbf{R}^N \cong \mathbf{C}^N$, with respect to the standard complex structure $J(u\,,\,v) = (-v\,,\,u)$ on $\mathbf{R}^N \oplus \mathbf{R}^N$. And in fact, the analogy goes further. As with minimal surfaces, the complex Gauss map for a minimal real Kähler submanifold will be a *holomorphic, isotropic map*; we will clarify in Theorem 1.1 below exactly what we mean by that. This will, at least in principle, allow us to find "Weierstrass representations" for minimal real Kähler submanifolds in general. In [D-G_2] for instance, Dajczer and Gromoll used such representations to describe the structure of *complete* minimal real Kähler submanifolds in codimension two. (For more on this case see the remarks below, and also Chapter 2.)

[2]Note: In most articles on these topics, we find that F is taken to be $\frac{1}{\sqrt{2}}\,(f + i\, f_{\pi/2})$. However, the image of the (complex) Jacobian F_* of F would then consist of *anti*–holomorphic vectors, whereas we want to work with holomorphic vectors here. Compare the remarks and the footnote after Theorem 1.1.

In the years after 1985, several articles were published concerning the structure of real Kähler submanifolds, in particular in low codimensions. Many of these results rely on the fact that, unless the submanifold is the image of a holomorphic map as described above (with respect to some complex structure on the Euclidean space), it usually has "plenty of (relative) nullity". Recall that the (relative) nullity space $\triangle_p$ of an isometric immersion $f : M \to \mathbf{R}^N$ at a point $p \in M$ is the degeneracy space of the second fundamental form α of M in the tangent space T_pM at p:

$$\triangle_p = \{v \in T_pM \,|\, \alpha(v, ...) = 0\} \,.$$

Its dimension is called the *index of (relative) nullity*: $\nu_f(p) = \dim\triangle_p$. This index is locally constant, and on every open subset U of M where it is constant, $\{\triangle_p \,|\, p \in U\}$ is a subbundle of TM. Moreover, it is well-known that on such an open set U, $\triangle$ forms an involutive distribution whose leaves are totally geodesic submanifolds of M, and furthermore that f maps these leaves into affine subspaces of $\mathbf{R}^N$; see e.g. [D_2], pages 67 to 70. For real Kähler submanifolds, $\triangle$ is often rather high-dimensional, if the map is not already holomorphic in the above mentioned sense. For example, Takahashi showed in [T] (see also [A]) that for a (not necessarily minimal) hypersurface immersion $f : M^{2n} \to \mathbf{R}^{2n+1}$ of a Kähler manifold M, we must have that $\nu_p(f) \geq 2n-2$ for all $p \in M$, which is as large as possible if f is not flat at p.

Perhaps the strongest result of this kind for codimension two was published by Dajczer in [D_1], where he uses the theory of flat bilinear forms developed by Moore (see [M]) to prove that if the nullity of a (not necessarily minimal) isometric immersion $f : M^{2n} \to \mathbf{R}^{2n+2}$ from a Kähler manifold M into Euclidean $(2n+2)$–space is everywhere less than $2n-4$, then f must be holomorphic with respect to some complex structure on $\mathbf{R}^{2n+2}$. This means that if such an immersion is non–holomorphic, then it must locally be an affine vectorbundle of rank at least $2n-4$ over an at most four–dimensional Kähler submanifold of M.

Shortly afterwards, Dajczer and Rodríguez where able to analyze the structure of such isometric immersions with $\nu_p(f) \geq 2n-4$ for all $p \in M$ in codimension two, given that they are *minimal* and that the underlying Kähler manifold M is *complete*; see [D-R_2]. The key to their result is that for *complete* M, one can always find one more "complex direction" in which M is ruled. This means that a complete isometric immersion

$f : M^{2n} \to \mathbf{R}^{2n+2}$ will either stem from a holomorphic map; or that f is a *cylinder*, i.e. $f = f_1 \times \mathrm{id}_{\mathbf{R}^{2n-4}}$, where $f_1 : \tilde{M}^4 \to \mathbf{R}^6$ is a minimal isometric immersion from a Kähler submanifold $\tilde{M}$ of M into Euclidean 6–space; or that f is *completely complex ruled*, i.e. M is an affine vectorbundle of rank $2n-2$ over a two–dimensional Kähler submanifold $\tilde{M}$ of M, and f maps the rulings into affine subspaces of $\mathbf{R}^{2n+2}$. But then the image of the submanifold $\tilde{M}$ of M under f is nothing but a minimal surface in $\mathbf{R}^{2n+2}$, and as Harvey and Osserman demonstrated in [H-O], most results concerning the structure of minimal surfaces in 3–spaces can be generalized verbatim to those in an arbitrary Euclidean space of dimension larger than 3; see also [L]. Finally, in [D-G_2] Dajczer and Gromoll use this idea to give Weierstrass representations for all minimal isometric immersions $f : M^{2n} \to \mathbf{R}^{2n+2}$ where M is *complete* and f is irreducible (i.e. not a cylinder) and non–holomorphic. Their method allows the explicit construction of examples for such immersions.

However, some questions remain open. For instance, it is not clear how non–holomorphic isometric immersions $f : M^{2n} \to \mathbf{R}^{2n+2}$ may look *locally* if M is *not complete*. And our nullity condition $\nu_p(f) \geq 2n-4$ tells us nothing if M is *four*–dimensional ($n = 2$), i.e. in the case of a minimal isometric immersion $f : M^4 \to \mathbf{R}^6$. In fact, to the knowledge of the author, only a few examples in this case were known up to this point; see [F] and [D-G_3]. Very recently, Arezzo, Pirola, and Solci were able to give entire series of examples (see [A-P-S]). But a classification of those submanifolds had not been established until this time.

The main goal of this dissertation is exactly to give a *complete local classification of four–dimensional, minimal real Kähler submanifolds in codimension two*, at least away from certain isolated "singularities" in the manifold. This will be established in Chapter 4, where we utilize a parametrization for two–dimensional isotropic subspaces in $\mathbf{C}^N$ that is based on "Weierstrass formulas" very similar to the one in (1). The latter will be developed in Chapter 3. Chapter 2 contains an "addendum" to Dajczer and Gromoll's work [D-G_2], namely that the methods developed in this article can be used almost verbatim to explicitly construct minimal Kähler hypersurfaces. One interesting consequence of this is that we will be able to explicitly write down formulas for so–called "superminimal surfaces" in Euclidean spheres, a topic that was first studied by E. Calabi in 1968 (see [C]) and is still an active area of research in algebraic geometry.

The following theorem contains the clarification promised on page 5, and is the backbone of the Weierstrass representation for minimal real Kähler submanifolds. It is well–known in the literature and can be proven in a straightforward (if tedious) fashion, e.g. by expressing all given conditions in a complex chart of the Kähler manifold. In their recent article mentioned above, Arezzo, Pirola, and Solci have given a very elegant proof of this theorem, using differential forms (see [A-P-S]).

Theorem 1.1: *Let $f : M^{2n} \to \mathbf{R}^N$ be a minimal isometric immersion from a Kähler manifold M into Euclidean N–space. Furthermore, let $(z_1, \ldots, z_n)$ be a complex chart of M on some open (and without loss of generality simply connected) subset U of M, and define the maps $\varphi_j : U \to \mathbf{C}^N$ for $j = 1, \ldots, n$ by*

$$\varphi_j := \sqrt{2}\,\frac{\partial f}{\partial z_j} = \frac{1}{\sqrt{2}}\left(\frac{\partial f}{\partial x_j} - i\,\frac{\partial f}{\partial y_j}\right),$$

where $z_j = x_j + i\,y_j$. Then these φ_j satisfy the following conditions:

(a) *For each point $p \in U$, the vectors $\varphi_1(p), \ldots, \varphi_n(p)$ are linearly independent in $\mathbf{C}^N$;*

(b) *φ_j is holomorphic for $j = 1, \ldots, n$;*

(c) span$\{\varphi_1, \ldots, \varphi_n\}$ *is an isotropic subspace of $\mathbf{C}^N$, i.e.*

$$\varphi_j \cdot \varphi_k = 0 \quad \textit{for all } 1 \le j\,,\, k \le n\,,$$

where "$\cdot$" is the standard symmetric *inner product in $\mathbf{C}^N$:*

$$(v_1, \ldots, v_N) \cdot (w_1, \ldots, w_N) := \sum_{j=1}^{N} v_j\, w_j\,;$$

(d) $\dfrac{\partial \varphi_j}{\partial z_k} = \dfrac{\partial \varphi_k}{\partial z_j}$ *for all $1 \le j\,,\, k \le n$ ("Integrability Conditions").*

Furthermore, if $F : U \to \mathbf{C}^N$ is a holomorphic representative of f on U as described in (3)*, then we have*

$$\frac{\partial F}{\partial z_j} = \varphi_j \quad \textit{for all } j = 1, \ldots, n\,. \tag{4}$$

Conversely, let U *be a simply connected open subset of* $\mathbf{C}^n$, *and* $\varphi_1, \ldots, \varphi_n : U \to \mathbf{C}^N$ *be maps that satisfy conditions* (a) *through* (d) *as above. Then there is a holomorphic map* $F : U \to \mathbf{C}^N$ *such that* (4) *is satisfied, and if* $f : M \to \mathbf{R}^N$ *is defined by*

$$f := \sqrt{2}\,\mathrm{Re}(F)\ ,$$

then $M := (U\, ,\, f^* <,>)$ *is a Kähler manifold and* f *is a minimal isometric immersion from* M *into Euclidean* N*–space whose holomorphic representative is* F.

The map $\varphi := \mathrm{span}\{\varphi_1, \ldots, \varphi_n\}$ from U into the complex Grassmannian $\mathrm{Gr}_n(\mathbf{C}^N)$ of all complex n–spaces in $\mathbf{C}^N$ is more correctly what one calls the **complex Gauss map**[3] of the Kähler manifold M over U. It is independent of the choice of the complex chart of M, and thus a holomorphic map on all of M. Note that we always have

$$\varphi(p) = F_* T_pM = f_* T_pM^{(1,0)}\ , \tag{5}$$

since $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a basis for the subspace of all *holomorphic* vectors in $(T_pM)^{\mathbf{C}} = T_pM \otimes_{\mathbf{R}} \mathbf{C}$. Therefore, (a) means exactly that the holomorphic representative F of our immersion f is itself immersive, regarded as a *holomorphic* map between *complex* manifolds.

To see how the factor $\sqrt{2}$ behaves in these formulas, let us briefly check (4). Using (2) and (3), we find

$$\begin{aligned}
\frac{\partial F}{\partial z_j} &= \frac{1}{\sqrt{2}} \left(\frac{\partial f}{\partial z_j} + i\,\frac{\partial f_{-\pi/2}}{\partial z_j} \right) \\
&= \frac{1}{2\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\,\frac{\partial f}{\partial y_j} - i\,f_* \left(J\frac{\partial}{\partial x_j} - i\,J\frac{\partial}{\partial y_j} \right) \right) \\
&= \frac{1}{2\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\,\frac{\partial f}{\partial y_j} - i\,\frac{\partial f}{\partial y_j} - i\left(i\,\frac{\partial f}{\partial x_j} \right) \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\,\frac{\partial f}{\partial y_j} \right) = \varphi_j\ .
\end{aligned}$$

[3]Again, usually one looks at the *antiholomorphic* Gauss map $\varphi = \mathrm{span}\{\frac{\partial F}{\partial \bar{z}_1}, \ldots, \frac{\partial F}{\partial \bar{z}_n}\} = f_* T_pM^{(0,1)}$ here. But since we will later work with this map quite extensively, we will prefer having a *holomorphic* Gauss map.

We will use Theorem 1.1 shortly to give simple examples of how to construct minimal real Kähler submanifolds. However, before we do this, we want to investigate isotropic subspaces in $\mathbf{C}^N$ a little closer.

First, in general we say that a subvectorspace V of $\mathbf{C}^N$ is **isotropic** if we have $V^2 := V \cdot V = \{0\}$, which, of course, means that

$$v \cdot w = \sum_{j=1}^{N} v_j \, w_j = 0$$

for all $v = (v_1, \ldots, v_N)$, $w = (w_1, \ldots, w_N) \in V$. Geometrically, the most important property of isotropic subspaces in $\mathbf{C}^N$ is that they are exactly the ones that "stem from orthonormal systems in $\mathbf{R}^N$". More exactly, we have:

Lemma 1.2: *Let V be an n–dimensional complex subvectorspace of $\mathbf{C}^N$. Then we have that*

$$Z_1 = \frac{1}{\sqrt{2}}(X_1 + i\,Y_1), \ldots, Z_n = \frac{1}{\sqrt{2}}(X_n + i\,Y_n)$$

(with X_j, $Y_j \in \mathbf{R}^N$) is an isotropic, *Hermitean orthonormal basis of V if and only if $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ is a Euclidean orthonormal system in $\mathbf{R}^N$.*

Note that this lemma immediately implies that the maximal dimension of an isotropic subspace of $\mathbf{C}^N$ is $[\frac{N}{2}]$, where $[x]$ is the largest integer less than or equal to x.

Proof: If $Z_1, \ldots, Z_n$ is an Hermitean orthonormal system, and if we denote the regular Hermitean product on $\mathbf{C}^N$ by $<.,.>$, we have that

$$\begin{aligned} \delta_{jk} &= <Z_j, Z_k> = Z_j \cdot \overline{Z_k} \\ &= \tfrac{1}{2}(<X_j, X_k> + <Y_j, Y_k>) + \tfrac{i}{2}(-<X_j, Y_k> + <Y_j, X_k>)\,. \end{aligned}$$

On the other hand, if V is isotropic, we have

$$0 = Z_j \cdot Z_k = \tfrac{1}{2}(<X_j, X_k> - <Y_j, Y_k>) + \tfrac{i}{2}(<X_j, Y_k> + <Y_j, X_k>)\,.$$

Adding these equations and then separating real from imaginary parts gives $<X_j, X_k> = \delta_{jk}$ and $<Y_j, X_k> = 0$, whereas subtracting leads to $<Y_j, Y_k> = \delta_{jk}$ and $<X_j, Y_k> = 0$, i.e $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ is an orthonormal system in $\mathbf{R}^N$. The converse is clear from the equations above.

Another fact that we will need later is that congruent minimal real Kähler immersions have congruent holomorphic representatives:

Lemma 1.3: *Let f and $\tilde{f} : M^n \to \mathbf{R}^N$ be two minimal real Kähler immersions from M into Euclidean N-space, and let F and $\tilde{F} : U \to \mathbf{C}^N$ be (local) holomorphic representatives of f and $\tilde{f}$, respectively. Assume that f and $\tilde{f}$ are congruent, i.e. $\tilde{f} = A \circ f + b$ for an $A \in O(N)$ and $b \in \mathbf{R}^N$. Then we have $\tilde{F} = A \circ F + c$, where we view A as a complex $N \times N$-matrix, and where $c \in \mathbf{C}^N$.*

Proof: Obviously, it suffices to show that $\tilde{F}_* = A \circ F_*$. By (3), we have that $F = \frac{1}{\sqrt{2}} \, (f + i\, f_{-\pi/2})$, and thus by (2) that $F_* = \frac{1}{\sqrt{2}} \, (f_* - i\, f_* \circ J)$, and analogously for $\tilde{F}$. But since $\tilde{f}_* = A \circ f_*$, we have

$$\tilde{F}_* = \frac{1}{\sqrt{2}} \left(\tilde{f}_* - i\, \tilde{f}_* \circ J\right) = A \circ F_* \,.$$

We will end this chapter with the promised simple examples of how we can use Theorem 1.1 to construct minimal real Kähler submanifolds.

Example 1.4: For some fixed integers $m > n \geq 1$ and $N \geq 2m$, choose a basis $\mathbf{X}_1, \ldots, \mathbf{X}_m$ of some *fixed* m-dimensional isotropic subspace of $\mathbf{C}^N$. Let $\mathbf{C}_0 \in \mathbf{C}^N$ be another constant vector. For $j = n+1, \ldots, m$, let $g_j(z_1, \ldots, z_n) : U \to \mathbf{C}$ be some holomorphic function in the n complex variables $z_1, \ldots, z_n$, defined on some open subset U of $\mathbf{C}^n$, and define the map $F : U \to \mathbf{C}^N$ by

$$F(z_1, \ldots, z_n) := \sum_{k=1}^{n} z_k \mathbf{X}_k \; + \sum_{k=n+1}^{m} g_k(z_1, \ldots, z_n)\, \mathbf{X}_k \; + \; \mathbf{C}_0 \,. \qquad (6)$$

Then we have that for $j = 1, \ldots, n$,

$$\frac{\partial F}{\partial z_j} = \mathbf{X}_j + \sum_{k=n+1}^{m} \frac{\partial g_k}{\partial z_j}(z_1, \ldots, z_n)\, \mathbf{X}_k \,,$$

and it is easy to see that the $\varphi_j := \frac{\partial F}{\partial z_j}$ indeed satisfy conditions (a) through (d) in Theorem 1.1. Therefore, $f := \sqrt{2}\,\mathrm{Re}(F) : M \to \mathbf{R}^N$, where the manifold $M := (U, f^*\langle\,,\rangle)$ has *real* dimension $2n$, is a minimal real Kähler immersion. Since we can view F as the "graph" of the holomorphic map $(g_{n+1}, \dots, g_m)$ "with respect to the isotropic basis $\mathbf{X}_1, \dots, \mathbf{X}_m$", we will say that such an immersion f is **generated by an isotropic graph**.

It is interesting to note that in *even codimensions*, we have basically only rediscovered the examples for minimal real Kähler submanifolds that we encountered on page 5, since we have the following

Proposition 1.5: *Let M^{2n} be a Kähler manifold, and $f : M \to \mathbf{R}^{2N}$ be an isometric immersion that is also a holomorphic map with respect to some complex structure on $\mathbf{R}^{2N}$ (so by the remarks on page 5: f is, in particular, a minimal immersion). Then for each point in M, there is a neighborhood U of this point and a complex chart $(z_1, \dots, z_n)$ on U such that with respect to this chart, $f|_U$ is generated by an isotropic graph.*

Conversely, every minimal real Kähler immersion $f : M^{2n} \to \mathbf{R}^{2N}$ that is generated by an isotropic graph in even codimension *is holomorphic with respect to a suitably chosen complex structure J on $\mathbf{R}^{2N}$.*

Proof: The main tool of this proof is a fact that Calabi discovered first for minimal surfaces in even–dimensional Euclidean spaces (see [C], or [L], page 165), and that Rigoli and Tribuzy generalized for minimal real Kähler immersions (see [R-T], Theorem 4 on page 517)[4]:

A minimal isometric immersion $f : M^{2n} \to \mathbf{R}^{2N}$ is holomorphic with respect to some complex structure J on $\mathbf{R}^{2N}$ if and only if there is a fixed *isotropic subspace V of $\mathbf{C}^{2N}$ such that the image of the complex Gauss map of f is contained in V, i.e.*

$$f_* T_p M^{(1,0)} \subset V \quad \textit{for all } p \in M .$$

Now assume that f is holomorphic with respect to some complex structure J on $\mathbf{R}^{2N}$. Let V be the isotropic subspace of $\mathbf{C}^N$ mentioned above, and let m denote the (complex) dimension of V (so: $n \leq m \leq N$). Taking an Hermitean orthonormal basis $\mathbf{X}_1, \dots, \mathbf{X}_m$ of V, we obviously can write any holomorphic representative $F : W \to \mathbf{C}^{2N}$ of f in any given complex coordinate system $(\tilde{z}_1, \dots, \tilde{z}_n)$ on $W \subset M$ as

[4] In their article, Rigoli and Tribuzy work with the *antiholomorphic* Gauss map $f_* T_p M^{(0,1)}$, so their theorem had to be slightly adapted here

$$F(\tilde{z}_1, \ldots, \tilde{z}_n) = \sum_{j=1}^{m} \tilde{g}_j(\tilde{z}_1, \ldots, \tilde{z}_n)\, \mathbf{X}_j + F(p_0)\,, \tag{7}$$

where the $\tilde{g}_j$ are certain holomorphic functions defined on W, and where p_0 is some fixed point in W. But since F is immersive (see (5)), the rank of the map $\tilde{G} := (\tilde{g}_1, \ldots, \tilde{g}_m) : W \to \mathbf{C}^m$ is also equal to n everywhere. Given a fixed point p in W, we can thus find indices $j_1, \ldots, j_n \in \{1, \ldots, m\}$ such that the $n \times n$–matrix

$$\left(\frac{\partial\, \tilde{g}_{j_k}}{\partial\, \tilde{z}_j}(p)\right)_{j,\, k=1,\ldots,n}$$

has rank n. Assume without loss of generality that $j_1 = 1, \ldots, j_n = n$. Now set

$$\Phi := (\tilde{g}_1, \ldots, \tilde{g}_n)^{-1}\,,$$

and let the domain of Φ be some open set $\tilde{U} \subset \mathbf{C}^n$ that Φ maps biholomorphically onto some open neighborhood U of p in W (the existence of such an open set is guaranteed by the "holomorphic" Inverse Mapping Theorem; see e.g. [G-R], page 17). Thus, $\Phi^{-1} : U \to \tilde{U}$ is a complex chart for M, and changing coordinates to $(z_1, \ldots, z_n) \in \tilde{U}$, we see that

$$\begin{aligned} \tilde{G} \circ \Phi(z_1, \ldots, z_n) &= (\tilde{g}_1(\Phi(z_1, \ldots, z_n)), \ldots, (\tilde{g}_m(\Phi(z_1, \ldots, z_n)) \\ &= (z_1, \ldots, z_n, g_{n+1}(z_1, \ldots, z_n), \ldots, g_m(z_1, \ldots, z_n))\,, \end{aligned}$$

for the holomorphic functions $g_j := \tilde{g}_j \circ \Phi$ $(j = n+1, \ldots, m)$. Inserting them into (7), we see that in the "complex coordinate system" $(z_1, \ldots, z_n)$, the holomorphic representative of f has the same form as in (6), which proves the first part of our claim.

Conversely, if f is given as in Example 1.4, then the complex subspace $V := \mathrm{span}\{\mathbf{X}_1, \ldots, \mathbf{X}_m\}$ of $\mathbf{C}^{2N}$ is isotropic, and by (5) and (6) it is clear that the image of the Gauss map of f always lies in V. Thus f is holomorphic with respect to some complex structure on $\mathbf{R}^{2N}$ by the criterion given above.

Another way to construct simple examples is to look at "isotropic cylinders" in the following sense:

Example 1.6: Take a *fixed* isotropic subspace V of $\mathbf{C}^N$ of dimension $n < [\frac{N}{2}]$. Further, take a basis $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of V. Denote the "isotropic orthogonal complement" of V by $V^{\perp^\bullet}$, where $A \perp^\bullet B$ for A, $B \in \mathbf{C}^N$ means by definition that $A \cdot B = 0$ with respect to the standard symmetric inner product in $\mathbf{C}^N$. Note that it is clear from the general theory of bilinear forms that since "$\cdot$" is *non–degenerate*, the dimension of $V^{\perp^\bullet}$ is $N - n$. However, since V is supposed to be isotropic, in our case we always have that $V \subset V^{\perp^\bullet}$.

Now choose any complement W of V in $V^{\perp^\bullet}$, so that $V^{\perp^\bullet} = V \oplus W$, and take any nowhere zero *isotropic* map $\mathbf{Z} : U \to W$ which is holomorphic in the one complex variable $w \in U \subset \mathbf{C}$. (It is not hard to show that any subspace of $\mathbf{C}^N$ of dimension at least two contains non-zero isotropic vectors, and that by our assumption on the dimension of V, $\dim W$ has to be at least two.) Then it is easy to check that the partial derivatives $\varphi_j := \frac{\partial F}{\partial z_j}$ and $\varphi_{n+1} := \frac{\partial F}{\partial w}$ of the map

$$F(z_1, \ldots, z_n, w) := \sum_{j=1}^{n} z_j \mathbf{X}_j \; + \; \int \mathbf{Z}(w)\, dw$$

satisfy conditions (a) through (d) in Theorem 1.1, and thus that the map $f := \sqrt{2}\,\mathrm{Re}(F) : M := (\mathbf{C}^n \times U\,,\, f^* <,>) \to \mathbf{R}^N$ is a minimal real Kähler immersion. Because F has a similar structure as a *cylinder* in a Euclidean space (where all coordinates and functions would be real), we will say that such an immersion f is **generated by an isotropic cylinder.**

Note that since we cannot assume that the $\mathbf{X}_j$ and $\mathbf{Z}$ are always contained in a *fixed* isotropic subspace of $\mathbf{C}^N$, we may very well obtain *non–holomorphic* immersions here (see the criterion in the proof of Proposition 1.5). For the same reason, Lemma 1.2 implies that we also cannot expect that f itself will split as a cylinder in $\mathbf{R}^N$.

The real reason why we introduce Examples 1.4 and 1.6 is that they will later reemerge naturally as special cases in our classification of four–dimensional minimal real Kähler submanifolds of $\mathbf{R}^6$.

2 Minimal real Kähler hypersurfaces

Locally, any real Kähler hypersurface in $\mathbf{R}^{2n+1}$ which is nowhere flat can be described through its *Gauss parametrization* $\psi : \Lambda \to \mathbf{R}^{2n+1}$, given by

$$\psi(z\,,\,w) = (h\cdot g + \nabla^V h)(z) + w\;, \tag{1}$$

where Λ is the normal bundle along a pseudoholomorphic isometric immersion $g : V^2 \to S^{2n}$ from a surface V into the $2n$–sphere, $h : V \to \mathbf{R}$ some support function on V, and $\nabla^V h$ its gradient in V (see [D-G$_1$], formula (2.4) and Theorem 2.5). The reason that we can find such a parametrization is that for every such hypersurface which is nowhere flat, the index of relative nullity must be constant and equal to $2n-2$ (as we mentioned on page 6), or – which is the same – that its Gauss map must have constant rank 2. In fact, V is nothing but the "Gauss image" of our hypersurface. If the hypersurface is to be *minimal*, we must require that, in addition, h is an eigenfunction of the Laplacian of V for the eigenvalue -2 ([D-G$_1$], Corollary 2.6):

$$\psi \text{ minimal } \iff \Delta^V h + 2\,h = 0 \text{ in } V.$$

Of course, it is neither very easy to find pseudoholomorphic surfaces in S^{2n} – which were first studied by Calabi [C] – nor to determine the eigenfunctions of the Laplacian of V. So the question arose if one could use techniques analogous to those in [D-G$_2$] to find examples of minimal real Kähler hypersurfaces via the Weierstrass representation, similar to the classical case of minimal surfaces in $\mathbf{R}^3$. In fact, this is easily possible by only slightly modifying the methods in [D-G$_2$], as we will now show.

Let $f : M^{2n} \to \mathbf{R}^{2n+1}$ be a minimal isometric immersion of a Kähler manifold M into $\mathbf{R}^{2n+1}$, which is also assumed to be nowhere flat. As mentioned above, the relative nullity bundle Δ of f must have rank $2n-2$ everywhere. Since this bundle is the kernel of the Jacobian of the (for us) holomorphic complex Gauss map of f (see page 9), it is in fact a holomorphic subbundle of the tangent bundle TM of M (compare [D-G$_2$], page 240). As mentioned on page 6, each leaf of Δ in M is totally geodesic, and furthermore, f maps any such leaf into an affine subspace of $\mathbf{R}^{2n+1}$.

Now, let $F : M \to \mathbf{C}^{2n+1}$ be a holomorphic representative of f as in formula (3) in Chapter 1 (so we either assume M to be simply connected, or we restrict ourselves to local considerations). The fact that f maps leaves of Δ into affine subspaces of $\mathbf{R}^{2n+1}$ means that F will map these leaves into complex $(n-1)$–dimensional subspaces of $\mathbf{C}^{2n+1}$ and will, thus, be *holomorphically ruled.* This implies that locally we can find complex coordinates $(z, w_1, \ldots, w_{n-1})$ on some open subset $U \times W \subset \mathbf{C} \times \mathbf{C}^{n-1}$ such that

$$\frac{\partial^2 F}{\partial w_j \, \partial w_k} = 0\,, \quad 1 \leq j\,, k \leq n-1$$

(recall that, by (4), $\frac{\partial F}{\partial z_j} = \sqrt{2}\,\frac{\partial f}{\partial z_j}$ for each complex coordinate system $(z_1, \ldots, z_n)$ on M). Furthermore, we know from Theorem 1.1 that the Jacobian of F spans an isotropic subspace in $\mathbf{C}^{2n+1}$, i.e.

$$\frac{\partial F}{\partial z_j} \cdot \frac{\partial F}{\partial z_k} = 0\,, \quad 1 \leq j\,, k \leq n\,,$$

for each complex coordinate system on M. As in [D-G$_2$], we see that this means that the Jacobian of F must be a $(2n+1) \times n$ matrix of rank n that has the form

$$F_* = \left(\beta(z) + \sum_{j=1}^{n-1} w_j\,\gamma_j'(z)\,,\ \gamma_1(z)\,, \ldots, \gamma_{n-1}(z)\right)\,, \tag{2}$$

where $\gamma'(z) = \frac{\partial \gamma}{\partial z}$, and where $\beta\,, \gamma_j : U \to \mathbf{C}^{2n+1}$ are holomorphic in z and satisfy the additional constraint that

$$\operatorname{span}\{\beta\,, \gamma_1\,, \gamma_1'\,, \ldots, \gamma_{n-1}\,, \gamma_{n-1}'\} \text{ is isotropic}\,. \tag{3}$$

Now, as noted on page 10, the largest possible dimension of an isotropic subspace in $\mathbf{C}^{2n+1}$ is n, and by (2) and (3) the holomorphic subbundle

$$E := \operatorname{span}\{\gamma_1\,, \ldots, \gamma_{n-1}\} \subset U \times \mathbf{C}^{2n+1}$$

is isotropic and already has rank $n-1$. This means that its osculating bundle $E' := \operatorname{span}\{\gamma_j\,, \gamma_j' \,|\, 1 \leq j \leq n-1\}$, which by (3) is also isotropic, can only have rank $\leq n$. But then Lemma 1 in [D-G$_2$] (page 239) tells us that either E contains a parallel subbundle (which means that f is reducible), or that there is a unique holomorphic line bundle $L \subset E$ such that $L^{(n-2)}$, the $(n-2)^{nd}$ osculating bundle of L, equals E. More explicitly, if in the latter case $\gamma : U \to \mathbf{C}^{2n+1}$ is a nowhere zero section in L, then

$$E = L^{(n-2)} = \operatorname{span}\{\gamma(z)\,, \gamma'(z)\,, \ldots, \gamma^{(n-2)}\}\,. \tag{4}$$

From now on we will assume that f is *irreducible*, so in particular we have (4). Note that, in this case, we can conclude further that rank $E' = n$, and we have

$$E' = L^{(n-1)} = \operatorname{span}\{\gamma(z)\,,\, \gamma'(z)\,, \ldots,\, \gamma^{(n-1)}\}\,.$$

Thus, (3) gives us that we always have $\beta \in E'$; i.e. we can always find holomorphic functions $b_0\,, \ldots,\, b_{n-1} : U \to \mathbf{C}$ such that

$$\beta(z) = \sum_{j=0}^{n-1} b_j(z)\, \gamma^{(j)}(z)\,, \tag{5}$$

where b_{n-1} is never 0. (Note: By [D-G$_2$] (9), this shows that an irreducible f can never be *completely* ruled, so in particular that *an irreducible, nowhere flat minimal real Kähler hypersurface can never be complete.* In fact, this is also true for *non-minimal* Kähler hypersurfaces, and was first proved by Abe [A]; see also Corollary 2.7 in [D-G$_1$]). With this β, (2) now becomes

$$F_* = \left(\sum_{j=0}^{n-1} b_j(z)\, \gamma^{(j)}(z) + \sum_{j=1}^{n-1} w_j\, \gamma^{(j)}(z)\,,\, \gamma(z)\,,\, \gamma'(z)\,, \ldots,\, \gamma^{(n-2)}(z) \right)\,,$$

and we have that

$$F(z\,, w_1\,, \ldots,\, w_{n-1}) = \sum_{j=0}^{n-1} \Big(\int b_j(z)\, \gamma^{(j)}(z)\, dz \Big) \;+\; \sum_{j=1}^{n-1} w_j\, \gamma^{(j-1)}(z)\,, \tag{6}$$

and $f(z\,, w_1\,, \ldots,\, w_{n-1}) = \sqrt{2}\ \mathrm{Re}(F)$. Writing $w_j = u_j + i\, v_j$, we thus obtain

$$\left.\begin{aligned} &\tfrac{1}{\sqrt{2}}\, f(z\,, u_1\,, \ldots,\, u_{n-1}\,, v_1\,, \ldots,\, v_{n-1}) \\ &= \mathrm{Re}\Big(\sum_{j=0}^{n-1} \int b_j(z)\, \gamma^{(j)}(z)\, dz \Big) + \sum_{j=1}^{n-1} \Big(u_j \mathrm{Re}\; \gamma^{(j-1)}(z) - v_j \mathrm{Im}\; \gamma^{(j-1)}(z) \Big)\,. \end{aligned}\right\} \tag{7}$$

This is a (local) **Weierstrass representation** of the minimal real Kähler hypersurface f.

Comparing this parametrization of f with the Gauss parametrization in (1), we see that, in these coordinates, the first term in (7) corresponds to $(h \cdot g + \nabla h)(z)$, and the second term to a normal vector to g in S^{2n}. Thus, if we have a convenient way to find such Weierstrass representations – i.e. maps γ as in (4) – we can also easily find (local) examples for pseudoholomorphic maps $g : U \to S^{2n}$: they are simply the normal vectors to $\text{span}\{\,\text{Re}\,\gamma^{(j)}(z)\,,\,\text{Im}\,\gamma^{(j)}(z)\,|\,0 \le j \le n-1\}$ in $\mathbf{R}^{2n+1}$, viewed as a function of $z = x + i\,y \cong (x\,,\,y)$, which represent *isothermal* coordinates on U (compare page 1).

Now it is in fact rather easy to find such a γ. We only have to mimic the construction on page 237 in [D-G$_2$]. Thus, let U be any simply connected domain in $\mathbf{C}$. Start with any *non-zero* holomorphic function[1] $\alpha_0 : U \to \mathbf{C}$, and let $\phi_0 := \int \alpha_0(z)\,dz$ (or just start with any non-constant holomorphic ϕ_0). Assuming that the maps $\alpha_r\,,\,\phi_r : U \to \mathbf{C}^{2r+1}$ have been defined for some $0 \le r \le n-1$, choose any *nowhere zero* function $\mu_{r+1} : U \to \mathbf{C}$, and set

$$\alpha_{r+1} := \mu_{r+1} \begin{pmatrix} \dfrac{1-\phi_r^2}{2} \\ i\,\dfrac{1+\phi_r^2}{2} \\ \phi_r \end{pmatrix}, \text{ and } \quad \phi_{r+1} := \int \alpha_{r+1}(z)\,dz\,, \qquad (8)$$

where $\phi_r^2 = \phi_r \cdot \phi_r$ with respect to the standard symmetric inner product in $\mathbf{C}^{2r+1}$. Then,

$$\gamma := \alpha_n$$

is the section of L for which we are looking; i.e. if we use it in (7) above, then the so defined f will be a minimal isometric immersion from the Kähler manifold $M := (U \times W,\, f^*\!<.\,,.>)$ into $\mathbf{R}^{2n+1}$, where W is some open subset of the origin in $\mathbf{C}^{n-1}$ and $<.\,,.>$ is the standard Euclidean metric in $\mathbf{R}^{2n+1}$.

The proof that this method works is *exactly* the same as for the case of codimension 2 described in [D-G$_2$]. In fact, it is not hard to show that *every* minimal real Kähler hypersurface *must locally* be of this form (up to renumbering the coordinates in $\mathbf{R}^{2n+1}$).

Note that the first step in (8) is *exactly* the classical way to find minimal surfaces in $\mathbf{R}^3$ via their Weierstrass representation (compage page 2). Thus, if we simply "continue this construction to higher dimensions" (as given in (8)), what we obtain are exactly minimal real Kähler hypersurfaces.

---

[1]The only difference to the procedure in [D-G$_2$] is that there, α_0 is a function with values in $\mathbf{C}^2$. Also, note that if $\alpha_0 \equiv 0$, we could choose all integration constants in (8) to be zero, and the hypersurface would be part of $\mathbf{R}^{2n} \times \{0\}$, and thus flat (and reducible).

Let us now consider two examples, both in the simplest possible case of a minimal real Kähler hypersurface in $\mathbf{R}^5$, so that $n = 2$.

Example 2.1: Start with $\alpha_0 := 1$, and thus $\phi_0 = z$, if we set the integration constant equal to zero. Now choose $\mu_1 := 6$, and obtain

$$\alpha_1 = \frac{\mu_1}{2} \begin{pmatrix} 1 - \phi_0^2 \\ i\,(1 + \phi_0^2) \\ 2\,\phi_0 \end{pmatrix} = \begin{pmatrix} 3 - 3\,z^2 \\ 3\,i\,(1 + z^2) \\ 6\,z \end{pmatrix} ,$$

and by integrating (and setting all integration constants equal to zero),

$$\phi_1 = \int \alpha_1(z)\,dz = \begin{pmatrix} 3\,z - z^3 \\ i\,(3\,z + z^3) \\ 3\,z^2 \end{pmatrix} .$$

For the next step, we need to calculate

$$\phi_1^2 = (3\,z - z^3)^2 + i^2(3\,z + z^3)^2 + (3\,z^2)^2 = -3\,z^4 .$$

Setting $\mu_2 := 2$, we obtain (since here $n = 2$)

$$\gamma = \alpha_2 = \frac{\mu_2}{2} \begin{pmatrix} 1 - \phi_1^2 \\ i\,(1 + \phi_1^2) \\ 2\,\phi_1 \end{pmatrix} = \begin{pmatrix} 1 + 3\,z^4 \\ i\,(1 - 3\,z^4) \\ 6\,z - 2\,z^3 \\ i\,(6\,z + 2\,z^3) \\ 6\,z^2 \end{pmatrix} .$$

If in (6) and (7) we now choose $b_0 := 0$ and $b_1 := 1$, we see that

$$f(z,\,w_1) = \sqrt{2}\ \operatorname{Re}\Big(\int 1 \cdot \gamma'(z)\,dz + w_1 \cdot \gamma(z)\Big) = \sqrt{2}\ \operatorname{Re}((1 + w_1) \cdot \gamma(z)) \quad (9)$$

(setting yet another integration constant equal to zero), and if we write $z = x + i\,y$ and $w_1 = w = u + i\,v$, we finally obtain that

$$f(x\,,\,y\,,\,u\,,\,v) = \sqrt{2}\,(1 + u)\operatorname{Re}\,\gamma(x + i\,y) - \sqrt{2}\,v\ \operatorname{Im}\,\gamma(x + i\,y)$$

$$= \sqrt{2}(1 + u) \cdot \begin{pmatrix} 1 + 3\,x^4 - 18\,x^2\,y^2 + 3\,y^4 \\ 12\,x^3\,y - 12\,x\,y^3 \\ 6\,x - 2\,x^3 + 6\,x\,y^2 \\ -6\,y - 6\,x^2\,y + 2\,y^3 \\ 6\,x^2 - 6\,y^2 \end{pmatrix} - \sqrt{2}\,v \cdot \begin{pmatrix} 12\,x^3\,y - 12\,x\,y^3 \\ 1 - 3\,x^4 + 18\,x^2\,y^2 - 3\,y^4 \\ 6\,y - 6\,x^2\,y + 2\,y^3 \\ 6\,x + 2\,x^3 - 6\,x\,y^2 \\ 12\,x\,y \end{pmatrix}$$

is a minimal real Kähler hypersurface in $\mathbf{R}^5$, defined for all (x, y, u, v) in some neighborhood $U \times W$ of the origin in $\mathbf{R}^4$ (U, W open in $\mathbf{R}^2$).

We can also use this example for f to obtain an example for a pseudoholomorhic surface $g : U \to S^4$ in the 4–sphere.[2] Note that since here

$$f_* T_{(z,0)} U = \operatorname{span}\{\mathrm{Re}\gamma(z), \mathrm{Im}\gamma(z), \mathrm{Re}\gamma'(z), \mathrm{Im}\gamma'(z)\},$$

we only need to calculate $\gamma'(z)$ and then determine a normal vector to f at $(z, 0)$. We have

$$\gamma'(z) = 6 \begin{pmatrix} 2z^3 \\ -2iz^3 \\ 1 - z^2 \\ i(1+z^2) \\ 2z \end{pmatrix}, \quad \text{and thus}$$

$$\tfrac{1}{6}\,\mathrm{Re}\,\gamma'(x, y) = \begin{pmatrix} 2x^3 - 6xy^2 \\ 6x^2y - 2y^3 \\ 1 - x^2 + y^2 \\ -2xy \\ 2x \end{pmatrix}, \quad \tfrac{1}{6}\,\mathrm{Im}\,\gamma'(x, y) = \begin{pmatrix} 6x^2y - 2y^3 \\ -2x^3 + 6xy^2 \\ -2xy \\ 1 + x^2 - y^2 \\ 2y \end{pmatrix}.$$

To find the required normal vector, one probably wants to use a computer algebra system (or otherwise plenty of time and patience). In any case, one finally arrives at the result

$$g(x, y) = \frac{1}{a^3 + 9a^2 + a + 1} \begin{pmatrix} 2(x^2 - y^2)(a+3) \\ 4xy(a+3) \\ 2x(3a^2 - 1) \\ 2y(3a^2 - 1) \\ a^3 - 9a^2 - a + 1 \end{pmatrix}$$

where $a := x^2 + y^2 = ||(x, y)||^2$.

[2] Actually, for a minimal surface in the 4-sphere, being pseudoholomorphic is equivalent to being *superminimal*. See [Loo], page 8, or [D-G$_1$], page 18.

Remark: Once that we have a candidate for a pseudoholomorphic map $g : U \longrightarrow S^{2n}$, it is not too difficult to check if it is indeed pseudoholomorphic. If $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$, where $(x\,,\,y)$ are isothermal coordinates on U, then according to Calabi [C] – or better: [Loo], page 7 – we only need to check that $\partial^j g \cdot \partial^j g = 0$ for $1 \leq j \leq n$. The author did, in fact, use a computer algebra system to successfully double-check our g above for pseudoholomorphicity.

Example 2.2: Here, we start with $\alpha_0 := \frac{1}{2}\,e^{\frac{z}{2}}$, and thus $\phi_0 = e^{\frac{z}{2}}$ and $\phi_0^2 = e^z$ (setting this and all following integration constants equal to zero again). Choosing $\mu_1 := 2$, we find

$$\alpha_1 = \frac{\mu_1}{2}\begin{pmatrix} 1-\phi_0^2 \\ i\,(1+\phi_0^2) \\ 2\,\phi_0 \end{pmatrix} = \begin{pmatrix} 1-e^z \\ i\,(1+e^z) \\ 2\,e^{\frac{z}{2}} \end{pmatrix}.$$

Integrate to obtain

$$\phi_1 = \int \alpha_1(z)\,dz = \begin{pmatrix} z-e^z \\ i\,(z+e^z) \\ 4\,e^{\frac{z}{2}} \end{pmatrix},$$

so that $\phi_1^2 = (z-e^z)^2 + i^2(z+e^z)^2 + (4\,e^{\frac{z}{2}})^2 = 16\,e^z - 4\,z\,e^z$. Setting $\mu_2 := 2$ gives

$$\gamma = \alpha_2 = \frac{\mu_2}{2}\begin{pmatrix} 1-\phi_1^2 \\ i\,(1+\phi_1^2) \\ 2\,\phi_1 \end{pmatrix} = \begin{pmatrix} 1-16\,e^z+4\,z\,e^z \\ i\,(1+16\,e^z-4\,z\,e^z) \\ 2\,z-2\,e^z \\ i\,(2\,z+2\,e^z) \\ 8\,e^{\frac{z}{2}} \end{pmatrix}.$$

Again, we choose $b_0 := 0$ and $b_1 := 1$, and have the same general form of f as in (9). Finally, we obtain another example for a minimal real Kähler manifold in $\mathbf{R}^5$, namely

$$
\begin{aligned}
f(x\,,\,y\,,\,u\,,\,v) &= \sqrt{2}\,(1+u)\ \operatorname{Re}\,\gamma(x+i\,y)-\sqrt{2}\ v\ \operatorname{Im}\,\gamma(x+i\,y)\\
&= \sqrt{2}\,(1+u)\cdot\begin{pmatrix} 1+e^{x}\,\cos y\,(4\,x-16)-4\,y\,e^{x}\,\sin y\\ 4\,y\,e^{x}\,\cos y+e^{x}\,\sin y\,(4\,x-16)\\ 2\,x-2\,e^{x}\,\cos y\\ -2\,y-2\,e^{x}\,\sin y\\ 8\,e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right)\end{pmatrix}\\
&\quad -\sqrt{2}\ v\cdot\begin{pmatrix} 4\,y\,e^{x}\,\cos y+e^{x}\,\sin y\,(4\,x-16)\\ 1-e^{x}\,\cos y\,(4\,x-16)+4\,y\,e^{x}\,\sin y\\ 2\,y-2\,e^{x}\,\sin y\\ 2\,x+2\,e^{x}\,\cos y\\ 8\,e^{\frac{x}{2}}\sin\left(\frac{y}{2}\right)\end{pmatrix}.
\end{aligned}
$$

As in Example 2.1, this f gives rise to another example of a pseudoholomorhic surface $g : U \to S^4$ in the 4–sphere. We have

$$
\gamma'(z)=\begin{pmatrix} -12\,e^{z}+4\,z\,e^{z}\\ i\,(12\,e^{z}-4\,z\,e^{z})\\ 2-2\,e^{z}\\ i\,(2+2\,e^{z})\\ 4\,e^{\frac{z}{2}}\end{pmatrix},
$$

and thus

$$
\operatorname{Re}\,\gamma'(x,\,y)=\begin{pmatrix} e^{x}\,\cos y\,(4\,x-12)-4\,y\,e^{x}\,\sin y\\ 4\,y\,e^{x}\,\cos y+e^{x}\,\sin y\,(4\,x-12)\\ 2-2\,e^{x}\,\cos y\\ -2\,e^{x}\,\sin y\\ 4\,e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right)\end{pmatrix}
$$

and

$$\operatorname{Im} \gamma'(x,\, y) = \begin{pmatrix} 4\,y\,e^{x}\,\cos y + e^{x}\,\sin y\,(4\,x - 12) \\ -e^{x}\,\cos y\,(4\,x - 12) + 4\,y\,e^{x}\,\sin y \\ -2\,e^{x}\,\sin y \\ 2 + 2\,e^{x}\,\cos y \\ 4\,e^{\frac{x}{2}}\sin\left(\frac{y}{2}\right) \end{pmatrix}.$$

It appears somewhat daunting to try to find the normal vector to the space $\operatorname{span}\{\operatorname{Re}\gamma(z)\,,\, \operatorname{Im}\gamma(z)\,,\, \operatorname{Re}\gamma'(z)\,,\, \operatorname{Im}\gamma'(z)\}$, but one can make this task slightly easier (even for a computer algebra system) by doing some linear algebra on this basis, thereby finding the following new basis vectors:

$$\begin{pmatrix} 1 - 4\,e^{x}\,\cos y \\ -4\,e^{x}\,\sin y \\ 2\,x - 2 \\ -2\,y \\ 4\,e^{\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} -4\,e^{x}\,\sin y \\ 1 + 4\,e^{x}\,\cos y \\ 2\,y \\ 2\,x - 2 \\ 4\,e^{\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} 2\,x - 6 \\ 2\,y \\ -1 + e^{-x}\,\cos y \\ e^{-x}\,\sin y \\ 2\,e^{-\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} 2\,y \\ -2\,x + 6 \\ -\,e^{-x}\,\sin y \\ 1 + e^{-x}\,\cos y \\ -2\,e^{-\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) \end{pmatrix}.$$

Again, we employ a computer algebra system to calculate the normal vector, and find

$$g(x\,,\, y) = \frac{2\,e^{\frac{x}{2}}}{4\,e^{2x} + e^{x}\,b(x,y) + 1} \begin{pmatrix} 2\,\cos(\frac{y}{2})\,(e^{x} + x - 2) - 2\,y\,\sin(\frac{y}{2}) \\ 2\,y\,\cos(\frac{y}{2}) + 2\,\sin(\frac{y}{2})\,(e^{x} + x - 2) \\ \cos(\frac{y}{2})\,(4\,x\,e^{x} - 8\,e^{x} - 1) + 4\,y\,e^{x}\,\sin(\frac{y}{2}) \\ -4\,y\,e^{x}\,\cos(\frac{y}{2}) + \sin(\frac{y}{2})\,(4\,x\,e^{x} - 8\,e^{x} - 1) \\ \frac{1}{2}\,e^{-\frac{x}{2}}\,(4\,e^{2x} - e^{x}\,b(x,y) + 1) \end{pmatrix},$$

where $b(x,y) := 4\,x^2 + 4\,y^2 - 16\,x + 17$. As in Example 2.1, the author double–checked that $\partial^j g \cdot \partial^j g = 0$ for $j = 1,2$. Actually, it took a fairly fast computer system several minutes to complete this task.

3 Minimal real Kähler surfaces and their Weierstrass representations

One key feature of the Weierstrass representation for minimal surfaces in $\mathbf{R}^N$ is that it uses a simple parametrization for isotropic vectors in $\mathbf{C}^N$ (see [H-O]). If one wants to find a similar representation for isometric immersions of a Kähler manifold M^{2n} into $\mathbf{R}^N$, one could, thus, try to first construct a similar parametrization for isotropic n-dimensional complex subspaces of $\mathbf{C}^N$. We will in fact do this for $n = 2$, i.e. for isotropic complex *planes* in $\mathbf{C}^N$ (for $N \geq 5$), and will then find a local characterization for all minimal (complex) *Kähler surfaces* in $\mathbf{R}^N$ (so: of *real* dimension 4). This characterization corresponds to the Weierstrass representation of a minimal surface (=complex *curve*) described on page 2. It is not only interesting in its own right, but also because all isometric immersions $f : M^{2n} \to \mathbf{R}^{2n+2}$ in codimension 2 that are *not* holomorphic with respect to some complex structure in $\mathbf{R}^{2n+2}$ must locally be affine vector bundles of (real) rank $2n-4$ over a 4-dimensional Kähler manifold $\tilde{M}^4$ (see our remarks on page 6). And the complex Gauss map of such an f restricted to $\tilde{M}$ can be viewed as a map from $\tilde{M}$ into the space of all isotropic complex planes in $\mathbf{C}^{2n+2}$.

In general, denote the space of all isotropic complex n-subspaces in $\mathbf{C}^N$ by $\mathbf{I}_n(\mathbf{C}^N)$, or $\mathbf{I}_n^N$ for short. It is well-known (see e.g. [R-T]) that $\mathbf{I}_n^N$ is a compact complex homogeneous space, namely

$$\mathbf{I}_n(\mathbf{C}^N) \cong {SO(N)}\Big/{\big(U(n) \times SO(N-2n)\big)} .$$

This implies for the (real) dimension of this manifold

$$\left.\begin{aligned} \dim_{\mathbf{R}}(\mathbf{I}_n(\mathbf{C}^N)) &= \tbinom{N}{2} - (n^2 + \tbinom{N-2n}{2}) = 2\,N\,n - 3\,n^2 - n \\ &= n \cdot (2\,N - 3\,n - 1)\,. \end{aligned}\right\} \qquad (1)$$

In fact, as a complex submanifold of the Grassmannian $\mathrm{Gr}_n(\mathbf{C}^N)$, which is a Kähler manifold with its standard (Fubini-Study like) metric (see [K-N], pages 133–134 and pages 160–161), $\mathbf{I}_n^N$ is itself a Kähler manifold. However, we will not use this last fact here.

We will now give a complex coordinate system for $\mathbf{I}_2^N$, whose *complex* dimension is, by (1), $\frac{1}{2} \cdot 2 \cdot (2\,N - 3 \cdot 2 - 1) = 2\,N - 7$. Note that since $\mathbf{I}_2^N$ is homogeneous and since we will be mainly interested in *local* considerations, any "small" coordinate system of $\mathbf{I}_2^N$ suffices for our purposes.

We will write the complex coordinates in $\mathbf{C}^{2N-7}$ as $(\xi\,,\,\zeta\,,\,\lambda)$, where $\xi=(\xi_1\,,\ldots,\,\xi_{N-4})$ and $\zeta=(\zeta_1\,,\ldots,\,\zeta_{N-4})$ are in $\mathbf{C}^{N-4}$ and λ is a complex number. Then set

$$\left.\begin{array}{ll} \mathbf{X} := (\tfrac{1}{2}\,,\,\tfrac{i}{2}\,,\,X)\,, & \text{where } X := \lambda\,(\tfrac{1-\xi^2}{2}\,,\, i\,\tfrac{1+\xi^2}{2}\,,\,\xi)\,, \\ \text{and} & \\ \mathbf{Y} := (-X\cdot Y\,,\, i\,X\cdot Y\,,\,Y)\,, & \text{where } Y := (\tfrac{1-\zeta^2}{2}\,,\, i\,\tfrac{1+\zeta^2}{2}\,,\,\zeta)\,. \end{array}\right\} \qquad (2)$$

Here, $X\cdot Y$ is the standard symmetric inner product in $\mathbf{C}^{N-2}$, and $\xi^2=\xi\cdot\xi$ and $\zeta^2=\zeta\cdot\zeta$ refer to the analogous inner product in $\mathbf{C}^{N-4}$ (see page 10). It is easy to check that $\mathbf{X}$ and $\mathbf{Y}$ span an *isotropic subspace* of $\mathbf{C}^N$, and that X and Y are isotropic *vectors* in $\mathbf{C}^{N-2}$. However, in general X and Y will *not* span an isotropic sub*space* of $\mathbf{C}^{N-2}$. Now, define the map $\Phi:\mathbf{C}^{2N-7}\to\mathbf{I}_2^N$ by

$$\Phi(\xi\,,\,\zeta\,,\,\lambda) := \operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\}\,,$$

with $\mathbf{X}$ and $\mathbf{Y}$ defined in terms of $(\xi\,,\,\zeta\,,\,\lambda)$ as in (2). Note that this map is well-defined, i.e. that $\operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\}$ is in fact always 2-dimensional; for let us assume that there were complex numbers α and β such that

$$0=\alpha\mathbf{X}+\beta\mathbf{Y}=\begin{pmatrix} \frac{\alpha}{2}-\beta\,X\cdot Y \\ i\,\frac{\alpha}{2}+i\,\beta\,X\cdot Y \\ \alpha\,X+\beta\,Y \end{pmatrix}.$$

Dividing the second component by i and adding it to the first component gives $\alpha=0$, so that in particular we must have $\beta\,Y=0$. But looking at the form of Y as in (2), we see that the first and the second component of Y can never be zero simultaneously. Thus, β must also be zero.

We will have to restrict the domain of Φ to obtain the (inverse of the) coordinate system for which we are looking. Thus, let

$$G\Phi := \{\,(\xi\,,\,\zeta\,,\,\lambda)\in\mathbf{C}^{2N-7}\mid \lambda\,(\xi-\zeta)^2\neq 0\,\}\,.$$

Note that the condition in this definition is equivalent to $X\cdot Y\neq 0$, since

$$\left.\begin{aligned} X\cdot Y &= \lambda\Big(\frac{(1-\xi^2)(1-\zeta^2)}{4}+i^2\frac{(1+\xi^2)(1+\zeta^2)}{4}+\xi\cdot\zeta\Big) \\ &= \lambda\big(-\tfrac{1}{2}\xi^2-\tfrac{1}{2}\zeta^2+\xi\cdot\zeta\big) = -\frac{\lambda}{2}\,(\xi-\zeta)^2\neq 0\,. \end{aligned}\right\} \qquad (3)$$

Thus, span$\{X\,,\,Y\}$ will, in fact, *never* be an isotropic subspace of $\mathbf{C}^{N-2}$ as long as $(\xi\,,\,\zeta\,,\,\lambda)\in G\Phi$. Obviously, $G\Phi$ is an open set in $\mathbf{C}^{2N-7}$, and it is easy to see that $G\Phi$ is connected. Then we have

Lemma 3.1: *The map* $\Phi: \quad G\Phi \longrightarrow \mathbf{I}_2^N$

$$(\xi\,,\,\zeta\,,\,\lambda) \longmapsto \operatorname{span}\{\mathbf{X}, \mathbf{Y}\}\,,$$

where $G\Phi := \{\,(\xi\,,\,\zeta\,,\,\lambda)\in\mathbf{C}^{2N-7} \mid \lambda\,(\xi-\zeta)^2\neq 0\,\}$, *and* $\mathbf{X}$ *and* $\mathbf{Y}$ *are given by* (2) *as*

$$\mathbf{X} := \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \lambda\,\frac{1-\xi^2}{2} \\ i\,\lambda\,\frac{1+\xi^2}{2} \\ \lambda\,\xi \end{pmatrix}, \quad \text{and} \quad \mathbf{Y} := \begin{pmatrix} \frac{\lambda}{2}\,(\xi-\zeta)^2 \\ -\frac{i\lambda}{2}\,(\xi-\zeta)^2 \\ \frac{1-\zeta^2}{2} \\ i\,\frac{1+\zeta^2}{2} \\ \zeta \end{pmatrix},$$

is an injective holomorphic immersion between complex manifolds of the same (complex) dimension $2N-7$, *and thus (the inverse of) a coordinate system for* $\mathbf{I}_2^N$.

We call the inverse chart Φ as in Lemma 3.1 a **Weierstrass coordinate system** for $\mathbf{I}_2^N$.

Proof: First, we prove that Φ is *injective* on $G\Phi$. Thus, suppose that

$$\operatorname{span}\{\mathbf{X}, \mathbf{Y}\} = \operatorname{span}\{\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}\}\,,$$

which means that there are complex numbers α, β, γ, and δ such that

$$\left.\begin{aligned} \tilde{\mathbf{X}} &= \alpha\,\mathbf{X} + \beta\,\mathbf{Y}\,, \\ \tilde{\mathbf{Y}} &= \gamma\,\mathbf{X} + \delta\,\mathbf{Y}\,. \end{aligned}\right\} \tag{4}$$

By (2), the first equation means

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} - \beta\,X\cdot Y \\ i\,\frac{\alpha}{2} + i\,\beta\,X\cdot Y \\ \alpha\,X + \beta\,Y \end{pmatrix}.$$

Dividing the second components by i and adding them to the first components on either side gives $\alpha = 1$, whereas subtracting these components gives $2\beta X \cdot Y = 0$. Since we are in $G\Phi$, we have by (3) that $X \cdot Y \neq 0$, and thus that $\beta = 0$. So, we have found that $\tilde{\mathbf{X}} = \mathbf{X}$, and thus, in particular, that $\tilde{X} = X$. By the second equation in (4) and by (2), we have

$$\begin{pmatrix} -X \cdot \tilde{Y} \\ i\, X \cdot \tilde{Y} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{2} - \delta\, X \cdot Y \\ i\,\frac{\gamma}{2} + i\,\delta\, X \cdot Y \\ \gamma X + \delta Y \end{pmatrix}.$$

Again, dividing the second components by i and adding them to the first, we immediately have $\gamma = 0$, and thus $\tilde{Y} = \delta Y$. But according to (2), this means that the first two components of $\tilde{Y}$ and δY give

$$\frac{1-\tilde{\zeta}^2}{2} = \delta\, \frac{1-\zeta^2}{2} \quad \text{and} \quad \frac{1+\tilde{\zeta}^2}{2} = \delta\, \frac{1+\zeta^2}{2}.$$

Adding these equations gives $\delta = 1$, and thus $\tilde{\mathbf{Y}} = \mathbf{Y}$. By (2), this obviously means that the coordinates (ξ, ζ, λ) and $(\tilde{\xi}, \tilde{\zeta}, \tilde{\lambda})$ generating the subspaces with which we started have to be the same. Thus, Φ is injective on $G\Phi$.

We will now show that Φ is *holomorphic and immersive* on $G\Phi$. For this, we need a complex chart for $\mathbf{I}_2^N$. Since $\mathbf{I}_2^N$ is a *compact* complex submanifold of the Grassmannian $\mathrm{Gr}_2(\mathbf{C}^N)$, its topology is the subspace topology of $\mathrm{Gr}_2(\mathbf{C}^N)$, and thus it suffices to use a complex chart φ_α of $\mathrm{Gr}_2(\mathbf{C}^N)$ as in the following diagram:

$$\begin{array}{ccccccc} G\Phi & \xrightarrow{\Phi} & \mathbf{I}_2^N & \hookrightarrow & \mathrm{Gr}_2(\mathbf{C}^N) & \xrightarrow{\varphi_\alpha} & \mathrm{Mat}((N-2)\times 2,\, \mathbf{C}) \\ (\xi, \zeta, \lambda) & \longmapsto & \underset{\mathbf{X},\mathbf{Y} \text{ as in (2)}}{\mathrm{span}\{\mathbf{X}, \mathbf{Y}\}}, & \longmapsto & \mathrm{span}\{\mathbf{X}, \mathbf{Y}\} & \longmapsto & \varphi_\alpha\big(\mathrm{span}\{\mathbf{X}, \mathbf{Y}\}\big). \end{array}$$

It is clear that if we can show that the map

$$\left.\begin{array}{rccc} \Phi_\alpha : & G\Phi & \longrightarrow & \mathrm{Mat}\big((N-2)\times 2,\, \mathbf{C}\big) \\ & (\xi, \zeta, \lambda) & \longmapsto & \varphi_\alpha(\mathrm{span}\{\mathbf{X}, \mathbf{Y}\}) \end{array}\right\} \quad (5)$$

is a holomorphic immersion, then the same has to be true for Φ. Of course, we need an explicit formula for φ_α. We will use the one described in Example 2.4 in [K-N], page 133 (where $p = 2$ and $p + q = N$). Thus, if $(z_1, \ldots, z_N)$

is the standard complex coordinate system in $\mathbf{C}^N$, consider the z_j as linear maps $\mathbf{C}^N \to \mathbf{C}$, and choose a set $\alpha := \{\alpha_1, \alpha_2\}$ of integers such that $1 \le \alpha_1 < \alpha_2 \le N$. Let U_α be the subset of all complex planes $S \subset \mathbf{C}^N$ such that $z_{\alpha_1}|S$ and $z_{\alpha_2}|S$ are linearly independent; i.e. they form basis of the dual space of S. Write the other coordinates as $z_{\alpha_3}, \ldots, z_{\alpha_N}$, where $\{\alpha_3, \ldots, \alpha_N\}$ is the complement of α in $\{1, \ldots, N\}$, written in increasing order. Then, we can find complex numbers s_{kj} such that

$$z_{\alpha_k}|S = \sum_{j=1}^{2} s_{kj}(z_{\alpha_j}|S) \quad \text{for} \quad k = 3, \ldots, N,$$

and the complex chart for which we are looking is given by

$$\begin{aligned} \varphi_\alpha : U_\alpha &\longrightarrow \operatorname{Mat}\big((N-2)\times 2, \mathbf{C}\big) \ . \\ S &\longmapsto (s_{kj})_{\substack{k=3,\ldots,N \\ j=1,2}} \ . \end{aligned}$$

Actually, it is not hard to see how we can write this chart in terms of a basis $v_1, v_2 \in \mathbf{C}^N$ of the given complex plane S, namely in the following way. Define the two projections $P_\alpha : \operatorname{Mat}(N \times 2, \mathbf{C}) \to \operatorname{Mat}(2\times 2, \mathbf{C})$ and $Q_\alpha : \operatorname{Mat}(N\times 2, \mathbf{C}) \to \operatorname{Mat}\big((N-2)\times 2, \mathbf{C}\big)$ by

$$P_\alpha\Big((a_{kj})_{\substack{k=1,\ldots,N \\ j=1,2}}\Big) := \begin{pmatrix} a_{\alpha_1,1} & a_{\alpha_1,2} \\ a_{\alpha_2,1} & a_{\alpha_2,2} \end{pmatrix} \text{ and } Q_\alpha\Big((a_{kj})_{\substack{k=1,\ldots,N \\ j=1,2}}\Big) := \begin{pmatrix} a_{\alpha_3,1} & a_{\alpha_3,2} \\ \vdots & \vdots \\ a_{\alpha_N,1} & a_{\alpha_N,2} \end{pmatrix}.$$

Then it is not hard to show that if v_1, v_2 is a basis of $S \in \operatorname{Gr}_2(\mathbf{C}^n)$, S is in U_α exactly if $\det\big(P_\alpha(v_1, v_2)\big) \neq 0$ (where (v_1, v_2) is regarded as the $N\times 2$–matrix whose columns are the components of v_1 and v_2, respectively). Furthermore, one can easily show that we have

$$\varphi_\alpha(S) = Q_\alpha(v_1, v_2)\cdot\Big(P_\alpha(v_1, v_2)\Big)^{-1}, \tag{6}$$

which is the form of φ_α with which we will work. Note that, in particular, this formula does not depend on the choice of the basis v_1, v_2 for $S \in U_\alpha$.

We will now use the above formulas in the special case $\alpha = \{1, 2\}$ and for the complex planes that are given by our map Φ. Substituting $v_1 = \mathbf{X}$ and $v_2 = \mathbf{Y}$, we see that

$$\det\Big(P_\alpha(\mathbf{X}, \mathbf{Y})\Big) = \det\begin{pmatrix} \frac{1}{2} & -X\cdot Y \\ \frac{i}{2} & i\,X\cdot Y \end{pmatrix} = i\,X\cdot Y\,,$$

which is different from zero *exactly* if span$\{\mathbf{X},\mathbf{Y}\} \in G\Phi$ (see (3)). This means that our map $\Phi_\alpha = \Phi_{\{1,2\}}$ in (5) is defined on all of $G\Phi$, and it is clear by (2) and (6) that its components are *rational* functions in the complex variables $(\xi, \zeta, \lambda) \in G\Phi$, and thus that Φ_α is a *holomorphic* map.

It remains to be shown that Φ_α is *immersive* on $G\Phi$. To establish this, we have to explicitly calculate how Φ_α looks in terms of (ξ, ζ, λ):

$$\begin{aligned}\Phi_\alpha(\xi, \zeta, \lambda) &= Q_\alpha(\mathbf{X},\mathbf{Y}) \cdot \big(P_\alpha(\mathbf{X},\mathbf{Y})\big)^{-1} \\ &= \begin{pmatrix} | & | \\ X & Y \\ | & | \end{pmatrix} \cdot \frac{1}{i\,X\cdot Y} \begin{pmatrix} i\,X\cdot Y & X\cdot Y \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix},\end{aligned}$$

since $\alpha = \{1,2\}$. Multiplying these matrices and substituting the expressions for X and Y according to (2) and for $X\cdot Y$ as in (3), we obtain

$$\Phi_\alpha(\xi, \zeta, \lambda) = \Big(X - \tfrac{1}{2\,X\cdot Y}\,Y \;,\; -i\,X - \tfrac{i}{2\,X\cdot Y}\,Y\Big)$$

$$= \begin{pmatrix} \frac{\lambda}{2}(1-\xi^2) + \frac{1}{2\lambda(\xi-\zeta)^2}(1-\zeta^2) & -\frac{i\lambda}{2}(1-\xi^2) + \frac{i}{2\lambda(\xi-\zeta)^2}(1-\zeta^2) \\ \frac{i\lambda}{2}(1+\xi^2) + \frac{i}{2\lambda(\xi-\zeta)^2}(1+\zeta^2) & \frac{\lambda}{2}(1+\xi^2) - \frac{1}{2\lambda(\xi-\zeta)^2}(1+\zeta^2) \\ \lambda\,\xi + \frac{1}{\lambda(\xi-\zeta)^2}\,\zeta & -i\,\lambda\,\xi + \frac{i}{\lambda(\xi-\zeta)^2}\,\zeta \end{pmatrix}.$$

We will now determine the partial derivatives of Φ_α in terms of the complex coordinates $(\xi_1, \dots, \xi_{N-4}, \zeta_1, \dots, \zeta_{N-4}, \lambda)$. Note that since

$$\xi^2 = \xi_1^2 + \ldots + \xi_{N-4}^2 \;,\quad \zeta^2 = \zeta_1^2 + \ldots + \zeta_{N-4}^2 \;,$$
$$\text{and}\quad (\xi-\zeta)^2 = (\xi_1-\zeta_1)^2 + \ldots + (\xi_{N-4}-\zeta_{N-4})^2 \,,$$

we have that, for all $1 \le j, k \le N-4$,

$$\frac{\partial\,\xi^2}{\partial\,\xi_j} = 2\,\xi_j \;,\quad \frac{\partial\,\xi^2}{\partial\,\zeta_k} = 0 \;,\quad \frac{\partial\,\zeta^2}{\partial\,\xi_j} = 0 \;,\quad \frac{\partial\,\zeta^2}{\partial\,\zeta_k} = 2\,\zeta_k \;,$$

$$\frac{\partial\,(\xi-\zeta)^2}{\partial\,\xi_j} = 2\,(\xi_j - \zeta_j) \;,\quad \text{and}\quad \frac{\partial\,(\xi-\zeta)^2}{\partial\,\zeta_k} = -2\,(\xi_k - \zeta_k)\,.$$

Also, it will be convenient to write

$$(\xi-\zeta)^4 := \left((\xi-\zeta)^2\right)^2 = \left((\xi_1-\zeta_1)^2+\ldots+(\xi_{N-4}-\zeta_{N-4})^2\right)^2,$$

and to denote the j^{th} coordinate vector in $\mathbf{C}^{N-4}$ by $\mathbf{e}_j$. Using these notations, we obtain for all $1 \le j, k \le N-4$:

$$\frac{\partial\,\Phi_\alpha}{\partial\,\xi_j} = \lambda \underbrace{\begin{pmatrix} -\xi_j & i\,\xi_j \\ i\,\xi_j & \xi_j \\ \mathbf{e}_j & -i\,\mathbf{e}_j \end{pmatrix}}_{=:\,A_j} - \frac{\xi_j-\zeta_j}{\lambda\,(\xi-\zeta)^4} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=:\,Z},$$

$$\frac{\partial\,\Phi_\alpha}{\partial\,\zeta_k} = \frac{1}{\lambda\,(\xi-\zeta)^2} \underbrace{\begin{pmatrix} -\zeta_k & -i\,\zeta_k \\ i\,\zeta_k & -\zeta_k \\ \mathbf{e}_k & i\,\mathbf{e}_k \end{pmatrix}}_{=:\,B_k} + \frac{\xi_k-\zeta_k}{\lambda\,(\xi-\zeta)^4} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=\,Z},$$

and finally

$$\frac{\partial\,\Phi_\alpha}{\partial\,\lambda} = \frac{1}{2} \underbrace{\begin{pmatrix} 1-\xi^2 & -i\,(1-\xi^2) \\ i\,(1+\xi^2) & 1+\xi^2 \\ 2\,\xi & -2\,i\,\xi \end{pmatrix}}_{=:\,\Xi} - \frac{1}{2\,\lambda^2\,(\xi-\zeta)^2} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=\,Z}.$$

Now, if we can show that the $(N-2)\times 2$–matrices A_j $(1 \le j \le N-4)$, B_k $(1 \le k \le N-4)$, Ξ, and Z defined as above are linearly independent in $\mathrm{Mat}\big((N-2)\times 2, \mathbf{C}\big)$ whenever $(\xi\,,\,\zeta\,,\,\lambda)$ is in $G\Phi$, then the same has to be true for the partial derivatives of Φ_α. For assume that we have complex numbers $a_j\,,\,b_j$ $(1 \le j \le N-4)$, and c such that

$$0 = \sum_{j=1}^{N-4} \left(a_j\,\frac{\partial\,\Phi_\alpha}{\partial\,\xi_j} + b_j\,\frac{\partial\,\Phi_\alpha}{\partial\,\zeta_j} \right) + c\,\frac{\partial\,\Phi_\alpha}{\partial\,\lambda}.$$

Replacing the partial derivatives of Φ_α as above and ordering the resulting terms, one finds that

$$0 = \sum_{j=1}^{N-4} \lambda\, a_j\, A_j + \sum_{j=1}^{N-4} \frac{b_j}{\lambda\,(\xi-\zeta)^2}\, B_j + \frac{c}{2}\; \Xi \; + \begin{pmatrix}\text{some}\\ \text{large sum}\end{pmatrix} Z\ .$$

Since we assume that the A_j, B_j, Ξ, and Z are linearly independent, we have, in particular, that

$$\lambda\, a_j = 0\ ,\quad \frac{b_j}{\lambda\,(\xi-\zeta)^2} = 0 \quad (1 \le j \le N-4)\ ,\quad \text{and} \quad c = 0\ .$$

Since we are in $G\Phi$, where $\lambda\,(\xi-\zeta)^2$ (and thus λ) is never zero, this means that all the a_j, b_j, and c must be zero, and thus that Φ_α is indeed *immersive* on all of $G\Phi$.

To show that the A_j, B_j, Ξ, and Z are linearly independent, we assume that there are complex numbers a_j, b_j $(1 \le j \le N-4)$, c, and d such that

$$0 = \sum_{j=1}^{N-4} \Big(a_j\, A_j + b_j\, B_j\Big) \; + \; c\, \Xi \; + \; d\, Z\ .$$

Combining the a_j and the b_j to the vectors

$$a := (a_1\,, \ldots,\, a_{N-4}) \quad \text{and} \quad b := (b_1\,, \ldots,\, b_{N-4}) \in \mathbf{C}^{N-4}\ ,$$

and using the definitions of our matrices, we can rewrite the above equation in terms of the entries of the matrices:

$(1,1)$–entry:	$0 = -a\cdot\xi - b\cdot\zeta + c\,(1-\xi^2) + d\,(1-\zeta^2)$	(a)
$(1,2)$–entry:	$0 = i\,\Big(a\cdot\xi - b\cdot\zeta - c\,(1-\xi^2) + d\,(1-\zeta^2)\Big)$	(b)
$(2,1)$–entry:	$0 = i\,\Big(a\cdot\xi + b\cdot\zeta + c\,(1+\xi^2) + d\,(1+\zeta^2)\Big)$	(c)
$(2,2)$–entry:	$0 = a\cdot\xi - b\cdot\zeta + c\,(1+\xi^2) - d\,(1+\zeta^2)$	(d)
rest of 1^{st} column:	$0 = a + b + 2\,c\,\xi + 2\,d\,\zeta$	(e)
rest of 2^{nd} column:	$0 = i\,(-a + b - 2\,c\,\xi + 2\,d\,\zeta)$	(f)

Dividing (f) by i and adding it to (e) gives, after dividing by 2, $b = -2\,d\,\zeta$, whereas subtracting gives $a = -2\,c\,\xi$. Dividing (b) by i, adding it to (a), and using the formula for b we just obtained gives

$$0 = -2\,\underbrace{(-2\,d\,\zeta)}_{=b}\cdot\zeta + 2\,d - 2\,d\,\zeta^2 = 2\,d + 2\,d\,\zeta^2\ ,$$

whereas subtracting and using the formula for a results in

$$0 = -2\underbrace{(-2\,c\,\xi)}_{=a}\cdot\xi + 2\,c - 2\,c\,\xi^2 = 2\,c + 2\,c\,\xi^2\,.$$

Performing similar operations on (c) and (d) gives

$$0 = 2\,c - 2\,c\,\xi^2 \quad \text{and} \quad 0 = 2\,d - 2\,d\,\zeta^2\,,$$

and adding the corresponding equations for c and d obviously gives $c = d = 0$. Thus, we also have that $a = b = 0$, which is what we set out to prove.

This finishes the proof of Lemma 3.1.

Before we proceed, a few more remarks about Weierstrass coordinate systems Φ as in Lemma 3.1 are in order. First, we have that Φ is actually holomorphic as a map from *all of* $\mathbf{C}^{2N-7}$ into $\mathbf{I}_2^n$.

[To see this, note that if $(\xi\,,\,\zeta\,,\,\lambda) \notin G\Phi$, i.e. according to (3) that $X\cdot Y = 0$ (with X and Y as in (2)), then the first two components of $\mathbf{Y}$ as in (2) are zero, i.e. $\mathbf{Y} = (\,0\,,\,0\,,\,Y)$. But since the first two components of Y are never zero simultaneously, we find that either

$$\det\begin{pmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_3 & \mathbf{Y}_3 \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & 0 \\ X_1 & Y_1 \end{pmatrix} = \frac{1}{2}\,Y_1 = \frac{1}{4}\,(1-\zeta^2) \neq 0\,,$$

or that

$$\det\begin{pmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_4 & \mathbf{Y}_4 \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & 0 \\ X_2 & Y_2 \end{pmatrix} = \frac{1}{2}\,Y_2 = \frac{i}{4}\,(1+\zeta^2) \neq 0\,.$$

This means that in a neighborhood of any point in $\mathbf{C}^{2N-7} - G\Phi$, we can repeat the procedure on pages 28 and 29 with $\alpha = \{1\,,3\}$ or $\alpha = \{1\,,4\}$, and we see that, again, we will obtain a map from this neighborhood to $\mathrm{Mat}\big((N-2)\times 2\,,\,\mathbf{C}\big)$, all of whose component functions are rational, and which, thus, is holomorphic. This makes Φ *holomorphic* in this neighborhood.]

On the other hand, we can*not* assume Φ to be *immersive* at points that are not in $G\Phi$.

[To see this, note e.g. that whenever $\lambda = 0$ and ζ is fixed, *any* ξ will give the same isotropic plane in $\mathbf{I}_2^N$ under Φ. Thus, Φ will *not* be locally injective in any neighborhood of such a point, and thus cannot be immersive there.]

The next Proposition gives the promised *local characterization of minimal real Kähler surfaces* (so: of *real* dimension 4) in Euclidean spaces:

Proposition 3.2: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal isometric immersion from a (real) 4–dimensional Kähler manifold M into $\mathbf{R}^N$ (where $N \geq 5$), and let $p \in M$ be any point in M. Further, let $F : W \to \mathbf{C}^N$ be a holomorphic representative of f, defined in a neighborhood W of p in M (see page 5). Then we can find a neighborhood U of p in W, a complex chart (u, v) of M defined on U, and holomorphic maps $\xi, \zeta : U \to \mathbf{C}^{N-4}$ and $\lambda : U \to \mathbf{C}$ such that on all of U*

$$\lambda\,(\xi - \zeta)^2 \neq 0\,, \tag{7}$$

and such that, up to isometry in $\mathbf{R}^N$, we have that on all of U

$$\left.\begin{aligned} F_u &= \left(\tfrac{1}{2}, \tfrac{i}{2}, X\right), \quad \textit{where } X := \lambda\left(\tfrac{1-\xi^2}{2}, i\,\tfrac{1+\xi^2}{2}, \xi\right), \\ &\textit{and} \\ F_v &= (-1, i, Y), \quad \textit{where } Y := \mu\left(\tfrac{1-\zeta^2}{2}, i\,\tfrac{1+\zeta^2}{2}, \zeta\right), \\ &\qquad\qquad \textit{with } \mu := -\tfrac{2}{\lambda\,(\xi-\zeta)^2}\,. \end{aligned}\right\} \tag{8}$$

In particular, we have that the complex Gauss map $\varphi := \mathrm{im}(F_*)$ of f (as defined on page 9) factors on U as

$$\varphi = \Phi \circ (\xi, \zeta, \lambda)\,. \tag{9}$$

Proof: Let (z_1, z_2) be some complex chart of M, without loss of generality defined on W. By Theorem 1.1, we know that $\varphi(q)$ is an isotropic complex plane in $\mathbf{C}^N$ for every $q \in M$. Let Φ denote the Weierstrass coordinate system for $\mathbf{I}_2^N$ as in Lemma 3.1. If we have that $\varphi(p) \notin \Phi(G\Phi)$ for the chosen point p, we can find an $A \in SO(N)$ such that $A(\varphi(p)) \in \Phi(G\Phi)$, since $\mathbf{I}_2^N$ is homogeneous. Note that by Lemma 1.3, $A \circ F$ is the holomorphic representative of $A \circ f : M \to \mathbf{R}^N$, which is congruent to f in $\mathbf{R}^N$. Thus, we can assume from now on that, without loss of generality, $\varphi(p) \in \Phi(G\Phi)$.

Now, the Gauss map $q \mapsto \varphi(q) = \mathrm{span}\{F_{z_1}(q), F_{z_2}(q)\}$ is easily seen to be a holomorphic map $W \to \mathbf{I}_2^N$ (compare the arguments on pages 28 and 29). Let $V \subset W$ be a simply connected neighborhood of p such that, for all $q \in V$, $\varphi(q) \in \Phi(G\Phi)$. Then the map $\Phi^{-1} \circ \varphi : V \to \mathbf{C}^{2N-7}$ is also holomorphic; i.e. with respect to the given chart (z_1, z_2) restricted to V, we have *holomorphic* functions ξ, ζ, and λ as in Proposition 3.2 such that (9) is true on all of V. Also, by the definition of $G\Phi$, we know that (7) is satisfied on all of V.

Let now $\mathbf{X}$, $\mathbf{Y}$, X and $\tilde{Y}$ (instead of "Y") be defined in terms of ξ, ζ, and λ as in (2); in particular, they are all holomorphic on V. Since we thus have

$$\Phi \circ (\xi\,,\, \zeta\,,\, \lambda) = \operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\} = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix}, \begin{pmatrix} -X\cdot\tilde{Y} \\ i\,X\cdot\tilde{Y} \\ \tilde{Y} \end{pmatrix} \right\}, \tag{10}$$

by (9) there must be holomorphic functions α, β, γ, $\delta : V \to \mathbf{C}$ such that

$$\left.\begin{aligned} F_{z_1} &= \alpha\mathbf{X} + \beta\mathbf{Y} \\ F_{z_2} &= \gamma\mathbf{X} + \delta\mathbf{Y} \end{aligned}\right\} \quad \text{and} \quad \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0\,. \tag{11}$$

Using the *integrability condition* $(F_{z_1})_{z_2} = (F_{z_2})_{z_1}$ in (11), we obtain

$$\begin{aligned} (F_{z_1})_{z_2} &= \Big((\tfrac{\alpha}{2} - \beta\, X\cdot\tilde{Y})_{z_2}\,,\, i\,(\tfrac{\alpha}{2} + \beta\, X\cdot\tilde{Y})_{z_2}\,,\, (\alpha\, X + \beta\tilde{Y})_{z_2} \Big) \\ &\,\| \\ (F_{z_2})_{z_1} &= \Big((\tfrac{\gamma}{2} - \delta\, X\cdot\tilde{Y})_{z_1}\,,\, i\,(\tfrac{\gamma}{2} + \delta\, X\cdot\tilde{Y})_{z_1}\,,\, (\gamma X + \delta\tilde{Y})_{z_1} \Big)\,. \end{aligned}$$

Dividing the second components on each side by i and then adding or subtracting them from the first ones, respectively, gives

$$\alpha_{z_2} = \gamma_{z_1} \quad \text{and} \quad (\beta\,(X\cdot\tilde{Y}))_{z_2} = (\delta\,(X\cdot\tilde{Y}))_{z_1}\,.$$

Since we chose V to be simply connected, the complex version of Poincaré's Lemma (see e.g. [W], page 49) tells us that we can find two holomorphic functions u, $v : V \to \mathbf{C}$ such that

$$du = \alpha\, dz_1 + \gamma\, dz_2 \quad \text{and} \quad dv = (\beta\,(X\cdot\tilde{Y}))\, dz_1 + (\delta\,(X\cdot\tilde{Y}))\, dz_2\,.$$

This means, in particular, that

$$\alpha = \frac{\partial\, u}{\partial\, z_1}\,,\ \gamma = \frac{\partial\, u}{\partial\, z_2}\,,\ \beta\,(X\cdot\tilde{Y}) = \frac{\partial\, v}{\partial\, z_1}\,, \ \text{ and }\ \delta\,(X\cdot\tilde{Y}) = \frac{\partial\, v}{\partial\, z_2}\,.$$

Thus, we obtain that

$$\det\begin{pmatrix} \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} \\ \frac{\partial v}{\partial z_1} & \frac{\partial v}{\partial z_2} \end{pmatrix} = \det\begin{pmatrix} \alpha & \gamma \\ \beta\,(X\cdot\tilde{Y}) & \delta\,(X\cdot\tilde{Y}) \end{pmatrix} = (X\cdot\tilde{Y})\,\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \neq 0\,,$$

by (3), (7), and (11). This means that in some neighborhood $U \subset V$ of p in M, the map $(u\,,\,v)$ is a *local biholomorphism* from U onto an open set in $\mathbf{C}^2$. Thus, $(u\,,\,v)$ is a complex chart of M on $U \ni p$. Changing to these new coordinates u and v, we find

$$F_{z_1} = \frac{\partial\,u}{\partial\,z_1}\,F_u + \frac{\partial\,v}{\partial\,z_1}\,F_v = \alpha\,F_u + \beta\,(X\cdot\tilde{Y})\,F_v\,,$$

and

$$F_{z_2} = \frac{\partial\,u}{\partial\,z_2}\,F_u + \frac{\partial\,v}{\partial\,z_2}\,F_v = \gamma\,F_u + \delta\,(X\cdot\tilde{Y})\,F_v\,.$$

By (11), this means that if we express F_u and $(X\cdot\tilde{Y})\,F_v$ on one hand, or $\mathbf{X}$ and $\mathbf{Y}$ on the other hand, in components of the basis F_{z_1} and F_{z_2} of $\operatorname{im}F_*$, then the corresponding components will be the same, namely the entries of the inverse of the matrix in (11). Thus, we have that

$$F_u = \mathbf{X} \quad\text{and}\quad F_v = \tfrac{1}{X\cdot\tilde{Y}}\,\mathbf{Y}\,.$$

Together with (10), this means that we have proved (8), by setting

$$Y := \tfrac{1}{X\cdot\tilde{Y}}\,\tilde{Y} = -\tfrac{2}{\lambda\,(\xi-\zeta)^2}\left(\tfrac{1-\zeta^2}{2}\,,\,i\,\tfrac{1+\zeta^2}{2}\,,\,\zeta\right) = \mu\left(\tfrac{1-\zeta^2}{2}\,,\,i\,\tfrac{1+\zeta^2}{2}\,,\,\zeta\right),$$

where we used the expression for $\tilde{Y}$ according to (2) (recall that we used "$\tilde{Y}$" here in place of "Y" in (2)). This completes the proof of Proposition 3.2.

Remark: Note that in the $(u\,,\,v)$–coordinates, Proposition 3.2 gives

$$F(u\,,\,v) = \left(\tfrac{u}{2} - v + C_1\,,\,i\,(\tfrac{u}{2} + v + C_2)\,,\,F_3(u\,,\,v)\,,\ldots,\,F_N(u\,,\,v)\right)$$

for some constants C_1 and C_2. Thus, after a translation in the first two coordinates, Proposition 3.2 essentially describes F locally as the *graph* of a map over the fixed "parameter plane"

$$\operatorname{span}\{\,\tfrac{u}{2}\,(1\,,\,i\,,\,0\,,\ldots,\,0) + v\,(-1\,,\,i\,,\,0\,,\ldots,\,0)\mid u\,,\,v\in\mathbf{C}\} \subset \mathbf{C}^N\,.$$

The next step is to give a procedure for how one can utilize Proposition 3.2 to construct local examples of minimal real Kähler surfaces. Since the maps $(\frac{1}{2}, \frac{i}{2}, X)$ and $(-1, i, Y)$ as in (8) always span an isotropic plane in $\mathbf{C}^2$, by Theorem 1.1 one only needs to find suitable holomorphic maps ξ, ζ, and λ as in (8) such that, in addition, it is guaranteed that $(\frac{1}{2}, \frac{i}{2}, X)$ and $(-1, i, Y)$ form the "gradient" of a map F with respect to some complex chart (u, v). By the complex Poincaré Lemma, this is obviously equivalent to requiring that, with respect to these coordinates, we have

$$X_v = Y_u\,. \tag{12}$$

Let us write this relation in terms of ξ, ζ, and λ. Since by (8)

$$X = \lambda\left(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\, \xi\right) \quad \text{and} \quad Y = \mu\left(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\, \zeta\right),$$

the integrability condition (12) can be rewritten as

$$\left((\lambda\,\tfrac{1-\xi^2}{2})_v\,,\, i\,(\lambda\,\tfrac{1+\xi^2}{2})_v\,,\, (\lambda\,\xi)_v\right) = \left((\mu\,\tfrac{1-\zeta^2}{2})_u\,,\, i\,(\mu\,\tfrac{1+\zeta^2}{2})_u\,,\, (\mu\,\zeta)_u\right).$$

By comparing components, we obtain

$$\begin{aligned} \lambda_v - (\lambda\,\xi^2)_v &= \mu_u - (\mu\,\zeta^2)_u\,, \\ \lambda_v + (\lambda\,\xi^2)_v &= \mu_u + (\mu\,\zeta^2)_u\,, \\ \text{and} \quad (\lambda\,\xi)_v &= (\mu\,\zeta)_u\,. \end{aligned}$$

By adding or subtracting the first two equations, respectively, we derive the following three equations, which are equivalent to the **integrability conditions** for F:

$$\left.\begin{aligned} (\lambda\,\xi^2)_v &= (\mu\,\zeta^2)_u\,, \\ \lambda_v &= \mu_u\,, \\ \text{and} \quad (\lambda\,\xi)_v &= (\mu\,\zeta)_u\,. \end{aligned}\right\} \tag{13}$$

Summing up the results we have obtained so far, we have the following

Proposition 3.3: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal isometric immersion from a (real) 4–dimensional Kähler manifold M into $\mathbf{R}^N$ (where $N \geq 5$). Then every point in M has a neighborhood U with a complex chart (u, v) defined on U such that the holomorphic representative of f on U is determined by* (8)*, where the holomorphic maps X and Y satisfy $X_v = Y_u$, or equivalently, where ξ, ζ, λ, and μ as defined in* (8) *satisfy* (13).

Conversely, if the holomorphic maps $\xi, \zeta : U \to \mathbf{C}^{N-4}$ *and* $\lambda : U \to \mathbf{C}$ *are defined on a simply connected open subset* U *of* $\mathbf{C}^2$*, and if they satisfy* (7) *and* (13) *on all of* U *(with* μ *defined as in* (8)*), then the* $\mathbf{C}^N$*-valued* 1*-form*

$$\omega := \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix} du + \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix} dv$$

with X *and* Y *as in* (8) *is exact on* U*, and if* $F : U \to \mathbf{C}^N$ *is a holomorphic map such that* $dF = \omega$*, then* $f := \sqrt{2}\,\mathrm{Re}(F) : M \to \mathbf{R}^N$ *is a minimal isometric immersion from the Kähler manifold* $M := (U, f^* \langle\, , \rangle)$ *into* $\mathbf{R}^N$.

We will call the triple of maps $((u, v), X, Y)$ that describes the local representation of a minimal real Kähler surface as given in this proposition a **Weierstrass representation** of the complex surface.

In the next chapter, we will use this local characterization of minimal real Kähler surfaces to give an explicit local construction method for such manifolds in codimension 2, i.e. in $\mathbf{R}^6$. But before we conclude the present chapter, let us first give two lemmata that work in all codimensions. The first one asserts that the "scaling function" λ in the Weierstrass representation is actually uniquely determined, up to a constant multiple, by the maps ξ and ζ. The second lemma shows that one can switch the roles played by the maps X and Y in 3.3 (at least up to a factor 2) by reparametrizing and reflecting the minimal real Kähler surface at a hyperplane.

Lemma 3.4: *Let* $\xi, \zeta : U \to \mathbf{C}^n$ *and* $\lambda : U \to \mathbf{C}$ *be holomorphic maps, defined on some simply connected open set* $U \subset \mathbf{C}^2$*; write the complex coordinates in* $\mathbf{C}^2$ *as* (u, v)*. Assume that, on all of* U*,* $\lambda(\xi - \zeta)^2 \neq 0$*, and thus that the holomorphic function*

$$\mu := -\frac{2}{\lambda(\xi - \zeta)^2}$$

is well defined on all of U*. Furthermore, assume that these maps satisfy the following partial differential equations:*

$$
\begin{aligned}
(\lambda\,\xi)_v &= (\mu\,\zeta)_u\,, && (13.a)\\
\lambda_v &= \mu_u\,, && (13.b)\\
\textit{and}\ \ (\lambda\,\xi^2)_v &= (\mu\,\zeta^2)_u\,. && (13.c)
\end{aligned}
$$

Then we have that, on all of U (which is simply connected!),

$$
\begin{aligned}
d(\log\lambda) &= -\frac{2}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot d\xi\\
&\left(= -\frac{2}{\sum_{j=1}^{n}(\xi_j-\zeta_j)^2}\,\sum_{j=1}^{n}(\xi_j-\zeta_j)\,d\xi_j\right).
\end{aligned}
$$

Proof: Expanding (13.a) and using (13.b) gives

$$\lambda_v\,\xi+\lambda\,\xi_v=\mu_u\,\zeta+\mu\,\zeta_u=\lambda_v\,\zeta+\mu\,\zeta_u\,,$$

which is equivalent to

$$\lambda_v\,(\xi-\zeta)=\mu\,\zeta_u-\lambda\,\xi_v\,.$$

Taking the symmetric inner product of the last equation with $(\xi+\zeta)$ and $(\xi-\zeta)$, respectively, gives

$$\lambda_v\,(\xi^2-\zeta^2)=\mu\,\xi\cdot\zeta_u+\mu\,\zeta\cdot\zeta_u-\lambda\,\xi\cdot\xi_v-\lambda\,\zeta\cdot\xi_v \tag{14}$$

and

$$\lambda_v\,(\xi-\zeta)^2=\mu\,(\xi-\zeta)\cdot\zeta_u-\lambda\,(\xi-\zeta)\cdot\xi_v\,. \tag{15}$$

Expanding (13.c) and using (13.b) gives

$$\lambda_v\,\xi^2+\lambda\,(2\,\xi\cdot\xi_v)=\lambda_v\,\zeta^2+\mu\,(2\,\zeta\cdot\zeta_u)\,,$$

which is equivalent to

$$\lambda_v\,(\xi^2-\zeta^2)=2\,\mu\,\zeta\cdot\zeta_u-2\,\lambda\,\xi\cdot\xi_v\,.$$

Subtracting the last equation from (14) and reordering terms results in

$$\mu\,(\xi-\zeta)\cdot\zeta_u=-\lambda\,(\xi-\zeta)\cdot\xi_v\,. \tag{16}$$

And using this last equation in (15) and dividing by $(\xi-\zeta)^2$ (which is never 0 by hypothesis) gives

$$\lambda_v = -\frac{2\,\lambda}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot\xi_v\,. \tag{17}$$

Now, by defintion of μ, we have that $\lambda\,\mu\,(\xi-\zeta)^2 = -2$, so that differentiating this equation with respect to u results in

$$0 = \lambda_u\,\mu\,(\xi-\zeta)^2 + \lambda\,\mu_u\,(\xi-\zeta)^2 + 2\,\lambda\,\mu\,(\xi-\zeta)\cdot(\xi_u-\zeta_u)\,.$$

Utilizing (13.b) again to replace μ_u by λ_v, using formula (17) for λ_v, and replacing $\mu\,(\xi-\zeta)\cdot\zeta_u$ by the right–hand side of (16), the last equation gives us that

$$0 = \lambda_u\,\mu\,(\xi-\zeta)^2 - \lambda\,\frac{2\,\lambda}{(\xi-\zeta)^2}\,((\xi-\zeta)\cdot\xi_v)\,(\xi-\zeta)^2 + 2\,\lambda\,\mu\,(\xi-\zeta)\cdot\xi_u + 2\,\lambda^2\,(\xi-\zeta)\cdot\xi_v\,.$$

As one sees, the second and fourth term on the right hand side cancel each other, so that after reorganizing and dividing by $\mu\,(\xi-\zeta)^2$ (which by hypothesis is never zero) we obtain

$$\lambda_u = -\frac{2\,\lambda}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot\xi_u\,.$$

Since λ is never zero and U simply connected, we have that

$$(\log\lambda)_u = \frac{\lambda_u}{\lambda} \quad\text{and}\quad (\log\lambda)_v = \frac{\lambda_v}{\lambda}\,,$$

which together with (17) and the above equation for λ_u proves Lemma 3.4.

Lemma 3.5: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal real Kähler immersion, and let $((u\,,v)\,,X\,,Y)$ be a Weierstrass representation of f that is defined on some open subset U of M. Furthermore, let $A : \mathbf{R}^N \to \mathbf{R}^N$ denote the reflection at the hyperplane given by all but the first coordinate in $\mathbf{R}^N$; i.e. the matrix of A is*

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in O(N)\,.$$

Let $\tilde{f}$ be the to f congruent isometric immersion $\tilde{f} := A \circ f : M \to \mathbb{R}^N$. Then the functions $\tilde{u} := 2\,v$ and $\tilde{v} := \frac{1}{2}\,u$ form a complex coordinate system on U (as an open submanifold of M), and the Weierstrass representation of $\tilde{f}$ with respect to the chart $(\tilde{u}\,,\,\tilde{v})$ on U is given by the maps

$$\tilde{X} = \tfrac{1}{2}\,Y \quad \text{and} \quad \tilde{Y} = 2\,X\,;$$

i.e., $\tilde{X}$ and $\tilde{Y}$ can be written in terms of suitably chosen holomorphic maps $\tilde{\xi}\,,\,\tilde{\zeta} : U \to \mathbb{C}^{N-4}$ and $\tilde{\lambda} : U \to \mathbb{C}$ as in (8).

This is the precise sense in which we may "switch X and Y in the Weierstrass representation of a minimal real Kähler surface", if we wish.

Proof: Let F be the holomorphic representative of f on U that is given by $((u\,,\,v)\,,\,X\,,\,Y)$. Then by Lemma 1.3, $\tilde{F} := A \circ F$ is the holomorphic representative of $\tilde{f}$ on U. Furthermore, we have by (8) that

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \tilde{X} \end{pmatrix} = \tilde{F}_{\tilde{u}} = A \circ \frac{\partial F}{\partial \tilde{u}} = A\Big(\underbrace{\frac{\partial u}{\partial \tilde{u}}}_{=0} \frac{\partial F}{\partial u} + \underbrace{\frac{\partial v}{\partial \tilde{u}}}_{=1/2} \frac{\partial F}{\partial v} \Big) = \frac{1}{2}\,A \circ F_v$$

$$= \frac{1}{2} \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \frac{1}{2}\,Y \end{pmatrix}.$$

Comparing components, we see that we have

$$\tilde{X} = \frac{1}{2}\,Y\,. \tag{18}$$

Similarly, we obtain that

$$\begin{pmatrix} -1 \\ i \\ \tilde{Y} \end{pmatrix} = \tilde{F}_{\tilde{v}} = A \circ \frac{\partial F}{\partial \tilde{v}} = 2\,A \circ F_u = \begin{pmatrix} -1 \\ i \\ 2\,X \end{pmatrix},$$

and thus

$$\tilde{Y} = 2\,X\,. \tag{19}$$

It remains to show that we can indeed express $\tilde{X}$ and $\tilde{Y}$ in terms of $\tilde{\xi}$, $\tilde{\zeta}$, and $\tilde{\lambda}$ as in (8) to establish that they do represent the Weierstrass representation of $\tilde{F}$ in the complex chart $(\tilde{u}\,,\,\tilde{v})$. To this end, set

$$\tilde{\xi} := \zeta \ , \ \ \tilde{\zeta} := \xi \ , \ \ \text{and} \ \ \tilde{\lambda} := \frac{\mu}{2} = -\frac{1}{\lambda\,(\xi-\zeta)^2} \, .$$

With these definitions, we obtain

$$\tilde{\lambda} \begin{pmatrix} \frac{1-\tilde{\xi}^2}{2} \\ i\,\frac{1+\tilde{\xi}^2}{2} \\ \tilde{\xi} \end{pmatrix} = \frac{\mu}{2} \begin{pmatrix} \frac{1-\zeta^2}{2} \\ i\,\frac{1+\zeta^2}{2} \\ \zeta \end{pmatrix} = \frac{1}{2}\,Y = \tilde{X} \, ,$$

where we used (18) in the last step. Next, according to Proposition 3.2, we have to set

$$\tilde{\mu} := -\frac{2}{\tilde{\lambda}\,(\tilde{\xi}-\tilde{\zeta})^2} \, ,$$

which by definition of $\tilde{\xi}$, $\tilde{\zeta}$, and $\tilde{\lambda}$ leads to

$$\tilde{\mu} = +\frac{2\,\lambda\,(\xi-\zeta)^2}{(\zeta-\xi)^2} = 2\,\lambda \, .$$

Using this last equation and (19), we obtain, in a similar fashion as above, that

$$\tilde{\mu} \begin{pmatrix} \frac{1-\tilde{\zeta}^2}{2} \\ i\,\frac{1+\tilde{\zeta}^2}{2} \\ \tilde{\zeta} \end{pmatrix} = 2\,\lambda \begin{pmatrix} \frac{1-\xi^2}{2} \\ i\,\frac{1+\xi^2}{2} \\ \xi \end{pmatrix} = 2\,X = \tilde{Y} \, .$$

This finishes the proof of Lemma 3.5.

4 Minimal real Kähler surfaces in $\mathbf{R}^6$

In this chapter, we will use the Weierstrass representation for minimal real Kähler surfaces that we described in the last chapter to give a classification and a local construction method in the special case of codimension 2, i.e. when the ambient Euclidean space is $\mathbf{R}^6$. The classification will consist of two non–trivial cases, which will be distinguished by the *rank of the second osculating bundle* F'' of the holomorphic representative F of the minimal real Kähler surface. Here,

$$F'' := \operatorname{span}\left\{ \frac{\partial F}{\partial z_j}, \frac{\partial^2 F}{\partial z_j \partial z_k} \,\middle|\, 1 \leq j, k \leq 2 \right\}$$

for any complex chart (z_1, z_2) on M. Note that F'' is independent of the choice of the complex chart. In fact, since F'' basically consists of vectors that define the complex Gauss map of f and its "first derivatives", it is also independent of the choice of F, and thus an *invariant* for our minimal real Kähler immersion f. In terms of a Weierstrass representation $((u, v), X, Y)$ of f (compare page 37 and Proposition 3.2), we obtain

$$F'' = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix}, \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ X_u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ X_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Y_u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ Y_v \end{pmatrix} \right\},$$

and by Laplace's determinant criterion, it is clear that we have

$$\operatorname{rank} F'' = 2 + \dim \operatorname{span}\{X_u, X_v = Y_u, Y_v\}. \tag{1}$$

Since F and thus X and Y are holomorphic (and M without loss of generality connected), this rank is, therefore, constant on M, perhaps except for some isolated points. For this reason, it makes sense to talk about "the rank" of the second osculating bundle of F.

The case $\operatorname{rank} F'' = 2$ is trivial, since this means that M is simply a (piece of a) 4-plane in $\mathbf{R}^6$. We will see soon that we must always have $\operatorname{rank} F'' \leq 4$, and that the "generic case" is $\operatorname{rank} F'' = 4$, since all these minimal real Kähler surfaces in $\mathbf{R}^6$ will (locally) emerge from the same construction method. The only remaining case, $\operatorname{rank} F'' = 3$, will lead to three classes of minimal real Kähler surfaces, one of which are the ones generated by isotropic cylinders (as in Example 1.6) with *one* fixed isotropic direction

$\mathbf{X}_1$, and another class being all the immersions $f : M \to \mathbf{R}^6$ that are *holomorphic* with respect to some complex structure on $\mathbf{R}^6$. Note that since the maximal dimension of an isotropic subspace of $\mathbf{C}^6$ is 3, the criterion mentioned on page 12 (by [R-T]) implies that the second osculating bundle of such an holomorphic map must necessarily have rank ≤ 3. Thus, our "generic case" will always lead to *non–holomorphic* maps.

To establish that the case $\operatorname{rank} F'' = 4$ is indeed "generic", we need the following proposition, which explores the case that the integrability condition $X_u = Y_v$ is trivially satisfied in detail.

Proposition 4.1: *Let $f : M^4 \to \mathbf{R}^6$ be a minimal real Kähler surface, and let $((u, v), X, Y)$ be a Weierstrass representation of f that is defined on some open subset U of M (compare page 37 and Proposition 3.2). Furthermore, assume that*

$$X_v = Y_u \equiv 0\,.$$

Then, f is holomorphic with respect to some complex structure on $\mathbf{R}^6$, or f is generated by an isotropic cylinder (see Example 1.6). In the latter case, we have more exactly that for each point p in U, we can find a neighborhood V of p in U, a coordinate system (z_1, w) on V, a fixed non–zero isotropic vector $\mathbf{X_1}$ in $\mathbf{C}^6$, and a holomorphic map $\mathbf{Z} = \mathbf{Z}(w)$ with values in a complementary subspace to $\mathbf{X_1}$ in the isotropic orthogonal complement of $\mathbf{X_1}$, such that we have on all of V that

$$f(z_1, w) = \sqrt{2}\ \mathrm{Re}\Big(z_1\,\mathbf{X}_1 + \int \mathbf{Z}(w)\,dw\Big) + b_0$$

for some fixed vector $b_0 \in \mathbf{R}^6$.

Proof: If $X = (X_1, \dots, X_{N-2})$, then we have by Proposition 3.2 that $\lambda = X_1 - i\,X_2$ and $\xi = \frac{1}{\lambda}(X_3, \dots, X_{N-2})$. Therefore, our hypothesis implies

$$\lambda_v = 0 \quad \text{and} \quad \xi_v = 0\,;$$

i.e. λ and ξ are functions "in u alone". Similarly, we find that μ and ζ are functions "in v alone"; i.e.

$$\mu_u = 0 \quad \text{and} \quad \zeta_u = 0\,.$$

So, we have $((\xi-\zeta)^2)_u = 2\,(\xi-\zeta)\cdot\xi_u$, and thus $((\xi-\zeta)^2)_{uv} = -2\,\xi_u\cdot\zeta_v$. Using that we always have $\mu = -\frac{2}{\lambda\,(\xi-\zeta)^2}$, we see that

$$\left(\frac{-2}{\lambda\,\mu}\right)_{uv} = -2\,\xi_u\cdot\zeta_v\,,$$

and noting that $\lambda_v = 0$ and $\mu_u = 0$, we find

$$\left(\frac{1}{\lambda}\right)_u \left(\frac{1}{\mu}\right)_v = \xi_u\cdot\zeta_v\,. \tag{2}$$

Here, we have to distinguish two cases: either we have that λ or μ is constant, and thus that $\xi_u\cdot\zeta_v\equiv 0$; or we have that $\lambda_u\neq 0$, $\mu_v\neq 0$, and $\xi_u\cdot\zeta_v\neq 0$ almost everywhere (except perhaps at some isolated points).

Case 1: λ or μ is constant, and $\xi_u\cdot\zeta_v\equiv 0$. We will show that in this case, f is *generated by an isotropic cylinder.*

First, we will prove that it suffices to show that either $\mathbf{X}$ or $\mathbf{Y}$ is constant, which by our hypothesis $X_v = Y_u = 0$ is equivalent to showing that $X_u = 0$ or $Y_v = 0$. For assume e.g. that $\mathbf{X}$ is constant. Then, by $0 = \mathbf{X}_v = (F_u)_v = (F_v)_u = \mathbf{Y}_u$, $\mathbf{Y}$ is a function "in v alone". And by integrating, we obtain that

$$F(u\,,\,v) = u\,\mathbf{X} + \tilde{\mathbf{Y}}(v) + \mathbf{C}_0$$

for some map $\tilde{\mathbf{Y}} = \tilde{\mathbf{Y}}(v)$ and a constant vector $\mathbf{C}_0 \in \mathbf{C}^6$. But then $F_v = \frac{d}{dv}\tilde{\mathbf{Y}}$, and hence

$$\frac{d}{dv}(\mathbf{X}\cdot\tilde{\mathbf{Y}}) = \mathbf{X}\cdot F_v = F_u\cdot F_v = 0\,.$$

This means that $\mathbf{X}\cdot\tilde{\mathbf{Y}}$ is constant, which implies that $\mathbf{X}\cdot(\tilde{\mathbf{Y}}-\tilde{\mathbf{Y}}(v_0)) = 0$, if we fix v_0 for some point $p_0\in U$ such that $v(p_0) = v_0$. Since we can write

$$F(u\,,\,v) = u\,\mathbf{X} + (\tilde{\mathbf{Y}}(v) - \tilde{\mathbf{Y}}(v_0)) + (\tilde{\mathbf{Y}}(v_0) + \mathbf{C}_0)\,,$$

we can thus assume, without loss of generality, that $\tilde{\mathbf{Y}}$ and $\mathbf{C}_0$ are chosen in a way such that

$$\mathbf{X}\cdot\tilde{\mathbf{Y}} = 0 \quad \text{on all of } U\,.$$

This means that $\tilde{\mathbf{Y}}$ is always in the "isotropic orthogonal complement" of $\mathbf{X}$ in $\mathbf{C}^6$, which is 5–dimensional (since "$\cdot$" is non–degenerate). Denote this complement by $\mathbf{X}^{\perp^{\bullet}}$ (compare page 14).

Now, $\mathbf{X}^{\perp^\bullet}$ obviously always contains $\mathbf{X}$. Thus, by choosing an complementary subspace W to $\mathbf{X}$ in $\mathbf{X}^{\perp^\bullet}$, we can write

$$\tilde{\mathbf{Y}}(v) = \eta(v)\,\mathbf{X} + \tilde{\mathbf{Z}}(v)\,,$$

where $\eta(v)$ is a holomorphic function and $\tilde{\mathbf{Z}}(v)$ a holomorphic map with values in the *fixed* subspace W. This means that F can be written as

$$F(u\,,\,v) = (u + \eta(v))\,\mathbf{X} + \tilde{\mathbf{Z}}(v) + \mathbf{C}_0\,.$$

Changing coordinates to $z_1 := u + \eta(v)$ and $w := v$, writing $\mathbf{Z} := \frac{d}{dw}\tilde{\mathbf{Z}}$ (which obviously takes values only in W), and adjusting the domain V of this new complex chart for M, if necessary, obviously gives us the desired form of f as in Example 1.6.

To establish our claim for Case 1, it remains to be shown that we indeed have that $X_u = 0$ or $Y_v = 0$. By our case assumption, we know that either λ or μ is constant, and by Lemma 3.5 we can, without loss of generality, assume that μ is constant. By our hypothesis $X_v = Y_u \equiv 0$, we obviously have that

$$\lambda_v = 0 \quad , \quad \xi_v = 0 \quad , \quad \text{and} \quad \zeta_u = 0\,. \tag{3}$$

Since μ is defined by $\mu = \frac{-2}{\lambda\,(\xi-\zeta)^2}$, the first of these equations implies that we also have

$$\frac{\partial}{\partial v}(\xi - \zeta)^2 = \frac{\partial}{\partial v}\,\frac{-2}{\lambda\,\mu} = \frac{2\,\lambda_v}{\lambda^2\mu} = 0\,. \tag{4}$$

If we write $\xi = (\xi_1\,,\,\xi_2)$ and $\zeta = (\zeta_1\,,\,\zeta_2)$, our case assumption furthermore gives that

$$0 = \xi_u \cdot \zeta_v = (\xi_1)_u\,(\zeta_1)_v + (\xi_2)_u\,(\zeta_2)_v\,.$$

Now assume that $(\zeta_2)_v \neq 0$. Then, we can rewrite the last equation as

$$(\xi_2)_u = -\frac{(\zeta_1)_v}{(\zeta_2)_v}\,(\xi_1)_u\,.$$

But (3) means in particular that ξ is a function "in u alone", whereas ζ is a function "in v alone". Thus, the last equation implies that $-\frac{(\zeta_1)_v}{(\zeta_2)_v}$ must be a constant $C \in \mathbf{C}$. It then follows that

$$(\xi_2)_u = C\,(\xi_1)_u \quad \text{and} \quad (\zeta_1)_v = -C\,(\zeta_2)_v\,,$$

which implies that there are two more constants A, $B \in \mathbf{C}$ such that

$$\xi_2 = C\,\xi_1 + A \quad \text{and} \quad \zeta_1 = -C\,\zeta_2 + B\,.$$

Using these equations, we can calculate that

$$\begin{aligned}(\xi-\zeta)^2 &= (\xi_1-\zeta_1)^2+(\xi_2-\zeta_2)^2 = (\xi_1+C\,\zeta_2-B)^2+(C\,\xi_1+A-\zeta_2)^2\\ &= \xi_1^2+C^2\zeta_2^2+B^2+2\,C\,\xi_1\,\zeta_2-2\,B\,\xi_1-2\,BC\,\zeta_2\\ &\qquad +C^2\xi_1^2+A^2+\zeta_2^2+2\,AC\,\xi_1-2\,C\,\xi_1\,\zeta_2-2\,A\,\zeta_2\\ &= (1+C^2)\,(\xi_1^2+\zeta_2^2)+2\,(AC-B)\,\xi_1-2\,(BC+A)\,\zeta_2+(A^2+B^2)\,.\end{aligned}$$

Inserting this expression into (4) gives (since $\xi_v = 0$, by (3))

$$\begin{aligned}0 &= 2\,(1+C^2)\,\zeta_2\,(\zeta_2)_v+0-2\,(BC+A)\,(\zeta_2)_v+0\\ &= 2\,(\zeta_2)_v\left((1+C^2)\,\zeta_2-(BC+A)\right).\end{aligned}$$

But since we assumed that $(\zeta_2)_v \neq 0$, and thus not constant, this implies that the term in parenthesis must be zero. This, in turn, would mean that ζ_2 is constant, unless

$$1+C^2=0 \quad \text{and} \quad BC+A=0\,.$$

So, we obtain that ($C=i$ and $A=-i\,B$) or ($C=-i$ and $A=i\,B$). In either case, we have that

$$A^2+B^2=0 \quad \text{and} \quad AC-B=0\,.$$

But this would mean that $(\xi-\zeta)^2=0$, which is impossible for a Weierstrass representation.

Thus, we must have that $(\zeta_2)_v = 0$, and since ζ_2 is a function "in v alone", this means that ζ_2 is constant. Using this in (4), we obtain

$$0 = \frac{\partial}{\partial v}(\xi-\zeta)^2 = -2\,(\xi_1-\zeta_1)\,(\zeta_1)_v\,.$$

So, either $(\zeta_1)_v = 0$ or $\xi_1 = \zeta_1$, which after differentiating with respect to v, and by (3), also gives $(\zeta_1)_v = 0$. In any case, ζ_1 is also constant, which makes ζ constant. Since we assumed that μ is constant, this means that Y

and thus also $\mathbf{Y}$ in the Weierstrass representation of f is constant, which is what we claimed in Case 1.

Case 2: $\lambda_u \neq 0$, $\mu_v \neq 0$, and $\xi_u \cdot \zeta_v \neq 0$ almost everywhere. We will show that in this case, f is *generated by an isotropic graph* (see Example 1.4), and is thus, by Proposition 1.5, *holomorphic* with respect to some complex structure on $\mathbf{R}^6$.

To simpify notation, write the components of ξ and ζ as $\xi = (s, t)$ and $\zeta = (p, q)$. Also, write

$$a := \frac{1}{(1/\lambda)_u} \quad \text{and} \quad b := \frac{1}{(1/\mu)_v},$$

so that, after multiplying by a and b (which by case assumption are almost never zero), equation (2) now reads

$$1 = a\,b\,(\xi_u \cdot \zeta_v) = a\,b\,(s_u\,p_v + t_u\,q_v) = (a\,s_u)(b\,p_v) + (a\,t_u)(b\,q_v)\,. \tag{5}$$

Note that in the last term, all the factors in parenthesis are either functions in u alone or functions in v alone, respectively. Taking the partial derivative with respect to u gives, after dividing by b,

$$0 = (a\,s_u)_u\,p_v + (a\,t_u)_u\,q_v\,.$$

Now *assume* that we have $(a\,t_u)_u \neq 0$ almost everywhere. Then the last equation gives that

$$q_v = -\frac{(a\,s_u)_u}{(a\,t_u)_u}\,p_v\,.$$

But since q_v and p_v are functions in v alone, $\frac{(a\,s_u)_u}{(a\,t_u)_u}$ must be constant. So there is a $C \in \mathbf{C}$ such that

$$q_v = C\,p_v\,.$$

Taking the derivative with respect to v in (5) gives, after dividing by a,

$$0 = s_u\,(b\,p_v)_v + t_u\,(b\,q_v)_v = s_u\,(b\,p_v)_v + t_u\,(b\,C\,p_v)_v = (s_u + C\,t_u)(b\,p_v)_v\,.$$

Again *assuming* that almost everywhere $(b\,p_v)_v \neq 0$, we obtain

$$s_u = -C\,t_u\,.$$

But this would result in

$$\xi_u \cdot \zeta_v = \begin{pmatrix} s_u \\ t_u \end{pmatrix} \cdot \begin{pmatrix} p_v \\ q_v \end{pmatrix} = s_u\, p_v + t_u\, q_v = -C\, t_u\, p_v + C\, t_u\, p_v \equiv 0\,,$$

in contradiction to our case assumption.

Thus, we must have that $(a\, t_u)_u \equiv 0$ or that $(b\, p_v)_v \equiv 0$. By (5), this implies in the first case that we also have that $(a\, s_u)_u \equiv 0$ or that $p_v \equiv 0$, and in the second case that we also have that $(b\, q_v)_v \equiv 0$ or that $t_u \equiv 0$. But if e.g. $p_v \equiv 0$, then (5) gives $1 = (a\, t_u)(b\, q_v)$, so that $(b\, q_v)_v \equiv 0$ and, of course, $(b\, p_v)_v \equiv 0$. In any case, we find that

$$a \begin{pmatrix} s_u \\ t_u \end{pmatrix} = \frac{1}{(1/\lambda)_u}\, \xi_u \text{ is constant, or that } b \begin{pmatrix} p_v \\ q_v \end{pmatrix} = \frac{1}{(1/\mu)_v}\, \zeta_v \text{ is constant,}$$

and by Lemma 3.5 we may, without loss of generality, assume that the former is the case. Thus, there is a constant vector $\xi_1 \in \mathbf{C}^2$ such that $\xi_u = (1/\lambda)_u\, \xi_1$, which implies that there is another constant vector $\xi_0 \in \mathbf{C}^2$ such that

$$\xi = \xi(u) = \frac{1}{\lambda}\, \xi_1 + \xi_0\,.$$

With this form of ξ, (2) gives $(1/\lambda)_u\, (1/\mu)_v = (1/\lambda)_u\, \xi_1 \cdot \zeta_v$, and since $\lambda_u \neq 0$, this means

$$\xi_1 \cdot \zeta_v = \left(\frac{1}{\mu}\right)_v\,.$$

Furthermore, substituting the form of ξ as above into $(\xi - \zeta)^2 = \frac{-2}{\lambda\mu}$, we obtain

$$\frac{-2}{\lambda\mu} = \left(\tfrac{1}{\lambda}\, \xi_1 + \xi_0 - \zeta\right)^2 = \tfrac{1}{\lambda^2}\, \xi_1^2 + \tfrac{2}{\lambda}\, \xi_1 \cdot (\xi_0 - \zeta) + (\xi_0 - \zeta)^2\,. \qquad (6)$$

Taking the partial derivative with respect to v and then substituting the expression we found for $\xi_1 \cdot \zeta_v$ results in

$$-\frac{2}{\lambda} \left(\frac{1}{\mu}\right)_v = -\frac{2}{\lambda}\, \underbrace{\xi_1 \cdot \zeta_v}_{=(1/\mu)_v} - 2\,(\xi_0 - \zeta) \cdot \zeta_v\,,$$

or, after simplifying,

$$0 = 2\,(\xi_0 - \zeta) \cdot \zeta_v = \frac{\partial}{\partial v}\,(\xi_0 - \zeta)^2\,.$$

But this means that $(\xi_0 - \zeta)^2$ is constant.

Now, taking the partial derivative with respect to u in (6) leads to

$$\frac{2}{\mu}\frac{\lambda_u}{\lambda^2} = -\frac{2\,\lambda_u}{\lambda^3}\,\xi_1^2 - \frac{2\,\lambda_u}{\lambda^2}\,\xi_1\cdot(\xi_0-\zeta)\,.$$

Since $\lambda_u \neq 0$ almost everywhere, this can be simplified to

$$\frac{1}{\mu} + \xi_1\cdot(\xi_0-\zeta) = -\frac{1}{\lambda}\,\xi_1^2\,. \tag{7}$$

Taking the derivative with respect to u again gives $0 = \frac{\lambda_u}{\lambda^2}\,\xi_1^2$, and since $\lambda_u \neq 0$, we must have that $\xi_1^2 = 0$, and thus that ξ_1 is *isotropic* in $\mathbf{C}^2$. But the only isotropic vectors in $\mathbf{C}^2$ are of the form $\kappa\,(1\,,\pm i)$ for some $\kappa \in \mathbf{C}$. Thus, we have found that ξ must have the form

$$\xi(u) = \frac{\kappa}{\lambda(u)}\begin{pmatrix}1\\ \pm i\end{pmatrix} + \xi_0\,,$$

where $\kappa \in \mathbf{C}$ and $\xi_0 \in \mathbf{C}^2$ are constant.

For (7), this means $\xi_1\cdot(\xi_0-\zeta) = -\frac{1}{\mu}$, and using all these relations in (6) gives

$$\tfrac{-2}{\lambda\,\mu} = \tfrac{2}{\lambda}\,\xi_1\cdot(\xi_0-\zeta) + (\xi_0-\zeta)^2 = \tfrac{-2}{\lambda\,\mu} + (\xi_0-\zeta)^2\,.$$

Thus, $(\xi_0-\zeta)^2 = 0$; i.e. $\xi_0 - \zeta$ is also isotropic in $\mathbf{C}^2$, and we can find a holomorphic function $\eta = \eta(v)$ such that

$$\zeta(v) = \eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix} + \xi_0\,.$$

On the other hand, we need

$$-\frac{1}{\mu} = \xi_1\cdot(\xi_0-\zeta) = \xi_1\cdot\left(-\eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix}\right) = -\kappa\,\eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix}\cdot\begin{pmatrix}1\\ \pm i\end{pmatrix}\,.$$

Since this expression can never be zero, κ cannot be zero, and we always must have the opposite signs in ξ_1 and ζ, so that we find, without loss of generality, that

$$\xi_1 = \kappa\begin{pmatrix}1\\ \pm i\end{pmatrix} \quad\text{and}\quad \zeta(v) = \eta(v)\begin{pmatrix}1\\ \mp i\end{pmatrix} + \xi_0\,.$$

This means that

$$\frac{1}{\mu} = \kappa\,\eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix}\cdot\begin{pmatrix}1\\ \mp i\end{pmatrix} = \kappa\,\eta\,(1-i^2) = 2\,\kappa\,\eta\,;$$

i.e. $\eta(v) = \frac{1}{2\,\kappa\,\mu(v)}$, and we have found that

$$\zeta(v) = \frac{1}{2\,\kappa\,\mu(v)}\begin{pmatrix}1\\ \mp i\end{pmatrix} + \xi_0$$

with the same $\kappa \neq 0$ and ξ_0 as above for ξ.

Let us now insert these expressions for ξ and ζ into X and Y. If $\xi_0 = (\alpha\,,\,\beta)$, then we have

$$\xi^2 = \frac{2\,\kappa}{\lambda}\,(\alpha \pm i\,\beta) + \xi_0^2 \quad\text{and}\quad \zeta^2 = \frac{1}{\kappa\,\mu}\,(\alpha \mp i\,\beta) + \xi_0^2\,.$$

Inserting this into X and Y gives, after some reordering,

$$X = \lambda\begin{pmatrix}\frac{1-\xi_0^2}{2}\\ i\,\frac{1+\xi_0^2}{2}\\ \xi_0\end{pmatrix} + \begin{pmatrix}-\kappa\,(\alpha \pm i\,\beta)\\ i\,\kappa\,(\alpha \pm i\,\beta)\\ \kappa\\ \pm i\,\kappa\end{pmatrix}$$

and

$$Y = \mu\begin{pmatrix}\frac{1-\xi_0^2}{2}\\ i\,\frac{1+\xi_0^2}{2}\\ \xi_0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2\kappa}\,(\alpha \mp i\,\beta)\\ \frac{i}{2\kappa}\,(\alpha \mp i\,\beta)\\ \frac{1}{2\kappa}\\ \mp\frac{i}{2\kappa}\end{pmatrix}.$$

Finally, for F_u and F_v this means

$$F_u = \lambda(u)\underbrace{\begin{pmatrix}0\\ 0\\ \frac{1-\xi_0^2}{2}\\ i\,\frac{1+\xi_0^2}{2}\\ \xi_0\end{pmatrix}}_{=:\mathbf{X_3}} + \underbrace{\begin{pmatrix}\frac{1}{2}\\ \frac{i}{2}\\ -\kappa\,(\alpha \pm i\,\beta)\\ i\,\kappa\,(\alpha \pm i\,\beta)\\ \kappa\\ \pm i\,\kappa\end{pmatrix}}_{=:\mathbf{X_1}}$$

and

$$F_v = \mu(v)\,\mathbf{X}_3 + \underbrace{\begin{pmatrix} -1 \\ i \\ -\frac{1}{2\kappa}\,(\alpha \mp i\,\beta) \\ \frac{i}{2\kappa}\,(\alpha \mp i\,\beta) \\ \frac{1}{2\kappa} \\ \mp\frac{i}{2\kappa} \end{pmatrix}}_{=:\mathbf{X}_2} .$$

Note that $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are *constant* in $\mathbf{C}^6$, and a straightforward calculation shows that they indeed span an *isotropic* subspace of $\mathbf{C}^6$. Integrating, we thus obtain

$$F(u\,,\,v) = u\,\mathbf{X}_1 + v\,\mathbf{X}_2 + \Big(\int \lambda(u)\,du \;+\; \int \mu(v)\,dv\Big)\;\mathbf{X}_3 + \mathbf{C}_0$$

for some constant vector $\mathbf{C}_0 \in \mathbf{C}$, and thus by Example 1.4 that f is generated by an isotropic graph, which is what we had to show in Case 2. This finally finishes the proof of Proposition 4.1.

Note that if $\mathbf{X} = F_u$ or $\mathbf{Y} = F_v$ is constant, we immediately have that $\operatorname{rank} F'' \leq 3$. The same is trivially true when f is holomorphic (see Remarks on page 43). Thus, the last proposition and Lemma 3.5 give us the following corollary, which we will need to establish our main result about the "generic case":

Corollary 4.2: *Let* $F : V \to \mathbf{C}^6$ *be a (local) holomorphic representative of a minimal real Kähler surface* $f : M^4 \to \mathbf{R}^6$ *. If the second osculating space* F'' *of* F *has rank at least* 4 *at a point* $p \in V$ *, then there is a Weierstrass representation* $((u\,,\,v)\,,\,X\,,\,Y)$ *on some neighborhood* U *of* p *in* V *such that the map* $X : V \to \mathbf{C}^4$ *has rank* 2*; i.e.* X_u *and* X_v *are* linearly independent *everywhere on* V*.*

We will now start to describe the **generic case**, i.e. that $\operatorname{rank} F'' \geq 4$. Working in a given Weierstrass representation $((u\,,\,v)\,,\,X\,,\,Y)$ of F, we first obtain that, since X and Y are isotropic maps (i.e. $X^2 = Y^2 = 0$),

$$X \cdot X_u = X \cdot X_v = Y \cdot Y_u = Y \cdot Y_v = 0\;.$$

But we have that $X \cdot Y = 1$, and so, since $X_v = Y_u$, the above equations imply

$$X_u \cdot Y = -X \cdot Y_u = -X \cdot X_v = 0 \quad \text{and} \quad X \cdot Y_v = -X_v \cdot Y = -Y_u \cdot Y = 0\,.$$

Therefore, all these equations together mean that

$$\operatorname{span}\{X_u\,,\, X_v = Y_u\,,\, Y_v\} \perp^{\bullet} \operatorname{span}\{X\,,\, Y\}\,, \tag{8}$$

where $A \perp^{\bullet} B$ for $A\,, B \in \mathbf{C}^n$ means by definition that $A \cdot B = 0$ with respect to the standard symmetric inner product in $\mathbf{C}^n$. But since X and Y are isotropic and $X \cdot Y = 1$, X and Y must be everywhere linearly independent, and thus $\operatorname{span}\{X\,,\, Y\}$ always a *non-degenerate* 2-dimensional subspace of $\mathbf{C}^4$. Since the standard symmetric inner product in $\mathbf{C}^4$ is non-degenerate, this means that its orthogonal complement in $\mathbf{C}^4$, $\{X\,,\, Y\}^{\perp^{\bullet}}$, must also be 2-dimensional and non-degenerate. Since (8) says exactly that

$$\operatorname{span}\{X_u\,,\, X_v = Y_u\,,\, Y_v\} \subset \{X\,,\, Y\}^{\perp^{\bullet}}\,,$$

we have shown by (1) that for the holomorphic representative F of a minimal real Kähler submanifold in $\mathbf{R}^6$, we must indeed have that

$$\operatorname{rank} F'' \leq 4\,.$$

Having established that the "generic case" is $\operatorname{rank} F'' = 4$, the last corollary tells us that we may assume that we have a Weierstrass representation of F on an open set $U \subset M$ such that

$$X_u \ \text{ and } \ X_v \in \mathbf{C}^4 \ \text{ are linearly independent on all of } U\,.$$

By (8), this means exactly that

$$\operatorname{span}\{X_u\,,\, X_v\} = \operatorname{span}\{X_u\,,\, X_v = Y_u\,,\, Y_v\} = \{X\,,\, Y\}^{\perp^{\bullet}}\,. \tag{9}$$

As we will show presently, this proves to be equivalent to saying that X, X_u, and X_v are linearly independent of all of U. This, in turn, is equivalent to saying that

$$\xi_u \ \text{ and } \ \xi_v \in \mathbf{C}^2 \ \text{ are linearly independent on all of } U\,. \tag{10}$$

The reason for this is the following. Write, as in Proposition 3.2,

$$X = \lambda \left(\frac{1-\xi^2}{2} \,,\, i\,\frac{1+\xi^2}{2} \,,\, \xi \right) ,$$

where ξ is a map from U into $\mathbf{C}^2$. Differentiation with respect to u and v gives

$$X_u = \frac{\lambda_u}{\lambda} X + \lambda \big(-\xi \cdot \xi_u \,,\, i\,\xi \cdot \xi_u \,,\, \xi_u \big) ,$$

and, likewise,

$$X_v = \frac{\lambda_v}{\lambda} X + \lambda \big(-\xi \cdot \xi_v \,,\, i\,\xi \cdot \xi_v \,,\, \xi_v \big) .$$

Now, it is clear that X_u and X_v are linearly independent if X, X_u, and X_v are. Next, assume that X_u and X_v are linearly independent, and that $\beta\,\xi_u + \gamma\,\xi_v = 0$ for some complex numbers β and γ. Then, by the above formulas for X_u and X_v, we have

$$\beta\, X_u + \gamma\, X_v = \left(\beta\,\frac{\lambda_u}{\lambda} + \gamma\,\frac{\lambda_v}{\lambda} \right) X + \lambda \begin{pmatrix} -\xi \cdot (\beta\,\xi_u + \gamma\,\xi_v) \\ i\,\xi \cdot (\beta\,\xi_u + \gamma\,\xi_v) \\ \beta\,\xi_u + \gamma\,\xi_v \end{pmatrix} ,$$

and the last term vanishes by our hypothesis. Taking the symmetric product of the resulting equation with X_u and X_v, respectively, we obtain by (9) that

$$\beta\, X_u^2 + \gamma\, X_u \cdot X_v = 0 \quad \text{and} \quad \beta\, X_u \cdot X_v + \gamma\, X_v^2 = 0 ,$$

or equivalently

$$\begin{pmatrix} X_u^2 & X_u \cdot X_v \\ X_u \cdot X_v & X_v^2 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = 0 .$$

But the matrix in this equation is the Gram matrix of the vectors X_u and X_v with respect to the symmetric inner product "$\cdot$", and it is well–known that the determinant of such a matrix is zero *exactly* if span$\{X_u \,,\, X_v\}$ is not 2–dimensional or is degenerate, neither of which we know to be the case here. Thus, we must have $\beta = \gamma = 0$, and so ξ_u and ξ_v are linearly independent.

Finally, assume that ξ_u and ξ_v are linearly independent, and that for some $\alpha \,,\, \beta \,,\, \gamma \in \mathbf{C}$,

$$0 = \alpha\, X + \beta\, X_u + \gamma\, X_v .$$

Using the forms of X, X_u, and X_v as above, this means that

$$0 = \underbrace{(\alpha\,\lambda + \beta\,\lambda_u + \gamma\,\lambda_v)}_{=\alpha'} \begin{pmatrix} \frac{1-\xi^2}{2} \\ i\,\frac{1+\xi^2}{2} \\ \xi \end{pmatrix} + \beta\,\lambda \begin{pmatrix} -\xi\cdot\xi_u \\ i\,\xi\cdot\xi_u \\ \xi_u \end{pmatrix} + \gamma\,\lambda \begin{pmatrix} -\xi\cdot\xi_v \\ i\,\xi\cdot\xi_v \\ \xi_v \end{pmatrix} .$$

But dividing the second components by i and adding the resulting equation to the one given by the first components of these vectors immediately gives $\alpha' = 0$, so that we necessarily must have that $\beta\,\xi_u + \gamma\,\xi_v = 0$, and thus that $\beta = \gamma = 0$. This, in turn, implies that α must also be zero, and we have proven the claimed equivalences.

Now, ξ is a holomorphic map from the complex 2–dimensional manifold $U \subset M$ into $\mathbf{C}^2$, which implies that our assumption (10) means that ξ is a *local biholomorphism* on U. If we thus make U slightly smaller, if necessary, we may assume that $\xi : U \to \mathbf{C}^2$ is a biholomorphism on all of U, and hence a *complex chart* of M on U. For convenience, we will write the component functions of this chart ξ as

$$s := \xi_1 \quad \text{and} \quad t := \xi_2\,,$$

and will now write X and Y "in the new coordinates $(s\,,\,t)$"; i.e. we are looking at the maps $X\circ\xi^{-1}$ and $Y\circ\xi^{-1}$, for which we will use the "classical" notation

$$\left.\begin{array}{l} X(s\,,\,t) = \lambda(s\,,\,t)\left(\dfrac{1-s^2-t^2}{2}\,,\; i\,\dfrac{1+s^2+t^2}{2}\,,\; s\,,\; t\right) \\ \text{and} \\ Y(s\,,\,t) = \mu(s\,,\,t)\left(\dfrac{1-\zeta^2(s\,,\,t)}{2}\,,\; i\,\dfrac{1+\zeta^2(s\,,\,t)}{2}\,,\; \zeta_1(s\,,\,t)\,,\; \zeta_2(s\,,\,t)\right). \end{array}\right\} \quad (11)$$

Remark: This change of coordinates essentially amounts to the following. Our assumption (9) means that the map $X : U \to \mathbf{C}^4$ is an immersion whose "Gauss map" $\text{span}\{X_u\,,\,X_v\}$ is basically identical to the complex Gauss map of f on U. Since X is also isotropic, we have the following commutative diagram:

$$\begin{array}{ccc} & X & \\ U & \longrightarrow & \mathbf{C}^4 - \{0\} \\ \tilde{X} \downarrow & & \downarrow \pi \\ Q_2 & \hookrightarrow & \mathbf{C}P^3 \end{array}$$

where $Q_2 = \{\pi(Z) \,|\, Z^2 = 0\}$ is the complex quadric in complex projective 3-space $\mathbf{C}P^3$, and $\pi : (\mathbf{C}^4 - \{0\}) \to \mathbf{C}P^3$ the canonical projection. Since our assumption that X_u and X_v are linearly independent is equivalent to X, X_u, and X_v being linearly independent, we have that $\tilde{X}$ is also an immersion, and since the (complex) dimension of Q_2 is 2 (thus the index), this implies that $\tilde{X}$ is a local biholomorphism. It is well–known that our ξ is nothing but a parametrization for Q_2 (see e.g. [H-O]), so that $\tilde{X}$ corresponds exactly to our coordinate system $\xi = (s\,,\,t)$. Thus, $X(s\,,\,t)$ and $Y(s\,,\,t)$ "*represent our minimal real Kähler submanifold in (complex) Gauss map coordinates*". This corresponds to the classical case where one takes the Gauss map of a minimal surface in 3–space to find isothermal coordinates for the minimal surface (see e.g. [S], page 385 and 386).

We see in formula (11) that changing to the new coordinates $(s\,,\,t)$ determines X up to the scaling factor $\lambda(s\,,\,t)$. We will find that Y is also *completely* determined by $\lambda(s\,,\,t)$ alone! Observe that by (11), X_s and X_t simply have the form

$$X_s = \frac{\lambda_s}{\lambda} X + \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0) \quad \text{and} \quad X_t = \frac{\lambda_t}{\lambda} X + \lambda\,(-t\,,\,i\,t\,,\,0\,,\,1)\,. \quad (12)$$

But by (9), we still have

$$X_s \cdot Y = \frac{\partial u}{\partial s} X_u \cdot Y + \frac{\partial v}{\partial s} X_v \cdot Y = 0\,,$$

and likewise $X_t \cdot Y = 0$. Replacing the expressions for X_s and X_t, as above, and the expression for Y in (11) gives (remembering that $X \cdot Y = 1$)

$$\begin{aligned} 0 = X_s \cdot Y &= \frac{\lambda_s}{\lambda} X \cdot Y + \lambda\,\mu\,(-s\,,\,i\,s\,,\,1\,,\,0) \cdot \left(\frac{1-\zeta^2}{2}\,,\,i\,\frac{1+\zeta^2}{2}\,,\,\zeta_1\,,\,\zeta_2\right) \\ &= \frac{\lambda_s}{\lambda} + \lambda\mu\left(\frac{-s + s\,\zeta^2 - s - s\,\zeta^2}{2} + \zeta_1\right) = \frac{\lambda_s}{\lambda} + \lambda\mu\,(\zeta_1 - s)\,, \end{aligned}$$

and in a completely analogous fashion we find that

$$\frac{\lambda_t}{\lambda} + \lambda\mu\,(\zeta_2 - t) = 0\,.$$

Using that

$$\lambda\,\mu = \frac{-2}{(\xi-\zeta)^2} = \frac{-2}{(s-\zeta_1)^2 + (t-\zeta_2)^2}\,,$$

we obtain

$$\zeta_1 - s = \frac{\lambda_s}{2\,\lambda}\left((\zeta_1 - s)^2 + (\zeta_2 - t)^2\right) \quad \text{and} \quad \zeta_2 - t = \frac{\lambda_t}{2\,\lambda}\left((\zeta_1 - s)^2 + (\zeta_2 - t)^2\right)\,.$$

Multiplying the first equation by λ_t, the second by λ_s, and then subtracting these equations from each other results in

$$\lambda_t\,(\zeta_1 - s) - \lambda_s\,(\zeta_2 - t) = 0\,. \tag{13}$$

Now, note that λ cannot be constant, since otherwise $\lambda_s = \lambda_t = 0$, and in this case the above equations give that $\zeta = (s\,,\,t) = \xi$, which cannot happen, since for a Weierstrass representation we have that $(\xi - \zeta)^2 \neq 0$. Therefore, we may assume, without loss of generality, that λ_s is not identically equal to zero, and thus, as a holomorphic function, only zero at some isolated points in U. By (13), we then have at all places where $\lambda_s \neq 0$ that

$$\zeta_2 - t = \frac{\lambda_t}{\lambda_s}\,(\zeta_1 - s)\,,$$

and inserting this expression into the one for $\zeta_1 - s$ gives

$$\zeta_1 - s = \frac{\lambda_s}{2\,\lambda}\,(\zeta_1 - s)^2\left(1 + \frac{\lambda_t^2}{\lambda_s^2}\right) = \frac{\lambda_s^2 + \lambda_t^2}{2\,\lambda\,\lambda_s}\,(\zeta_1 - s)^2\,.$$

But if $\zeta_1 - s = 0$ at some point where $\lambda_s \neq 0$, then by (13) we also would have that $\zeta_2 - t = 0$, which again would lead to $(\xi - \zeta)^2 = 0$, and is thus impossible. Consequently, we can divide by $\zeta_1 - s$ (almost everywhere), which leads to the following result:

λ has no singularities in U, and we have

$$\left.\zeta_1 = s + \frac{2\,\lambda\,\lambda_s}{\lambda_s^2 + \lambda_t^2} \quad \text{and} \quad \zeta_2 = t + \frac{2\,\lambda\,\lambda_t}{\lambda_s^2 + \lambda_t^2}\,.\right\} \tag{14}$$

Note that these formulas are still true at the isolated points $q \in U$ where $\lambda_s(q) = 0$; by what we showed above, we must then necessarily have that $\lambda_t(q) \neq 0$. These are the promised formulas that give ζ in terms of λ alone. Now we can also calculate μ in terms of λ alone:

$$\mu = \frac{-2}{\lambda\,((s-\zeta_1)^2 + (t-\zeta_2)^2)} = \frac{-2\,(\lambda_s^2 + \lambda_t^2)^2}{\lambda\,((2\,\lambda\,\lambda_s)^2 + (2\,\lambda\,\lambda_t)^2)},$$

which gives

$$\mu = -\frac{\lambda_s^2 + \lambda_t^2}{2\,\lambda^3}\,. \tag{15}$$

For later reference, we also need to calculate Y_s and Y_t in terms of λ alone. This can be done directly by using the formulas above, but the following approach is much shorter. By our assumption (9), we know that Y_s and Y_t are always in the span of X_s and X_t, so that we can find holomorphic functions a, b, c, and d such that

$$Y_s = a\,X_s + b\,X_t \quad \text{and} \quad Y_t = d\,X_s + c\,X_t\,.$$

Taking the symmetric inner product of either equation with X_s and X_t, and recalling that $Y \cdot X_s = 0$ and $Y \cdot X_t = 0$, we obtain the equations

$$\left.\begin{aligned} Y_s \cdot X_s &= -Y \cdot X_{ss} = a\,X_s^2 + b\,X_s \cdot X_t\,, \\ Y_s \cdot X_t &= -Y \cdot X_{st} = a\,X_s \cdot X_t + b\,X_t^2\,, \\ Y_t \cdot X_s &= -Y \cdot X_{ts} = d\,X_s^2 + c\,X_s \cdot X_t\,, \\ Y_t \cdot X_t &= -Y \cdot X_{tt} = d\,X_s \cdot X_t + c\,X_t^2\,. \end{aligned}\right\} \tag{16}$$

Let us calculate the expressions for X_s^2, X_t^2, and $X_s \cdot X_t$. By (12), and since $X \cdot X_s = 0$, we have

$$\begin{aligned} X_s^2 &= \left(\tfrac{\lambda_s}{\lambda}\,X + \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0)\right) \cdot X_s \\ &= \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0) \cdot \left[\lambda_s\left(\tfrac{1-s^2-t^2}{2}\,,\; i\,\tfrac{1+s^2+t^2}{2}\,,\; s\,,\; t\right) + \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0)\right] \\ &= \lambda\,\lambda_s\left(\tfrac{-s+s^3+s\,t^2-s-s^3-s\,t^2}{2} + s\right) + \lambda^2\,(s^2 - s^2 + 1) \\ &= \lambda^2\,, \end{aligned}$$

and in the same way we find

$$X_t^2 = \lambda^2\,.$$

Further,

$$\begin{aligned} X_s \cdot X_t &= \left(\tfrac{\lambda_s}{\lambda} X + \lambda(-s, is, 1, 0)\right) \cdot X_t \\ &= \lambda(-s, is, 1, 0) \cdot \left[\lambda_t \left(\tfrac{1-s^2-t^2}{2}, i\,\tfrac{1+s^2+t^2}{2}, s, t\right) + \lambda(-t, it, 0, 1)\right] \\ &= \lambda\lambda_t \left(\tfrac{-s+s^3+st^2-s-s^3-st^2}{2} + s\right) + \lambda^2 (st - st) \\ &= 0\,. \end{aligned}$$

By (16), these equations immediately give that

$$a = -\frac{Y \cdot X_{ss}}{\lambda^2}\ ,\ \ b = d = -\frac{Y \cdot X_{st}}{\lambda^2}\ ,\ \ \text{and}\ \ c = -\frac{Y \cdot X_{tt}}{\lambda^2}\,. \tag{17}$$

Furthermore, we have that

$$X_{ss} = \left(\frac{\lambda_s}{\lambda}\right)_s X + \frac{\lambda_s}{\lambda} X_s + \lambda_s(-s, is, 1, 0) + \lambda(-1, i, 0, 0)\,.$$

Taking the symmetric inner product of X_{ss} with Y, and remembering that $X \cdot Y = 1$ and $X_s \cdot Y = 0$, results in

$$\begin{aligned} X_{ss} \cdot Y &= \left(\tfrac{\lambda_s}{\lambda}\right)_s X \cdot Y + \tfrac{\lambda_s}{\lambda} X_s \cdot Y \\ &\qquad + \lambda_s(-s, is, 1, 0) \cdot \mu\left(\tfrac{1-\zeta^2}{2}, i\,\tfrac{1+\zeta^2}{2}, \zeta_1, \zeta_2\right) \\ &\qquad\quad + \lambda(-1, i, 0, 0) \cdot \mu\left(\tfrac{1-\zeta^2}{2}, i\,\tfrac{1+\zeta^2}{2}, \zeta_1, \zeta_2\right) \\ &= \left(\tfrac{\lambda_s}{\lambda}\right)_s + \mu\lambda_s\left(\tfrac{-s+s\zeta^2-s-s\zeta^2}{2} + \zeta_1\right) + \lambda\mu\,\tfrac{-1+\zeta^2-1-\zeta^2}{2} \\ &= \left(\tfrac{\lambda_s}{\lambda}\right)_s + \mu\left(\lambda_s(\zeta_1 - s) - \lambda\right)\,. \end{aligned}$$

But by (14) and (15), $\zeta_1 - s$ and μ in the second term can be replaced, to give

$$\begin{aligned} X_{ss} \cdot Y &= \left(\tfrac{\lambda_s}{\lambda}\right)_s - \tfrac{\lambda_s^2+\lambda_t^2}{2\lambda^3}\left(\lambda_s \tfrac{2\lambda\lambda_s}{\lambda_s^2+\lambda_t^2} - \lambda\right) = \tfrac{2\lambda_{ss}\lambda - 2\lambda_s^2}{2\lambda^2} - \tfrac{2\lambda_s^2-(\lambda_s^2+\lambda_t^2)}{2\lambda^2} \\ &= \frac{2\lambda_{ss}\lambda - 3\lambda_s^2 + \lambda_t^2}{2\lambda^2}\,. \end{aligned}$$

Completely analogously, one will find that

$$X_{tt} \cdot Y = \frac{2\,\lambda_{tt}\,\lambda + \lambda_s^2 - 3\,\lambda_t^2}{2\,\lambda^2} \, .$$

Finally, since

$$X_{st} = \left(\frac{\lambda_s}{\lambda}\right)_t X + \frac{\lambda_s}{\lambda} X_t + \lambda_t\,(-s\,,\, i\,s\,,\, 1\,,\, 0)\,,$$

we find that

$$\begin{aligned} X_{st} \cdot Y \;&=\; \left(\tfrac{\lambda_s}{\lambda}\right)_t X \cdot Y + \tfrac{\lambda_s}{\lambda} X_t \cdot Y \\ &\qquad + \lambda_t\,(-s\,,\, i\,s\,,\, 1\,,\, 0) \cdot \mu \left(\tfrac{1-\zeta^2}{2}\,,\, i\,\tfrac{1+\zeta^2}{2}\,,\, \zeta_1\,,\, \zeta_2\right) \\ &=\; \left(\tfrac{\lambda_s}{\lambda}\right)_t + \lambda_t\,\mu\,(\zeta_1 - s) = \frac{\lambda_{st}\,\lambda - \lambda_s\,\lambda_t}{\lambda^2} - \frac{\lambda_t\,(\lambda_s^2 + \lambda_t^2)}{2\,\lambda^3}\,\frac{2\,\lambda\,\lambda_s}{\lambda_s^2 + \lambda_t^2} \\ &=\; \frac{2\,\lambda_{st}\,\lambda - 4\,\lambda_s\,\lambda_t}{2\,\lambda^2} \, . \end{aligned}$$

Using these expressions in (17) gives us a, b, and c, and hence Y_s and Y_t, in terms of λ alone, namely:

$$\left.\begin{aligned} &\textit{We have } Y_s = a\,X_s + b\,X_t \;\textit{ and }\; Y_t = b\,X_s + c\,X_t\,, \textit{ where} \\ &a = \frac{-2\,\lambda_{ss}\,\lambda + 3\,\lambda_s^2 - \lambda_t^2}{2\,\lambda^4}\,, \quad b = \frac{-2\,\lambda_{st}\,\lambda + 4\,\lambda_s\,\lambda_t}{2\,\lambda^4}\,, \\ &\qquad \textit{and} \quad c = \frac{-2\,\lambda_{tt}\,\lambda - \lambda_s^2 + 3\,\lambda_t^2}{2\,\lambda^4}\,. \end{aligned}\right\} \quad (18)$$

Note that we have just shown that *in "$(s\,,\,t)$-coordinates", X and Y are essentially determined by λ alone.* The question remains whether *every* function λ is possible. We will see shortly that this is basically the case, the only essential restriction being that λ has *no singularities* (cf. (14)).

To this end, note that our integrability condition $X_v = Y_u$ in $(u\,,\,v)$-coordinates looks as follows in $(s\,,\,t)$-coordinates, if we use (18) (and write $s_u = \frac{\partial s}{\partial u}$, etc.):

$$\begin{aligned} X_v \;&=\; \frac{\partial s}{\partial v} X_s + \frac{\partial t}{\partial v} X_t = Y_u = \frac{\partial s}{\partial u} Y_s + \frac{\partial t}{\partial u} Y_t \\ &=\; (a\,s_u + b\,t_u)\,X_s + (b\,s_u + c\,t_u)\,X_t \, . \end{aligned}$$

Since in our generic case X_s and X_t must be everywhere linearly independent, this leads to the following condition, which is equivalent to the integrability condition $X_v = Y_u$:

$$s_v = a\,s_u + b\,t_u \quad \text{and} \quad t_v = b\,s_u + c\,t_u\,. \tag{19}$$

But recall that we have (in "classical notation") $(s\,,\,t) = \xi(u\,,\,v)$, and thus $(u\,,\,v) = \xi^{-1}(s\,,\,t)$. By virtue of the Inverse Mapping Theorem, this gives

$$\begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix} = d(\xi^{-1}) = (d\,\xi)^{-1} \circ \xi^{-1} = \begin{pmatrix} s_u & s_v \\ t_u & t_v \end{pmatrix}^{-1} \circ \xi^{-1}$$

$$= \left(\frac{1}{\det(d\,\xi)} \begin{pmatrix} t_v & -s_v \\ -t_u & s_u \end{pmatrix} \right) \circ \xi^{-1}\,,$$

and comparing the entries of these matrices gives

$$\begin{aligned} s_u &= \det(d\,\xi)\,(v_t \circ \xi) \quad , \quad & s_v &= -\det(d\,\xi)\,(u_t \circ \xi)\ , \\ t_u &= -\det(d\,\xi)\,(v_s \circ \xi) \quad , \quad & t_v &= \det(d\,\xi)\,(u_s \circ \xi)\ . \end{aligned}$$

Replacing these expressions in (19) leads to

$$-(\det(d\,\xi) \circ \xi^{-1})\,u_t = (\det(d\,\xi) \circ \xi^{-1})\,(a\,v_t - b\,v_s)$$

and

$$(\det(d\,\xi) \circ \xi^{-1})\,u_s = (\det(d\,\xi) \circ \xi^{-1})\,(b\,v_t - c\,v_s)\,,$$

so that canceling the common (non-zero) factor results in the following *integrability condition in terms of* $(s\,,\,t)$*-coordinates* :

$$X_v = Y_u \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} u_s = -c\,v_s + b\,v_t \\ \text{and} \\ u_t = b\,v_s - a\,v_t\,. \end{array} \right\} \tag{20}$$

Note that the coefficients a, b, and c are determined by $\lambda(s\,,\,t)$ alone as in (18), and that the equations in (20) determine u (as a function in $(s\,,\,t)$-coordinates) in terms of v alone, up to a constant. But for the equations in (20) to be (locally) integrable, they in turn need to satisfy the integrability condition $(u_s)_t = (u_t)_s$; i.e. by (20):

$$(u_s)_t = -c_t\,v_s - c\,v_{st} + b_t\,v_t + b\,v_{tt} = (u_t)_s = b_s\,v_s + b\,v_{ss} - a_s\,v_t - a\,v_{st}\,,$$

or equivalently:

$$b\,v_{ss} + (c-a)\,v_{st} - b\,v_{tt} + (b_s + c_t)\,v_s - (b_t + a_s)\,v_t = 0\,. \tag{21}$$

This is a second order *linear* homogeneous partial differential equation in v (as a function in $(s\,,\,t)$-coordinates) whose coefficients are holomorphic in $(s\,,\,t)$ and completely determined by λ as in (18). By the Cauchy–Kowalewski Theorem (see [Hö], pages 348 to 350), such a partial differential equation always has (local) solutions, where we even have the freedom to choose functions in one complex variable along certain complex curves, the so–called "characteristics" of the equation (see our Example 4.4 below).

Since all the necessary conditions for the integrability of the functions involved are locally sufficient, we have, therefore, proven the following

Theorem 4.3: *Let* $\tilde\lambda : V \to \mathbf{C}$ *be any* nowhere zero *holomorphic function in the two complex variables* $(s\,,\,t)$ *on an open subset* V *of* $\mathbf{C}^2$, *and assume that* $\tilde\lambda$ *has* **no singularities** *on* V. *Define the holomorphic functions* a, b, $c : V \to \mathbf{C}$ *by*

$$a := \frac{-2\,\tilde\lambda_{ss}\,\tilde\lambda + 3\,\tilde\lambda_s^2 - \tilde\lambda_t^2}{2\,\tilde\lambda^4}\quad,\quad b := \frac{-2\,\tilde\lambda_{st}\,\tilde\lambda + 4\,\tilde\lambda_s\,\tilde\lambda_t}{2\,\tilde\lambda^4}\,,$$

$$\text{and}\quad c := \frac{-2\,\tilde\lambda_{tt}\,\tilde\lambda - \tilde\lambda_s^2 + 3\,\tilde\lambda_t^2}{2\,\tilde\lambda^4}\,.$$

Next, solve the second order linear homogeneous partial differential equation

$$b\,v_{ss} + (c-a)\,v_{st} - b\,v_{tt} + (b_s + c_t)\,v_s - (b_t + a_s)\,v_t = 0$$

on some simply connected open subset $\tilde V$ *of* V. *Then there is a function* $u : \tilde V \to \mathbf{C}$ *such that*

$$u_s = -c\,v_s + b\,v_t \quad\text{and}\quad u_t = b\,v_s - a\,v_t\,.$$

At some fixed point in $\tilde V$ **where the map** $(u\,,\,v) : \tilde V \to \mathbf{C}^2$ **has (complex) rank 2**, *calculate the inverse map* $\xi = \xi(u\,,\,v)$ *of the map* $(u\,,\,v) = \big(u(s\,,\,t)\,,\,v(s\,,\,t)\big)$, *which is defined on some open, simply connected subset* U *of* $\mathbf{C}^2$, *and view* $\xi = (\xi_1\,,\,\xi_2)$ *as a map in the two complex variables* $(u\,,\,v) \in U \subset \mathbf{C}^2$. *Define* $\lambda := \tilde\lambda \circ \xi$, *and view* λ *as a function in the two complex variables* $(u\,,\,v) \in U \subset \mathbf{C}^2$.

Now, define the following maps: $\zeta := (\zeta_1, \zeta_2) : U \to \mathbf{C}^2$, *where*

$$\zeta_1 := \xi_1 + \left(\frac{2\,\tilde\lambda\,\tilde\lambda_s}{\tilde\lambda_s^2 + \tilde\lambda_t^2}\right) \circ \xi \quad \textit{and} \quad \zeta_2 := \xi_2 + \left(\frac{2\,\tilde\lambda\,\tilde\lambda_t}{\tilde\lambda_s^2 + \tilde\lambda_t^2}\right) \circ \xi \,,$$

$$\mu := \frac{-2}{\lambda\,(\xi - \zeta)^2} = -\left(\frac{\tilde\lambda_s^2 + \tilde\lambda_t^2}{2\,\tilde\lambda^3}\right) \circ \xi : U \to \mathbf{C}\,,$$

and further

$$\mathbf{X} := (\tfrac{1}{2}\,,\,\tfrac{i}{2}\,,\,X)\,, \quad \textit{where } X := \lambda\,(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\, \xi)\,,$$

and

$$\mathbf{Y} := (-1\,,\,i\,,\,Y)\,, \quad \textit{where } Y := \mu\,(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\, \zeta)\,.$$

Then the $\mathbf{C}^6$*–valued* 1*-form*

$$\omega := \mathbf{X}\,du + \mathbf{Y}\,dv$$

is exact on U*, and if* $F : U \to \mathbf{C}^6$ *is a holomorphic map such that* $dF = \omega$, *then* $f := \sqrt{2}\,\mathrm{Re}(F) : M \to \mathbf{R}^6$ *is a minimal isometric immersion of the Kähler manifold* $M := (U\,,\, f^* <,>)$ *into* $\mathbf{R}^6$.

Conversely, if $F : W \to \mathbf{C}^6$ *is a holomorphic representative of a minimal isometric immersion* $f : M^4 \to \mathbf{R}^6$ *from a* 4*-dimensional Kähler manifold* M *into* $\mathbf{R}^6$*, and if* p *is a point in* M *where* $\mathrm{rank}\,F'' = 4$, *then we can find a complex coordinate system* $(u\,,\,v)$ *such that in a neighborhood* U *of* p *in* M*,* $\mathbf{X} = F_u$ *and* $\mathbf{Y} = F_v$ *are given, up to isometry, in terms of a function* $\lambda : U \to \mathbf{C}$ *without singularities as described above.*

Example 4.4: Let

$$\tilde\lambda(s\,,\,t) := t\,.$$

Then, we have $\tilde\lambda_s = 0$ and $\tilde\lambda_t = 1$, and all second partial derivatives of $\tilde\lambda$ vanish. By (18), we obtain

$$a(s\,,\,t) = \frac{0 + 3\cdot 0 - 1^2}{2\,t^4} = -\frac{1}{2\,t^4}\,, \quad b(s\,,\,t) = \frac{0 + 4\cdot 0\cdot 1}{2\,t^4} = 0\,,$$

$$\text{and} \quad c(s\,,\,t) = \frac{0 - 0 + 3\cdot 1^2}{2\,t^4} = \frac{3}{2\,t^4}\,.$$

Thus, (21) collapses to

$$0 \cdot v_{ss} + \left(\frac{3}{2\,t^4} + \frac{1}{2\,t^4}\right) v_{st} - 0 \cdot v_{tt} + \left(0 + \left(\frac{3}{2\,t^4}\right)_t\right) v_s - \left(0 + \left(-\frac{1}{2\,t^4}\right)_s\right) v_t$$
$$= \frac{2}{t^4}\, v_{st} - \frac{6}{t^5}\, v_s = 0\,,$$

which is equivalent to the partial differential equation

$$(v_s)_t = \frac{3}{t}\, v_s\,.$$

If $z := v_s$ is not constantly zero, we can separate this equation to $\frac{z_t}{z} = \frac{3}{t}$, or equivalently

$$(\log z)_t = (3\,\log t)_t = (\log t^3)_t\,.$$

Integrating this gives that for every function $g = g(s)$ in s alone,

$$z = g(s)\, t^3$$

is a solution for the partial differential equation given above. Choose

$$g(s) := e^{s+C}\,,$$

where C is a constant that will be determined later. Since then $z = v_s = e^{s+C}\, t^3$, integrating with respect to s (and setting the integration constant equal to zero) gives

$$v(s\,,\,t) = e^{s+C}\, t^3\,.$$

Using this function in (20) leads to the following "gradient" of u:

$$u_s = -c\, v_s + 0 \cdot v_t = -\frac{3}{2\,t^4}\, e^{s+C}\, t^3 = -\frac{3}{2\,t}\, e^{s+C}\,,$$
$$u_t = 0 \cdot v_s - a\, v_t = \frac{1}{2\,t^4}\, 3\, e^{s+C}\, t^2 = \frac{3}{2\,t^2}\, e^{s+C}\,.$$

Integrating the first equation gives $u(s\,,\,t) = -\frac{3}{2t}\, e^{s+C} + h(t)$ with a function h in t alone. Inserting this into the second equation results in

$$\frac{3}{2\,t^2}\, e^{s+C} + h'(t) = \frac{3}{2\,t^2}\, e^{s+C}\,,$$

which means that $h'(t) = 0$; i.e. h is constant. Choosing this constant to be zero, we thus find

$$u(s\,,\,t) = -\frac{3}{2\,t}\,e^{s+C}\,.$$

One can easily check that the Jacobian of $(u\,,\,v)$ as a holomorphic map in $(s\,,\,t)$ is never zero (since we must have that $t \neq 0$).

We now have to find the inverse of this map; i.e. we have to express s and t as functions of u and v. To this end,

$$\frac{v}{u} = \frac{t^3\,e^{s+C}}{-\frac{3}{2t}\,e^{s+C}} = -\frac{2\,t^4}{3}\,.$$

Using the function $z^{\frac{1}{4}} = \exp(\frac{1}{4}\,\log z)$ on some branch of the complex logarithm gives

$$t = \xi_2(u\,,\,v) = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,.$$

Thus, we have that

$$e^{s+C} = -\frac{2\,t}{3}\,u = -\frac{2}{3}\left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,u\,,$$

or, taking the logarithm on both sides,

$$s = -C + \log\left(-\frac{2}{3}\right) + \frac{1}{4}\left[\log\left(-\frac{3}{2}\right) + \log v - \log u\right] + \log u\,.$$

Choosing C in a way such that all constant terms in this expression cancel out, we finally obtain

$$s = \xi_1(u\,,\,v) = \frac{1}{4}\,\log(v\,u^3)\,.$$

Note also that we have

$$\lambda = \tilde{\lambda} \circ (s(u\,,\,v)\,,\,t(u\,,\,v)) = t(u\,,\,v) = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,,$$

and by (14) and (15), we obtain

$$\zeta_1 = s + \frac{2\,t\cdot 0}{0^2 + 1^2} = s = \frac{1}{4}\,\log(v\,u^3)\,,$$

$$\zeta_2 = t + \frac{2t \cdot 1}{0^2 + 1^2} = 3t = 3\left(\frac{-3v}{2u}\right)^{\frac{1}{4}},$$

and

$$\mu = -\frac{0^2 + 1^2}{2t^3} = -\frac{1}{2}\left(\frac{-3v}{2u}\right)^{-\frac{3}{4}}.$$

Inserting these maps $\xi = (\xi_1, \xi_2)$, $\zeta = (\zeta_1, \zeta_2)$, λ, and μ into our equations for X and Y gives, according to Theorem 4.3, the (complex Gauss map of an) isometric immersion from a minimal real Kähler manifold with respect to the complex chart (u, v) into $\mathbf{R}^6$.

We will now investigate the **remaining case** that $\operatorname{rank} F'' = 3$. If we exclude the case that f is holomorphic or generated by an isotropic cylinder (as in Proposition 4.1), then we can assume that for all $p \in U$ (staying away from some isolated points),

$$Z(p) := X_v(p) = Y_u(p) \neq 0.$$

Since by (1) X_u and Y_v have to point in the same complex direction as Z, we can find holomorphic functions α and $\beta : U \to \mathbf{C}$ such that

$$X_u = \alpha Z \quad \text{and} \quad Y_v = \beta Z.$$

Using the integrability conditions for X and Y, we must have

$$(X_u)_v = (\alpha Z)_v = \alpha_v Z + \alpha Z_v = (X_v)_u = Z_u$$

and

$$(Y_v)_u = (\beta Z)_u = \beta_u Z + \beta Z_u = (Y_u)_v = Z_v.$$

Inserting Z_u as in the first equation into the second equation, and vice versa for Z_v, results in

$$\alpha_v Z + \alpha(\beta_u Z + \beta Z_u) = Z_u \quad \text{and} \quad \beta_u Z + \beta(\alpha_v Z + \alpha Z_v) = Z_v,$$

or equivalently

$$(\alpha_v + \alpha\beta_u) Z = (1 - \alpha\beta) Z_u \quad \text{and} \quad (\beta_u + \beta\alpha_v) Z = (1 - \alpha\beta) Z_v. \quad (22)$$

At this point, we have to distinguish two cases: either that the factor $1 - \alpha\beta$ is identically zero, or that it is different from zero almost everywhere (recall that α and β are holomorphic).

Case 1: $\alpha\beta \neq 1$ almost everywhere. In this case, (22) gives that Z_u and Z_v are always linearly dependent, and thus that the map Z has rank ≤ 1 everywhere. But then the Rank Theorem (see e.g. [B-J], pages 45–48[1]) gives us that, away from isolated singularities, Z can be factored as

$$Z(u, v) = \tilde{Z} \circ \gamma(u, v),$$

where $\tilde{Z} = \tilde{Z}(w)$ is a holomorphic map in one complex variable w, and $\gamma : U \to \mathbf{C}$ is some holomorphic function on U. Using this more specific form of Z in our integrability conditions for X and Y results in (writing $\tilde{Z}' = \frac{d}{dw}\tilde{Z}$)

$$(X_v)_u = Z_u = \gamma_u\,(\tilde{Z}' \circ \gamma) = (X_u)_v = (\alpha\,Z)_v = \alpha_v\,(\tilde{Z} \circ \gamma) + \alpha\,\gamma_v\,(\tilde{Z}' \circ \gamma)$$

and

$$(Y_u)_v = Z_v = \gamma_v\,(\tilde{Z}' \circ \gamma) = (Y_v)_u = (\beta\,Z)_u = \beta_u\,(\tilde{Z} \circ \gamma) + \beta\,\gamma_u\,(\tilde{Z}' \circ \gamma),$$

so that, after simplifying,

$$\alpha_v\,(\tilde{Z} \circ \gamma) = (\gamma_u - \alpha\,\gamma_v)\,(\tilde{Z}' \circ \gamma) \quad \text{and} \quad \beta_u\,(\tilde{Z} \circ \gamma) = (\gamma_v - \beta\,\gamma_u)\,(\tilde{Z}' \circ \gamma). \quad (23)$$

Now, *assume* that $\tilde{Z}$ and $\tilde{Z}'$ would be linearly independent on some open set in $\mathbf{C}$. Then, by (23), we would clearly have

$$\alpha_v = 0\;,\quad \beta_u = 0\;,\quad \gamma_u = \alpha\,\gamma_v\;,\quad \text{and} \quad \gamma_v = \beta\,\gamma_u\,.$$

The last two equations give that $\gamma_u = \alpha\,\beta\,\gamma_u$ and $\gamma_v = \alpha\,\beta\,\gamma_v$. But we cannot have that γ_u and γ_v are both zero everywhere, since in this case γ would be constant, and we could replace $\tilde{Z}$ by a constant vector, in contradiction to $\tilde{Z}$ and $\tilde{Z}'$ being linearly independent almost everywhere. But then we would find that $\alpha\,\beta = 1$ everywhere, in contradiction to our case assumption.

Thus, $\tilde{Z}$ and $\tilde{Z}'$ are *everywhere linearly dependent*. But that means that we must have a *constant*, non–zero vector $Z_0 \in \mathbf{C}^4$ and a function $\delta = \delta(w)$ in w such that

$$\tilde{Z}(w) = \delta(w)\,Z_0$$

(and δ is a solution of a certain differential equation; but we will not need to know δ in more detail here).

[1]There, one can find a proof for *real* manifolds. However, the only tool that is necessary is the Inverse Mapping Theorem, and this theorem is also true in the holomorphic case.

Since $X_v = Y_u = Z = \tilde{Z} \circ \gamma = (\delta \circ \gamma)\, Z_0$, we can integrate to obtain

$$X = g(u\,,\,v)\, Z_0 + X_0 \quad \text{and} \quad Y = h(u\,,\,v)\, Z_0 + Y_0\,,$$

for some holomorphic functions $g\,,\,h : U \to \mathbf{C}$, and some constant vectors $X_0\,,\,Y_0 \in \mathbf{C}^4$. Using our integrability condition once more, we find

$$X_v = g_v\, Z_0 = Y_u = h_u\, Z_0\,, \tag{24}$$

so since $Z_0 \neq 0$, we must have $g_v = h_u \neq 0$. Thus (after making U smaller if necessary), there is a holomorphic function $\eta : U \to \mathbf{C}$ without singularities such that

$$g = \eta_u \quad \text{and} \quad h = \eta_v\,.$$

Now, inserting X and Y into the partial derivatives for F results in

$$F_u = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \eta_u\, Z_0 + X_0 \end{pmatrix} \quad \text{and} \quad F_v = \begin{pmatrix} -1 \\ i \\ \eta_v\, Z_0 + Y_0 \end{pmatrix},$$

and integrating these equations, we see that the holomorphic representative F of the minimal real Kähler immersion we are looking for, in this case, must have the form

$$F(u\,,\,v) = u \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X_0 \end{pmatrix} + v \begin{pmatrix} -1 \\ i \\ Y_0 \end{pmatrix} + \eta(u\,,\,v) \begin{pmatrix} 0 \\ 0 \\ Z_0 \end{pmatrix} + C_0$$

for some constant vector $C_0 \in \mathbf{C}^6$. Since $g_v = h_u \neq 0$ and thus g and h non–constant, (8) and (24) immediately give that

$$X_0 \cdot Z_0 = Y_0 \cdot Z_0 = Z_0^2 = 0\,.$$

And since we need $X \cdot Y = 1$, the equations above also give $X_0 \cdot Y_0 = 1$. This means that the vectors in the formula for F indeed span an isotropic subspace of $\mathbf{C}^6$, and thus that $f = \sqrt{2}\,\mathrm{Re}(F)$ *is generated by the graph of the holomorphic function* $\eta(u\,,\,v)$. Proposition 1.5 then says that **such an f must be holomorphic with respect to some complex structure on $\mathbf{R}^6$**. So, Case 1 leads to our "trivial examples" for minimal real Kähler submanifolds in $\mathbf{R}^6$.

Case 2: $\alpha\beta = 1$ everywhere. This means that $\beta = \dfrac{1}{\alpha}$, where α can never be zero, and since Z is (almost) never zero either, (22) implies that

$$\alpha_v = -\alpha\,\beta_u = -\alpha\left(\frac{1}{\alpha}\right)_u = \alpha\,\frac{\alpha_u}{\alpha^2}\,,$$

and hence that $\alpha = \alpha(u\,,\,v)$ must be a solution of the (non–linear) partial differential equation

$$\alpha_u = \alpha\,\alpha_v = \frac{1}{2}\,(\alpha^2)_v\,. \tag{25}$$

This of course restricts the choice of α greatly. Furthermore, we also have that

$$X_u = \alpha\,Z = \alpha\,X_v \quad \text{and} \quad Y_u = Z = \alpha\left(\frac{1}{\alpha}\,Z\right) = \alpha\,(\beta\,Z) = \alpha\,Y_v\,;$$

i.e. α and the component functions of X and Y satisfy the same *linear* partial differential equation

$$\phi_u = \alpha\,\phi_v\,.$$

But from the theory of linear partial differential equations (see the Proposition in the Appendix), we know that, in this case, there must be holomorphic maps $\tilde{X} = \tilde{X}(w)$ and $\tilde{Y} = \tilde{Y}(w)$ in one complex variable w such that

$$X = \tilde{X}\circ\alpha \quad \text{and} \quad Y = \tilde{Y}\circ\alpha\,.$$

Note that α is a "fundamental system of solutions", since it is never zero and non–constant (see below). Using these relations in our integrability condition, and noting (25), we find that the former is equivalent to

$$X_v = (\tilde{X}\circ\alpha)_v = \alpha_v\,(\tilde{X}'\circ\alpha) = Y_u = (\tilde{Y}\circ\alpha)_u = \alpha_u\,(\tilde{Y}'\circ\alpha) = \alpha_v\,(\alpha\,(\tilde{Y}'\circ\alpha))\,,$$

where $\tilde{X}'$ again means $\frac{d}{dw}\tilde{X}$, and where α_v is not constantly zero since we assumed that $X_v = Y_u$ is almost nowhere zero. Therefore, we must have that

$$0 = \tilde{X}'\circ\alpha - \alpha\,(\tilde{Y}'\circ\alpha) = (\tilde{X}' - w\,\tilde{Y}')\circ\alpha\,,$$

and since $\tilde{X}$ and $\tilde{Y}$ are holomorphic, this implies that we always have

$$\tilde{X}' = w\,\tilde{Y}'\,.$$

Using our knowledge about the form of X and Y in a Weierstrass representation, we can write this *ordinary* differential equation in w as

$$\begin{pmatrix} \left(\tilde\lambda \frac{1-\tilde\xi^2}{2}\right)' \\ \left(i\,\tilde\lambda \frac{1+\tilde\xi^2}{2}\right)' \\ (\tilde\lambda\,\tilde\xi)' \end{pmatrix} = w \begin{pmatrix} \left(\tilde\mu \frac{1-\tilde\zeta^2}{2}\right)' \\ \left(i\,\tilde\mu \frac{1+\tilde\zeta^2}{2}\right)' \\ (\tilde\mu\,\tilde\zeta)' \end{pmatrix},$$

where here $\tilde\lambda$, $\tilde\mu$, $\tilde\xi$, and $\tilde\zeta$ are all holomorphic functions in the one complex variable w. Performing by now familiar operations, the last equation can be simplified to the system

$$\left.\begin{aligned} \tilde\lambda' &= w\,\tilde\mu'\,, \\ (\tilde\lambda\,\tilde\xi^2)' &= w\,(\tilde\mu\,\tilde\zeta^2)'\,, \\ \text{and}\quad (\tilde\lambda\,\tilde\xi)' &= w\,(\tilde\mu\,\tilde\zeta)'\,. \end{aligned}\right\} \qquad (26)$$

Note that the first of these equations *determines* $\tilde\mu$ *in terms of* $\tilde\lambda$, up to a constant.

Let us now write $\tilde\xi = (s\,,\,t)$ and $\tilde\zeta = (p\,,\,q)$, where s, t, p, and q are holomorphic functions in w. Since $\tilde\mu$ is also given in terms of $\tilde\lambda$, $\tilde\xi$, and $\tilde\zeta$ as $\tilde\mu = \frac{-2}{\tilde\lambda\,(\tilde\xi-\tilde\zeta)^2}$, we can write

$$(\tilde\xi - \tilde\zeta)^2 = (s-p)^2 + (t-q)^2 = \frac{-2}{\tilde\lambda\,\tilde\mu} =: \tilde g(w)\,.$$

Note that by the last paragraph, the so–defined holomorphic function $\tilde g$ is actually almost entirely determined by $\tilde\lambda$ alone. But we know that $(\tilde\xi - \tilde\zeta)^2$ is never zero. Thus, we can *locally* find a well–defined, holomorphic "square-root" of $\tilde g$, i.e. a holomorphic function $g(w)$ such that

$$(s-p)^2 + (t-q)^2 = \tilde g(w) = g^2(w)\,.$$

Since the complex trigonometric functions are surjective (see [Ah], page 47), we can therefore conclude that, locally, there is a holomorphic function $h(w)$ such that we can write

$$s(w) = p(w) + g(w)\,\cos(h(w)) \quad \text{and} \quad t(w) = q(w) + g(w)\,\sin(h(w))\,. \quad (27)$$

Let us now break the last equation in (26) into first and second components. The first components give

$$(\tilde{\lambda}\, s)' = \tilde{\lambda}'\, s + \tilde{\lambda}\, s' = w\,(\tilde{\mu}\, p)' = w\,\tilde{\mu}'\, p + w\,\tilde{\mu}\, p' \,.$$

Using the first equations of (26) and (27), this can be expanded to

$$\tilde{\lambda}'\, p + \tilde{\lambda}'\, g\,\cos h + \tilde{\lambda}\, p' + \tilde{\lambda}\,(g\,\cos h)' = \tilde{\lambda}'\, p + w\,\tilde{\mu}\, p' \,.$$

Cancelling the first term on both sides and reordering, we obtain

$$(w\,\tilde{\mu} - \tilde{\lambda})\, p' = (\tilde{\lambda}\, g\,\cos h)' \,.$$

But $w\,\tilde{\mu} - \tilde{\lambda}$ can only be zero at isolated points, since otherwise, we would have that $\tilde{\lambda} \equiv w\,\tilde{\mu}$. Differentiating this and then using the first equation in (26) would give

$$\tilde{\lambda}' = \tilde{\mu} + w\,\tilde{\mu}' = \tilde{\mu} + \tilde{\lambda}' \,,$$

and thus $\tilde{\mu} = 0$, which cannot happen. Therefore, we can solve the equation above for p' almost everywhere; and in completely analogous fashion, we can obtain for q':

$$p' = \frac{(\tilde{\lambda}\, g\,\cos h)'}{w\,\tilde{\mu} - \tilde{\lambda}} \quad \text{and} \quad q' = \frac{(\tilde{\lambda}\, g\,\sin h)'}{w\,\tilde{\mu} - \tilde{\lambda}} \,. \tag{28}$$

Note that this means that p and q, and thus by (27) also s and t, are (almost) completely determined by $\tilde{\lambda}$ and h alone.

Finally, let us examine the second equation in (26). We claim that this equation must necessarily be satisfied. To see this, we will replace $\tilde{\xi}$ and $\tilde{\zeta}$ by their component functions, and use the relations we worked out above in the following way. By (27), we find that

$$\begin{aligned}
\tilde{\xi}^2 &= s^2 + t^2 = (p + g\,\cos h)^2 + (q + g\,\sin h)^2 \\
&= p^2 + 2\,p\,g\,\cos h + g^2\,\cos^2 h + q^2 + 2\,q\,g\,\sin h + g^2\,\sin^2 h \\
&= \underbrace{p^2 + q^2}_{=\tilde{\zeta}^2} + 2\,g\,(p\,\cos h + q\,\sin h) + g^2 \,.
\end{aligned}$$

Thus, we have

$$(\tilde{\lambda}\,\tilde{\xi}^2)' = \underbrace{\tilde{\lambda}'}_{=w\,\tilde{\mu}'}\,\tilde{\zeta}^2 + 2\,\tilde{\lambda}\,(\tilde{\zeta}\cdot\tilde{\zeta}') + (2\,\tilde{\lambda}\, g\,(p\,\cos h + q\,\sin h) + \tilde{\lambda}\, g^2)' \,. \tag{29}$$

Let us work on the last term on the right–hand side. Recalling (28) and the fact that $g^2 = \tilde{g} = \frac{-2}{\tilde{\lambda}\,\tilde{\mu}}$, we find that this term equals

$$\begin{aligned}
&2\,(\tilde{\lambda}\,g\,\cos h)'\,p + 2\,(\tilde{\lambda}\,g\,\sin h)'\,q + 2\,\tilde{\lambda}\,g(p'\,\cos h + q'\,\sin h) - \left(\frac{2}{\tilde{\mu}}\right)' \\
&\quad = 2\,(w\,\tilde{\mu} - \tilde{\lambda})\,p'\,p + 2\,(w\,\tilde{\mu} - \tilde{\lambda})\,q'\,q \\
&\qquad\qquad + 2\,\frac{(\tilde{\lambda}\,g\,\cos h)(\tilde{\lambda}\,g\,\cos h)' + (\tilde{\lambda}\,g\,\sin h)(\tilde{\lambda}\,g\,\sin h)'}{w\,\tilde{\mu} - \tilde{\lambda}} - \left(\frac{2}{\tilde{\mu}}\right)' \\
&\quad = 2\,(w\,\tilde{\mu} - \tilde{\lambda})\,\underbrace{(p\,p' + q\,q')}_{=\tilde{\zeta}\cdot\tilde{\zeta}'} + \frac{(\tilde{\lambda}^2\,g^2)'}{w\,\tilde{\mu} - \tilde{\lambda}} - \left(\frac{2}{\tilde{\mu}}\right)' \\
&\quad = -\,2\,\tilde{\lambda}\,(\tilde{\zeta}\cdot\tilde{\zeta}') + w\,\tilde{\mu}\,(\tilde{\zeta}^2)' + \frac{-2\left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)'}{w\,\tilde{\mu} - \tilde{\lambda}} - 2\left(\frac{1}{\tilde{\mu}}\right)' .
\end{aligned}$$

But the last two terms on the last line cancel each other, since, using the first equation of (26) once more, we find that

$$\frac{\tilde{\lambda}'\,\tilde{\mu} - \tilde{\lambda}\,\tilde{\mu}'}{\tilde{\mu}^2\,(w\,\tilde{\mu} - \tilde{\lambda})} = \frac{w\,\tilde{\mu}'\,\tilde{\mu} - \tilde{\lambda}\,\tilde{\mu}'}{\tilde{\mu}^2\,(w\,\tilde{\mu} - \tilde{\lambda})} = \frac{\tilde{\mu}'}{\tilde{\mu}^2} = -\left(\frac{1}{\tilde{\mu}}\right)' .$$

This means that (29) takes the form

$$(\tilde{\lambda}\,\tilde{\xi}^2)' = w\,\tilde{\mu}'\tilde{\zeta}^2 + 2\,\tilde{\lambda}\,(\tilde{\zeta}\cdot\tilde{\zeta}') - 2\,\tilde{\lambda}\,(\tilde{\zeta}\cdot\tilde{\zeta}') + w\,\tilde{\mu}\,(\tilde{\zeta}^2)' = w\,(\tilde{\mu}\,\tilde{\zeta}^2)' ,$$

which is exactly the second equation in (26).

This finishes the investigation of the case that $\operatorname{rank} F'' = 3$. Summarizing our results (including Proposition 4.1), we thus have proven the following theorem, which together with Theorem 4.3 gives the promised complete local classification of all minimal real Kähler surfaces in Euclidean 6–space, away from the isolated "singularities" where the rank of the second osculating bundle F'' is smaller than on the rest of the manifold.

Theorem 4.5: *Let $F : W \to \mathbf{C}^6$ be a holomorphic representative of a minimal isometric immersion $f : M^4 \to \mathbf{R}^6$ from a 4-dimensional Kähler manifold M into $\mathbf{R}^6$, and assume that* **the rank of the second osculating bundle F'' of F equals three** *on all of W. Then f is holomorphic with respect to some complex structure on $\mathbf{R}^6$, or f is generated by an isotropic cylinder (see Example 1.6), or we can describe f locally in the following way:*

Let $((u,v), X, Y)$ be a Weierstrass representation of F in the neighborhood of a point $p \in W$ (see page 37 and Proposition 3.2). Then in a (perhaps smaller) neighborhood U of p in M, we can find a solution $\alpha : U \to \mathbf{C}$ of the non–linear partial differential equation

$$\alpha_u = \alpha\, \alpha_v = \frac{1}{2}\,(\alpha^2)_v$$

and two holomorphic functions $\tilde\lambda(w)$ and $h(w)$ in one complex variable w such that we have

$$X = (\tilde\lambda \circ \alpha)\left(\frac{1-\tilde\xi^2}{2}\,,\, i\,\frac{1+\tilde\xi^2}{2}\,,\, \tilde\xi\right) \circ \alpha$$

and

$$Y = (\tilde\mu \circ \alpha)\left(\frac{1-\tilde\zeta^2}{2}\,,\, i\,\frac{1+\tilde\zeta^2}{2}\,,\, \tilde\zeta\right) \circ \alpha\,,$$

where $\tilde\mu := \int \frac{\tilde\lambda'}{w}\, dw$, *$g$ is a function such that* $g^2 = \frac{-2}{\tilde\lambda\,\tilde\mu}$,

$$\tilde\zeta := \left(\int \frac{(\tilde\lambda g \cos h)'}{w\,\tilde\mu - \tilde\lambda}\, dw\,,\, \int \frac{(\tilde\lambda g \sin h)'}{w\,\tilde\mu - \tilde\lambda}\, dw\right)\,,$$

and

$$\tilde\xi := \tilde\zeta + g\,(\cos h\,,\, \sin h)\,.$$

Conversely, for any choice of a non–constant α, a nowhere zero $\tilde\lambda$, and arbitrary h as above, the map $f := \sqrt{2}\ \mathrm{Re}(F)$ that we obtain through a Weierstrass representation $((u,v), X, Y)$ as defined above gives a minimal real Kähler immersion that is defined on some small neighborhood of $\mathbf{C}^2$.

Remark 4.6: Note that we have found another way to construct minimal real Kähler hypersurfaces in $\mathbf{R}^5$: Simply set $h := 0$, and (28) gives that q has to be a constant. Choosing this constant to be zero, (27) gives that t also has to be zero. Hence, the last components of X and Y and consequentially also the ones of F_u and F_v will be zero, which means that $f = \sqrt{2}\ \mathrm{Re}(F)$ can be considered as a map into $\mathbf{R}^5$.

Example 4.7: By a separation ansatz, one finds (and can easily check) that for all constants $A, B \in \mathbf{C}$,

$$\alpha(u, v) := -\frac{v+A}{u+B}$$

is a solution of (25). To keep things simple, let us set $\tilde{\lambda} := -1$, and let us determine h a bit later in such a way that the example stays simple. Since $\tilde{\lambda}' = w\,\tilde{\mu}'$, this means that $\tilde{\mu}$ also has to be constant. Let us choose $\tilde{\mu} = 2$, since then $\tilde{g} = \frac{-2}{(-1)\cdot 2} = 1$, and we can choose $g = 1$. Then, the equations for p' and q' have the form

$$p' = \frac{(-\cos h(w))'}{2w+1} = \frac{h'(w)\sin(h(w))}{2w+1}$$

and

$$q' = \frac{(-\sin h(w))'}{2w+1} = \frac{-h'(w)\cos(h(w))}{2w+1}.$$

So, if we set $h(w) := w^2 + w$, then

$$p' = \sin(w^2+w) \quad \text{and} \quad q' = -\cos(w^2+w).$$

Setting all integration constants equal to zero and using (27), we finally obtain

$$\tilde{\zeta} = \begin{pmatrix} \int \sin(w^2+w)\,dw \\ -\int \cos(w^2+w)\,dw \end{pmatrix} \quad \text{and} \quad \tilde{\xi} = \tilde{\zeta} + \begin{pmatrix} \cos(w^2+w) \\ \sin(w^2+w) \end{pmatrix}.$$

Inserting these α, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\xi}$, and $\tilde{\zeta}$ into the Weierstrass representation formulas gives the complex Gauss map of a minimal real Kähler surface whose holomorphic representative has a second osculating bundle of rank three everywhere.

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Appendix

In this appendix, we give the proof of a result that is well–known in the theory of *real* partial differential equations, but somewhat hard to find in the literature for several complex variables. We used this proposition on page 68.

Proposition: *Let* $\alpha : U \to \mathbf{C}$ *be a non–zero holomorphic function that is defined on some open subset of* $\mathbf{C}^2$*, where we denote the complex variables in* $\mathbf{C}^2$ *by* $(u\,,\,v)$*. Furthermore, let* $\phi : U \to \mathbf{C}$ *be a non–constant holomorphic function that is a solution of the linear partial differential equation*

$$\phi_u = \alpha\,\phi_v\,. \tag{1}$$

If $\psi : U \to \mathbf{C}$ *is another solution of this partial differential equation, then for every point* $p_0 = (u_0\,,\,v_0) \in U$ *where* ϕ *has no singularity (i.e.,* $d\phi(p_0) \neq 0$*), there is a neighborhood* V *of* p_0 *in* U *and a holomorphic function* $h(w)$ *in one complex variable* w *such that, on all of* V*,*

$$\psi = h \circ \phi\,.$$

Proof: Pick a point p_0 where ϕ is non–singular, and assume, without loss of generality, that for all points p in a neighborhood V of p_0, $\phi_v(p) \neq 0$. Let C be any of the values that ϕ takes on V, say $C = \phi(p_1)$, where $p_1 = (u_1\,,\,v_1) \in V$. By the Implicit Mapping Theorem (see [G-R], page 16), there is a holomorphic function $g_C(w)$ with $g_C(u_1) = v_1$ such that

$$\phi(u\,,\,g_C(u)) \equiv C$$

for all u in some neighborhood of u_0 in $\mathbf{C}$. Taking the derivative with respect to u and using (1), we obtain (with $g' := \frac{\partial}{\partial w}\,g$)

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial u}\,\phi(u\,,\,g_C(u)) = \phi_u(u\,,\,g_C(u)) + \phi_v(u\,,\,g_C(u))\,g_C'(u) \\ &= \phi_v(u\,,\,g_C(u))\,[\alpha(u\,,\,g_C(u)) + g_C'(u)]\,. \end{aligned}$$

But we assumed that $\phi_v(p) \neq 0$ for all $p \in V$, and so $g_C(u)$ must satisfy the *ordinary* differential equation

$$g_C'(u) = -\alpha(u\,,\,g_C(u))\,.$$

Note that this differential equation is *completely independent* of ϕ. Thus, if we would repeat the above considerations for our second solution ψ, we would obtain holomorphic functions $\tilde{g}_{\tilde{C}}$ that satisfy the same ordinary differential equation with the same initial condition $\tilde{g}_{\tilde{C}}(u_0) = v_0$. But by the theory of ordinary (complex) differential equations, this means that close to u_0, $g_C(u) = \tilde{g}_{\tilde{C}}(u)$ (see e.g. [Wa], page 110). Therefore, we have shown that the *level curves* of ϕ and ψ must be the same, and thus that (locally) there is a function $h(w)$ such that $\psi = h \circ \phi$; namely, h transforms the values associated with the level curves of ϕ into the values associated with the level curves of ψ.

It remains to show that h is holomorphic. But since $\phi_v(u_1, v_1) \neq 0$, the function $\eta(v) := \phi(u_1, v)$ is invertible in a neighborhood of v_1, and the inverse is holomorphic. Since

$$\psi(u_1, v) = h(\phi(u_1, v)) = h(\eta(v)),$$

we have

$$h(w) = \psi(u_1, \eta^{-1}(w)),$$

and thus that h is holomorphic.