# Homotopy type of Symplectomorphism Groups of $S^2 \times S^2$

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# Abstract of the Dissertation

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Let $M_\lambda$ be the symplectic manifold $(S^2 \times S^2, \omega_\lambda = (1+\lambda)\sigma_0 \oplus \sigma_0)$ where the 2-form $\sigma_0$ has total area equal to 1 and $\lambda \geq 0$. This work calculates the homotopy type of the group of symplectomorphisms of $M_\lambda$, $G_\lambda$, when $0 < \lambda \leq 1$. It turns out that if $\lambda$ is in this range, $G_\lambda$ contains two finite dimensional Lie groups that generate its homotopy, and the group is homotopy equivalent to the product $X = S^1 \times SO(3) \times SO(3) \times L$ where $L$ is the loop space on the suspension of the smash product $S^1 \wedge SO(3)$. A key step in this work is calculating the mod 2 homology of $G_\lambda$. Although this

homology has a finite number of generators with respect to the Pontryagin product, it is unexpected large because it contains a free noncommutative ring on 3 generators. Our arguments involve a study of the space of $\omega_\lambda$-compatible almost complex structures on $M_\lambda$.

*To my parents*

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# Chapter 1

# Introduction

In general symplectomorphism groups are thought to be intermediate objects between Lie groups and full groups of diffeomorphisms. In 1985, Gromov showed, among several other results, that the symplectomorphism group of a product of two 2-dimensional spheres that have the same area has the homotopy type of a Lie group. More precisely, let $M_\lambda$ be the symplectic manifold $(S^2 \times S^2, \omega_\lambda = (1+\lambda)\sigma_0 \oplus \sigma_0)$ where $0 \leq \lambda \in \mathbb{R}$ and $\sigma_0$ is the standard area form on $S^2$ with total area equal to 1. Denote by $G_\lambda$ the group of symplectomorphisms of $M_\lambda$ that act as the identity on $H_2(S^2 \times S^2; \mathbb{Z})$. Gromov proved that $G_0$ is homotopy equivalent to its subgroup of standard isometries $SO(3) \times SO(3)$. He also showed that this would no longer hold when one sphere is larger than the other, and in [9] McDuff constructed explicitly an element of infinite order in $H_1(G_\lambda)$, $\lambda > 0$. Abreu and McDuff in [2] calculated the rational cohomology of these symplectomorphism groups and confirmed that these groups could not be homotopic to Lie groups. In this paper we show that when $0 < \lambda \leq 1$ the topology of $G_\lambda$ is generated in some way by its subgroup of isometries $SO(3) \times SO(3)$ and by this new element of infinite order. In particular we will calculate the homotopy type of $G_\lambda$:

**Theorem 1.0.1.** *If $0 < \lambda \leq 1$, $G_\lambda$ is homotopy equivalent to the product $X = L \times S^1 \times SO(3) \times SO(3)$ where $L$ is the loop space of the suspension of the smash product $S^1 \wedge SO(3)$.*

In this product, $X$, of $H$-spaces, $S^1$ corresponds to the generator in $H_1(G_\lambda)$ described by McDuff, one of the $SO(3)$ factors corresponds to rotation in one of the spheres and the other $SO(3)$ factor represents the diagonal in $SO(3) \times SO(3)$. This new element of infinite order represents a $S^1$-action that commutes with the diagonal action of $SO(3)$, but not with rotations in each one of the spheres. The loop space $L = \Omega\Sigma(S^1 \wedge SO(3))$ appears as the result of that non-commutativity.

Although this space $X$ is an $H$-space, its multiplication is not the same as on $G_\lambda$. This can be seen by comparing the Pontryagin products on integral homology.

The main steps in the proof of this theorem determine the organization of the paper. Therefore in §2 we have the first main result which is the calculation of the mod 2 homology ring $H_*(G_\lambda; \mathbb{Z}_2)$. Recall that the product structure in $H_*(G_\lambda; \mathbb{Z}_2)$, called Pontryagin product, is induced by the product in $G_\lambda$. Denote by $\Lambda(y_1, ..., y_n)$ the exterior algebra over $\mathbb{Z}_2$ with generators $y_i$ where this means that $y_i^2 = 0$ and $y_i y_j = y_j y_i$ for all $i, j$, and by $\mathbb{Z}_2\langle x_1, ..., x_n\rangle$ the free noncommutative algebra over $\mathbb{Z}_2$ with generators $x_j$. Recall that $H_*(SO(3); \mathbb{Z}_2) \cong \Lambda(y_1, y_2)$.

**Theorem 1.0.2.** *If* $0 < \lambda \leq 1$ *then*

$$H_*(G_\lambda; \mathbb{Z}_2) = \Lambda(y_1, y_2) \otimes \mathbb{Z}_2\langle t, x_1, x_2 \rangle / R$$

*where* $\deg y_i = \deg x_i = i$ *and* $R$ *is the set of relations* $\{t^2 = x_i^2 = 0, x_1 x_2 = x_2 x_1\}$.

The notation implies that $y_i$ commutes with $t$ and $x_i$. We see that $H_*(G_\lambda; \mathbb{Z}_2)$ contains $\Lambda(x_1, x_2)$ which appears from rotation in the first sphere, $\Lambda(y_1, y_2)$ which represents the diagonal in $SO(3) \times SO(3)$ plus the new generator $t$. The proof is based on the fact that the space $\mathcal{J}_\lambda$, of almost complex structures on $S^2 \times S^2$ compatible with $\omega_\lambda$, is a stratified space with two strata $U_0$ and $U_1$, where $U_0$ is the open subset of $\mathcal{J}_\lambda$ consisting of all $J \in \mathcal{J}_\lambda$ for which the homology classes $E = [S^2 \times \{pt\}]$ and $F = [\{pt\} \times S^2]$ are both represented by $J$-holomorphic spheres and $U_1$ is its complement ( for a proof of this fact see [1] and [8] ). More precisely, $U_1$ is a non-empty closed, codimension 2 submanifold of $\mathcal{J}_\lambda$, consisting of all $J \in \mathcal{J}_\lambda$ for which the homology class of the antidiagonal $E - F$ is represented by an $J$-holomorphic sphere. Let $J_0 = j_0 \oplus j_0 \in \mathcal{J}_\lambda$ be the standard split compatible complex structure and $J_2$ be the Hirzebruch integrable complex structure. Denote by $\mathrm{Aut}(J_{2i})$ the stabilizer of $J_{2i}$ in $G_\lambda$ and let $K_i$ be the identity component of $\mathrm{Aut}(J_{2i})$ with $i = 0$ or 1. Thus $K_0$ is the subgroup $SO(3) \times SO(3)$ of isometries and $K_1$ is isomorphic to the subgroup $S^1 \times SO(3)$ where the $SO(3)$ factor is the diagonal in $SO(3) \times SO(3)$ and the $S^1$ factor corresponds to the element of infinite order in $H_1(G_\lambda)$, $\lambda > 0$. Each of the $U_i$ is homotopy equivalent to a homogeneous space of the group $G_\lambda$:

**Proposition 1.0.3.** *$U_i$ is homotopic to the quotient $G_\lambda/\mathrm{Aut}(J_{2i})$ where $i = 0$ or 1.*

Thus we have that $U_0$ is homotopy equivalent to $G_\lambda/(SO(3) \times SO(3))$ and $U_1$ to $G_\lambda/(SO(3) \times S^1)$ (see proof of the above proposition in [2]). Although the mod 2 homology has a finite number of generators with respect to the Pontryagin product we can see it is very large because it contains a free non-commutative ring on 3 generators. From the inclusion of $K_0$ in $G_\lambda$ we have generators $x_1, x_2, x_3 \in H_*(G_\lambda; \mathbb{Z}_2)$ in dimensions 1,2 and 3 respectively, representing the rotation in the first factor. Denote by $t$ the new generator in $H_1(G_\lambda)$, $\lambda > 0$. It does not commute with $x_i$, and so we have a nonzero class defined as the commutator and represented by $x_i t + t x_i$ for $i = 1, 2, 3$. It is easy to understand what these generators are in homotopy. For example, $x_1$ is a spherical class, so it represents an element in $\pi_1(G_\lambda)$ and $x_1 t + t x_1$ corresponds to the Samelson product $[t, x_1] \in \pi_2(G_\lambda)$. This is given by the map

$$S^2 = S^1 \times S^1 / S^1 \vee S^1 \to G_\lambda$$

induced by the commutator

$$S^1 \times S^1 \to G_\lambda : (s, u) \mapsto t(s) x_1(u) t(s)^{-1} x_1(u)^{-1}.$$

In §3 we relate the mod 2 homology to the integer homology using the Bockstein spectral sequence and the fact that this spectral sequence is multiplicative for an $H$-space.

Finally in §4 we use the results from the previous sections to define a map

$f$ between $G_\lambda$ and the product $X = L \times S^1 \times SO(3) \times SO(3)$ and to prove that it induces isomorphisms on homology with integer coefficients. The existence of such map together with some standard results in algebraic topology prove theorem 1.0.1.

# Chapter 2

# The Pontryagin ring $H_*(G_\lambda;\mathbb{Z}_2)$

Recall that for any group $G$ the product $\phi : G \times G \to G$ induces a product in homology

$$H_*(G;\mathbb{Z}_2) \otimes H_*(G;\mathbb{Z}_2) \xrightarrow{\times} H_*(G \times G;\mathbb{Z}_2) \xrightarrow{\phi_*} H_*(G;\mathbb{Z}_2)$$

called the Pontryagin product, that we will denote by ".". Every time it is clear from the context we will suppress this for simplicity of notation. In this section we will compute the ring structure on $H_*(G_\lambda;\mathbb{Z}_2)$ induced by this product. Unless noted otherwise we assume $\mathbb{Z}_2$ coefficients throughout.

### 2.0.1 Geometric Description

First we give a brief geometric description of the $S^1$-action corresponding to the element of infinite order in $\pi_1(G_\lambda)$, when $\lambda > 0$ (for a complete description see [2]). As we mentioned in the introduction, if $\lambda > 0$ the space of almost complex structures compatible with $\omega_\lambda$, $\mathcal{J}_\lambda$, is a stratified space with two strata $U_i$, $i = 0$ or $1$. $U_i$ contains the Hirzebruch integrable complex structure

$J_{2i}$ with a holomorphic sphere $C_{J_{2i}}$ of self-intersection $-2i$. In this case $J_2$ is tamed by $\omega_\lambda$ and $S^2 \times S^2$ is diffeomorphic to the underlying manifold in the projectivization $\mathbf{P}(\mathcal{O}(2) \oplus \mathbb{C})$ over $S^2$. Here $\mathcal{O}(2)$ is a complex line bundle over $S^2$ with first Chern class 2. This bundle has two natural sections, $\mathbf{P}(\{0\} \oplus \mathbb{C})$ and $\mathbf{P}(\mathcal{O}(2) \oplus \{0\})$, in classes $E + F$ and $E - F$ respectively. $E$ is the class $[S^2 \times \{pt\}]$ and $F$ is the class of the fiber $[\{pt\} \times S^2]$.

The element of infinite order in $\pi_1(G_\lambda)$ acts on this fibration by rotation on the fibers and leaving fixed the sections corresponding to the classes of the diagonal and antidiagonal. We see that this element is in the stabilizer of $J_2$ in $G_\lambda$, because it fixes each point of the $J_2$-holomorphic representative for the class $E - F$. Therefore for each $J \in U_0$ in a neighborhood of $U_1$ the action of $t \in \pi_1(G_\lambda)$ in $J$ gives a loop around $U_1$ which represents the link of $U_1$ in $U_0$.

## 2.1 Relation between $H_*(G_\lambda)$ and $H_*(U_i)$: additive version

As we mentioned before $U_1$ is a codimension 2 submanifold of $\mathcal{J}_\lambda$, the space of all complex structures in $S^2 \times S^2$. This implies that $U_0 = \mathcal{J}_\lambda - U_1$ is connected. Hence

$$H_0(U_0; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H_0(U_1; \mathbb{Z}_2).$$

Just as M.Abreu showed in [1] we still have for $p \geq 1$,

$$H_p(U_0; \mathbb{Z}_2) \cong H_{p-1}(U_1; \mathbb{Z}_2). \tag{2.1}$$

This already implies that $H_1(U_0;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Now consider the following principal fibrations

$$\begin{array}{ccc} K_0 \xrightarrow{i_0} G_\lambda & \qquad & K_1 \xrightarrow{i_1} G_\lambda \\ \downarrow p_0 & & \downarrow p_1 \\ U_0 & & U_1 \end{array} \tag{2.2}$$

where $K_i$ is the identity component of $\mathrm{Aut}(J_{2i})$. As we stated before $K_0$ is the subgroup $SO(3) \times SO(3)$ and $K_1$ is isomorphic to $S^1 \times SO(3)$.

We need to prove that the following proposition is true with $\mathbb{Z}_2$ coefficients:

**Proposition 2.1.1.** *Let* $\mathrm{Diff}_0(S^2 \times S^2)$ *denote the group of diffeomorphisms of* $S^2 \times S^2$ *that act as the identity on* $H_2(S^2 \times S^2, \mathbb{Z})$. *The inclusion*

$$i : K_0 = SO(3) \times SO(3) \longrightarrow \mathrm{Diff}_0(S^2 \times S^2) \ .$$

*is injective in homology.*

*Proof.* As in [1] we define a map

$$F : \mathrm{Diff}_0(S^2 \times S^2) \longrightarrow \mathrm{Map}_1(S^2) \times \mathrm{Map}_1(S^2)$$

where $\mathrm{Map}_1(S^2)$ is the space of all orientation preserving self-homotopy equivalences of $S^2$. Given $\varphi \in \mathrm{Diff}_0(S^2 \times S^2)$ we define a self map of $S^2$, denoted by $\tilde{\varphi}_1$, via the composite

$$\tilde{\varphi}_1 : S^2 \xrightarrow{i_1} S^2 \times S^2 \xrightarrow{\varphi} S^2 \times S^2 \xrightarrow{\pi_1} S^2,$$

where $i_1$, respectively $\pi_1$, denote inclusion into, respectively projection onto, the first $S^2$ factor of $S^2 \times S^2$. Because $\varphi$ acts as the identity on $H_2(S^2 \times S^2, \mathbb{Z})$, $\tilde{\varphi}_1$ is an orientation preserving self homotopy equivalence of $S^2$, i.e., $\tilde{\varphi}_1 \in \mathrm{Map}_1(S^2)$. Defining $\tilde{\varphi}_2$ in analogous way using the second $S^2$ factor of $S^2 \times S^2$, we have thus constructed the desired map given by

$$\varphi \mapsto \tilde{\varphi}_1 \times \tilde{\varphi}_2.$$

It is clear from the construction that $F$ restricted to $SO(3) \times SO(3)$ is just the inclusion

$$SO(3) \times SO(3) \longrightarrow \mathrm{Map}_1(S^2) \times \mathrm{Map}_1(S^2)$$

Now we use the following theorem ( see [4] )

**Theorem 2.1.2.** *The space of orientation preserving self-homotopy equivalences on the 2-sphere has the homotopy type of* $SO(3) \times \Omega$, *where* $\Omega = \tilde{\Omega}^2_0(S^2)$ *is the universal covering space for the component in the double loop space on* $S^2$ *containing the constant based map.*

This proves that $SO(3)$ is not homotopy equivalent to $\mathrm{Map}_1(S^2)$ but we have, using the Künneth formula, with field coefficients,

$$H_*(SO(3) \times \Omega) \cong H_*(SO(3)) \otimes H_*(\Omega) \cong H_*(\mathrm{Map}_1(S^2))$$

thus the map

$$i_* : H_*(SO(3)) \longrightarrow H_*(\mathrm{Map}_1(S^2))$$

induced by injection is injective for any field coefficients. □

It is proved by D.McDuff in [9] that the generator of the $\mathbb{Z}$ factor in $\pi_1(G_\lambda)$ lies in $\pi_1(K_1)$. This means that the generator of the $S^1$-action in $\pi_1(K_1)$ maps to a generator of infinite order in $\pi_1(G_\lambda)$. Thus the map

$$i_{1*} : H_*(K_1) \longrightarrow H_*(G_\lambda)$$

induced by inclusion is injective. Since we are working over a field, the cohomology is the dual of homology, thus from the above and prop 2.1.1 the maps

$$i_0^* : H^*(K_0) \longrightarrow H^*(G_\lambda)$$

and

$$i_1^* : H^*(K_1) \longrightarrow H^*(G_\lambda)$$

induced by inclusions $i_0$ and $i_1$ are surjective.

By the Leray-Hirsch Theorem, we know that the spectral sequences of the fibrations collapse at the $E_2$-term, and we have the following vector space isomorphisms

$$H^*(G_\lambda) \cong H^*(U_0) \otimes H^*(K_0) \tag{2.3}$$

$$H^*(G_\lambda) \cong H^*(U_1) \otimes H^*(K_1) \tag{2.4}$$

Passing to the dual we get the homology isomorphisms as vector spaces

$$H_*(G_\lambda) \cong H_*(U_0) \otimes H_*(K_0) \tag{2.5}$$

$$H_*(G_\lambda) \cong H_*(U_1) \otimes H_*(K_1). \tag{2.6}$$

## 2.2 The elements $x_i, y_i, t$ and $w_i$

Denote by $t$ the generator of infinite order in $H_1(G_\lambda;\mathbb{Z})$, $\lambda > 0$. We know that $H_*(SO(3)) = \Lambda(x_1, x_2)$ where $\Lambda$ is the exterior algebra on generators $x_i$ of degree $i$. Thus $H_*(K_0) = \Lambda(x_1, x_2, z_1, z_2)$, where $x_i, z_i$ represent rotation in first and second factors respectively. The homology of the $SO(3)$ factor in $K_1 \cong SO(3) \times S^1$ is generated by $y_i$, and we explain in the next lemma the relation of these generators with the generators $x_i$ and $z_i$.

**Lemma 2.2.1.** *The homology ring of the diagonal in $SO(3) \times SO(3)$ is given by $H_*(SO_d(3)) = \Lambda(y_1, y_2)$ where*

$$\begin{aligned} y_1 &= x_1 + z_1 \\ y_2 &= x_2 + z_2 + x_1 z_1 \\ y_3 &= x_3 + z_3 + x_1 z_2 + x_2 z_1 \end{aligned}$$

*where $x_i$ and $z_i$ are the generators of the homology ring of $SO(3) \times SO(3)$.*

*Proof.* It is clear that $y_i$ has terms $x_i + z_i$, just looking at the cell structure. Note that we define the cup product using the diagonal map $d : SO(3) \to SO(3) \times SO(3)$. If $\alpha \in H^*(SO(3))$ then $(\alpha \cup \alpha)(y_2) = d^*(\alpha \times \alpha)(y_2)$. We know that the cup product of $\hat{x}_1$ and $\hat{z}_1$ does not vanish, so we have $0 \neq (\hat{x}_1 \cup \hat{z}_1)(y_2)$.

Thus we also have

$$\begin{aligned}(\hat{x}_1 \cup \hat{z}_1)(y_2) &= d^*(\hat{x}_1 \times \hat{z}_1)(y_2) \\ &= (\hat{x}_1 \times \hat{z}_1)(d_* y_2) \neq 0\end{aligned}$$

Therefore we see that $d_* y_2$ must have a component in $H_1(SO(3)) \otimes H_1(SO(3))$. The only element like that is $x_1 z_1$, so $y_2$ must involve this element. Similarly we prove that $y_3$ must involve an element in $H_2(SO(3)) \otimes H_1(SO(3))$ and $H_1(SO(3)) \otimes H_2(SO(3))$ and those are $x_2 z_1$ and $x_1 z_2$.

□

It follows that the generators $y_i$ commute with generators $x_i$ and $z_i$. From injections $i_{0*}$ and $i_{1*}$ we have elements $t, x_i, z_i$ and $y_i$ in $H_*(G_\lambda)$. From isomorphisms (2.1) and (2.5) we know that the rank of $H_1(G_\lambda)$ is 3 and as we just showed we have elements $t, x_1$ and $y_1$ in $H_1(G_\lambda)$. Clearly these are linearly independent.

Looking at (2.5) and (2.6) we see that $t$ must have a nonzero image in $H_1(U_0)$. On the other hand, since the homology of the $SO(3)$ factor in $K_1$ is generated by $y_i$, $x_i$, for example, must have a nonzero image in $H_i(U_1)$. The class $x_1$ must correspond , by (2.1), to a class in $H_2(U_0)$ and we will see in lemma 2.3.1 below that this class is the image of $x_1 t$ in $U_0$. $x_1$ is a spherical representative of the first $SO(3)$ factor in $H_1(K_0)$. Therefore, since $K_0$ acts on $\mathcal{J}_\lambda$ by multiplication on the left there is a well defined 2-cycle $x_1 t$ in $U_0$. More precisely, if $x_1$ is represented by

$$S^1 \to G_\lambda : u \mapsto x_1(u)$$

and $t$ by

$$S^1 \to G_\lambda : v \mapsto t(v)$$

we define a 2-cycle in $G_\lambda$ given by the map

$$S^2 = S^1 \times S^1/S^1 \vee S^1 \to G_\lambda$$

induced by the commutator

$$S^1 \times S^1 \to G_\lambda : (v, u) \mapsto t(v)x_1(u)t(v)^{-1}x_1(u)^{-1}.$$

Let's recall that for any group $G$ the Samelson product $[x, y] \in \pi_{p+q}(G)$ of elements $x \in \pi_p(G)$ and $y \in \pi_q(G)$ is represented by the commutator

$$S^{p+q} = S^p \times S^q/S^p \vee S^q \to G : (u, v) \mapsto x(u)y(v)x(u)^{-1}y(v)^{-1}$$

The Samelson product in $\pi_*(G)$ is related to the Pontryagin product in $H_*(G : \mathbb{Z})$ by the formula

$$[x, y] = xy - (-1)^{|x||y|}yx,$$

where we suppressed the Hurewicz homomorphism $\rho : \pi_*(G) \to H_*(G)$ to simplify the expression. Therefore we see that this 2-cycle is given by the commutator $[x_1, t]$, so in homology, i.e., in $H_2(G_\lambda, \mathbb{Z}_2)$ is simply given by $x_1t + tx_1$. Similarly we define a cycle in $H_4(G_\lambda, \mathbb{Z}_2)$ that in homotopy is given by the commutator $[t, x_3]$. Although $x_2$ is not a spherical class, i.e., $x_2 \notin \pi_2(G_\lambda)$ we can consider a cycle in degree 3 given by $x_2t + tx_2$ in $H_*(G_\lambda, \mathbb{Z}_2)$.

**Definition 2.2.2.** *We define elements $w_i \in H_{i+1}(G_\lambda, \mathbb{Z}_2)$ to be the commu-*

*tators $x_it + tx_i$ with $i = 1, 2, 3$. For a word in the $w_i'$s we use the notation $w_I = w_{i_1}...w_{i_n}$ with $I = (i_1, ..., i_n)$.*

The reason why we use these generators $x_it + tx_i$ instead of simply $x_it, tx_i$ is first because they project simultaneously to additive generators in $H_*(U_1)$ and $H_*(U_0)$ so it is easier to see the correspondence between elements in isomorphisms (2.5) and (2.6). Secondly their dual in cohomology represent a new generator in the ring $H^*(G_\lambda)$, this meaning that they are not in the subalgebra generated by the duals of $t$, $x_1$ and $y_1$. We show this fact in the next lemma. First we define the duals of these elements in $H^1(G_\lambda)$. $\hat{t}$ is the element in $H^1(G_\lambda)$ such that $\hat{t}(t) = 1$ and $\hat{t}(x_1) = \hat{t}(y_1) = 0$. We define $\hat{x_1}$ and $\hat{y_1}$ in the obvious way. We also have $\hat{x_i} = (\hat{x_1})^i$ and $\hat{y_i} = (\hat{y_1})^i$.

**Lemma 2.2.3.** *We have $(\hat{t} \cup \hat{x}_i)([x_i, t]) = 0$, where $\hat{t}$ and $\hat{x}_i$ represent the dual of $t$ and $x_i$ in $H^*(G_\lambda)$ respectively.*

*Proof.* Although in this section we are working with $\mathbb{Z}_2$ coefficients we will prove a stronger result by showing that the statement is true also over $\mathbb{Z}$. Note that $(\hat{t} \cup \hat{x}_i)([x_i, t]) = (\hat{t} \cup \hat{x}_i)(x_it + tx_i) = (\hat{t} \cup \hat{x}_i)(x_it) + (\hat{t} \cup \hat{x}_i)(tx_i)$ and we show that $(\hat{t} \cup \hat{x}_i)(tx_i) = (\hat{x}_i \cup \hat{t})(x_it) = -(\hat{t} \cup \hat{x}_i)(x_it) = 1$. For example, in the case when $i = 1$ consider $f : S^1 \times S^1 \to G_\lambda : (t, s) \mapsto \varphi_t\psi_s$, where $S^1 \to G_\lambda : t \mapsto \varphi_t$ and $S^1 \to G_\lambda : s \mapsto \psi_s$ represent the cycles $t$ and $x_1$ respectively. Then

$$\begin{aligned}(\hat{t} \cup \hat{x_1})(tx_1) &= f^*(\hat{t} \cup \hat{x_1})[S^1 \times S^1] \\ &= f^*(\hat{t}) \cup f^*(\hat{x_1})[S^1 \times S^1] \\ &= f^*(\hat{t})[S^1]f^*(\hat{x_1})[S^1] = 1\end{aligned}$$

Thus $(\hat{t} \cup \hat{x_1})([x_1, t]) = -1 + 1 = 0.$ □

The idea is to work in terms of basis for each group $H^*(U)$ or $H_*(U)$, because in this case we have a canonical identification between $H^*(U)$ and $H_*(U)$, this meaning that if $\{c_\alpha\}$ is a basis for $H_*(U)$ then $\hat{c}_\alpha$ is the element in $H^*(U)$ such that $\hat{c}_\alpha(c_\beta) = \delta_{\alpha\beta}$.

We choose a normalized set of elements in the subring of $H_*(G_\lambda)$ generated by $t, x_i, y_j$ with $i, j = 1, 2, 3$.

**Lemma 2.2.4.** *Any word in the $t, x_i, y_j$ with $i, j = 1, 2, 3$ is a sum of elements of the form*

$$w_I t^{\epsilon_t} x_i^{\epsilon_i} y_j^{\eta_j}, \tag{2.7}$$

*where $\epsilon_t, \epsilon_i, \eta_j = 0$ or $1$, $I = (i_1, ..., i_k)$ and $i, j = 1, 2$ or $3$ ($x_3 = x_1 x_2, y_3 = y_1 y_2$).*

*Proof.* We know that $y_j$ commutes with all other elements and we have equations

$$x_i w_j = w_i x_j \text{ if } (i, j) \neq (1, 2) \text{ or } (2, 1)$$

$$x_i w_j = w_i x_j + w_3 \text{ if } (i, j) = (1, 2) \text{ or } (2, 1).$$

We also know that $x_i t = t x_i + w_i$, $t$ commutes with $w_i$ for $i = 1, 2, 3$ and $t^2 = 0$. These facts together with the two equations imply that we can always bring any copy of $x_i$ to the right of the $w_i's$, adding, if necessary, words on the $w_i's$. □

## 2.3 A generating set for $H_*(G_\lambda)$

In this subsection the aim is to show that the elements $x_i, y_j, t$ generate the ring $H_*(G_\lambda; \mathbb{Z}_2)$. In order to do that we give a geometric description of the isomorphism $H_{p+1}(U_0) \cong H_p(U_1)$.

We have projections $p_{i*}$ to $H_*(U_i)$ with $i = 0$ or 1. Since $x_i$ has image in $H_i(K_0)$ we see that $p_{0*}([x_i, t]) = p_{0*}(x_i t)$ in $H_*(U_0)$ and $p_{1*}([x_i, t]) = p_{1*}(tx_i)$ in $H_*(U_1)$ because $t$ has image in $H_1(K_1)$. We write $t$ for $p_{0*}(t) \in H_*(U_0)$ and $x_i$ for $p_{1*}(x_1) \in H_*(U_1)$. However it will be convenient to distinguish notationally between the different incarnations of $w_1$,$w_2$,$w_3$ on the different spaces. We will denote by $v_i = p_{0*}(w_i)$ the generators in $H_*(U_0)$ and by $u_i = p_{1*}(w_i)$ the generators in $H_*(U_1)$ where $i = 1, 2$ or 3. Let $v_I = p_{0*}(w_I)$ and $u_I = p_{1*}(w_I)$ where $w_I$ is given as in definition 2.2.2. This way we give meaning to expressions as $v_i v_j = p_{0*}(w_i w_j)$, $v_i t = p_{0*}(w_i t)$ and $u_i u_j = p_{1*}(w_i w_j)$. We can write $v_i t$ or $tv_i$ to refer to the same element because $t$ commutes with $w_i$ in $H_*(G_\lambda)$. Note that $H_*(U_i)$ is a left $H_*(G_\lambda)$-module, so $H_*(G_\lambda)$ acts on $H_*(U_i)$ by multiplication on the left. Using this module action we have $v_{I'} = w_i.v_I$ and $u_{I'} = w_i.u_I$ for $I' = (i, I)$.

We can choose right inverses $s_i : H_*(U_i) \to H_*(G_\lambda)$ such that $s_0(t) = t$ , $s_0(v_i) = w_i$, $s_1(x_i) = x_i$, $s_1(u_i) = w_i$ and $p_{i*} \circ s_i = id$. They exist because of isomorphisms (2.5) and (2.6). Moreover we can choose $s_i$ such that $s_0$ preserves multiplication by $t, w_i$ and $s_1$ preserves multiplication by $w_i$.

**Lemma 2.3.1.** *The isomorphism $H_{p+1}(U_0) \cong H_p(U_1)$ is given by the map*

$$\psi : H_p(U_1) \to H_{p+1}(U_0) : c \mapsto p_{0*}(s_1(c)t)$$

*Proof.* Note that since $U_1$ is a codimension 2 submanifold of $\mathcal{J}_\lambda$, there is a circle bundle $\partial\mathcal{N}_{U_1}$ where $\partial\mathcal{N}_{U_1}$ is a neighborhood of $U_1$ in $U_0$:

$$\begin{array}{ccc} S^1 & \longrightarrow & \partial\mathcal{N}_{U_1} \\ & & \downarrow \pi \\ & & U_1 \end{array} \tag{2.8}$$

Thus for any map $c : C \to U_1$ representing a cycle in $H_p(U_1)$ we can obtain a cycle in $H_{p+1}(U_0)$ by lifting $c$ to $\partial\mathcal{N}_{U_1}$. This is the geometric interpretation of the isomorphism $H_p(U_1) \to H_{p+1}(U_0; \mathbb{Z}_2)$ stated in (2.1). More precisely, using the section $s_1$ we can lift $c$ to a compact set $s_1(c) \in G_\lambda$. Then for each $g \in s_1(c)$ there is a map to $U_0$ given by $g \mapsto g_*J$, where we can choose $J \in U_0$ close to $U_1$. In fact, we can choose $J \in \mathcal{N}_{U_1}$ so close to $U_1$ such that $g_*J \in \mathcal{N}_{U_1}$ also. Then a cycle in $\mathcal{N}_{U_1} \subset U_0$ is obtained by the action on $s_1(c)$ of the $S^1$-action represented by $t$. For each $g \in s_1(c)$ we have a loop around $U_1$ given by $g_*(t_*J) = (gt)_*J$, so the cycle $c$ lifts to $p_{0*}(s_1(c)t)$ in $U_0$. □

**Remark 2.3.2.** *Using the notation introduced before we can say that* $\psi(x_i) = p_{0*}(s_1(x_i)t) = x_i t = v_i$, $\psi(u_i) = p_{0*}(s_1(u_i)t) = w_i t = v_i t$, $\psi(u_I x_i) = w_I.v_i = v_{I'}$ *with* $I' = (I, i)$ *and* $\psi(u_I) = v_I t$.

*The map* $\psi^D$ *that gives the corresponding isomorphism in cohomology,* $\psi^D : H^p(U_1) \to H^{p+1}(U_0)$, *is the composite of the restriction*

$$i^* : H^*(U_0) \longrightarrow H^*(\partial\mathcal{N}_{U_1})$$

*with integration over the fiber of the projection* $\pi : \partial\mathcal{N}_{U_1} \to U_1$, *of the fibration* (2.8) *( see [1]).*

Now we use the lemma to prove the following proposition.

**Proposition 2.3.3.** *The generators of the Pontryagin ring $H_*(G_\lambda)$ are $t, x_i$, $y_j$ with $i, j = 1, 2$.*

*Proof.* We know we have elements $t, x_i, y_j$ in $H_*(G_\lambda)$ because of injections $i_{0*}$ and $i_{1*}$. Let $R_* \subset H_*(G_\lambda)$ be the subring generated by $t, x_i, y_j$. Suppose there is an element of minimal degree in $H_*(G_\lambda) - R_*$. From isomorphism (2.5) we can conclude that such an element would be mapped to a sum of elements

$$\sum_l c_l \otimes k_l \in \bigoplus_l (H_{d-l}(U_0) \otimes H_l(K_0))$$

with $0 \leq l \leq 6$. For some $l$, $c_l$ is not a polynomial in the $v_I, t$. Take the largest such $l$. By the isomorphism in lemma (2.3.1) and remark 2.3.2 this would create an element in $H_{d-l-1}(U_1)$ that is not a polynomial in $u_I$ and $x_i$. But this is impossible because this would give rise to a new generator in $H_{d-l-1}(G_\lambda)$ corresponding to this new element in $H_{d-l-1}(U_1) \otimes H_0(K_1)$ and this contradicts the minimality of $d$. □

## 2.4 Main theorem

We first show that we have isomorphisms $H_*(G_\lambda) \cong H_*(U_i) \otimes H_*(K_i)$ given by Pontryagin product. More precisely, we can define maps

$$\varphi_i : H_*(U_i) \otimes H_*(K_i) \to H_*(G_\lambda) : c \otimes k \mapsto s_i(c).k \tag{2.9}$$

with $i = 0$ or 1. Since $K_i \subset G_\lambda$ and $i_i$ is injective in homology we denote $i_{i*}(k)$ simply by $k$. It is clear that $p_{i*}(s_i(c).k) = 0$ if $k \in H_*(K_i)$, with $* > 0$. Clearly the product $s_0(c)k$ is an element in the normalized set we defined in lemma 2.2.4, because $s_0(c)$ is a product of $w_i's$ and $t$ and $k$ is a product of $x_i$ and $y_j$. Then we prove that these maps are isomorphisms.

**Proposition 2.4.1.** *The maps $\varphi_i : H_*(U_i) \otimes H_*(K_i) \to H_*(G_\lambda) : c \otimes k \mapsto s_i(c).k$ given by Pontryagin product are isomorphisms.*

*Proof.* Consider the elements of the form $v_I t^{\epsilon_t}$, with $\epsilon_t = 0, 1$ in $H_*(U_0)$. If they are not linearly independent, choose a maximal linearly independent subset $B = \{c_\alpha\}$. It follows from proposition 2.3.3 that this is a basis for $H_*(U_0)$. Now consider the image in $H_*(G_\lambda)$ of $B$. This is given by $B' = \{s_0(c_\alpha)\}$ with $c_\alpha \in B$. These are elements of the form $w_I t^{\epsilon_t}$, $\epsilon_t = 0, 1$ and the set $B'$ is linearly independent. Therefore it is an additive basis for the space spanned by elements of the form $w_I t^{\epsilon_t}$. Note that $H_*(G_\lambda)$ has a subalgebra isomorphic to $H_*(K_0)$ and an additive basis for this is $D = \{k_\gamma\} = \{x_i^{\epsilon_i} y_j^{\eta_j}\}$ where $\epsilon_i$ and $\eta_j$ are equal to 0 or 1, so an additive basis for $H_*(G_\lambda)$, will contain all elements of this form. To prove the theorem in the case $i = 0$ we need to show that the set $B'' = \{s_0(c_\alpha).k_\gamma\}$ where $s_0(c_\alpha) \in B'$ and $k_\gamma \in D$ is an additive basis of $H_*(G_\lambda)$. First we will prove that these elements generate additively $H_*(G_\lambda)$. Suppose we have an element $a \in H_*(G_\lambda)$. From proposition 2.3.3 and lemma 2.2.4 we know that every element in $H_*(G_\lambda)$ is a sum of elements of the form (2.7). Thus

$$a = \sum_\alpha w_{J_\alpha} t^{\epsilon_\alpha} x_{i_\alpha}^{\epsilon_i} y_{j_\alpha}^{\eta_j}.$$

We know that $x_{i_\alpha}^{\epsilon_i} y_{j_\alpha}^{\eta_j}$ is in $D$ and if $w_{J_\alpha} t^{\epsilon_\alpha}$ is not in $B'$ we can write it as sum of elements in $B'$. Thus $a$ is a sum of elements in $B''$.

Now we need to prove that the elements in $B''$ are linearly independent. We know that for a fixed degree $d$, the dimension of $H_d(G_\lambda)$ is given by $\sum_{l=0}^{d} \dim H_l(U_0) \times \dim H_{d-l}(K_0)$, because of the vector space isomorphism (2.5). But this is precisely the number of elements in $B''$ of degree $d$. So they must be linearly independent, otherwise their span would not be the space $H_*(G_\lambda)$. This means that the set $B'' = \{s_0(c_\alpha).k_\gamma\}$ defined above is an additive basis for $H_*(G_\lambda)$. Therefore $\varphi_0$ is an isomorphism.

In the case $i = 1$, $\varphi_1$ maps $c \otimes k$ to $s_1(c).k$ and this is not in the form (2.7). However we can prove a result analogous to lemma 2.2.4 stating that any word in the $x_i, y_j, t$ is a sum of elements of the form $w_I x_i^{\epsilon_i} t y_j^{\eta_j}$. This is clear because $w_I t x_i y_j = w_I x_i t y_j + w_I w_i t y_j$ for all $I, i$ and $j$. Now repeating the steps for the case $i = 0$ and using isomorphism (2.6), it follows easily that $\varphi_1$ is also an isomorphism. □

We are now in position to calculate the algebra structure on $H_*(G_\lambda)$.

**Theorem 2.4.2.** *If* $0 < \lambda \leq 1$ *then*

$$H_*(G_\lambda; \mathbb{Z}_2) = \Lambda(y_1, y_2) \otimes \mathbb{Z}_2\langle t, x_1, x_2\rangle / R$$

*where* $\deg y_i = \deg x_i = i$ *and* $R$ *is the set of relations* $\{t^2 = x_i^2 = 0, x_1 x_2 = x_2 x_1\}$.

*Proof.* We already know from proposition 2.3.3 that the generators of the Pontryagin ring are $t, x_i, y_j$. Now we need to prove that the only relations between them are the ones in $R$. We will prove by induction on the dimension that the elements of the form (2.7) give an additive basis of $H_*(G_\lambda)$. The induction hypothesis is that this statement is true up to dimension $d-1$. This implies that elements of form (2.7) are linearly independent, thus we have no relations between these elements up to dimension $d-1$. Suppose there was some relation of minimal degree $d$ in $H_d(G_\lambda)$. The first step is to show that the relation would be between the $w_i's$ only. Assume the relation was given by a finite sum of the type

$$\sum_k w_{I_k} A_k = 0$$

where $w_{I_k}$ is a word on the $w_i's$ and $A_k = t^{\epsilon_k} b_k$ where $b_k$ is an element in $H_*(K_0)$ and $\epsilon_k$ equals 0 or 1. Then from proposition 2.4.1 with $i = 0$ we can conclude that we must have

$$\sum_k w_{I_k} t^{\epsilon_k} \otimes b_k = 0.$$

We can group together the terms in which $b_k$ is the same, thus we can write the relation as

$$\sum_k (\sum_{l_k} w_{l_k} t^{\epsilon_{l_k}}) \otimes b_k = 0$$

where now $b_k$ runs over a set of basis elements of $H_*(K_0)$. This implies that we have a relation of the type

$$\sum_l w_{I_l} t^{\epsilon_l} = 0.$$

Using proposition 2.4.1 with $i = 1$ we show that the relation is between the

$w_i's$, because

$$\sum_l w_{I_l} \otimes t^{\epsilon_l} = \sum_{l'} w_{I_{l'}} \otimes t + \sum_{l''} w_{I_{l''}} \otimes 1 = 0$$

implies

$$\sum_{l'} w_{I_{l'}} = 0 \text{ and } \sum_{l''} w_{I_{l''}} = 0.$$

A relation in the $w_i's$ projects, under the map $p_{0*}$, to a relation in the $v_i's$ in $H_d(U_0)$. Using isomorphism (2.1) this would give a relation in degree $d-1$ between the $u_i's$ and $x_i's$ in $H_{d-1}(U_1)$. But this contradicts the induction hypothesis because such relation implies one in $H_*(G_\lambda)$ with $*$ at most equal to $d-1$.

Since there are no relations between the $w_i's$ the Pontryagin ring $H_*(G_\lambda)$ contains a free noncommutative ring on 3 generators, namely $w_1, w_2, w_3$. □

So we proved also the following proposition

**Proposition 2.4.3.** *An additive basis for $H_*(G_\lambda)$ is given by*

$$w_I t^{\epsilon_t} x_i^{\epsilon_i} y_j^{\eta_j}, \tag{2.10}$$

*where $\epsilon_t, \epsilon_i, \eta_j = 0$ or $1$, $I = (i_1, ..., i_n)$ and $i, j = 1, 2$ or $3$ ($x_3 = x_1x_2, y_3 = y_1y_2$).*

## 2.5 Relation between cohomology and homology

Proving isomorphisms (2.5) and (2.6) does not imply that we have algebra isomorphisms. That is proved in the next lemma.

**Lemma 2.5.1.** *The following isomorphisms hold as algebra isomorphisms.*

$$H^*(G_\lambda) \cong H^*(U_i) \otimes H^*(K_i) \text{ with } i = 0, 1 \tag{2.11}$$

*Proof.* The proof is based in the argument used by Abreu in [1] with some necessary changes. $H^*(G_\lambda)$ has subalgebras $p_i{}^*(H^*(U_i)) \cong H^*(U_i)$. From theorem 2.1.2 we know that $\mathrm{Map}_1(S^2)$ is homotopy equivalent to $SO(3) \times \Omega$ where $\Omega$ denotes the universal covering space of $\mathrm{Map}_{1*}(S^2)$. Therefore we have a map $\mathrm{Map}_1(S^2) \times \mathrm{Map}_1(S^2) \to SO(3) \times SO(3)$. The composite of $G_\lambda \to \mathrm{Map}_1(S^2) \times \mathrm{Map}_1(S^2)$ with the previous map gives us a map $p : G_* \to K_0$. Thus $H^*(G_\lambda)$ has a subalgebra $p^*(H^*(K_0)) \cong H^*(K_0)$. Composing these inclusions of $H_*(U_0)$ and $H^*(K_0)$ as subalgebras of $H^*(G_\lambda)$ with cup product multiplication in $H^*(G_\lambda)$ we get a map

$$\nu_0 : H^*(U_0) \otimes H^*(K_0) \to H^*(G_\lambda).$$

$\nu_0$ is an algebra homomorphism because $H^*(G_\lambda)$ is commutative and it is compatible with filtrations ( the obvious one on $H_*(U_0) \otimes H^*(K_0)$ and the filtration $F$ on $H^*(G_\lambda)$ coming from the fibration on the left in (2.2)). The degeneration of the spectral sequence at the $E_2$-term implies that $\nu_0$ is an algebra isomorphism. This proves isomorphism (2.11) in the case $i = 0$. For the case $i = 1$ note that the map $i_1^* : H^*(G_\lambda) \to H^*(K_1)$ is surjective, so there are $\hat{t}$ and $\hat{y}$ in $H^*(G_\lambda)$ such that $i_1^*(\hat{t})$ and $i_1^*(\hat{y})$ generate the ring $H^*(K_1)$, where $i_1^*(\hat{t})$ is the generator of the cohomology of $S^1$ and $\hat{t}$ is such that $\hat{t}(x_1) = 0$. $i_1^*(\hat{y})$ is the generator of the cohomology of the $SO(3)$ factor. Now we need to prove that $\hat{t}^2 = 0$ in $H^*(G_\lambda)$ in order to claim that the subalgebra of $H^*(G_\lambda)$ generated by $\hat{t}$ and $\hat{y}$ is isomorphic to $H^*(K_1)$.

**Lemma 2.5.2.** *$\hat{t}^2 = 0$ in $H^*(G_\lambda)$*

*Proof.* Using isomorphisms (2.1), (2.5) and (2.6) we can show that the rank of $H_2(G_\lambda)$ is 6. But in $H_2(G_\lambda)$ the cycles $x_2, y_2, tx_1, ty_1, x_1y_1, w_1$ are linearly independent. We will show that $\hat{t}^2$ evaluated on all these classes is 0. The only one at which is not obviously 0 is $w_1$. Let the map $\alpha : S^2 = S^1 \times S^1 / S^1 \vee S^1 \to G_\lambda$ represent the 2-cycle $w_1$. Then $\hat{t}^2(w_1) = \alpha^*(\hat{t}^2)[S^2] = (\alpha^*(\hat{t})[S^2])^2$ and this vanishes because $w_1$ is a spherical class,i.e., $\alpha^*(\hat{t}) \in H^1(S^2) = 0$. □

Again composing these inclusions of $H^*(K_1)$ and $H_*(U_1)$ as subalgebras of $H^*(G_\lambda)$ with cup product multiplication we get a map

$$\nu_1 : H^*(U_1) \otimes H^*(K_1) \to H^*(G_\lambda)$$

which is an algebra isomorphism. □

From the isomorphisms in the previous lemma and in proposition 2.4.1 we would be tempted to think that the diagram

$$\begin{array}{ccc} H_*(U_0)\otimes H_*(K_0) & \xrightarrow{\cdot} & H_*(G_\lambda) \\ \downarrow & & \downarrow \\ H^*(U_0)\otimes H^*(K_0) & \xrightarrow{\cup} & H^*(G_\lambda) \end{array}$$

commutes, where the vertical arrows are given by the dual isomorphisms with respect to basis in $H_*(U_0)$, $H_*(G_\lambda)$. Actually this diagram does not commute. To see that first recall that for a topological group we have maps

$$G \xrightarrow{d} G\times G \xrightarrow{p} G$$

that give rise to a Hopf algebra $(H_*(G,\mathbb{Z}_2), p_*, d_*)$ where $p_*$ is the Pontryagin product. The dual algebra is $(H^*(G,\mathbb{Z}_2), \cup, p^*)$. To give a complete description of $H_*(G_\lambda;\mathbb{Z}_2)$ as a Hopf algebra we will also need to compute the ring homomorphism $d_*$. The dual of $H_n(G_\lambda;\mathbb{Z}_2)$ is given by $H^n(G_\lambda;\mathbb{Z}_2) = \mathrm{Hom}(H_n(G_\lambda;\mathbb{Z}_2))$. The value of a homomorphism $c'$ on $c$ will be denoted by $\langle c', c\rangle$. It is understood that $\langle c', c\rangle = 0$ unless $\dim c' = \dim c$.

**Lemma 2.5.3.** *The following formulas hold.*

$$d_*(x_k) = \sum_{i=0}^{k} x_i \otimes x_{k-i}$$

$$d_*(w_k) = \sum_{i=0}^{k-1} (x_i \otimes w_{k-i} + w_{k-i}\otimes x_i)$$

*with* $k = 1, 2, 3$ *and assuming* $x_0 = 1$.

*Proof.* Recall that $H^*(SO(3)) = \mathbb{Z}_2[\hat{x_1}]/\{\hat{x_1}^4 = 0\}$ and $\hat{x_i} = \hat{x_1}^i$ when $i = 2, 3$. Denote $\hat{x_1}$ simply by $\hat{x}$. Because degree of $x_1$ is 1 we obtain $d_*(x_1) = 1 \otimes x_1 + x_1 \otimes 1$.

$$\begin{aligned}
\langle \hat{x}^n \otimes \hat{x}^m, d_*(x_2) \rangle &= \langle d^*(\hat{x}^n \otimes \hat{x}^m), x_2 \rangle = \\
&= \langle \hat{x}^n \cup \hat{x}^m, x_2 \rangle = \\
&= \langle \hat{x}^{n+m}, x_2 \rangle = \\
&= \begin{cases} 1 & \text{if } n + m = 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

So this proves that

$$d_*(x_2) = x_2 \otimes 1 + x_1 \otimes x_1 + 1 \otimes x_2$$

and

$$d_*(x_3) = d_*(x_1 x_2) = d_*(x_1) d_*(x_2).$$

Thus

$$d_*(x_3) = 1 \otimes x_3 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes x_3.$$

We also have $d_*(w_i) = d_*(x_i t + t x_i)$ therefore

$$\begin{aligned}
d_*(w_1) &= 1 \otimes w_1 + w_1 \otimes 1 \\
d_*(w_2) &= 1 \otimes w_2 + x_1 \otimes w_1 + w_1 \otimes x_1 + w_2 \otimes 1 \\
d_*(w_3) &= 1 \otimes w_3 + x_1 \otimes w_2 + w_2 \otimes x_1 + x_2 \otimes w_1 + w_1 \otimes x_2 + w_3 \otimes 1
\end{aligned}$$

□

Now consider the following example.

**Example 2.5.4.** *If we had a commutative diagram then we would have* $\langle \hat{w}_1 \cup \widehat{x_1 y_2}, w_1 x_1 y_2 \rangle = 1$ *and* $\hat{w}_1 \cup \widehat{x_1 y_2}$ *evaluated at all other elements would be 0. But this does not happen as we can see in the following calculations. We have*

$$\langle \hat{w}_1 \cup \widehat{x_1 y_2}, w_1 x_1 y_2 \rangle \; = 1$$

*and*

$$\begin{aligned}
\langle \hat{w}_1 \cup \widehat{x_1 y_2}, w_2 y_2 \rangle &= \langle \hat{w}_1 \otimes \widehat{x_1 y_2}, d_*(w_2 y_2) \rangle \; = \\
&= \langle \hat{w}_1 \otimes \widehat{x_1 y_2}, d_*(w_2) d_*(y_2) \rangle \; = \\
&= \langle \hat{w}_1 \otimes \widehat{x_1 y_2}, y_2 \otimes w_2 + x_1 y_2 \otimes w_1 + w_1 y_2 \otimes x_1 + \\
&\quad + w_2 y_2 \otimes 1 + y_1 \otimes w_2 y_1 + x_1 y_1 \otimes w_1 y_1 + \\
&\quad + w_1 y_1 \otimes x_1 y_1 + w_2 y_1 \otimes y_1 + 1 \otimes w_2 y_2 + \\
&\quad + x_1 \otimes w_1 y_2 + w_1 \otimes x_1 y_2 + w_2 \otimes y_2 \rangle \; = 1
\end{aligned}$$

# Chapter 3

# Torsion in $H_*(G_\lambda;\mathbb{Z})$

We can repeat the argument of the previous section with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients, with $p$ prime and $\neq 2$. In this case the homology of $SO(3)$ is given by a single generator in dimension 3, so the generators of the homology ring of $G_\lambda$ are simply $t, x_3$ and $y_3$. An additive basis for the homology is given by elements of the form $w_3^k t^{\epsilon_t} x_3^{\epsilon} y_3^{\eta}$, where $\epsilon_t, \epsilon, \eta = 0$ or 1 and $w_3$ is obtained as the commutator of $x_3$ and $t$. The results are the same if we consider $\mathbb{Q}$ coefficients or $Z_p$ coefficients with $p \neq 2$. Therefore we can conclude that $H_*(G_\lambda;\mathbb{Z})$ has no $p$-torsion if $p \neq 2$.

In this section we will prove that in fact $H_*(G_\lambda;\mathbb{Z})$ contains no torsion elements of order greater than 2.

## 3.1 The James construction

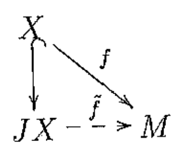
For a pointed topological space $(X,*)$, let $J_k(X) = X^k/\sim$ where

$$(x_1, ..., x_{j-1}, *, x_{j+1}, ..., x_k) \sim (x_1, ..., x_{j-1}, x_{j+1}, *, ..., x_k)$$

The James construction on $X$, denoted $J(X)$ is defined by

$$J(X) = \lim_{\vec{k}} J_k(X),$$

where $J_k(X) \subset J_{k+1}(X)$ by adding $*$ in last component. There is a canonical inclusion $X = J_1(X) \hookrightarrow J(X)$. $J(X)$ is a topological monoid and any map from $X$ to a topological monoid $M$ extends uniquely to a morphism $J(X) \to M$ of topological monoids. That is, $X \hookrightarrow J(X)$ is universal with respect to maps from $X$ to topological monoids, i.e., if $f : X \to M$ is given there is a unique $\tilde{f}$ such that




the diagram commutes. $\tilde{f}$ is defined by $\tilde{f}(x_1, ..., x_k) = f(x_1)...f(x_k)$. From the definition, $J^k X / J^{k-1} X = X \wedge X \wedge ... \wedge X$ and since we have a filtration

$$JX \supset ... \supset J^k X \supset J^{k-1} X \supset ...$$

applying the Künneth Theorem we conclude that ( see [14] )

**Theorem 3.1.1.**

$$H_*(J^k X; \mathbb{Z}_2) = H_*(J^{k-1} X; \mathbb{Z}_2) \oplus H_*(X \wedge X \wedge ... \wedge X; \mathbb{Z}_2)$$

*and*

$$\tilde{H}_*(JX; \mathbb{Z}_2) = \oplus_k \tilde{H}_*(X; \mathbb{Z}_2) \otimes ... \otimes \tilde{H}_*(X; \mathbb{Z}_2) = T(\tilde{H}_*(X; \mathbb{Z}_2))$$

*where given a vector space $H$, $T(H)$ is the tensor algebra on $H$ and the last isomorphism is an isomorphism of Pontryagin rings.*

Now note the following theorem ( see proof in [12] )

**Theorem 3.1.2 (James).** *If $X$ has the homotopy type of a connected CW-complex then $JX$ and $\Omega\Sigma X$ are homotopy equivalent.*

Thus we can conclude that

$$\tilde{H}_*(\Omega\Sigma X;\mathbb{Z}_2) \cong T(\tilde{H}_*(X;\mathbb{Z}_2))$$

so, in particular, if $X = S^1 \wedge SO(3)$ we get

$$\tilde{H}_*(\Omega\Sigma(S^1 \wedge SO(3));\mathbb{Z}_2) \cong T(\tilde{H}_*(S^1 \wedge SO(3);\mathbb{Z}_2)) \cong \mathbb{Z}_2\langle w_1, w_2, w_3\rangle,$$

where $w_1, w_2, w_3$ are generators in dimension $2, 3, 4$ respectively. So we see that the homology with $\mathbb{Z}_2$ coefficients of this space is isomorphic to a subalgebra of $H_*(G_\lambda;\mathbb{Z}_2)$. We are going to use this to prove that $H_*(G_\lambda;\mathbb{Z})$ has only $\mathbb{Z}_2$-torsion.

**Theorem 3.1.3.** *The homology of the space $\Omega\Sigma(S^1 \wedge SO(3))$ has only $\mathbb{Z}_2$ torsion.*

*Proof.* Let $q : P^*(B) \to B$ be the fibration where the fiber of $q$ is the *Moore loop space* , $\Omega^*(B)$, defined by

$$\Omega^*(B) = \{(s,\omega) \in \mathbb{R}^+ \times B^{\mathbb{R}^+} \mid \omega(0) = * \text{ and } \omega(t) = * \text{ for } t \geq s\},$$

and $P^*(B)$ is the *Moore path space* on B defined by

$$P^*(B) = \{(s, \omega) \in \mathbb{R}^+ \times B^{\mathbb{R}^+} \mid \omega(0) = * \text{ and } \omega(t) = \omega(s) \text{ for } t \geq s\}.$$

We think of elements of $\Omega^*(B)$ and $P^*(B)$ as paths parameterized by $[0, s]$. Such paths have a strictly associative multiplication under which the product of paths of length $s$ and $s'$ has length $s + s'$. We introduce these related spaces with strictly associative multiplication, because multiplication of paths in $\Omega(B)$ and $P(B)$ is only homotopy associative. The operation of multiplication of paths defines an associative left action of $\Omega^*(B)$ on the fibration $q$.

Suppose now that $B$ is the suspension of a space $W$. We have the fibration $P^*(\Sigma W) \to \Sigma W$ with fiber $F = \Omega^*(\Sigma W)$. Let us consider $W$ as a space with base point and $F$ as a free space. Then the quotient $W \times F / * \times F$ is just the reduced join $W \wedge F$. As we can see in [14](p321), in this case the total space is contractible and the Wang sequence of the fibration reduces to a family of isomorphisms

$$H_q(W \wedge F) \cong H_q(F)$$

for all $q \geq 0$. This allows a recursive calculation of the homology groups of $F$. For, by the Künneth Theorem

$$H_q(F) \cong H_q(W \wedge F) \quad (3.1)$$

$$\cong \bigoplus_{r+s=q} H_r(W,*) \otimes H_s(F) \bigoplus_{r+s=q-1} \mathrm{Tor}\{H_r(W,*), H_s(F)\} \quad (3.2)$$

If $W$ is 0-connected, $H_r(W,*) = 0$ for $\leq 0$, and therefore the right-hand side of the above formula involves only the homology groups of $F$ in dimensions less then q. It is also proved in [14] ( Corollary 2.19 of Chapter III) that $\Omega(\Sigma W)$ and $\Omega^*(\Sigma W)$ have the same homotopy type. Therefore they have isomorphic homology and cohomology groups, and even isomorphic cohomology rings. That they have isomorphic Pontryagin rings follows from the observation that the homotopy equivalence $h : \Omega(\Sigma W) \to \Omega^*(\Sigma W)$ is an $H$-map. If we put $W = S^1 \wedge SO(3)$ and using the fact that the homology of $W$ has only $\mathbb{Z}_2$-torsion, i.e.,

$$\begin{aligned} H_0(S^1 \wedge SO(3); \mathbb{Z}) &= \mathbb{Z} \\ H_1(S^1 \wedge SO(3); \mathbb{Z}) &= 0 \\ H_2(S^1 \wedge SO(3); \mathbb{Z}) &= \mathbb{Z}_2 \\ H_3(S^1 \wedge SO(3); \mathbb{Z}) &= 0 \\ H_4(S^1 \wedge SO(3); \mathbb{Z}) &= \mathbb{Z} \end{aligned}$$

We can conclude from (3.2) that $H_*(F, \mathbb{Z})$ has only $\mathbb{Z}_2$-torsion. Therefore we proved that $H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z})$ has only $\mathbb{Z}_2$-torsion. □

## 3.2 The Bockstein Spectral Sequence

Let $C$ be a chain complex of free abelian groups and let $p$ be a positive integer. The short exact sequence of groups

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$$

yields a short exact sequence of chain complexes

$$0 \to C \otimes \mathbb{Z}_p \to C \otimes \mathbb{Z}_{p^2} \to C \otimes \mathbb{Z}_p \to 0.$$

The boundary map $\beta_p : H_n(C; \mathbb{Z}_p) \to H_{n-1}(C; \mathbb{Z}_p)$ from the corresponding long exact sequence is called the mod $p$ Bockstein.When $p$ is clear from the context, it is dropped in the notation. $\beta_p$ factors as the composition

$$H_*(C; \mathbb{Z}_p) \xrightarrow{\partial} H_{*-1}(C; \mathbb{Z}) \xrightarrow{j} H_{*-1}(C; \mathbb{Z}_p) .$$

There is a graded exact couple in which $D_s = H_s(C; \mathbb{Z})$ and $E_s = H_s(C; \mathbb{Z}_p)$ with the maps induced from the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

The corresponding spectral sequence is called the mod $p$ *Bockstein spectral sequence* of $C$. It is clear from the definitions that $\beta$ is the $d^1$-differential of this spectral sequence. The $d^r$-differential is written $\beta^{(r)}$ and is called the $r$th Bockstein modulo $p$. Here are some key facts about the Bockstein spectral sequence.

**Fact 3.2.1.** (see [3] and [12])

1. *If $C$ is an arcwise connected $H$-space with $H_i(C)$ finitely generated for each $i$, then*

$$E^{(\infty)} \cong (H_*(C)/\text{Torsion}) \otimes \mathbb{Z}_p$$

*as Hopf algebras.*

2. $E^1_* \cong H_*(C;\mathbb{Z}_p)$.

3. *The spectral sequence of $C$ collapses at $E^r$* iff *$H_*(C)$ has no elements of order $p^m$ for $m \geq r$.*

4. *Let $(\ )_{(p)}$ denote localization at $p$. If $H_*(C)$ is finitely generated then $H_*(C)_{(p)}$ can be reconstructed from its Bockstein spectral sequence. Suppose $H_n(C)_{(p)} = Z^s_{(p)} \oplus \mathbb{Z}_{p^{t_1}} \oplus ... \oplus \mathbb{Z}_{p^{t_k}}$ for some integers $s, t_1, t_2, ...,$ $t_k$. Corresponding to a summand $\mathbb{Z}_{(p)}$ there will be a basis element $x \in H_n(C;\mathbb{Z}_p) = E^1_n$ such that $\beta^{(r)}(x) = 0$ for all $r$, and corresponding to a summand $\mathbb{Z}_{p^t}$ there will be a pair of basis elements $x \in E^1_n$, $y \in E^1_{n+1}$ such that $\beta^{(r)}(x) = 0$ for all $r$, $\beta^{(r)}(y) = 0$ for $r < t$ and $\beta^{(t)}(y) = x$.*

For an $H$-space this spectral sequence is multiplicative which means that the following equality is valid

$$\beta^{(r)}(ab) = \beta^{(r)}(a)b + (-1)^{|a|}a\beta^{(r)}(b)$$

with $a, b \in E^r_*$.

Since $\Omega\Sigma(S^1 \wedge SO(3))$ is an $H$-space, we can look at its mod 2-Bockstein spectral sequence. It will degenerate at the second term, by Theorem 3.1.3 and fact 3, i.e.,

$$E^2_* = E^{(\infty)} \cong (H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z})/\text{Torsion}) \otimes \mathbb{Z}_2.$$

We have

$$(\tilde{H}_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z})/\text{Torsion}) \otimes \mathbb{Q} \cong \tilde{H}_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Q})$$
$$\cong T(\tilde{H}_*(S^1 \wedge SO(3); \mathbb{Q})) = \mathbb{Q}[w_3]$$

This implies that the differentials $\beta^{(r)}$ must satisfy $\beta^{(r)}(w_3) = 0$ for all $r > 0$, because $w_3$ is the generator of the algebra $\tilde{H}_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Q})$. Because the mod-2 Bockstein spectral sequence of this space collapses at the second term, $\beta^{(r)} = 0$ if $r \geq 2$ and the following lemma is true.

**Lemma 3.2.2.** *If $w \in H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2)$ and $\beta(w) = 0$, either $w$ is in the subalgebra $W_0 = \mathbb{Z}_2[w_3]$ or there is $\tilde{w} \in \mathbb{Z}_2\langle w_1, w_2, w_3\rangle$ such that $\beta(\tilde{w}) = w$.*

In particular, since $w_2$ and $w_1$ must disappear and there's nothing in dimension 1, we have $\beta(w_1) = 0$ and therefore $\beta(w_1^2) = 0$. Thus $\beta(w_2) = w_1$.

Looking now at the homology of $G_\lambda$

$$H_*(G_\lambda, \mathbb{Z}_2) \cong \Lambda(y_1, y_2) \otimes \mathbb{Z}_2\langle t, x_1, x_2\rangle / R,$$

with relations $R = \{t^2 = x_1^2 = x_2^2 = 0\}$, we see that the differential $\beta$ of the mod 2-Bockstein spectral sequence of $G_\lambda$ must satisfy

$$\beta(w_1) = \beta(tx_1 + x_1 t) = t\beta(x_1) + \beta(x_1)t = 0, \tag{3.3}$$
$$\beta(w_2) = \beta(tx_2 + x_2 t) = tx_1 + x_1 t = w_1, \tag{3.4}$$
$$\beta(w_3) = 0, \tag{3.5}$$

because $\beta(x_1) = 0$, $\beta(x_2) = x_1$, $\beta(t) = 0$ and $w_3$ corresponds to the generator of the rational homology.

We know that in isomorphism

$$H_*(G_\lambda, \mathbb{Z}_2) \cong H_*(U_0, \mathbb{Z}_2) \otimes H_*(K_0, \mathbb{Z}_2)$$

the elements $w_1, w_2, w_3$ correspond to the elements $v_1, v_2, v_3$ in $H_*(U_0, \mathbb{Z}_2)$. From (3.3), (3.4), (3.5) we can conclude, because $\beta$ is natural, that

$$\begin{aligned}
\beta(v_1) &= \beta(p_{0*}(w_1)) = p_{0*}(\beta(w_1)) = 0, \\
\beta(v_2) &= \beta(p_{0*}(w_2)) = p_{0*}(\beta(w_2)) = p_{0*}(w_1) = v_1, \\
\beta(v_3) &= \beta(p_{0*}(w_3)) = p_{0*}(\beta(w_3)) = 0
\end{aligned}$$

where $\beta$ is the $d^1$-differential of the spectral sequence of $U_0$. Comparing the spectral sequence of $U_0$ with the one of $\Omega\Sigma(S^1 \wedge SO(3))$ we see that lemma 3.2.2 must be true also for every $v \in \mathbb{Z}_2\langle v_1, v_2, v_3\rangle$ where $v_i = p_{0*}(w_i) \in H_*(U_0, \mathbb{Z}_2)$ . On the other hand , we proved in the previous section that an additive basis for $H_*(U_0, \mathbb{Z}_2)$ is given by $v_I t^{\epsilon_t}$ with $\epsilon_t = 0, 1$. Denote a word on $v_i's$ simply by $v$. Then $\beta(vt) = \beta(v)t$, because $\beta(t) = 0$. This implies that the spectral sequence of $U_0$ collapses at the second term and therefore $H_*(U_0, \mathbb{Z}_2)$ has no elements of order greater then 2. This means that $H_*(U_0, \mathbb{Z}_2)$ has only $\mathbb{Z}_2$-torsion. Using isomorphism (2.5) together with the fact that $H_*(K_0, \mathbb{Z}_2)$ has only $\mathbb{Z}_2$-torsion we prove the following theorem.

**Theorem 3.2.3.** *The integer homology of the group $G_\lambda$, $H_*(G_\lambda; \mathbb{Z})$, contains no torsion elements of order greater than 2.*

## 3.3 The Integer Homology of $G_\lambda$

Now we can use the previous result to describe the integer homology. We know there is no $p$-torsion if $p \neq 2$ , so we just have to check which cycles $w$ in $H_*(G_\lambda;\mathbb{Z}_2)$ represent also cycles with $\mathbb{Z}$ coefficients. $\beta$ is the composition

$$H_*(G_\lambda;\mathbb{Z}_2) \xrightarrow{\partial} H_{*-1}(G_\lambda;\mathbb{Z}) \xrightarrow{j} H_{*-1}(G_\lambda;\mathbb{Z}_2)\ ,$$

where $j$ is reduction mod 2 and $\mathrm{im\,j} = \mathrm{H_*(G_\lambda;\mathbb{Z})} \otimes \mathbb{Z}_2$. We have inclusions

$$\mathrm{im}\,\beta \subset \mathrm{im\,j} \subset \ker\beta.$$

On the other hand, since we know that $x_3, y_3$ and $t$ correspond to generators of the rational homology, if $w$ is in the subalgebra $\Lambda(y_3) \otimes \mathbb{Z}_2\langle t, x_3\rangle$ where $y_3 = y_1 y_2$ and $x_3 = x_1 x_2$, then it represents a cycle in integer homology. Otherwise we proved that if $\beta(w) = 0$, then there is $\tilde{w}$ such that $\beta(\tilde{w}) = w$. So we conclude that $\ker\beta \subset \mathrm{im}\,\beta$ in $H_*(G_\lambda;\mathbb{Z}_2) - \Lambda(y_3) \otimes \mathbb{Z}_2\langle t, x_3\rangle$. Therefore we have

**Lemma 3.3.1.** *The image of $\beta$ in $H_n(G_\lambda;\mathbb{Z}_2)$ equals the image under $j$ : $H_n(G_\lambda;\mathbb{Z}) \to H_n(G_\lambda;\mathbb{Z}_2)$ of the torsion subgroup of $H_n(G_\lambda;\mathbb{Z})$ under the obvious inclusion* $\mathrm{Tor}\, H_n(G_\lambda;\mathbb{Z}) \hookrightarrow H_n(G_\lambda;\mathbb{Z}_2)$.

**Remark 3.3.2.** Note that this injective image of $j$ is the full 2-torsion subgroup of $H_n(G_\lambda;\mathbb{Z})$, because $H_n(G_\lambda;\mathbb{Z})$ contains no elements of order greater than 2.

Thus the classes in the $\mathbb{Z}_2$ homology that lift to torsion classes in the integer homology are precisely the ones that are in the image of $\beta$.

## 3.4 An algebraic proof of $\operatorname{im}\beta = \ker\beta$ on $\mathbb{Z}_2\langle w_1, w_2, w_3\rangle - W_0$

We gave a topological proof that $H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z})$ has only $\mathbb{Z}_2$-torsion. Now we will see an algebraic proof of this fact. We know that $H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2) = \mathbb{Z}_2\langle w_1, w_2, w_3\rangle$, and what we want to show is that $\operatorname{im}\beta = \ker\beta$ on $\mathbb{Z}_2\langle w_1, w_2, w_3\rangle - W_0$, where $W_0$ is the subalgebra generated by $w_3$, i.e., $W_0 = \mathbb{Z}_2[w_3]$ and $\beta$ is the Bockstein homomorphism

$$H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2) \longrightarrow H_{*-1}(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2).$$

We have to exclude $W_0$ because $w_3$ corresponds to a generator in the rational homology, so the elements $w_3^i$ never get killed in the Bockstein spectral sequence. From now on we will assume that we are working with $\mathbb{Z}_2$ coefficients. First consider the following simplified problem:

**Problem 3.4.1.** *Suppose we have a free non-commutative algebra generated by $w_1, w_2$ and a differential $d : A \to A$ such that $dw_1 = 0$ and $dw_2 = w_1$. We want to show that $\operatorname{im} d = \ker d$ and describe this ring ( In our case $A = \mathbb{Z}_2\langle w_1, w_2\rangle$ and $d = \beta$ ).*

This is equivalent to solving the problem:

**Problem 3.4.2.** *Let $V$ be a vector space over $\mathbb{Z}_2$ spanned by $w_1, w_2$ and consider $V^{\otimes n}$ ( this space is isomorphic to words on $w_1, w_2$ of length $n$ ). Assume* *there is a differential d acting by the Leibnitz rule. Then we want to:*

1. *Prove that* $\operatorname{im} d = \ker d$.

2. *Calculate graded dimension of* $\ker d$ *( assuming wlog that* $\deg w_1 = 1$ *and* $\deg w_2 = -1$ *).*

The solution is given by considering graded $d$-modules, or vector spaces with action of the differential $d$ and $\mathbb{Z}$-grading. Assume $d = 2$.

**Example 3.4.3.** *Let* $V(a)$ *be a vector space spanned by* $u, v$ *of degree* $a - 1$ *and* $a + 1$ *respectively, such that* $du = v$ *and* $dv = 0$.

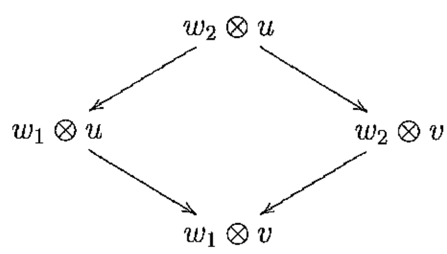
$$u \xrightarrow{\ d\ } v$$

$$\deg a - 1 \qquad \deg a + 1$$

We claim that

**Lemma 3.4.4.** $V(0) \otimes V(a) \cong V(a-1) \oplus V(a+1)$, *where* $V(0) = V = span\{w_1, w_2\}$

*Proof.* This is easy to prove looking at the following diagram




The element in the first row has degree $a - 2$, the elements in the second row have degree $a$ and in the third row degree $a+2$ and $d(w_2 \otimes u) = w_1 \otimes u + w_2 \otimes v$, $d(w_1 \otimes u) = w_1 \otimes v$. $\square$

Moreover we have

**Corollary 3.4.5.** $V^{\otimes n} = \bigoplus_{k=0}^{n-1} \binom{n-1}{k} V(2k-(n-1))$

*Proof.* We give a proof by induction. If $n = 1$ then we obtain

$$V = \bigoplus_{k=0}^{0} \binom{0}{k} V(2k) = V(0).$$

Now assume the result is true for n. Then

$$\begin{aligned}
V^{\otimes(n+1)} &= V(0) \otimes V^{\otimes n} \\
&= V(0) \otimes \left\{ \bigoplus_{k=0}^{n-1} \binom{n-1}{k} V(2k-(n-1)) \right\} \\
&= \bigoplus_{k=0}^{n-1} \binom{n-1}{k} (V(2k-n) \oplus V(2k-(n-2))) \\
&= V(-n) \oplus \left\{ \bigoplus_{k=1}^{n-1} \left( \binom{n-1}{k-1} + \binom{n-1}{k} \right) V(2k-n) \right\} \oplus V(n) \\
&= V(-n) \oplus \left\{ \bigoplus_{k=1}^{n-1} \binom{n}{k} V(2k-n) \right\} \oplus V(n) \\
&= \bigoplus_{k=0}^{n} \binom{n}{k} V(2k-n)
\end{aligned}$$

$\square$

It's clear that for each $V(a)$ we have $\ker d_{|V(a)} = \operatorname{im} d_{|V(a)}$ and $\dim\ker d_{|V(a)} = 1$, so from the previous result we can conclude that $\ker d_{|V^{\otimes n}} = \operatorname{im} d_{|V^{\otimes n}}$ and $\dim\ker d_{|V^{\otimes n}} = \sum_{k=0}^{n-1} \binom{n-1}{k} \dim\ker d_{|V(2k-(n-1))} = \sum_{k=0}^{n-1} \binom{n-1}{k}$. This shows that $\ker\beta = \operatorname{im}\beta$ in $\mathbb{Z}_2\langle w_1, w_2\rangle$ when the differential $\beta$ acts in this algebra satisfying $\beta(w_2) = w_1$ and $\beta(w_1) = 0$. Moreover the dimension of the kernel restricted to the space of words on $w_1$ and $w_2$ of length n is $\sum_{k=0}^{n-1}\binom{n-1}{k}$.

Now we can introduce the generator $w_3$, by considering the space $\hat{V}(0) = V(0) \oplus W_0^1$, where $W_0^i$ is the vector space over $\mathbb{Z}_2$ spanned by $w_3^i$, i.e., $W_0^i = span\{w_3^i\}$. We assume that $dw_3 = 0$ and $\deg w_3 = 0$. Then the space $\hat{V}^{\otimes n} = \hat{V}(0)^{\otimes n}$ is isomorphic to the space of words on $w_1, w_2$ and $w_3$ of length n. Similarly we define $\hat{V}(a) = V(a) \oplus W_0^1$. It is clear from lemma 3.4.4 that

$$\hat{V}(0) \otimes \hat{V}(a) \cong V(a-1) \oplus V(a+1) \oplus V(0) \oplus V(a) \oplus W_0^2,$$

therefore we can prove the following lemma.

**Lemma 3.4.6.** $\hat{V}^{\otimes n} \cong \bigoplus_{k=1}^{n}\binom{n}{k} V^{\otimes k} \oplus W_0^n$

*Proof.* The proof in given by induction on $n$. If $n = 1$ we obtain $\hat{V} = V \oplus W_0^1$. Now assume the result is true for $n$. Then

$$\begin{aligned}
\hat{V}^{\otimes n+1} &\cong \hat{V}^{\otimes n} \otimes \hat{V} \\
&\cong \bigoplus_{k=1}^{n}\binom{n}{k} V^{\otimes k+1} \oplus V \oplus \bigoplus_{k=1}^{n}\binom{n}{k} V^{\otimes k} \oplus W_0^{n+1} \\
&\cong \bigoplus_{k=1}^{n}\binom{n}{k} V^{\otimes k+1} \oplus \bigoplus_{k=0}^{n-1}\binom{n}{k+1} V^{\otimes k+1} \oplus V \oplus W_0^{n+1} \\
&\cong \bigoplus_{k=1}^{n-1}\left\{\binom{n}{k} + \binom{n}{k+1}\right\} V^{\otimes k+1} \oplus V^{\otimes n+1} \oplus (n+1)V \oplus W_0^{n+1} \\
&\cong \bigoplus_{k=1}^{n-1}\binom{n+1}{k+1} V^{\otimes k+1} \oplus V^{\otimes n+1} \oplus (n+1)V \oplus W_0^{n+1}
\end{aligned}$$

$$\cong \bigoplus_{k=1}^{n} \binom{n+1}{k+1} V^{\otimes k+1} \oplus (n+1)V \oplus W_0^{n+1}$$

$$\cong \bigoplus_{k=2}^{n+1} \binom{n+1}{k} V^{\otimes k} \oplus (n+1)V \oplus W_0^{n+1}$$

$$\cong \bigoplus_{k=1}^{n+1} \binom{n+1}{k} V^{\otimes k} \oplus W_0^{n+1}$$

□

We already proved that $\ker d_{|V^{\otimes n}} = \operatorname{im} d_{|V^{\otimes n}}$. Therefore it is clear that the restriction of the differential $d$ to the subspace $W = \bigoplus_{k=1}^{n} \binom{n}{k} V^{\otimes k}$ of $\hat{V}^{\otimes n}$ also satisfies $\ker d_{|W} = \operatorname{im} d_{|W}$. This means that $\ker d_{|\hat{V}^{\otimes n}} = \ker d_{|W} + \ker d_{|W_0^n} = \operatorname{im}_{|W} + \ker d_{|W_0^n}$. This shows that the differential $\beta$ satisfies $\operatorname{im}\beta = \ker\beta$ on $\mathbb{Z}_2\langle w_1, w_2, w_3\rangle - W_0$. Moreover we can compute the dimension of the kernel of $\beta$ restricted to the space of words on $w_1, w_2$ and $w_3$ of length $n$ on $\mathbb{Z}_2\langle w_1, w_2, w_3\rangle - W_0$. From lemma 3.4.6 and corollary 3.4.5 this dimension is given by $\sum_{k=1}^{n} \sum_{l=1}^{k-1} \binom{n}{k}\binom{k-1}{l}$.

# Chapter 4

# Homotopy type of $G_\lambda$

Let $L = \Omega\Sigma(S^1 \wedge SO(3))$ and consider the space $X = L \times S^1 \times SO(3) \times SO(3)$. We will show that $G_\lambda$ is homotopy equivalent to $X$. It is known (see [5] , Cor.3.37) that

**Proposition 4.0.7.** *A map $f : X \to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients iff it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all primes $p$.*

We will define a map $f$ from $X$ to $G_\lambda$ and we will prove that induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients, for all primes $p$. We have an inclusion map

$$i : S^1 \times SO(3) \to G_\lambda$$

given by

$$(x, y) \mapsto i_1(x)i_0(y)i_1(x)^{-1}i_0(y)^{-1} \tag{4.1}$$

where $i_0$ and $i_1$ are the inclusions given in section 2. More precisely, in this formula $i_1$ is the restriction of the inclusion $K_1 \hookrightarrow G_\lambda$ to the $S^1$ factor and

$i_0$ is the restriction of the inclusion $K_0 \hookrightarrow G_\lambda$ to the first $SO(3)$ factor. The restriction to $S^1 \vee SO(3)$ of $i$ is the identity so there is an induced map

$$h : S^1 \wedge SO(3) \to G_\lambda.$$

This map induces the right correspondence between generators in homology

$$h_* : H_*(S^1 \wedge SO(3); \mathbb{Z}_2) \to H_*(G_\lambda; \mathbb{Z}_2),$$

this meaning that the three generators of $H_*(S^1 \wedge SO(3); \mathbb{Z}_2)$ are mapped to $w_1, w_2, w_3 \in H_*(G_\lambda; \mathbb{Z}_2)$, because as we saw before these generators in $H_*(G_\lambda; \mathbb{Z}_2)$ are obtained as commutators as in (4.1). Moreover there is a unique map $\tilde{h}$ that extends $h$ to $\Omega\Sigma(S^1 \wedge SO(3))$ as in section 3.1. Therefore the map

$$\tilde{h}_* : H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2) \to H_*(G_\lambda; \mathbb{Z}_2) \tag{4.2}$$

takes the generators of $H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2)$ to the elements $w_1, w_2, w_3$ in $H_*(G_\lambda; \mathbb{Z}_2)$. Now consider the map $f : L \times S^1 \times SO(3) \times SO(3) \to G_\lambda$ given by

$$(w, t, y, x) \mapsto \tilde{h}(w) i_1(t, y) i_0(x),$$

where $w \in L = \Omega\Sigma(S^1 \wedge SO(3))$.

**Lemma 4.0.8.** *The map $f$ defined above induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all primes $p$.*

*Proof.* We see that $f$ restricted to $S^1 \times SO(3)$ or the second $SO(3)$ factor is just the inclusion in $G_\lambda$. Moreover, using the Künneth formula for homology with coefficients in a field F, we obtain that

$$H_n(X;F) \cong \bigoplus_{p+q+l=n} H_p(L;F) \otimes H_q(S^1 \times SO(3);F) \otimes H_l(SO(3);F). \quad (4.3)$$

Let $F$ be $\mathbb{Q}$ or $\mathbb{Z}_p$ with $p \neq 2$. Note that in this case $H_*(SO(3);F)$ has only a generator in dimension 3. Therefore an additive basis for $H_*(G_\lambda;F)$ is given by

$$w_3^k t^{\epsilon_t} x_3^{\epsilon} y_3^{\eta}$$

where $\epsilon_t, \epsilon, \eta = 0$ or 1. Thus comparing equation (4.3) and an additive basis for $H_*(G_\lambda;F)$ we see that the homology groups of $X$ and $G_\lambda$ are the same. We just need to show that $f$ induces those isomorphisms. The elements $t$, $x_3$ and $y_3$ are the images of the generators of $H_*(S^1 \times SO(3);F)$ and $H_*(SO(3);F)$ under the injective maps

$$i_{1*} : H_*(S^1 \times SO(3);F) \to H_*(G_\lambda;F)$$

and

$$i_{0*} : H_*(SO(3);F) \to H_*(G_\lambda;F)$$

induced by inclusions $i_0$ and $i_1$. We also know that the restriction of $f$ to $L$ is given by the map $h_*$ and we know that $\tilde{h}_*$ maps the 4-dimensional generator of

$$H_*(\Omega\Sigma(S^1 \wedge SO(3));F) \cong T(\tilde{H}_*(S^1 \wedge SO(3);F)) \cong F[w_3]$$

to the element in $H_4(G_\lambda;F)$ obtained as the Samelson product of $t$ and $x_3$. This proves that $f$ induces an isomorphism in homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all primes $p$ with $p \neq 2$.

If $F = \mathbb{Z}_2$ then an additive basis for $H_*(G_\lambda; \mathbb{Z}_2)$ is given by (2.7). Thus from equation (4.3) we can conclude that the homology groups $H_*(G_\lambda; \mathbb{Z}_2)$ and $H_*(X; \mathbb{Z}_2)$ are isomorphic. The elements $t, y_1, y_2$ are the images of the generators of $H_*(S^1 \times SO(3); \mathbb{Z}_2)$ and $x_1, x_2$ are the images of the generators of $H_*(SO(3); \mathbb{Z}_2)$. In this case the map $\tilde{h}_*$ mentioned in (4.2) takes the generators of $H_*(\Omega\Sigma(S^1 \wedge SO(3)); \mathbb{Z}_2)$ to the elements $w_1, w_2, w_3$ which are the three generators of the free noncommutative subalgebra of $H_*(G_\lambda; \mathbb{Z}_2)$ . Therefore, again we have an isomorphism in homology. □

Now we have the conditions of 4.0.7 satisfied, thus $f$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients. Finally, we know that $G_\lambda$ is an H-space and $X$ is also an H-space, because is the product of H-spaces. Therefore for both spaces $\pi_1$ acts trivially on all $\pi_n$'s (see [14] , pp 119). Now we can apply the following corollary of Whitehead's theorem ( [5] prop 4.48):

**Corollary 4.0.9.** *If $X$ and $Y$ are CW-complexes that are abelian ($\pi_1$ acts trivially on all $\pi_n$'s) then a map $f : X \to Y$ that induces isomorphisms in homology is a homotopy equivalence.*

Therefore we proved the following theorem:

**Theorem 4.0.10.** *If $0 < \lambda \leq 1$, $G_\lambda$ is homotopy equivalent to the product $\Omega\Sigma(S^1 \wedge SO(3)) \times S^1 \times SO(3) \times SO(3)$.*

**Remark 4.0.11.** Although the spaces $G_\lambda$ and $\Omega\Sigma(S^1 \wedge SO(3)) \times S^1 \times SO(3) \times SO(3)$ are homotopy equivalent the above homotopy equivalence is not an $H$-map, because it does not preserve the product structure. This can be seen by comparing the Pontryagin products on integral homology. It would be a interesting question to find an easy understandable $H$-space with a Pontryagin ring isomorphic to the one of $G_\lambda$.

**Remark 4.0.12 (The Künneth formula with integer coefficients).**

*Looking at the Künneth formula with integer coefficients we can check in low dimensions that the homology groups of this spaces are in fact the same.*

$$\begin{aligned} H_n(X;\mathbb{Z}) \cong & \bigoplus_{p+q+l=n} H_p(S^1;\mathbb{Z}) \otimes H_q(SO(3)\times SO(3);\mathbb{Z}) \otimes H_l(L;\mathbb{Z}) \oplus \\ & \bigoplus_{p+q=n-1} \mathrm{Tor}\,(H_p(S^1 \times SO(3) \times SO(3);\mathbb{Z}); H_q(L;\mathbb{Z})) \end{aligned}$$

*First we need to know what is the integer homology of $SO(3) \times SO(3)$. Again using the Künneth formula*

$$\begin{aligned} H_n(SO(3)\times SO(3);\mathbb{Z}) \cong & \bigoplus_{p+q=n} H_p(SO(3);\mathbb{Z}) \otimes H_q(SO(3);\mathbb{Z}) \oplus \\ & \bigoplus_{p+q=n-1} \mathrm{Tor}\,(H_p(SO(3);\mathbb{Z}); H_q(SO(3);\mathbb{Z})) \end{aligned}$$

*we get*

| $n$ | $H_n(SO(3) \times SO(3); \mathbb{Z})$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1 | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ |
| 4 | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| 5 | $0$ |
| 6 | $\mathbb{Z}$ |

*The generator of the* $\mathbb{Z}_2$ *factor in the homology of degree 3 is* $x_1y_2 + x_2y_1$, *where* $x_1, x_2, y_1, y_2$ *are the generators of the* $\mathbb{Z}_2$*-homology of the two* $SO(3)$ *components. Although* $x_2, y_2$ *represent cycles only with* $\mathbb{Z}_2$ *coefficients, because* $\partial x_2 = 2x_1$ *and* $\partial y_2 = 2y_1$, *and they do not represent integer cycles, we get*

$$\partial(x_1y_2 + x_2y_1) = -2x_1y_1 + 2x_1y_1 = 0,$$

*since* $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$*,where* $a, b$ *are homology classes. Looking at the homology of* $X$ *we shall see that it is the existence of torsion elements of the same order in the homology of* $S^1 \times SO(3) \times SO(3)$ *and* $L$ *which prevents the homology of* $S^1 \times SO(3) \times SO(3) \times L$ *from being just the product of the homology of* $S^1 \times SO(3) \times SO(3)$ *and* $L$. *Therefore using the Künneth formula* *again we see that the generators of the homology groups $H_n(X;\mathbb{Z})$ are*

| | |
|---|---|
| 1 | $t, x_1, y_1$ |
| 2 | $tx_1, ty_1, w_1, x_1y_1$ |
| 3 | $tw_1, x_1w_1, y_1w_1, x, y, tx_1y_1, x_1y_2 + x_2y_1$ |
| 4 | $tx_1w_1, ty_1w_1, tx, ty, t(x_1y_2 + x_2y_1), x_1y_1w_1,$ |
| | $x_1y, xy_1, w_1^2, w_3, x_1w_2 + x_2w_1, y_1w_2 + y_2w_1$ |

*where the last two generators are coming from the torsion term, because if $n = 4$ then*

$$\bigoplus_{p+q=3} \mathrm{Tor}\,(H_p(S^1 \times SO(3) \times SO(3);\mathbb{Z}); H_q(L;\mathbb{Z}))$$

*contributes with two $\mathbb{Z}_2$ factors since $H_1(S^1 \times SO(3) \times SO(3);\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ and $H_2(L;\mathbb{Z}) = \mathbb{Z}_2$. We can check that in fact $x_1w_2 + x_2w_1, y_1w_2 + y_2w_1$ represent integer cycles, because, for example, $\partial(x_1w_2 + x_2w_1) = -x_1\partial w_2 + 2x_1w_1 = -2x_1w_1 + 2x_1w_1 = 0$. This agrees with the fact, proved in section 3.3, that in $H_*(G_\lambda;\mathbb{Z}_2)$ the generators that lift to torsion classes in $H_*(G_\lambda;\mathbb{Z})$ are the ones in the image of $\beta$ and we have $\beta(x_2w_2) = x_1w_2 + x_2w_1$.*

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