Dynamics of Cubic Siegel Polynomials

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Abstract of Dissertation

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In this dissertation we study the family of cubic polynomials in the complex plane which have a Siegel disk of a fixed rotation number of Brjuno type. One of the main results of this work is the theorem that when the rotation number is of bounded type, the boundary of the Siegel disk of a cubic polynomial is a quasicircle of Hausdorff dimension greater than 1 and contains one or both critical points. This generalizes an earlier result of Douady, Ghys, Herman and Shishikura for quadratic polynomials. In the last part of this work, we sketch how to generalize these results to Siegel polynomials of higher degree.

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PREFACE

This dissertation is based on part of my research in Holomorphic Dynamics during 1995-1999 at Stony Brook. Some version of this work will appear in *Communications in Mathematical Physics* [**Z3**]. My other related works on rational maps with Siegel disks can be found in [**Z2**], the joint work [**YZ**], and the work in progress [**Z4**].

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Back in Iran, there are two mathematicians to whom I owe a great deal. G. Khosrovshahi who restlessly encouraged me to choose Mathematics when I was about to pursue a career in Electrical Engineering and keep Mathematics only as a hobby. Also S. Shahshahani who taught my first lessons in Dynamical Systems and introduced me to some world-class research in this area. To me, both have been a source of moral support throughout these years.

And a word of appreciation to my great parents, brother and sister to whom I owe everything I have accomplished in my life. I am deeply grateful to my parents for helping me turn my many dreams into reality since I was a little child, and for their devotion and sacrifice especially during the difficult years of war. Last but not least, I would like to thank my future wife Zohreh for her love, support and understanding. She has been there for me through thick and thin.

-S. Z.

To my parents

1. Introduction

Let f be a polynomial of degree $d \geq 2$ in the complex plane and consider the following statements:

• $(\mathbf{A}_d)$ "If f has a fixed Siegel disk Δ of bounded type rotation number, then $\partial\Delta$ is a quasicircle passing through some critical point of f."

• (B_d) "If f has a fixed Siegel disk Δ such that $\partial\Delta$ is a quasicircle passing through some critical point of f, then the rotation number of Δ is bounded type."

Statement $(\mathbf{A}_2)$ is a theorem of Douady, Ghys, Herman and Shishikura, $(\mathbf{B}_d)$ is open, even for $d = 2$, and one of the main corollaries of this work is $(\mathbf{A}_3)$:

Theorem. *Let P be a cubic polynomial which has a fixed Siegel disk Δ of rotation number θ. Let θ be of bounded type. Then the boundary of Δ is a quasicircle which contains one or both critical points of P.*

In fact, we study the one-dimensional slice in the cubic parameter space which consists of all cubics with a fixed Siegel disk of a given rotation number. Many of the results apply to general rotation numbers of Brjuno type. In the last part of this work we sketch how to prove $(\mathbf{A}_d)$ for $d \geq 4$.

Siegel disks provide examples of quasiperiodic dynamics. Let p be an *irrationally indifferent* fixed point of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. This means that $f(p) = p$ and the *multiplier* $f'(p)$ is of the form $e^{2\pi i\theta}$, where the *rotation number* $0 < \theta < 1$ is irrational. When f is linearizable near p, the largest domain Δ on which the linearization is possible is simply-connected and is called the *Siegel disk* of f centered at p. Every punctured Siegel disk $\Delta \smallsetminus \{p\}$ is foliated by dynamically-defined real-analytic invariant curves. However, as we get close to $\partial\Delta$, these invariant curves may become more wiggly, and in the limit we lose control of their distortion. So, *a priori*, we do not even know if $\partial\Delta$ is a Jordan curve. The topology and geometry of the boundary of Siegel disks is a current field of research in Holomorphic Dynamics.

It was conjectured by Douady and Sullivan in the early 80's that the boundary of every Siegel disk of a rational map has to be a Jordan curve (see [**D1**]). To this date, this has remained an open problem, even for polynomials, even when the degree is 2. Even worse, there are very few explicit examples of polynomials for which we can effectively verify this conjecture. For instance, it is easy to see that local-connectivity of the Julia set implies the boundary of a Siegel disk to be a Jordan curve, but except for one case in the quadratic family [**Pe**], we do not know how to check local-connectivity of the Julia set of a rational map which has a Siegel disk (and even in that single case, the boundary being a Jordan curve is proved as a first step in the proof of local-connectivity). On the other hand, there are examples of non locally-connected quadratic Julia sets whose Siegel disks are bounded by quasicircles [**H3**] or indifferent linearizable germs with non locally-connected "hedgehogs" whose Siegel disks are bounded by smooth or even quasianalytic Jordan curves [**Pr2**]. It is known that in any counterexample to the Douady-Sullivan conjecture, the boundary of the Siegel disk must either be very complicated (an indecomposable continuum) or very simple (a circle with infinitely many topologist's sine curves planted on it) [**R**].

Let $[a_1, a_2, \ldots, a_n, \ldots]$ be the continued fraction expansion of the rotation number θ and $p_n/q_n = [a_1, a_2, \ldots, a_n]$ be its n-th rational approximation, where every a_n is a positive integer. According to the theorem of Brjuno-Yoccoz [**Y**], every holomorphic germ with an indifferent fixed point of multiplier $e^{2\pi i\theta}$ is linearizable if and only if θ satisfies the condition

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty. \tag{1.1}$$

Such θ is called of *Brjuno type*. It is not hard to show that this set has full measure in the unit interval. The set of irrational numbers of Brjuno type contains two important arithmetic subsets: (1) numbers of *Diophantine type*, the set of all $0 < \theta < 1$ for which there exist positive constants C and ν such that $|\theta - p/q| > C/q^{\nu}$ for every rational number $0 < p/q < 1$; and (2) numbers of *bounded type*, the set of all $0 < \theta < 1$ for which $\sup_n a_n < +\infty$.

Another issue is the existence of critical points on the boundary of Siegel disks. This problem was first studied by Ghys under the assumption that the boundary is a Jordan curve and the rotation number is Diophantine [**G**]. Later Herman improved the result by showing that when the rotation number is Diophantine and the action on the boundary is injective, there must be a critical point on the boundary [**H1**]. A very short proof of this theorem is now possible with knowledge of "Siegel compacts" as recently introduced by Perez-Marco [**Pr1**] (see [**Z2**] for such a proof). In the case of quadratic polynomials, no critical point on the boundary of the Siegel disk automatically implies that the map acts injectively on this boundary. Hence one concludes that for θ of Diophantine type, the critical point of $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ is on the boundary of the Siegel disk centered at 0. Later Herman gave the first example of a θ of Brjuno type for which the boundary of the Siegel disk for Q_θ is disjoint from the entire orbit of the critical point [**H3**].

The most significant example in which one can explicitly show that the boundary of a Siegel disk is a Jordan curve containing a critical point is the quadratic map $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$, with θ of bounded type. The idea, originally due to Ghys but utilized by Douady, Herman and Shishikura, is to consider the degree 3 Blaschke product

$$f_\theta(z) = e^{2\pi i t(\theta)} z^2 \left(\frac{z-3}{1-3z}\right)$$

which has a double critical point at 1 and $0 < t(\theta) < 1$ is chosen such that the rotation number of the restriction of f_θ to the unit circle is θ. Using a theorem of Świątek and Herman on quasisymmetric linearization of critical circle maps ([**Sw**], [**H2**]), one can redefine f_θ on the unit disk to make it quasiconformally conjugate to the rigid rotation by angle θ. After modifying the conformal structure of the sphere on the unit disk and all its preimages, one applies the Measurable Riemann Mapping Theorem of Morrey-Ahlfors-Bers to prove that the resulting map is quasiconformally conjugate to a quadratic polynomial Q. But the image of the unit disk has to be a Siegel disk of rotation number θ for Q, and there is only one such quadratic up to an affine conjugacy, so Q must be conjugate to Q_θ, which proves ($\mathbf{A}_2$). The Julia set of Q_θ for the golden mean $\theta = (\sqrt{5} - 1)/2$ and its self-similar properties was studied empirically by physicists in the early 80's (see [**MN**], [**W**]). For general θ of bounded type, it has been a subject of recent rigorous studies by mathematicians (see for example [**Pe**], [**GJ**], [**Mc3**], [**YZ**]). In a very recent work in progress [**Z4**], using a non-quasiconformal surgery on f_θ, we find explicit arithmetical conditions on unbounded type rotation numbers θ which guarantee the Siegel disk of Q_θ is a Jordan curve passing through the critical point.

In any attempt to generalize ($\mathbf{A}_2$) to higher degrees, one must address several problems. In fact, the main difficulty is not the surgery which can be performed in all degrees in a similar way, provided that one has the appropriate Blaschke products in hand. Instead, we have to face a different set of questions such as parametrization of the candidate Blaschke products by their critical points, combinatorics of various "drops" of their Julia sets, continuity of the surgery, and surjectivity of this operation. None of these issues arises in degree 2, where the corresponding parameter spaces are single points.

In this work we address these questions in detail for cubic polynomials and later we sketch how to extend them to higher degrees. We introduce the parameter space $\mathcal{P}_3^{cm}(\theta)$ of critically marked cubic polynomials with a Siegel disk of a given rotation number θ of Brjuno type, which is canonically isomorphic to the punctured plane. The connectedness locus $\mathcal{M}_3(\theta) \subset \mathcal{P}_3^{cm}(\theta)$ (the analogue of the Mandelbrot set for the quadratic family) is the set of all cubics with both critical orbits bounded (see Fig. 1). In the interior of $\mathcal{M}_3(\theta)$, every cubic is either *hyperbolic-like*, for which the free critical point approaches an attracting cycle, or *capture*, for which the free critical point eventually maps into the Siegel disk, or of neither type, in which case it is called *queer* (there may be no queer components. In any case, no example is known). The presence of hyperbolic-like cubics in $\mathcal{P}_3^{cm}(\theta)$ implies the existence of copies of the Mandelbrot set all over $\mathcal{M}_3(\theta)$, while captures appear as components in $\mathcal{M}_3(\theta)$ which look like Siegel disks in the dynamical plane. The most significant property of queer cubics is that their Julia sets support invariant line fields and in particular have positive Lebesgue measure (Theorem 3.4).

Motivated by the Douady-Ghys-Herman-Shishikura approach, we introduce an auxiliary family of degree 5 critically marked Blaschke products which serve as models for cubics in $\mathcal{P}_3^{cm}(\theta)$ in the same way f_θ does for the quadratic Q_θ. We show that these Blaschke products can be parametrized by their critical points (Theorem 7.1) and we use this parametrization to define the parameter space $\mathcal{B}_5^{cm}(\theta)$ which is also homeomorphic to the punctured plane. A connectedness locus $\mathcal{C}_5(\theta) \subset \mathcal{B}_5^{cm}(\theta)$ can be defined similarly. When θ is of bounded type, one can perform a quasiconformal surgery on Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ in order to obtain critically marked cubics in $\mathcal{P}_3^{cm}(\theta)$. The result of this surgery does not depend on various choices we make along the way (Proposition 9.2), hence it gives rise to a well-defined *surgery map* $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$. Continuity of $\mathcal{S}$ is far from being straightforward and depends on the fact that the parameter spaces have one complex dimension (Theorem 11.1). In fact, in higher degrees, this continuity step is the only part in which our techniques for cubics polynomials fail to apply.

Various evidence suggest that the connectedness loci $\mathcal{C}_5(\theta)$ and $\mathcal{M}_3(\theta)$ are in fact homeomorphic. One can go even farther as to speculate that $\mathcal{S}$ is a homeomorphism. Although we provide some evidence to support this, we only need to show that $\mathcal{S}$ is surjective (Theorem 13.6) in order to get the desired results in the dynamical plane of cubics. Surjectivity follows from an injectivity result (Theorem 13.3) which in particular shows that $\mathcal{S}$ induces a homeomorphism between the complementary components of $\mathcal{C}_5(\theta)$ and $\mathcal{M}_3(\theta)$. The proof of the injectivity result relies on various tools developed along the way, especially a renormalization scheme to "extract" Q_θ from some cubics in $\mathcal{P}_3^{cm}(\theta)$ and f_θ from some Blaschke products in $\mathcal{B}_5^{cm}(\theta)$. Surjectivity of $\mathcal{S}$ proves $(\mathbf{A}_3)$. As another consequence, we obtain the following

Theorem. *For θ of bounded type, the boundary of the Siegel disk of $P \in \mathcal{P}_3^{cm}(\theta)$ is a continuous function of P in the Hausdorff topology.*

It is interesting to contrast this result with the fact that the Julia set of P undergoes drastic implosions near the boundary of $\mathcal{M}_3(\theta)$, especially near the set of cubics with both critical points on the boundary of their Siegel disk. We study this locus and describe its topology:

Theorem. *For θ of bounded type, the set Γ of all cubics in $\mathcal{P}_3^{cm}(\theta)$ with both critical points on the boundary of their Siegel disk is a Jordan curve.*

Fig. 18 shows the Jordan curve Γ. We give a topological characterization of this set as the common boundary of the two complementary components of $\mathcal{M}_3(\theta)$ (Theorem 14.4).

Finally, we sketch how the results of this work can be generalized to arbitrary degrees, suggesting a near-to-finish program to prove $(\mathbf{A}_d)$ for any $d \geq 4$. In this case, we can define the parameter space $\mathcal{P}_d^{cm}(\theta)$ consisting of degree d critically marked Siegel polynomials with rotation number θ, and a similar space $\mathcal{B}_{2d-1}^{cm}(\theta)$ of degree $2d-1$ critically marked Blaschke products. We show that both spaces are naturally isomorphic to the product of $d-2$ copies of the punctured plane. A similar surgery map $\mathcal{S} : \mathcal{B}_{2d-1}^{cm}(\theta) \to \mathcal{P}_d^{cm}(\theta)$ can be defined when θ is bounded type. Assuming the continuity of this map, and by generalizing the arguments for the cubic case, one can prove that this map is surjective, which implies $(\mathbf{A}_d)$. But at the moment, continuity of this map for $d \geq 4$ seems to be open.

2. A Cubic Parameter Space

We begin by considering the space of all cubic polynomials which have a fixed Siegel disk of multiplier $\lambda = e^{2\pi i\theta}$ centered at the origin. Here $0 < \theta < 1$ is a given irrational number of Brjuno type satisfying the condition (1.1). By the theorem of Brjuno-Yoccoz [**Y**], every holomorphic germ $z \mapsto e^{2\pi i\theta}z + O(z^2)$ with θ of Brjuno type is holomorphically linearizable near 0. In particular, every cubic polynomial of the form

$$z \mapsto \lambda z + a_2 z^2 + a_3 z^3, \tag{2.1}$$

with $(a_2, a_3) \in \mathbb{C} \times \mathbb{C}^*$ has a Siegel disk centered at the origin. We are not directly interested in the rather big space of all such cubics. Instead, we would like to consider the space of affine conjugacy classes of these cubics together with a marking of their critical points. A few words on the notion of "marking" is in order; however, we will hardly refer to the following formal definition in the rest of this work.

Roughly speaking, a *marking* of the critical points of a cubic P of the form (2.1) is a choice of labeling these critical points. It can be thought of as a surjective function $\mathbf{m}$ from the set $\{1, 2\}$ to the set of critical points of P. Two such *critically marked* cubics $(P, \mathbf{m})$ and $(Q, \mathbf{m}')$ are affinely conjugate if there is a dilation $\varphi : z \mapsto \alpha z$ such that $\varphi \circ P = Q \circ \varphi$ and $\mathbf{m}' = \varphi \circ \mathbf{m}$. In other words, an affine conjugacy must also respect the markings. We denote the space of affine conjugacy classes of such critically marked cubics by $\mathcal{P}_3^{cm}(\theta)$.

One way to parametrize $\mathcal{P}_3^{cm}(\theta)$ is as follows: In each conjugacy class we choose the unique critically marked cubic $(P, \mathbf{m})$ whose second critical point $\mathbf{m}(2)$ is located at $z = 1$ in the complex plane. The first critical point $\mathbf{m}(1)$ will then be located at some point $c \neq 0$. It is easy to see that such a cubic has the form

$$P_c : z \mapsto \lambda z \left(1 - \frac{1}{2}(1 + \frac{1}{c})z + \frac{1}{3c}z^2\right). \tag{2.2}$$

Note that using this normal form, every cubic comes automatically with a marking of its critical points. Thus (2.2) provides us with an identification $\mathcal{P}_3^{cm}(\theta) \simeq \mathbb{C}^*$. Under this identification, the natural $\mathbb{Z}_2$-action on $\mathcal{P}_3^{cm}(\theta)$ (swapping the markings of the critical points) corresponds to the involution $c \mapsto 1/c$. By an abuse of notation, we often identify the cubic $P_c \in \mathcal{P}_3^{cm}(\theta)$ with the parameter $c \in \mathbb{C}^*$.

The parameter space $\mathcal{P}_3^{cm}(\theta)$ has two very special points: P_1 which corresponds to the conjugacy class of cubics of the form (2.1) with one critical point, and P_{-1} which corresponds to the conjugacy class of those cubics whose critical points are centered. The pair $\{P_1, P_{-1}\}$ coincides with the set of fixed points of the natural $\mathbb{Z}_2$-action on $\mathcal{P}_3^{cm}(\theta)$.

To understand the implication of marking the critical points, let us also consider the space $\mathcal{P}_3(\theta)$ of affine conjugacy classes of cubics of the form (2.1), this time with no particular marking. Every cubic in (2.1) can be conjugated to a monic cubic of the form

$$z \mapsto \lambda z + az^2 + z^3,$$

where $a \in \mathbb{C}$, and this polynomial is uniquely determined by $\pm a$. In other words, the space $\mathcal{P}_3(\theta)$ is parametrized by the invariant $\zeta = a^2 \in \mathbb{C}$, hence it can be identified with the complex plane. Consider the map which sends every critically marked cubic in $\mathcal{P}_3^{cm}(\theta)$ to its unique monic representative in $\mathcal{P}_3(\theta)$. This amounts to "forgetting" the markings of the critical points. It is easy to check that in the coordinates we have chosen, this map $\mathcal{P}_3^{cm}(\theta) \to \mathcal{P}_3(\theta)$ is given by

$$\zeta = \frac{3}{4}\lambda c \left(1 + \frac{1}{c}\right)^2.$$

It follows that $\mathcal{P}_3^{cm}(\theta)$ is a double cover of $\mathcal{P}_3(\theta)$, branched over the points $c = \pm 1$. Note that by the above formula $\zeta(c) = \zeta(1/c)$, as expected.

Notation and Terminology. Throughout this work, the Siegel disk of the cubic P_c centered at the origin is denoted by Δ_c. When we do not want to emphasize the dependence on c, we denote the Siegel disk of a cubic P by Δ_P. By the *grand orbit* $GO(\Delta_P)$ we mean the set of all points in the plane which eventually map to the Siegel disk under the iteration of P. In other words,

$$GO(\Delta_P) = \bigcup_{k \geq 0} P^{-k}(\Delta_P).$$

Remark. From classical Fatou-Julia theory, we know that every point on the boundary of the Siegel disk Δ_c must be in the closure of the orbit of either c or 1 ([**M1**], Corollary 11.4). According to Herman [**H1**], $P_c|_{\partial \Delta_c}$ has a dense orbit. It follows that the orbit of either c or 1 must accumulate on the entire boundary of Δ_c.

The "size" of a Siegel disk can be measured by the following invariant:

Definition (Conformal Capacity). Consider the Siegel disk Δ_c for $c \in \mathbb{C}^*$ and the unique linearizing map $h_c : \mathbb{D}(0, r_c) \xrightarrow{\simeq} \Delta_c$, with $h_c(0) = 0$ and $h_c'(0) = 1$. The radius $r_c > 0$ of the domain of h_c is called the *conformal capacity* of Δ_c and is denoted by $\kappa(\Delta_c)$.

Alternatively, $\kappa(\Delta_c)$ can be described as the derivative $\varphi_c'(0)$ of the unique linearizing map $\varphi_c : \mathbb{D} \xrightarrow{\simeq} \Delta_c$ normalized by $\varphi_c(0) = 0$ and $\varphi_c'(0) > 0$. Naturally, one is interested in the behavior of the function $c \mapsto \kappa(\Delta_c)$. This function is upper semicontinuous [**Y**], so *a priori* it can jump to a lower value, meaning that the Siegel disk Δ_c can shrink by a very small perturbation of the cubic

P_c. Later we will see that for θ of bounded type, the closed Siegel disk $\overline{\Delta}_c$ is a quasidisk which moves continuously in the Hausdorff topology on compact subsets of the plane (see Theorem 13.9). Therefore, in that case $\kappa(\Delta_c)$ is actually continuous as a function of c. On the other hand, for arbitrary θ of Brjuno type, I do not know if $c \mapsto \kappa(\Delta_c)$ is continuous. However, we have the following general theorem of Yoccoz [**Y**]:

Theorem 2.1. *Let $0 < \theta < 1$ be an irrational number of Brjuno type, and set $W(\theta) = \sum_{n=1}^{\infty} (\log q_{n+1})/q_n < \infty$. Let $S(\theta)$ be the space of all univalent functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = e^{2\pi i \theta}$, with the maximal Siegel disk $\Delta_f \subset \mathbb{D}$. Finally, define $\kappa(\theta) = \inf_{f \in S(\theta)} \kappa(\Delta_f)$. Then, there is a universal constant $C > 0$ such that $|\log(\kappa(\theta)) + W(\theta)| < C$.*

We obtain the following statement which will be used in Theorem 5.3.

Corollary 2.2. *In the family $\{P_c\}$ of cubic polynomials in (2.2), the conformal capacity function $c \mapsto \kappa(\Delta_c)$ is locally bounded away from 0.*

Definition. We define the *cubic connectedness locus* $\mathcal{M}_3(\theta)$ as the set of all critically marked cubics $P \in \mathcal{P}_3^{cm}(\theta)$ whose Julia sets $J(P)$ are connected. It follows from classical Fatou-Julia theory that $P \in \mathcal{M}_3(\theta)$ if and only if both critical points of P have bounded orbits ([M1], Theorem 17.3):

$$\begin{aligned} \mathcal{M}_3(\theta) &= \{c \in \mathbb{C}^* : \text{The Julia set } J(P_c) \text{ is connected}\} \\ &= \{c \in \mathbb{C}^* : \text{Both sequences } \{P_c^{\circ k}(c)\} \text{ and } \{P_c^{\circ k}(1)\} \text{ are bounded}\}. \end{aligned}$$

Since P_c and $P_{1/c}$ are affinely conjugate as maps when we neglect markings of their critical points, $\mathcal{M}_3(\theta)$ as a subset of the c-plane is invariant under the mapping $c \mapsto 1/c$. Fig. 1 shows the connectedness locus $\mathcal{M}_3(\theta)$ for the golden mean $\theta = (\sqrt{5} - 1)/2 = 0.61803399...$ and Fig. 2 shows the details of the same set near the unit circle.

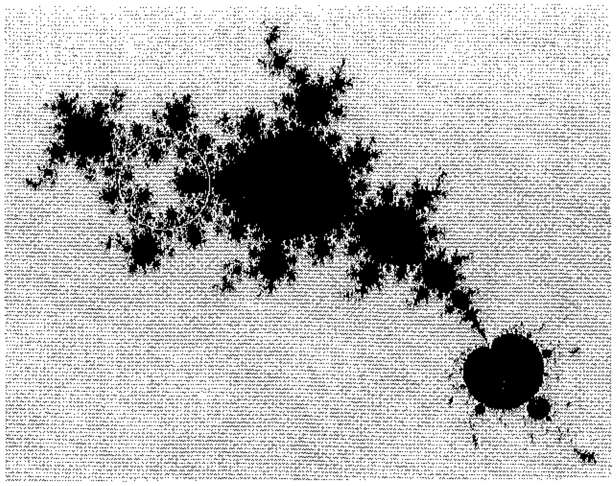

FIGURE 1. *The connectedness locus $\mathcal{M}_3(\theta)$*

Proposition 2.3.

(a) *$\mathcal{M}_3(\theta)$ is compact and contained in the open annulus $\mathbb{A}(\frac{1}{30}, 30)$.*

(b) *The complement $\mathbb{C}^* \smallsetminus \mathcal{M}_3(\theta)$ has exactly two connected components Ω_{ext} and Ω_{int} which are mapped to one another by $c \mapsto 1/c$. Moreover,*

$$\Omega_{ext} = \{c \in \mathbb{C}^* : P_c^{\circ k}(c) \to \infty \text{ as } k \to \infty\},$$
$$\Omega_{int} = \{c \in \mathbb{C}^* : P_c^{\circ k}(1) \to \infty \text{ as } k \to \infty\}.$$

Later we will prove that Ω_{ext} (hence Ω_{int}) is homeomorphic to a punctured disk. This will show that $\mathcal{M}_3(\theta)$ is a connected set (Theorem 6.1).

Proof. (a) $\mathcal{M}_3(\theta)$ is clearly closed. Let

$$m_c = (4.38)\max\{|c|, 1\}. \tag{2.3}$$

If $|z| \geq m_c$, then

$$\begin{aligned} |P_c(z)| &\geq (\frac{1}{|c|}(\frac{1}{3}|z| - \frac{1}{4.38}|z|)|z| - 1)|z| \\ &\geq (0.46|z| - 1)|z| \\ &\geq 1.0148\ |z|, \end{aligned}$$

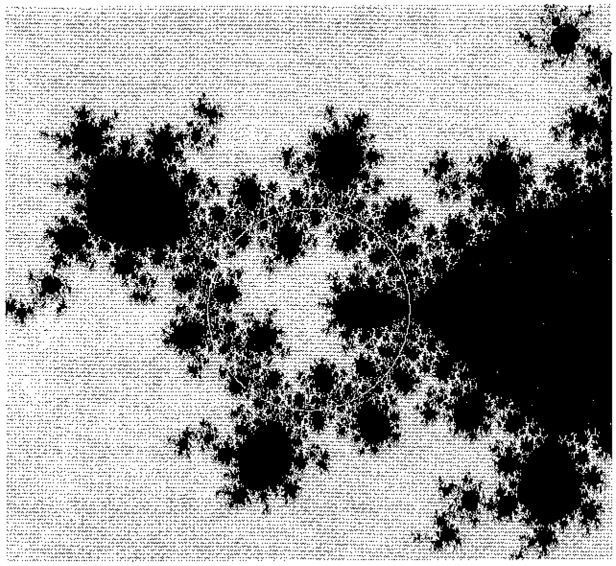

FIGURE 2. *Details of $\mathcal{M}_3(\theta)$ near the unit circle*

from which it follows that

$$K(P_c) \subset \mathbb{D}(0, m_c), \tag{2.4}$$

where $K(P_c)$ is the filled Julia set of P_c. Now if $|c| \geq 30$, then

$$|P_c(c)| = |\frac{1}{6}c - \frac{1}{2}||c| \geq (4.5)|c| > m_c,$$

which implies $P_c^{\circ k}(c) \to \infty$ as $k \to \infty$. Therefore $\mathcal{M}_3(\theta) \subset \mathbb{D}(0, 30)$, hence by symmetry $\mathcal{M}_3(\theta) \subset \mathbb{A}(\frac{1}{30}, 30)$.

(b) Let Ω_{ext} be the unbounded connected component of $\mathbb{C}^* \smallsetminus \mathcal{M}_3(\theta)$. Since $\mathcal{M}_3(\theta)$ is invariant under $c \mapsto 1/c$, there exists a corresponding component Ω_{int} of the complement of $\mathcal{M}_3(\theta)$ containing a punctured neighborhood of the origin. By the proof of (a), we have $\Omega_{ext} \subset \{c \in \mathbb{C}^* : P_c^{\circ k}(c) \to \infty \text{ as } k \to \infty\}$ and similarly $\Omega_{int} \subset \{c \in \mathbb{C}^* : P_c^{\circ k}(1) \to \infty \text{ as } k \to \infty\}$.

Suppose that there exists a bounded connected component U of $\mathbb{C}^* \setminus \mathcal{M}_3(\theta)$ which is not Ω_{int}. Then

$$0 < \sup_{c \in \partial U} |c| = R < +\infty.$$

If $c \in \partial U$, it follows from (2.4) that for each $k \geq 0$, $|P_c^{\circ k}(c)|$ and $|P_c^{\circ k}(1)|$ are not greater than m_c, and

$$\sup_{c \in \partial U} m_c \leq (4.38) \max\{R, 1\} < +\infty.$$

Since $U \neq \Omega_{int}$, we have $\partial U \subset \partial \mathcal{M}_3(\theta)$ and both $P_c^{\circ k}(c)$ and $P_c^{\circ k}(1)$ are holomorphic in U as functions of c. It follows from the Maximum Principle that the iterates $P_c^{\circ k}(c)$ and $P_c^{\circ k}(1)$ are uniformly bounded throughout U, which is a contradiction. □

3. Components of the Interior of $\mathcal{M}_3(\theta)$

First we give the following dynamical characterization of the boundary of the connectedness locus $\mathcal{M}_3(\theta)$, which is reminiscent of the similar property of the Mandelbrot set. For terminology and basic results on holomorphic motions and J-stability, see for example [**Mc2**].

Theorem 3.1 (Boundary of $\mathcal{M}_3(\theta)$ is Unstable). *The boundary $\partial\mathcal{M}_3(\theta)$ is the set of parameters for which the corresponding cubics are not J-stable in $\mathcal{P}_3^{cm}(\theta)$.*

Proof. A polynomial $P_{c_0} \in \mathcal{P}_3^{cm}(\theta)$ is J-stable if and only if both sequences $\{P_c^{\circ k}(c)\}$ and $\{P_c^{\circ k}(1)\}$ are normal for c in a neighborhood of c_0 ([**Mc2**], Theorem 4.2). If $c_0 \in \Omega_{ext}$, then c_0 escapes to infinity under iterations of P_{c_0}, while 1 has bounded orbit. For c close to c_0, the orbit of c under P_c will still converge to infinity while 1 will have bounded orbit, with a bound given by m_c in (2.3). It follows from Montel's theorem that both sequences are normal throughout a neighborhood of c_0. Hence c_0 is J-stable. Similarly, every P_{c_0} with $c_0 \in \Omega_{int}$ is J-stable. If c_0 belongs to the interior of $\mathcal{M}_3(\theta)$, then both c_0 and 1 will have orbits contained in $\mathbb{D}(0, m_{c_0})$ and the same holds for all c sufficiently close to c_0. Again both sequences $\{P_c^{\circ k}(c)\}$ and $\{P_c^{\circ k}(1)\}$ are normal in a neighborhood of c_0. Finally, if c_0 belongs to the boundary of $\mathcal{M}_3(\theta)$, then a small perturbation will make either c or 1 escape to infinity. Hence at least one of the sequences $\{P_c^{\circ k}(c)\}$ or $\{P_c^{\circ k}(1)\}$ fails to be normal in any neighborhood of c_0. $\square$

Corollary 3.2. *Let $P_{c_0} \in \mathcal{P}_3^{cm}(\theta)$ have an indifferent periodic orbit other than the fixed point at the origin. Then $c_0 \in \partial\mathcal{M}_3(\theta)$.*

Proof. Otherwise c_0 will be a J-stable parameter by the above theorem. But any stable indifferent cycle has to be persistent ([**Mc2**], Theorem 4.2). So the indifferent cycle $z(c_0) \mapsto P_{c_0}(z(c_0)) \mapsto \cdots \mapsto P_{c_0}^{\circ k-1}(z(c_0)) \mapsto z(c_0)$ can be continued analytically to the whole plane as a function of c and the multiplier function $c \mapsto (P_c^{\circ k})'(z(c))$ remains constant. This is clearly impossible, since for example when $c = 3 - 6\overline{\lambda}$, $P_c(c) = c$ is a superattracting fixed point, hence there cannot be any indifferent periodic point other than 0. $\square$

Definition (Types of Components). A component U of the interior of $\mathcal{M}_3(\theta)$ is called *hyperbolic-like* if for every $c \in U$, the orbit of either c or 1 under P_c converges to an attracting cycle. U is called a *capture* component if for every $c \in U$, either c or 1 eventually maps into the Siegel disk Δ_c. In case U is neither hyperbolic-like nor capture, we call it a *queer* component. We say that P_c is hyperbolic-like, capture, or queer if the corresponding parameter c belongs to such a component.

For example, there is a hyperbolic-like component in the form of the main cardioid of a large copy of the Mandelbrot set on the lower right corner of Fig. 1. For every c in this component, the orbit of the critical point c of P_c converges to an attracting fixed point. On the other hand, the large component which is attached on the right side of the unit circle to $c = 1$ is a capture, consisting of all c for which $P_c(c)$ belongs to Δ_c. Fig. 3-Fig. 7 show examples of the filled Julia sets of cubics in $\mathcal{P}_3^{cm}(\theta)$ for $\theta = (\sqrt{5}-1)/2$. Fig. 3 is the filled Julia set of a hyperbolic-like cubic. The large topological disk in black is the immediate basin of attraction of an attracting fixed point. Fig. 4 is the filled Julia set of a capture, with a critical point in the large preimage of the Siegel disk on the right. The cubic in Fig. 5 is located at the "cusp" of the large cardioid in the right lower corner of Fig. 1, hence it has a parabolic fixed point. Fig. 6 has two critical points on the boundary of its Siegel disk. Finally, the cubic in Fig. 7 belongs to Ω_{ext} so it has a disconnected Julia set. There are countably many components each quasiconformally homeomorphic

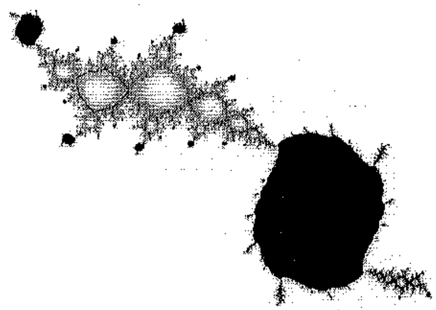

FIGURE 3

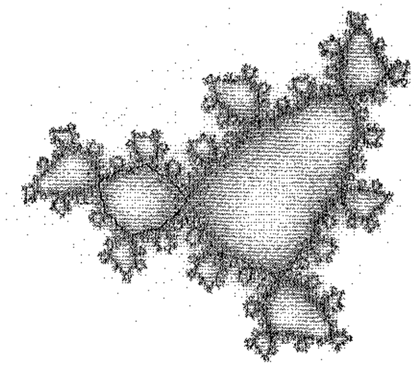

FIGURE 4

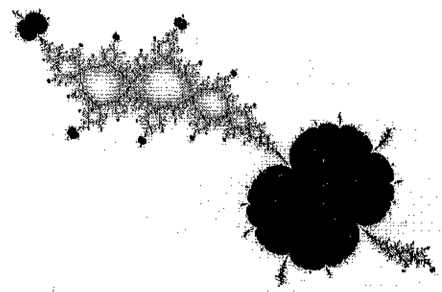

FIGURE 5

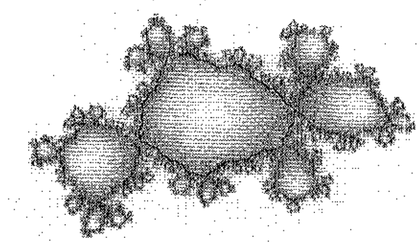

FIGURE 6

to the quadratic Siegel filled Julia set with the same rotation number. The uncountably many remaining components are single points.

In the above definition, we tacitly assumed that hyperbolic-like or capture cubics define components of the interior of $\mathcal{M}_3(\theta)$. The condition of being hyperbolic-like is clearly open. It is also closed in the interior of $\mathcal{M}_3(\theta)$ since by Theorem 3.1 a cubic P in the interior of $\mathcal{M}_3(\theta)$ is J-stable, so in a small neighborhood of it the number of attracting cycles remains constant ([**Mc2**], Theorem 4.2). This number is 1 if P is accumulated by hyperbolic-like cubics.

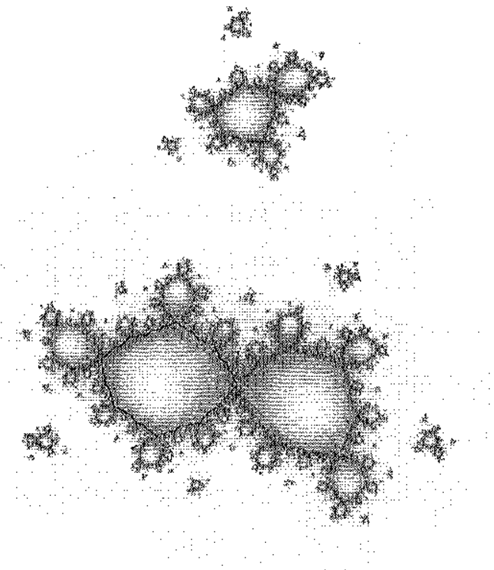

FIGURE 7

Now consider the property of being capture for $P \in \mathcal{P}_3^{cm}(\theta)$. It follows from Theorem 3.1 that when a capture cubic P belongs to the interior of $\mathcal{M}_3(\theta)$, there is an open neighborhood of P consisting of captures. Let V be the component of the interior of the set of capture cubics containing P. Similarly, define U to be the component of the interior of $\mathcal{M}_3(\theta)$ containing P. Clearly $V \subset U$. If they are not equal, choose a cubic $Q \in \partial V \cap U$. Since Q is J-stable, for all Q' in a small neighborhood of Q, a critical point of Q' belongs to the Fatou set of Q' if and only if the corresponding critical point of Q belongs to the Fatou set of Q. If we choose $Q' \in V$, there is a critical point of Q' which hits the Siegel disk $\Delta_{Q'}$. It follows that the same is true for Q, hence Q is capture, which contradicts $Q \in \partial V$. This proves $V = U$. In other words, when a capture cubic P belongs to the interior of $\mathcal{M}_3(\theta)$, the entire component of the interior of $\mathcal{M}_3(\theta)$ containing P consists of captures, hence the name "capture component."

However, the above argument does not rule out the possibility of a capture being on the boundary of the connectedness locus $\mathcal{M}_3(\theta)$. In fact, that the capture condition is open follows from a different type of argument which is standard in deformation theory of rational maps (see Theorem 5.3).

Conjecturally, queer components do not exist. But if they do, every cubic in a queer component exhibits an outstanding property: It admits an invariant line field on its Julia set, and in particular, its Julia set has positive Lebesgue measure. The proof of this fact depends on the harmonic λ-lemma of Bers and Royden [**BR**] as well as the elementary observation of Sullivan [**Su2**] that if the boundary of a Siegel disk moves holomorphically in a family of rational maps, then there is a choice of holomorphically varying Riemann maps for the Siegel disks (also see the new expanded version [**McS**]). There is a technical difficulty showing up in the proof: For a general θ of Brjuno type, it is not known whether the boundary of the Siegel disk of a $P \in \mathcal{P}_3^{cm}(\theta)$ is a Jordan curve. For this reason, the extension of holomorphic motions to the grand orbits of Siegel disks will require some extra work.

We will repeatedly use the following lemma of L. Bers [**B**], [**DH2**]:

Lemma 3.3 (Bers Sewing Lemma). *Let $E \subset \mathbb{C}$ be closed and U and V be two open neighborhoods of E. Let $\varphi : U \xrightarrow{\simeq} \varphi(U)$ and $\psi : V \xrightarrow{\simeq} \psi(V)$ be two homeomorphisms such that*

- *φ is K_1-quasiconformal,*
- *$\psi|_{V \smallsetminus E}$ is K_2-quasiconformal,*
- *$\varphi|_{\partial E} = \psi|_{\partial E}$.*

Then the map $\varphi \amalg \psi$ defined on V by

$$(\varphi \amalg \psi)(z) = \begin{cases} \varphi(z) & z \in E \\ \psi(z) & z \in V \smallsetminus E \end{cases}$$

is a K-quasiconformal homeomorphism with $K = \max\{K_1, K_2\}$. Moreover, $\overline{\partial}(\varphi \amalg \psi) = \overline{\partial}\varphi$ almost everywhere on E.

Theorem 3.4 (Invariant Line Fields for Queer Cubics). *Let U be a queer component of the interior of $\mathcal{M}_3(\theta)$. Then for any $c \in U$, the Julia set $J(P_c)$ has positive Lebesgue measure and supports an invariant line field.*

Proof. Fix some $c_0 \in U$. We first note that every Fatou component of P_{c_0} eventually maps to the Siegel disk Δ_{c_0} and the mapping is a conformal isomorphism: There cannot be further attracting cycles (since P_{c_0} is not hyperbolic-like) or indifferent periodic orbits (see Corollary 3.2). In particular, $K(P_{c_0}) = \overline{GO(\Delta_{c_0})}$.

Choose some $c \in U$ with $c \neq c_0$, and let

$$\varphi_c : \mathbb{C} \smallsetminus K(P_{c_0}) \xrightarrow{\simeq} \mathbb{C} \smallsetminus K(P_c)$$

be the conformal conjugacy given by composition of the Böttcher maps of P_{c_0} and P_c. A brief computation using the normal form (2.2) shows that $\varphi_c(z) = \sqrt{c/c_0}\,z + O(1)$ and we can choose the branch of the square root near c_0 for which $\varphi_{c_0}(z) = z$. Since φ_c depends holomorphically on c, it defines a holomorphic motion of $\mathbb{C} \smallsetminus K(P_{c_0})$. By the harmonic λ-lemma [**BR**], this motion extends to a *unique* holomorphic motion $\widehat{\varphi}_c$ of the entire plane, which is now defined only for c in a small neighborhood V of c_0, with the following properties:

- For every $c \in V$, $\widehat{\varphi}_c$ is a quasiconformal homeomorphism of the plane.
- For every $c \in V$, the Beltrami differential $\dfrac{\overline{\partial}\widehat{\varphi}_c}{\partial\widehat{\varphi}_c}\dfrac{d\overline{z}}{dz}$ is harmonic in $GO(\Delta_{c_0})$.

It is easy to see that uniqueness of this extended motion implies that $\widehat{\varphi}_c$ conjugates P_{c_0} to P_c on the entire plane (compare [**McS**]). In fact, one can replace $\widehat{\varphi}_c$ by $P_c^{-1} \circ \widehat{\varphi}_c \circ P_{c_0}$ on $GO(\Delta_{c_0})$, which also extends φ_c, where the branch of

P_c^{-1} is determined uniquely by the values of $\widehat{\varphi}_c$ on the Julia set $J(P_{c_0})$. Hence $\widehat{\varphi}_c = P_c^{-1} \circ \widehat{\varphi}_c \circ P_{c_0}$ by uniqueness.

Next, we want to show that the restriction $\widehat{\varphi}_c : GO(\Delta_{c_0}) \to GO(\Delta_c)$ is a *conformal* conjugacy. As Sullivan observes in **[Su2]**, the fact that the boundary of Δ_c moves holomorphically for $c \in U$ (Theorem 3.1) implies that there is a choice of the Riemann map $\zeta_c : \mathbb{D} \to \Delta_c$ such that $\zeta_c(0) = 0$ and $c \mapsto \zeta_c$ is holomorphic in c. Define a conformal conjugacy $\psi_c : \Delta_{c_0} \to \Delta_c$ by $\psi_c = \zeta_c \circ \zeta_{c_0}^{-1}$, and extend it to a conformal conjugacy $\psi_c : GO(\Delta_{c_0}) \to GO(\Delta_c)$ by taking pull-backs as follows. Take any component W of $P_{c_0}^{-n}(\Delta_{c_0})$ and let $W_c = \widehat{\varphi}_c(W)$ be the corresponding component of $P_c^{-n}(\Delta_c)$. Define $\psi_c : W \to W_c$ by $\psi_c = P_c^{-n} \circ \psi_c \circ P_{c_0}^{\circ n}$. Since $c \mapsto \psi_c$ is holomorphic and $\psi_c = \text{id}$ on $GO(\Delta_{c_0})$ when $c = c_0$, it follows that ψ_c defines a holomorphic motion of $GO(\Delta_{c_0})$. By the harmonic λ-lemma, ψ_c extends to a unique holomorphic motion $\widehat{\psi}_c$ of the entire plane which is defined for c in a neighborhood V' of c_0 and has harmonic Beltrami differential on $\mathbb{C} \smallsetminus K(P_{c_0})$. By an argument similar to the one we used for $\widehat{\varphi}_c$, it follows that $\widehat{\psi}_c$ respects the dynamics, i.e., it conjugates P_{c_0} to P_c on the entire plane. In particular, it sends the marked critical point c_0 of P_{c_0} to the marked critical c of P_c. Let us assume for example that the forward orbit of c_0 accumulates on the boundary of Δ_{c_0}. Then the same is true for c and Δ_c. Since $\widehat{\varphi}_c$ was also a conjugacy to begin with, for all $c \in V \cap V'$ we have $\widehat{\psi}_c(c_0) = c = \widehat{\varphi}_c(c_0)$, and by induction $\widehat{\psi}_c(P_{c_0}^{\circ k}(c_0)) = P_c^{\circ k}(c) = \widehat{\varphi}_c(P_{c_0}^{\circ k}(c_0))$ for all k. Since every point on the boundary of Δ_{c_0} is in the closure of the forward orbit of c_0, we conclude that $\widehat{\psi}_c$ and $\widehat{\varphi}_c$ agree on $\partial\Delta_{c_0}$. Evidently this shows that $\widehat{\psi}_c$ and $\widehat{\varphi}_c$ agree on the boundary of every bounded Fatou component of P_{c_0}, hence on the entire Julia set $J(P_{c_0})$. It follows then from the Bers Sewing

Lemma 3.3 that $\varphi_c \amalg \psi_c$ defined by

$$(\varphi_c \amalg \psi_c)(z) = \begin{cases} \widehat{\varphi}_c(z) & z \in \mathbb{C} \smallsetminus GO(\Delta_{c_0}) \\ \widehat{\psi}_c(z) & z \in GO(\Delta_{c_0}) \end{cases}$$

is a quasiconformal homeomorphism which has harmonic Beltrami differential in $\mathbb{C} \smallsetminus J(P_{c_0})$. Note that $\varphi_c \amalg \psi_c$ is an extension of both φ_c and ψ_c. By uniqueness, we conclude that $\widehat{\varphi}_c \equiv \widehat{\psi}_c$. In particular, when $c \in V \cap V'$, $\widehat{\varphi}_c$ is conformal away from the Julia set $J(P_{c_0})$.

Now, if the Julia set $J(P_{c_0})$ had measure zero, $\widehat{\varphi}_c$ would have been conformal, contradicting $c \neq c_0$. So $J(P_{c_0})$ has positive measure. The desired invariant line field is then given by $\widehat{\varphi}_c^*(\sigma_0)$, the pull-back of the standard conformal structure σ_0 on the plane by $\widehat{\varphi}_c$. $\square$

The existence of holomorphic motions in the above proof was the crucial fact which made the conformal extensions possible. In the case we have "static" quasiconformal conjugacies, such conformal extensions are still possible once we assume that the boundaries of Siegel disks are Jordan curves. Let Δ be a Jordan domain containing the origin and $R_t : z \mapsto e^{2\pi i t} z$ be the rigid rotation on the unit circle. Let $\zeta : \Delta \to \mathbb{D}$ be any conformal isomorphism with $\zeta(0) = 0$. Then the homeomorphism $h^t_\Delta : \partial\Delta \to \partial\Delta$ defined by $h^t_\Delta = \zeta^{-1} \circ R_t \circ \zeta$ is the *intrinsic rotation of* $\partial\Delta$ *by angle* t. By Schwarz Lemma, h^t_Δ is independent of the choice of ζ. Now suppose Δ_1 and Δ_2 are two Jordan domains containing 0 and t is irrational. Let $\varphi : \partial\Delta_1 \to \partial\Delta_2$ be a homeomorphism satisfying $\varphi \circ h^t_{\Delta_1} = h^t_{\Delta_2} \circ \varphi$. Then two points $a_1 \in \Delta_1$ and $a_2 \in \Delta_2$ have the same *conformal position* with respect to φ if $\zeta_1(a_1) = \zeta_2(a_2)$, where the $\zeta_j : \Delta_j \to \mathbb{D}$ are conformal isomorphisms with $\zeta_j(0) = 0$ and $\zeta_1 = \zeta_2 \circ \varphi$ on $\partial\Delta_1$.

Lemma 3.5 (Extending QC Conjugacies). *Let P and Q be two cubics in $\mathcal{P}^{cm}_3(\theta)$ such that the boundaries of the Siegel disks Δ_P and Δ_Q are Jordan curves.*

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal homeomorphism whose restriction $\mathbb{C} \smallsetminus GO(\Delta_P) \to \mathbb{C} \smallsetminus GO(\Delta_Q)$ conjugates P to Q. Then

(a) *If P is not capture, there exists a quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ which conjugates P and Q, which is conformal on $GO(\Delta_P)$ and agrees with φ on $\mathbb{C} \smallsetminus GO(\Delta_P)$.*

(b) *If P is capture, we can construct a ψ as in (a) if and only if the captured images of the critical points of P and Q in Δ_P and Δ_Q have the same conformal position with respect to φ.*

Proof. (a) Fix some $b_1 \in \partial\Delta_P$ and let $b_2 = \varphi(b_1)$. Consider conformal isomorphisms $\zeta_1 : \Delta_P \xrightarrow{\simeq} \mathbb{D}$ and $\zeta_2 : \Delta_Q \xrightarrow{\simeq} \mathbb{D}$, with $\zeta_1(0) = 0 = \zeta_2(0)$ and $\zeta_1(b_1) = 1 = \zeta_2(b_2)$, which conjugate P on Δ_P and Q on Δ_Q to the rigid rotation $R_\theta : z \mapsto e^{2\pi i\theta}z$ on $\mathbb{D}$. Since the boundaries of Δ_P and Δ_Q are Jordan curves, ζ_1 and ζ_2 extend homeomorphically to the closures. The composition $\psi = \zeta_2^{-1} \circ \zeta_1 : \Delta_P \to \Delta_Q$ is conformal and conjugates P on Δ_P to Q on Δ_Q. Also $\psi(b_1) = \varphi(b_1) = b_2$ and by induction $\psi(P^{\circ k}(b_1))) = Q^{\circ k}(b_2) = \varphi(P^{\circ k}(b_1))$. Since the orbit of b_1 is dense on the boundary of Δ_P, we have $\psi|_{\partial\Delta_P} = \varphi|_{\partial\Delta_P}$. Therefore, ψ gives the required extension of φ to the Siegel disk Δ_P. It is now easy to extend ψ to the grand orbit $GO(\Delta_P)$: $P^{\circ k}$ maps any component of $P^{-k}(\Delta_P)$ isomorphically onto Δ_P. Hence we can define ψ on any such component as the composition $Q^{-k} \circ \psi|_{\Delta_P} \circ P^{\circ k}$, where the branch of Q^{-k} is determined by the values of φ on the Julia set $J(P)$. Clearly this composition is conformal inside this component and agrees with φ on its boundary. ψ defined this way is a quasiconformal homeomorphism by the Bers Sewing Lemma 3.3, with $U = V = \mathbb{C}$ and $E = \mathbb{C} \smallsetminus GO(\Delta_P)$.

(b) Now let P be capture. The construction of ψ goes through as in case (a) except for the last part where we want to extend ψ by taking pull-backs. Suppose that there exists a positive integer k such that the critical point c_1

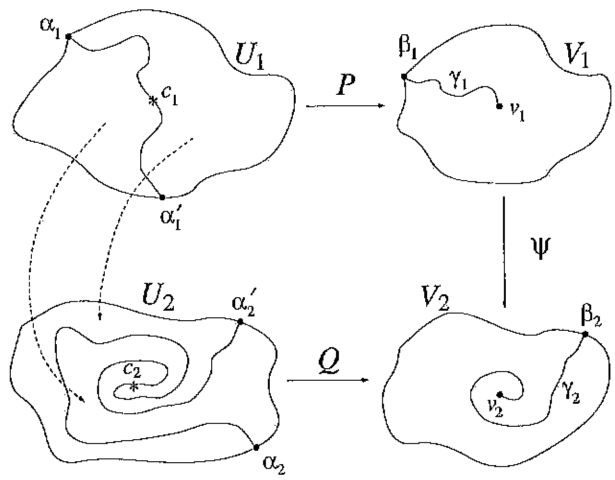

FIGURE 8. *Extending φ in the capture case.*

of P belongs to the component U_1 of $P^{-k}(\Delta_P)$. Let $V_1 = P(U_1)$ and let $v_1 = P(c_1)$ be the critical value in V_1. Since $P : \partial U_1 \to \partial V_1$ is a double covering and φ conjugates P to Q on the Julia sets, there must be a critical point c_2 of Q in a component U_2 of $Q^{-k}(\Delta_Q)$, with $\partial U_2 = \varphi(\partial U_1)$. Similarly define V_2 and v_2. By the proof of part (a) we can define ψ inductively up to the $(k-1)$-th preimages of Δ_P, including V_1. This gives us a conformal isomorphism $\psi : V_1 \to V_2$ which necessarily maps v_1 to v_2, because by our assumption $P^{\circ k}(c_1)$ and $Q^{\circ k}(c_2)$ have the same conformal position in Δ_P and Δ_Q and so one gets mapped to the other by $\psi|_{\Delta_P}$. Choose any simple arc γ_1 in V_1 connecting v_1 to some boundary point β_1. The simple arc $\gamma_2 = \psi(\gamma_1)$ in V_2 connects v_2 to the boundary point $\beta_2 = \psi(\beta_1)$. Pull γ_1 back by P to get two branches of a simple arc passing through the critical point c_1 with two distinct endpoints α_1 and α_1' on the boundary of U_1. Similarly we consider the pull-back of γ_2 by Q and we get two endpoints on the boundary of U_2, which we label as $\alpha_2 = \varphi(\alpha_1)$ and $\alpha_2' = \varphi(\alpha_1')$ (see Fig. 8). Now the inverse Q^{-1} can be defined analytically over $V_2 \smallsetminus \gamma_2$ and has two branches which take values in two different connected components of $U_2 \smallsetminus Q^{-1}(\gamma_2)$. Define ψ on U_1 as the composition $Q^{-1} \circ \psi \circ P$, where the boundary orientation tells us which of the two branches of Q^{-1} has to be taken. This way we extend ψ to U_1 and ψ can then be defined on further preimages of Δ_P similar to the case (a). $\square$

4. Renormalizable Cubics

This section briefly studies the class of renormalizable cubics in $\mathcal{P}_3^{cm}(\theta)$. These are the cubics with disjoint critical orbits from which one can extract the quadratic $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ by straightening. From a different point of view, one may consider a renormalizable cubic with connected Julia set as the result of "intertwining" the quadratic Q_θ with another quadratic with connected Julia set (compare [**EY**]). For background on polynomial-like maps, straightening and hybrid classes, see for example [**DH2**].

Definition. A cubic $P \in \mathcal{P}_3^{cm}(\theta)$ is called *renormalizable* if there exists a pair of Jordan domains U and V, with $0 \in U \Subset V$, such that the restriction $P|_U : U \to V$ is a quadratic-like map hybrid equivalent to $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$.

When θ is irrational of bounded type, it follows from the work of Douady-Ghys-Herman-Shishikura [**D2**] that the boundary of the Siegel disk of Q_θ is a quasicircle passing through the critical point. Hence the same is true for the Siegel disk Δ_P when P is renormalizable.

To prove the next theorem, we need the following useful lemma of Kiwi in [**K**]. This lemma in particular shows that each indifferent cycle for a cubic $P \in \mathcal{P}_3^{cm}(\theta)$ must attract its own critical point.

Lemma 4.1 (Separation Lemma). *Let P be a polynomial with connected Julia set. Then there exists a finite collection of closed preperiodic external rays, separating the plane into disjoint open simply-connected sets $\{U_j\}$, such that:*

- *Each U_j contains at most one non-repelling periodic point or periodic Fatou component of P.*

- *If* $z_1 \mapsto \cdots \mapsto z_p \mapsto z_1$ *is a non-repelling cycle meeting* $U_{i_1} \mapsto \cdots \mapsto U_{i_p} \mapsto U_{i_1}$, *then* $\bigcup_{j=1}^{p} U_{i_j}$ *contains the entire orbit of at least one critical point of* P.

Theorem 4.2. *A cubic* $P \in \mathcal{P}_3^{cm}(\theta)$ *is renormalizable if either of the following conditions holds:*

(a) *P has a non-repelling periodic orbit other than* 0 *which is not parabolic.*

(b) *P has disconnected Julia set.*

Proof. First assume that we are in case (a) so that $J(P)$ is connected. Let $\mathcal{R}$ be the finite collection of the closed preperiodic external rays given by the Separation Lemma 4.1. Let V be the component of $\mathbb{C} \smallsetminus \mathcal{R}$ which contains 0, cut off by an equipotential of $K(P)$. Finally, let U be the component of $P^{-1}(V)$ containing 0. Since all the rays in $\mathcal{R}$ are preperiodic, $P(\mathcal{R}) \subset \mathcal{R}$, hence $U \subset V$. U necessarily contains a critical point of P since otherwise Schwarz lemma and $|P'(0)| = 1$ would imply that $U = V$ and $P|_U : U \to V$ is a conformal isomorphism conjugate to a rotation. This would contradict the fact that U intersects the basin of attraction of infinity for P. The other critical point of P has to stay away from V because by the second part of the Separation Lemma its entire orbit lives in the cycle of components of $\mathbb{C} \smallsetminus \mathcal{R}$ which contains the non-repelling periodic orbit of P.

Since by our assumption the non-repelling cycle of P is not parabolic, the landing points of the external rays in $\mathcal{R}$ must all be repelling. Therefore, by a simple "thickening" procedure (see for example [**M3**]), we can assume that $\overline{U} \subset V$, so that $P|_U : U \to V$ is a quadratic-like map. Up to affine conjugation, there is only one quadratic polynomial which has a fixed Siegel disk of rotation number θ, so this quadratic-like map has to be hybrid equivalent to $Q_\theta : z \mapsto e^{2\pi i\theta} z + z^2$.

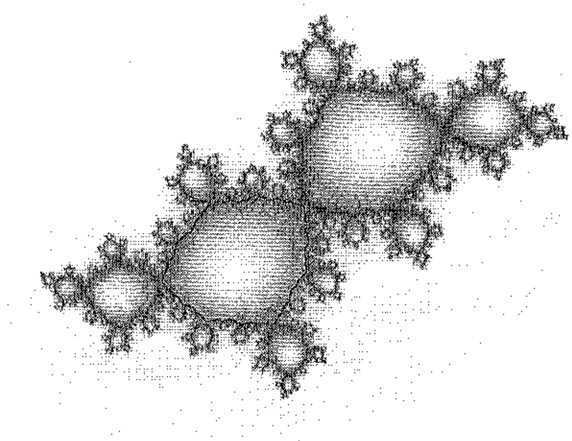

FIGURE 9. *Filled Julia set of the quadratic* $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ *for* $\theta = (\sqrt{5} - 1)/2$.

Now suppose that $J(P)$ is disconnected. For $\epsilon > 0$, let U_ϵ be the connected component of $\{z \in \mathbb{C} : G_P(z) < \epsilon\}$ containing the Siegel disk Δ_P, where $G_P : \mathbb{C} \to \{x \in \mathbb{R} : x \geq 0\}$ is the Green's function of $K(P)$. It is not hard to see that for small ϵ, $P|_{U_\epsilon} : U_\epsilon \to U_{3\epsilon}$ is a quadratic-like map, necessarily hybrid equivalent to Q_θ. □

Fig. 3 and Fig. 7 demonstrate the above theorem. In either example, one can see the filled Julia set of the quadratic-like restriction $P|_U : U \to V$ given by the above theorem, which is quasiconformally homeomorphic to the filled Julia set of $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ in Fig. 9.

Remark. When $P \in \mathcal{P}_3^{cm}(\theta)$ has a parabolic cycle, we can no longer expect to extract Q_θ from it by renormalization. However, there must be a homeomorphic embedding $K(Q_\theta) \to K(P)$, conformal in the interior of $K(Q_\theta)$, which conjugates Q_θ to P. This can be proved directly when θ is of bounded type, and in the general case by using the parabolic surgery recently introduced in [**Ha**].

Corollary 4.3. *Let θ be an irrational number of bounded type. Let $P \in \mathcal{P}_3^{cm}(\theta)$ be hyperbolic-like or have disconnected Julia set $J(P)$. Then $J(P)$ has Lebesgue measure zero.*

Proof. Let $P|_U : U \to V$ be the quadratic-like restriction given by Theorem 4.2 and let K be its filled Julia set. Since this restriction is hybrid equivalent to $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ whose Julia set has measure zero by the theorem of Petersen [**Pe**], we simply conclude that ∂K has Lebesgue measure zero.

It is well-known that the forward orbit of almost every point $z \in J(P)$ accumulates on the ω-limit set of the critical points of P ([**Ly**], Proposition 1.14), which in this case is just $\partial\Delta_P$ union the attracting periodic orbit (resp. $\partial\Delta_P$) if P is hyperbolic-like (resp. with disconnected Julia set). So the orbit of almost every $z \in J(P)$ accumulates on $\partial\Delta_P$. This implies that for all $n \geq N = N(z)$, $P^{\circ n}(z) \in V$. This can happen only if $P^{\circ N}(z) \in \partial K$ or equivalently $z \in P^{-N}(\partial K)$. We conclude that, up to a set of measure zero, $J(P) = \bigcup_{N \geq 0} P^{-N}(\partial K)$. But the right-hand side has measure zero because ∂K does. This proves that $J(P)$ has Lebesgue measure zero as well. $\square$

5. Quasiconformal Conjugacy Classes

In this section we characterize the quasiconformal conjugacy classes in $\mathcal{P}_3^{cm}(\theta)$. A central role is played by the following:

Theorem 5.1 (Parametrization of QC Conjugacy Classes). *Let P_{c_0}, P_{c_1} be distinct cubics in $\mathcal{P}_3^{cm}(\theta)$ and let $\varphi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal homeomorphism which conjugates P_{c_0} to P_{c_1}, i.e., $\varphi \circ P_{c_0} = P_{c_1} \circ \varphi$ and $\varphi(c_0) = c_1$. Then there exists a holomorphic map $t \mapsto c_t$ from an open disk $\mathbb{D}(0,r)$ $(r > 1)$ into $\mathbb{C}^*$ which maps 0 to c_0 and 1 to c_1, such that for every $t \in \mathbb{D}(0,r)$, P_{c_0} is conjugate to P_{c_t} by a K_t-quasiconformal homeomorphism $\varphi_t : \mathbb{C} \to \mathbb{C}$. Moreover, $K_t \to 1$ as $t \to 0$.*

Proof. The idea of the proof is standard in complex dynamics (see [**Su2**], [**DH2**]); however, we briefly sketch it here because similar arguments appear again in the rest of this work. Define a conformal structure σ on $\mathbb{C}$ by $\sigma = \varphi^* \sigma_0$, where, as usual, σ_0 is the standard conformal structure on $\mathbb{C}$. (To simplify the notation, in what follows we identify a conformal structure on $\mathbb{C}$ with its associated Beltrami differential.) Since P_{c_1} is holomorphic, P_{c_0} has to preserve σ. Since φ is quasiconformal, $\|\sigma\|_\infty < 1$. Define a one-parameter family $\{\sigma_t\}$ of complex-analytic deformations of σ by $\sigma_t = t\sigma$, where $t \in \mathbb{D}(0,r)$ and $r > 1$ is chosen such that $r\|\sigma\|_\infty < 1$. By the Measurable Riemann Mapping Theorem [**AB**], there exists a unique quasiconformal homeomorphism φ_t of the plane which solves the Beltrami equation $\varphi_t^* \sigma_0 = \sigma_t$ and fixes 0, 1 and ∞. Define $P^t = \varphi_t \circ P_{c_0} \circ \varphi_t^{-1}$. Since P_{c_0} is holomorphic, it acts as a pure rotation on Beltrami differentials. Hence $P_{c_0}^* \sigma = \sigma$ implies $P_{c_0}^* \sigma_t = \sigma_t$ and therefore P^t is a quasiregular self-map of the plane which preserves σ_0 and is conjugate to a cubic polynomial. It is then easy to see that P^t itself is a cubic polynomial with a fixed Siegel disk of rotation number θ centered at 0 with a marked critical point at $z = 1$.

Note that $t \mapsto \sigma_t$ is holomorphic, so the same is true for $t \mapsto \varphi_t$ and hence $t \mapsto P^t$ by the analytic dependence of the solutions of the Beltrami equation on parameters [**AB**]. Therefore the map $t \mapsto c_t$ which defines the second critical point of P^t so that $P^t = P_{c_t}$ is holomorphic. It is easy to see that c_t has all the required properties. □

Corollary 5.2. *Quasiconformal conjugacy classes in $\mathcal{P}_3^{cm}(\theta)$ are either single points or open and connected. In particular, cubics on the boundary $\partial\mathcal{M}_3(\theta)$ are quasiconformally rigid, i.e., their conjugacy classes are single points.*

Theorem 5.3 (Capture is an open condition). *Let P_{c_0} be a capture cubic. Then there is an open neighborhood $U \subset \mathcal{P}_3^{cm}(\theta)$ of c_0 such that for every $c \in U$, P_c is also capture. In particular, capture cubics belong to the interior of the connectedness locus $\mathcal{M}_3(\theta)$.*

Proof. To fix the ideas, let us assume that $P_{c_0}^{\circ k}(c_0) \in \Delta_{c_0}$ and $k \geq 1$ is the smallest such integer. First assume that $P_{c_0}^{\circ k}(c_0) \neq 0$. Let $A \subset \Delta_{c_0}$ be the annulus bounded by $\partial\Delta_{c_0}$ and the analytic invariant curve in Δ_{c_0} passing through $P_{c_0}^{\circ k}(c_0)$. Take a conformal isomorphism $\psi : A \xrightarrow{\simeq} \mathbb{A}(1, \epsilon)$, with $\epsilon = e^{2\pi \operatorname{mod}(A)} > 1$, which conjugates P_{c_0} on A to the rotation on $\mathbb{A}(1, \epsilon)$. Postcompose ψ with a (non-conformal) dilation $\mathbb{A}(1, \epsilon) \to \mathbb{A}(1, \epsilon^2)$ to get a quasiconformal homeomorphism $\varphi : A \to \mathbb{A}(1, \epsilon^2)$ conjugating P_{c_0} to the rotation. Define a P_{c_0}-invariant conformal structure σ on $\mathbb{C}$ by putting $\sigma = \varphi^*\sigma_0$ on A and pulling it back by the inverse branches of P_{c_0} to the entire grand orbit of A. Set $\sigma = \sigma_0$ elsewhere. As in the proof of Theorem 5.1, we define $\sigma_t = t\sigma$ for $t \in \mathbb{D}(0, r)$ for some $r > 1$, solve the Beltrami equation $\varphi_t^*\sigma_0 = \sigma_t$ and set $P^t = \varphi_t \circ P_{c_0} \circ \varphi_t^{-1}$. Then P^t is a capture cubic in $\mathcal{P}_3^{cm}(\theta)$ and $P^0 = P_{c_0}$. The holomorphic mapping $t \mapsto P^t$ is not constant because $\operatorname{mod}(\varphi_1(A))$ is the same as the modulus of A equipped with the conformal structure σ, which in turn is $(1/2\pi)\log(\epsilon^2) = 2 \mod(A)$. Hence $P^1 \neq P^0$ and the mapping $t \mapsto P^t$ is open.

Now consider the case where $P_{c_0}^{\circ k}(c_0) = 0$. In this case, by Corollary 2.2, the conformal capacity of Δ_c has a positive lower bound for all c sufficiently close to c_0. It follows that there exists an $\epsilon > 0$ such that for all c close to c_0, $\Delta_c \supset \mathbb{D}(0, \epsilon)$. Hence a small perturbation of P_{c_0} will still be a capture cubic. □

By a *center* of a hyperbolic-like component $U \subset \mathcal{M}_3(\theta)$ we mean a cubic $P_c \in U$ with one of the critical points c or 1 being periodic. Similarly, a center of a capture component will be a cubic with one critical point eventually mapped to the indifferent fixed point at the origin.

Lemma 5.4 (Existence of Centers). *Every hyperbolic-like or capture component of the interior of $\mathcal{M}_3(\theta)$ has a center.*

By the remark after the proof, centers of hyperbolic-like or capture components are unique when θ is of bounded type.

Proof. First let U be a hyperbolic-like component. For every $c \in U$, consider the multiplier $m(c)$ of the unique attracting periodic orbit of P_c. The mapping $c \mapsto m(c)$ from U into $\mathbb{D}$ is easily seen to be proper and holomorphic. Hence it vanishes at a finite number of points in U.

Now let U be capture. To be more specific, let us assume that for every $c \in U$, $P_c^{\circ k}(c)$ belongs to the Siegel disk Δ_c, and let k be the smallest such integer. Since P_c is J-stable by Theorem 3.1, the boundary of Δ_c moves holomorphically. Then, as in the proof of Theorem 3.4, there is a holomorphically varying choice of the Riemann maps $\zeta_c : \mathbb{D} \to \Delta_c$ with $\zeta_c(0) = 0$. Define a map $m : U \to \mathbb{D}$ by

$$m(c) = \zeta_c^{-1}(P_c^{\circ k}(c)).$$

Clearly m is holomorphic. Let $c_n \in U$ be any sequence which converges to $c \in \partial U$ as $n \to \infty$. For simplicity, put $\zeta_{c_n} = \zeta_n$. Let $z_n = P_{c_n}^{\circ k}(c_n) \in \Delta_{c_n}$ and $w_n = m(c_n) = \zeta_n^{-1}(z_n) \in \mathbb{D}$. If w_n does not converge to the unit circle, we can find a subsequence $w_{n(j)}$ such that $w_{n(j)} \to w \in \mathbb{D}$ as $j \to \infty$. Since the family of univalent functions $\{\zeta_n : \mathbb{D} \to \mathbb{C}\}$ is normal, by passing to a further subsequence if necessary, we may assume that $\zeta_{n(j)} \to \zeta$ locally uniformly on $\mathbb{D}$. Clearly $\zeta(\mathbb{D}) \subset \Delta_c$. Therefore, $\zeta(w) = \lim_j \zeta_{n(j)}(w_{n(j)}) = \lim_j z_{n(j)} = P_c^{\circ k}(c) \in \Delta_c$. But this means that P_c is capture, which contradicts $c \in \partial U$. This proves that w_n converges to the unit circle. Hence m is a proper map. Now, as before, $m^{-1}(0)$ has to be non-vacuous and finite. □

Remark. To show *uniqueness* of centers, by Theorem 5.1 it would be enough to prove that any two centers for a component are quasiconformally conjugate. When the rotation number θ is of bounded type, this can be proved by a pull-back argument similar to Lemma 3.5 since in this case the boundary of Δ_P for $P \in \mathcal{P}_3^{cm}(\theta)$ is a Jordan curve by Theorem 13.7 (compare [**Mc1**] or [**M2**], where uniqueness of centers is shown for every hyperbolic component in the space of polynomial maps).

Theorem 5.5 (QC Conjugacy Classes in $\mathcal{P}_3^{cm}(\theta)$). *Quasiconformal conjugacy classes in $\mathcal{P}_3^{cm}(\theta)$ are given by the following list:*

(a) *Hyperbolic-like or capture components of the interior of $\mathcal{M}_3(\theta)$ with the center(s) removed.*

(b) *The two components Ω_{ext} and Ω_{int}.*

(c) *Queer components of the interior of $\mathcal{M}_3(\theta)$.*

(d) *Centers of hyperbolic-like or capture components.*

(e) *Single points on the boundary of $\mathcal{M}_3(\theta)$.*

Proof. Corollary 5.2 shows that no conjugacy class intersects two distinct members of the above list. It also proves that (d) and (e) are in fact conjugacy classes. Also the proof of Theorem 3.4 shows that every queer component is a conjugacy class. That (a) and (b) are quasiconformal conjugacy classes follows from the fact that over the components of type (a) or (b), the family $\{P_c\}$ has no critical orbit relations ([**McS**], Theorem 2.7). □

6. Connectivity of $\mathcal{M}_3(\theta)$

In this section we prove that $\mathcal{M}_3(\theta)$ is connected. This amounts to showing that each of its complementary components Ω_{ext} and Ω_{int} are homeomorphic to the punctured disk. One way to do this is to mimic the standard Douady-Hubbard proof of connectivity of the Mandelbrot set [**DH1**]: We can construct a holomorphic branched covering $\Phi : \Omega_{ext} \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$ by assigning to each $P_c \in \Omega_{ext}$ the position of the critical point c in the Böttcher coordinate of P_c. Φ extends holomorphically to infinity with $\Phi^{-1}(\infty) = \infty$. The degree of this map is 3, so to prove that Ω_{ext} is a punctured disk we must show that Φ has no critical point other than ∞. (This additional difficulty does not show up in the case of the Mandelbrot set where the similar map has degree 1.) To prove that Φ is locally injective, one can start with two nearby polynomials in the same fiber of Φ and define a *conformal* conjugacy between them near infinity by composing their Böttcher coordinates. This conjugacy can be conformally extended using the dynamics to the entire basin of attraction of infinity. Then a delicate argument is necessary to prove that one can extend the conjugacy further to the complex plane in a holomorphic way, proving that the two polynomials are identical (see [**Z3**] for details of such a proof).

However, to prove that Ω_{ext} is a punctured disk, it would be much easier to use methods of Teichmüller theory of rational maps as developed in [**McS**]. (There one can also find a different proof for connectivity of the Mandelbrot set.) Let $P \in \mathcal{P}_3^{cm}(\theta)$. By definition, the *Teichmüller space* $\text{Teich}(P)$ consists of all pairs $(Q, [\varphi])$, where $Q \in \mathcal{P}_3^{cm}(\theta)$ and $\varphi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal conjugacy between P and Q, i.e., $P \circ \varphi = \varphi \circ Q$. Here $[\varphi]$ means that we only consider the isotopy class of φ. The *modular group* $\text{Mod}(P)$ is the group of isotopy classes of quasiconformal homeomorphisms commuting with P. $\text{Mod}(P)$ acts on $\text{Teich}(P)$ properly discontinuously by $[\psi](Q, [\varphi]) = (Q, [\psi \circ \varphi])$. The quotient Teich($P$)/Mod($P$), also called the *moduli space of* P, is isomorphic to the quasiconformal conjugacy class of P in $\mathcal{P}_3^{cm}(\theta)$.

More generally, one can define the Teichmüller space Teich(U, P), where U is an open set invariant under P. It consists of all triples $(V, Q, [\varphi])$, where V is open and invariant under Q, and the quasiconformal homeomorphism $\varphi : V \to U$ conjugates P and Q. But now $[\varphi]$ denotes the isotopy class of φ *rel ideal boundary of* V.

Theorem 6.1. *The connectedness locus* $\mathcal{M}_3(\theta)$ *is connected.*

Proof. Let $P = P_c \in \Omega_{ext}$. Then $J(P)$ is disconnected and the critical point c belongs to the basin of attraction of infinity. Let γ be the equipotential of the Green's function of $K(P)$ passing through c. Topologically γ is a figure eight with the double point at c (see Fig. 10). Let $\widehat{J}(P)$ be the union of $J(P)$ together with the backward orbit of the fixed point 0 as well as the union of all forward and backward images of γ. In other words, $\widehat{J}(P)$ is the closure of the grand orbits of all periodic points and critical points of P. The complement $U = \mathbb{C} \setminus \widehat{J}(P)$ consists of countably many annuli A_i of finite modulus (contained in the basin of attraction of ∞) and countably many punctured disks (corresponding to the Siegel disk and its preimages). On U the grand orbit equivalence relation is clearly indiscrete. By [**McS**], Theorem 6.2.,

$$\text{Teich}(P) \simeq \text{Teich}(U, P) \times M_1(J(P), P),$$

where $M_1(J(P), P)$ is the unit ball in the space of all P-invariant Beltrami differentials supported on $J(P)$. This factor is trivial by the following

Lemma 6.2. *The Julia set of a cubic polynomial outside the connectedness locus* $\mathcal{M}_3(\theta)$ *admits no invariant line field.*

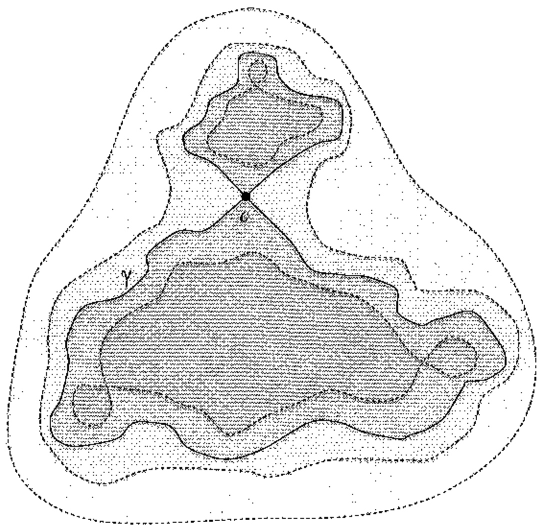

FIGURE 10

Note that for arbitrary θ of Brjuno type, it is not known whether this Julia set has measure zero (compare Corollary 4.3).

Proof. By Theorem 4.2(b), such a cubic is renormalizable. By straightening, an invariant line field on its Julia set gives rise to an invariant line field, or equivalently an invariant Beltrami differential σ, on the Julia set of $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$. Now, as in the proof of Theorem 5.1, by deforming σ to $\sigma_t = t\sigma$ we can get a holomorphic family Q^t of normalized quadratic polynomials all quasiconformally conjugate to Q_θ. But Q_θ belongs to the boundary of the Mandelbrot set, hence admits no non-trivial deformations, implying that $Q^t = Q_\theta$ for all t. So the normalized quasiconformal homeomorphisms φ_t which solve the Beltrami equation $\varphi_t^*\sigma_0 = \sigma_t$ must all commute with Q. Now for any periodic point $z \in J(Q_\theta)$ of period n, $t \mapsto \varphi_t(z)$ is a continuous path in the finite set of all period-n points in $J(Q_\theta)$. Since $\varphi_0(z) = z$, we must have $\varphi_t(z) = z$ for all t. Such points z are dense in the Julia set, so $\varphi_t|_{J(Q_\theta)}$ must be the identity. Since $\sigma_t = 0$ off the Julia set, it follows from the Bers Sewing

Lemma 3.3 that $\overline{\partial}\varphi_t = 0$ almost everywhere in the plane. This implies that σ_t, or equivalently σ, vanishes almost everywhere, which is a contradiction. □

Now by Theorem 5.5, Ω_{ext} coincides with the quasiconformal conjugacy class of P. It follows that

$$\Omega_{ext} \simeq \text{Teich}(P)/\text{Mod}(P).$$

By [**McS**], Theorem 6.1, Teich$(P) \simeq$ Teich(U, P) is isomorphic to the upper half-plane $\mathbb{H}$. Finally, every quasiconformal self-conjugacy ψ of P preserves grand orbits of the distinguished points 0 and c, hence it fixes the boundaries of all the annuli A_i pointwise. In particular, ψ is the identity on the Julia set $J(P)$. Hence the action of $[\psi] \in$ Mod(P) is identity except in the annuli A_i where it is possibly a power of a Dehn twist. So Mod(P) is at most $\mathbb{Z}$. Since Ω_{ext} is not simply-connected, Mod$(P) = \mathbb{Z}$. It follows that Ω_{ext} is homeomorphic to a punctured disk. □

7. Critical Parametrization of Blaschke Products

This section is the beginning of a digression in the study of cubic Siegel polynomials. We look at certain Blaschke products which will serve as models for the cubics in $\mathcal{P}_3^{cm}(\theta)$. We will introduce these model maps in Section 8 and return to their relation with the cubics in Section 9.

Let us consider the following space of degree 5 normalized Blaschke products:

$$\widehat{\mathcal{B}} = \{B : z \mapsto \tau z^3 \left(\frac{z-p}{1-\overline{p}z}\right)\left(\frac{z-q}{1-\overline{q}z}\right) : B(1) = 1 \text{ and } |p| > 1, |q| > 1\}, \tag{7.1}$$

where the rotation factor τ on the unit circle $\mathbb{T}$ is chosen so as to achieve the normalization $B(1) = 1$. Each $B \in \widehat{\mathcal{B}}$ has superattracting fixed points at 0 and ∞ and four other critical points counted with multiplicity. We are interested in the open subset $\mathcal{B} \subset \widehat{\mathcal{B}}$ of those normalized Blaschke products of the form (7.1) whose four critical points other than 0 and ∞ are of the form

$$c_1,\ c_2,\ \frac{1}{\overline{c_1}},\ \frac{1}{\overline{c_2}}$$

with $|c_1| > 1, |c_2| > 1$. Our goal is to parametrize elements of $\mathcal{B}$ by their critical points c_1 and c_2. The following theorem provides this "critical parametrization" for $\mathcal{B}$:

Theorem 7.1 (Critical Parametrization). *Let c_1 and c_2 be two points outside the closed unit disk in the complex plane. Then there exists a unique normalized Blaschke product $B \in \mathcal{B}$ whose critical points are located at* $0, \infty, c_1, c_2, \frac{1}{\overline{c_1}}, \frac{1}{\overline{c_2}}$.

The proof of this theorem will be given after the following two supporting lemmas. It would be interesting to find a conceptual proof of this fact which can be generalized to higher degrees (compare a similar situation in [**Z1**], where such a proof is given).

The space $\widehat{\mathcal{B}}$ of all Blaschke products of the form (7.1) can be identified with the set of all unordered pairs $\{p,q\}$ of points outside the closed unit disk. This is homeomorphic to the symmetric product of two copies of the punctured plane. The latter can be identified with the space of all degree 2 monic polynomials

$$w \mapsto (w-w_1)(w-w_2) = w^2 - (w_1+w_2)w + w_1w_2$$

with $w_1w_2 \neq 0$. It follows that $\widehat{\mathcal{B}}$ is homeomorphic to $\mathbb{C}\times\mathbb{C}^*$. In particular, it is an open topological manifold of real dimension 4.

In the same way, we may consider the space $\mathcal{C}$ of all unordered pairs $\{c_1,c_2\}$ of points outside the closed unit disk, which has a completely similar description.

We consider the continuous map

$$\Psi : \mathcal{B} \to \mathcal{C}$$

which sends a normalized Blaschke product $B \simeq \{p,q\}$ with critical points $\{0,\infty,c_1,c_2,\dfrac{1}{\overline{c_1}},\dfrac{1}{\overline{c_2}}\}$ to the unordered pair $\{c_1,c_2\}$.

Lemma 7.2. *Ψ is a proper map.*

Proof. Let $B_n \simeq \{p_n,q_n\}$ be a sequence of normalized Blaschke products in $\mathcal{B}$ which leaves every compact subset of $\mathcal{B}$. Then, a priori we have the following three possibilities:

- Some critical point of B_n accumulates on the unit circle, or
- After relabeling, p_n goes to ∞, or
- After relabeling, p_n accumulates on the unit circle (later we show that this cannot be the case; see Lemma 7.4).

In the first two cases, it is easy to see that $\Psi(B_n)$ leaves every compact subset of $\mathcal{C}$. In the third case, there is a subsequence of B_n which converges locally uniformly on $\mathbb{C} \setminus \mathbb{T}$ to a Blaschke product of degree < 5. It follows that the corresponding subsequence of $\Psi(B_n)$ has to leave every compact subset of $\mathcal{C}$. $\square$

Lemma 7.3. *Ψ is injective.*

Proof. Let A and B be two normalized Blaschke products in $\mathcal{B}$ with the same critical points $\{0, \infty, c_1, c_2, \frac{1}{\overline{c_1}}, \frac{1}{\overline{c_2}}\}$. Let

$$A : z \mapsto \tau_A z^3 \left(\frac{z - p_1}{1 - \overline{p_1}z}\right) \left(\frac{z - q_1}{1 - \overline{q_1}z}\right),$$

$$B : z \mapsto \tau_B z^3 \left(\frac{z - p_2}{1 - \overline{p_2}z}\right) \left(\frac{z - q_2}{1 - \overline{q_2}z}\right),$$

If $p_1 = p_2$ or $p_1 = q_2$, or if one of the critical points c_1, c_2 coincides with one of the zeros p_i, q_i, then a straightforward computation shows that $A = B$. So let us assume that $p_1 \neq p_2$ and $p_1 \neq q_2$ and consider the rational function

$$R(z) = \frac{A(z)}{B(z)}.$$

Clearly $\deg R = 4$ and hence R has 6 critical points counted with multiplicity. We have

$$A'(z) = (\text{const.})\frac{z^2 \prod(z - c_j)(1 - \overline{c_j}z)}{(1 - \overline{p_1}z)^2(1 - \overline{q_1}z)^2} \;,\; B'(z) = (\text{const.})\frac{z^2 \prod(z - c_j)(1 - \overline{c_j}z)}{(1 - \overline{p_2}z)^2(1 - \overline{q_2}z)^2}$$

from which it follows that

$$R'(z) = (\text{const.})\frac{1}{z} \prod(z - c_j)(1 - \overline{c_j}z) \left\{ \frac{\sum(-1)^j(z - p_j)(z - q_j)(1 - \overline{p_j}z)(1 - \overline{q_j}z)}{(z - p_2)^2(z - q_2)^2(1 - \overline{p_1}z)^2(1 - \overline{q_1}z)^2} \right\}.$$

(Note that all the sums and products are taken over $j = 1, 2$.) From the above expression, R has already 4 critical points at the c_j and $1/\overline{c_j}$. So the rational function in the braces could have at most 2 zeros. Since this fraction is irreducible (by our assumption $p_1 \neq p_2$ and $p_1 \neq q_2$), the numerator should have degree ≤ 2. But that implies

$$p_1 q_1 = p_2 q_2,$$

$$\overline{p_1}(1+|q_1|^2)+\overline{q_1}(1+|p_1|^2)=\overline{p_2}(1+|q_2|^2)+\overline{q_2}(1+|p_2|^2)$$

from which it follows that $p_1 = p_2$ or $p_1 = q_2$, hence $q_1 = q_2$ or $q_1 = p_2$, which contradicts our assumption. □

Proof of Theorem 7.1 (Critical Parametrization). By Lemma 7.2 and Lemma 7.3, Ψ is a covering map of degree 1. Hence, it is a homeomorphism $\mathcal{B} \xrightarrow{\simeq} \mathcal{C}$. □

In particular, the theorem shows that $\mathcal{B}$ is also homeomorphic to the product $\mathbb{C} \times \mathbb{C}^*$.

Lemma 7.4. *Let* $B : z \mapsto \tau z^3\,(z-p)(z-q)/((1-\overline{p}z)(1-\overline{q}z))$ *be any normalized Blaschke product in* $\mathcal{B}$. *Then* $|p| > 2$ *and* $|q| > 2$.

Proof. Write $B(z) = \rho z^3/R(z)$, where $|\rho| = 1$ and

$$R(z) = \left(\frac{z-\alpha}{1-\overline{\alpha}z}\right)\left(\frac{z-\beta}{1-\overline{\beta}z}\right)$$

is a degree 2 Blaschke product preserving the unit disk having zeros at $\alpha = 1/\overline{p}$ and $\beta = 1/\overline{q}$. We look at the logarithmic derivative $LD(z) = d(\log R(z))/d(\log z) = zR'(z)/R(z)$ on the unit circle $\mathbb{T}$. A brief computation shows that for $z \in \mathbb{T}$,

$$LD(z) = \frac{1-|\alpha|^2}{|z-\alpha|^2} + \frac{1-|\beta|^2}{|z-\beta|^2},$$

which is strictly positive. It is easy to see that

$$\max_{z\in\mathbb{T}} LD(z) \geq \max\left\{\frac{1+|\alpha|}{1-|\alpha|}, \frac{1+|\beta|}{1-|\beta|}\right\}.$$

Hence if either $|\alpha| > 1/2$ or $|\beta| > 1/2$, the maximum value of LD on $\mathbb{T}$ will be greater than 3. On the other hand, R induces a 2-to-1 covering map of the unit circle, so the average value of $|R'| = LD$ on $\mathbb{T}$ will be 2. Putting these two facts together, it follows that if $|\alpha| > 1/2$ or $|\beta| > 1/2$, then

$$\min_{z\in\mathbb{T}} LD(z) \leq 2 < 3 < \max_{z\in\mathbb{T}} LD(z).$$

This simply implies that when $|\alpha| > 1/2$ or $|\beta| > 1/2$, there are at least two points on $\mathbb{T}$ where LD takes on the value 3. Now $B(z) = \rho z^3/R(z)$ gives

$$B'(z) = \rho \frac{3z^2 R(z) - z^3 R'(z)}{R(z)^2} = \rho z^2 \frac{3 - LD(z)}{R(z)}.$$

Hence by the above argument, B has at least two critical points on the unit circle as soon as $|p| < 2$ or $|q| < 2$. Certainly this cannot happen since by definition $B \in \mathcal{B}$ means the critical points of B are off the unit circle. □

Corollary 7.5. *Given any two points c_1 and c_2 in the plane, with $|c_1| \geq 1$ and $|c_2| \geq 1$, there exists a unique normalized Blaschke product B in the closure $\overline{\mathcal{B}}$ with critical points $\{0, \infty, c_1, c_2, \frac{1}{\overline{c_1}}, \frac{1}{\overline{c_2}}\}$.*

In other words, critical parametrization is possible even if one or both critical points c_1, c_2 belong to the unit circle.

Proof. Take a sequence $\{c_1^n, c_2^n\}$ of pairs of points outside the closed unit disk such that $c_1^n \to c_1$ and $c_2^n \to c_2$ as $n \to \infty$. The zeros p_n, q_n of the corresponding normalized Blaschke products $\Psi^{-1}(\{c_1^n, c_2^n\})$ stay away from the unit circle by Lemma 7.4. Therefore, $\Psi^{-1}(\{c_1^n, c_2^n\})$ has a subsequence which converges to a normalized Blaschke product which, by continuity of Ψ, has critical points at $\{0, \infty, c_1, c_2, \frac{1}{\overline{c_1}}, \frac{1}{\overline{c_2}}\}$.

To see uniqueness, it is enough to note that the proof of Lemma 7.3 can be repeated word by word even if we assume $|c_1| = 1$ or $|c_2| = 1$. □

We conclude with the following proposition, the proof of which is quite straightforward.

Proposition 7.6. *Every $B \in \mathcal{B}$ induces a real-analytic diffeomorphism of the unit circle. Consequently, if $B \in \overline{\mathcal{B}} \setminus \mathcal{B}$, the restriction of B to the unit circle will be a real-analytic homeomorphism with one (or two) critical point(s).*

8. A Blaschke Parameter Space

Now we focus on a certain class of degree 5 Blaschke products. These are the maps B with the following two properties:

(i) B has the form

$$B : z \mapsto e^{2\pi i t} z^3 \left(\frac{z-p}{1-\overline{p}z} \right) \left(\frac{z-q}{1-\overline{q}z} \right), \qquad |p|>1, |q|>1 \tag{8.1}$$

where p and q are chosen such that B has a double critical point on the unit circle $\mathbb{T}$ and a pair $(c, 1/\overline{c})$ of symmetric critical points which may or may not be on $\mathbb{T}$.

(ii) t is the unique number in $[0,1]$ for which the rotation number of $B|_{\mathbb{T}}$ is equal to θ, with $0<\theta<1$ being a given irrational number.

The number t in (ii) is unique because the rotation number of B in (8.1) is a continuous and increasing function of t which is strictly increasing at all irrational values (see for example [**KH**], Proposition 11.1.9).

From the above description, it follows that every B which satisfies (i) and (ii) can be represented as a normalized Blaschke product in $\overline{\mathcal{B}} \smallsetminus \mathcal{B}$ followed by a unique rotation which adjusts the rotation number to θ. As a consequence, Corollary 7.5 shows that every such B is uniquely determined by the position of its critical points.

The rotation group **rot**$= \{R_\rho : z \mapsto \rho z \ \text{ with } \ |\rho|=1\}$ acts on the set of all such Blaschke products by conjugation. In fact,

$$R_\rho^{-1} \circ B \circ R_\rho : z \mapsto e^{2\pi i t} \rho^4 z^3 \left(\frac{z-p\overline{\rho}}{1-\overline{p}\rho z} \right) \left(\frac{z-q\overline{\rho}}{1-\overline{q}\rho z} \right).$$

We would like to understand the topology of the space $\mathcal{B}_5^{cm}(\theta)$ of all "critically marked" Blaschke products satisfying (i) and (ii) modulo the action of **rot**. Here by a marking of the critical points of such a Blaschke product B we mean a surjective function $\mathbf{m}$ from the set $\{1,2\}$ to the set of finite critical points of B outside the open unit disk. Two critically marked Blaschke products $(B, \mathbf{m})$

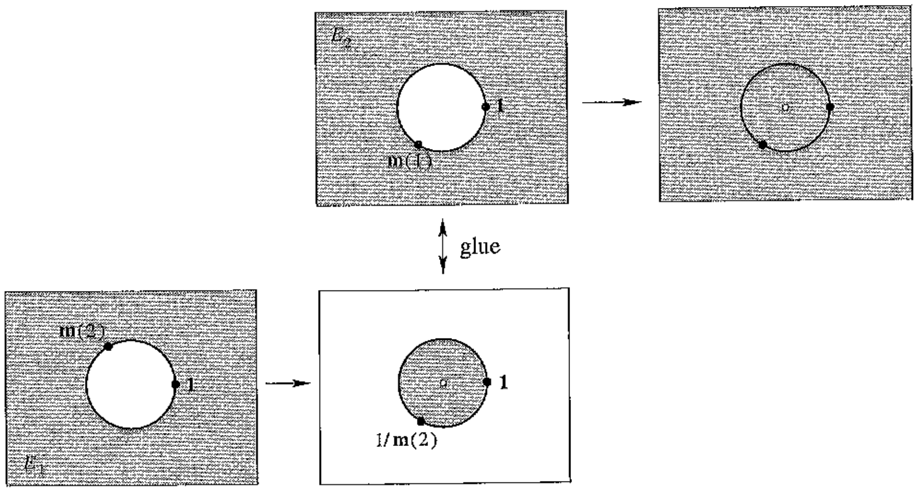

FIGURE 11. *Topology of the parameter space* $\mathcal{B}_5^{cm}(\theta)$.

and $(A, \mathbf{m}')$ are equivalent under the action of **rot** if there exists an R_ρ such that $R_\rho \circ B = A \circ R_\rho$ and $\mathbf{m}' = R_\rho \circ \mathbf{m}$.

Here is how we parametrize the space $\mathcal{B}_5^{cm}(\theta)$: For $j = 1, 2$, consider the closed set E_j consisting of all conjugacy classes in $\mathcal{B}_5^{cm}(\theta)$ for which the critical point $\mathbf{m}(j)$ belongs to the unit circle. In each class in E_1, we choose the unique representative $(B, \mathbf{m})$ for which $\mathbf{m}(1) = 1$. It follows from Corollary 7.5 that E_1 can be parametrized by the location of the second critical point $\mathbf{m}(2) \in \mathbb{C} \smallsetminus \mathbb{D}$. Similarly, in each class in E_2, pick up the unique representative $(B, \mathbf{m})$ for which $\mathbf{m}(2) = 1$. This shows that E_2 can be parametrized by the location of the first critical point $\mathbf{m}(1) \in \mathbb{C} \smallsetminus \mathbb{D}$. Now on the common boundary $E_1 \cap E_2$, consisting of all Blaschke products with two double critical points on $\mathbb{T}$, we have two different coordinates which must correspond to the same conjugacy class. This simply yields the identification $\mathbf{m}(1) = 1/\mathbf{m}(2)$ between the two copies of $\mathbb{C} \smallsetminus \mathbb{D}$ along their boundary circles. Consequently, $\mathcal{B}_5^{cm}(\theta)$ can be identified with the punctured plane (see Fig. 11).

It is easy to see that this gluing corresponds to choosing the uniformizing parameter $\mu = \mathbf{m}(1)/\mathbf{m}(2) \in \mathbb{C}^*$ for the space $\mathcal{B}_5^{cm}(\theta)$. Here is the concrete interpretation of this identification $\mathcal{B}_5^{cm}(\theta) \simeq \mathbb{C}^*$: For $\mu \in \mathbb{C}^*$ with $|\mu| \geq 1$, the corresponding Blaschke product B_μ has marked critical points at $\mathbf{m}(1) = \mu, \mathbf{m}(2) = 1$. Similarly, if $|\mu| \leq 1$, B_μ is the unique Blaschke product with marked critical points at $\mathbf{m}(1) = 1, \mathbf{m}(2) = 1/\mu$. Note that $B_\mu = B_{1/\mu}$ as maps, if we forget the markings of the critical points.

As in the case of the cubic parameter space $\mathcal{P}_3^{cm}(\theta)$, the Blaschke space $\mathcal{B}_5^{cm}(\theta)$ also has two very special points: $\mu = 1$ which corresponds to the conjugacy class of Blaschke products with a critical point of local degree 5 on $\mathbb{T}$, and $\mu = -1$, which corresponds to the conjugacy class of Blaschke products with two centered double critical points on $\mathbb{T}$.

The identification with $\mathbb{C}^*$ puts the following topology on $\mathcal{B}_5^{cm}(\theta)$: If $|\mu| \neq 1$ so that B_μ has only one double critical point on $\mathbb{T}$, then $B_{\mu_n} \to B_\mu$ simply means uniform convergence on compact subsets of the plane respecting the convergence of the marked critical points. On the other hand, if $|\mu| = 1$ so that B_μ has two double critical points on the unit circle, then $B_{\mu_n} \to B_\mu$ means that in the topology of local uniform convergence, $\{B_{\mu_n}\}$ can only accumulate on B_μ or its conjugate $R_\mu^{-1} \circ B_\mu \circ R_\mu$.

For the future reference, we need to analyze the structure of the invariant set $\bigcup_{k \geq 0} B^{-k}(\mathbb{T})$ for a Blaschke product $B \in \mathcal{B}_5^{cm}(\theta)$. For similar descriptions in a family of degree 3 Blaschke products, see [**Pe**] or [**YZ**].

Definition (Skeletons). Let $B \in \mathcal{B}_5^{cm}(\theta)$. Define $T_0 = \mathbb{T}$ and $T_1 = \overline{B^{-1}(T_0) \smallsetminus T_0}$. In general, for $k \geq 2$ we define T_k inductively as $T_k = B^{-1}(T_{k-1})$. We call the closed set T_k the *k-skeleton* of B. Note that B commutes with the reflection $I : z \mapsto 1/\overline{z}$. Therefore, every T_k is invariant under I.

Fig. 12 shows different possibilities for the 1-skeleton of a $B \in \mathcal{B}_5^{cm}(\theta)$.

The next proposition gives basic properties of k-skeletons. The proofs are straightforward and will be omitted.

Proposition 8.1 (Structure of the k-Skeleton).

(a) *For $k \geq 1$, the k-skeleton T_k is the union of finitely many piecewise analytic Jordan curves $\{T_k^1, \cdots, T_k^m\}$ which intersect one another at finitely many points and do not cross the unit circle $\mathbb{T}$. None of the T_k^i encloses $\mathbb{T}$. For any T_k^i in this family, the reflected copy $I(T_k^i)$ also belongs to this family.*

(b) *With the notation of (a), let D_k^i denote the bounded component of $\mathbb{C} \smallsetminus T_k^i$ for $k \geq 1$. For $k = 0$, D_0^i could mean either $\mathbb{D}$ or $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. Then for $k \geq 1$, B maps D_k^i onto some D_{k-1}^j. The mapping is either a conformal isomorphism or a 2-to-1 branched covering. As a result, $B^{\circ k}$ is a proper holomorphic map from D_k^i onto $\mathbb{D}$ or $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$.*

(c) *If $k \geq 1$ and $i \neq j$, we have $D_k^i \cap D_k^j = \emptyset$.*

(d) *For $k > \ell \geq 1$, either D_k^i and D_ℓ^j are disjoint or $D_k^i \subset D_\ell^j$. Conversely, if $D_k^i \subset D_\ell^j$, we necessarily have $k \geq \ell$.*

Every D_k^i is called a k-*drop* or simply a *drop* of B. In other words, k-drops are the open topological disks bounded by the Jordan curves in the decomposition of the k-skeleton of B. For $k = 0$, we have slightly changed the notion of drops. The unit circle $\mathbb{T}$ is the only Jordan curve in the 0-skeleton of B, *and we agree to call any of the two topological disks $\mathbb{D}$ or $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ a 0-drop.* The integer k is called the *depth* of D_k^i.

Definition (Nucleus of a Drop). Let D_k^i be a drop. We define the *nucleus* N_k^i of D_k^i as the set of all points in D_k^i which are not accumulated by any other drop of B. The nuclei of k-drops are said to have depth k.

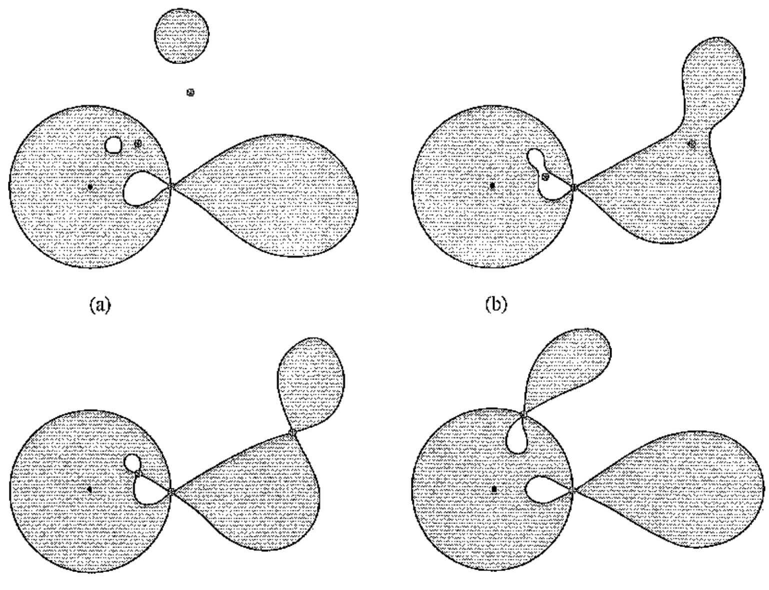

FIGURE 12. *Four different configurations for $B^{-1}(\mathbb{T})$, where $B \in \mathcal{B}_5^{cm}(\theta)$. The shaded regions are components of $B^{-1}(\mathbb{D})$. The shaded subregion of $\mathbb{D}$ is mapped to $\mathbb{D}$ by a 3-to-1 branched covering with a superattracting fixed point at the origin. There is a critical point at $z = 1$ and the other critical point(s) (marked by an asterisk) are symmetric with respect to the unit circle.*

It follows from Proposition 8.1(c) that

$$N_k^i = D_k^i \smallsetminus \overline{\bigcup_{\ell \neq k} \bigcup_j D_\ell^j}.$$

Clearly every nucleus is open. It is also non-empty because every drop contains an open set which eventually maps to the immediate basin of attraction of 0 or ∞, and this open set cannot intersect the closure of any other drop of B.

We have two nuclei of depth zero: N_0, which is the nucleus of $\mathbb{D}$ and contains the immediate basin of attraction of 0, and N_∞, which is the nucleus of $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ and contains the immediate basin of attraction of ∞. Obviously $N_\infty = I(N_0)$.

It is not hard to see that both N_0 and N_∞ are invariant under B:

$$B(N_0) \subset N_0, \qquad B(N_\infty) \subset N_\infty. \tag{8.2}$$

This of course implies that N_0 and N_∞ are subsets of the Fatou set of B.

It follows from Proposition 8.1(b) that B maps every nucleus of depth k onto some nucleus of depth $k-1$ and the mapping is either a conformal isomorphism or a 2-to-1 branched covering. We include the following lemma for completeness:

Lemma 8.2. *Let N_k^i be the nucleus of a drop D_k^i which eventually maps to the unit disk $\mathbb{D}$. Then*

(a) *No point in the orbit*

$$N_k^i = N_k^{i_0} \xrightarrow{B} N_{k-1}^{i_1} \xrightarrow{B} \cdots \xrightarrow{B} N_1^{i_{k-1}} \xrightarrow{B} N_0$$

can intersect any of the reflected nuclei $I(N_{k-j}^{i_j})$, $0 \leq j \leq k$.

(b) *For $z \in N_k^i$, $B^{\circ k}$ is the first iterate of B which sends z to N_0.*

Proof. (a) B commutes with I, so there is a reflected orbit

$$I(N_k^i) = I(N_k^{i_0}) \xrightarrow{B} I(N_{k-1}^{i_1}) \xrightarrow{B} \cdots \xrightarrow{B} I(N_1^{i_{k-1}}) \xrightarrow{B} N_\infty.$$

Now any point in both orbits would have to map to a point in N_0 and N_∞ simultaneously, which is impossible since $N_0 \cap N_\infty = \emptyset$.

(b) This is obvious if $k = 1$. Suppose that $k > 1$ and that for some $0 < \ell < k$, $B^{\circ \ell}(z) \in N_0$. Then by (8.2), $B^{\circ k-1}(z) \in N_0 \subset \mathbb{D}$. But $B^{\circ k-1}(z) \in B^{\circ k-1}(D_k^i)$ and $B^{\circ k-1}(D_k^i)$ is a 1-drop which does not intersect $\mathbb{D}$. $\square$

Remark. If $z \in N_k^i$, it is *not* true that $B^{\circ k}$ is the first iterate of B which sends z to the unit disk. In fact, the orbit of z can pass through $\mathbb{D}$ several times before it maps to N_0.

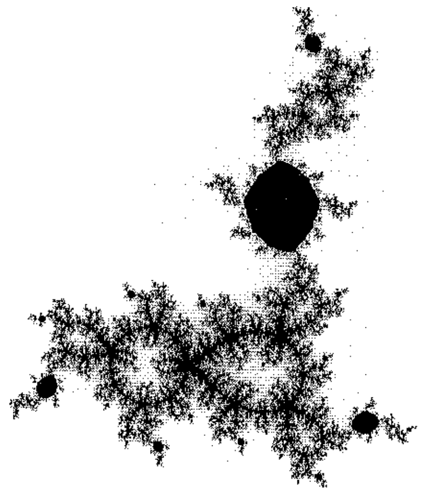

FIGURE 13

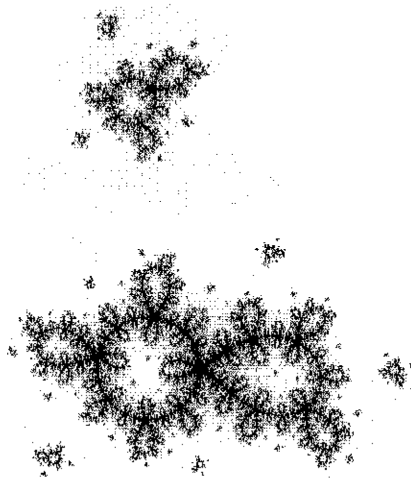

FIGURE 14

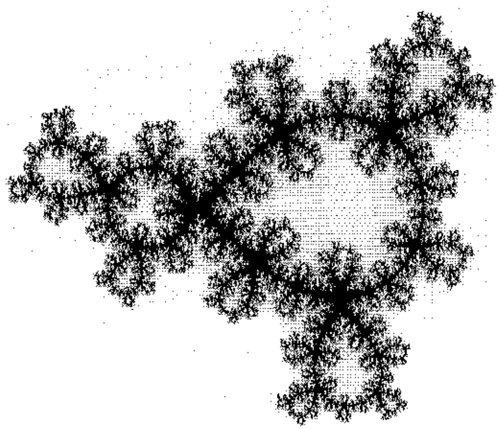

FIGURE 15

Proposition 8.3.

(a) *Distinct nuclei are disjoint.*

(b) *The map $B^{\circ k}$ from N_k^i onto N_0 or N_∞ is either a conformal isomorphism or a 2-to-1 branched covering.*

Proof. (a) Let N_k^i and N_ℓ^j be two distinct nuclei which intersect. By Proposition 8.1(c), we have $k \neq \ell$. Without loss of generality, we assume that $k > \ell$ and the iterate $B^{\circ\ell}$ maps N_ℓ^j onto N_0. So for every z in the intersection $N_k^i \cap N_\ell^j$, $B^{\circ\ell}(z)$ will belong to N_0. This contradicts Lemma 8.2(b).

(b) Since by (a) distinct nuclei are disjoint, an orbit

$$N_k^i = N_k^{i_0} \xrightarrow{B} N_{k-1}^{i_1} \xrightarrow{B} \cdots \xrightarrow{B} N_1^{i_{k-1}} \xrightarrow{B} N_0 \text{ or } N_\infty$$

can hit every critical point of B at most once. Since the critical point $z = 1$ of B does not belong to any nucleus, the above orbit can only hit the pair of critical points c and $1/\bar{c}$, with $|c| \neq 1$. By Lemma 8.2(a), these critical points cannot belong to the above orbit simultaneously. This means that $B^{\circ k} : N_k^i \to N_0$ or N_∞ is either a conformal isomorphism or a 2-to-1 branched covering. □

Fig. 13-Fig. 15 show the Julia sets of some Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ for $\theta = (\sqrt{5} - 1)/2$. In Fig. 13 there are two symmetric attracting cycles in the nuclei N_0 and N_∞ whose basins of attraction consist of the topological disks in black. Fig. 14 shows the Julia set of a map outside of the connected locus $\mathcal{C}_5(\theta)$ (see Section 10). In Fig. 15 there is a critical point in the nucleus of the large 1-drop attached to the unit disk at $z = 1$ which maps into N_0. Hence this nucleus contains the zeros p and q. Surgery (see Section 9 below) will turn the first Blaschke product into a hyperbolic-like cubic, while sends the second to a cubic in Ω_{ext} and the last one to a capture cubic in $\mathcal{P}_3^{cm}(\theta)$.

9. The Surgery

For the rest of the paper, unless otherwise stated, *we assume that θ is an irrational number of bounded type.* We describe a surgery on Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ to obtain cubic polynomials in $\mathcal{P}_3^{cm}(\theta)$. A similar surgery has been done in the case of quadratic polynomials [**D2**] using the following theorem of Świątek and Herman (see [**Sw**] or [**H2**]). Recall that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is called *k-quasisymmetric*, or simply quasisymmetric, if

$$0 < k^{-1} \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq k < +\infty$$

for all x and all $t > 0$. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is k-quasisymmetric if its lift to $\mathbb{R}$ has this property.

Theorem 9.1 (Linearization of Critical Circle Maps). *Let $f : \mathbb{T} \to \mathbb{T}$ be a real-analytic homeomorphism with finitely many critical points and rotation number θ. Then there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which conjugates f to the rigid rotation $R_\theta : z \mapsto e^{2\pi i\theta}z$ if and only if θ is an irrational number of bounded type. Moreover, if f belongs to a compact family of real-analytic homeomorphisms with rotation number θ, then h is k-quasisymmetric, where the constant k only depends on the family and not on the choice of f.*

Let us briefly sketch what this surgery does on a Blaschke product $B \in \mathcal{B}_5^{cm}(\theta)$. By Proposition 7.6, the restriction $B|_{\mathbb{T}}$ is a real-analytic homeomorphism with one (or two) critical point(s). When the rotation number of this circle map is of bounded type, by Theorem 9.1 one can find a unique k-quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ with $h(1) = 1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{B} & \mathbb{T} \\ \Big\downarrow h & & \Big\downarrow h \\ \mathbb{T} & \xrightarrow{R_\theta} & \mathbb{T} \end{array}$$

Moreover, the family $\{B|_{\mathbb{T}}\}_{B\in\mathcal{B}_5^{cm}(\theta)}$ is compact (see Theorem 12.3), hence h is $k(\theta)$-quasisymmetric, where the constant $k(\theta)$ only depends on the family $\mathcal{B}_5^{cm}(\theta)$. We can extend h to a $K(\theta)$-quasiconformal homeomorphism $H : \mathbb{D} \to \mathbb{D}$ whose dilatation depends only on $k(\theta)$. Possible extensions are given by the theorem of Beurling and Ahlfors [**A**] or Douady and Earle [**DE**] (which has the advantage of being conformally invariant). Define a *modified Blaschke product* $\widetilde{B}$ as follows:

$$\widetilde{B}(z) = \begin{cases} B(z) & |z| \geq 1 \\ (H^{-1} \circ R_\theta \circ H)(z) & |z| < 1 \end{cases} \tag{9.1}$$

This amounts to cutting out the unit disk and gluing in a Siegel disk instead. Note that the two definitions match along $\mathbb{T}$ by the above commutative diagram. Now define a conformal structure σ on the plane as follows: On $\mathbb{D}$, let σ be the pull-back $H^*\sigma_0$ of the standard conformal structure σ_0. Since R_θ preserves σ_0, $\widetilde{B}$ will preserve σ on $\mathbb{D}$. For every $k \geq 1$, pull $\sigma|_{\mathbb{D}}$ back by $\widetilde{B}^{\circ k} = B^{\circ k}$ on $\widetilde{B}^{-k}(\mathbb{D}) \smallsetminus \mathbb{D}$ (which consists of all the *maximal* k-drops of B; see Section 10). Since $B^{\circ k}$ is holomorphic, this does not increase the dilatation of σ. Finally, let $\sigma = \sigma_0$ on the rest of the plane. By the construction, σ has bounded dilatation and is invariant under $\widetilde{B}$. Therefore, by the Measurable Riemann Mapping Theorem, we can find a quasiconformal homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi^*\sigma_0 = \sigma$. Set

$$P = \varphi \circ \widetilde{B} \circ \varphi^{-1}. \tag{9.2}$$

Then P is a quasiregular self-map of the sphere which preserves σ_0, hence it is holomorphic. Also P is proper of degree 3 since $\widetilde{B}$ has the same properties. Therefore P is a cubic polynomial. Evidently, $\varphi(\mathbb{D})$ is a Siegel disk for P whose boundary $\varphi(\mathbb{T})$ is a quasicircle passing through the critical point $\varphi(1)$.

To mark the critical points of P, hence getting an element of $\mathcal{P}_3^{cm}(\theta)$, we must normalize φ carefully. Recall from Section 8 that $\mathcal{B}_5^{cm}(\theta)$ is uniformized by the parameter $\mu \in \mathbb{C}^*$ as follows: If $|\mu| \geq 1$, B_μ has marked critical points at $\mathbf{m}(1) = \mu, \mathbf{m}(2) = 1$, while for $|\mu| \leq 1$, B_μ has marked critical points at $\mathbf{m}(1) = 1, \mathbf{m}(2) = 1/\mu$. In the first case, we normalize φ such that $\varphi(H^{-1}(0)) = 0$ and $\varphi(1) = 1$. Call $\varphi(\mu) = c$ and mark the critical points of P by declaring $P = P_c$ as in Section 2. In the case $|\mu| \leq 1$, we normalize φ similarly by putting $\varphi(H^{-1}(0)) = 0$ and $\varphi(1/\mu) = 1$, but this time we call $\varphi(1) = c$ and set $P = P_c$. It is easy to see that when $|\mu| = 1$, both normalizations produce the same critically marked cubic polynomial in $\mathcal{P}_3^{cm}(\theta)$.

Let us denote the polynomial P constructed this way by $\mathcal{S}_H(B)$. We will see that for two quasiconformal extensions H and H', the cubics $\mathcal{S}_H(B)$ and $\mathcal{S}_{H'}(B)$ are quasiconformally conjugate and the conjugacy is conformal everywhere except on the grand orbit of the Siegel disk centered at the origin. When $\mathcal{S}_H(B)$ is capture, we can certainly end up with two different cubics if we choose the extensions arbitrarily. In fact, let k be the first moment the orbit of the critical point c of B hits the unit disk, and let $w = B^{\circ k}(c)$. Then for two quasiconformal extensions H and H', the captured images of the critical points of $\mathcal{S}_H(B)$ and $\mathcal{S}_{H'}(B)$ have the same conformal position in their corresponding Siegel disks if and only if $H(w) = H'(w)$. It follows that $\mathcal{S}_H(B) \neq \mathcal{S}_{H'}(B)$ as soon as we choose two different extensions H, H' with $H(w) \neq H'(w)$.

The following proposition has a very non-trivial content in case the result of the surgery is a cubic whose Julia set has positive measure (say, in a queer component). It is the Bers Sewing Lemma which makes the proof work.

Proposition 9.2. *Let $P = \mathcal{S}_H(B)$ and H' be any other quasiconformal extension of the circle homeomorphism h which linearizes $B|_{\mathbb{T}}$. Then, if P is not capture, $\mathcal{S}_H(B) = \mathcal{S}_{H'}(B)$. On the other hand, when P is capture, $\mathcal{S}_H(B) = \mathcal{S}_{H'}(B)$ if and only if $H(w) = H'(w)$, where $w \in \mathbb{D}$ is the captured image of the critical point of B.*

Proof. Let $Q = \mathcal{S}_{H'}(B)$ and φ_H and $\varphi_{H'}$ denote the quasiconformal homeomorphisms which satisfy $P = \varphi_H \circ \widetilde{B}_H \circ \varphi_H^{-1}$ and $Q = \varphi_{H'} \circ \widetilde{B}_{H'} \circ \varphi_{H'}^{-1}$ as in (9.2). The homeomorphism φ defined by

$$\varphi(z) = \begin{cases} (\varphi_{H'} \circ \varphi_H^{-1})(z) & z \in \mathbb{C} \smallsetminus GO(\Delta_P) \\ (\varphi_{H'} \circ B^{-k} \circ H'^{-1} \circ H \circ B^{\circ k} \circ \varphi_H^{-1})(z) & z \in P^{-k}(\Delta_P) \end{cases}$$

is quasiconformal and conjugates P to Q. By Lemma 3.5, one can find a quasiconformal conjugacy $\psi : \mathbb{C} \to \mathbb{C}$ between P and Q which is conformal on the grand orbit $GO(\Delta_P)$ and agrees with φ everywhere else. By the Bers Sewing Lemma, $\overline{\partial}\psi = \overline{\partial}\varphi$ almost everywhere on $\mathbb{C} \smallsetminus GO(\Delta_P)$. But the latter generalized partial derivative vanishes almost everywhere on $\mathbb{C} \smallsetminus GO(\Delta_P)$ because the surgery does not change the conformal structures outside $\bigcup_{k\geq 0} \widetilde{B}^{-k}(\mathbb{D})$. Hence $\overline{\partial}\psi = 0$ almost everywhere on $\mathbb{C}$, which means ψ is conformal. This shows $P = Q$. $\square$

Convention. For the rest of this paper, we always choose the Douady-Earle extension of circle homeomorphisms to perform surgery. By the above proposition, this is really a "choice" only in the capture case. We can therefore neglect the dependence on H and call

$$\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$$

the *surgery map.*

As an immediate corollary of the normalization of φ and the construction of $\mathcal{S}$, we have the following:

Corollary 9.3. *Let $\mu \in \mathbb{C}^*$ and $P_c = \mathcal{S}(B_\mu)$ be the cubic obtained by performing the above surgery.*

- *If $|\mu| > 1$, then $1 \in \partial\Delta_c$ and $c \notin \partial\Delta_c$.*
- *If $|\mu| < 1$, then $c \in \partial\Delta_c$ and $1 \notin \partial\Delta_c$.*
- *If $|\mu| = 1$, then both c and $1 \in \partial\Delta_c$.*

10. THE BLASCHKE CONNECTEDNESS LOCUS $\mathcal{C}_5(\theta)$

Suggested by the case of cubic polynomials, we define the *Blaschke connectedness locus* $\mathcal{C}_5(\theta)$ by

$$\mathcal{C}_5(\theta) = \{B \in \mathcal{B}_5^{cm}(\theta) : \text{ The Julia set } J(B) \text{ is connected}\}.$$

The following theorem provides a useful characterization of $\mathcal{C}_5(\theta)$ in terms of the critical orbits.

Theorem 10.1. *$B \in \mathcal{C}_5(\theta)$ if and only if one of the following holds:*

- *The orbit of c, the critical point of B in $\mathbb{C} \smallsetminus \mathbb{D}$ other than 1, eventually hits $\overline{\mathbb{D}}$.*
- *The orbit of c never hits $\overline{\mathbb{D}}$, but remains bounded.*

The proof of this theorem depends on an alternative dynamical description for Julia sets of Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ which is obtained by taking pull-backs along a certain type of drops called maximal drops. This description will be useful later in the proof of Theorem 13.1.

Definition. Let D_k^i be a k-drop of $B \in \mathcal{B}_5^{cm}(\theta)$. We call D_k^i a *maximal drop* if $D_k^i = \mathbb{D}$, or if $D_k^i \cap \mathbb{D} = \emptyset$ and D_k^i is not contained in any other ℓ-drop of B for $\ell \geq 1$.

It follows in particular that maximal drops of B are disjoint.

Proposition 10.2. *Let $B \in \mathcal{B}_5^{cm}(\theta)$ and let $P = \mathcal{S}(B) = \varphi \circ \widetilde{B} \circ \varphi^{-1}$ as in (9.2). Then*

(a) *D_k^i is a maximal drop of B if and only if $\varphi(D_k^i)$ is a Fatou component of P which eventually maps to the Siegel disk Δ_P.*

(b) *φ maps the nucleus N_∞ of B onto $\widehat{\mathbb{C}} \smallsetminus \overline{GO(\Delta_P)}$.*

(c) *The boundary of the immediate basin of attraction of infinity for B is precisely the closure of the union of the boundaries of all maximal drops of B. Under φ this set maps to the Julia set $J(P)$.*

Proof. (a) and (b) are easy consequences of the definitions. For (c), just note that under φ, the boundary of the immediate basin of attraction of infinity for B corresponds to the similar boundary for P, and the closure of the union of the boundaries of all maximal drops of B corresponds to the Julia set $J(P)$ by (a). □

Lemma 10.3 (Alternative description for Julia Sets). *Let $B \in \mathcal{B}_5^{cm}(\theta)$ and let J_0 be the boundary of the immediate basin of attraction of infinity for B. Define a sequence of compact sets $J_n = J_n(B)$ inductively by*

$$J_n = \overline{\bigcup_{D_k^i \text{ maximal}} B^{-k}(IJ_{n-1} \cap \mathbb{D}) \cap D_k^i}, \tag{10.1}$$

Then

$$J(B) = \overline{\bigcup_{n \geq 0} J_n}. \tag{10.2}$$

Proof. Each J_n is compact and contained in $J(B)$. By Lemma 10.2(c), $J_0 \subset J_1$ and it follows by induction on n that $J_n \subset J_{n+1}$ for $n \geq 0$. Put

$$J_\infty = \overline{\bigcup_{n \geq 0} J_n}.$$

Clearly J_∞ is compact and contained in the Julia set $J(B)$, and it is not hard to see that it is invariant under the reflection I. We will show that J_∞ is totally invariant under B, i.e., $B^{-1}(J_\infty) = J_\infty$. This will prove that $J_\infty = J(B)$.

First we prove that J_∞ is forward invariant. For any n, it follows from (10.1) that $B(J_n \setminus \mathbb{D}) \subset J_n \subset J_\infty$. On the other hand, $B(J_n \cap \overline{\mathbb{D}}) = B(IJ_{n-1} \cap \overline{\mathbb{D}}) = IB(J_{n-1} \setminus \mathbb{D}) \subset IJ_\infty = J_\infty$. These two inclusions show that $B(J_n) \subset J_\infty$, hence $B(J_\infty) \subset J_\infty$.

To prove backward invariance, first note that for any n, $B^{-1}(J_n) \setminus \mathbb{D} \subset J_n \subset J_\infty$ by (10.1). To obtain the same kind of inclusion for $B^{-1}(J_n) \cap \overline{\mathbb{D}}$, we distinguish two cases: First, $B^{-1}(J_n \cap \overline{\mathbb{D}}) \cap \overline{\mathbb{D}} = B^{-1}(IJ_{n-1} \cap \overline{\mathbb{D}}) \cap \overline{\mathbb{D}} \subset I(B^{-1}(J_{n-1} \setminus \mathbb{D})) \subset IJ_{n-1} \cup J_n \subset J_\infty$. Second, $B^{-1}(J_n \setminus \mathbb{D}) \cap \overline{\mathbb{D}} = I(B^{-1}(IJ_n \cap \overline{\mathbb{D}}) \setminus \mathbb{D}) \subset I(B^{-1}(J_{n+1}) \setminus \mathbb{D}) \subset IJ_{n+1} \subset J_\infty$. Altogether, these three inclusions show that $B^{-1}(J_n) \subset J_\infty$ for all n. Hence $B^{-1}(J_\infty) \subset J_\infty$ and this proves (10.2). □

Proof of Theorem 10.1. One direction is quite easy to see: If the orbit of c never hits the closed unit disk and escapes to infinity, one can easily show that $J(B)$ is disconnected in a way identical to the polynomial case by considering the Böttcher map of the immediate basin of attraction of ∞ for B (see for example [**M1**], Theorem 17.3). Conversely, suppose that the orbit of the critical point c either hits $\overline{\mathbb{D}}$ or stays bounded in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then the Julia set $J(P)$ is connected, where $P = \mathcal{S}(B)$. Consider the sequence of compact sets J_n in (10.1). By Proposition 10.2(c), J_0 is connected and it follows by induction on n that each J_n defined by (10.1) is connected. Therefore (10.2) shows that $J(B)$ is connected. Hence $B \in \mathcal{C}_5(\theta)$. □

In what follows, we prove that the connectedness locus $\mathcal{C}_5(\theta)$ is compact. Other facts, e.g. having only two complementary components, or connectivity, will be proved later using surgery (see Corollary 13.4 and Corollary 13.5). We would like to remark that unlike the case of cubic polynomials, it is often difficult to prove anything about the topology of the Blaschke connectedness locus, partly because of the complicated way these Blaschke products depend on their critical points, but more importantly because of the fact that the family $\mu \mapsto B_\mu$ does not depend holomorphically on μ.

Lemma 10.4. *Let $\{B_{\mu_n}\}$ be an arbitrary sequence of Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ and $h_n : \mathbb{T} \to \mathbb{T}$ be the unique normalized quasisymmetric homeomorphism which conjugates $B_{\mu_n}|_{\mathbb{T}}$ to the rigid rotation R_θ. Let H_n denote the Douady-Earle extension of h_n. Then the sequence $\{H_n\}$ has a subsequence which converges locally uniformly to a quasiconformal homeomorphism of $\mathbb{D}$.*

It follows in particular that the sequence $\{H_n^{-1}(0)\}$ stays in a compact subset of the unit disk.

Proof. This follows from the facts that the space of all uniformly quasisymmetric normalized homeomorphisms of the circle is compact ([**Le**], Lemma 5.1) and the Douady-Earle extension depends continuously on the circle homeomorphism [**DE**]. □

Corollary 10.5. *Let $B \in \mathcal{B}_5^{cm}(\theta)$ and $\varphi_B : \mathbb{C} \to \mathbb{C}$ be the quasiconformal homeomorphism which conjugates the modified Blaschke product $\widetilde{B}$ to the cubic $P = \mathcal{S}(B)$ as in (9.2): $P = \varphi_B \circ \widetilde{B} \circ \varphi_B^{-1}$. Then the family $\mathcal{F} = \{\varphi_B\}_{B \in \mathcal{B}_5^{cm}(\theta)}$ is normal.*

Proof. By the surgery construction as described in Section 9, $\mathcal{F}$ is uniformly quasiconformal. Choose a sequence $\{B_{\mu_n}\}$ in $\mathcal{B}_5^{cm}(\theta)$ and let $\varphi_n = \varphi_{B_{\mu_n}}$ denote the corresponding sequence in $\mathcal{F}$. Choose a subsequence, still denoted by B_{μ_n}, such that $|\mu_n| \geq 1$ for all n (the case $|\mu_n| \leq 1$ is similar). By the way we normalized φ_n,

$$\varphi_n(H_n^{-1}(0)) = 0, \quad \varphi_n(1) = 1, \quad \varphi_n(\infty) = \infty.$$

But $\{H_n^{-1}(0)\}$ lives in a compact subset of $\mathbb{D}$ by the previous lemma. Hence the three points $H_n^{-1}(0)$, 1 and ∞ has mutual spherical distance larger than some positive constant independent of n. This implies equicontinuity of $\{\varphi_n\}$ by a standard theorem on quasiconformal mappings ([**Le**], Theorem 2.1). □

Proposition 10.6. *The surgery map $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ is proper.*

Proof. Let the sequence $\{B_{\mu_n}\}$ leave every compact set in $\mathcal{B}_5^{cm}(\theta)$ and consider the corresponding cubics $P_{c_n} = \mathcal{S}(B_{\mu_n}) = \varphi_n \circ \widetilde{B}_{\mu_n} \circ \varphi_n^{-1}$. To be more specific, let us assume that the critical point μ_n tends to infinity. Clearly $c_n = \varphi_n(\mu_n)$. Since $\{\varphi_n\}$ is normal by the above corollary, we simply conclude that $c_n \to \infty$. □

Proposition 10.7. *The Blaschke connectedness locus $\mathcal{C}_5(\theta)$ is compact and invariant under $\mu \mapsto 1/\mu$. As a result, there exists an unbounded component Λ_{ext} of $\mathbb{C}^* \smallsetminus \mathcal{C}_5(\theta)$ which contains a punctured neighborhood of ∞ and a corresponding component Λ_{int} which is mapped to it by $\mu \mapsto 1/\mu$.*

Proof. The invariance follows from the definition of $\mathcal{B}_5^{cm}(\theta)$ and its identification with $\mathbb{C}^*$. Note that the unit circle $\mathbb{T} \subset \mathcal{B}_5^{cm}(\theta)$ is contained in $\mathcal{C}_5(\theta)$ by Theorem 10.1. So Λ_{ext} and Λ_{int} are actually distinct components of $\mathbb{C}^* \smallsetminus \mathcal{C}_5(\theta)$.

$\mathcal{C}_5(\theta)$ is clearly closed by Theorem 10.1. Let us prove it is bounded. Assuming the contrary, there is a sequence $B_{\mu_n} \in \mathcal{C}_5(\theta)$ with $\mu_n \to \infty$ as in the above proof. It follows from Proposition 10.2(c) and Theorem 10.1 that the corresponding polynomials $P_{c_n} = \mathcal{S}(B_{\mu_n}) = \varphi_n \circ \widetilde{B}_{\mu_n} \circ \varphi_n^{-1}$ have connected Julia sets. By Proposition 2.3, $1/30 \leq |c_n| \leq 30$. This contradicts properness of $\mathcal{S}$. □

11. Continuity of the Surgery Map

This section is devoted to the proof of continuity of the surgery map $\mathcal{S}$ which depends strongly on the cubic parameter space being one-dimensional. We point out that the situation is similar to Douady-Hubbard's proof of the continuity of the "straightening map" in their study of the space of quadratic-like maps [**DH2**]. One additional difficulty here is the lack of complete information on quasiconformal conjugacy classes in the non-holomorphic family $\mathcal{B}_5^{cm}(\theta)$ (the analogue of Theorem 5.5; see however Theorem 12.4).

The idea of the proof is as follows: Given a sequence $B_{\mu_n} \in \mathcal{B}_5^{cm}(\theta)$ such that $B_{\mu_n} \to B = B_\mu$, we prove that there exists a subsequence $\{B_{\mu_{n(j)}}\}$ such that $\mathcal{S}(B_{\mu_{n(j)}}) \to \mathcal{S}(B)$ in $\mathcal{P}_3^{cm}(\theta)$. The topology of the parameter space $\mathcal{P}_3^{cm}(\theta)$ is local uniform convergence respecting the markings of the critical points. The same is true for $\mathcal{B}_5^{cm}(\theta)$ with one exception (see Section 9): If μ has absolute value 1, i.e., if B has two double critical points on the unit circle, then $B_{\mu_n} \to B$ means that every subsequence of $\{B_{\mu_n}\}$ has a further subsequence which either converges locally uniformly to B or to its conjugate $R_\mu^{-1} \circ B \circ R_\mu$. From the construction of $\mathcal{S}$ it is easy to see that $\mathcal{S}(B) = \mathcal{S}(R_\mu^{-1} \circ B \circ R_\mu)$. Therefore, in order to prove continuity of $\mathcal{S}$, all we have to show is that $B_{\mu_n} \to B$ locally uniformly on $\mathbb{C}$ (respecting the markings of the critical points) implies that for some subsequence $\{B_{\mu_{n(j)}}\}$, $\mathcal{S}(B_{\mu_{n(j)}}) \to \mathcal{S}(B)$ locally uniformly on $\mathbb{C}$ (again, respecting the markings of the critical points).

So let h_n and h be the unique $k(\theta)$-quasisymmetric homeomorphisms which fix $z = 1$ and conjugate $B_{\mu_n}|_{\mathbb{T}}$ and $B|_{\mathbb{T}}$ to the rigid rotation R_θ. It is easy to see that $h_n \to h$ uniformly on $\mathbb{T}$. Consider the Douady-Earle extensions H_n and H, which are $K(\theta)$-quasiconformal homeomorphisms of the unit disk. By the construction of these extensions, H_n and H are real-analytic in $\mathbb{D}$ and $H_n \to H$ locally uniformly in C^∞ topology [**DE**]. In particular, the partial derivatives ∂H_n and $\overline{\partial} H_n$ converge locally uniformly in $\mathbb{D}$ to the corresponding derivatives ∂H and $\overline{\partial} H$. This shows that $\sigma_n|_{\mathbb{D}} \to \sigma|_{\mathbb{D}}$ locally uniformly, where $\sigma_n = H_n^* \sigma_0$ and $\sigma = H^* \sigma_0$ are the conformal structures we constructed in the course of surgery for B_{μ_n} and B (see Section 9).

At this point, the main problem is to prove that $B_{\mu_n} \to B$ and $\sigma_n|_{\mathbb{D}} \to \sigma|_{\mathbb{D}}$ implies $\sigma_n \to \sigma$ in the L^1-norm on $\mathbb{C}$, for this would show that the normalized solutions $\varphi_n = \varphi_{H_n}$ of the Beltrami equations $\varphi_n^* \sigma_0 = \sigma_n$ converge locally uniformly on $\mathbb{C}$ to the normalized solution φ of the equation $\varphi^* \sigma_0 = \sigma$. This would simply mean that $\mathcal{S}(B_{\mu_n}) \to \mathcal{S}(B)$ as $n \to \infty$.

Unfortunately, we cannot prove $\sigma_n \to \sigma$ in $L^1(\mathbb{C})$ in all cases. So, following [**DH2**], we take a slightly different approach by splitting the argument into two cases depending on whether $\mathcal{S}(B)$ is quasiconformally rigid or not. In the former case, we show continuity directly using the rigidity. In the latter case, however, we prove $\varphi_n \to \varphi$ using the fact that $\mathcal{S}(B)$ admits non-trivial deformations.

Theorem 11.1. *The surgery map* $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ *is continuous.*

Proof. Consider $B_{\mu_n}, B \in \mathcal{B}_5^{cm}(\theta)$ and start with the same construction as above to get a sequence $\{\sigma_n\}$ of conformal structures on the plane with uniformly bounded dilatation and the corresponding sequence $\{\varphi_n\}$ of normalized solutions of $\varphi_n^* \sigma_0 = \sigma_n$. Since $\{\varphi_n\}$ is a normal family by Corollary 10.5, it has a subsequence, still denoted by $\{\varphi_n\}$, which converges locally uniformly to a quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$.

Set $P_{c_n} = \varphi_n \circ \widetilde{B}_{\mu_n} \circ \varphi_n^{-1} = \mathcal{S}(B_{\mu_n})$, $P = \varphi \circ \widetilde{B} \circ \varphi^{-1} = \mathcal{S}(B)$, and $Q = \psi \circ \widetilde{B} \circ \psi^{-1}$. All these maps are cubic polynomials in $\mathcal{P}_3^{cm}(\theta)$. Also P is quasiconformally conjugate to Q, and $P_{c_n} \to Q$ as $n \to \infty$. We will show that $P = Q$ and this will prove continuity at B.

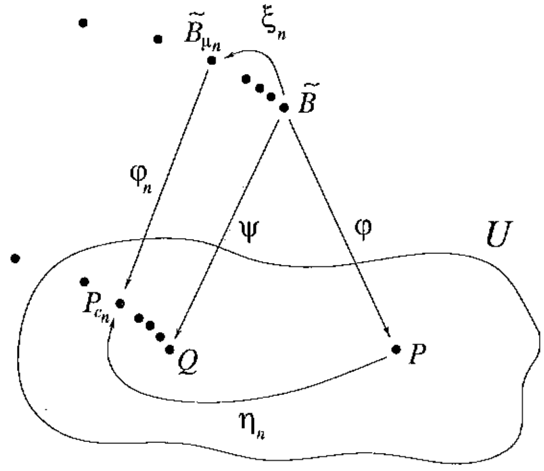

FIGURE 16. *Sketch of the proof of continuity of $\mathcal{S}$.*

For the rest of the argument, we distinguish two cases: If $P = \mathcal{S}(B)$ is quasiconformally rigid, then automatically $P = Q$ and we are done. Otherwise, P is not rigid, so the quasiconformal conjugacy class of P is a non-empty open set $U \subset \mathcal{P}_3^{cm}(\theta)$ by Corollary 5.2. Assume by way of contradiction that $P \neq Q$. Since $P_{c_n} \to Q$ as $n \to \infty$, $P_{c_n} \in U$ for large n. Hence P_{c_n} is quasiconformally conjugate to P for large n, i.e., there exists a normalized quasiconformal homeomorphism $\eta_n : \mathbb{C} \to \mathbb{C}$ such that $\eta_n \circ P = P_{c_n} \circ \eta_n$. Observe that the dilatation of η_n is uniformly bounded, since by Theorem 5.1 the dilatation of $(\psi \circ \varphi^{-1}) \circ \eta_n^{-1}$ goes to 1 as n goes to ∞ (see Fig. 16). By "lifting" η_n, we can find a quasiconformal conjugacy $\xi_n = \varphi_n^{-1} \circ \eta_n \circ \varphi$ between the modified Blaschke products $\widetilde{B}$ and $\widetilde{B}_{\mu_n}$, i.e.,

$$\xi_n \circ \widetilde{B} = \widetilde{B}_{\mu_n} \circ \xi_n. \tag{11.1}$$

Again, note that the dilatation of ξ_n is uniformly bounded.

We prove that the sequence of conformal structures $\{\sigma_n\}$ converges in $L^1(\mathbb{C})$ to σ. This, by a standard theorem on quasiconformal mappings (see for example [**Le**], Theorem 4.6) will show that $\varphi_n \to \varphi$ locally uniformly, hence $P_{c_n} \to P$, hence $P = Q$, which contradicts our assumption.

To this end, we introduce the following sequences of conformal structures (where, as usual, we identify a conformal structure with its associated Beltrami differential):

$$\sigma_n^k(z) = \begin{cases} \sigma_n(z) & \text{when } z \in \bigcup_{i=0}^{k} \widetilde{B}_n^{-i}(\mathbb{D}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sigma^k(z) = \begin{cases} \sigma(z) & \text{when } z \in \bigcup_{i=0}^{k} \widetilde{B}^{-i}(\mathbb{D}) \\ 0 & \text{otherwise} \end{cases}$$

Note that $\sigma^k \to \sigma$ in $L^1(\mathbb{C})$ as $k \to \infty$ and for every fixed k, $\sigma_n^k \to \sigma^k$ in $L^1(\mathbb{C})$ as $n \to \infty$.

Lemma 11.2. *The L^1-norm $\|\sigma_n - \sigma\|_1$ goes to zero as $n \to \infty$ if the area of the open set $\bigcup_{i=k}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D})$ goes to zero uniformly in n as $k \to \infty$.*

Proof. For a given $\epsilon > 0$, take k_0 so large that $k > k_0$ implies $\text{area}(\bigcup_{i=k}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D})) < \epsilon$ for all n. Then for a fixed large $k > k_0$ and n large enough,

$$\begin{aligned} \|\sigma_n - \sigma\|_1 &\leq \|\sigma_n - \sigma_n^k\|_1 + \|\sigma_n^k - \sigma^k\|_1 + \|\sigma^k - \sigma\|_1 \\ &\leq \|\sigma_n - \sigma_n^k\|_1 + 2\epsilon \\ &= \int_{\bigcup_{i=k+1}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D})} |\sigma_n| \, dxdy + 2\epsilon \\ &< 3\epsilon. \end{aligned}$$

This completes the proof of the lemma. □

So it remains to prove that the area of $\bigcup_{i=k}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D})$ goes to zero uniformly in n as $k \to \infty$. Clearly $\text{area}(\bigcup_{i=k}^{\infty} \widetilde{B}^{-i}(\mathbb{D})) \to 0$ as $k \to \infty$. Since $\{\xi_n\}$ is uniformly quasiconformal, there is a constant $C \geq 1$ such that

$$C^{-1} \ \text{area}(E) \leq \text{area}(\xi_n(E)) \leq C \ \text{area}(E)$$

for every n and every measurable set $E \subset \bigcup_{i=0}^{\infty} \widetilde{B}^{-i}(\mathbb{D})$. By (11.1),

$$\bigcup_{i=k}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D}) = \xi_n(\bigcup_{i=k}^{\infty} \widetilde{B}^{-i}(\mathbb{D})),$$

so $\text{area}(\bigcup_{i=k}^{\infty} \widetilde{B}_{\mu_n}^{-i}(\mathbb{D})) \leq C \ \text{area}(\bigcup_{i=k}^{\infty} \widetilde{B}^{-i}(\mathbb{D}))$ and this proves that the left side goes to zero uniformly in n. □

12. Renormalizable Blaschke Products

Here we consider those Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ from which one can "extract" the standard degree 3 Blaschke product f_θ to be defined below. The importance of this particular Blaschke product lies in the fact that it provides a model for the dynamics of the quadratic polynomial $Q_\theta : z \mapsto e^{2\pi i\theta} z + z^2$. It will be convenient to define renormalizable Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ as ones which after the surgery give rise to renormalizable cubics in $\mathcal{P}_3^{cm}(\theta)$ (see Section 4). In what follows we will have to work with a *symmetrized* version of the notion of a quadratic-like map in order to show that any renormalizable Blaschke product is quasiconformally conjugate near the Julia set of its renormalization to the standard map f_θ. The proof of this fact resembles the proof of [**DH2**] that every hybrid class of polynomial-like maps contains a polynomial.

First we include the following simple fact for completeness.

Proposition 12.1. *Let $0 < \theta < 1$ be a given irrational number and $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a degree 3 Blaschke product with a superattracting fixed point at the origin and a double critical point at $z = 1$. Let the rotation number of $f|_{\mathbb{T}}$ be θ. Then there exists a unique $0 < t(\theta) < 1$ such that*

$$f(z) = f_\theta(z) = e^{2\pi i t(\theta)} z^2 \left(\frac{z-3}{1-3z} \right). \tag{12.1}$$

Proof. Clearly $f(z) = e^{2\pi i t} z^2 \left(\dfrac{z-a}{1-\overline{a}z} \right)$, with $|a| > 1$ and $0 < t < 1$. The fact that $f'(1) = 0$ implies $a = 3$. Since the rotation number of $f|_{\mathbb{T}}$ as a function of t is continuous and strictly monotone at all irrational values, there exists a unique t for which this rotation number is θ. □

Remark. Computer experiments give the value $t(\theta) \approx 0.613648\cdots$ for the golden mean $\theta = (\sqrt{5}-1)/2$. Fig. 17 shows the Julia set of f_θ for this value of

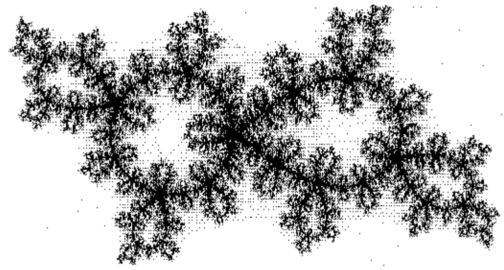

FIGURE 17. *Julia set of f_θ for $\theta = (\sqrt{5}-1)/2$.*

θ. This standard degree 3 Blaschke product was introduced by Douady, Ghys, Herman and Shishikura as a model for the quadratic $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ in the case θ is irrational of bounded type [**D2**]. It was also used by Petersen [**Pe**] to prove that the Julia set of Q_θ is locally-connected and has measure zero.

Definition. A Blaschke product $B \in \mathcal{B}_5^{cm}(\theta)$ is called *renormalizable* if $\mathcal{S}(B) \in \mathcal{P}_3^{cm}(\theta)$ is a renormalizable cubic, as defined in Section 4.

Theorem 12.2. *Let $B \in \mathcal{B}_5^{cm}(\theta)$ be renormalizable. Then there exists a pair of annuli $W' \Subset W$, both containing the unit circle and symmetric with respect to it, and a quasiconformal homeomorphism $\varphi_B : \mathbb{C} \to \mathbb{C}$ such that:*

(a) *$B : \partial W' \to \partial W$ is a degree 2 covering map,*

(b) *$\varphi_B \circ I = I \circ \varphi_B$,*

(c) *$(\varphi_B \circ B)(z) = (f_\theta \circ \varphi_B)(z)$ for all $z \in W'$.*

Moreover, one can arrange $\overline{\partial}\varphi_B = 0$ on $K(B) = \bigcap_{n\geq 0} B^{-n}(W')$.

Proof. Consider the cubic $P = \mathcal{S}(B) = \varphi \circ \widetilde{B} \circ \varphi^{-1} \in \mathcal{P}_3^{cm}(\theta)$ which is renormalizable. Consider the quadratic-like restriction $P|_U : U \to V$ and the corresponding regions $U_1 = \varphi^{-1}(U)$ and $V_1 = \varphi^{-1}(V)$. Clearly $U_1 \Subset V_1$ and both contain the closed unit disk. Define the symmetrized regions

$$W' = U_1 \cap I(U_1), \qquad W = V_1 \cap I(V_1)$$

which are topological annuli with $W' \Subset W$. Note that B sends $\partial W'$ to ∂W in a 2-to-1 fashion.

Now extend $B|_{W'}$ to the whole complex plane by gluing it to the polynomial $z \mapsto z^2$ near 0 and ∞ as follows: Let $r > 1$ and $\omega : \mathbb{C} \smallsetminus W' \to \mathbb{C} \smallsetminus \mathbb{A}(r^{-1}, r)$ be a diffeomorphism such that

$$\omega \circ I = I \circ \omega,$$
$$\omega(B(z)) = \omega(z)^2, \quad z \in \partial W'.$$

Define the extension of $B|_{W'}$ by

$$F(z) = \begin{cases} B(z) & z \in W' \\ \omega^{-1}(\omega(z)^2) & z \notin W' \end{cases}$$

Note that F is a quasiregular degree 3 self-map of the sphere, $F \circ I = I \circ F$, and every point outside W' will converge to 0 or ∞ under the iteration of F.

Define a conformal structure σ on the plane as follows: Put $\sigma = \omega^* \sigma_0$ on $\mathbb{C} \smallsetminus W'$, and pull it back by $F^{\circ n}$ to all the components of $F^{-n}(\mathbb{C} \smallsetminus W') \cap W'$. Finally, on $K(B)$ set $\sigma = \sigma_0$. It is easy to see that σ has bounded dilatation on the plane, is symmetric with respect to the unit circle, and $F^*(\sigma) = \sigma$. By the Measurable Riemann Mapping Theorem, there exists a unique quasiconformal homeomorphism φ_B of the plane which fixes $0, 1, \infty$, such that $\varphi_B^*(\sigma_0) = \sigma$. The conjugate map $f = \varphi_B \circ F \circ \varphi_B^{-1}$ is easily seen to be a degree 3 rational map on the sphere. The quasiconformal homeomorphism $I \circ \varphi_B \circ I$ also fixes $0, 1, \infty$ and pulls σ_0 back to σ because σ is symmetric with respect to $\mathbb{T}$. By uniqueness, $\varphi_B = I \circ \varphi_B \circ I$. This implies that f commutes with I, hence it is a Blaschke product. By Proposition 12.1, $f = f_\theta$, and we are done. □

While the above theorem establishes a direct connection between some Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ and f_θ, it is curious to note the following entirely different relation:

Theorem 12.3. *Let B_{μ_n} be any sequence in $\mathcal{B}_5^{cm}(\theta)$ such that $\mu_n \to \infty$ as $n \to \infty$. Then $B_{\mu_n} \to f_\theta$ locally uniformly on $\mathbb{C}^*$ as $n \to \infty$.*

In other words, f_θ can be regarded as the point at infinity of the parameter space $\mathcal{B}_5^{cm}(\theta)$.

Proof. As in Section 8, let

$$B_{\mu_n} : z \mapsto e^{2\pi i t_n} z^3 \left(\frac{z - p_n}{1 - \overline{p_n} z}\right)\left(\frac{z - q_n}{1 - \overline{q_n} z}\right).$$

The first and second logarithmic derivatives

$$\frac{B'_{\mu_n}}{B_{\mu_n}} \quad \text{and} \quad \frac{B_{\mu_n} B''_{\mu_n} - (B'_{\mu_n})^2}{(B_{\mu_n})^2}$$

both vanish at $z = 1$. A brief computation shows that these two conditions translate into

$$\frac{|p_n|^2 - 1}{|p_n - 1|^2} + \frac{|q_n|^2 - 1}{|q_n - 1|^2} = 3, \tag{12.2}$$

and

$$\frac{(p_n - \overline{p_n})(|p_n|^2 - 1)}{|p_n - 1|^4} + \frac{(q_n - \overline{q_n})(|q_n|^2 - 1)}{|q_n - 1|^4} = 0. \tag{12.3}$$

Since $\mu_n \to \infty$, both p_n and q_n cannot stay bounded. Hence, after relabeling, $p_n \to \infty$ (compare Theorem 7.1). Then (12.2) shows that $(|q_n|^2 - 1)/|q_n - 1|^2 \to 2$, or equivalently, $|q_n - 2| \to 1$ but q_n stays away from $z = 1$ by Lemma 7.4. On the other hand, (12.3) shows that $(q_n - \overline{q_n})(|q_n|^2 - 1)/|q_n - 1|^4 \to 0$, hence $(q_n - \overline{q_n})/|q_n - 1|^2 \to 0$. Since q_n does not accumulate on $z = 1$, this implies that $(q_n - \overline{q_n}) \to 0$. Near the circle $|z-2| = 1$ this can happen only if $q_n \to 3$. Since the rotation number depends continuously on the circle map, it is easy to see that $B_{\mu_n} \to f_\theta$ locally uniformly on $\mathbb{C}^*$. $\square$

Consider a sequence B_{μ_n} going off to infinity as in the previous theorem. Consider the cubics $P_{c_n} = \mathcal{S}(B_{\mu_n}) = \varphi_n \circ \widetilde{B}_{\mu_n} \circ \varphi_n^{-1}$ as in (9.2). By the previous theorem, $B_{\mu_n} \to f_\theta$ locally uniformly on $\mathbb{C}^*$, so $\widetilde{B}_{\mu_n} \to \widetilde{f}_\theta$ locally uniformly on $\mathbb{C}$. Here $\widetilde{f}_\theta$ denotes the modified Blaschke product for f_θ, defined in a way similar to (9.1). Since $\{\varphi_n\}$ is normal by Corollary 10.5, by passing to a subsequence if necessary, φ_n converges to a quasiconformal homeomorphism φ. Since the surgery map is proper by Proposition 10.6, $c_n \to \infty$. By examining the normal form (2.2), we see that $P_{c_n} \to Q$, where $Q : z \mapsto \lambda z(1 - 1/2z)$ is affinely conjugate to $Q_\theta : z \mapsto e^{2\pi i\theta} z + z^2$. Hence, $Q = \varphi \circ \widetilde{f}_\theta \circ \varphi^{-1}$ and we recover the surgery introduced by Douady and others. We conclude that the surgery map $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ extends continuously to the points at infinity of both parameter spaces, and the extension is also a surgery.

The next theorem is the analogue of Theorem 5.1 for Blaschke products. It will be more convenient to formulate it for a general Blaschke product since we would like to use it for f_θ as well as the elements of $\mathcal{B}_5^{cm}(\theta)$.

Theorem 12.4 (Paths of QC Conjugacies). *Let A and B be two Blaschke products of degree d and let Φ be a quasiconformal homeomorphism which fixes $0, 1, \infty$ such that $\Phi \circ I = I \circ \Phi$ and $\Phi \circ A = B \circ \Phi$. Then there exists a path $\{\Phi_t\}_{0 \leq t \leq 1}$ of quasiconformal homeomorphisms, with $\Phi_0 = \mathrm{id}$ and $\Phi_1 = \Phi$, such that $A_t = \Phi_t \circ A \circ \Phi_t^{-1}$ is a Blaschke product for every $0 \leq t \leq 1$. In particular, either A is quasiconformally rigid or its conjugacy class is non-trivial and path-connected.*

Proof. The proof is almost identical to that of Theorem 5.1. Consider $\sigma = \Phi^*\sigma_0$, which is invariant under A, and take the *real* perturbations $\sigma_t = t\sigma$, $0 \leq t \leq 1$. Let Φ_t be the unique quasiconformal homeomorphism which fixes $0, 1, \infty$ and satisfies $\Phi_t^*\sigma_0 = \sigma_t$. The map $A_t = \Phi_t \circ A \circ \Phi_t^{-1}$ is easily seen to be a degree d rational map. By uniqueness, $I \circ \Phi_t \circ I = \Phi_t$ since the left-hand side also pulls σ_0 back to σ_t and fixes $0, 1, \infty$. Hence A_t commutes with I. So it is a Blaschke product. □

We will need the next lemma in the proof of Theorem 13.3.

Lemma 12.5 (Rigidity on the Julia Set). *Let ψ be a quasiconformal homeomorphism defined on an open annulus containing the Julia set $J(f_\theta)$ of the Blaschke product f_θ defined in (12.1). Suppose that ψ commutes with I and conjugates f_θ to itself. Then $\psi|_{J(f_\theta)}$ is the identity.*

Proof. Extend ψ to a quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ which commutes with I and conjugates f_θ to itself. By the previous theorem, there exists a path $t \mapsto \psi_t$ of quasiconformal homeomorphisms, with $0 \leq t \leq 1$ and $\psi_0 = \mathrm{id}, \psi_1 = \psi$, such that $\psi_t \circ f_\theta \circ \psi_t^{-1}$ is a degree 3 Blaschke product quasiconformally conjugate to f_θ. By Proposition 12.1, this Blaschke product has to be f_θ itself, so ψ_t commutes with f_θ. Now, that $\psi|_{J(f_\theta)}$ must be the identity map follows from an argument similar to the proof of Lemma 6.2. □

13. Surjectivity of the Surgery Map

In this section we prove that the surgery map $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ is surjective. We do this by showing that $\mathcal{S}$ is injective on the set of Blaschke products which map to $\mathbb{C}^* \smallsetminus \mathcal{M}_3(\theta)$ or to hyperbolic-like cubics. The proof of this fact is based on the combinatorics of drops and their nuclei as developed in Section 8. Here is the outline of the proof: If $\mathcal{S}(A) = \mathcal{S}(B)$ for some $A, B \in \mathcal{B}_5^{cm}(\theta)$, there exists a quasiconformal homeomorphism of the plane which conjugates the modified Blaschke products $\widetilde{A}$ and $\widetilde{B}$, which is conformal everywhere except on the union of the maximal drops. A careful analysis will then show that when $\mathcal{S}(A)$ is not capture, one can redefine this homeomorphism on all the drops of the two Blaschke products to get a conjugacy between A and B everywhere. A pull-back argument together with the Bers Sewing Lemma at each step shows that this conjugacy is conformal away from the Julia sets (Theorem 13.1). When $\mathcal{S}(A)$ is hyperbolic-like or has disconnected Julia set, one can use the renormalization scheme of Section 12 and the rigidity on the Julia sets (Lemma 12.5) to conclude that the conjugacy between A and B is in fact conformal (Theorem 13.3). Surjectivity of $\mathcal{S}$, Theorem 13.7 and some corollaries will follow immediately.

Theorem 13.1. *Let $A, B \in \mathcal{B}_5^{cm}(\theta)$ and $\mathcal{S}(A) = \mathcal{S}(B) = P$. Suppose that P is not capture. Then there exists a quasiconformal homeomorphism $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes $0, 1, \infty$, commutes with I and conjugates A to B. Moreover, Φ is conformal on the Fatou set $\widehat{\mathbb{C}} \smallsetminus J(A)$.*

Proof. Following the notation of (9.2), we assume that $P = \varphi \circ \widetilde{A} \circ \varphi^{-1} = \varphi' \circ \widetilde{B} \circ \varphi'^{-1}$ for some quasiconformal homeomorphisms φ and φ'. Consider the quasiconformal homeomorphism $\Phi_0 = \varphi'^{-1} \circ \varphi$ which conjugates $\widetilde{A}$ to $\widetilde{B}$ on the entire plane and is conformal (i.e., $\overline{\partial}\Phi_0 = 0$) everywhere except on $\bigcup_{k \geq 0} \widetilde{A}^{-k}(\mathbb{D})$.

Note that by Proposition 10.2(b) the open set $\widehat{\mathbb{C}} \smallsetminus \overline{\bigcup_{k\geq 0} \widetilde{A}^{-k}(\mathbb{D})}$ is precisely the nucleus N_∞ as defined in Section 8. Also, $\bigcup_{k\geq 0} \widetilde{A}^{-k}(\mathbb{D})$ is the *disjoint* union of the maximal drops of A (which by Proposition 10.2(a) correspond to the bounded Fatou components of P which map to the Siegel disk Δ_P). Similar correspondence holds for the open set $\bigcup_{k\geq 0} \widetilde{B}^{-k}(\mathbb{D})$. Therefore, for any maximal k-drop $D_k^i(A)$, there corresponds a unique maximal k-drop $D_k^i(B) = \Phi_0(D_k^i(A))$. Finally, note that for any such maximal drops, $A^{\circ k} : D_k^i(A) \to \mathbb{D}$ and $B^{\circ k} : D_k^i(B) \to \mathbb{D}$ are conformal isomorphisms since by our assumption P is not capture.

In what follows we construct a sequence of quasiconformal homeomorphisms $\Phi_n : \mathbb{C} \to \mathbb{C}$ which preserve the unit circle $\mathbb{T}$ and another sequence Υ_n by *symmetrizing* each Φ_n:

$$\Upsilon_n(z) = \begin{cases} \Phi_n(z) & |z| \geq 1 \\ (I \circ \Phi_n \circ I)(z) & |z| < 1 \end{cases}$$

We have already constructed Φ_0, hence Υ_0. Consider the sequences of compact sets $\{J_n(A)\}$ and $\{J_n(B)\}$ as in Lemma 10.3. Note that $\Phi_0 \circ A = B \circ \Phi_0$ on $J_0(A)$. The next step is to define Φ_1: Let $\Phi_1 = \Upsilon_0$ everywhere except on the maximal drops of A. On any maximal k-drop $D_k^i(A)$ we define $\Phi_1 : D_k^i(A) \to D_k^i(B)$ by $B^{-k} \circ \Upsilon_0 \circ A^{\circ k}$. (When $k = 0$, the only maximal 0-drop is $\mathbb{D}$ and by this definition $\Phi_1|_{\mathbb{D}} = \Upsilon_0|_{\mathbb{D}}$.) Observe that the two definitions match along the common boundary. Hence Φ_1 is in fact a quasiconformal homeomorphism by the Bers Sewing Lemma. Note that $\Phi_1|_{J_0(A)} = \Phi_0|_{J_0(A)}$ and by definition of $J_1(A)$ in (10.1), $\Phi_1 \circ A = B \circ \Phi_1$ on $J_1(A)$. The homeomorphism Υ_1 is then obtained by symmetrizing Φ_1.

Continuing inductively, we define Φ_n to be equal to Υ_{n-1} everywhere except on the maximal drops of A and then on the maximal drops we define it by taking pull-backs. In other words, $\Phi_n : D_k^i(A) \to D_k^i(B)$ will be defined by $B^{-k} \circ \Upsilon_{n-1} \circ A^{\circ k}$.

Lemma 13.2. *The sequence of quasiconformal homeomorphisms* $\{\Phi_n\}$ *has the following properties:*

$$\Phi_n|_{J_{n-1}(A)} = \Phi_{n-1}|_{J_{n-1}(A)}, \tag{13.1}$$

and

$$(\Phi_n \circ A)(z) = (B \circ \Phi_n)(z) \qquad z \in J_n(A). \tag{13.2}$$

Proof. Both properties follow by induction on n. Let us prove (13.1) first. We have already seen (13.1) for $n = 1$. Assume (13.1) is true and let $z \in J_n(A)$. We distinguish three cases:

• *Case 1:* $z \in J_n(A) \cap \overline{\mathbb{D}}$. Then $I(z) \in J_{n-1}(A)$ and we have $\Phi_{n+1}(z) = \Upsilon_n(z) = (I \circ \Phi_n \circ I)(z) = (I \circ \Phi_{n-1} \circ I)(z)$ by the induction hypothesis. The latter is clearly equal to $\Upsilon_{n-1}(z) = \Phi_n(z)$.

• *Case 2:* $z \in J_n(A) \setminus \overline{\mathbb{D}}$ and $A^{\circ k}(z) \in \overline{\mathbb{D}}$ for some $k \geq 1$. $A^{\circ k}(z) \in IJ_{n-1}$ and hence $(I \circ A^{\circ k})(z) \in J_{n-1}(A)$. So $\Phi_{n+1}(z) = (B^{-k} \circ \Upsilon_n \circ A^{\circ k})(z) = (B^{-k} \circ I \circ \Phi_n \circ I \circ A^{\circ k})(z) = (B^{-k} \circ I \circ \Phi_{n-1} \circ I \circ A^{\circ k})(z)$ by the induction hypothesis. Again, the latter is equal to $(B^{-k} \circ \Upsilon_{n-1} \circ A^{\circ k})(z) = \Phi_n(z)$.

• *Case 3:* $z \in J_n(A) \setminus \overline{\mathbb{D}}$ and z is accumulated by points of the form *Case 2*. Then, clearly, $\Phi_{n+1}(z) = \Phi_n(z)$ by continuity.

Altogether the three steps show that $\Phi_{n+1}|_{J_n(A)} = \Phi_n|_{J_n(A)}$, which completes the induction step and the proof of (13.1).

To prove (13.2) we have to work a little bit more. We have already seen (13.2) for $n = 1$. Assume (13.2) is true and let $z \in J_{n+1}(A)$. We split the induction step into the following cases:

• *Case 1:* $z \in J_{n+1}(A) \setminus \overline{\mathbb{D}}$ and $A(z) \notin \overline{\mathbb{D}}$. Then $(\Phi_{n+1} \circ A)(z) = (B \circ \Phi_{n+1})(z)$ automatically since Φ_{n+1} is defined by pull-backs.

• *Case 2:* $z \in J_{n+1}(A) \setminus \overline{\mathbb{D}}$ but $A(z) \in \overline{\mathbb{D}}$. Then $(\Phi_{n+1} \circ A)(z) = (\Upsilon_n \circ A)(z) = (B \circ B^{-1} \circ \Upsilon_n \circ A)(z) = (B \circ \Phi_{n+1})(z)$.

• *Case 3:* $z \in J_{n+1}(A) \cap \overline{\mathbb{D}}$ and $A(z) \in \overline{\mathbb{D}}$. Then $(\Phi_{n+1} \circ A)(z) = (\Upsilon_n \circ A)(z) = (I \circ \Phi_n \circ I)(A(z)) = (I \circ \Phi_n \circ A)(I(z))$. But $I(z) \in J_n(A)$ so by the induction hypothesis, $(I \circ \Phi_n \circ A)(I(z)) = (I \circ B \circ \Phi_n)(I(z)) = (B \circ I \circ \Phi_n)(I(z)) = (B \circ \Upsilon_n)(z) = (B \circ \Phi_{n+1})(z)$.

• *Case 4:* $z \in J_{n+1}(A) \cap \overline{\mathbb{D}}$ but $A(z) \notin \overline{\mathbb{D}}$. Then $I(z) \in J_n(A)$. Let $w = A(z)$. Since $A(I(z)) = I(w) \in \mathbb{D}$, we have $I(w) \in IJ_{n-1}(A)$, hence $w \in J_{n-1}(A)$. By (13.1), one has $\Phi_{n+1}(w) = \Phi_n(w) = \Phi_{n-1}(w) = \Upsilon_{n-1}(w) = (I \circ \Upsilon_{n-1} \circ I)(w) = (I \circ \Phi_n \circ I)(w) = (I \circ \Phi_n \circ I)(A(z)) = (I \circ \Phi_n \circ A)(I(z)) = (I \circ B \circ \Phi_n)(I(z))$ by the induction hypothesis. The latter is equal to $(B \circ I \circ \Phi_n)(I(z)) = (B \circ \Upsilon_n)(z) = (B \circ \Phi_{n+1})(z)$. □

Back to the proof of Theorem 13.1. By the Bers Sewing Lemma, the symmetrization $\Phi_n \longrightarrow \Upsilon_n$ does not increase the dilatation. On the other hand, the modification $\Upsilon_n \longrightarrow \Phi_{n+1}$ achieved by pull-backs along the maximal drops does not increase the dilatation either, simply because A and B are holomorphic. So we may assume that $\{\Phi_n\}$ is uniformly quasiconformal. Since all the

Φ_n fix $0, 1, \infty$, it follows that some subsequence $\Phi_{n(j)}$ converges locally uniformly to a quasiconformal homeomorphism Φ. Lemma 10.3 and Lemma 13.2 imply that $\Phi \circ A = B \circ \Phi$ on $J(A)$.

In particular, this shows that Φ sends *all* the drops of A bijectively to the drops of B (before we only had a correspondence between the *maximal* drops of A and B).

It is easy to check that Φ obtained this way is conformal on the union $N = \bigcup_{i,k} N_k^i(A)$ of all the nuclei of drops of A at all depths as defined in Section 8 and in fact conjugates A to B there. Since N is clearly disjoint from the Julia set $J(A)$ by (8.2), it remains to show that every Fatou component of A is contained in N.

Consider a component U of the Fatou set of A. Under the iteration of A, U visits both $\mathbb{D}$ and $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ either finitely many times or infinitely often. In the first case, U has to map eventually into the nucleus $N_0(A)$ or $N_\infty(A)$, hence it has to be contained in N. We prove that the second case cannot occur. In fact, suppose that the orbit of U visits $\mathbb{D}$ and $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ infinitely often. According to Sullivan [**Su1**], U eventually maps to a periodic Fatou component of A which is either an attracting or parabolic basin or a Siegel disk or a Herman ring. It follows that this cycle of periodic Fatou components intersects both $\mathbb{D}$ and $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$, so in either case a critical point of A has to enter $\mathbb{D}$ and leave it infinitely often, which is impossible since $\mathcal{S}(A)$ is not a capture. This shows that $N = \widehat{\mathbb{C}} \smallsetminus J(A)$ and proves that Φ is a conjugacy between A and B everywhere and is conformal on $\widehat{\mathbb{C}} \smallsetminus J(A)$. It is easy to see that Φ constructed this way commutes with I. □

Theorem 13.3. *Let $A, B \in \mathcal{B}_5^{cm}(\theta)$ and $\mathcal{S}(A) = \mathcal{S}(B)$. If $\mathcal{S}(A)$ is hyperbolic-like or has disconnected Julia set, then $A = B$.*

Proof. A and B are renormalizable by Theorem 4.2. Consider the quasiconformal homeomorphism Φ given by Theorem 13.1. By Theorem 12.2, there exists a pair of annuli $W'_A \Subset W_A$ (resp. $W'_B \Subset W_B$) and a quasiconformal homeomorphism φ_A (resp. φ_B) which conjugates A (resp. B) to f_θ on W'_A (resp. W'_B). Since $\mathcal{S}(A) = \mathcal{S}(B)$, we can assume that $W'_B = \Phi(W'_A)$ and $W_B = \Phi(W_A)$. The quasiconformal homeomorphism $\psi = \varphi_B \circ \Phi \circ \varphi_A^{-1} : \varphi_A(W'_A) \to \varphi_B(W'_B)$ is a self-conjugacy of f_θ near its Julia set which commutes with I. By Lemma 12.5, we must have $\psi|_{J(f_\theta)} = \text{id}$. It follows from the Bers Sewing Lemma that the $\overline{\partial}$-derivative of ψ is zero almost everywhere on $J(f_\theta)$. Since by Theorem 12.2(b) φ_A (resp. φ_B) has zero $\overline{\partial}$-derivative on $K(A)$ (resp. $K(B)$), we conclude that $\overline{\partial}\Phi = 0$ almost everywhere on $K(A)$. But, as in the proof of Corollary 4.3, up to a set of measure zero, $J(A) = \bigcup_{n\geq 0} A^{-n}(K(A))$. Therefore, $\overline{\partial}\Phi$ has to be zero almost everywhere on the Julia set $J(A)$. Hence Φ is conformal, so $A = B$. □

Remark. We believe that the surgery map is a homeomorphism, at least outside of the capture components where it might have branching. This would imply that the connectedness loci $\mathcal{C}_5(\theta)$ and $\mathcal{M}_3(\theta)$ are actually homeomorphic, a conjecture that is strongly supported by computer experiments.

Corollary 13.4. *The surgery map $\mathcal{S}$ restricts to a homeomorphism $\Lambda_{ext} \xrightarrow{\simeq} \Omega_{ext}$. Similar conclusion holds for Λ_{int} and Ω_{int}. In particular, the connectedness locus $\mathcal{C}_5(\theta)$ is connected.*

Proof. Clearly $\mathcal{S}$ maps Λ_{ext} into Ω_{ext} injectively by the previous theorem. Since $\mathcal{S}$ is a proper map by Proposition 10.6, it extends to a continuous injection $\Lambda_{ext} \cup \{\infty\} \hookrightarrow \Omega_{ext} \cup \{\infty\}$. We claim that this injection is onto. To this end, we show that for any sequence $B_{\mu_n} \in \Lambda_{ext}$ which converges to the boundary of the connectedness locus $\mathcal{C}_5(\theta)$, the sequence $P_{c_n} = \mathcal{S}(B_{\mu_n}) \in \Omega_{ext}$ converges to the boundary of $\mathcal{M}_3(\theta)$. If not, there is a subsequence of B_{μ_n} which converges to $B \in \partial\mathcal{C}_5(\theta)$ but the corresponding subsequence of P_{c_n} converges to some $P \in \Omega_{ext}$. By continuity, $P = \mathcal{S}(B)$. But B has connected Julia set while $J(P)$ is disconnected. This is impossible by Theorem 10.1. $\square$

Corollary 13.5. *The connectedness locus $\mathcal{C}_5(\theta)$ has only two complementary components Λ_{ext} and Λ_{int}.*

Proof. Let U be a bounded component of $\mathbb{C}^* \smallsetminus \mathcal{C}_5(\theta)$ which is not Λ_{int}. Without loss of generality, we assume that U maps into Ω_{ext} by $\mathcal{S}$. Take $A \in U$. By the previous corollary, there exists a $B \in \Lambda_{ext}$ such that $\mathcal{S}(A) = \mathcal{S}(B)$. By Theorem 13.3, $A = B$ and this is a contradiction. $\square$

Corollary 13.6. *The surgery map $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ is surjective.*

Proof. Compactify $\mathcal{B}_5^{cm}(\theta)$ and $\mathcal{P}_3^{cm}(\theta)$ by adding points at 0 and ∞ to get topological 2-spheres. $\mathcal{S}$ extends to a continuous map between these spheres by Proposition 10.6. This map has topological degree $\neq 0$ because it is a homeomorphism $\Lambda_{ext} \xrightarrow{\simeq} \Omega_{ext}$ and $\mathcal{S}^{-1}(\Omega_{ext}) = \Lambda_{ext}$. Therefore it has to be surjective. $\square$

Since the boundary of the Siegel disk of a cubic which comes from the surgery is a quasicircle passing through some critical point, we have proved the following:

Theorem 13.7 (Bounded type cubic Siegel disks are quasidisks). *Let P be a cubic polynomial which has a fixed Siegel disk Δ of rotation number θ. Let θ be of bounded type. Then the boundary of Δ is a quasicircle which contains one or both critical points of P.*

By a recent theorem of Graczyk and Jones [**GJ**], we have

Corollary 13.8. *Under the assumptions of Theorem 13.7, the boundary of the Siegel disk Δ has Hausdorff dimension greater than* 1.

A recent result of McMullen [**Mc3**] implies the following interesting fact: The Hausdorff dimension of $\partial\Delta_c$ is equal to the Hausdorff dimension $1 < \delta(\theta) < 2$ of the boundary of the Siegel disk of $Q_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ whenever P_c is renormalizable. It follows from Theorem 4.2 that the function $c \mapsto \mathrm{HD}(\partial\Delta_c)$ takes on the single value $\delta(\theta)$ on Ω_{ext}, Ω_{int} as well as on all the hyperbolic-like components of $\mathcal{M}_3(\theta)$. (One can actually find more rigorous estimates for the value of $\delta(\theta)$ when $\theta = (\sqrt{5}-1)/2$; see [**BOS**].)

Now it is possible to show that despite all the bifurcations taking place near the boundary of the connectedness locus $\mathcal{M}_3(\theta)$, which give rise to discontinuity of the Julia sets, the boundaries of the Siegel disks move continuously.

Theorem 13.9 (Boundary of Siegel disks move continuously). *The boundary $\partial\Delta_c$ of the Siegel disk of $P_c \in \mathcal{P}_3^{cm}(\theta)$ centered at* 0 *is a continuous function of $c \in \mathbb{C}^*$ in the Hausdorff topology.*

Proof. Let us fix some $P \in \mathcal{P}_3^{cm}(\theta)$. If $P \notin \partial\mathcal{M}_3(\theta)$, Theorem 3.1 shows that $J(P)$, hence $\partial\Delta_P$, moves holomorphically in a neighborhood of P and continuity at P is obvious. So let us assume that $P \in \partial\mathcal{M}_3(\theta)$ and consider a sequence $P_{c_n} \in \mathcal{P}_3^{cm}(\theta)$ which converges to P as $n \to \infty$. Since the surgery map is surjective, there exists a sequence $B_{\mu_n} \in \mathcal{B}_5^{cm}(\theta)$ such that $\mathcal{S}(B_{\mu_n}) = P_{c_n}$. By properness (Proposition 10.6), some subsequence which we still denote by B_{μ_n} converges to some $B \in \mathcal{B}_5^{cm}(\theta)$, which by continuity maps to P. Now consider the representations $P_{c_n} = \varphi_n \circ \widetilde{B}_{\mu_n} \circ \varphi_n^{-1}$ as in (9.2). Then the boundary $\partial\Delta_{P_{c_n}}$ is just the image $\varphi_n(\mathbb{T})$. Since $\{\varphi_n\}$ is normal by Corollary 10.5, some further subsequence, still denoted by $\{\varphi_n\}$, converges to a quasiconformal homeomorphism ψ. The map $Q = \psi \circ \widetilde{B} \circ \psi^{-1} \in \mathcal{P}_3^{cm}(\theta)$ is quasiconformally conjugate to P. Since P is rigid by Theorem 5.5, $P = Q$. Now, as $n \to \infty$, $\partial\Delta_{P_{c_n}} = \varphi_n(\mathbb{T})$ converges in the Hausdorff topology to $\psi(\mathbb{T}) = \partial\Delta_Q = \partial\Delta_P$. □

Remark. We can actually make this theorem stronger in the following sense: Let $P_{c_0} \in \mathcal{P}_3^{cm}(\theta)$ be a cubic for which one of the critical points c_0 or 1 is *off* the boundary $\partial\Delta_{c_0}$ (this happens if P_{c_0} is off the Jordan curve Γ studied in the next section). Then the boundary $\partial\Delta_c$ moves holomorphically as a function of c in a neighborhood of c_0. To see this, assume for example that for all c sufficiently close to c_0 we have $c \in \partial\Delta_c$ but $1 \notin \partial\Delta_c$. Evidently the critical orbit $\{P_c^{\circ k}(c)\}_{k\geq 0}$ moves holomorphically as a function of c, and we can extend this motion to the closure of this critical orbit by the λ-lemma. But this closure is precisely the boundary $\partial\Delta_c$ if c is close to c_0.

14. Siegel Disks with Two Critical Points on Their Boundary

In this section we characterize those cubics in $\mathcal{P}_3^{cm}(\theta)$ which have both critical points on the boundary of their Siegel disk. In Theorem 14.3 we will prove that the set of all such cubics is a Jordan curve Γ in $\mathcal{P}_3^{cm}(\theta)$. The proof of this theorem will use the fact that the quasiconformal conjugacy classes in $\mathcal{B}_5^{cm}(\theta)$ are path-connected (Theorem 12.4). We then show that when there are no queer components, Γ is in fact the common boundary of Ω_{ext} and Ω_{int} (Theorem 14.4).

Consider the set Γ which consists of all cubics $P \in \mathcal{P}_3^{cm}(\theta)$ such that both critical points of P belong to the boundary of the Siegel disk Δ_P. Fig. 18 shows this set in the parameter space $\mathcal{P}_3^{cm}(\theta)$.

Since the surgery map $\mathcal{S} : \mathcal{B}_5^{cm}(\theta) \to \mathcal{P}_3^{cm}(\theta)$ is surjective by Corollary 13.6, every $P \in \Gamma$ is of the form $\mathcal{S}(B_\mu)$ with B_μ having two double critical points on the circle. Corollary 9.3 shows that μ must belong to the unit circle $\mathbb{T} \subset \mathbb{C}^* \simeq \mathcal{B}_5^{cm}(\theta)$. Therefore, we simply have

$$\Gamma = \mathcal{S}(\mathbb{T}).$$

In particular, Γ is a closed path in $\mathcal{P}_3^{cm}(\theta) \simeq \mathbb{C}^*$. Suggested by Fig. 18, we want to prove that Γ is a Jordan curve. This would follow immediately if one could prove that $\mathcal{S}|_{\mathbb{T}}$ is injective. However, I have not been able to show this. In fact, I do not know how to prove that Blaschke products on the boundary of the connectedness locus $\mathcal{C}_5(\theta)$ are quasiconformally rigid. So we take a slightly different approach by showing that the fibers of $\mathcal{S}|_{\mathbb{T}} : \mathbb{T} \to \Gamma$ are connected.

Lemma 14.1. *Let $A, B \in \mathcal{B}_5^{cm}(\theta)$ and $\mathcal{S}(A) = \mathcal{S}(B) = P$. Suppose that P is not capture. Then there exists a path $t \mapsto A_t \in \mathcal{B}_5^{cm}(\theta)$ of Blaschke products for $0 \leq t \leq 1$, with $A_0 = A$, $A_1 = B$, such that $\mathcal{S}(A_t) = P$ for all t.*

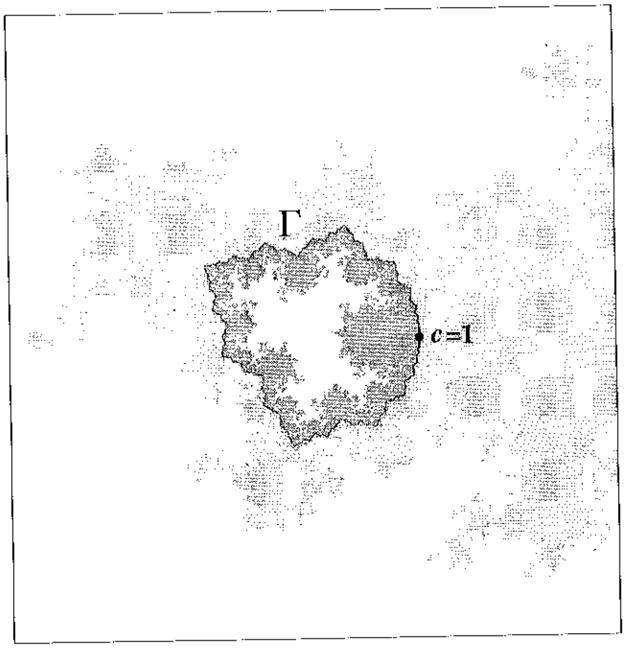

FIGURE 18. *The Jordan curve* Γ, *the locus of all critically marked cubics in* $\mathcal{P}_3^{cm}(\theta)$ *which have both critical points on the boundary of their Siegel disk. Topologically it can be described as the common boundary of the complementary regions* Ω_{ext} *and* Ω_{int}. *Note that* Γ *is invariant under* $c \mapsto 1/c$.

Proof. Since P is not capture, by Theorem 13.1 there exists a quasiconformal homeomorphism Φ which conjugates A to B, which is conformal away from the Julia set $J(A)$. By Theorem 12.4 there exists a path $\{\Phi_t\}_{0\leq t\leq 1}$ connecting the identity map to Φ and a corresponding path $\{A_t = \Phi_t \circ A \circ \Phi_t^{-1}\}_{0\leq t\leq 1}$ of elements of $\mathcal{B}_5^{cm}(\theta)$ connecting A to B. Note that by the definition of Φ_t, these quasiconformal homeomorphisms are all conformal away from $J(A)$.

It remains to show that $\mathcal{S}(A_t) = P$ for all $0 \leq t \leq 1$. Consider the Douady-Earle extension $H : \mathbb{D} \to \mathbb{D}$ used in the definition of $\mathcal{S}(A)$ in Section 9. Recall that $H|_{\mathbb{T}}$ conjugates $A|_{\mathbb{T}}$ to the rigid rotation R_θ. Hence, the quasiconformal homeomorphism $H_t = H \circ \Phi_t^{-1} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ will conjugate $A_t|_{\mathbb{T}}$ to the rigid rotation as well. Note that H_t is not in general the Douady-Earle extension of the linearizing homeomorphism $h_t : \mathbb{T} \to \mathbb{T}$ for A_t. Nevertheless, $\mathcal{S}_{H_t}(A_t) = \mathcal{S}(A_t)$ by Proposition 9.2. Consider the modified Blaschke products

$$\widetilde{A}(z) = \begin{cases} A(z) & |z| \geq 1 \\ (H^{-1} \circ R_\theta \circ H)(z) & |z| < 1 \end{cases}$$

and

$$\widetilde{A}_t(z) = \begin{cases} A_t(z) & |z| \geq 1 \\ (H_t^{-1} \circ R_\theta \circ H_t)(z) & |z| < 1 \end{cases}$$

Note that $\Phi_t \circ \widetilde{A} = \widetilde{A}_t \circ \Phi_t$.

Define the corresponding conformal structures $\sigma = H^*\sigma_0$ and $\sigma_t = H_t^*\sigma_0$ as in Section 9. It is easy to see that

$$\sigma = \Phi_t^* \sigma_t. \tag{14.1}$$

Here we use that fact that Φ_t is conformal away from $J(A)$. Consider the normalized solutions φ and φ_t of the Beltrami equations

$$\varphi^* \sigma_0 = \sigma, \qquad \varphi_t^* \sigma_0 = \sigma_t.$$

By (14.1) and uniqueness, we have

$$\varphi_t = \varphi \circ \Phi_t^{-1}.$$

Hence, by Proposition 9.2,

$$\begin{aligned} \mathcal{S}(A_t) &= \varphi_t \circ \widetilde{A}_t \circ \varphi_t^{-1} \\ &= \varphi \circ \Phi_t^{-1} \circ \widetilde{A}_t \circ \Phi_t \circ \varphi^{-1} \\ &= \varphi \circ \widetilde{A} \circ \varphi^{-1} \\ &= \mathcal{S}(A). \end{aligned}$$

This completes the proof of the lemma. □

Corollary 14.2. *The fibers of $\mathcal{S}|_{\mathbb{T}} : \mathbb{T} \to \Gamma$ are connected.*

Proof. Let $A, B \in \mathbb{T} \subset \mathcal{B}_5^{cm}(\theta)$ and $\mathcal{S}(A) = \mathcal{S}(B)$. Apply the previous lemma to A and B. Note that $A_t \in \mathbb{T}$ for all $0 \leq t \leq 1$, since A_t is quasiconformally conjugate to A, hence has two double critical points on the unit circle. □

Theorem 14.3. *Γ is a Jordan curve.*

Proof. Consider $\mathcal{S}|_{\mathbb{T}} : \mathbb{T} \to \Gamma$ whose fibers are closed and connected by Corollary 14.2. By general topology, Γ is homeomorphic to $\mathbb{T}/\sim$, where $A \sim B$ means $\mathcal{S}(A) = \mathcal{S}(B)$. Since each equivalence class of $\sim$ is a closed connected proper subset of $\mathbb{T}$, it follows that $\mathbb{T}/\sim$ is homeomorphic to the circle. □

Finally, we find a topological characterization of Γ in $\mathcal{P}_3^{cm}(\theta)$ under the assumption that there are no queer components in the interior of $\mathcal{M}_3(\theta)$.

Theorem 14.4 (Topological Characterization of Γ). *Γ is a subset of the boundary $\partial\mathcal{M}_3(\theta)$ and it contains $\partial\Omega_{ext} \cap \partial\Omega_{int}$. If there are no queer components in the interior of $\mathcal{M}_3(\theta)$, then $\Gamma = \partial\Omega_{ext} \cap \partial\Omega_{int}$.*

Proof. First let us show that $\partial\Omega_{ext} \cap \partial\Omega_{int} \subset \Gamma$. Let $P_c \in \partial\Omega_{ext} \cap \partial\Omega_{int}$ and assume that $P_c \notin \Gamma$. Choose $B_\mu \in \mathcal{B}_5^{cm}(\theta)$ such that $\mathcal{S}(B_\mu) = P_c$. We can assume without loss of generality that $|\mu| > 1$. Choose a sequence $P_{c_n} \in \Omega_{int}$ converging to P_c and a sequence $B_{\mu_n} \in \Lambda_{int}$ such that $\mathcal{S}(B_{\mu_n}) = P_{c_n}$. By passing to a subsequence, we may assume that $B_{\mu_n} \to B_{\mu'}$ as $n \to \infty$, where $|\mu'| \leq 1$. By continuity, $\mathcal{S}(B_{\mu'}) = P_c$ and by our assumption $P_c \notin \Gamma$, so we must have $|\mu'| < 1$. Since P_c is not capture by Corollary 5.3, Lemma 14.1 shows that there is a path $t \mapsto A_t$ of quasiconformally conjugate Blaschke products in $\mathcal{B}_5^{cm}(\theta)$ connecting B_μ to $B_{\mu'}$, all of which are mapped to P_c. Since this path must intersect $\mathbb{T}$ somewhere, we conclude that $P_c \in \Gamma$ which is a contradiction.

Now we prove that $\Gamma \subset \partial\mathcal{M}_3(\theta)$. Fix some $P \in \Gamma$. Since P has both critical points on $\partial\Delta_P$, it cannot belong to any hyperbolic-like or capture component. Also, P cannot be in a queer component U of the interior of $\mathcal{M}_3(\theta)$, since

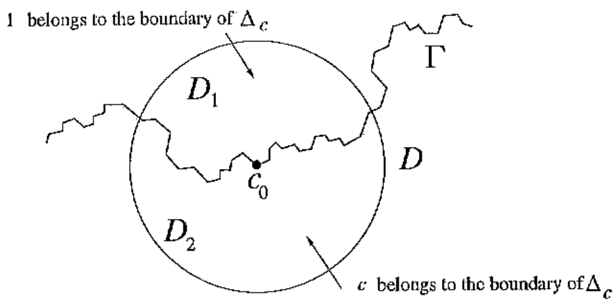

FIGURE 19

otherwise every $Q \in U$ would have to be quasiconformally conjugate to P by Theorem 5.5, which would imply that Q has two critical points on $\partial\Delta_Q$, which would show $U \subset \Gamma$. But this is evidently impossible because U is open and Γ is a Jordan curve. Therefore, P has to lie in $\partial\mathcal{M}_3(\theta) = \partial\Omega_{ext} \cup \partial\Omega_{int}$.

Now assume that there are no queer components in the interior of $\mathcal{M}_3(\theta)$. To show that $\Gamma = \partial\Omega_{ext} \cap \partial\Omega_{int}$, let $P_{c_0} \in \Gamma$ and assume by way of contradiction that $c_0 \in \partial\Omega_{ext} \smallsetminus \partial\Omega_{int}$. Since c_0 has positive distance from Ω_{int}, for all c in a neighborhood D of c_0 the sequence $\{P_c^{\circ n}(1)\}$ has to be normal. Assuming that D is a small disk, the Jordan curve Γ cuts D into two topological disks D_1 and D_2 such that for every $c \in D_1$, $1 \in \partial\Delta_c$ and $c \notin \partial\Delta_c$, and for every $c \in D_2$, $c \in \partial\Delta_c$ and $1 \notin \partial\Delta_c$ (see Fig. 19).

Clearly $D_2 \cap \partial\Omega_{ext} = D_2 \cap \partial\Omega_{int} = \emptyset$. So D_2 has to be a subset of a component U of the interior of $\mathcal{M}_3(\theta)$. Since there are no queer components by the assumption, U is either hyperbolic-like or capture.

For every $c \in D_1$, we have $1 \in \partial\Delta_c$ and the restriction $P_c|_{\partial\Delta_c}$ is conjugate to the rigid rotation by angle θ. Therefore, $P_c^{\circ q_n}(1) \to 1$ for all $c \in D_1$, where the q_n are the denominators of the rational approximations of θ. Since $\{P_c^{\circ n}(1)\}$ is normal in D, for a subsequence $\{q_{n(j)}\}$ we must have $P_c^{\circ q_{n(j)}}(1) \to 1$ throughout D. In particular, if $c \in D_2$, the critical point 1 of P_c must be recurrent. This is impossible if U is hyperbolic-like or capture, since over D_2, $c \in \partial\Delta_c$ and hence 1 either gets attracted to the attracting cycle or eventually maps to the Siegel disk Δ_c. $\square$

15. The Higher-Degree Case

The techniques developed in the previous sections can be used to study Siegel polynomials of higher degrees. When appropriately formulated, many of the results of this work for cubic Siegel polynomials generalize directly without significant extra effort. However, there is one particular step in this program which seems problematic. In this section we shall try to sketch, without proofs, how this generalization can be achieved and what that particular snag seems to be.

Fix an integer $d \geq 4$ and an irrational number $0 < \theta < 1$ of Brjuno type. Consider the space $\mathcal{P}_d^{cm}(\theta)$ of affine conjugacy classes of critically marked polynomials of degree d which have a fixed Siegel disk of multiplier $\lambda = e^{2\pi i\theta}$ centered at the origin. Recall that a marking in this context is a surjective function $\mathbf{m}$ from the set $\{1, 2, \ldots, d-1\}$ to the set of critical points, and that affine conjugacies between critically marked polynomials are dilations which respect the markings. Each conjugacy class in $\mathcal{P}_d^{cm}(\theta)$ has a unique representative $P_{\mathbf{c}}$ with marked critical points located at

$$\mathbf{c} = (\mathbf{m}(1) = c_1, \ldots, \mathbf{m}(d-2) = c_{d-2}, \mathbf{m}(d-1) = 1),$$

where $c_j \in \mathbb{C}^*$. It is easy to see that

$$P_{\mathbf{c}} : z \mapsto A \int_0^z (\zeta - c_1) \cdots (\zeta - c_{d-2})(\zeta - 1)\, d\zeta, \quad \text{where } A = \frac{(-1)^{d-1}\lambda}{c_1 \cdots c_{d-2}}. \tag{15.1}$$

It follows that $\mathcal{P}_d^{cm}(\theta)$ is isomorphic to the product $(\mathbb{C}^*)^{d-2}$ of $d-2$ copies of the punctured plane, with coordinates $(c_1, \ldots, c_{d-2})$.

The full symmetric group S_{d-1} acts on $\mathcal{P}_d^{cm}(\theta)$ in a canonical way (permuting the markings of the critical points). In fact, in the above coordinates for

$\mathcal{P}_d^{cm}(\theta)$, the transposition $i \leftrightarrow j$ for $1 \leq i < j \leq d-2$ acts as

$$L_{i,j} : (c_1, \ldots, c_i, \ldots, c_j, \ldots, c_{d-2}) \mapsto (c_1, \ldots, c_j, \ldots, c_i, \ldots, c_{d-2}),$$

while the transposition $j \leftrightarrow d-1$ acts as the biholomorphic involution

$$L_{j,d-1} : (c_1, \ldots, c_j, \ldots, c_{d-2}) \mapsto \left(\frac{c_1}{c_j}, \ldots, \frac{1}{c_j}, \ldots, \frac{c_{d-2}}{c_j} \right).$$

We define the connectedness locus $\mathcal{M}_d(\theta)$ as the subset of $\mathcal{P}_d^{cm}(\theta)$ consisting of all polynomials with connected Julia set. It is not hard to show that the filled Julia set of $P_{\mathbf{c}} \in \mathcal{P}_d^{cm}(\theta)$ is contained in the disk $\mathbb{D}(0, m_{\mathbf{c}})$, where $m_{\mathbf{c}} = r_d \max\{|c_1|, \ldots, |c_{d-2}|, 1\}$ and r_d is a constant only depending on the degree d. It follows from this fact that $\mathcal{M}_d(\theta)$ is a compact set in $\mathcal{P}_d^{cm}(\theta)$. Moreover, $\mathcal{M}_d(\theta)$ has many symmetries since it is invariant under the action of the biholomorphic involutions $L_{i,j}$. It should be possible to show, for example using an argument similar to [**La**], that the connectedness locus $\mathcal{M}_d(\theta)$ is connected, although the rest of the arguments will not depend on this fact.

The definitions of hyperbolic-like and capture polynomials should now be modified as follows: A polynomial $P_{\mathbf{c}} \in \mathcal{P}_d^{cm}(\theta)$ is called *hyperbolic-like* if all but one of the critical points $c_1, \ldots, c_{d-2}, 1$ get attracted to attracting cycles. On the other hand, $P_{\mathbf{c}}$ is called *capture* if at least one of the critical points eventually hits the Siegel disk $\Delta_{\mathbf{c}}$ of $P_{\mathbf{c}}$ centered at the origin *and* other critical points which do not hit $\Delta_{\mathbf{c}}$, if any, get attracted to attracting cycles.

It can be shown, using the same technique of holomorphic motions as in Section 3, that hyperbolic-like and capture polynomials form components of the interior of the connectedness locus $\mathcal{M}_d(\theta)$. We call all the possible remaining components *queer*. Again, one can show that the Julia set of every queer polynomial in $\mathcal{P}_d^{cm}(\theta)$ admits an invariant line field and has positive measure; this requires an adaptation of the argument in Theorem 3.4.

Outside the connectedness locus, at least one of the critical points gets attracted to infinity. We are particularly interested in the domains

$$\Omega_j = \{P \in \mathcal{P}_d^{cm}(\theta) : \text{All critical points of } P \text{ except } \mathbf{m}(j) \text{ get attracted to } \infty\}.$$

There is a principal domain Ω_{d-1}, which in coordinates $(c_1, \dots, c_{d-2})$ intersects every neighborhood of $(\infty, \dots, \infty)$ of the form

$$\{|c_1| \geq R\} \times \cdots \times \{|c_{d-2}| \geq R\},$$

and plays the role of Ω_{ext} in the cubic family. Other copies $\Omega_j = L_{j,d-1}(\Omega_{d-1})$ for $1 \leq j \leq d-2$ are just biholomorphic images of this principal domain (analogues of Ω_{int} in the cubic case). Note, however, that there is a significant difference between the domains Ω_j and the topological disks Ω_{ext} and Ω_{int}: For $d \geq 4$, Ω_j is *not* a component of $\mathcal{P}_d^{cm}(\theta) \smallsetminus \mathcal{M}_d(\theta)$.

Every $P \in \Omega_j$ is renormalizable in the sense that it has a quadratic-like restriction hybrid equivalent to the polynomial $Q_\theta : z \mapsto e^{2\pi i\theta} z + z^2$. The Julia set of such P consists of countably many components each quasiconformally homeomorphic to the Julia set of Q_θ, as well as uncountably many points. It has Lebesgue measure zero if θ is bounded type.

Inside a hyperbolic-like or capture component, we expect to have different critical orbit relations. These are defined by a finite number of algebraic subvarieties of codimension at least 1 (the generalization of "centers" in the cubic case) which do not disconnect the component.

In order to prove conjecture $(\mathbf{A}_d)$ of the introduction for arbitrary $d \geq 4$, we introduce a similar family of degree $2d-1$ Blaschke products. Let

$$\widehat{\mathcal{B}} = \left\{ B : z \mapsto \tau z^d \left(\frac{z - p_1}{1 - \overline{p_1} z} \right) \cdots \left(\frac{z - p_{d-1}}{1 - \overline{p_{d-1}} z} \right) : |p_j| > 1 \text{ for all } 1 \leq j \leq d-1 \right\},$$

where the rotation factor $\tau \in \mathbb{T}$ is chosen so as to achieve the normalization $B(1) = 1$. Clearly, $\widehat{\mathcal{B}}$ is homeomorphic to the symmetric product of $d-1$

copies of $\mathbb{C} \setminus \overline{\mathbb{D}}$, or equivalently, the punctured plane. Therefore, it can be identified with the space of all monic polynomials of the form

$$w \mapsto (w - w_1) \cdots (w - w_{d-1}) = w^{d-1} - \sigma_1 w^{d-2} + \cdots + (-1)^{d-1} \sigma_{d-1},$$

where $w_j \in \mathbb{C}^*$ and the σ_j are symmetric elementary functions on $\{w_1, \ldots, w_{d-1}\}$. It follows that

$$\widehat{\mathcal{B}} \simeq \mathbb{C}^{d-2} \times \mathbb{C}^*.$$

We are interested in the subset $\mathcal{B} \subset \widehat{\mathcal{B}}$ of those normalized Blaschke products of the above form whose critical points other than 0 and ∞ are of the form

$$c_1, \ldots, c_{d-1}, \frac{1}{\overline{c_1}}, \ldots, \frac{1}{\overline{c_{d-1}}}$$

with $|c_j| > 1$ for all $1 \leq j \leq d - 1$. The following critical parametrization result is a generalization of Theorem 7.1:

Theorem 15.1. *Let $c_1, \ldots, c_{d-1}$ be points outside the closed unit disk in the complex plane. Then there exists a unique normalized Blaschke product $B \in \mathcal{B}$ whose critical points are located at*

$$0, \infty, c_1, \ldots, c_{d-1}, \frac{1}{\overline{c_1}}, \ldots, \frac{1}{\overline{c_{d-1}}}.$$

As a result, $\mathcal{B}$ is also homeomorphic to $\mathbb{C}^{d-2} \times \mathbb{C}^*$. A straightforward generalization of Lemma 7.4 (with the constant 2 replaced by $(d+1)/(d-1)$) shows that this theorem holds even if some of the c_j belong to the unit circle.

Now consider the space of Blaschke products B of degree $2d - 1$ subject to the following two conditions:

(i) B has the form

$$B : z \mapsto e^{2\pi i t} z^d \left(\frac{z - p_1}{1 - \overline{p_1} z} \right) \cdots \left(\frac{z - p_{d-1}}{1 - \overline{p_{d-1}} z} \right), \quad |p_j| > 1 \text{ for all } 1 \leq j \leq d - 1,$$

where the points p_j are chosen such that B has a double critical point on the unit circle $\mathbb{T}$ and $d-2$ pairs $(c_j, 1/\overline{c_j})$ of symmetric critical points which may or may not be on $\mathbb{T}$.

(ii) $0 \le t \le 1$ is chosen so that the rotation number of the analytic homeomorphism $B|_{\mathbb{T}}$ is equal to the given irrational number θ.

By definition, $\mathcal{B}^{cm}_{2d-1}(\theta)$ is the space of all critically marked Blaschke products satisfying (i) and (ii), modulo the action of the rotation group. Here a marking of the critical points of B is a surjective function $\mathbf{m}$ from the set $\{1, 2, \ldots, d-1\}$ to the set of finite critical points of B outside the open unit disk. To understand the topology of $\mathcal{B}^{cm}_{2d-1}(\theta)$, for $1 \le j \le d-1$ we define $E_j \subset \mathcal{B}^{cm}_{2d-1}(\theta)$ as the set of all conjugacy classes $(B, \mathbf{m})$ of critically marked Blaschke products with $\mathbf{m}(j) \in \mathbb{T}$. In each E_j one can choose a unique representative with $\mathbf{m}(j) = 1$ and hence by the remark after Theorem 15.1 use the other $d-2$ critical points $\mathbf{m}(i) \in \mathbb{C} \smallsetminus \mathbb{D}$ $(i \neq j)$ as coordinates for E_j. It follows that E_j is isomorphic to $(\mathbb{C} \smallsetminus \mathbb{D})^{d-2}$. On the common boundary $\partial E_i \cap \partial E_j \simeq (\mathbb{C} \smallsetminus \mathbb{D})^{d-3} \times \mathbb{T}$, a point has two coordinates

$$(\mathbf{m}(1), \ldots, \mathbf{m}(i) = 1, \ldots, \mathbf{m}(j), \ldots, \mathbf{m}(d-1)) \in \partial E_i$$

and

$$\left(\frac{\mathbf{m}(1)}{\mathbf{m}(j)}, \ldots, \frac{\mathbf{m}(i)}{\mathbf{m}(j)}, \ldots, 1, \ldots, \frac{\mathbf{m}(d-1)}{\mathbf{m}(j)} \right) \in \partial E_j$$

which we must identify. The space $\mathcal{B}^{cm}_{2d-1}(\theta)$ is therefore homeomorphic to the disjoint union of $d-1$ copies of $(\mathbb{C} \smallsetminus \mathbb{D})^{d-2}$ glued along the boundaries by the above identification. Fortunately, there is an easy way to identify the topology of this quotient. It suffices to consider the map $\Phi : \coprod_{j=1}^{d-1} (\mathbb{C} \smallsetminus \mathbb{D})^{d-2} \to (\mathbb{C}^*)^{d-2}$ defined by

$$\Phi : (\mathbf{m}(1), \ldots, \mathbf{m}(d-1)) \mapsto \left(\mu_1 = \frac{\mathbf{m}(1)}{\mathbf{m}(2)}, \mu_2 = \frac{\mathbf{m}(1)}{\mathbf{m}(3)}, \ldots, \mu_{d-2} = \frac{\mathbf{m}(1)}{\mathbf{m}(d-1)} \right).$$

An easy exercise shows that the fibers of Φ are precisely the equivalence classes defined by the above gluing and hence Φ descends to a homeomorphism between $\mathcal{B}^{cm}_{2d-1}(\theta)$ and $(\mathbb{C}^*)^{d-2}$. As a result, the $\mu_i \in \mathbb{C}^*$ $(1 \leq i \leq d-2)$ can be used as coordinates of $\mathcal{B}^{cm}_{2d-1}(\theta)$.

Theorem 15.2. *For every $d \geq 4$, both parameter spaces $\mathcal{P}^{cm}_d(\theta)$ and $\mathcal{B}^{cm}_{2d-1}(\theta)$ are homeomorphic to the product of $d-2$ copies of the punctured plane $\mathbb{C}^*$.*

For an element $B \in \mathcal{B}^{cm}_{2d-1}(\theta)$ one can define the notion of a drop, the nucleus of a drop, ... in a similar way. All the basic properties of these objects developed in Section 8 remain valid when appropriately restated.

Now a similar surgery map $\mathcal{S} : \mathcal{B}^{cm}_{2d-1}(\theta) \to \mathcal{P}^{cm}_d(\theta)$ can be defined in a way identical to the cubic case. An argument similar to the one in Proposition 10.6 shows that this map is proper.

Inside the Blaschke space $\mathcal{B}^{cm}_{2d-1}(\theta)$ we have the connectedness locus $\mathcal{C}_{2d-1}(\theta)$ consisting of all maps with connected Julia sets. This locus is compact and we can consider in its complement the domains

$$\Lambda_j = \{B \in \mathcal{B}^{cm}_{2d-1}(\theta) : \text{All critical points of } P \text{ except } \mathbf{m}(j) \text{ get attracted to } \infty\}.$$

It is easy to see that $\mathcal{S}(\Lambda_j) \subset \Omega_j$ for $1 \leq j \leq d-1$. In particular, the Blaschke products in Λ_j are renormalizable so that one can extract the standard map f_θ from them by straightening (the analogue of Theorem 12.2).

Now consider the following:

Statement. *The surgery map $\mathcal{S} : \mathcal{B}^{cm}_{2d-1}(\theta) \to \mathcal{P}^{cm}_d(\theta)$ is continuous.*

Assume for a moment that this statement is true. Consider the compactification

$$(\mathbb{C}^*)^{d-2} \hookrightarrow \widehat{\mathbb{C}}^{d-2}$$

into a product of 2-spheres for both parameter spaces $\mathcal{P}_d^{cm}(\theta)$ and $\mathcal{B}_{2d-1}^{cm}(\theta)$. The added points in these compactifications have well-defined dynamical meanings. For example, it is easy to check using the normal form (15.1) that when k of the critical point c_j tend to infinity (or zero), the polynomial P_c converges locally uniformly on $\mathbb{C}$ to a critically marked Siegel polynomial of degree $d-k$, i.e., an element of $\mathcal{P}_{d-k}^{cm}(\theta)$. Similarly, using the parameters μ_i for $\mathcal{B}_{2d-1}^{cm}(\theta)$, one can check that when k of the critical points μ_i tend to infinity (or zero), the corresponding Blaschke product converges locally uniformly on $\mathbb{C}^*$ to a Blaschke product in $\mathcal{B}_{2(d-k)-1}^{cm}(\theta)$.

Knowing this fact and properness of $\mathcal{S}$, it is not hard to show that $\mathcal{S}$ extends to a continuous map $\widehat{\mathbb{C}}^{d-2} \to \widehat{\mathbb{C}}^{d-2}$, so it has a well-defined topological degree. We show that this degree is non-zero, which proves $\mathcal{S}$ must be surjective. To this end, it suffices to note that the proof of Theorem 13.1 extends word-by-word to the case of Blaschke products in $\mathcal{B}_{2d-1}^{cm}(\theta)$. In particular, it follows that $\mathcal{S}$ injects Λ_{d-1} into Ω_{d-1}. We claim that the map $\mathcal{S} : \Lambda_{d-1} \to \Omega_{d-1}$ is a homeomorphism. Let a sequence B_n of Blaschke products in Λ_{d-1} converge to $B \in \partial\Lambda_{d-1}$ and the image sequence $P_n = \mathcal{S}(B_n)$ converge to some $P \in \Omega_{d-1}$. Then either B has a lower degree so that it belongs to $\mathcal{B}_{2s-1}^{cm}$ for some $1 < s < d$, or $B \in \mathcal{B}_{2d-1}^{cm}(\theta)$ but at least two of the critical points of B have bounded orbits. If the surgery map is continuous on the compactified space $\widehat{\mathbb{C}}^{d-2}$, it follows that either P has degree less than d or at least two distinct critical points of P have bounded orbits. Either possibility would contradict $P \in \Omega_{d-1}$, proving that the restriction $\mathcal{S}|_{\Lambda_{d-1}} : \Lambda_{d-1} \to \Omega_{d-1}$ is a proper injective map. Hence it has to be a covering map of degree 1, or in other words a homeomorphism. Since $\mathcal{S}^{-1}(\Omega_{d-1}) = \Lambda_{d-1}$, the topological degree of $\mathcal{S} : \widehat{\mathbb{C}}^{d-2} \to \widehat{\mathbb{C}}^{d-2}$ cannot be zero, and this shows surjectivity of $\mathcal{S}$. Conjecture $(\mathbf{A}_d)$ follows immediately.

So all the ingredients for a generalization of $(\mathbf{A}_3)$ to any $(\mathbf{A}_d)$ is available using the techniques developed in this work, except for one missing link which is the above continuity statement for $\mathcal{S}$. The proof given for the cubic case in Theorem 11.1 relies strongly on the fact that nontrivial quasiconformal conjugacy classes in $\mathcal{P}_3^{cm}(\theta)$ are open. This definitely fails in $\mathcal{P}_d^{cm}(\theta)$ when $d \geq 4$. But still we believe that it should be possible to show continuity by a different method.

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