Fixed Points of Non-Hamiltonian Symplectomorphisms

A Dissertation, Presented

by

Luisa D. Stelling

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

December 1999

State University of New York

at Stony Brook

The Graduate School

Luisa D. Stelling

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Dusa McDuff
Distinguished Professor of Mathematics
Dissertation Director

Claude LeBrun
Professor of Mathematics
Chairman of Dissertation

Leon Takhtajan
Professor of Mathematics

Santiago Simanca
Research Scientist
Department of Applied Mathematics and Statistics
Outside Member

This dissertation is accepted by the Graduate School

Graduate School

Abstract of the Dissertation,

Fixed Points of Non-Hamiltonian Symplectomorphisms

by

Luisa D. Stelling

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

1999

In this work we study an analogue of the Arnol'd conjecture. We give a lower bound for the number of fixed points of a non-Hamiltonian symplectomorphism which is isotopic to the identity through symplectomorphisms on a closed, semi-positive symplectic manifold M. The result holds for symplectomorphisms whose Calabi invariant is sufficiently small. The bound on the size of the Calabi invariant depends only on the geometry of the symplectic manifold M.

to my father and to Heberto.

Contents

Acknowledgements

I would like to thank my advisor Dusa McDuff for her patience and all the advice I received from her throughout my graduate years. And of course, for the many useful comments on the previous drafts of this work. I enjoyed her seminar, and learnt a great deal of mathematics by attending and participating in it.

I am also very grateful to Alain Etcheberry and Enrique Planchart who made me feel confident about myself, and whose classes I enjoyed deeply. From them I learnt much more than just mathematics.

I would also want to thank my friends Jennifer, Joe, Olga, Silvia, Jose, Santiago and Haydee for the many good times we spent together. Thanks to them I will have real good memories of these five years.

I owe special thanks to my husband Heberto for making me so happy and for listening to me so many times.

Finally I want to thank my family for the support and encouragement I received from them all my life. I specially want to thank Francia for being a mother to me and making my life so much better.

Chapter 1

Introduction

A symplectic structure ω on a manifold M provides us with a one-to-one correspondence between closed 1-forms and vector fields X satisfying $\mathcal{L}_X\omega = 0$. A vector field is called Hamiltonian if it corresponds to an exact 1-form, and a symplectomorphism φ on M is Hamiltonian if it is the time-1 map of a time-dependent Hamiltonian vector field. The Arnol'd conjecture states that the number of fixed points of a Hamiltonian symplectomorphism on a compact symplectic manifold can be estimated below by the sum of the Betti numbers of M provided that all the fixed points are non-degenerate.

The present work generalizes the paper by L. Vân and K. Ono where they consider an analogue of the Arnol'd conjecture for non-Hamiltonian symplectomorphisms in the identity component. In this case the fixed point set may be empty, for example, an irrational rotation on an even-dimensional torus with the standard symplectic structure. For this reason, it is necessary to consider the Novikov homology instead of ordinary homology. The aim of this work is to prove the following theorem

Theorem 1. *Let (M, ω) be a closed, semi-positive symplectic manifold of di-*

mension $2n$. Suppose φ is a symplectomorphism on (M, ω) which is isotopic to the identity through symplectomorphisms. There exists $e > 0$ such that if all the fixed points of φ are non-degenerate, and its Calabi invariant $[\theta]$ has a representative that satisfies $|\theta|_{C^1} < e$, then the number of fixed points of φ is at least the sum of the Betti numbers of the Novikov homology over $\mathbb{Z}_2$ associated to the Calabi invariant.

Here we suppose that M is equipped with some Riemannian metric g and that the norm $|\theta|_{C^1}$ is defined via ω. The quantity e depends only on the geometry of (M, g) (see remark 5).

If M is not compact, the symplectic structure should satisfy some conditions implying reasonable behaviour at infinity. For the main theorem to hold we could have required (M, ω) to be tame (geometrically bounded). This means that there exists an almost complex structure J on M, such that $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a complete Riemannian metric whose sectional curvature is bounded and whose injectivity radius is bounded away from zero.

It is well known (see [12]) that the Novikov homology groups are isomorphic for almost all cohomology classes in $H^1(M, \mathbb{R})$ and that the rank of these groups is minimal in the class of Novikov homology groups associated to all the classes in $H^1(M, \mathbb{R})$. This enables us to estimate the number of fixed points in terms of the Novikov homology for generic 1-forms.

The main steps towards the proof of the previous theorem are the following: first we define Floer homology in this setting (this was already done in [12]). Then we relate Floer homology, for the case where the Calabi invariant is sufficiently small, to the Novikov homology associated to that Calabi invariant. This is achieved following the ideas of S. Piunikhin, D. Salamon and M. Schwarz (see [8]), not by the traditional method of proving that the connecting orbits used in the definition of Floer homology, for small enough time-independent Hamiltonian, are independent of the circle variable, and hence correspond to ordinary gradient flow lines. Finally we prove that the Floer homology groups are isomorphic under deformations that preserve the Calabi invariant. This proof is just a reinterpretation of the proof given in [12] adjusted to our setting.

It is important to note that we chose to use Seidel's approach (see [11]) to Floer homology and rephrased everything in terms of bundles over cylinders, discs and spheres. This has the advantage that we work then with pseudoholomorphic sections of certain symplectic bundles instead of perturbed curves on a non-compact symplectic manifold. In order to compute the Floer homology groups, we need to consider deformations that change the Calabi invariant. Since we will be working on a non-compact manifold, we have an additional problem. We need to prove that the image of the sections we are working with lie in some compact subset of the total space of the bundle. This will enable us to have that the 2-form, on the relevant subspace of the total space, is non-degenerate (and hence symplectic), so we can apply the weak compactness argument. To prove the latter we rely on a lower bound on the energy of these sections (monotonicity). Here is where we need the Calabi invariant to be small enough (in terms of the geometry of (M, ω, J)), so that we can obtain certain results by comparing our situation to the one that considers (unperturbed) $J-$holomorphic curves on a symplectic manifold, or equivalently, sections of trivial bundles with the product structures.

We also want to point out that the semi-positive hypothesis in the main theorem is there just to avoid dealing with pseudoholomorphic curves of negative Chern number. We expect the same result to hold for arbitrary closed symplectic manifolds. Recent results by Fukaya-Ono, Tian-Li, and Tian-Liu show that this hypothesis is no longer necessary to define or compute the Floer homology groups, but the compactification of the moduli spaces we deal with is much easier to understand, otherwise we would have to deal with the "virtual moduli cycle" (see [6] for an explanation of how to deal with these spheres). Note that L. Van and K. Ono cannot remove the hypothesis on their main theorem because, unless the symplectic form and the first Chern class are related on spheres (more precisely, $c_1|_{\pi_2(M)} = \lambda\omega|_{\pi_2(M)}$ for $\lambda \neq 0$ and if $\lambda < 0$, the minimal Chern number N satisfies $N > n-3$) they cannot compare the Floer homology groups for different Calabi invariants, and this is the key ingredient in the proof of their main theorem (see theorem 5.4 of [12]). One of the problems in this situation is that the ring they use changes when they rescale the Calabi invariant.

Finally, it should be fairly easy to improve the bound imposed to the Calabi invariant just by being more efficient when doing the estimates of chapter 4.

Chapter 2

Floer homology and Floer continuations

In this chapter we will define Floer homology and explain Floer continuations in the bundle setting for non-hamiltonian symplectomorphisms. All of this material is well known.

First of all recall that (M, ω) is semi-positive if one of the following three conditions is satisfied

1. $\langle [\omega], A \rangle = \lambda \langle c_1, A \rangle$ for every $A \in \pi_2(M)$ where $\lambda > 0$ (M is monotone).
2. $\langle c_1, A \rangle = 0$ for every $A \in \pi_2(M)$.
3. The minimal Chern number $N \geq 0$ defined by $\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}$ is greater or equal to $n - 2$.

Here $c_1 = c_1(TM, J)$ is the first Chern class of the tangent bundle with an almost complex structure J which is compatible with ω in the sense that

$$g_J(v, w) = \omega(v, Jw)$$

defines a Riemannian metric on M. These assumptions guarantee that for generic compatible J there are no $J-$holomorphic spheres with negative Chern number. Although this condition is no longer necessary to define Floer Homology, we use it to avoid the complications that dealing with these spheres present.

Let φ be an element in $Symp_0(M,\omega)$, and $\{\varphi_t\}$ for $0 \leq t \leq 1$ a path connecting the identity with $\varphi = \varphi_1$. Let X_t be the vector field generating the flow $\{\varphi_t\}$, and θ_t the family of 1-forms defined by $\theta_t(Y) = -\omega(X_t, Y)$. The $Flux$ homomorphism is a map

$$Flux : \widetilde{Symp_0}(M,\omega) \to H^1(M,\mathbb{R})$$

defined by $Flux(\widehat{\varphi}) = \left[\int_0^1 \theta_t dt\right]$. Here $\widetilde{Symp_0}(M,\omega)$ is the universal cover of $Symp_0(M,\omega)$. The Calabi invariant of $\widehat{\varphi} = [\{\varphi_t\}]$ is the image of $\widehat{\varphi}$ under this homomorphism.

The following lemma was proved by L. Vân and K. Ono (see [12]).

Lemma 1. Let $[\theta]$ be the Calabi invariant of an element $\widehat{\varphi}$. Then there exists a smooth path $\{\varphi_t\}$ in $Symp_0(M,\omega)$, joining the identity with φ, and a periodic Hamiltonian H_t on M such that $\theta_t = \theta + dH_t$ for all t.

2.1 Floer homology

Suppose we have $\{\varphi_t\}$ such that $\varphi_0 = id$, $\varphi_1 = \varphi$ and the family of 1-forms that generates this path satisfies the condition of the deformation lemma, that is, $\theta_t = \theta + dH_t$, for some periodic Hamiltonian H_t on M.

Let $p : \widetilde{M} \longrightarrow M$, be the smallest abelian cover such that the form θ is exact, that is $p^*\theta = df$. We will take $\widetilde{\omega} = p^*\omega$ to be the symplectic form on $\widetilde{M}$. The covering transformation group is isomorphic to the quotient group

$$\Gamma_1 = \frac{\pi_1(M)}{Ker\ I_\theta}$$

where $I_\theta(\delta) = \int_\delta \theta$ is the period homomorphism of θ. Since the θ_t satisfy the condition of the deformation lemma we have that for each $t \in [0,1]$ $p^*\theta_t$ is an exact 1-form, that is $p^*\theta_t = d\widetilde{H}_t$ where

$$\widetilde{H}_t = f + p^*H_t.$$

The time-dependent Hamiltonian flow on $(\widetilde{M}, \widetilde{\omega})$ generated by $\widetilde{H}_t$, say $\{\widetilde{\varphi}_t\}$, is the pullback of the original flow on M, and as before the time-1 map of this flow $\widetilde{\varphi}_1$ is denoted by $\widetilde{\varphi}$. We will assume that φ is small enough as to have $\varphi_t(p) = p$ for all $p \in fix\{\varphi\}$. This is not necessary in order to define Floer homology (see remark 9), but it is necessary when relating this homology to the Novikov homology.

We will denote by $\mathcal{P}$ the set of equivalence classes $\widetilde{y} = [(t, \widetilde{y}_0); u_0]$, where $\widetilde{y}_0$ is a fixed point of $\widetilde{\varphi}$, and u_0 is a spheres that represents a class in the image of $\pi_2(\widetilde{M}) \to H_2(\widetilde{M})$. We say that two of these spherical classes are equivalent if they have the same evaluation on ω and on c_1. The points in $\mathcal{P}$ will be the vertices of our Floer complex (associated to $\widetilde{\varphi}$). The group $\Gamma = \Gamma_1 \oplus \Gamma_2$ acts on $\widetilde{y} \in \mathcal{P}$ via

$$(\delta, A)\#\widetilde{y} = [(t, \delta \cdot \widetilde{y}_0); A + u_0]$$

where $\Gamma_2 = \frac{\pi_2(M)}{Ker\ I_\omega \cap Ker I_{c_1}}$. We will actually use the subgroup of Γ_2 defined by $\Gamma_0 = \frac{Ker I_{c_1}}{Ker\ I_\omega \cap Ker I_{c_1}}$, and we denote by $\Gamma' = \Gamma_1 \oplus \Gamma_0$.

In the usual definition of the Floer complex associated to a family of closed 1-forms θ_t (see [12], section 4) the vertices of the complex will be obtained from the elements of $\mathcal{P}(\theta_t) = \{\text{contractible 1-periodic orbits of } X_{\theta_t}\}$. Denote by $\mathcal{P}(\widetilde{H}_t)$ the lift of these loops to $\widetilde{M}$ (using p), then the vertices of the Floer complex will be equivalence classes of loops $\widetilde{x}(t) \in \mathcal{P}(\widetilde{H}_t)$ together with the discs $\widetilde{v}$ they bound. We will denote the elements by $[\widetilde{x}(t); \widetilde{v}]$, and the set by $\widetilde{\mathcal{P}(\widetilde{H}_t)}$. In fact, we have the following diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{L}\widetilde{M}} & \longrightarrow & \mathcal{L}\widetilde{M} & \longrightarrow & \widetilde{M} \\ \downarrow \widetilde{P} & & \downarrow P & & \downarrow p \\ \widetilde{\mathcal{L}}M & \xrightarrow{j} & \mathcal{L}M & \xrightarrow{e} & M \end{array}$$

Here e is just the evaluation map $x(t) \to x(0)$, and j denotes the projection associated to the action of the homomorphisms $I_\omega, I_{c_1} : \pi_2(M) \longrightarrow R$ defined by evaluation of ω and c_1 respectively. So Γ_1 is the covering transformation group associated to $p : \widetilde{M} \longrightarrow M$, and Γ_2 is the one associated to $j : \widetilde{\mathcal{L}}M \longrightarrow \mathcal{L}M$. Then $\mathcal{P}(\theta_t) \in \mathcal{L}M$, $\mathcal{P}(\widetilde{H}_t) \in \mathcal{L}\widetilde{M}$ and $\widetilde{\mathcal{P}(\widetilde{H}_t)} \in \widetilde{\mathcal{L}\widetilde{M}}$.

There is a natural grading for the Floer complex, which is given by the Conley-Zehnder index (see [4] for the definition and properties). Recall that the Conley-Zehnder index for a non-degenerate contractible 1-periodic orbit $x(t)$ bounding a disc v depends only on the trivialization on the induced complex bundle v^*TM and the linearized flow along $x(t)$. Therefore, the Conley-Zehnder index of $[\widetilde{x}(t); \widetilde{v}]$ is Γ_1-invariant, that is $\mu(g \cdot [\widetilde{x}(t); \widetilde{v}]) = \mu([\widetilde{x}(t); \widetilde{v}])$

for all $g \in \Gamma_1$. Moreover, it satisfies the following identity

$$\mu([\widetilde{x}(t); A\#\widetilde{v}]) - \mu([\widetilde{x}(t); \widetilde{v}]) = -2c_1(A) \text{ for } A \in \pi_2(M).$$

In our case θ_t is such that $\mathcal{P}(\theta_t)$ consists only of points (the zeros of θ_t). So there is a 1-1 correspondence between the elements of $\mathcal{P}$ and $\widetilde{\mathcal{P}(\widetilde{H}_t)}$ given by $\widetilde{y} = [(t, \widetilde{y}_0); u_0] \longleftrightarrow [\widetilde{y}_0; u_0]$, and we can use the Conley-Zehnder index for the elements of $\mathcal{P}$.

The ring we use is a suitable completion of the group ring $\mathbb{Z}_2[\Gamma']$ adapted to this setting, we will denote it by $\Lambda_{\theta,\omega}$.

Definition 1. The elements of $\Lambda_{\theta,\omega}$ are formal sums of the form

$$\lambda = \sum_{(\delta,A)\in\Gamma'} \lambda_{\delta,A}(\delta, A)$$

for $\lambda_{\delta,A} \in \mathbb{Z}_2$, with the following finiteness condition,

$$\#\left\{(\delta, A) \ / \ \lambda_{\delta,A} \neq 0, \ \int_A \omega > N_1 \text{ or } \int_\delta \theta \leq N_2\right\} < \infty$$

for all N_1, N_2.

Definition 2. We define the elements of $CF_*(\widetilde{H}_t)$ as the formal sums of the form

$$z = \sum_{\widetilde{y}\in\mathcal{P}} \xi_{\widetilde{y}}\widetilde{y}$$

where $\xi_{\widetilde{y}} \in \mathbb{Z}_2$ and such that for all N_1, N_2

$$\#\left\{\widetilde{y} \ / \ \xi_{\widetilde{y}} \neq 0, \ \int u_0{}^*\omega \leq N_1 \text{ and } f(\widetilde{y}_0) > N_2\right\} < \infty$$

We can easily deduce the following lemma.

Lemma 2. The chain complex $CF_*(\widetilde{H}_t)$ is a torsion-free module (of rank $\#fix\{\varphi\}$) over the Novikov ring $\Lambda_{\theta,\omega}$, via

$$\lambda * z = \sum_{\widetilde{y}\in\mathcal{P}}\left(\sum_{\delta,A}\lambda_{\delta,A}\xi_{(\delta,A)\#\widetilde{y}}\right)\widetilde{y}$$

Now, as observed by P. Seidel in [11], the idea is to define a symplectic bundle such that solutions $u : \mathbb{R}\times S^1 \longrightarrow \widetilde{M}$ of

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - \widetilde{X}_t(u)\right) = 0 \tag{2.1}$$

which are used in the definition of the boundary operator, become holomorphic sections of that symplectic bundle with respect to some suitable almost complex structure (tamed by the 2-form on the total space). Such a bundle is defined as follows

$$\begin{array}{c} T_{\widetilde{\varphi}} = (\mathbb{R}\times[0,1]\times\widetilde{M})/\sim \\ \downarrow \\ \mathbb{R}\times S^1 \end{array}$$

where the equivalence relation is given by $(s,1,x)\equiv(s,0,\widetilde{\varphi}(x))$. We can equip this bundle with an almost complex structure $J_{\widetilde{\varphi}} = \widetilde{\varphi}_t^* J_t + j$, and a 2-form

$\Omega_{\widetilde{\varphi}}$ defined by the pull-back of $\widetilde{\omega}$ to $\mathbb{R} \times [0,1] \times \widetilde{M}$ (this form descends to the quotient $T_{\widetilde{\varphi}}$). Although this form is not symplectic we have the following

Lemma 3. The form $\Omega_{\widetilde{\varphi}}$ is non-negative on $J_{\widetilde{\varphi}}$-holomorphic sections, that is, $\Omega_{\widetilde{\varphi}}(v, J_{\widetilde{\varphi}}v) \geq 0$ for all v.

Proof. Let $v = (\alpha, \beta, \xi)$ be a vector tangent to $T_{\widetilde{\varphi}}$ at a point $p = (s, t, x)$, then $\Omega_{\widetilde{\varphi}}(v, J_{\widetilde{\varphi}}v) = \Omega_{\widetilde{\varphi}}((\alpha, \beta, \xi), (-\beta, \alpha, \widetilde{\varphi}_t^* J_t \xi)) = \widetilde{\omega}(\xi, \widetilde{\varphi}_t^* J_t \xi) \geq 0$. □

It is easy to verify the following lemma.

Lemma 4. For each solution $u : \mathbb{R} \times S^1 \longrightarrow \widetilde{M}$ of (2.1) the map $\sigma(s,t) = (s, t, \widetilde{\varphi}_t^{-1} u(s,t))$ is a $J_{\widetilde{\varphi}}$–holomorhic section of $T_{\widetilde{\varphi}}$ and conversely.

Observe that the bundle $T_{\widetilde{\varphi}} \longrightarrow \mathbb{R} \times S^1$ is isomorphic to the trivial bundle via the map

$$\alpha : T_{\widetilde{\varphi}} \longrightarrow \mathbb{R} \times S^1 \times \widetilde{M} \text{ given by } \alpha(s,t,x) = (s, t, \widetilde{\varphi}_t(x))$$

which is such that $\alpha(\mathbb{R} \times S^1 \times \{p\}) = \mathbb{R} \times S^1 \times \{p\}$ for any $p \in fix\{\widetilde{\varphi}\}$. In fact, $\alpha \circ \sigma = graph(u)$.

Let $S_0(\sigma)$ be the sphere in $\widetilde{M}$ defined by $S_0(\sigma) = p_F \circ \alpha(\sigma)$ where $p_F : \mathbb{R} \times S^1 \times \widetilde{M} \longrightarrow \widetilde{M}$ is just the projection onto the fiber.

Now for $\widetilde{x} = [(t, \widetilde{x}_0); w_0]$ and $\widetilde{y} = [(t, \widetilde{y}_0); v_0]$ we define

$$\mathcal{M}(\widetilde{x}, \widetilde{y}) = \left\{ \begin{array}{c} J_{\widetilde{\varphi}} - \text{holomorphic sections } \sigma : \mathbb{R} \times S^1 \longrightarrow T_{\widetilde{\varphi}}, \text{ such that} \\ \sigma \underset{s \to -\infty}{\longrightarrow} \widetilde{x},\ \sigma \underset{s \to \infty}{\longrightarrow} \widetilde{y} \text{ and } v_0 = w_0 + [S_0(\sigma)] \end{array} \right\}$$

We denote by $\widehat{\mathcal{M}}(\widetilde{x},\widetilde{y}) = \mathcal{M}(\widetilde{x},\widetilde{y})/\mathbb{R}$. Here two sections σ and σ' are equivalent iff $\sigma' = \rho_r^*\sigma$ for some $r \in \mathbb{R}$, where $\rho_r : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times S^1$ is defined by $\rho_r(s,t) = (s+r,t)$. This equivalence relation comes from the fact that we can reparametrize solutions of (2.1) so we don't want to distinguish between $\sigma(s,t) = (s,t,\widetilde{\varphi}_t^{-1}u(s,t))$ and $\sigma'(s,t) = (s,t,\widetilde{\varphi}_t^{-1}u(s+r,t))$. We are assuming that the parameters are regular (in the sense of theorems6 and 7 of the appendix) so that the moduli spaces are manifolds of the right dimension.

Definition 3. The boundary operator $\partial_F : CF_*(\widetilde{H}_t) \longrightarrow CF_*(\widetilde{H}_t)$ is defined as follows

$$\partial_F(\widetilde{x}) = \sum_{\widetilde{y}\in\mathcal{P}} \#\{\textit{isolated points in } \widehat{\mathcal{M}}(\widetilde{x},\widetilde{y})\}\ \widetilde{y}$$

In order to see that this map is well defined we need to check that

1. There are finitely many isolated points in $\widehat{\mathcal{M}}(\widetilde{x},\widetilde{y})$ for every pair of $\widetilde{x},\widetilde{y} \in \mathcal{P}$.
2. The finiteness condition holds and $\partial_F(\widetilde{x}) \in CF_*(\widetilde{H}_t)$. More explicitly, we need to see that for all N_1, and N_2 the set

$$\left\{\widetilde{y} \ / \ \xi_{\widetilde{y}} \neq 0,\ \int u_0{}^*\omega \leq N_1 \text{ and } f(\widetilde{y}_0) > N_2\right\}$$

has a finite number of elements. Here

$$\xi_{\widetilde{y}} = \#\{\textit{isolated points in } \widehat{\mathcal{M}}(\widetilde{x},\widetilde{y})\}.$$

For a given section we define the section energy by

$$E_s(\sigma) = \int \sigma^* \Omega_{\widetilde{\varphi}}.$$

This does not correspond to the usual definition of energy of a pseudoholomorphic curve because the form, $\Omega_{\widetilde{\varphi}}$, on the top manifold is not symplectic, so $\Omega_{\widetilde{\varphi}}(\cdot, J_{\widetilde{\varphi}}\cdot)$ is not a metric. Now for any $\sigma \in \mathcal{M}(\widetilde{x}, \widetilde{y})$ observe that

$$E_s(\sigma) = \int \sigma^* \Omega_{\widetilde{\varphi}} = \int_{S_0(\sigma)} \widetilde{\omega} = -\int_{w_0} \widetilde{\omega} + \int_{v_0} \widetilde{\omega}$$

is fixed by its ends.

The main issue in this argument is to prove compactness for the relevant moduli spaces, in this case $\widehat{\mathcal{M}}(\widetilde{x}, \widetilde{y})$, in order to conclude that, whenever the dimension is zero, they consist of a finite number of elements for every pair $\widetilde{x}, \widetilde{y} \in \mathcal{P}$. There are two ways of doing this. The first one is to use lemma 4 and consider the elements of the moduli spaces as curves in $\widetilde{M}$. This is not hard because although we are working with perturbed $J-$holomorphic curves, the perturbation doesn't change with the parameter s. More precisely, we are dealing with curves $u : \mathbb{R} \times S^1 \longrightarrow \widetilde{M}$ that are solutions of

$$\frac{\partial u}{\partial s} + J_t(u)(\frac{\partial u}{\partial t} - \widetilde{X}_t(u)) = 0$$

such that $u \underset{s\to-\infty}{\longrightarrow} \widetilde{x}_0$, $u \underset{s\to\infty}{\longrightarrow} \widetilde{y}_0$ and $v_0 = w_0 + [u]$. We have that the (usual) energy of such a curve is

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left(\left|\frac{\partial u}{\partial s}\right|_{J_t}^2 + \left|\frac{\partial u}{\partial t} - \widetilde{X}_t(u)\right|_{J_t}^2 \right) dtds \tag{2.2}$$

$$= \int u^*\widetilde{\omega} + \int_0^1 \left(\widetilde{H}_t(\widetilde{x}_0) - \widetilde{H}_t(\widetilde{y}_0)\right) dt$$

$$= -\int_{w_0}\widetilde{\omega} + \int_{v_0}\widetilde{\omega} + \int_0^1 \left(\widetilde{H}_t(\widetilde{x}_0) - \widetilde{H}_t(\widetilde{y}_0)\right) dt$$

and this last quantity is fixed by the ends. Since we are assuming the parameters to be regular, the result follows from the weak compactness argument, see theorem 8 in the appendix. We just need to observe that $p \circ u \in \mathcal{M}(x_0, y_0; \theta_t, J)$, where $x_0 = p(\widetilde{x}_0)$ and $y_0 = p(\widetilde{y}_0)$.

We also need to check that $\partial_F(\widetilde{x}) \in CF_*(\widetilde{H}_t)$, that is we need to verify the finiteness condition. Suppose we have $u_\nu \in \mathcal{M}(\widetilde{x}, \widetilde{y}_\nu)$ where $\widetilde{y}_\nu = [(t, \widetilde{y}_0^\nu); v_\nu]$ are such that $\int_{v_\nu}\widetilde{\omega} \leq N_1$ and $f(\widetilde{y}_0^\nu) > N_2$ for some N_1 and N_2. Then

$$E(u_\nu) = \int u_\nu^*\widetilde{\omega} + \int_0^1 \left(\widetilde{H}_t(\widetilde{x}_0) - \widetilde{H}_t(\widetilde{y}_0^\nu)\right) dt$$

$$= -\int_{w_0}\widetilde{\omega} + \int_{v_\nu}\widetilde{\omega} + \int_0^1 \left(\widetilde{H}_t(\widetilde{x}_0) - \widetilde{H}_t(\widetilde{y}_0^\nu)\right) dt$$

$$\leq -\int_{w_0}\widetilde{\omega} + N_1 + \int_0^1 \widetilde{H}_t(\widetilde{x}_0)dt - f(\widetilde{y}_0^\nu) - \int_0^1 H_t \circ p(\widetilde{y}_0^\nu)dt$$

Using that $-f(\widetilde{y}_0^\nu) < -N_2$ and the fact that H_t has compact support (so it is bounded) we get a uniform bound on the energy. Now by the weak compactness argument (see theorem 8) we conclude that the sequence u_ν has a convergent subsequence. This means that the ends of the curves have to be the same for sufficiently large values of ν, so we have only finitely many different $\widetilde{y}_\nu$ as we wanted (for more details see section 4 of [12]).

The second approach that we now describe is easier to adapt to the more general situation of Floer continuations considered below. We argue as before but now looking at the elements of $\mathcal{M}(\widetilde{x}, \widetilde{y})$ as holomorphic sections. Since $\Omega_{\widetilde{\varphi}}$ is non-negative on sections, that is, $\int \sigma^* \Omega_{\widetilde{\varphi}} \geq 0$ we can make this form symplectic by adding the pull-back of the area form of the base. The only problem would be that the base has infinite area. Take a sequence σ_n of elements in $\mathcal{M}(\widetilde{x}, \widetilde{y})$ we want to see that it has a convergent subsequence. First observe that since $\int \sigma_n^* \Omega_{\widetilde{\varphi}} = a$ is fixed for all n (uniformly bounded would be enough) we have that the image of these sections remains in a piece of $T_{\widetilde{\varphi}}$ with compact fiber. This is an easy consequence of the monotonicity lemma (see section 4.1).

Let $\widehat{\Omega}_{\widetilde{\varphi}} = \Omega_{\widetilde{\varphi}} + ds \wedge dt$ be the symplectic form on $T_{\widetilde{\varphi}}$. Cut the cylinder at $[-2N, 2N]$ in order to have a uniform bound on the area of this sections. For each N we have $\int_{[-2N,2N]\times S^1} \sigma_n^* \widehat{\Omega}_{\widetilde{\varphi}} \leq \int_{S_0(\sigma_n)} \widetilde{\omega} + 4N = a + 4N$ is uniformly bounded, so by the weak compactness argument we end up with a subsequence (still denoted by σ_n) so that $\sigma_n|_{[-N,N]}$ converges. Now we use this subsequence and repeat the above process but now cutting the cylinder at a larger value of N. In the limit we will have a subsequence of σ_n (on the whole cylinder) that converges uniformly with all derivatives on compact sets to a holomorphic section.

We also need to check that $\partial_F(\widetilde{x}) \in CF_*(\widetilde{H}_t)$, that is we need to check the finiteness condition. Suppose we have distinct elements $\sigma_n \in \mathcal{M}(\widetilde{x}, \widetilde{y}_n)$, and $\widetilde{y}_n$ for $n \in \mathbb{Z}$ are such that $\int_{v_n} \widetilde{\omega} \leq N_1$ and $f(\widetilde{y}_0^n) > N_2$ for some N_1 and N_2. Observe that since $\int_{v_n} \widetilde{\omega} \leq N_1$, we have a bound on $\int_{S_0(\sigma_n)} \widetilde{\omega}$ independent of

n, that is

$$\int_{S_0(\sigma_n)} \widetilde{\omega} = -\int_{w_0} \widetilde{\omega} + \int_{v_n} \widetilde{\omega} \leq -\int_{w_0} \widetilde{\omega} + N_1$$

where $\widetilde{x} = [(t, \widetilde{x}_0); w_0]$, and $\widetilde{y}_n = [(t, \widetilde{y}_0^n); v_n]$. We can use the previous discussion to conclude that these trajectories converge (they converge up to bubbling, but the hypothesis of the main theorem assure that we can avoid bubbles generically for dimensional reasons), so the ends have to be the same for large enough n and we get a contradiction.

Remark 1. As we observed above there are two ingredients involved in getting the necessary compactness of the moduli spaces of sections. They are

1. Controlling the section energy of σ, that is $\int \sigma^*\Omega_{\widetilde{\varphi}}$, where $\Omega_{\widetilde{\varphi}}$ is non-negative. The bounds depend only on the end points of the sections.

2. Knowing that the images of these sections lie in some subset, of the total manifold of the bundle, that has compact fiber. This set is also determined by the ends.

Finally, to see that $\partial_F^2 = 0$ we use the usual gluing and compactness arguments (see [12]). Also note that since ∂_F is invariant under the action of Γ' we can extend it to be a $\Lambda_{\theta,\omega}$−linear map.

The homology of the complex $\left(CF_*(\widetilde{H}_t), \partial_F\right)$ is the Floer homology associated to (M, ω, θ_t, J), which we will denote by $HF_*(M, \omega, \theta_t, J; \mathbb{Z}_2)$.

2.2 Floer continuations

As we did with the boundary operator, we will now work out an alternative interpretation of Floer continuations (taken from [11] and adapted to our situation). This will enable us to compare the Floer homology groups obtained for different choices of parameters.

Let $\widetilde{\gamma} : \mathbb{R} \rightarrow Ham(\widetilde{M}, \widetilde{\omega})$ be a smooth path of hamiltonian symplectomorphisms of $(\widetilde{M}, \widetilde{\omega})$ which is constant outside some compact set. We will work basically with two kinds of paths. First let us consider the family of exact 1-forms $p^*\theta_{s,t} = d\widetilde{H}_{s,t}$, where $\widetilde{H}_{s,t} = \rho(s)(f + p^*H_t)$ for H_t any periodic time-dependent Hamiltonian on M and ρ is a smooth function such that

$$\rho(s) = \begin{cases} 0 & s \leq 1 \\ 1 & s \geq 2 \end{cases}$$

For each fixed s we denote by $\{\widetilde{\varphi}_{s,t}\}$ the Hamiltonian flow of $\widetilde{H}_{s,t}$ for $0 \leq t \leq 1$, and $\widetilde{\gamma}_s \in Ham(\widetilde{M}, \widetilde{\omega})$ denotes the time-1 map of this flow, that is, $\widetilde{\gamma}_s = \widetilde{\varphi}_{s,1}$. Note that in this case, the Calabi invariant of the corresponding path in M changes with s (since $\theta_{s,t} = \rho(s)(\theta + dH_t)$).

Let $u : \mathbb{R} \times S^1 \longrightarrow \widetilde{M}$ be a cylinder satisfying

$$\frac{\partial u}{\partial s} + J_{s,t}(u)(\frac{\partial u}{\partial t} - \widetilde{X}_{s,t}(u)) = 0 \tag{2.3}$$

where $\widetilde{X}_{s,t}$ is the Hamiltonian vector field induced by $\widetilde{H}_{s,t}$, and $J_{s,t}$ is a 2-parameter family of almost complex structures that is constant outside some compact set. This is the kind of connecting orbit generally used when relating the Floer homology groups for two different sets of parameters (in this case we relate $(0, J)$ to $(\widetilde{H}_t, J_t)$). As before we want to construct a symplectic bundle so that solutions of (2.3) will become holomorphic sections of that symplectic bundle with respect to some suitable almost complex structure. Such a bundle is defined as follows

$$\begin{array}{c} E_{\widetilde{\gamma}} = (\mathbb{R} \times [0,1] \times \widetilde{M})/\sim \\ \downarrow \\ \mathbb{R} \times S^1 \end{array}$$

where the equivalence relation is given by $(s, 1, x) \equiv (s, 0, \widetilde{\gamma}_s(x))$. This bundle has a connection whose projections onto the vertical spaces are

$$\Pi_{s,t,x}(\alpha, \beta, \xi) = \xi - \alpha Y_{s,t,x}$$

where Y is the vector field defined by

$$Y_{s,t,x} = \frac{\partial \widetilde{\varphi}_{s,t}^{-1}}{\partial s}(\widetilde{\varphi}_{s,t}(x))$$

Using this we can define an almost complex structure as follows

$$J_{\widetilde{\gamma}}(\alpha, \beta, \xi) = (-\beta, \alpha, \widetilde{\varphi}_{s,t}^{*} J_{s,t}\,(\xi - \alpha Y) - \beta Y).$$

This almost complex structure agrees with $\widetilde{\varphi}_{s,t}^{*} J_{s,t}$ on the vertical subspaces.

Now let σ be a section of $E_{\widetilde{\gamma}}$ given by

$$\sigma(s,t) = (s,t,\widetilde{\varphi}_{s,t}^{-1}u(s,t))$$

Then σ is $J_{\widetilde{\gamma}}$–holomorphic iff

$$D\widetilde{\varphi}_{s,t}(\frac{\partial\sigma}{\partial t}) = J_{s,t}D\widetilde{\varphi}_{s,t}(\frac{\partial\sigma}{\partial s} - Y)$$

Using

$$D\widetilde{\varphi}_{s,t}(\frac{\partial\sigma}{\partial t}) + \frac{\partial\widetilde{\varphi}_{s,t}}{\partial t}(\sigma) = \frac{\partial u}{\partial t}, \qquad D\widetilde{\varphi}_{s,t}(\frac{\partial\sigma}{\partial s} - Y(\sigma)) = \frac{\partial u}{\partial s}$$

we get

Lemma 5. A section $\sigma(s,t) = (s,t,\widetilde{\varphi}_{s,t}^{-1}u(s,t))$ of $E_{\widetilde{\gamma}} \longrightarrow \mathbb{R}\times S^1$ is $J_{\widetilde{\gamma}}$-holomorphic iff $u : \mathbb{R}\times S^1 \longrightarrow \widetilde{M}$ satisfies (2.3).

We want to equip $E_{\widetilde{\gamma}}$ with a closed 2-form that extends $\widetilde{\omega}$ using our connection. First, let $\widetilde{\omega}' = \widetilde{\omega} - ds \wedge i_Y\widetilde{\omega}$. However this form is not closed,

$$d\widetilde{\omega}' = ds \wedge dt \wedge i_{\frac{\partial Y}{\partial t}}\widetilde{\omega}$$

but since our maps $\widetilde{\varphi}_{s,t}$ are actually Hamiltonian on $\widetilde{M}$, we have that

$$\frac{\partial}{\partial t}(i_Y\widetilde{\omega}) = dK_{s,t}. \qquad (2.4)$$

On $\mathbb{R}\times\mathbb{R}\times\widetilde{M}$ we have $ds \wedge dt \wedge \frac{\partial}{\partial t}(i_Y\widetilde{\omega}) = d(K_{s,t}ds \wedge dt)$, and we can choose $K_{s,t}$ so that it descends to $E_{\widetilde{\gamma}}$. So we have a function $K : E_{\widetilde{\gamma}} \longrightarrow \mathbb{R}$ defined by

$K[(s,t,x)] = K_{s,t}(x)$. Finally our closed 2-form is

$$\Omega_{\widetilde{\gamma}} = \widetilde{\omega} - ds \wedge i_Y\widetilde{\omega} - K_{s,t}ds \wedge dt \tag{2.5}$$

Note that the functions $K_{s,t}$ are not bounded. They do not pull back from M because the Calabi invariant of γ_s (the path of time-1 maps in M) varies with s. This implies that we cannot make $\Omega_{\widetilde{\gamma}}$ symplectic, that is, there is no way to make this form non-degenerate by adding a large enough multiple of the form on the base.

Now let us consider the case where $\widetilde{\gamma}$ is the pull back of a path $\gamma_s \in Symp_0(M,\omega)$, such that the Calabi invariant doesn't change with s. More precisely, γ_s is the family of time-one maps of the flow on M induced by $\eta_{s,t} = \theta_t + \rho(s)dR_t$. Then $\widetilde{\gamma}_s$ is the corresponding family (on $\widetilde{M}$) induced by $\widetilde{L}_{s,t} = \widetilde{H}_t + \rho(s)p^*R_t$, where $p^*\eta_{s,t} = d\widetilde{L}_{s,t}$. We can repeat the previous construction now using this kind of path to get

$$\begin{array}{c} E_{\widetilde{\gamma}} = (\mathbb{R} \times [0,1] \times \widetilde{M})/\sim \\ \downarrow \\ \mathbb{R} \times S^1 \end{array}$$

where the equivalence relation is given by $(s,1,x) \equiv (s,0,\widetilde{\gamma}_s(x))$. By the previous process we obtain an almost complex structure $J_{\widetilde{\gamma}}$, and a closed 2-form $\Omega_{\widetilde{\gamma}}$.

Holomorphic sections of $E_{\widetilde{\gamma}}$ are in one-to-one correspondence to solutions

$u : \mathbb{R} \times S^1 \longrightarrow \widetilde{M}$ of

$$\frac{\partial u}{\partial s} + J_{s,t}(u)(\frac{\partial u}{\partial t} - X_{\widetilde{H}_t}(u) - X_{\rho(s)p^* R_t}(u)) = 0$$

Observe that in this case the form we add to $\widetilde{\omega} - ds \wedge i_{Y^\gamma}\widetilde{\omega}$ in order to get the closed 2-form $\Omega_{\widetilde{\gamma}}$, say $-K^{\gamma}_{s,t} ds \wedge dt$, is such that $K^{\gamma}_{s,t}$ is bounded (it is the pull back of functions on M). Hence this form can be made symplectic by adding a large enough multiple of the area form of the base.

Remark 2. In conclusion, for each smooth path $\widetilde{\gamma} : \mathbb{R} \rightarrow Ham(\widetilde{M}, \widetilde{\omega})$ of hamiltonian symplectomorphisms on $(\widetilde{M}, \widetilde{\omega})$ which is constant outside some compact set of $\mathbb{R}$ we can construct a bundle as above. The holomorphic sections of the bundle will be in 1-to-1 correspondence with perturbed holomorphic curves, where the perturbation is determined by $\widetilde{\gamma}$.

Chapter 3

Novikov homology

In this chapter we will briefly define the Novikov homology and list a few useful facts. For more details and proofs we refer the reader to the appendix C of [12].

Let θ be a closed 1-form on M which is not exact. Let $p : \widetilde{M} \longrightarrow M$ be the smallest abelian cover on which θ is exact. Then there exists a function $f : \widetilde{M} \longrightarrow \mathbb{R}$ such that $p^*\theta = df$. For a generic Riemannian metric μ on M, the gradient flow of f with respect to $p^*\mu$ is of Morse-Smale type and the Novikov complex $CN_*(\theta, g)$ is defined in the same way as the Morse complex. An element of the nth Novikov chain group associated to θ is a formal sum of the form

$$x = \sum_{x_0 \in crit\{f\}} a_{x_0} x_0$$

where the index of the Hessian of f at x_0 is n and the set

$$\{x_0 \mid a_{x_0} \neq 0 \text{ and } f(x_0) > N\}$$

is finite for all values of N. The boundary operator is defined using the trajectories of the gradient flow of f, that is, we count the isolated points in

$$\mathcal{M}(x_0, y_0) = W^u(x_0) \cap W^s(y_0)$$

where $y_0 \in crit\{f\}$. Here $W^u(x_0)$ and $W^s(x_0)$ denote the unstable and stable manifolds of x_0 respectively. The Novikov ring Λ_θ is defined as the completion of the group ring of the covering transformation group of $p : \widetilde{M} \longrightarrow M$ with respect to the period homomorphism I_θ. Its elements are formal sums of the form

$$\lambda = \sum_{\delta \in \Gamma_1} \lambda_\delta \delta, \text{ where } \lambda_\delta \in \mathbb{Z}_2$$

subject to the condition that the set

$$\left\{ \delta \mid \lambda_\delta \neq 0 \text{ and } \int_\delta \theta \leq N \right\}$$

is finite for all values of N. The complex $CN_*(\theta, \mu)$ is a graded module over Λ_θ via the action

$$\lambda * x = \sum_{x_0 \in crit\{f\}} \left(\sum_{\delta \in \Gamma_1} \lambda_\delta a_{\delta \cdot x_0} \right) x_0$$

The homology groups $Nov_*(\theta, \mu)$ of $CN_*(\theta, \mu)$ is called the Novikov homology associated to θ.

Some facts about the Novikov homology:

1. $Nov_*(\theta, \mu)$ is independent of the choice of Riemannian metric μ for which

the flow of f is of Morse-Smale type.

2. $Nov_*(\theta)$ depends only on the projective class of the cohomology class of θ.

3. If h is a Morse function on M, then p^*h is a Morse function on $\widetilde{M}$. The Novikov homology can be computed from the Morse complex of p^*h, that is, $Nov_*(\theta) = H_*(C_*(p^*h) \otimes_{\mathbb{Z}_2[\Gamma_1]} \Lambda_\theta)$. We will refer to the ranks of this groups over the Novikov ring Λ_θ, as the Novikov-Betti numbers.

Now we want to consider the Novikov complex defined as a module over the ring $\Lambda_{\theta,\omega}$ defined in chapter 2. More explicitly, we think of elements of $CN_*(\theta,\omega)$ as formal sums of the form

$$\xi = \sum_{(x_0,A)} \xi_{x_0,A}(x_0,A)$$

with coefficients $\xi_{x_0,A} \in \mathbb{Z}_2$, where $x_0 \in crit(f)$ and $A \in \Gamma_2$. The index of (x_0,A) is defined by $\mu(x_0,A) = ind_f(x_0) - 2c_1(A)$. We impose the finiteness condition

$$\{(x_0,A) \ / \ \xi_{x_0,A} \neq 0, \ \int_A \omega \leq N_1 \text{ and } f(x_0) > N_2\} < \infty$$

for all N_1, N_2.

We define the boundary operator ∂_N as follows

$$\partial_N(x_0,A) = \sum_{y_0} n(x_0,y_0)(y_0,A)$$

where $n(x_0, y_0)$ counts the number of isolated trajectories of the negative gradient flow of f, that is the elements of $\mathcal{M}(x_0, y_0)$. As groups we have that $Nov_*(\theta, \omega) \cong Nov_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega}$.

We have Künneth's formula in Novikov homology theory

Theorem 2. *Let M and N be closed manifolds and ξ and η closed 1-forms on M and N respectively. If the kernel of the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$ is the direct sum of the kernels of the weight homomorphism for ξ and η, we have*

$$Nov_*(M \times N; \pi_1^*\xi + \pi_2^*\eta) \cong (Nov_*(M;\xi) \otimes_{\mathbb{Z}_2} Nov_*(N;\eta)) \otimes_{\Lambda_\xi \otimes \Lambda_\eta} \Lambda_{\pi_1^*\xi + \pi_2^*\eta}.$$

Theorem 3. *The Euler number of the Novikov homology, that is, the alternating sum of the ranks of the Novikov homology groups over Λ_θ, equals the Euler number of the ordinary homology of M.*

Chapter 4

Energy Bounds

The computations of this chapter are the main ingredient needed to have the necessary compactness of the moduli spaces we use. The estimate in section 4.1 is obtained by perturbing a standard result. However, the estimate in section 4.3 is new.

4.1 Monotonicity lemma

Here we will give a lower bound for the energy of holomorphic sections of certain bundles over discs. This bundles are constructed using a path in $Ham(\widetilde{M}, \widetilde{\omega})$, and the result will be achieved only for small paths, where the notion of small will be clear by the end of this section.

Let $E_0 = (\mathbb{R} \times [0,1] \times \widetilde{M}) / \sim_0$ where the equivalence relation is given by $(s,1,x) \equiv (s,0,x)$, and $E_{\widetilde{\gamma}}$ as before, be bundles over $\mathbb{R} \times S^1$.

The following two lemmas are taken from [1]. The proofs are included so that the reader can observe that controlling the injectivity radius of is the basic ingredient in order to have the desired result, namely proposition 9.

Let (W, ω, J) be a tame (geometrically bounded) symplectic manifold, and μ is the metric obtained from ω and J. For a $J-$curve S and a point x on S, let us consider the intersection of S with the ball $B(x, \varepsilon)$ in W.

Definition 4. For $x \in S$, we will say that the curve S is properly embedded in the ball $B(x, \varepsilon)$ if $S \cap B(x, \varepsilon)$ is compact and its boundary is contained in the boundary $\partial B(x, \varepsilon)$.

Since we assume W to have bounded geometry we have that all the constants mentioned in the following lemmas do not depend on the considered points x, y...

Lemma 6. (Monotonicity lemma). There are constants c_0 and ε_0 such that for every $\varepsilon \leq \varepsilon_0$, every $J-$holomorphic curve S, and every $x \in S$ we have

$$area_\mu(S \cap B(x, \varepsilon)) \geq \frac{\pi}{1 + c_0\varepsilon}\varepsilon^2$$

provided that S is properly embedded in the ball $B(x, \varepsilon)$.

This result will be an easy consequence of

Lemma 7. There are constants c_1 and ε_1 such that for every compact J-holomorphic curve S (with boundary) contained in a ball $B(x, \varepsilon)$ with $\varepsilon < \varepsilon_1$, the area a and the length ℓ of the boundary verify

$$\ell^2 \geq \frac{4\pi}{1 + c_1\varepsilon}a$$

Proof. We will give the ball $B(x, \varepsilon)$ two sets of structures, for small enough ε. The first set is just (ω, J, μ) where μ is the metric obtained from ω and

J. The second set will be denoted by (ω_x, J_x, μ_x) and consists of the constant structures obtained on T_xW by translation, so if we choose $\varepsilon_1^{'}$ small enough the exponential map allows us to carry all these structures onto $B(x, \varepsilon_1^{'})$. In what follows the norm notation without a lower index x will denote the use of the metric μ, and similarly the norm notation with a lower index x will mean the use of μ_x.

We have at every $y \in B(x, \varepsilon_1^{'})$ the following inequalities:

1. $||\mu(y) - \mu_x(y)|| \leq c_\mu dist(x, y)$

2. $||J(y) - J_x(y)|| \leq c_J dist(x, y)$

for some constants c_μ and c_J independent of x and y. This two inequalities will enable us to compare the lengths and areas of μ_x to those of μ. In fact there is a constant c (c is such that $1 + c\varepsilon \geq \sqrt{1 + c_\mu \varepsilon}$) so that for every curve contained in the ball $B(x, \varepsilon)$ for $\varepsilon \leq \varepsilon_1^{'}$, the respective lengths ℓ_x and ℓ verify:

$$(1 + c\varepsilon)\ell \geq \ell_x$$

The next step is to find a relation between the area a of S and the integral $\int_S \omega_x$. Consider an orthonormal (for μ) frame (U, JU) of T_yS. Using the previous inequalities (repeatedly) plus the following relation

$$|\omega_x(U, JU - J_xU)| \leq ||J - J_x||_x \, ||U||_x^2$$

we can obtain:

$$\omega_x(U, JU) \geq 1 - cdist(x, y)$$

which implies that for any $J-$curve S as in the lemma, contained in some $B(x, \varepsilon)$ with $\varepsilon < \varepsilon_1^{'}$, there is c such that

$$\int_S \omega_x \geq (1 - c\varepsilon)a.$$

Finally a homological argument and Wirtinger's inequality allow us to relate the preceding integral to the length ℓ as follows. Let us go back to $T_x W$ and consider there the image of S (still denoted by S). Choose a surface $S^{'}$ which has the same boundary as S and whose μ_x-area $a_x^{'}$ verifies the isoperimetric inequality $\ell_x^2 \geq 4\pi a_x^{'}$. For a hand-made solution see [1].

Applying Wirtinger's inequality to $S^{'}$ and using the previous estimate we obtain (for suitable c)

$$\begin{aligned}(1 + c\varepsilon)\ell^2 &\geq \ell_x^2 \geq 4\pi a_x^{'} \geq 4\pi \int_{S^{'}} \omega_x = 4\pi \int_S \omega_x \\ &\geq 4\pi(1 - c\varepsilon)a\end{aligned}$$

Then if we choose ε_1 so that $\varepsilon_1 \leq \varepsilon_1^{'}$ and $2c\varepsilon_1 \leq 1$ we get

$$\ell^2 \geq \left(\frac{4\pi}{1 + c_1\varepsilon}\right) a.$$

□

Proof. (of the monotonicity lemma) Let $A(\varepsilon)$ be the area of the intersection

$S \cap B(x,\varepsilon)$. Its derivative with respect to ε is related to the length $L(\varepsilon)$ of the boundary $\partial(S \cap B(x,\varepsilon))$ by $A'(\varepsilon) \geq L(\varepsilon)$. The lemma yields for every $\varepsilon \leq \varepsilon_1$

$$\begin{aligned} \left(A'(\varepsilon)\right)^2 &\geq L^2(\varepsilon) \geq \frac{4\pi}{1+c_1\varepsilon} A(\varepsilon) \\ &\Longleftrightarrow \frac{A'(\varepsilon)}{\sqrt{A(\varepsilon)}} \geq \sqrt{\frac{4\pi}{1+c_1\varepsilon}} \end{aligned}$$

Integration from 0 to ε together with Taylor's formula gives

$$A(\varepsilon) \geq \pi\varepsilon^2 \left(1 - \frac{c_1}{2}\varepsilon\right)$$

and hence the result by taking for example $c_0 = c_1$ and $c_1\varepsilon_0 \leq 1$, $\varepsilon_0 \leq \varepsilon_1$. $\square$

Consider (E_0, g_0) where g_0 is the metric induced by $\widetilde{\omega} + ds \wedge dt$ and $J_0 = j + J_t$. Denote by $c_0, \varepsilon_0 > 0$ the constants obtained from the monotonicity lemma for (M, ω, J). Actually what we denote by c_0 is the quantity $\frac{\pi}{1+c_0\varepsilon_0}$ obtained from the lemma. Recall that $J-$holomorphic curves in $\widetilde{M}$ correspond to holomorphic sections of E_0. So we have the following monotonicity result, now for sections of E_0.

Corollary 8. For every $\varepsilon \leq \varepsilon_0$, every holomorphic section $\widehat{S}$ and every $p \in \widehat{S}$ we have

$$area_{g_0}(\widehat{S} \cap \widehat{B}(p,\varepsilon)) \geq c_0\varepsilon^2$$

provided that $\widehat{S}$ is properly embedded in the ball $\widehat{B}(p,\varepsilon)$.

For each family of 1-forms $\eta_t = \theta + dL_t$, where L_t is any time-dependent Hamiltonian on M, we have a bundle $E_\eta = (\mathbb{R} \times [0,1] \times \widetilde{M})/\sim$. Here the equivalence relation is given by $(s, 1, x) \equiv (s, 0, \widetilde{\psi}_{s,1}(x))$ where $\widetilde{\psi}_{s,t}$ denotes the flow generated by $\widetilde{L}_{s,t} = \rho(s)(f + p^* L_t)$ for $0 \leq t \leq 1$. We have a closed 2-form Ω_η and an almost complex structure J_η associated to these bundles. The bundles constructed in this way are isomorphic to the product bundle via the map $\beta_\eta : E_0 \longrightarrow E_\eta$ defined by $\beta_\eta(s, t, x) = (s, t, \widetilde{\psi}_{s,t}^{-1}(x))$. This construction is done in detail in chapter 2 (Floer continuations).

Let

$$V_0 = (\mathbb{R} \times [0, 1] \times \overline{\Delta}_0)/ \sim_0$$

where $\overline{\Delta}_0$ is the closure of Δ_0, a fundamental domain of the cover $p : \widetilde{M} \to M$. We denote by $V_\eta = \beta(V_0)$ the corresponding subset of E_η which has compact fiber. In the following proposition we will work only with $s \in [1, 2]$, so even though we don't change the notation, we are working with the bundles restricted to $[1, 2] \times S^1$. Since V_η have compact fiber we can make the form Ω_η symplectic there. Let's denote this form by

$$\Omega_{V_\eta} = \Omega_\eta + k_\eta g(s) ds \wedge dt$$

where g is a cut-off function on $\mathbb{R}$ with $g(s) = 1$ for $s \in [1, 2]$. We can assume that $k_\eta = 1$ for $|\eta|$ near zero, so $\Omega_{V_0} = \widetilde{\omega} + ds \wedge dt$. We can actually have that the change in the parameter s occurs for $s \in [1 + \delta, 2 - \delta]$ for very small $\delta > 0$, so that g will have support contained in $[1, 2]$ and $1 \geq m = \int_{\mathbf{R}} g(s) ds$. This V_η acts as a "fundamental domain" in E_η, under the obvious action of Γ_1 induced from that on E_0.

Remark 3. Since

$$(V_0, g_0|_{V_0}) \subset (E_0, g_0),$$

we have that for every J_0-holomorphic section $\widehat{S}$ in V_0, and for $x \in \widehat{S}$ the following holds

$$area_{g_0}(\widehat{S} \cap \widehat{B}(p, \varepsilon_0)) \geq c_0 \varepsilon_0^2$$

provided that $\widehat{S}$ is properly embedded in the ball $\widehat{B}(p, \varepsilon_0)$. The values of c_0, ε_0 are the same ones obtained for (E_0, g_0).

The idea is to transfer this estimate to the bundles E_η for small enough η.

Let h_η be the metric induced by Ω_{V_η} and J_η on V_η. Also $|\cdot|_{C^1}$ is the C^1 norm induced by g (the metric on M) on 1-forms via ω.

Proposition 9. There exists $e > 0$ such that if $|\eta_t|_{C^1} \leq e$ then the minimal energy result holds in (V_η, h_η) for $c = \frac{c_0}{2}, \varepsilon = \frac{\varepsilon_0}{2}$. That is,

$$area_{h_\eta}(\widehat{S} \cap \widehat{B}(p, \varepsilon)) \geq c\varepsilon^2$$

Proof. By examining the proof of the previous two lemmas, we observe that to be able to relate the size of the constants c_0 and ε_0 (of the monotonicity lemma) obtained for some differentiable manifold, in this case V_0, with different metrics, g_0 and $\beta_\eta^* h_\eta = g_\eta$, we need to relate the following quantities:

1. the injectivity radius of both metrics,

2. the constants c_{g_0} and c_J that appear in the inequalities that make possible the comparison of the lengths and areas of the metric g_0 (or g_η) with the constant one g_0^x (or g_η^x).

To relate the injectivity radius of the different metrics on V_0 we note that as $e \to 0$ we have that $\Omega_{V_\eta} \to \Omega_{V_0}$ and $J_\eta \to j + J_t$ in the C^2-topology (since we have a C^1 control over η_t). Now we use the well known fact that in a compact manifold the map *injectivity radius* is continuous if we consider the set of metrics on that manifold with the C^2-topology. Let our manifold be V_0, and consider the space of metrics on that manifold with the C^2-topology. Then for each $\delta > 0$, there exists a neighbourhood of g_0, say $U_{g_0}(\delta)$, such that if $h \in U_{g_0}(\delta)$ then the injectivity radius of h lies in a $\delta-$ball around the injectivity radius of g_0. So we get our result because for $e = 0$ the metrics g_0 and $\beta_\eta^* h_\eta$ are the same.

So for each fixed $\delta > 0$, there exists e small enough as to have $\beta_\eta^* h_\eta = g_\eta \in U_{g_0}(\delta)$. To relate the constants c_η and c_{J_η} for the different metrics (in this neighbourhood of g_0) we will basically rely on the fact that there exists a constant $R > 1$ such that $\frac{1}{R} g_\eta \leq g_0 \leq R g_\eta$, this will enable us to compare the different measurements (lengths and areas) obtained with the two metrics. Note that $R \to 1$ as $\delta \to 0$, so we can compare the constants c_{g_0} and c_J (that give c_0) with the constants c_{g_η} and c_{J_η} obtained using a metric g_η in a neighbourhood of g_0. $\square$

4.2 Setup

In this subsection we present all the necessary elements involved in the construction of the maps that will allow us to compare the Floer Homology to the Novikov homology. We saw that for any choice of family of 1-forms $\eta_t = \theta + dL_t$, where L_t is any time-dependent Hamiltonian on M, we have a bundle $E_\eta = (\mathbb{R} \times [0,1] \times \widetilde{M})/\sim$. We fix a choice of η_t such that it satisfies proposition 9. Using these η_t we can perform the following constructions.

Definition 5. Let

$$E' = D^2 \times \widetilde{M} \cup (\mathbb{R}^+ \times [0,1] \times \widetilde{M})/\sim$$

$$\downarrow$$

$$\Sigma' = D^2 \cup \mathbb{R}^+ \times S^1$$

with the structures Ω' and J' obtained from the ones in E_η and the product structures on the trivial bundle over the disc. Here we assume that a collar neighbourhood of $\partial D^2 = S^1$ looks like $(-1,0] \times S^1$. Also note that the bundle $E_\eta|_{[-1,1]\times S^1}$ is trivial. So we can define the bundle E' by gluing $D^2 \times \widetilde{M}$ and E_η over $(-1,0] \times S^1 \times \widetilde{M}$ using the identity. The form Ω' on the gluing region is just $\widetilde{\omega}$. The almost complex structure J' is Ω'–compatible on each fiber, it agrees with J_η on the cylindrical end, and makes the projection onto Σ' a (j, J')–holomorphic map.

We will also work with the bundle $\overline{E}_\eta \longrightarrow \mathbb{R} \times S^1$ constructed in the same way as $E_\eta \longrightarrow \mathbb{R} \times S^1$ (see chapter 2 for the construction) but the equivalence relation defining this bundle is given by $(s, 0, x) \equiv (s, 1, \widetilde{\psi}^{-1}_{-s,1}(x))$ where $\widetilde{\psi}_{s,t}$ denotes the flow generated by $\widetilde{L}_{s,t} = \rho(s)(f + p^* L_t)$ for $0 \leq t \leq 1$.

Definition 6. Let

$$\begin{aligned}\overline{E}' &= D^2 \times \widetilde{M} \cup (\mathbb{R}^+ \times [0,1] \times \widetilde{M}) / \sim' \\ &\downarrow \\ \overline{\Sigma}' &= \mathbb{R}^- \times S^1 \cup D^2\end{aligned}$$

with the structures $\overline{\Omega}'$ and $\overline{J}'$ obtained from the ones in $\overline{E}_\eta$ and the product structures on the trivial bundle over the disc. Here we assume that a collar neighbourhood of $\partial D^2 = S^1$ looks like $[0,1) \times S^1$. Also note that the bundle $\overline{E}_\eta|_{[-1,1]\times S^1}$ is trivial. So we can define the bundle $\overline{E}'$ by gluing $D^2 \times \widetilde{M}$ and $\overline{E}_\eta$ over $[0,1) \times S^1 \times \widetilde{M}$ using the identity. The form $\overline{\Omega}'$ on the gluing region is just $\widetilde{\omega}$. The almost complex structure $\overline{J}'$ is $\overline{\Omega}'$−compatible on each fiber, it agrees with $\overline{J}_\eta$ on the cylindrical end, and makes the projection onto $\overline{\Sigma}'$ a $(j, \overline{J}')$−holomorphic map.

We will assume that $\Gamma_1 \simeq \mathbb{Z}$ and τ denotes the generator of this group. Later we prove that this assumption posses no restriction at all.

Definition 7. Let V' be defined by

$$V' = \begin{cases} V_\eta & \text{over} \quad [0,\infty) \times S^1 \\ D^2 \times \overline{\Delta}_0 & \text{over} \quad D^2 \end{cases}$$

Note that the translates of V' by deck transformations cover E'. We will call such a translate $\tau V'$ a domain for E'.

Similarly we can define a domain for $\overline{E}' \longrightarrow \overline{\Sigma}'$ as follows, let

$$\overline{V}' = \begin{cases} \overline{V}_\eta & \text{over} \quad (-\infty, 0] \times S^1 \\ D^2 \times \overline{\Delta}_0 & \text{over} \quad D^2 \end{cases}$$

where $\overline{V}_\eta$ is constructed in the same way as V_η.

Remark 4. When the image of a section σ (or just a point p) lies in the subset of E' given by $\bigcup_{i=-\infty}^{r} \tau^i V'$ we will say that σ (respectively p) lies below $\tau^r V'$. Similarly if $\mathrm{Im}(\sigma)$ lies in $\bigcup_{i=r}^{\infty} \tau^i V'$ we will say that σ lies above $\tau^r V'$.

Another notion we will use is the minimal width of a domain V', which is given by twice the injectivity radius of (V', h). Since the injectivity radius of (V', h) is greater or equal to one half of the injectivity radius of (V_0', g_0) (see proof of proposition 9), and we can set things so that the latter is 1, we have the minimal width of a domain V' is 1. In fact, since we want everything independent of the covering we will proceed as follows. Let $\Gamma = \{$free part of $\pi_1(M)\}$, and $\widetilde{M}^u \longrightarrow M$ be the covering with this transformation group. Denote by Δ^u the fundamental domain associated to this cover. Now, for any $[\theta] \in H_1(M; \mathbb{R})$ the cover associated to this form is smaller than $\widetilde{M}$, that is, $\widetilde{M}^u \longrightarrow \widetilde{M} \longrightarrow M$, and Δ_0 is a union of Δ^u. We set everything so that the minimal width of Δ^u is 1.

We need one more piece of information, and that is related to the behaviour of the form Ω', from definition 5, under deck transformations. If we recall how this form was constructed (see 2.5), we see that the part of the 2-form that is not invariant under the deck transformations is the term $K_{s,t} ds \wedge dt$. We can easily deduce, from the way they were constructed (see 2.4), the following lemma

Lemma 10. The function $K : E' \longrightarrow \mathbb{R}$ satisfies the following

$$K(p) + \lambda_s r = K(\tau^r p)$$

for any $p = [(s,t,x)] \in E'|_{[1,2]\times S^1}$ and $r > 0$. Here $\lambda = \inf_{\delta \in \Gamma_1^*} \left|\int_\delta \theta\right|$ for $\Gamma_1^* = \Gamma_1 \backslash \{0\}$, and $\lambda_s = \frac{d\rho}{ds}\lambda$. If $s \notin [1,2]$ this function is zero.

Observe that we can choose ρ so that $\frac{d\rho}{ds} \leq 1$ for every $s \in \mathbb{R}$, so we can assume that $\lambda_s \leq \lambda$.

4.3 Control over number of domains a section enters

Our next task will be to control the number of domains (in the sense of definition 7) a section of E' enters. This point is crucial in getting the appropriate compactness result for the moduli spaces we use.

For the following argument to work, we will assume that θ is such that $\lambda < \frac{c\varepsilon}{2}$, where $\lambda = \inf_{\delta \in \Gamma_1^*} \left|\int_\delta \theta\right|$, and the constants ε and c were obtained from proposition 9.

Let σ be any holomorphic section of $E' \longrightarrow \Sigma'$ that ends at $\widetilde{y} = [(t, \widetilde{y}_0); u_0]$ and is such that $\sigma(0) \in \{0\} \times W^u(x_0)$ (the unstable manifold of x_0). We can assume without loss of generality that x_0 lies in V'. We define the section energy of σ as follows

$$E_s(\sigma) = \int_{\Sigma'} \sigma^* \Omega'.$$

Suppose that $\int_{\Sigma'} \sigma^*\Omega' = a$ is fixed for any such σ. We will see in the next section that this is the case, and that the value a only depends on the end, that is, on $\widetilde{y}$. We will split the base manifold Σ' into two pieces, namely, $\Sigma' = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1 = D^2 \cup ([0,2] \times S^1)$ and $\Sigma_2 = [2,\infty) \times S^1$.

Lemma 11. Since $\Omega'|_{\Sigma_2} = \widetilde{\omega}$ we have that Ω' is non-negative on sections restricted to Σ_2, that is, $\int_{\Sigma_2} \sigma^*\Omega' \geq 0$. This implies that $\int_{\Sigma_1} \sigma^*\Omega' \leq a$.

Let us work with the bundle over Σ_1 first. Since Σ_1 is compact then $\sigma|_{\Sigma_1}$ enters a finite number of domains V' of E', so we can make the form Ω' symplectic there by adding a large enough multiple of the area form on the base. For sufficiently large k the 2-form

$$\Omega_{V'} = \Omega' + kj(s)ds \wedge dt$$

is symplectic on $V'_\varepsilon|_{\Sigma_1}$. Here V'_ε is an $\frac{\varepsilon}{2}$-neighbourhood of V'. Also $j : \Sigma' \longrightarrow \mathbb{R}$ is a function with compact support that takes the value 1 on Σ_1, and whose integral is $m_1 = \int j(s)ds$.

Let $l_1 + 2$ be the number of domains that $\sigma|_{\Sigma_1}$ enters, for $l_1 \geq 0$. Then, by lemma 10, the form

$$\Omega_{l_1} = \Omega_{V'} + l_1\lambda_s g(s)ds \wedge dt \tag{4.1}$$

will be symplectic on the part of $E'|_{\Sigma_1}$ that lies below $\tau^{l_1}V'$, where $g(s)$ is a cut-off function that takes the value 1 for $s = [1,2]$. Although the form Ω_{l_1} is defined on all of E' it will only be symplectic on the region of $E'|_{\Sigma_1}$ below $\tau^{l_1}V'|_{\Sigma_1}$. Recall that Ω_{l_1} over Σ_2 is non-negative, in fact, it equals $\widetilde{\omega}$ for sufficiently large s ($s = 3$ will do).

Since $\sigma|_{\Sigma_1}$ enters $l_1 + 2$ domains we can find N disjoint balls of radius ε with centers in $\sigma(\Sigma_1)$. We have at least $\frac{l_1}{2\varepsilon}$ of these balls. To have this last statement we used that the minimal width of (V', h) is 1 (recall that $\frac{1}{2} \geq \frac{\varepsilon_0}{2} = \varepsilon$, see remark 4).

For each ball $B(\varepsilon)$, there is r with $0 \leq |r| \leq l_1$ such that $\tau^r B(\varepsilon) \subseteq V'_\varepsilon$. Let $\overline{B(\varepsilon)} = \sigma^{-1}(B(\varepsilon) \cap \sigma(\Sigma_1)) \subset \Sigma_1$.

The key point in this argument is that we can relate the Ω_{l_1}-area of $\sigma|_{B(\varepsilon)}$ with the $\Omega_{V'}$-area of $\tau^r\sigma|_{\tau^r B(\varepsilon)}$ in the following way.

$$\int_{\overline{B(\varepsilon)}} \sigma^*\Omega_{l_1} - \int_{\overline{B(\varepsilon)}} (\tau^r\sigma)^*\Omega_{V'} = \iint_{\overline{B(\varepsilon)}} -K_{s,t}(\sigma) + K_{s,t}(\tau^r\sigma) + l_1\lambda_s g(s) dt ds$$

For $s \notin [1,2]$ the right hand side in this equation is positive because $K_{s,t} \equiv 0$. When $r \geq 0$ we have that $-K_{s,t}(\sigma) + K_{s,t}(\tau^r\sigma) \geq 0$ so

$$\int_{\overline{B(\varepsilon)}} \sigma^*\Omega_{l_1} - \int_{\overline{B(\varepsilon)}} (\tau^r\sigma)^*\Omega_{V'} \geq 0$$

Now for $s \in [1,2]$ and $r < 0$ ($-r > 0$) we have

$$-K_{s,t}(\tau^{-r}\sigma) - r\lambda_s = -K_{s,t}(\sigma)$$

(see lemma 10 to recall how K behaves under deck transformations).

Therefore

$$\int_{\overline{B(\varepsilon)}} \sigma^*\Omega_{l_1} \geq \int_{\overline{B(\varepsilon)}} (\tau^{-r}\sigma)^*\Omega_{V'} + \iint (l_1 - r)\lambda_s dtds \geq \int_{\overline{B(\varepsilon)}} (\tau^{-r}\sigma)^*\Omega_V$$

in this case as well.

Now

$$\begin{aligned} \int_{\Sigma_1} \sigma^*\Omega_{l_1} &> \sum_{i=1}^{N} \int_{\overline{B_i(\varepsilon)}} \sigma^*\Omega_{l_1} \geq \sum_{i=1}^{\frac{l_1}{2\varepsilon}} \int_{\overline{B_i(\varepsilon)}} (\tau^{r_i}\sigma)^*\Omega_{V'} \\ &\geq \sum_{i=1}^{\frac{l_1}{2\varepsilon}} c\varepsilon^2 = \frac{l_1 c\varepsilon}{2} \end{aligned}$$

The last inequality follows from proposition 9. On the other hand, by 4.1

$$\begin{aligned} \int_{\Sigma_1} \sigma^*\Omega_{l_1} &= \int_{\Sigma_1} \sigma^*\Omega' + \iint (kj(s) + l_1\lambda_s g(s))dtds \\ &\leq a + km_1 + l_1\lambda m \leq a + km_1 + l_1\lambda \end{aligned}$$

To get the first inequality we used that $\int_{\Sigma_1} \sigma^*\Omega' \leq a$ (see lemma 11), $m_1 = \int j(s)ds$, $m = \int g(s)ds$ and $\lambda_s \leq \lambda$. For the second inequality we just used that $m \leq 1$.

Now combining all the previous observations we can estimate the quantity l_1 as follows

$$a + km_1 + l_1\lambda \geq \frac{l_1 c\varepsilon}{2} \iff l_1 \leq \frac{a + km_1}{(\frac{c\varepsilon}{2} - \lambda)}$$

So we get an upper bound for the number of domains that $\sigma|_{\Sigma_1}$ enters given by $l_1 \leq \frac{a+km_1}{(\frac{c\varepsilon}{2}-\lambda)}$.

Now if $x_0 \in \tau^{r_0}V'$ (instead of V') we conclude that $\sigma|_{\Sigma_1}$ lies below $\tau^{r_0+l_1}V'$, in particular, $\sigma(2,t)$ lies below this level.

Lemma 12. For any such σ we have $\int_{\Sigma_1} \sigma^*\Omega' \geq -L$, where $L \geq 0$ only depends on x_0 and $\widetilde{y}$.

Proof. The form $\Omega = \Omega' + (r_0 + l_1)j(s)ds \wedge dt$ is symplectic below $\tau^{r_0+l_1}V'|_{\Sigma_1}$. So

$$0 \leq \int_{\Sigma_1} \sigma^*\Omega = \int_{\Sigma_1} \sigma^*\Omega' + (r_0 + l_1)m_1$$

and the conclusion follows. □

To control the number of domains that $\sigma|_{\Sigma_2}$ enters we will regard the section over Σ_2 as a perturbed holomorphic curve u into $\widetilde{M}$ (see lemma 4). Using equation (2.2) we have that

$$0 \leq E(u|_{\Sigma_2}) = \int_{\Sigma_2} u^*\widetilde{\omega} + \int_0^1 \left[\widetilde{H}_t(u(2,t)) - \widetilde{H}_t(\widetilde{y}_0)\right] dt$$

Since $\int_{\Sigma_1} \sigma^*\Omega' + \int_{\Sigma_2} \sigma^*\Omega' = a$ we use the previous lemma to conclude that

$$\int_{\Sigma_2} \sigma^*\Omega' \leq a - L,$$

but $\Omega'|_{\Sigma_2} = \widetilde{\omega}$ so $\int_{\Sigma_2} \sigma^*\Omega' = \int_{\Sigma_2} \sigma^*\widetilde{\omega} = \int_{\Sigma_2} u^*\widetilde{\omega} \leq a - L$. This together with the fact that $\sigma(2,t)$ lies below $\tau^{r_0+l_1}V'$ (which implies that $u(2,t)$ lies below a certain level in $\widetilde{M}$) will give us the uniform bound on $E(u|_{\Sigma_2})$. Suppose u enters l_2 fundamental domains (of the projection $p : \widetilde{M} \longrightarrow M$, which we denoted by Δ_0). When a curve u enters a fundamental domain it uses certain (fixed) amount of energy (this is a consequence of monotonicity on $\widetilde{M}$), so since we have a uniform bound for the total energy that gives us a bound on l_2.

Now combining the previous estimates we get the following

Proposition 13. If σ is a holomorphic section of $E^{'} \longrightarrow \Sigma^{'}$ with

1. $\int_{\Sigma'} \sigma^* \Omega^{'} = a$
2. ends at $\widetilde{y} = [(t, \widetilde{y}_0); u_0]$,
3. and such that $\sigma(0) \in \{0\} \times W^u(x_0)$,

for $x_0 \in \tau^{r_0} V^{'}$ and $\widetilde{y}_0 \in \tau^{r_1} V^{'}$ then

$$\mathrm{Im}(\sigma) \subset Q = \bigcup_{r=-l}^{l} \tau^r V^{'}$$

where l is determined by r_0, r_1, l_1 and l_2. In fact,

$$Q = \bigcup_{r=-l_2-l_1+r_1}^{\max\{r_0+l_1, r_1+l_2\}} \tau^r V^{'}.$$

It is important to remember that the quantities l_1 and l_2 depend only on the value a (determined by u_0), on x_0 and $\widetilde{y}_0$, and the geometry of M.

By the same arguments we can get a bound on the number of domains a section of $\overline{E}^{'} \longrightarrow \overline{\Sigma}^{'}$ enters. As before this bound will only depend on the ends of the section.

Now we can say precisely how "small" θ has to be in order for the bounds of this section to hold. First observe that for each value of $e > 0$, there exists a $\nu > 0$ small enough so that if $|\theta|_{C^1} < e$ then $|\eta_t|_{C^1} \leq e$, where $\eta_t = \theta + \nu dH_t$ can be chosen to be regular (in the sense of theorems 6 and 7 of the appendix).

Remark 5. The bound on the size of the Calabi invariant, that is, the size of e such that $|\theta|_{C^1} < e$, imposed to get the result of the main theorem will depend only on the geometry of (M, ω, J) and is such that

1. The flow of the vector field X_θ (the dual under ω to θ) will not have any non-constant 1-periodic orbits.

2. $\lambda < \frac{c\varepsilon}{2}$, recall that $\lambda = \inf_{\delta \in \Gamma_1^*} \left| \int_\delta \theta \right|$, $\varepsilon = \frac{\varepsilon_0}{2}$, $c = \frac{c_0}{2}$. Here ε_0 and c_0 are the constants obtained from corollary 8.

3. The result of proposition 9 holds for (V_η, h_η), where $\eta_t = \theta + \nu dH_t$ as above.

Chapter 5

Proof of the main theorem

We will now relate the Novikov homology to the Floer homology by defining a chain map such that the resulting map in homology is injective. In section 5.1 we define the chain maps

$$\Psi : CN_*(\theta, \omega) \longrightarrow CF_*(\widetilde{L}_t)$$

and

$$\overline{\Psi} : CF_*(\widetilde{L}_t) \longrightarrow CN_*(f, \omega).$$

In section 5.2 we will see that the composition of these maps is chain homotopic to the identity, so the induced map in homology, namely Ψ_*, is injective.

5.1 Maps between the two homologies

To define the map

$$\Psi : CN_*(\theta,\omega) \longrightarrow CF_*(\widetilde{L}_t)$$

we will study spaces of sections of $E' \longrightarrow \Sigma'$ (see definition 5). We always assume the parameters to be chosen generically (in the sense of theorems 6 and 7 of the appendix) so that all the moduli spaces are manifolds of the right dimension, and all intersections are transverse.

Observe that since the bases of these bundles are just discs, we have that they are isomorphic to the trivial bundle $\Sigma' \times \widetilde{M}$ (via the map β defined at the beginning of chapter 4). Therefore $\pi_2(E') \simeq \pi_2(\widetilde{M})$.

Definition 8. Given a section σ of $E' \longrightarrow \Sigma'$ with end $\widetilde{y} = [(t,\widetilde{y}_0);u_0)]$ we denote by $S(\sigma)$ be the sphere in $\widetilde{M}$ obtained as follows. Define the flat section $\sigma_{\widetilde{y}_0} : \Sigma' \to E'$ by $\sigma_{\widetilde{y}_0}(z) = (z,\widetilde{y}_0)$. Then $\sigma\# - \sigma_{\widetilde{y}_0}$ is a sphere in E', so it is homotopic to a sphere in $\widetilde{M}$. We denote this last by $S(\sigma)$. To be more precise, we have the following diagram

$$\begin{array}{ccc} E' & \xrightarrow{\beta} & \Sigma' \times \widetilde{M} \\ \downarrow & & \downarrow p_F \\ \Sigma' & & \widetilde{M} \end{array}$$

and $S(\sigma) = p_F \circ \beta(\sigma\# - \sigma_{\widetilde{y}_0})$.

Now, for $\widetilde{y} = [(t, \widetilde{y}_0); u_0)]$ let

$$\mathcal{M}(\widetilde{y}) = \left\{ \begin{array}{c} J' - \text{holomorphic sections } \sigma : \Sigma' \to E', \text{ such that} \\ \sigma \underset{s \to \infty}{\longrightarrow} \widetilde{y} \text{ and } [S(\sigma)] - u_0 = 0 \in \pi_2(\widetilde{M}) \end{array} \right\}$$

Let $W^u(x_0)$ denote the unstable manifold of $x_0 \in crit\{f\}$. Now $\mathcal{M}(x_0, \widetilde{y})$ is just the intersection of $\mathcal{M}(\widetilde{y})$ with $\{0\} \times W^u(x_0)$, that is, $\sigma \in \mathcal{M}(x_0, \widetilde{y})$ if $\sigma \in \mathcal{M}(\widetilde{y})$ and $\sigma(0) \in \{0\} \times W^u(x_0)$.

Finally our map $\Psi : CN_*(\theta, \omega) \longrightarrow CF_*(\widetilde{L}_t)$ is defined as follows

$$\Psi(x_0, A) = \sum_{\widetilde{y} \in \mathcal{P}} \#\{isolated\ points\ in\ \mathcal{M}(x_0, (-A)\#\widetilde{y})\}\ \widetilde{y}$$

As always, in order for Ψ to be well defined we need

1. The spaces of connecting orbits, namely the isolated points in $\mathcal{M}(x_0, \widetilde{y})$ for any $x_0 \in crit\{f\}$ and $\widetilde{y} \in \mathcal{P}$, to be compact. So they consist of finitely many points.

2. The finiteness condition holds, therefore $\Psi(x_0, A) \in CF_*(\widetilde{L}_t)$. More explicitly, we need to see that for all N_1, and N_2 the set

$$\left\{ \widetilde{y}\ /\ \xi_{\widetilde{y}} \neq 0,\ \int {u_0}^* \omega \leq N_1 \text{ and } f(\widetilde{y}_0) > N_2 \right\}$$

has a finite number of elements. Here

$$\xi_{\widetilde{y}} = \#\{isolated\ points\ in\ \mathcal{M}(x_0, (-A)\#\widetilde{y})\}.$$

The main problem in proving this is that if we think of these connecting orbits as (perturbed) holomorphic curves in $\widetilde{M}$, we cannot bound the energy unless the curves remain in a compact piece of $\widetilde{M}$ (determined only by x_0 and $\widetilde{y}$). On the other hand, if we think of them as holomorphic sections of $E' \to \Sigma'$, the problem is that the closed 2-form Ω' we have on the total space cannot be made non-degenerate unless these sections remain in a subspace of E' with compact fiber.

By the same arguments that led to the proof of proposition 13, we have the following result

Lemma 14. If $x_0 \in \tau^{r_0}V'$ and $\widetilde{y}_0 \in \tau^{r_1}V'$ then any $\sigma \in \mathcal{M}(x_0, \widetilde{y})$ is such that $\mathrm{Im}(\sigma) \subset Q = \bigcup_{r=-l}^{l} \tau^r V'$ where l is determined by r_0, r_1, l_1 and l_2.

Proof. We just need to observe that every $\sigma \in \mathcal{M}(\widetilde{y})$ has the same section energy. To see this, note that $S(\sigma) \simeq \sigma\# - \sigma_{\widetilde{y}_0}$ (recall definition 8), and any element in this moduli space satisfies $[S(\sigma)] - u_0 = 0$ we get .

$$\int \sigma^*\Omega' - \int \sigma_{\widetilde{y}_0}^*\Omega' = \int_{\sigma\# - \sigma_{\widetilde{y}_0}} \Omega' = \int_{S(\sigma)} \Omega' = \int_{S(\sigma)} \widetilde{\omega} = \int_{u_0} \widetilde{\omega}$$

so

$$E_s(\sigma) = \int_{\Sigma'} \sigma^*\Omega' = \int_{u_0} \widetilde{\omega} + \int_{\Sigma'} \sigma_{\widetilde{y}_0}^*\Omega' = a$$

Now we just apply proposition 13 to get the result. $\square$

Observe that we have $\mathcal{M}(x_0, \widetilde{y}) = \emptyset$ for many choices of x_0 and $\widetilde{y}$, for example if $r_0 + l_1 + l_2 < r_1$ the moduli space will be empty.

Remark 6. We should have in mind that the bound on the section energy (denoted by a) determines the quantities l_1 and l_2. The position of the ends, that is, the values r_0 and r_1 such that $x_0 \in \tau^{r_0}V'$ and $\widetilde{y}_0 \in \tau^{r_1}V'$, determine the position of our compact set Q (in E' relative to V').

Lemma 15. The map Ψ is well defined.

Proof. First we verify that the zero dimensional component of $\mathcal{M}(x_0, \widetilde{y})$ for any $x_0 \in crit\{f\}$ and $\widetilde{y} \in \mathcal{P}$ is compact.

We use lemma 14 to conclude that σ remains on a subset Q of E' with compact fiber (this set is determined by x_0 and $\widetilde{y}$). We can make the form Ω' symplectic in $Q|_{D^2 \cup [0,2] \times S^1}$ by adding a suitable multiple of the form on the base, in this case we can use $\Omega_l = \Omega' + (kj(s) + l\lambda g(s))ds \wedge dt$, where $kj(s)ds \wedge dt$ is what we needed to add to Ω' to make it symplectic on $V'|_{D^2 \cup [0,2] \times S^1}$, and g is a cut-off function with support contained on $[1, 2]$ (see equation 4.1). The sections have uniform energy bound, the 2-form Ω_l on the total space is non-negative, and the sections lie in a piece of E' with compact fiber, so we are in the same situation of remark 1. Therefore we can use the same compactness argument to conclude that the number of isolated points in $\mathcal{M}(x_0, \widetilde{y})$ is finite. To be more precise, we take a sequence of elements in $\mathcal{M}(x_0, (-A)\#\widetilde{y})$ we cut the cylindrical end at some large value of s, say R, and consider the sections over this base. We can easily check that the area (we add to Ω_l the pull-back of a suitable form from the base, so as to make it symplectic on Q) of such sections is uniformly bounded, also recall that their images lie in some compact piece of E', so by the usual compactness arguments we have that there is a convergent subsequence. We repeat this process using the subsequence, but now cutting the cylindrical end at a larger value of s. We now let $R \to \infty$, and end up with a convergent subsequence on the whole of Σ'. Bubbling does not occur for regular parameters because of the semi-positive condition imposed on M.

Now we have to verify the finiteness condition. Suppose, by contradiction, that we have infinitely many different elements $\sigma_n \in \mathcal{M}(x_0, (-A)\#\widetilde{y}_n)$, where the $\widetilde{y}_n$ are such that $\int_{u_n} \widetilde{\omega} \leq N_1$, and $f(\widetilde{y}_0^n) > N_2$ for some N_1 and N_2. Since $f(\widetilde{y}_0^n) > N_2$, this implies that these curves ends lie above certain level (so the function $-K_{s,t}(p)$ has a supremum in this region). This together with the hypothesis on the homology class bounds the section energy of σ_n. Namely, since $[S(\sigma_n)] - (u_n - A) = 0$ and $\int_{u_n} \widetilde{\omega} \leq N_1$, we get from equation 2.5 that

$$\begin{aligned} E_s(\sigma_n) &= \int \sigma_n^* \Omega' = \int_{u_n} \widetilde{\omega} - \int_A \widetilde{\omega} + \int \sigma_{\widetilde{y}_0^n}^* \Omega' \\ &\leq N_1 - \int_A \widetilde{\omega} - \int_0^1 \int_1^2 K_{s,t}(\widetilde{y}_0^n) ds dt = a \end{aligned}$$

where a is independent of n.

Suppose $f(x_0) = c_0$. Since f decreases along the gradient trajectories we have that $f(\sigma_n(0)) \leq c_0$. This means that $\sigma_n(0)$ lies below a certain level in $E'_{\widetilde{\gamma}}$ (this level is determined by x_0). Using lemma 14 for trajectories in $\mathcal{M}(x_0, (-A)\#\widetilde{y}_n)$ we can conclude that $\widetilde{y}_0^n$ lies at most l domains above the level where x_0 is. Therefore $\widetilde{y}_0^n$ remain in a compact piece of E', so this set is finite. Again by compactness arguments we conclude that the $\widetilde{y}_n = [(t, \widetilde{y}_0^n); u_n]$ are finite, what implies by our previous result that there are finitely many σ_n, this gives a contradiction. $\square$

Remark 7. Observe that having sections σ of $E' \longrightarrow \Sigma'$ with a uniform bound on $\int \sigma^* \Omega'$, and such that their images remain in a piece of E' with compact fiber is the same as having a uniform bound on the (usual) energy of the perturbed curves u in $\widetilde{M}$ used to construct these sections, so we are in the same situation as in [8] (since the image of the curves lies in a compact piece of $\widetilde{M}$).

Lemma 16. Ψ is a chain map.

Proof. We want to see that Ψ commutes with the boundaries. For this we will use the usual gluing and compactness arguments. Using the definitions of the operators $\partial_N, \partial_F, \Psi$ and fixing (x_0, A) we have that

$$(\partial_F \circ \Psi + \Psi \circ \partial_N)(x_0, A) = \sum_{\widetilde{z} \in \mathcal{P}} \# \{S((x_0, A); \widetilde{z})\} \, \widetilde{z}$$

where $S((x_0, A); \widetilde{z})$ is the set consisting of one of the following

1. The set of pairs consisting of an isolated point $\nu \in \widehat{\mathcal{M}}(x_0, y_0)$, for some $y_0 \in crit\{f\}$, and an isolated element $\sigma \in \mathcal{M}(y_0, (-A)\#\widetilde{z})$. Here $\widehat{\mathcal{M}}(x_0, y_0)$ is the space of unparametrized gradient trajectories of f (the trajectories used to construct the boundary in the Novikov homology).

2. The set of pairs consisting of an isolated point $\sigma \in \mathcal{M}(x_0, (-A)\#\widetilde{y})$, for some $\widetilde{y} \in \mathcal{P}$, and an isolated element $\sigma_0 \in \widehat{\mathcal{M}}(\widetilde{y}, \widetilde{z})$. Here $\widehat{\mathcal{M}}(\widetilde{y}, \widetilde{z})$ consists of isolated sections of $T_{\widetilde{\varphi}}$ (the sections used to define the Floer boundary).

We have to show that for any $\widetilde{z} = ((t, \widetilde{z}_0); v_0) \in \mathcal{P}$, the set $S((x_0, A); \widetilde{z})$ has an even number of elements. The proof follows the gluing-compactness argument used in [2] to prove the invariantness of Floer homology.

To see this, fix $\widetilde{z} = ((t, \widetilde{z}_0); v_0)$ and study the space $\mathcal{N}$ of trajectories in the 1-dimensional components of $\mathcal{M}(x_0, (-A)\#\widetilde{z})$. Using gluing arguments (see [8]) we can show that any element of $S((x_0, A); \widetilde{z})$ can be considered as a compactifying point for exactly one end of $\mathcal{N}$. We already proved that any trajectory in this space remains in a compact piece of $E'_{\widetilde{\gamma}}$ that can be made into a tame symplectic manifold. Also by the same arguments as before we have that no bubbling-off occurs as we move along $\mathcal{N}$. So the appropriate version of the compactness theorem claims that $\mathcal{N}$ has finitely many components, and every end of $\mathcal{N}$ can be uniquely compactified in one of the following ways

1. The set of pairs consisting of an isolated point $\nu \in \widehat{\mathcal{M}}(x_0, y_0)$, for some $y_0 \in crit\{f\}$, and an isolated element $\sigma \in \mathcal{M}(y_0, (-A)\#\widetilde{z})$.

2. The set of pairs consisting of an isolated point $\sigma \in \mathcal{M}(x_0, (-A)\#\widetilde{y})$, for some $\widetilde{y} = [(t, \widetilde{y}_0); u_0] \in \mathcal{P}$, and an isolated element $\sigma_0 \in \widehat{\mathcal{M}}(\widetilde{y}, \widetilde{z})$.

The proof of this is standard except that in 2 we need to make sure that the conditions on $S(\sigma)$ and $S_0(\sigma_0)$ are satisfied. Let us see;

Suppose $\sigma_n \longrightarrow (\sigma, \sigma_0)$, and observe that $S(\sigma\#\sigma_0) = \lim_{n\to\infty} S(\sigma_n) = v_0 - A$, since $\sigma_n \in \mathcal{M}(x_0, (-A)\#\widetilde{z})$. We will prove that $[S(\sigma\#\sigma_0)] = [S(\sigma)] + [S_0(\sigma_0)]$. From this it follows easily that $[S(\sigma)] - u_0 = 0$ and $[S_0(\sigma_0)] + u_0 = v_0$, which are the appropriate conditions on $S(\sigma)$ and $S_0(\sigma_0)$ respectively. Let us see;

$$
\begin{aligned}
S(\sigma\#\sigma_0) &= p_F \circ \beta(\sigma\#\sigma_0\# - \sigma_{\widetilde{z}_0}) = p_F \circ \beta(\sigma\# - \sigma_{\widetilde{y}_0}\#\sigma_{\widetilde{y}_0}\#\sigma_0\# - \sigma_{\widetilde{z}_0}) \\
&= p_F \circ \beta(\sigma\# - \sigma_{\widetilde{y}_0})\#p_F \circ \beta(\sigma_{\widetilde{y}_0})\#p_F \circ \beta(\sigma_0)\#p_F \circ \beta(-\sigma_{\widetilde{z}_0}) \\
&\simeq S(\sigma) + S_0(\sigma_0).
\end{aligned}
$$

To get the last relation just observe that $\beta|_{[2,\infty)\times S^1} = \alpha$ and that $p_F \circ \beta(\sigma_{\widetilde{y}_0}) \equiv \widetilde{y}_0$ and $p_F \circ \beta(\sigma_{\widetilde{z}_0}) \equiv \widetilde{z}_0$.

We just saw that there is a one-to-one correspondence between elements in $S((x_0, A); \widetilde{z})$ and the ends of the 1-dimensional components of $\mathcal{M}(x_0, (-A)\#\widetilde{z})$ which are even in number (and finitely many). We get that $\sum_{\widetilde{z}\in\mathcal{P}} S((x_0, A); \widetilde{z})\widetilde{z} = 0$. It follows that Ψ is a chain map. $\square$

Our next task is to define a map in the other direction ("an inverse"), that is,

$$\overline{\Psi} : CF_*(\widetilde{L}_t) \longrightarrow CN_*(f, \omega).$$

We will do this by studying sections of the bundle $\overline{E}' \longrightarrow \overline{\Sigma}'$, which is constructed in the same way as $E' \longrightarrow \Sigma'$, but using the equivalence relation $(s, 0, x) \equiv (s, 1, \widetilde{\gamma}_{-s}^{-1}(x))$ on the cylindrical end (see definition 6). We can also define an almost complex structure and a closed 2-form, by the same method used in chapter 2. Note that for sufficiently large s we have that the almost complex structures and the closed forms on the two bundles agree, in fact since $Y_{s,t,x} = K_{s,t,x} = \overline{Y}_{-s,t,x} = \overline{K}_{-s,t,x} = 0$ unless $s \in [1, 2]$, we have that $\Omega'_{\widetilde{\gamma}}(s, t, x) = \overline{\Omega}'_{\widetilde{\gamma}}(s, t, x) = \widetilde{\omega}_x$ if $|s| \geq 2$, similarly for the almost complex structures. Actually we can think of sections of $\overline{E}' \longrightarrow \overline{\Sigma}'$ as pseudoholomorphic sections with a cylindrical end to the left converging to some critical point.

Definition 9. Given a section σ of $\overline{E}' \longrightarrow \overline{\Sigma}'$ with end $\widetilde{y} = [(t, \widetilde{y}_0); u_0)]$ we denote by $\overline{S}(\sigma)$ the sphere in $\widetilde{M}$ obtained as follows. Define the flat section $\overline{\sigma}_{\widetilde{y}_0} : \overline{\Sigma}' \to \overline{E}'$ by $\sigma_{\widetilde{y}_0}(z) = (z, \widetilde{y}_0)$. Then $\sigma \# -\overline{\sigma}_{\widetilde{y}_0}$ is a sphere in E' so it is homotopic to a sphere in $\widetilde{M}$. We denote this last by $\overline{S}(\sigma)$. To be more precise,

$\overline{S}(\sigma) = p_F \circ \overline{\beta}(\sigma\# - \overline{\sigma}_{\widetilde{y}_0})$, where $\overline{\beta}$ is the isomorphism between $\overline{E}'$ and the trivial bundle over $\overline{\Sigma}'$.

For $\widetilde{y} = [(t, \widetilde{y}_0); u_0]$ let

$$\mathcal{M}^-(\widetilde{y}) = \left\{ \begin{array}{c} \overline{J}' - \text{holomorphic sections } \sigma : \overline{\Sigma}' \to \overline{E}', \text{ such that} \\ \sigma \underset{s\to\infty}{\longrightarrow} \widetilde{y} \text{ and } [\overline{S}(\sigma)] - u_0 = 0 \in \pi_2(\widetilde{M}) \end{array} \right\}$$

Now $\mathcal{M}^-(\widetilde{y}, z_0)$ is just the intersection of $\mathcal{M}^-(\widetilde{y})$ with $\{0\} \times W^s(z_0)$, that is, $\sigma \in \mathcal{M}^-(\widetilde{y}, z_0)$ if $\sigma \in \mathcal{M}^-(\widetilde{y})$ and $\sigma(0) \in W^s(z_0)$. Now define the map $\overline{\Psi}$ as follows,

$$\overline{\Psi}(\widetilde{y}) = \sum_{(z_0, A)} \#\{\textit{isolated points in } \mathcal{M}((-A)\#\widetilde{y}, z_0)\}(z_0, A)$$

We can reproduce the previous arguments to show that $\overline{\Psi}$ is well defined, and that it is a chain map.

5.2 Composite maps

Now to see that $\overline{\Psi} \circ \Psi$ is chain homotopic to the identity, we will glue together the two bundles in order to define the map that realizes the chain homotopy. This is exactly the case when you have a homotopy (or continuation) of homotopies, usually used when proving that the Floer homologies for different choices of regular parameters are isomorphic. We proceed as follows

$$\begin{array}{ccc} E' & & \overline{E}' \\ \downarrow & & \downarrow \\ D_\kappa = D^2 \cup [0, \kappa+2] \times S^1 & & [-\kappa-2, 0] \times S^1 \cup D^2 = \overline{D}_\kappa \end{array}$$

For $\kappa \geq 1$ we cut the ends at $\kappa + 2$. Choose an orientation reversing diffeomorphism $g : \partial D_\kappa \to \partial \overline{D}_\kappa$ and form the connected sum $E' \# \overline{E}' = E' \cup_{g \times id} \overline{E}'$ (this uses the trivializations of $E'|_{\partial D_\kappa}$ and $\overline{E}'|_{\partial \overline{D}_\kappa}$). The resulting bundles over the spheres S^κ will be

$$\begin{array}{c} P_\kappa = E' \cup_{g \times id} \overline{E}' \\ \downarrow \\ S^\kappa \end{array}$$

Note that since the symplectic forms and the almost complex structures defined on the bundles over the disc agree on the gluing region, we can equip P_κ with a closed 2-form Ω_κ that restricts to $\widetilde{\omega}$ on the fibers. We also have and an almost complex structure J_κ that restrict to J' and $\overline{J}'$ over D_κ and $\overline{D}_\kappa$ respectively.

For $\kappa \in [0, 1]$, we use the time one map of the flow generated by $\kappa \widetilde{L}_{s,t}$, say $\widetilde{\psi}^\kappa_{s,1} = \widetilde{\gamma}^\kappa_s$, and perform an analogous construction, to build bundles over the sphere for these values of κ as well. Therefore we have a one-parameter family of bundles over the sphere that are all isomorphic to the trivial bundle. To study pseudoholomorphic sections of these bundles is equivalent to the study of perturbed Cauchy-Riemann equations, where the perturbation and the almost complex structure change with an extra parameter κ.

Because of the nature of $\widetilde{\gamma}_s$ (they are all in $Ham(\widetilde{M},\widetilde{\omega})$) it is easy to see that these bundles over the sphere are isomorphic to the trivial bundle. Let $\chi_\kappa : P_\kappa \longrightarrow S^\kappa \times \widetilde{M}$ denote these isomorphisms (these maps χ_κ are constructed using the maps β and $\overline{\beta}$ that trivialize the bundles E' and $\overline{E}'$ over $D^2 \cup [0, \kappa+2] \times S^1$ and $[-\kappa-2, 0] \times S^1 \cup D^2$ respectively). As a consequence we have that $\pi_2(P_\kappa) \simeq \pi_2(\widetilde{M}) \times \pi_2(S^2)$. Let p_F denotes the projection onto $\widetilde{M}$, and σ be a J_κ−holomorphic section of $P_\kappa \longrightarrow S^\kappa$. Then we have the following

Definition 10. We set $S_\kappa(\sigma) = S(\kappa, \sigma)$ as the sphere in $\widetilde{M}$ obtained by $S_\kappa(\sigma) = p_F \circ \chi_\kappa(\sigma)$.

Given (x_0, A) and (y_0, B) in $CN_*(\theta, \omega)$ we define

$$\mathcal{M}_\kappa((x_0, y_0); B - A) = \left\{ \begin{array}{c} J_\kappa\text{-hol sections } \sigma : S^\kappa \to P_\kappa / [S_\kappa(\sigma)] = B - A \\ \sigma(0) \in W^u(x_0) \text{ and } \sigma(\infty) \in W^s(y_0) \end{array} \right\}$$

Now let

$$\widetilde{\mathcal{M}}((x_0, y_0); B - A) = \{(\kappa, \sigma) \ / \ \sigma \in \mathcal{M}_\kappa((x_0, y_0); B - A) \text{ and } \kappa \in [0, \infty)\}.$$

Finally we define the map $\Theta : CN_*(\theta, \omega) \longrightarrow CN_*(\theta, \omega)$ as follows

$$\Theta(x_0, A) = \sum_{(y_0, B)} \#\{\textit{isolated points in } \widetilde{\mathcal{M}}((x_0, z_0); B - A)\}(y_0, B)$$

To check that this map is well defined, first observe that since σ is a section

$p_B(\chi_\kappa \circ \sigma(S^\kappa)) = S^\kappa$, where p_B denotes the projection from $S^\kappa \times \widetilde{M}$ onto S^κ. This implies that $\chi_\kappa[\sigma] = [S_\kappa(\sigma)] + [S^\kappa]$. By construction, the sections we are using are such that $\int_{[S^\kappa]}(\chi_\kappa \circ \sigma)^*\Omega_\kappa = 0$ for all κ, enabling us to have a bound on the section energy $E_s(\sigma) = \int \sigma^*\Omega_\kappa$ independent of κ. That is, $\int \sigma^*\Omega_\kappa = \int_{B-A} \widetilde{\omega}$. We now show that this implies that we will have a bound on the number of domains the sections (of the different bundles) enter, independently of the bundle. This is equivalent to having a uniform bound on the (usual) energy of perturbed pseudoholomorphic curves, where the perturbation and the almost complex structure change with the extra parameter κ.

Lemma 17. Every $\sigma \in \mathcal{M}_\kappa((x_0, y_0); B-A)$ is such that its image is contained in Q_κ, a compact subset of P_κ. Moreover, if $x_0 \in \tau^{r_0}V_\kappa$ and $y_0 \in \tau^{r_1}V_\kappa$ then

$$Q_\kappa = \bigcup_{r=r_1-l_1-l_2}^{r_0+l_1+l_2} \tau^r V_\kappa$$

where l_1 and l_2 are determined by (x_0, A) and (y_0, B) independently of κ.

Proof. First of all we have that the section energy of any such σ is given by $\int_{S_\kappa} \sigma^*\Omega_\kappa = \int_{B-A} \widetilde{\omega} = a$ (a fixed value). Recall that we denoted by D_κ and $\overline{D}_\kappa$ the two discs that make up the manifold S_κ. Observe that $\sigma|_{D_\kappa}$ is a section of $E'|_{D_\kappa}$, and $\sigma|_{\overline{D}_\kappa}$ is a section of $\overline{E}'|_{\overline{D}_\kappa}$. Since $a = \int_{S_\kappa} \sigma^*\Omega_\kappa = \int_{D_\kappa} \sigma^*\Omega' + \int_{\overline{D}_\kappa} \sigma^*\overline{\Omega}'$ then one of the two quantities on the right hand side is bounded above by a. Suppose without loss of generality that $\int_{D_\kappa} \sigma^*\Omega' \leq a$. Then by the arguments previous to lemma 12 we get a bound on the number of domains that $\sigma|_{D_\kappa}$ enters, say l_1. So if $x_0 \in \tau^{r_0}V_\kappa$ then $\sigma|_{D_\kappa}$ lies below $\tau^{r_0+l_1}V_\kappa$. With this we can conclude, just as in lemma 12 , that $\int_{D_\kappa} \sigma^*\Omega' \geq -L$ where L depends only on the ends. This implies $\int_{\overline{D}_\kappa} \sigma^*\overline{\Omega}' \leq a + L$. So we can bound the number of domains, say l_2, that $\sigma|_{\overline{D}_\kappa}$ enters as well. Using that $\sigma(0) \in W^u(x_0)$, $\sigma(\infty) \in W^s(y_0)$ and the bounds for l_1 and l_2 we can conclude that σ remains on a compact subset Q_κ of P_κ. In fact $Q_\kappa = \bigcup_{r=r_1-l_1-l_2}^{r_0+l_1+l_2} \tau^r V_\kappa$. By the way these sets are constructed we easily see that they are determined by (x_0, A) and (y_0, B) independently of κ as we wanted. □

Lemma 18. The map Θ is well defined.

Proof. Since there are diffeomorphisms $\widehat{\chi}_\kappa : P_\kappa \longrightarrow P_0$, where $\widehat{\chi}_\kappa = F_\kappa \circ \chi_\kappa$ for $F_\kappa : S^\kappa \times \widetilde{M} \longrightarrow S^0 \times \widetilde{M} = P_0$ the obvious map, we can use P_0 as the reference manifold, with a family of symplectic and almost complex structures indexed by κ. As before we have the fundamental domain $V_0 = S^0 \times \overline{U}_{\Delta_0}$ of P_0 and we define $V_\kappa = \widehat{\chi}_\kappa^{-1}(V_0)$. We can also think of V_κ as made out by the gluing $V' \cup_{g\times id} \overline{V}'$, here the two domains are taken from the corresponding bundles E' and $\overline{E}'$ cutting the cylindrical ends at $\kappa + 2$ and $-\kappa - 2$ respectively (see section 4.2 for all these definitions).

Now recall that every $(\kappa, \sigma) \in \widetilde{\mathcal{M}}((x_0, y_0); B - A)$ has the same section energy, namely $\int \sigma^*\Omega_\kappa = \int_{B-A} \widetilde{\omega}$. By the previous lemma we can conclude that σ remains on a compact subset Q_κ of P_κ. In fact this set is given by $Q_\kappa = \bigcup_{r=i_o}^{i_1} \tau^r V_\kappa$ where i_0 and i_1 are determined by (x_0, A) and (y_0, B) independently of κ. Now we use that $\chi_\kappa(\tau^r V_\kappa) = \tau^r V_0 \Rightarrow \chi_\kappa\left(\bigcup_{r=i_o}^{i_1} \tau^r V_\kappa\right) = \bigcup_{r=i_o}^{i_1} \tau^r V_0 = Q_0$ to conclude that, for any κ and any $\sigma \in \mathcal{M}_\kappa((x_0, y_0); B - A)$, $\widehat{\chi}_\kappa(\sigma)$ remains in the fixed compact subset Q_0 of P_0. So again we are in the usual situation (see remark 1). That is, if we think of the elements of the moduli space as curves in $\widetilde{M}$, we can uniformly bound the energy of this curves independently of κ. By the weak compactness argument (see theorem 8) we have that the number of isolated points in $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$ is finite. If we think of these elements as sections, we argue as in the definition of Ψ to make the 2-form on the appropriate subspace symplectic, and prove the compactness of the moduli space in question.

Observe that we cannot have $\kappa_n \to \infty$ because this will imply having a broken trajectory $\sigma = (\sigma_+, \sigma_-)$ as limit, but this is impossible for dimensional reasons. More precisely, suppose we have as a limit of (κ_n, σ_n), such a pair of isolated trajectories $\sigma = (\sigma_+, \sigma_-)$, where $\sigma_+ \in \mathcal{M}(x_0, (-A)\#\widetilde{z})$ and $\sigma_- \in \mathcal{M}((-B)\#\widetilde{z}, y_0)$, for some $\widetilde{z} \in \mathcal{P}$. Then $\mu(\sigma_+) = 0$ and $\mu(\sigma_-) = 0$, but also since $\mu((\kappa_n, \sigma_n)) = 0$ we have that $\mu(\sigma_n) = -1$ (see remark below), this gives a contradiction.

To check the finiteness condition we use an analogous argument. Suppose that $\sigma_n \in \mathcal{M}_{\kappa_n}((x_0, y_n); B_n - A)$ and (y_n, B_n) are such that for some N_1 and N_2 we have $\int_{B_n} \widetilde{\omega} \leq N_1$ and $f(y_n) > N_2$. Observe that as before we have uniform bound on the section energy, since

$$\int \sigma_n^* \Omega_{\kappa_n} = \int_{S_{\kappa_n}(\sigma_n)} \widetilde{\omega} = \int_{B_n - A} \widetilde{\omega} \leq N_1 - \int_A \widetilde{\omega}$$

so we have a maximum length, say $\widehat{l}$, for all σ_n. Now using that $\sigma_n(0) \in W^u(x_0)$, $\sigma_n(\infty) \in W^s(y_n)$, the fact that $f(y_n) > N_2$ and the previous lemma we can conclude that $\widehat{\chi}_{\kappa_n}(\sigma_n)$ remains in a compact subset Q_0 of P_0. Remember that $\sigma_n(0) \in W^u(x_0)$ implies that $f(x_0) \geq f(\sigma_n(0))$, which means that all σ_n have at least one point below the level where x_0 lies. Similarly $\sigma_n(\infty) \in W^u(y_n)$ implies that $N_2 < f(y_n) \leq f(\sigma_n(\infty))$, which means that all σ_n have at least one point above certain level. These observations, together with the fact that

$\int \sigma_n^* \Omega_{\kappa_n}$ is uniformly bounded gives that the image of $\widehat{\chi}_{\kappa_n}(\sigma_n)$ is contained in Q_0 for all n. There are only finitely many different (y_n, B_n) with these characteristics. This is precisely the finiteness condition we require. $\square$

Remark 8. Due to the extra parameter $\kappa \in \mathbb{R}^+$ incorporated in the moduli space $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$, we have that $\mu(\kappa, \sigma) = \mu(\sigma) + 1$. Here $\mu(\kappa, \sigma)$ represents the index of (κ, σ) viewed as a trajectory in $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$, and $\mu(\sigma)$ is index of σ as a trajectory in $\mathcal{M}_\kappa((x_0, y_0); B - A)$. So, for example, when we take sections in the 1-dimensional part of $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$, this means that the ends of such sections have the same index.

Proposition 19. $\overline{\Psi} \circ \Psi$ is chain homotopic to the identity. In fact, $\overline{\Psi} \circ \Psi + id = \Theta \circ \partial_N + \partial_F \circ \Theta$.

Proof. The idea in this proof is the same as in lemma 16. To see that $\overline{\Psi} \circ \Psi + id = \Theta \circ \partial_N + \partial_F \circ \Theta$ we study the 1-dimensional components of the space $\widetilde{\mathcal{M}}((x_0, z_0); B - A)$, and use gluing-compactness argument to get the desired result. Fixing (x_0, A) we have that

$$(\overline{\Psi} \circ \Psi + id + \partial_N \circ \Theta + \Theta \circ \partial_N)(x_0, A) = \sum_{z_0, B} \# \{S((x_0, A); (z_0, B))\} (z_0, B)$$

where $S((x_0, A); (z_0, B))$ is the set consisting of one of the following

1. The set of pairs consisting of an isolated point $x \in \mathcal{M}(x_0, y_0)$, for some $y_0 \in crit(f)$, and an isolated element $(\kappa, \sigma) \in \widetilde{\mathcal{M}}((y_0, z_0); B - A)$, where $\mathcal{M}(x_0, y_0)$ is the space of negative gradient trajectories of f (the trajectories used to construct the boundary in the Novikov homology).

2. The set of pairs consisting of an isolated point $(\kappa, \sigma) \in \widetilde{\mathcal{M}}((x_0, y_0); B - A)$, for some $y_0 \in crit(f)$, and an isolated element $x \in \mathcal{M}(y_0, z_0)$, where $\mathcal{M}(y_0, z_0)$ is the space of negative gradient trajectories of f (the trajectories used to construct the boundary in the Novikov homology).

3. The set of pairs consisting of isolated continuation trajectories $\sigma_+ \in \mathcal{M}(x_0, (-A)\#\widetilde{y})$ and $\sigma_- \in \mathcal{M}((-B)\#\widetilde{y}, z_0)$, for some $\widetilde{y} \in \mathcal{P}$.

4. The set which contains exactly one element when $x_0 = z_0$ and $A = B$, and is empty otherwise.

The gluing theorem (see [8]) shows that any element of $S((x_0, A); (z_0, B))$ can be considered as a compactifying point for exactly one end of the 1-dimensional part of $\widetilde{\mathcal{M}}((x_0, z_0); B - A)$, (where (4) is identified with the set of isolated sections the trivial bundle over the sphere, that is the elements of $\mathcal{M}_0((x_0, y_0); B - A)$. This space has just one element for dimensional reasons). Since no bubbling occurs as we move along the 1-dimensional part of $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$ we can use the appropriate version of compactness to claim that the 1-dimensional part of $\widetilde{\mathcal{M}}((x_0, y_0); B - A)$ has finitely many connected components, and every end of this space can be uniquely compactified in one of the ways described above. To have 3. we only need to check that $\sigma_+ \in \mathcal{M}(x_0, (-A)\#\widetilde{y})$ and $\sigma_- \in \mathcal{M}((-B)\#\widetilde{y}, z_0)$, for some $\widetilde{y} \in \mathcal{P}$, that is, $S(\sigma_+)$ and $S(\sigma_-)$ represent the appropriate classes. This will easily follow from

$$\begin{aligned} B - A &= S(\kappa_n, \sigma_n) = S(\sigma_+ \# \sigma_-) = p_F \circ \chi(\sigma_+ \# \sigma_-) \\ &\simeq p_F \circ \beta(\sigma_+) \# p_F \circ \overline{\beta}(\sigma_-) \\ &= p_F \circ \beta(\sigma_+) \# p_F \circ \beta(-\sigma_{\widetilde{y}_0}) \# p_F \circ \overline{\beta}(-\overline{\sigma}_{\widetilde{y}_0}) \# p_F \circ \overline{\beta}(\sigma_-) \\ &= S(\sigma_+) + \overline{S}(\sigma_-) \end{aligned}$$

we just need to recall that $\chi_\kappa|_{E'} = \beta$ and $\chi_\kappa|_{\overline{E}'} = \overline{\beta}$ for all κ, and that $p_F \circ \chi(\sigma_{\widetilde{y}_0} \# \overline{\sigma}_{\widetilde{y}_0}) = \widetilde{y}_0$. □

Our next task is to see that the Floer homology does not depend on the choice of generic pair $(\{\theta_t\}, J)$ with prescribed Calabi invariant θ. So we can denote this homology by $HF_*(\theta)$.

We now consider the case where $\widetilde{\gamma}$ is the pull back of a path γ_s, $\gamma_s \in Symp_0(M, \omega)$, such that the Calabi invariant doesn't change with s. That is, γ_s is the family of time-one maps of the flow on M induced by $\eta_{s,t} = \theta_t^\alpha + \rho(s) dR_t$. Then $\widetilde{\gamma}_s$ is the corresponding family (on $\widetilde{M}$) induced by $\widetilde{L}_{s,t} = \widetilde{H}_t^\alpha + \rho(s) p^* R_t$ (where $p^*\eta_{s,t} = d\widetilde{L}_{s,t}$). We will use the bundles defined in chapter 2, namely

$$\begin{array}{c} E_{\widetilde{\gamma}} = (\mathbb{R} \times [0,1] \times \widetilde{M}) / \sim \\ \downarrow \\ \mathbb{R} \times S^1 \end{array} \tag{5.1}$$

where the equivalence relation is given by $(s, 1, x) \equiv (s, 0, \widetilde{\gamma}_s(x))$. Recall that this bundle is equipped with an almost complex structure $J_{\widetilde{\gamma}}$, and a closed 2-form $\Omega_{\widetilde{\gamma}}$. Since holomorphic sections of $E_{\widetilde{\gamma}}$ are in 1-to-1 correspondence with solutions $u : \mathbb{R} \times S^1 \longrightarrow \widetilde{M}$ of

$$\frac{\partial u}{\partial s} + J_{s,t}(u)(\frac{\partial u}{\partial t} - X_{\widetilde{H}_t}(u) - X_{\rho(s)p^* R_t}(u)) = 0 \tag{5.2}$$

the following result is equivalent to theorem 4.3 in [12].

Theorem 4. *For generic pairs $(\theta_t^\alpha, J^\alpha)$, $(\theta_t^\beta, J^\beta)$ such that $\theta_t^\alpha = \theta_t^\beta + dR_t$, there exists a natural $\Lambda_{\theta,\omega}-$ module isomorphism*

$$HF^{\beta,\alpha} : HF_*(\theta_t^\alpha, J^\alpha) \longrightarrow HF_*(\theta_t^\beta, J^\beta)$$

that preserves degree.

Proof. The proof of this theorem can be carried out using arguments similar to the ones used when defining all the previous maps. Namely, we construct a chain homomorphism with sections of bundles over the cylinder, where the key problem, which is the control of the energy of such sections, can be easily solved in this situation. We do not need to control the length of sections in this case. The main difference between this bundles and the ones used when defining Ψ and $\overline{\Psi}$ is that the form $\Omega_{\widetilde{\gamma}}$ can be made symplectic by adding a suitable multiple of the area form on the base. The only thing we need to further explain is how to define $S(\sigma)$ for a section σ of the bundle 5.1 that has $\widetilde{x} = [(t,\, \widetilde{x}_0); u_0]$ and $\widetilde{y} = [(t,\, \widetilde{y}_0); v_0]$ as ends. Observe that in this case $\widetilde{x}_0 \in fix\{\widetilde{\varphi}\}$, but $\widetilde{y}_0 \in fix\{\widetilde{\varphi}'\}$, where $\widetilde{\gamma}_s = \widetilde{\varphi}$ for $s \leq 1$ and $\widetilde{\gamma}_s = \widetilde{\varphi}'$ for $s \geq 2$). We do this by capping-off the ends of the cylinder with constant discs. That is, let $\sigma_p(z) = (z, p)$, where $z \in D^2$ and $p \in \widetilde{M}$. Then $\sigma_{\widetilde{x}_0} \# \sigma \# - \sigma_{\widetilde{y}_0}$ is homotopic to a sphere in $\widetilde{M}$ which we denote by $S(\sigma)$. So, as before, the topological condition on the space of sections we use to define the chain homomorphism will be $[S(\sigma)] + u_0 = v_0$.

The chain map is defined, as usual, by counting isolated sections of the bundle 5.1, or equivalently solutions of 5.2. By the usual gluing and compactness arguments we get that this map is well defined, that it is a chain map, and that it has an inverse. To prove all this just reproduce a simplified version of the arguments given when we verified similar results for Ψ. The arguments will be simpler because the closed 2-form on the total space of the bundle is already non-negative, so we do not need all the estimates of chapter 4. □

Remark 9. We would like to point out that we can use this capping off procedure to get a sphere homologous to $S_0(\sigma_0)$ for sections $\sigma_0 : \mathbb{R}\times S^1 \longrightarrow T_{\widetilde{\varphi}}$ (the bundle used to define boundary operator ∂_F). This will have the advantage that we do not need $\widetilde{\varphi}_t(p) = p$ for all $p \in fix\{\widetilde{\varphi}\}$ in order to have the sphere $S_0(\sigma_0)$, so we can use it to define Floer homology even if $\widetilde{\varphi}$ is not close to the identity.

Theorem 5. *Let (M, ω) be a symplectic manifold of dimension $2n$ satisfying the conditions of the main theorem. If $\theta \in H_1(M;\mathbb{Z})$ then there exists $e = e(M, \omega, J) > 0$ such that if $|\theta|_{C^1} < e$ the sum of ranks of $HF_*(\eta_t)$ is greater than or equal to that of $Nov_*(\theta)$.*

Proof. The proof of this theorem is an immediate corollary of proposition 19. Since $\left(\overline{\Psi} \circ \Psi\right)_* = id$ the map $\Psi_* : Nov_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega} \longrightarrow HF_*(\eta_t)$ is injective, and the conclusion follows. The appropriate value of e such that all the bounds of section 4 hold, having then that the maps Ψ, $\overline{\Psi}$ and Θ are well defined is explained in remark 5. □

Remark 10. The same kind of arguments should enable one to show that in fact Ψ_* is an isomorphism. We will not deal with this situation here. The only difficulty that one finds when doing this is constructing the correct bundle that corresponds to the gluing of a section of $\overline{\Psi}$ with one of Ψ, so that $(\Psi \circ \overline{\Psi})_*$ is the identity of the Floer homology (see to get an idea of how this argument should be).

Finally observe that whenever $\Gamma_1 \simeq \mathbb{Z}$ we have

$$\#fix\{\varphi\} \geq rank_{\Lambda_{\theta,\omega}}(HF_*(\theta_t)) = rank_{\Lambda_{\theta,\omega}}(HF_*(\eta_t)) \geq rank_{\Lambda_\theta}(Nov_*(\theta)).$$

Here the first inequality is given by construction (recall that $\theta_t = \theta + dH_t$), the next equality follows from theorem 4, and the last inequality from the previous theorem.

Now, when $\Gamma_1 \simeq \mathbb{Z}^k$ we can use a continuation argument. It is easy to see that if $\theta \in H_1(M;\mathbb{Z})$ then $\Gamma_1 \simeq \mathbb{Z}$. If $\theta \in H_1(M;\mathbb{Q})$ then there exists a q such that $q\theta \in H_1(M;\mathbb{Z})$ and we are in the above situation. If $\theta \in H_1(M;\mathbb{R})$ we can approximate it by any sequence of rational forms α_n, in a small enough neighbourhood of θ as to have that $zeros\{\theta\} = zeros\{\alpha_n\}$. This is possible because the quantity e that bounds the size of θ only depends on the geometry of M.

The main theorem follows.

Chapter 6

Examples

Theorem 3 tells us that the Euler number of the Novikov homology of a free abelian covering of a manifold W is the same as that of the original manifold. Thus we have the Lefschetz fixed point formula for non-Hamiltonian symplectomorphisms in a neighbourhood of the identity. Here we will give examples of semi-positive symplectic manifolds such that the sum of the Novikov-Betti numbers corresponding to certain free abelian coverings of W is strictly greater than the Euler number of W, and such that the symplectic form and the first Chern class are not related.

Example 1. Take $W = \Sigma_g \times M^6$ with the product symplectic structure. Here M is any 6−dimensional symplectic manifold such that: the first Chern class is not a multiple of the symplectic form, the minimal Chern number $N \geq 2$, and $b_3 \neq 0$. Σ_g is a Riemman surface of genus $g > 1$.

An example of such a manifold will be for g and $g^{'} \geq 2$ let

$$(W, \Omega) = (\Sigma_g \times \Sigma_{g'} \times S^2 \times S^2, \omega_g \oplus \omega_{g'} \oplus \omega_0 \oplus \lambda\omega_0).$$

Here ω_g and $\omega_{g'}$ are any non-degenerate 2-forms on Σ_g and $\Sigma_{g'}$ respectively. This manifold is not monotone, in fact the symplectic form and the first Chern class are not related. It is semi-positive because the minimal Chern number is 2. Take any 1-form θ such that the minimal free abelian cover that makes it exact is $\widetilde{W} = \widetilde{\Sigma}_g \times \Sigma_{g'} \times S^2 \times S^2$ or $\widetilde{W} = \Sigma_g \times \widetilde{\Sigma}_{g'} \times S^2 \times S^2$ (for example $\theta = (\theta_1, 0)$ or $\theta = (0, \theta_1) \in H^1(W; \mathbb{R}) = H^1(\Sigma_g; \mathbb{R}) \times H^1(\Sigma_{g'} \times S^2 \times S^2; \mathbb{R})$). Using theorem 3 we see that the Novikov-Betti numbers of $\widetilde{\Sigma}_g$ (respectively $\widetilde{\Sigma}_{g'}$) are 0, $2g-2$, 0 (respectively 0, $2g'-2$, 0). Using theorem 2 we see that the sum of the Novikov-Betti numbers of $\widetilde{W}$ is $(2g-2)(8g'+8)$ whereas the Euler number is equal to $(2g-2)(8g'-8)$.

Example 2. Let $M = \Sigma_g \times X_5$ with $g \geq 2$, where X_5 is the hypersurface in $\mathbb{C}P^4$ defined by $\{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\}$. It is a well known fact that $c_1 = 0$, and that its Betti numbers are $b_1 = 0$, $b_2 = 1$, and $b_3 = 204$. The maximal free abelian covering of M is $\widetilde{M} = \widetilde{\Sigma}_g \times X_5$, where $\widetilde{\Sigma}_g$ denotes the maximal free abelian cover of Σ_g. We have, using theorem 2 and theorem 3, that the sum of the Novikov-Betti numbers of $\widetilde{M}$ is $416(g-1)$ while the Euler number is $400(g-1)$.

Chapter 7

Appendix: on transversality and compactness

In this appendix we will state some useful results. The proof of the following theorems can be found in complete detail in [3], see also [4].

Given any smooth periodic 1-form $\theta_t = \theta + dH_t$ we denote by $\mathcal{U}_\delta(\theta_t)$ the set of all periodic 1-form $\theta_t = \theta + dH_t'$ with $\left\|H_t - H_t'\right\|_\varepsilon$, where the norm $\|h\|_\varepsilon$ is defined as follows

$$\|h\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|h\|_{C^k(S^1 \times M)}$$

Where ε_k is a rapidly decreasing sequence (see [12]).

We choose a generic almost complex structure compatible with ω. The weak monotonicity condition yields the non-existence of J-holomorphic spheres of negative Chern number for generic J. Moreover, we denote by $M_k(c; J)$ the set of points $p \in M$ for which there exists a non-constant $J-$holomorphic sphere $v : S^2 \to M$ with $c_1(v) \leq k$, $\omega(v) \leq c$ and $p \in v(S^2)$. The set $M_0(\infty; J)$ is then a subset of M of codimension 4 and the set $M_1(\infty; J)$ has codimension 2 [4]. Recall that we are working with M semi-positive.

Theorem 6. *There exists a dense subset $\Theta_0 \subset \mathcal{U}_\delta(\theta_t)$ such that the following holds for $\theta_t \in \Theta_0$*

(i) every periodic solution $x \in \mathcal{P}(\theta_t)$ is non-degenerate;

(ii) $x(t) \notin M_1(\infty; J)$ for every $x \in \mathcal{P}(\theta_t)$ and every $t \in \mathbb{R}$.

By the previous theorem there exists a periodic 1-form $\theta_t = \theta + dH_t$ in a prescribed cohomology class $[\theta]$ such that every element in $\mathcal{P}(\theta_t)$ is non-degenerate and does not intersect $M_1(\infty; J)$. Choose disjoint compact neighbourhoods $U_1, U_2, ..., U_m \subset S^1 \times M$ of the graphs of the finitely many contractible periodic solutions of $\dot{z}(t) = X_{\theta_t}(z(t))$. We denote by $\mathcal{V}_\delta(\theta_t)$ the set of all periodic 1-form $\theta_t = \theta + dH_t^{'}$ with $\left\|H_t - H_t^{'}\right\|_\varepsilon < \delta$ and $H_t = H_t^{'}$ on U_j for $j = 1, 2, ..., m$. If δ is sufficiently small then there are no contractible 1-periodic solutions of the previous equation outside the sets U_j for $\theta_t^{'} \in \mathcal{V}_\delta(\theta_t)$.

Theorem 7. *There is a generic set $\Theta_1 \subset \mathcal{V}_\delta(\theta_t)$ containing θ_t such that the following holds for $\theta_t^{'} \in \Theta_1$*

(i) the moduli space $\mathcal{M}(x^-, x^+; \theta_t^{'}, J)$ of connecting orbits is a finite dimensional manifold for all $x^\pm \in \mathcal{P}(\theta_t)$.

(ii) $u(s,t) \notin M_0(\infty; J)$ for every $u \in \mathcal{M}(x^-, x^+; \theta_t^{'}, J)$ with $\mu(u) \leq 2$ and every $(s,t) \in \mathbb{R} \times S^1$.

Here $\mu(u)$ is the local dimension of $\mathcal{M}(x^-, x^+; \theta_t^{'}, J)$ near u, that is the Fredholm index of the operator obtained by linearizing the perturbed Cauchy-Riemman equation.

Theorem 8. *Suppose (θ_t, J) are regular parameters (in the sense of theorem 6 and theorem 7). Then the parts of $\mathcal{M}(x^-, x^+; \theta_t, J)$ of dimension less than* *or equal to 2 with a uniform bound on the (usual) energy are compact, up to splittings.*

Proof. (taken from [4]) Let u_ν be a sequence of elements of $\mathcal{M}(x^-, x^+; \theta'_t, J)$ such that $\mu(u_\nu) \leq 2$ and $E(u_\nu) \leq c$. Assume without loss of generality that $E(u_\nu)$ converges. Using a standard argument as in [9] one can show that there exists a subsequence (still denoted by u_ν), periodic solutions $x^- = x^0, x^1, ..., x^{l-1}, x^l = x^+$ (not necessarily distinct), and connecting orbits $u^j \in \mathcal{M}(x^{j-1}, x^j; \theta'_t, J)$ for $j = 1, 2, ..., l$ with total energy $\sum_{j=1}^{l} E(u^j) \leq c$ such that the following holds. Given any sequence $s_\nu \in \mathbb{R}$ the sequence $v_\nu(s,t) = u_\nu(s+s_\nu, t)$ has a subsequence that converges modulo bubbling either to $u^j(s + s^j, t)$ for some s^j or to $x^j(t)$ for some j. Here convergence modulo bubbling means that there exist finitely many points in $\mathbb{R} \times S^1$ such that u_ν converges with its derivatives uniformly on compact subsets of the complements of these points. Moreover, every u^j is such a limit and no other connecting orbit can be approximated by u_ν in this way.

We prove that bubbling cannot occur. There are only finitely many holomorphic spheres that can bubble off in our limit process. We denote this spheres by $v^1, v^2, ..., v^m$. We get that

$$\sum_{j=1}^{l} E(u^j) + \sum_{j=1}^{m} E(v^j) = \lim_{\nu \to \infty} E(u_\nu) \leq c$$

and

$$\sum_{j=1}^{l} \mu(u^j) + \sum_{j=1}^{m} 2c_1(v^j) = 2$$

Since there is no J–holomorphic sphere of negative Chern number this implies that $\mu(u^j) \leq 2$ for every j. The key point in our argument is that the spheres v^j together with the connecting orbits u^j and the periodic solutions x^j form a connected family. So if bubbling occurs then one of the spheres v^j must intersect one of the connecting orbits u^j or one of the periodic solution x^j. Since the connecting orbits u^j avoid spheres of Chern number 0 there must be a j with $c_1(v^j) > 0$. This implies $\sum_{j=1}^{l} \mu(u^j) \leq 0$. But for regular parameters there is no nonconstant connecting orbit u with $\mu(u) \leq 0$ (see lemma 3.5 of [12]). Hence $x^- = x^+$, $l = 1$, and one of the spheres must intersect the periodic solution $x^\pm$ contradicting the fact that holomorphic spheres of Chern number 1 do not intersect the periodic solutions. The same argument works for $\mu(u_\nu) = 1$ and this proves the theorem. $\square$

For a regular homotopy J_s ($s \in \mathbb{R}$) of almost complex structures constant outside some compact set, we have that the set

$$\mathcal{M}_s(A; \{J_s\}) = \{(s, v) \mid v \in \mathcal{M}_s(A; J_s)\}$$

is a manifold of dimension $2n + 2c_1(A) + 1$ for a generic family $\{J_s\}$. This space will be empty in the case $c_1(A) < 3 - n$. In the case $c_1(A) = 0$ the set of pairs (s, p) such that p is a point in a J_s-holomorphic curve in class A is roughly speaking a set of codimension 4 in $\mathbb{R} \times M$. So the previous theorems remain valid if we consider perturbations of the Cauchy-Riemann equations that depend also on the real parameter s. That is, we can still avoid bubbling when we take limits of sequences of connecting orbits of index 0 or 1 for dimensional reasons.

Theorem 9. *Let (W, J_n, μ_n) be uniformly tame and such that J_n converge to J_∞. Let S be a closed real surface (not necessarily connected) and j_n a sequence of complex structures on S. Let $f_n : (S, j_n) \longrightarrow (W, J_n)$ be a sequence of J_n-holomorphic maps. Assume that the area(f_n) is bounded by some constant A and that the image $f_n(S)$ meet a fixed compact set. Then there is a subsequence which converges to a cusp curve.*

For a proof of this theorem see [1].

Bibliography

[1] Michèle Audin and Jacques Lafontaine, editors. *Holomorphic curves in symplectic geometry.* Birkhäuser Verlag, Basel, 1994.

[2] Yu. V. Chekanov. Lagrangian intersections, symplectic energy, and areas of holomorphic curves. *Duke Math. J.*, 95(1):213–226, 1998.

[3] Andreas Floer, Helmut Hofer, and Dietmar Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 1995.

[4] H. Hofer and D. A. Salamon. Floer homology and Novikov rings. In *The Floer memorial volume*, pages 483–524. Birkhäuser, Basel, 1995.

[5] D. McDuff. Elliptic methods in symplectic geometry. *Bull. A.M.S.*, 23:311–358, 1990.

[6] D. McDuff. The virtual moduli cycle. preprint, 1998.

[7] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford University Press, 1995.

[8] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer-Donaldson theory and quantum cohomology. In *Contact and symplectic geometry*

(Cambridge, 1994), pages 171–200. Cambridge Univ. Press, Cambridge, 1996.

[9] D. Salamon. Morse theory, Conley index and Floer homology. *Bull. L.M.S*, 22:113–140, 1990.

[10] M. Schwarz. *Cohomology operations from S^1-cobordisms in Floer homology.* PhD thesis, Swiss Federal Inst. Techn. Zurich, 1995.

[11] P. Seidel. π_1 of symplectic automorphism groups and invertibles in quantum homology rings. preprint, March 1996.

[12] Lê Hông Vân and Kaoru Ono. Symplectic fixed points, the Calabi invariant and Novikov homology. *Topology*, 34(1):155–176, 1995.