Seiberg-Witten Invariants of Non-Simple Type

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Abstract of the Dissertation,

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We construct examples of four dimensional manifolds with Spin^c-structures, whose moduli spaces represent a non-trivial homology class of positive dimension. Our work relies on the analysis of the Seiberg-Witten equations on manifolds with cylindrical ends, which correspond, somehow, to certain connected sums. As an application of the results above, we show the existence of infinitely many non-homeomorphic compact oriented 4-manifolds with predetermined Euler characteristic and signature that do not carry Einstein metrics.

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Chapter 1

Introduction

After the work of S. K. Donaldson in 1980 it became clear that gauge-theoretic invariants of principal bundles and connections were an important tool in the study of smooth four dimensional manifolds. Donaldson showed the importance of the moduli space of anti-self-dual connections. Computation of Donaldson's polynomial invariants for a wide class of four dimensional manifolds, especially algebraic surfaces produced many powerful topological consequences, such as, for example, the diffeomorphism classification of elliptic surfaces. In 1994, N. Seiberg and E. Witten introduced a different gauge-theoretic invariant. These new invariants are easier than those of Donaldson because they involve principal bundles with structure group $U(1)$, instead of the non-abelian structure groups in Donaldson's theory. The new invariant are so powerful that several long standing conjectures that were proven immediately after their discovery.

Once again, the moduli space of solutions to the Seiberg-Witten equations plays an essential rôle in the theory. The homology class of this moduli space is an invariant of the *smooth structure* of the manifold. The equations defining the moduli space are elliptic modulo the gauge group. A generic perturbation of the equations leads to a smooth orientable moduli space, whose dimension can be computed by the Atiyah-Singer index theorem. One special property of this theory is the compactness of the moduli space of solutions, a consequence of *a priori* bounds for the point-wise norms of solutions to the equations. This result has no analogue in Donaldson's theory. The Seiberg-Witten moduli spaces are compact and vary by a compact bordism as we vary the metric.

In order to define these invariants we have to consider a Riemannian 4-manifold (M, g) and a Spin^c-structure $\mathfrak{c}$ on M. The Seiberg-Witten equations are equations that involves a $U(1)$-connection A on the determinant line bundle $L_{\mathfrak{c}}$ associated to the Spin^c-structure and a section ϕ of the self-dual spinors. The configuration space consists of all pairs (A, ϕ). A pair is called irreducible if $\phi \neq 0$. The configuration space of irreducible pairs has the same homotopy type of $\mathbb{CP}^{\infty} \times T^{b_1}$ where b_1 is the first betti number of M. The invariants are constructed pairing the moduli space with a cohomology class of the configuration space of irreducible solutions. Originally, these invariants were defined in a slightly more restrictive fashion. Namely, they were defined as zero if the moduli space was odd dimensional, and otherwise, they were defined by pairing the moduli space with the appropriate power of the generator U of the $\mathbb{CP}^{\infty}$ factor.

If the Seiberg-Witten invariant is non zero we say that $L_{\mathfrak{c}}$ is a basic class. If all the basic classes are zero dimensional, then M is said to be of simple type. The first examples of four dimensional manifolds with Spin^c-structures, whose moduli space represented a non-trivial homology class of positive dimension, were constructed recently by P. Ozsváth and Z. Szabó [11]. They do this by finding a cohomology class with non-trivial pairing with the moduli space. This cohomology class is not a power of U, and the proof of the non-triviality of the pairing involves a well-known *gluing argument* that is known for Donaldson's theory and presumably extendable to Seiberg-Witten theory, but to our knowledge such an extension has not been published in the literature.

As a consequence of our work, we reproduce some of theirs examples relying on an entirely different technique, and therefore, bypassing the gluing argument above. Our work relies on the analysis of the Seiberg-Witten equations on manifolds with cylindrical ends, which correspond, somehow, to certain connected sums. These manifolds with cylindrical ends can be thought out as the limit of suitable conformal deformations of a fixed Riemannian manifold. We then prove that the analysis of the limiting manifold can be carried out through certain conformal properties of the Seiberg-Witten equations. More precisely, let (M, J, g) be a Kähler surface with canonical line bundle K_M of positive degree, and let $\mathcal{M}_{g_f}$ be the SW-moduli space of $g_f = e^{2f}g$. We show that $\mathcal{M}_{g_f}$ consists of a single *smooth* point. Furthermore after proving that if a manifold admits a Kähler metric, then, without changing the Kähler class, it also admits a Kähler metric that is flat nearby any finite collection of points, we produce a sequence of Riemannian manifolds (M, g_l) conformally equivalent to (M, g), that converges in the $\mathcal{CO}$-topology to a Riemannian manifold with finitely many cylindrical ends (M_∞, g_∞). Since the induced sequence $\mathcal{M}_{g_l}$ of moduli spaces consists of a single smooth point, we then show that the SW-moduli space $\mathcal{M}_{g_\infty}$ consists of a single *smooth* point as well.

The manifold (M_∞, g_∞) is not compact and therefore, in the proof of the result above, we cannot use the usual analytical tools. Instead of working with the Sobolev spaces L^p_q, we are forced to use the weighted Sobolev spaces $L^p_{q,\epsilon}$ which permit us to overcome the lack of compactness on M_∞. The introduction of these weighted Sobolev spaces will ensure that the involved operator is Fredholm, but it does not give us enough information to compute its index. We do so by showing that we can *extend* a solution of the SW-equations on (M_∞, g_∞) to a solution of a special kind of perturbation to the SW-equations on (M, g). The index in the later case is therefore the index of the operator on the non-compact manifold. For a compact oriented 4-manifold M, the connected sum $M\#(S^1 \times S^3)$ can be thought of as a manifold with two cylindrical ends where we cut the ends at some finite length and identify the boundary. We may then use the result above, and the notion of holonomy, to conclude that if M is a Kähler manifold with $\deg(K_M) > 0$, then the SW-moduli space of any Riemannian metric on $M\#(S^1 \times S^3)$ represents a one dimensional, non-trivial, bordism class.

As an application of the results above we show the existence of infinitely many non-homeomorphic compact oriented 4-manifolds with predetermined Euler characteristic and signature that do not carry Einstein metrics. These examples are of the form $M\#k\,\overline{\mathbb{CP}^2}\#l(S^1 \times S^3)$, where M, k and l depend on the predefined Euler characteristic and signature. The fact that the examples constructed do not carry Einstein metrics does not follow from Hitchin-Thorpe's or Gromov's obstruction theorems [5, 4]. Rather, this follows from careful estimates of the ingredients in the Gauss-Bonnet formula obtained using SW-invariants and generalizing C. LeBrun's ideas [8]. Our examples have free fundamental group. Similar examples with very complicated fundamental group have being obtained by A. Sambusetti [12] using connected sums with real or complex hyperbolic 4-manifolds.

Chapter 2

Clifford Algebras and Spin Groups

Definition 1. Let $(V, \langle , \rangle)$ be an inner-product real vector space of dimension n. The *real Clifford algebra* $Cl(V)$ is the algebra generated by $\{e_1, \ldots, e_n\}$, with relations $e_i^2 = -1$, $e_i e_j = -e_j e_i$ for $i \neq j$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V.

An alternative definition may be given as follows: Let $T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ and consider the two sided ideal $I(V)$ generated by $v \otimes v + |v|^2$, $v \in V$. Then $Cl(V) = T(V)/I(V)$. It follows from this last definition, that $Cl(V)$ is independent of the choice of orthonormal basis.

There is a natural map $F : \bigwedge^*(V) \to Cl(V)$, where $\bigwedge^*(V)$ is the exterior algebra of V. F is linear, but not multiplicative. $F(e_i \wedge \cdots \wedge e_k) = e_i \cdot \ldots \cdot e_k$. There is a $\mathbb{Z}_2$ grading on $Cl(V)$: $Cl(V) = Cl_0(V) \oplus Cl_1(V)$, where

$$Cl_0(V) = F\left(\oplus_{i\geq 0} \bigwedge^{2i}(V)\right), \quad Cl_1(V) = F\left(\oplus_{i\geq 0} \bigwedge^{2i+1}(V)\right)$$

In the case when $n = 4$, the structure is richer. Indeed, let $\Omega = -e_1e_2e_3e_4$, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis for $\mathbb{R}^4$. It is easy to see that the *volume element* Ω is independent of the choice of orthonormal basis. Note that Ω is in the center of $Cl_0(V)$. Since $\Omega^2 = 1$, we get a decomposition

$$Cl_0(\mathbb{R}^4) = Cl_0^+(\mathbb{R}^4) \oplus Cl_0^-(\mathbb{R}^4),$$

where $Cl_0^+(\mathbb{R}^4)$ and $Cl_0^-(\mathbb{R}^4)$ are the $+1$ and -1 eigenspaces of Clifford multiplication by Ω.

Definition 2. $Pin(V) \subset Cl(V)$ is the multiplicative group generated by elements $v \in V$ with $|v|^2 = 1$, and $\mathrm{Spin}(V) = Pin(V) \cap Cl_0(V)$.

$\mathrm{Spin}(V)$ acts on $Cl(V)$ by conjugation: $\sigma \circ c = \sigma \cdot c \cdot \sigma^{-1}$. The action of any $\sigma \in \mathrm{Spin}(V)$ preserves the subspace $V \subset Cl(V)$. Since this action preserves the norm and orientation of V, it induces map $\mu : \mathrm{Spin}(V) \to SO(V)$.

Lemma 1. $\mathrm{Spin}(V)$ *is connected and* $\mu : \mathrm{Spin}(V) \to SO(V)$ *gives a double cover of* $SO(V)$.

Since $\pi_1(SO(n)) = \mathbb{Z}_2$, it follows that $\mathrm{Spin}(V)$ is the universal cover of $SO(n)$ for $n \geq 3$.

Definition 3. The group $\mathrm{Spin}^c(V)$ is the subgroup of the multiplicative group of units of $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $\mathrm{Spin}(V)$ and the unit circle of complex scalars.

Lemma 2. *There is an isomorphism*

$$\mathrm{Spin}^c(V) \cong \mathrm{Spin}(V) \times_{\{\pm 1\}} U(1) \underset{2:1}{\longrightarrow} SO(V) \times U(1)$$

Consider the map $\mathrm{Spin}^c(V) \to SO(n)$ given by dividing out by the center, and ask when a principal $SO(n)$-bundle $P \to M$ lifts to a principal $\mathrm{Spin}^c(n)$-bundle. The homomorphism $\mathrm{Spin}^c(n) \to U(1)$ given by dividing out by $\mathrm{Spin}(n)$ determines a complex line bundle $L \to M$ associated to any principal $\mathrm{Spin}^c(n)$-bundle. This is the *determinant line bundle* of the $\mathrm{Spin}^c(n)$-bundle. If this $\mathrm{Spin}^c(n)$-bundle lifts a principal $SO(n)$-bundle $P \to M$, then it is easy to see that this determinant line bundle has its first Chern class $c_1(L)$ which agrees mod 2 with $w_2(P)$, the second Stieffel-Witney class of P. Conversely, given any line bundle $L \to M$ whose first Chern class satisfies this mod 2 equation, there is a $\mathrm{Spin}^c(n)$ lifting of P with determinant line bundle isomorphic to L.

Given a $\mathrm{Spin}^c(n)$-lifting $\mathfrak{c}$ of a principal $SO(n)$-bundle then we can construct the associated Spin^c-bundle

$$S(\mathfrak{c}) = \mathfrak{c} \times_{\mathrm{Spin}^c(n)} S_{\mathbb{C}}(\mathbb{R}^n),$$

where $S_{\mathbb{C}}(\mathbb{R}^n)$ is an irreducible, finite dimensional, representation of $Cl(\mathbb{R}^n)$. In fact $S(\mathfrak{c})$ is a complex vector bundle.

When $n = 4$, $S_{\mathbb{C}}(\mathbb{R}^4)$ decomposes into $S^{\pm}_{\mathbb{C}}(\mathbb{R}^4)$ under the action of Ω. This decomposition is a decomposition of modules over $Cl_0(\mathbb{R}^4) \otimes \mathbb{C}$. Clifford multiplication induces isomorphisms

$$(Cl_0(\mathbb{R}^4) \otimes \mathbb{C})^{\pm} \cong \mathrm{End}_{\mathbb{C}}(S^{\pm}_{\mathbb{C}}(\mathbb{R}^4))$$

Varying the $\mathrm{Spin}^c(n)$-lifting by a class $\alpha \in H^2(M;\mathbb{Z})$ has the effect of replacing $S(\mathfrak{c})$ by $S(\mathfrak{c}) \otimes L_\alpha$ where L_α is the complex line bundle with first Chern class α.

Definition 4. Given a smooth compact oriented n-manifold M, a $\mathrm{Spin}^c(n)$-lifting $\mathfrak{c}$, of the frame bundle associated to a Riemannian metric g on M will be called a Spin^c-structure on M.

Example 1. Let $(V, \langle , \rangle)$ be a finite dimensional real inner product space, with a complex structure $J : V \to V$. We assume that this complex structure is compatible with the inner product in the sense that J is an orthogonal transformation. Since $J^2 = -I$, the space $V \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into the i and the $-i$ eigenspaces for the complexification of the action of J. We denote these eigensubspaces by $V_{\mathbb{C}}^{1,0}$, $V_{\mathbb{C}}^{0,1}$ and we denote the projections onto them by $\pi^{1,0}$ and $\pi^{0,1}$ respectively. Notice that $\pi^{1,0}$ is complex linear. This decomposition induces a bigrading of the complex exterior algebra $\bigwedge_{\mathbb{C}}^*(V \otimes \mathbb{C})$. We denote by $\bigwedge_{\mathbb{C}}^* V$ the subalgebra of the exterior powers of $V_{\mathbb{C}}^{1,0}$. We define an action of V on $\bigwedge_{\mathbb{C}}^* V$ by

$$v \cdot (\alpha^1 \wedge \ldots \wedge \alpha^l) = \sqrt{2}(\pi^{1,0}(v) \wedge (\alpha^1 \wedge \ldots \wedge \alpha^l) - \pi^{1,0}(v) \lrcorner (\alpha^1 \wedge \ldots \wedge \alpha^l))$$

where

$$\pi^{1,0}(v) \lrcorner (\alpha^1 \wedge \ldots \wedge \alpha^l) = \sum_{i=1}^{l} (-1)^{i-1} \langle \alpha^i, \pi^{1,0}(v) \rangle \alpha^1 \wedge \ldots \wedge \hat{\alpha^i} \wedge \ldots \wedge \alpha^l$$

and the inner product $\langle \alpha^i, \pi^{0,1}(v) \rangle$, is the Hermitian inner product on $V \otimes \mathbb{C}$

extending the given inner product on V.

Next we will construct an embedding ρ of the unitary group $U(V)$ into $\mathrm{Spin}^c(V)$. For any $A \in U(V)$ there is an unitary frame $e_1, \ldots e_n$ for V, considered as a complex vector space, in which A is diagonal; say $A(e_k) = e^{i\theta_k} e_k$, for $1 \leq k \leq n$. We associate to A the element in the complexification of the Clifford algebra of V given by

$$\Pi_{k=1}^{n} e^{\frac{i\theta_k}{2}} (\cos(\frac{\theta_k}{2}) + \sin(\frac{\theta_k}{2}) e_k \cdot J e_k).$$

If (M, J) is an almost complex $2n$-manifold, the complex structure and the Riemannian metric determine a reduction of the frame bundle of the tangent bundle to $U(n)$. That is to say we have a principal $U(n)$-bundle $P_{U(n)} \to M$ and an isomorphism $P_{U(n)} \times_{U(n)} SO(2n)$ with the orthogonal frame bundle $P_{SO(2n)}$ of M. We form

$$\tilde{P}_M = P_{U(n)} \times_{U(n)} \mathrm{Spin}^c(2n)$$

using the embedding $\rho : U(n) \to \mathrm{Spin}^c(2n)$ constructed above. Clearly,

$$\tilde{P}_M / S^1 = P_{U(n)} \times_{U(n)} SO(2n)$$

where the representation $U(n) \to SO(2n)$ is the quotient of ρ. By construction this quotient is the natural embedding. This proves that $\tilde{P}_M$ is a Spin^c-structure on M. The determinant line bundle of this Spin^c-structure is $\tilde{P}_M / \mathrm{Spin}(2n) = P_{U(n)} \times_{U(n)} S^1$ via the determinant map $\det : U(n) \to S^1$. That is to say the determinant line bundle of $\tilde{P}_M$ is the determinant line bundle of $P_{U(n)}$. This is the inverse line bundle of $P^*_{U(n)}$, the unitary frame bundle for the cotangent bundle. The determinant line bundle of the cotangent bundle is of course the canonical line bundle of $(0, n)$-forms.

Chapter 3

Spin Bundles and the Dirac Operator

There is a preferred connection on the frame (principal $SO(n)$) bundle associated to a Riemmanian metric g, the Levi-Civita connection ∇. Now let us fix an $U(1)$-connection A, on the determinant line bundle, $L_{\mathfrak{c}}$, of $\mathfrak{c}$. This gives a product connection on the principal $SO(n) \times U(1)$-bundle. Pulling back the product connection to the double cover gives a $\text{Spin}^c(n)$-connection ∇_A on $\mathfrak{c}$ and on $S(\mathfrak{c})$.

Definition 5. We define the Dirac operator D_A induced by the $U(1)$-connection A,

$$D_A : \mathcal{C}^\infty(S(\mathfrak{c})) \to \mathcal{C}^\infty(S(\mathfrak{c})),$$

by

$$D_A(\phi) = \sum_{i=1}^{n} e_i \cdot (\nabla_A)_{e_i}\phi,$$

where e_i is a local orthonormal coordinate system for TM, $\phi \in \mathcal{C}^\infty(S(\mathfrak{c}))$,

$(\nabla_A)_{e_i}$ is the covariant derivative in the e_i direction and $\cdot$ is the Clifford multiplication.

Note that D_A is independent of the choice of the local coordinate system. When $n = 4$ the Dirac operator D_A decomposes in the following way

$$D_A : \mathcal{C}^\infty(S^{\pm}(\mathfrak{c})) \to \mathcal{C}^\infty(S^{\mp}(\mathfrak{c})),$$

Example 2. Let (M, J, g) be a Kähler $2n$-manifold. We take the Spin^c-structure induced from the complex structure J (see Example 1). The determinant line bundle is identified with K_M^{-1}, the inverse of the canonical line bundle. Of course, the complex structure on M determines a holomorphic structure (or equivalently a $(0,1)$-connection on K_M^{-1}). The metric g on M determines a Hermitian metric on K_M^{-1}. There is a unique Hermitian connection A on K_M^{-1} which is compatible with the holomorphic structure in the sense that the $(0,1)$-part of the connection is the holomorphic $(0,1)$-connection. As we have seen (Example 1), the complex spin bundle of the Spin^c-structure is identified with the bundle of $(0,q)$-forms. Let ϕ be a $(0,k)$-form, let $(z_1, \ldots, z_n)$ be holomorphic coordinate near a point $p \in M$, so that setting $x_{2i-1} = \Re(z_i)$ and $x_{2i} = \Im(z_i)$ the Riemannian metric is standard to second order at the point p. We let e_i be the unit tangent vector at p in the x_i-direction. By definition for any section ϕ of the complex spin bundle we have

$$D_A(\phi)(p) = \sum_{i=1}^{2n} e_i \cdot (\nabla_A)_{e_i}(\phi)(p).$$

since the metric is standard to second order at the point, the connection A

is the product connection at the point and $(\nabla_A)_{e_i} = \partial_i$. Thus in a local trivialization,

$$D_A(\phi)(p) = \sum_{i=1}^{2n} e_i \cdot \partial_i(\phi)(p)$$

which in turn is given by

$$\sqrt{2}\sum_{i=1}^{2n} \pi^{0,1}(dx_i) \wedge \partial_i(\phi)(p) - \pi^{0,1}(dx_i) \lrcorner \partial_i(\phi)(p).$$

Of course $\pi^{0,1}(dx_{2k-1}) = d\overline{z}_k/2$ and $\pi^{0,1}(dx_{2k}) = id\overline{z}_k/2$. Thus, we have

$$\begin{aligned} D_A(\phi)(p) =& \sqrt{2}\sum_{k=1}^{n} d\overline{z}_k \wedge \frac{1}{2}(\partial_{2k-1}(\phi)(p) + i\partial_{2k}(\phi)(p)) \\ & - \sqrt{2}\sum_{i=1}^{2n} \pi^{0,1}(dx_i) \lrcorner \partial_i(\phi)(p) \end{aligned}$$

which is

$$\sqrt{2}\sum_{k=1}^{n} d\overline{z}_k \wedge \frac{\partial}{\partial \overline{z}_k}(\phi) - \sqrt{2}\sum_{i=1}^{2n} \pi^{0,1}(dx_i) \lrcorner \partial_i(\phi).$$

Since contraction is complex anti-linear in the first variable we can rewrite this last sum as

$$\begin{aligned} D_A(\phi)(p) =& \sqrt{2}\sum_{k=1}^{n} d\bar{z}_k \wedge \frac{\partial}{\partial \overline{z}_k}(\phi) \\ & - \sqrt{2}\sum_{k=1}^{n} (d\overline{z}_k) \lrcorner (\partial_{2k-1}(\phi)(p) - i\partial_{2k}(\phi)(p)). \end{aligned}$$

This is the same as

$$D_A(\phi)(p) = \sqrt{2}\left(\overline{\partial}(\phi)(p) - \sum_{k=1}^{n} d\overline{z}_k \lrcorner \frac{\partial}{\partial z_k}(\phi)(p)\right).$$

Which can be rewritten as

$$D_A(\phi)(p) = \sqrt{2}\left(\overline{\partial}(\phi)(p) + \overline{\partial}^*(\phi)(p)\right).$$

Chapter 4

SW-Moduli Space

Definition 6. Let $(M, \mathfrak{c})$ be a smooth compact oriented 4-manifold with a Spin^c-structure $\mathfrak{c}$. Let $L_\mathfrak{c} = \det(\mathfrak{c})$ be the determinant line bundle associated to $\mathfrak{c}$. Fix a Riemannian metric g on M. The configuration space $\mathcal{C}(\mathfrak{c})$ consist of pairs (A, ϕ) , where A is an $U(1)$-connection on $L_\mathfrak{c}$ and $\phi \in \mathcal{C}^\infty(S^+(\mathfrak{c}))$. We say that (A, ϕ) satisfy the Seiberg-Witten equations (SW-equations) if and only if

$$D_A\phi = 0$$
$$F_A^+ = q(\phi),$$

where $q(\phi) = \phi \otimes \phi^* - \frac{|\phi|^2}{2}\mathrm{Id}$.

Example 3. Recall from Example 2 that an orthogonal almost complex structure $J : TM \to TM$ on a Riemannian 4-manifold induces a Spin^c-structure $\mathfrak{c}$ whose determinant line bundle is K_M^{-1}, the inverse of the canonical line bundle of the almost complex structure. The spin bundles are given by

$$S^+(\mathfrak{c}) = \Omega^0(M;\mathbb{C}) \oplus \Omega^{0,2}(M;\mathbb{C})$$
$$S^-(\mathfrak{c}) = \Omega^{0,1}(M;\mathbb{C}).$$

Clifford multiplication by a one-form $a \in \Omega^1(M;\mathbb{C})$ is given by the sum of wedge product and minus contraction with $\sqrt{2}\pi^{0,1}(a) \in \Omega^{0,1}(M;\mathbb{C})$ of a (see Example 1). Furthermore, if the almost complex structure is in fact a complex structure for which the Riemannian metric is a Kähler metric, then the Dirac operator on self-dual spinors associated to the Spin^c-structure and the natural holomorphic, Hermitian connection on K_M^{-1} is (see Example 2)

$$\sqrt{2}(\overline{\partial} + \overline{\partial}^*) : \Omega^0(M;\mathbb{C}) \oplus \Omega^{0,2}(M;\mathbb{C}) \to \Omega^{0,1}(M;\mathbb{C}).$$

Any other Spin^c-structure $\tilde{\mathfrak{c}}$ differs from $\mathfrak{c}$ by tensoring with some $U(1)$-bundle $P_{U(1)} \to M$. Let L_0 be the complex line bundle associated to $P_{U(1)}$. Then the spin bundles for $\tilde{\mathfrak{c}}$ are given by

$$S^+(\tilde{\mathfrak{c}}) = S^+(\mathfrak{c}) \otimes L_0 = \Omega^0(M;L_0) \oplus \Omega^{0,2}(M;L_0)$$
$$S^-(\tilde{\mathfrak{c}}) = S^-(\mathfrak{c}) \otimes L_0 = \Omega^{0,1}(M;L_0).$$

As before, Clifford multiplication by $a \in \Omega^1(M;\mathbb{C})$ is given by the sum of wedge product and minus contraction with $\sqrt{2}\pi^{0,1}(a) \in \Omega^{0,1}(M;\mathbb{C})$. Furthermore, the determinant of $\tilde{\mathfrak{c}}$ is identified with $K_M^{-1} \otimes L_0^2$, or in other words $L_0 = \sqrt{K_M \otimes L_{\tilde{\mathfrak{c}}}}$ where $L_{\tilde{\mathfrak{c}}}$ is the determinant line bundle of $\tilde{\mathfrak{c}}$. To each $U(1)$-

connection A_0 on L_0 there is a canonical connection on $\tilde{\mathfrak{c}}$ which is the lift of the product of the holomorphic connection on M (induced by the Kähler metric) with A_0 via the covering map on Lemma 2. This canonical connection on $\tilde{\mathfrak{c}}$ induces a $U(1)$-connection A on $L_{\tilde{\mathfrak{c}}}$. To summarize

Proposition 3. *There is a one-to-one relationship between $U(1)$-connections on $L_{\tilde{\mathfrak{c}}}$ and $U(1)$-connections on L_0.*

We now have a good description of the Dirac Equation: A spinor field ϕ, has two components $\phi = (\alpha, \beta) \in \Omega^0(M, L_0) \oplus \Omega^{0,2}(M; L_0)$, and the Dirac equation is

$$\sqrt{2}(\overline{\partial}(\alpha) + \overline{\partial}^*(\beta)) = 0.$$

Let us write the equation for the curvature. Let ω be the Kähler form, it is a nowhere zero, self-dual real 2-form of type $(1,1)$. The complex-valued self-dual 2-forms on a Kähler manifold split as

$$\Omega^0(M;\mathbb{C})\omega \oplus (\Omega^{2,0}(M;\mathbb{C}) \oplus \Omega^{0,2}(M;\mathbb{C})).$$

The purely imaginary self-dual 2-forms are then

$$\Omega^0(M; i\mathbb{R})\omega \oplus \{\mu - \overline{\mu} \,|\, \mu \in \Omega^{0,2}(M;\mathbb{C})\}.$$

Hence, the self-dual part of the curvature of the unitary connection A on L can be written as $F_A^+ = if\omega + \mu - \overline{\mu}$ for some real-valued function f on M and some complex-valued $(0,2)$-form μ on M. As an endomorphism on $S^+(\mathfrak{c})$

this can be written

$$\begin{pmatrix} 2f & *2(\mu \wedge \overline{(\cdot)}) \\ 2\mu \wedge (\cdot) & -2f \end{pmatrix}.$$

On the other hand the matrix representation for $q(\phi)$ is given by

$$\begin{pmatrix} \frac{|\alpha|^2-|\beta|^2}{2} & \alpha\overline{\beta} \\ \overline{\alpha}\beta & \frac{|\beta|^2-|\alpha|^2}{2} \end{pmatrix}.$$

Thus, we see that the curvature equation is equivalent to the following equations

$$(F_A^+)^{1,1} = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega \tag{4.1}$$
$$F_A^{0,2} = \frac{\overline{\alpha}\beta}{2}$$

The solution space for the SW-equations is usually infinite dimensional (in case is not empty) and one has to mod out by an appropriate infinite dimensional group, the gauge group, to get a finite dimensional moduli space.

Definition 7. The *gauge group* $\mathcal{G}(\mathfrak{c})$ consists of smooth automorphisms of the principal $\mathrm{Spin}^c(4)$-bundle $\mathfrak{c}$ that cover the identity on M.

Remark. Note that the center of $\mathrm{Spin}^c(4)$ is S^1 and the action of this S^1 on $\mathrm{Spin}^c(4)$ covers the identity on $\mathrm{SO}(4)$. Now is easy to see that $\mathcal{G}(\mathfrak{c})$ consists of smooth maps $\sigma : M \to S^1$.

Note that any $\sigma \in \mathcal{G}(\mathfrak{c})$ acts on $L_\mathfrak{c}$, $S^+(\mathfrak{c})$ and $S^-(\mathfrak{c})$. Now suppose that $\phi \in C^\infty(S^+(\mathfrak{c}))$ and A is a $U(1)$-connection on $L_\mathfrak{c}$. Then there is an induced action $(A,\phi) \to (\sigma^* A, \sigma^{-1}\phi)$.

Definition 8. We will denote by $\mathcal{B}(\mathfrak{c})$ the equivalence classes under this action *i.e* $\mathcal{B}(\mathfrak{c}) = \mathcal{C}(\mathfrak{c})/\mathcal{G}(\mathfrak{c})$.

Lemma 4. *The Seiberg-Witten solution space is invariant under the action of the gauge group* $\mathcal{G}(\mathfrak{c})$.

Lemma 5. *The stabilizer in* $\mathcal{G}(\mathfrak{c})$ *of an element* $(A,\phi) \in \mathcal{C}(\mathfrak{c})$ *is trivial unless* $\phi = 0$ *in which case the stabilizer is the group consisting of the constant maps of* M *to* S^1.

Definition 9. We say that an element (A,ϕ) is irreducible if $\phi \neq 0$, otherwise it is reducible. We denote by $\mathcal{C}^*(\mathfrak{c})$ the open subset of irreducible configurations and by $\mathcal{B}^*(\mathfrak{c})$ the open subset of irreducible equivalence classes.

The naive definition of the Seiberg-Witten moduli space would be:

$$\mathcal{M}_g(\mathfrak{c}) = \{(A,\phi) \in \mathcal{C}(\mathfrak{c}) |\ D_A\phi = 0,\ F_A^+ = q(\phi)\}/\mathcal{G}(\mathfrak{c}),$$

but in order to use the usual analytical tools, one has to extend the $\mathcal{C}^\infty$ objects to appropriate Sobolev spaces. From now on we extend the configuration space $\mathcal{A}(\mathfrak{c})$ and the gauge group $\mathcal{G}(\mathfrak{c})$ by requiring A and ϕ to be in $L_2^2(M,g)$ and σ to be in $L_3^2(M,g)$. The SW-equations and the gauge actions make sense in this context also and we define:

Definition 10. The Seiberg-Witten moduli space is:

$$\mathcal{M}_g(\mathfrak{c}) = \{(A,\phi) \in \mathcal{C}(\mathfrak{c}) |\ D_A\phi = 0,\ F_A^+ = q(\phi)\}/\mathcal{G}(\mathfrak{c}),$$

where $\mathcal{A}(\mathfrak{c})$ and $\mathcal{G}(\mathfrak{c})$ are the extended configuration space and gauge group. The formal dimension of this moduli space is

$$d(\mathfrak{c}) = \frac{c_1^2(\mathfrak{c}) - (2e(M) + 3\sigma(M))}{4}.$$

In general there is no reason to expect that the moduli space form a smooth manifold. The best we can hope for is that *generically* it does. The next Theorem guarantees that this is the case. For the proof see [10].

Theorem 6. *Suppose that $b_2^+ > 0$. Fix a metric g on M. Then for a generic C^∞ self-dual 2-form h on M the following holds. For any* Spin^c*-structure $\mathfrak{c}$ on M the moduli space $\mathcal{M}_g(\mathfrak{c}, h) \subset \mathcal{B}(\mathfrak{c})$ of gauge equivalence classes of pairs $[A, \phi]$ which are solutions to the perturbed SW-equations*

$$D_A\phi = 0$$
$$F_A^+ - q(\phi) = ih$$

form a smooth compact submanifold of $\mathcal{B}^*(\mathfrak{c})$ of dimension $d(\mathfrak{c})$.

Proposition 7. *Consider a fixed $U(1)$-connection A on $L_\mathfrak{c}$. Let $[A_i, \phi_i]$ be solutions to the SW-equations, and let (A_i, ϕ_i) be the unique representatives such that $A_i - A$ is co-closed (gauge fixing condition, see [10]), for $i = 1, 2$. If $\phi_1 = \phi_2$ then $A_1 = A_2$.*

Proof. The first thing to notice is that $A_2 = A_1 + \theta$, where θ is a co-closed

1-form. Since (A_1, ϕ_1) and (A_2, ϕ_2) are solutions to the SW-equations we have

$$\begin{aligned} F_{A_1}^+ &= q(\phi_1) \\ &= q(\phi_2) \\ &= F_{A_2}^+. \end{aligned}$$

Therefore

$$\begin{aligned} F_{A_2}^+ - F_{A_1}^+ = 0 &\Leftrightarrow (d\theta)^+ = 0 \\ &\Leftrightarrow *d\theta = -d\theta \\ &\Rightarrow d * d\theta = -dd\theta = 0 \\ &\Leftrightarrow *d * d\theta = 0 \\ &\Leftrightarrow \delta d\theta = 0. \end{aligned}$$

This last statement and the fact $\delta\theta = 0$ implies that

$$\begin{aligned} \Delta\theta &= d\delta\theta + \delta d\theta \\ &= \delta d\theta \\ &= 0. \end{aligned}$$

Since (A_i, ϕ_i) $i = 1, 2$ are solutions to the Seiberg-Witten equations we have

$$\begin{aligned} 0 &= D_{A_2}\phi_2 \\ &= D_{A_1+\theta}\phi_1 \\ &= D_{A_1}\phi_1 + \theta \cdot \phi_1 \\ &= \theta \cdot \phi_1, \end{aligned}$$

multiplying by θ both sides of the equality we get that $|\theta|^2\phi_1 = 0$. Taking the point-wise norm we will have $|\theta|^2|\phi_1| = 0$. If we denote by $Z_{|\theta|^2}$ and $Z_{|\phi_1|}$ the set of points where $|\theta|^2$ and $|\phi_1|$ vanish respectively, and we denote by $Z^c_{|\theta|^2}$ and $Z^c_{|\phi_1|}$ their corresponding complements, we will have that $Z^c_{|\phi_1|} \subset Z_{|\theta|^2}$, therefore if $[A_1, \phi_1]$ is not a reducible solution then $Z^c_{|\phi_1|}$ is a non-empty open set. By a result of N. Aronszajn ([1]) we will have that $\theta = 0$, since it vanishes in an open set. □

Since $\mathcal{C}(\mathfrak{c})$ is an affine space it is contractible. Also the space of reducible configurations $\mathcal{A}(\mathfrak{c}) \times \{0\}$ is contractible and has infinite codimension in $\mathcal{C}(\mathfrak{c})$. Since $\mathcal{C}^*(\mathfrak{c})$ is open in $\mathcal{C}(\mathfrak{c})$ and it is the complement of $\mathcal{A}(\mathfrak{c}) \times \{0\}$ then it is contractible. $\mathcal{B}^*(\mathfrak{c}) = \mathcal{C}^*(\mathfrak{c})/\mathcal{G}(\mathfrak{c})$ is the classifying space of $\mathcal{G}(\mathfrak{c}) = Map(M, S^1)$ since $\mathcal{G}(\mathfrak{c})$ acts freely on $\mathcal{C}^*(\mathfrak{c})$.

Moreover,

$$Map(M, S^1) \sim Map(M, S^1)_o \times \pi_0(Map(M, S^1)),$$

where $Map(M, S^1)_o$ denotes homotopically constant maps. $Map(M, S^1)_o$ can be identified with S^1, therefore $Map(M, S^1) \sim S^1 \times H^1(M;\mathbb{Z})$, so the classifying space for $Map(M, S^1)$ is weakly homotopically equivalent to $\mathbb{CP}^\infty \times \frac{H^1(M;\mathbb{R})}{H^1(M'\mathbb{Z})}$, and

$$H^*(\mathcal{B}^*(\mathfrak{c});\mathbb{Z}) \cong \mathbb{Z}[U] \otimes \Omega^* H^1(M;\mathbb{Z}), \tag{4.2}$$

where U is a generator for $H^*(\mathbb{CP}^\infty;\mathbb{Z})$.

Definition 11. The Seiberg-Witten invariant $SW(\mathfrak{c})$ for the Spin^c-structure $\mathfrak{c}$ is defined as follows

$$SW(\mathfrak{c}) = \begin{cases} \langle U^{d(\mathfrak{c})/2}, \mathcal{M}(\mathfrak{c})\rangle|_{\mathcal{B}^*(\mathfrak{c})} & \text{if } d(\mathfrak{c}) \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that this invariant is a cobordism invariant of the moduli space $\mathcal{M}(\mathfrak{c})$, therefore it does not depend on the metric we used to define the Dirac operator, it does define an invariant of the smooth manifold M.

From this definition it is easy to see that we are loosing information about the moduli space. For example if the moduli space is odd dimensional this invariant is zero, even though the moduli itself may not represent a trivial bordism class in $\mathcal{B}^*(\mathfrak{c})$.

Definition 12. Let $(M, \mathfrak{c})$ be a smooth compact oriented 4-manifold with a Spin^c-structure $\mathfrak{c}$. We will say that $\mathfrak{c}$ is a *B-class* if for some (then for any) Riemannian metric g on M, the moduli space $\mathcal{M}_g(\mathfrak{c})$ of irreducible solutions to the SW-equations is a smooth manifold of dimension $d(\mathfrak{c}) \geq 0$ that represents a non-trivial bordism class in $\mathcal{B}^*(\mathfrak{c})$, *i.e.* there exists $\eta \in H^*(\mathcal{B}^*(\mathfrak{c});\mathbb{Z})$ of degree

$d(\mathfrak{c})$ such that

$$\langle \eta, \mathcal{M}(\mathfrak{c}) \rangle |_{\mathcal{B}^*(\mathfrak{c})} \neq 0.$$

Chapter 5

SW-Equations and Conformal Structures

It is easy to see that conformal changes on the metric can be lifted to a fixed Spin^c-structure, and one can study the associated change in the Dirac operator. A basic important fact is that *the Dirac operator remains essentially invariant under all conformal changes of the metric.*

We now make this statement precise. Let $(M, \mathfrak{c})$ be a fixed smooth compact oriented n-manifold with a fixed Spin^c-structure $\mathfrak{c}$ and a fixed Hermitian structure h on the determinant line bundle $L_{\mathfrak{c}}$. Fix a Riemannian metric g on M and consider the conformally related metric $g_f = e^{2f}g$, where f is a smooth function on M. To each g-orthonormal tangent frame $\{e_i\}_{i=1...n}$ we can associate the g_f-orthonormal frame $\{e_i'\}_{i=1...n}$, where $e_i' = \psi_f(e_i) = e^{-f}e_i$ for each i. This map induces a bundle isometry between the bundles $S(\mathfrak{c})$ and $S'(\mathfrak{c})$. Let $\Psi_f = e^{-\frac{n-1}{2}f}\psi_f$. The resulting map is a bundle isomorphism which is conformal on each fiber.

Proposition 8. *Let D_A and D_A' be the Dirac operators (induced by the $U(1)$ connection A) defined over the conformally related Riemannian manifolds*

(M, g) and (M, g_f) respectively. Then

$$\Psi_f \circ D_A = D'_A \circ \Psi_f$$

Proof. Since we are not changing the $U(1)$-connection A on $L_{\mathfrak{c}}$ the proof is the same given by Lawson and Michelsohn, (see [7], pages 132 – 134) which we reproduce it here for the sake of completeness. We have two metrics g and g_f defined on the same vector bundle TM. Let ∇ and ∇' respectively denote the associated canonical Riemannian connections. It is easy to check that for any two vectors X and Y we have

$$\nabla'_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X, Y)\nabla f.$$

Suppose now that $\{e_i\}_{i=1...n}$ and $\{e'_i\}_{i=1...n}$ are local orthonormal frame fields for g and g_f respectively, and let $\omega_{ij} = g(\nabla e_i, e_j)$ and $\omega'_{ij} = g_f(\nabla' e_i, e_j)$ be the associated 1-forms. It is easy to check that for any vector X we have

$$w'_{ij}(X) = w_{ij}(X) + e_i(f)g(X, e_j) - e_j(f)g(X, e_i).$$

The local tangent frame field $\{e_i\}_{i=1...n}$ determines a local frame field $\{\sigma_i\}_{i=1...n}$ for $S(\mathfrak{c})$. Similarly, $\{e'_i\}_{i=1...n}$ determines a frame field $\{\sigma'_i\}_{i=1...n}$ for $S'(\mathfrak{c})$ where $\sigma'_j = \psi_f(\sigma_j)$ for each j. And the induced connections on $S(\mathfrak{c})$ and $S'(\mathfrak{c})$ are related as follows:

$$\begin{aligned}
\nabla'_X \sigma'_\alpha &= \frac{1}{4}\sum_{i,j} \omega'_{ji}(X) e'_i e'_j \sigma'_\alpha \\
&= \frac{1}{4}\psi_f\{\sum_{i,j} \omega'_{ji}(X) e_i e_j \sigma_\alpha \\
&= \frac{1}{4}\psi_f\{\sum_{i,j} (\omega_{ji}(X) + e_i(f)\langle X, e_j\rangle - e_j(f)\langle X, e_i\rangle) e_i e_j \sigma_\alpha\} \\
&= \psi_f\{\nabla_X \sigma_\alpha + \frac{1}{4}(\nabla f \cdot X - X \cdot \nabla f)\sigma_\alpha\}.
\end{aligned}$$

Since $\nabla f \cdot X = -X \cdot \nabla f - 2\langle \nabla f, X\rangle$, we conclude that

$$\nabla'_X = \psi_f \circ \{\nabla_X - \frac{1}{2} X \cdot \nabla f - \frac{1}{2} X(f)\} \circ \psi_f^{-1}.$$

If D_A denotes the Dirac operator on $S(\mathfrak{c})$ and D'_A denotes the Dirac operator on $S'(\mathfrak{c})$ then we have

$$D'_A = \psi_f \circ \{D_A + \frac{1}{2}(n-1)\nabla f\} \circ \psi_f^{-1}.$$

Finally, for every constant α we have

$$\begin{aligned}
D_A(e^{\alpha f}\sigma) &= e^{\alpha f}(D_A\sigma + \alpha \sum_j e_j(f) e_j \sigma) \\
&= e^{\alpha f}(D_A \sigma + \alpha \nabla f \cdot \sigma),
\end{aligned}$$

and therefore

$$\begin{aligned}\Psi_f \circ D_A \circ \Psi_f^{-1} &= e^{-\frac{n-1}{2}f}\psi_f \circ D_A \circ (e^{\frac{n-1}{2}f}\psi_f^{-1}) \\ &= \psi_f \circ (D_A + \frac{1}{2}(n-1)\nabla f) \circ \psi_f^{-1} \\ &= D'_A.\end{aligned}$$

□

Corollary 9. *There is bijection between* $\ker D_A$ *and* $\ker D'_A$.

Let $(M, \mathfrak{c})$ be a fixed smooth compact oriented 4-manifold with a fixed Spin^c-structure $\mathfrak{c}$. We want to relate the moduli spaces $\mathcal{M}(\mathfrak{c})$ and $\mathcal{M}'(\mathfrak{c})$ for two Riemannian metrics g and g_f (respectively) in the same conformal class. It is well known (see [10]) that both moduli spaces represent the same bordism class (in $\mathcal{B}(\mathfrak{c})$), but when one of the metrics is Kähler, both moduli spaces are diffeomorphic. This last statement will be proven in the next section.

Proposition 10. *Let* $(M, \mathfrak{c})$ *be a fixed smooth compact oriented* 4*-manifold with a fixed* Spin^c*-structure* $\mathfrak{c}$. *Let* g *be a fixed Riemannian metric on* M *and consider the conformal metric* $g_f = e^{2f}g$. *Solutions to the Seiberg-Witten equation for the metric* g_f *are in one-to-one correspondence with solutions of the following pair of equations:*

$$\begin{aligned} D_A\phi &= 0 \\ F_A^+ &= e^{-f}q(\phi). \end{aligned} \qquad (SW_f)$$

The one-to-one correspondence is given by the map $(A, \phi) \mapsto (A, \Psi_f\phi)$.

Proof. This is a consequence of Proposition 8, the expression for q (see Definition 6) and that $\star'|_{\bigwedge^2} = \star|_{\bigwedge^2}$, where $\star$ and $\star'$ are the Hodge operators of g and g_f, respectively. $\square$

Chapter 6

SW-Moduli Space and Conformally Kähler Surfaces

The principal result in this section is that for any two representatives of the conformal class of a Kähler metric the corresponding moduli spaces are diffeomorphic, more specifically, if the *degree* of K_M is negative then the moduli spaces are empty *i.e.* the only solutions to (SW_f) are reducible, but if the *degree* of K_M is positive then the moduli spaces consist of only one point. This section is entirely based on [10].

Lemma 11. *Let (M, g) be a smooth compact oriented Kähler surface, fix a* Spin^c*-structure $\mathfrak{c}$ and a smooth function f on M. Let $L_\mathfrak{c}$ be the determinant line bundle for $\mathfrak{c}$ and set $L_0 = \sqrt{K_M \otimes L_\mathfrak{c}}$. Let (A, ϕ) be a solution of (SW_f). Let write $\phi = (\alpha, \beta)$ with $\alpha \in \Omega^0(M; L_0)$ and $\beta \in \Omega^{0,2}(M; L_0)$. If $\deg(L_\mathfrak{c}) \leq 0$, then we have $\beta = 0$, and if $\deg(L_\mathfrak{c}) \geq 0$ then $\alpha = 0$. Furthermore, A induces a holomorphic structure on $L_\mathfrak{c}$. With respect to the induced holomorphic structure on L_0, the section α is holomorphic and $\overline{\beta}$ is a holomorphic section of $K_M \otimes L_0^{-1}$.*

Proof. Let us begin by proving that $\overline{\alpha}\beta$ is zero. We have the harmonic spinor equation (see Example 3)

$$\sqrt{2}\left(\overline{\partial}_{A_0}(\alpha) + \overline{\partial}^*_{A_0}(\beta)\right) = 0, \tag{6.1}$$

where A_0 is the connection on $\sqrt{K_M \otimes L_\mathfrak{c}}$ which is the *square root* of the natural connection induced by a holomorphic connection on K_M and the connection A on $L_\mathfrak{c}$ (see Proposition 3). Applying $\sqrt{2}^{-1}\overline{\partial}_{A_0}$ to this equation we get

$$\overline{\partial}_{A_0}\overline{\partial}_{A_0}(\alpha) + \overline{\partial}_{A_0}\overline{\partial}^*_{A_0}(\beta) = 0. \tag{6.2}$$

Of course, $\overline{\partial}^2_{A_0}(\alpha) = F^{0,2}_{A_0} \cdot \alpha$. It is clear from equation (4.1) that

$$F^{0,2}_{A_0} = \frac{1}{2}F^{0,2}_{A} = \frac{1}{4}e^{-f}\overline{\alpha}\beta.$$

Plugging this into equation (6.2) gives

$$\frac{1}{4}e^{-f}|\alpha|^2\beta + \overline{\partial}_{A_0}\overline{\partial}^*_{A_0}(\beta) = 0.$$

Taking the L^2-inner product with β yields

$$\int_M \frac{1}{2}e^{-f}|\alpha|^2|\beta|^2 d\mu_g + \|\overline{\partial}^*_{A_0}(\beta)\|_2^2 = 0.$$

Since each of this terms is non-negative, it follows that they both vanish. Of course, $|\overline{\alpha}\beta|^2 = |\alpha|^2|\beta|^2$, so that we conclude that $\overline{\alpha}\beta = 0$. This means that $F^{0,2}_{A} = 0$, and hence that A is holomorphic connection. It follows that A_0 is also a holomorphic connection. We also see that $\overline{\partial}^*_{A_0}(\beta) = 0$. This implies that β is an anti-holomorphic section or equivalently that $\overline{\beta}$ is a holomorphic two-form with values in $\overline{L_0} = L_0^{-1}$. From equation (6.1) it now follows that α is a holomorphic section of L_0. In particular, since M is connected, if either α or β vanish on an open subset of M, then it vanishes identically on M. Thus, we see that one of α and β is identically zero since their product is identically zero. All that remains is to show that the sign of $\deg(L_\mathfrak{c})$ determines which of α and β is zero. We have

$$(F_A^+)^{1,1} = \frac{i}{4} e^{-f}(|\alpha|^2 - |\beta|^2)\omega.$$

Thus, we see that

$$\deg(L_\mathfrak{c}) = \int_M c_1(L_\mathfrak{c}) \wedge \omega = \frac{1}{8\pi} \int_M e^{-f}(|\beta|^2 - |\alpha|^2) d\mu_g.$$

Since at least one of α and β is zero, we see that if $\deg(L_\mathfrak{c})$ is non-negative, then $\alpha = 0$ and if $\deg(L_\mathfrak{c})$ is non-positive, then $\beta = 0$. □

Corollary 12. *If $\deg(L_\mathfrak{c})$ is non-positive, then any solution of (SW_f) consists of a holomorphic, Hermitian connection A on $L_\mathfrak{c}$ and a holomorphic section α of L_0 with*

$$(F_A^+)^{1,1} = \frac{i}{4} e^{-f}|\alpha|^2\omega.$$

Two such pairs (A, α) and (A', α') determine the same point in the moduli space if and only if there is a holomorphic, Hermitian isomorphism between *the holomorphic structures on $L_{\mathfrak{c}}$ induced by A and A', such that the induced holomorphic isomorphism on L_0 carries α to α'.*

Proof. In the previous Lemma we have seen that the conditions on (A, α) are equivalent to the fact that this pair yields a solution to equation (SW_f) and that any solution to equation (SW_f) arises in this way provided that the degree of $L_{\mathfrak{c}}$ is non-positive. The uniqueness statement is clear. □

There is a similar result in the case that the degree of $L_{\mathfrak{c}}$ is non-negative.

Corollary 13. *If* $\deg(L_{\mathfrak{c}})$ *is non-negative, then a solution of (SW_f) consists of a holomorphic, Hermitian connection A on $L_{\mathfrak{c}}$ and a holomorphic section $\overline{\beta}$ of $K_M \otimes L_0^{-1}$ with*

$$(F_A^+)^{1,1} = -\frac{i}{4} e^{-f} |\beta|^2 \omega.$$

Two such pairs (A, β) and (A', β') determine the same point in the moduli space if and only if there is a holomorphic, Hermitian isomorphism between the holomorphic structures on $L_{\mathfrak{c}}$ induced by A and A', such that the induced holomorphic isomorphism on $K_M \otimes L_0^{-1}$ carries β to β'.

Lemma 14. *Suppose that* $\deg(L_{\mathfrak{c}})$ *is negative, A is a Hermitian, holomorphic connection on $L_{\mathfrak{c}}$, and finally that α is a non-zero holomorphic section of L_0 (with respect to the holomorphic structure defined by A_0). Then there exists another Hermitian structure h' on $L_{\mathfrak{c}}$ such that for the connection A' which is Hermitian with respect to h' and which defines the same holomorphic structure* *on $L_{\mathfrak{c}}$ as A does, we have*

$$F_{A'}^{1,1} = \frac{i}{4} e^{-f} |\alpha|_{h'}^2 \omega \tag{6.3}$$

where $|\alpha|_{h'}$ means the norm measured with respect to the Hermitian structure on L_0 determined by h'.

Proof. Let us denote by h the given Hermitian inner product on $L_{\mathfrak{c}}$. A new Hermitian structure h' on $L_{\mathfrak{c}}$ is given by $e^t h$ for some smooth real-valued function t. Of course $|\alpha|_{h'}^2 = e^t |\alpha|_h^2$. The curvature of the holomorphic connection A' which is Hermitian with respect to h' is given by

$$F_{A'} = F_A + \overline{\partial} \partial t.$$

Thus, the equation that we need to solve for t is

$$F_A^+ + (\overline{\partial}\partial t)^+ = \frac{i}{4} e^t e^{-f} |\alpha|^2 \omega,$$

or equivalently

$$F_A \wedge \omega + \overline{\partial}\partial t \wedge \omega = \frac{i}{4} e^t e^{-f} |\alpha|^2 \omega \wedge \omega. \tag{6.4}$$

Of course, since the metric is Kähler we have

$$\overline{\partial}\partial t \wedge \omega = \frac{2}{i} \left(-\frac{\partial^2 t}{\partial z_1 \overline{\partial} z_1} - \frac{\partial^2 t}{\partial z_2 \overline{\partial} z_2} \right) d\mu_g = \frac{2}{i} \Delta t d\mu_g.$$

Also, since $\deg(L_{\mathfrak{c}})$ is negative, we have

$$\int_M iF_a \wedge \omega < 0.$$

Thus, we can rewrite equation (6.4) as

$$\Delta t + e^{-f}|\alpha|^2 e^t + C = 0 \tag{6.5}$$

where C is the smooth function with $Cd\mu_g = \frac{i}{2}F_A \wedge \omega$. Because of the degree condition on $L_{\mathfrak{c}}$ we have

$$\int_M C d\mu_g < 0.$$

According to [6], equation (6.5) has a unique solution. □

Corollary 15. *Let (M, g) be a smooth compact oriented Kähler surface, fix a* Spin^c*-structure $\mathfrak{c}$ and fix a smooth function f on M. Suppose that the degree of the determinant line bundle $L_{\mathfrak{c}}$ is negative. Fix a pair $(\overline{\partial}, \alpha_0)$, where $\overline{\partial}$ is a holomorphic structure on $L_{\mathfrak{c}}$ and α_0 is a non-zero holomorphic section of $L_0 = \sqrt{K_M \otimes L_{\mathfrak{c}}}$. Then there is a solution (A, α) to (SW_f) as in Corollary 12 with the following properties:*

1. *A determines a holomorphic structure on $L_{\mathfrak{c}}$ which is isomorphic to the holomorphic structure $\overline{\partial}$, and*

2. *there is a holomorphic isomorphism from the structure determined by A and $\overline{\partial}$ which sends α to α_0.*

Such a solution (A, α) is unique up to gauge equivalence. In this way any pair $(\overline{\partial}, \alpha_0)$ as above determines a point in the moduli space $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$. All points of $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ arise in this way. Two pairs $(\overline{\partial}, \alpha)$ and $(\overline{\partial}', \alpha')$ determines the same point in $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ if and only if the holomorphic structures on $L_\mathfrak{c}$ are isomorphic and the induced holomorphic isomorphism of L_0 carries the holomorphic sections to constant scalar multiples of each other.

Proof. Let us denote by h the given Hermitian metric on $L_\mathfrak{c}$. We begin with a pair $(\overline{\partial}, \alpha_0)$ as in the statement of the Corollary. By Lemma 14 there is a Hermitian metric h' on $L_\mathfrak{c}$ such, letting A' be the h'-Hermitian connection inducing the holomorphic structure $\overline{\partial}$, we have

$$F_{A'}^+ = \frac{i}{4} e^{-f} |\alpha_0|_{h'}^2 \omega.$$

Let $\rho : L_\mathfrak{c} \to L_\mathfrak{c}$ be a $\mathcal{C}^\infty$ complex linear isomorphism with $\rho^*(h') = h$. Let $A = \rho^*(A')$. Then A is a h-Hermitian connection inducing a holomorphic structure on $L_\mathfrak{c}$ which is isomorphic to the holomorphic structure $\overline{\partial}$. Clearly, $\alpha = \rho^{-1}(\alpha_0)$ is a holomorphic section of L_0 and

$$F_A^+ = \frac{i}{4} e^{-f} |\alpha|^2 \omega.$$

This proves the first statement of the Corollary.

Now let us show that the resulting solution (A, α) is unique up to gauge equivalence. The point is that the function t which scales the metric is itself unique (see Lemma 14). This means that the isomorphism ρ is unique up to an S^1 change of gauge. From this the uniqueness of (A, α) up to change of gauge is clear.

Notice that if we replace α by a constant complex multiple $\lambda_0\alpha$ with $\lambda_0 \neq 0$ then the resulting solution to equation (SW_f) is gauge equivalent to the one produced by α. By Lemma 11 any solution to equation (SW_f) arises in this way.

Lastly, we need to see when two pairs $(\overline{\partial}, \alpha_0)$ and $(\overline{\partial}', \alpha_0')$ as in the statement of the Corollary determine gauge equivalent solutions of equation(SW_f). If they determine gauge equivalent solutions, then the holomorphic structures $\overline{\partial}$ and $\overline{\partial}'$ are isomorphic. Thus, we may assume that $\overline{\partial} = \overline{\partial}'$. Clearly, if the pairs determine gauge equivalent solutions to equation (SW_f) then there must be a holomorphic isomorphism of $\overline{\partial}$ which sends α_0 to α_0'. But the only holomorphic isomorphisms of a holomorphic line bundle are multiplication by constant complex scalars. This completes the proof. □

There is a completely analogous result when the degree of $L_{\mathfrak{c}}$ is positive.

Corollary 16. *Let (M, g) be a smooth compact oriented Kähler surface, fix a* Spin^c*-structure $\mathfrak{c}$ and fix a smooth function f on M. Suppose that the degree of the determinant line bundle $L_{\mathfrak{c}}$ is positive. Fix a pair $(\overline{\partial}, \beta_0)$, where $\overline{\partial}$ is a holomorphic structure on $L_{\mathfrak{c}}$ and β_0 is a non-zero holomorphic section of $K_M \otimes L_0^{-1} = \sqrt{K_M \otimes L_{\mathfrak{c}}^{-1}}$. Then there is a solution (A, β) to (SW_f) as in Corollary 13 with the following properties:*

1. *A determines a holomorphic structure on $L_{\mathfrak{c}}$ which is isomorphic to the holomorphic structure $\overline{\partial}$, and*
2. *there is a holomorphic isomorphism from the structure determined by A to $\overline{\partial}$ which sends β to β_0.*

Such a solution (A, β) is unique up to gauge equivalence. In this way any pair $(\overline{\partial}, \beta_0)$ as above determines a point in the moduli space $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$. All points of $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ arise in this way. Two pairs $(\overline{\partial}, \alpha)$ and $(\overline{\partial}', \alpha')$ determines the same point in $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ if and only if the holomorphic structures on $L_\mathfrak{c}$ are isomorphic and the induced holomorphic isomorphism of L_0 carries the holomorphic sections to constant scalar multiples of each other.

Finally we have to consider the case when the degree of $L_\mathfrak{c}$ is zero. Since the degree of $L_\mathfrak{c}$ is both non-positive and non-negative we conclude (using Corollary 12 and Corollary 13) that the spinor field (α, β) vanishes identically. This yields the following result.

Corollary 17. *Let (M, g) be a smooth compact oriented Kähler surface, fix a* Spin^c*-structure $\mathfrak{c}$ and fix a smooth function f on M. Suppose that the degree of the determinant line bundle $L_\mathfrak{c}$ is zero, then any solution to (SW_f) consists of an anti-self dual connection A on $L_\mathfrak{c}$ and a trivial spinor field. This identifies the moduli space $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ with the space of gauge equivalence classes of anti-self dual connections on $L_\mathfrak{c}$.*

Proposition 18. *Let (M, g) be a Kähler surface with Kähler metric g. Then for any smooth function $f : M \to \mathbb{R}$*

- *If the degree of K_M is negative the only solutions to (SW_f) are reducible, i.e. $\mathcal{M}_{e^{2f}g}(\mathfrak{c}) = \emptyset$.*
- *Let $\mathfrak{c}$ be the* Spin^c*-structure determined by the complex structure. If the degree of K_M is positive then $\#\mathcal{M}_{e^{2f}g}(\mathfrak{c}) = 1$.*

Proof. Let us first consider the case when K_M has negative degree with respect to the Kähler form. Fix a Spin^c-structure $\mathfrak{c}$. Let us first consider the case when the determinant line bundle $L_\mathfrak{c}$ is non-positive. According to Lemma 11, in this case $\beta = 0$ and α is holomorphic section of L_0. If $\alpha \neq 0$, then we see that the degree of L_0 is ≥ 0. But this is a contradiction since $L_0^2 = K_M \otimes L_\mathfrak{c}$ clearly has negative degree. It follows that in this case $\alpha = 0$ also and the solution is reducible.

If the degree of $L_\mathfrak{c}$ is non-negative, then according to Lemma 11 the section $\alpha = 0$ and $\overline{\beta}$ is holomorphic section of $K_M \otimes L_0^{-1}$. If $\beta \neq 0$, this implies that this bundle has non-negative degree. But this bundle is isomorphic to a square root of $K_M \otimes L_\mathfrak{c}^{-1}$ which implies that it has negative degree. This contradiction shows that $\beta = 0$ and hence in this case as well there are only reducible solutions to equations (SW_f). This completes the proof of the first item in the statement of the Proposition.

Now let us suppose that the degree of K_M is positive and that the Spin^c-structure $\mathfrak{c}$ that we are considering is the one induced by the complex structure. Of course the determinant line bundle $L_\mathfrak{c}$ is equal to K_M^{-1} and hence has negative degree. This is the Spin^c-structure for which L_0 is trivial as a C^∞ complex line bundle. Suppose that we have a solution (A, α) to (SW_f). The holomorphic structure on L_0 induced by the A_0 has a non-trivial holomorphic section α. Since the bundle is topological trivial, this holomorphic section must be nowhere zero and α is a constant section. This proves that the moduli space $\mathcal{M}_{e^{2f}g}(\mathfrak{c})$ is a single point.

Next we show that this point is a smooth point by computing the differentials in the elliptic complex associated to the solution (A, α).

The complex is

$$0 \longrightarrow \Omega^0(M; i\mathbb{R}) \xrightarrow{D_1} \begin{matrix} \Omega^1(M; i\mathbb{R}) \\ \oplus \\ \Omega^0(M;\mathbb{C}) \oplus \Omega^{0,2}(M;\mathbb{C}) \end{matrix} \xrightarrow{D_2} \begin{matrix} \Omega^2_+(M; i\mathbb{R}) \\ \oplus \\ \Omega^{0,1}(M;\mathbb{C}) \end{matrix} \longrightarrow 0$$

where

$$D_1(if) = \begin{pmatrix} 2idf \\ -if \cdot \alpha \end{pmatrix}$$

$$D_2 \begin{pmatrix} i\lambda \\ (a,b) \end{pmatrix} = \begin{pmatrix} d^+(i\lambda) - e^{-f}((i\Re(a\overline{\alpha})/2)\omega + (\alpha\overline{b} - \overline{\alpha}b)/2) \\ \sqrt{2}\overline{\partial}(a) + \sqrt{2}\overline{\partial}^*(b) + \pi^{0,1}(i\lambda) \cdot \alpha/\sqrt{2} \end{pmatrix}$$

Since α is a constant, non-zero section , it is clear that the kernel of D_1 is trivial. Let us consider the kernel of D_2. Suppose that

$$D_2 \begin{pmatrix} i\lambda \\ (a,b) \end{pmatrix} = 0.$$

Applying $\overline{\partial}$ to the second coordinate of $D_2\left(\begin{smallmatrix} i\lambda \\ (a,b) \end{smallmatrix}\right)$ and using the fact that $\overline{\partial}^2 = 0$ and that $\overline{\partial}\alpha = 0$, we conclude that

$$\frac{1}{2}\overline{\partial}(\pi^{0,1}(i\lambda)) \cdot \alpha + \overline{\partial}\overline{\partial}^*(b) = 0.$$

We also have

$$\overline{\partial}(\pi^{0,1}(i\lambda)) = d(i\lambda)^{0,2} = e^{-f}\frac{\overline{\alpha}b}{2}.$$

(The second equality uses the fact that the first coordinate of $D_2\left(\begin{smallmatrix} i\lambda \\ (a,b) \end{smallmatrix}\right)$ is zero.) Plugging this in gives

$$\frac{1}{4}e^{-f}|\alpha|^2 b + \overline{\partial}\,\overline{\partial}^* b = 0.$$

Taking the L^2 inner product with b we find that

$$\frac{1}{4}\|e^{-f/2}\alpha b|_2^2 + \|\overline{\partial}^* b\|_2^2 = 0,$$

and hence $\overline{\alpha}b = 0$, implying that $b = 0$.

We write $i\lambda = \overline{\xi} - \xi$ for some $\overline{\xi} \in \Omega^{0,1}(M;\mathbb{C})$. The equations telling us that the element is in the kernel of D_2 now become

$$\sqrt{2}\overline{\partial}a + \frac{1}{\sqrt{2}}\overline{\xi}\cdot\alpha = 0 \tag{6.6}$$

$$(\partial\overline{\xi} - \overline{\partial}\xi)_+ = \frac{i}{2}e^{-f}\Re(a\overline{\alpha})\omega. \tag{6.7}$$

We write $a = (u + iv)\alpha$ with u and v being real-valued functions. By adding $D_1(iv)$ to $(i\lambda, a, b)$ we arrange that in fact $a = u\alpha$ with u a real-valued function. Equation (6.6) now reads

$$\sqrt{2}\overline{\partial}u\cdot\alpha + \frac{1}{\sqrt{2}}\overline{\xi}\cdot\alpha = 0$$

from which we conclude

$$\overline{\xi} = -2\overline{\partial}u.$$

Using this, equation (6.7) is equivalent to

$$8\Delta u + e^{-f}u = 0.$$

Since Δ has a non-negative spectrum, this implies that $u = 0$. We conclude that $i\lambda = 0$ and that $\alpha = 0$. This proves that any element in the kernel of D_2 is in the image of D_1 and hence that the first cohomology of the elliptic complex is trivial.

Lastly we need to compute the second cohomology of the elliptic complex. But we know that the index of the complex is given by

$$(K_M^2 - (2\xi(M) + 3\sigma(M)))/4.$$

Since M is a Kähler manifold, it follows that this index is zero. Since $H^0 = H^1 = 0$, it follows that the second cohomology is zero as well. This completes the proof that the unique solution to (SW_f) is a smooth point of the moduli space and hence that the Seiberg-Witten invariant of the Spin^c-structure is $\mathfrak{c}$ is ± 1 when computed with respect to the metric $e^{2f}g$. □

Remark. Note that $\#\mathcal{M}_{e^{2f}g}(\mathfrak{c}) = 1$ is stronger than $SW_{e^{2f}g}(\mathfrak{c}) = 1$, which we already knew (see [10]).

Chapter 7

SW-Moduli Space of a Manifold with a Cylindrical End

In the previous Section we proved that if (M, g) is a Kähler surface with $\deg(K_M) < 0$ the Seiberg-Witten moduli space for any metric $g_f = e^{2f}g$ in the same conformal class of g consists of a single point. In this Section we extend this result to a manifold with finitely many cylindrical ends.

Definition 13. We will say that (M_∞, g_∞) is a *manifold with a cylindrical end modeled on* $\mathbb{R}^+ \times S^3$, if M_∞ is diffeomorphic to $M - \{p\}$ where M is a closed manifold, and $F : U_p - \{p\} \to \mathbb{R}^+ \times S^3$ where $F(x) = (\log(|x|^{-1}), x/|x|)$ is a diffeomorphism such that $(g_\infty)|_{U_p - \{p\}}$ is the F-pull-back of the standard product metric $dt^2 + g_{S^3}$ on $\mathbb{R}^+ \times S^3$ and U_p is a neighborhood of p.

If (M, g) is a Riemannian manifold such that g is flat in a δ-neighborhood of p, where $\delta < \mathrm{inj}(M, g)$, there is a canonical way to produce a manifold with a cylindrical end using the conformal class of g. Here $\mathrm{inj}(M, g)$ denotes the injectivity radius of (M, g). Choose a function $\lambda_t : (0, 1] \to [1, \infty)$ which satisfies

$$\lambda_l(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq e^{-l}\delta^3 \\ \delta^2/r & \text{if } e^{-l}\delta^2 \leq r \leq \delta^2 \\ 1 & \text{if } r \geq \delta. \end{cases}$$

Consider the sequence of functions $\{f_l\}$, where $e^{f_l(x)} = \lambda_l(|x|)$ and the sequence of metrics $g_l = e^{2f_l}g$. This sequence of metrics converges in the compact-open topology on $M - \{p\}$ to a metric g_∞. The pair $(M - \{p\}, g_\infty)$ is a manifold with a cylindrical end. We will denote by Ψ_l the associated conformal isomorphism defined above Proposition 8.

The SW-equations make perfectly good sense on a manifold with a cylindrical end, but in order to use the usual analytical tools, one has to extend the C^∞ objects to appropriate weighted Sobolev spaces (see [9]). From now on every time we work on a manifold with finitely many cylindrical ends we extend the configuration space $\mathcal{A}(\mathfrak{c})$ and the gauge group $\mathcal{G}(\mathfrak{c})$ by requiring A and ϕ to be in $L^2_{2,\epsilon}(M_\infty, g_\infty)$ and σ to be in $L^2_{3,\epsilon}(M_\infty, g_\infty)$. The $L^p_{q,\epsilon}(M_\infty, g_\infty)$ norm is defined as

$$||h||_{p,q,\epsilon} = ||e^{\tilde{\epsilon}t}h||_{p,q},$$

where $\tilde{\epsilon}$ is a smooth non-decreasing function with bounded derivatives, $\tilde{\epsilon} : M \to [0, \epsilon]$, such that $\tilde{\epsilon}(x) \equiv 0$ for $x \notin B_\delta(p)$ and $\tilde{\epsilon}(x) \equiv \epsilon > 0$ for $x \in B_{\delta^2}(p)$.

Here we choose the weight $\epsilon < 1$ because we want to produce solutions on the manifold with cylindrical end from solutions on the manifold (M, g) via the conformal process $(g_l \to g_\infty)$ using Proposition 10.

Proposition 19. *Let (M, g) be any Riemannian 4-manifold, where g is flat in the neighborhood of some point $p \in M$. If (A, ϕ) is a solution of the SW-equations on (M, g) then $(A, \Psi_\infty \phi)$ is a solution of the SW-equations on (M_∞, g_∞), such that $(A, \Psi_\infty \phi) \in L^2_{1,\epsilon}(M_\infty, g_\infty)$.*

Proof. The fact that $(A, \Psi_\infty \phi)$ satisfies the SW-equations follows from Proposition 10. We just need to show that $(A, \Psi_\infty \phi) \in L^2_{1,\epsilon}(M_\infty, g_\infty)$. In order to do this, we will use the metric g as the background metric.

$$
\begin{aligned}
||\Psi_\infty \phi||^2_{2,1,\epsilon} = ||e^{\tilde{\epsilon} t} \Psi_\infty \phi||^2_{2,1} \\
&\int_{M - B_\delta(p)} (|\phi|^2 + |\nabla \phi|^2) d\mu + \\
&\int_{\mathbb{R}^+ \times S^3} (|e^{\tilde{\epsilon} t} \Psi_\infty \phi|^2_\infty + |e^{\tilde{\epsilon} t} \partial_t \nabla \Psi_\infty \phi|^2_\infty) dt d\mu_{S^3} \\
&= \int_{M - B_\delta(p)} (|\phi|^2 + |\nabla \phi|^2) d\mu + \\
&\int_{B_\delta(p) - \{p\}} (|r^{-\epsilon + 3/2} \phi|^2 + |r^{-\epsilon + 1 + 3/2} \partial_r \nabla \phi|^2) \frac{1}{r} dr d\mu_{S^3} \\
&= \int_{M - B_\delta(p)} (|\phi|^2 + |\nabla \phi|^2) d\mu + \\
&\int_{B_\delta(p) - \{p\}} r^{-2\epsilon - 1} (|\phi|^2 + |r \partial_r \nabla \phi|^2) r^3 dr d\mu_{S^3} \\
&\leq C ||\phi||^2_{2,1}.
\end{aligned}
$$

To prove that $A \in L^2_{1,\epsilon}(M_\infty, g_\infty)$ we need to recall that

$$\star_{g_f}|_{\wedge^p} = e^{(n-2p)f} \star_g |_{\wedge^p}$$

where $g_f = e^{2f}g$. The computation is very similar to the one above. □

Corollary 20. *For any compact oriented Kähler surface (M, g) with canonical line bundle K_M of positive degree, where g is flat in a neighborhood of some point, the* induced *manifold with a cylindrical end (M_∞, g_∞) admits solutions to the SW-equations.*

Remark. We will see later (Proposition 24) that there is no loss of generality in assuming that a Kähler metric g is flat in a neighborhood of some point.

In order to prove that the Seiberg-Witten moduli space of a manifold with a cylindrical end consists of only one point if $\deg(K_M) < 0$, we will need the following

Proposition 21. *Let (M_∞, g_∞) be a 4-manifold with a cylindrical end. If $(A_\infty, \phi_\infty) \in \mathcal{C}^\infty \bigcap L^2_{k,\epsilon}(M_\infty, g_\infty)$ is a solution of the SW-equations on the manifold with cylindrical end (M_∞, g_∞), then $(A_\infty, \Psi_\infty^{-1}\phi_\infty)$ extends to a smooth solution of (SW_f) on (M, g), replacing the strictly positive function e^{-f} by the non-negative function*

$$\lambda_\infty(x) = \begin{cases} |x|/\delta^2 & \text{if } |x| < \delta^2 \\ 1 & \text{if } |x| > \delta \end{cases}$$

Proof. It is easy to see that $(A_\infty, \Psi_\infty^{-1}\phi_\infty) \in L^2(M, g)$, as it is to see that $(A_\infty, \Psi_\infty\phi_\infty)$ is a solution of (SW_f) with function λ_∞ replacing e^{-f}. The first equation in (SW_f) tell us that $\Psi_\infty^{-1}\phi_\infty$ is a holomorphic section on $M - \{p\}$. Using Hartog's Theorem we can extend this to a holomorphic section on M. All the analysis done in Section 6 can be carry out if we replace the strictly positive function e^{-f} in (SW_f) by a non-negative function λ_∞ whose zero set has measure zero. □

Corollary 22. *Let (M, g) be a compact oriented Kähler surface with canonical line bundle K_M of positive degree, where g is flat in a neighborhood of some point. Then there exists a solution $(A_\infty, \phi_\infty) \in C^\infty \bigcap L^2_{k,\epsilon}(M_\infty, g_\infty)$ of the SW-equations on (M_∞, g_∞). This solution is unique up to gauge equivalence.*

Proof. The existence is a consequence of Corollary 20 and uniqueness is obtained using Proposition 21 and Proposition 18 □

Chapter 8

SW-Invariant, Holonomy and Connected Sums with $S^1 \times S^3$

Remark. Consider the diffeomorphism

$$F : \mathbb{R}^4 - \{0\} \to \mathbb{R} \times S^3, \quad F(x) = \left(\log |x|, \frac{x}{|x|} \right).$$

It is easy to see that the pull-back of the standard product metric g on $\mathbb{R} \times S^3$ under this diffeomorphism is given by

$$F^* g(\xi, \eta) = \frac{1}{|x|^2} \langle \xi, \eta \rangle$$

for $|x| \leq 1$. Fix $\delta > 0$ and choose a function $\lambda_l : (0, 1] \to [1, \infty)$ as in (7) and consider the metric

$$g_l(\xi, \eta) = \lambda_l(|x|)^2 \langle \xi, \eta \rangle.$$

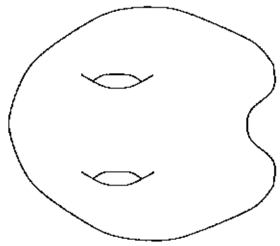

Figure 8.1: Manifold M with a flat metric around two points

Note that for $e^{-l}\delta^2 \leq |x| \leq \delta^2$ this metric agrees with the above pull-back metric F^*g.

It is convenient to think of the connected sum $M\#(S^1 \times S^3)$ as follows. Let M be a smooth compact oriented 4-manifold. Fix two points $p_1, p_2 \in M$, and choose a metric g on M which is flat in a δ-neighborhood of p_i (see Fig. 8.1). For every $l \in \mathbb{N}$ consider the $e^{-l-1}\delta^2$-neighborhood of p_i (with respect to g) $B_{p_i}(e^{-l-1}\delta^2)$, and denote by M_l the open subset of M given by the complement of $\overline{B_{p_1}(e^{-l-1}\delta^2)} \cup \overline{B_{p_2}(e^{-l-1}\delta^2)}$. If we denote by $T_i = T_i(e^{-l}\delta^2, e^{-l-1}\delta^2)$ the annulus centered at p_i with radii $e^{-l-1}\delta^2$ and $e^{-l}\delta^2$, it is easy to see that there exist a diffeomorphism (orientation reversing) that takes T_1 into T_2 and if we define $g_l = \lambda_l^2 g$, such diffeomorphism becomes a g_l-isometry. Since we have observed that T_1 and T_2 are g_l-isometric we can identify T_1 with T_2 , and call them T_l, to obtain a Riemannian manifold $(M\#_l(S^1 \times S^3), g_l)$ (see Fig. 8.2). This manifold is simply the manifold M with two cylindrical ends of length l obtained by conformally rescaling the metric g and identifying the annuli. It is easy to see that such manifold is diffeomorphic to the connected sum $M\#(S^1 \times S^3)$.

Even though the process above described can be realized on any smooth

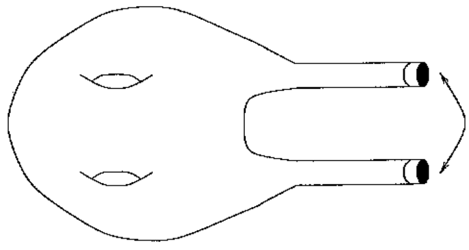

Figure 8.2: Manifold $M\#(S^1 \times S^3)$ with a neck of length l

4-manifold the following results are only valid when M is a Kähler surface, because to prove them, we (strongly) use that on a given conformal class of metrics, the moduli spaces of solutions of the SW-equations for any two representatives are diffeomorphic, and this was proved for Kähler surfaces on Section 6.

Our next task is to explain how a Spin^c-structure on M transforms into a Spin^c-structure on $M\#(S^1 \times S^3)$ under the process above described. The following Proposition will be very useful to explain it.

Proposition 23. *There is a canonical projection map $\pi : M\#(S^1 \times S^3) \to M$. It has the following properties:*

1. *The induced maps in cohomology*

$$\pi^* : H^i(M; \mathbb{F}) \to H^i(M\#(S^1 \times S^3); \mathbb{F})$$

   are injective. Here $\mathbb{F} = \mathbb{Z}_2$ or $\mathbb{Z}$. In particular for $i = 0, 2, 4$, π^* is an isomorphism.

2. $\pi^*(w_2(M)) = w_2(M\#(S^1 \times S^3))$.

Since we start with a Spin^c-structure $\mathfrak{c}$ on M we know that $c_1(L_\mathfrak{c}) \equiv w_2(M)$ mod 2. Consider the complex line bundle $\tilde{L}_\mathfrak{c}$ on $M\#(S^1 \times S^3)$ with $c_1(\tilde{L}_\mathfrak{c}) = \pi^*(c_1(L_\mathfrak{c}))$. By 2 in the Proposition above we know that $c_1(\tilde{L}_\mathfrak{c}) \equiv w_2(M\#(S^1 \times S^3))$ mod 2, therefore there exists a Spin^c-structure on $M\#(S^1 \times S^3)$ whose determinant line bundle is $\tilde{L}_\mathfrak{c}$. The uniqueness of this Spin^c-structure is due to the fact that when we restrict this Spin^c-structure to $M_l \hookrightarrow M\#(S^1 \times S^3)$ it has to be $\mathfrak{c}$, and when we restrict it to T_l it has to be the trivial one. We will denote this Spin^c-structure by $\mathfrak{c}_{0,1}$. It is not difficult to show that the formal dimension of the moduli space associated to $\mathfrak{c}_{0,1}$ is $d(\mathfrak{c}_{0,1}) = d(\mathfrak{c}) + 1$.

To explain the extra unit in the dimension above we need to recall the concept of *holonomy*. Let $P_G \to M$ be a principal G-bundle over M, with a connection A. Let $x \in M$ and denote by $C(x)$ the loop space at x. For each $\gamma \in C(x)$ the parallel displacement along γ is an isomorphism of the fiber $\approx G$ onto itself and we will denote it by $\mathrm{hol}_\gamma(A)$. The set of all such isomorphisms forms a group, the *holonomy group of A with reference point x.*

Once and for all for each $l > 0$ we will choose $p_l \in T_1$, $q_l \in T_2$ and a path $\Gamma_l : I \to M$ from p_l to q_l such that after identifying T_1 with T_2 we obtain and embedding $\gamma_l : S^1 \to M\#_l(S^1 \times S^3)$. It is not difficult to observe that for all $l > 0$ $[\gamma_l] \nsim 0 \in \pi_1(M\#(S^1 \times S^3))$, and in fact γ_l represents the S^1 factor of the connected sum.

If A is a $U(1)$-connection on the determinant line bundle $L_\mathfrak{c}$, we can trivialize $L_\mathfrak{c}$ along Γ_l so that the parallel transport along Γ_l induces the identity from the fiber at p_l to the fiber at q_l. When we identify T_1 with T_2 we still have the extra degree of freedom of how to identify the fiber at p_l with the fiber at q_l, and this is measured by $\mathrm{hol}_\gamma(A)$, where A is the *glued* connection. If we change of gauge, $\mathrm{hol}_\gamma(A)$ remains unchanged because the structure group $U(1)$ is Abelian. In this section we will prove that when M is a Kähler surface then every solution to the Seiberg-Witten equations for a Spin^c-structure $\mathfrak{c}$, induces an S^1 family of solutions to the SW-equations for the Spin^c-structure $\mathfrak{c}_{0,1}$ on $M\#(S^1\times S^3)$, but before we have to prove the following *non-obstruction* result.

Proposition 24. *Let (M^{2n}, g) be a Kähler 2n-manifold with Kähler metric g and induced Kähler form ω . There is no local obstruction to finding a Kähler metric on M, flat in a neighborhood of a point (a finite collection of points) without changing the Kähler class of ω.*

Proof. Let $p \in M$. The existence of such metric is equivalent to finding a neighborhood U of p, and a Kähler form ω' in the same Kähler class of ω, such that $\omega'|_U = \omega_0 = \sum_{i=1}^n dz^i \wedge d\overline{z^i}$. It is well known that there exist an ϵ-neighborhood U_p of p and a function $f : U_p \to \mathbb{R}$ such that $\omega|_{U_p} = i\partial\overline{\partial}(z\overline{z} + f(z)) > 0$, where $|f(z)| \sim o(|z|^4)$ and $|z|$ denotes the distance (using the Kähler metric g) on U_p to p. Let $\mathcal{K}^\infty(f)$ be the space of smooth functions on M that satisfy

$$\mathcal{K}^\infty(f) = \{h_{s,t} \in \mathcal{C}^\infty(M)|\, h(z) = -f(z) \text{ if } |z| < s,\, h(z) = 0 \text{ if } t < |z|\}$$

where $0 < s < t \leq \epsilon$, depend on h. Observe that if f is zero we do not have anything to prove, otherwise $0 \notin \mathcal{K}^\infty(f)$, but $0 \in \mathcal{K}^{3+\alpha}(f)$, where $\mathcal{K}^{3+\alpha}(f)$ denotes the completion of $\mathcal{K}^\infty(f)$ in the $\mathcal{C}^{3+\alpha}$ topology. To see this consider

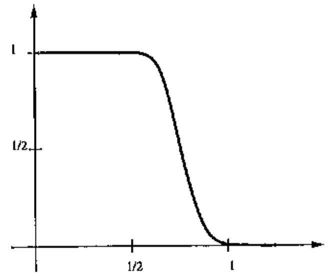

Figure 8.3: $\rho : \mathbb{R}^+ \to \mathbb{R}^+$

the one-parameter family of functions $h_k(z) = -\rho(k|z|)f(z)$, where ρ (see Fig. 8.3) is a smooth non-negative non-increasing function such that

$$\rho(r) = \begin{cases} 1 & \text{if } 0 < r < 1/2 \\ 0 & \text{if } 1/2 < r < 1. \end{cases}$$

All these functions are in $\mathcal{K}^\infty(f)$ and satisfy

$$|h_k(z)| \sim o(|z|^4)$$
$$|\nabla h_k(z)| \sim o(|z|^3)$$
$$|\nabla^2 h_k(z)| \sim o(|z|^2)$$
$$|\nabla^3 h_k(z)| \sim o(|z|)$$
$$|\nabla^4 h_k(z)| \sim o(1).$$

It is not difficult to see that $h_k \to 0$ in the $\mathcal{C}^{3+\alpha}$ topology. It is important to recall that the set $\mathcal{P}(\omega)$ of smooth functions h such that $\omega_h = \omega + i\partial\overline{\partial}h > 0$, is open in the $\mathcal{C}^\infty$ topology. This two facts allow us to find $h_{s,t} \in \mathcal{K}^\infty(f) \bigcap \mathcal{P}(\omega)$,

$\mathcal{C}^{3+\alpha}$ close to 0, such that

$$\omega_{h_{s,t}} = \omega + i\partial\overline{\partial}h_{s,t} > 0$$
$$= \omega_0 + i\partial\overline{\partial}(f + h_{s,t}),$$

therefore we have

$$\omega_{h_{s,t}}\big|_{B_s(p)} = \omega_0,$$

where $B_s(p) = \{z \in U_p \,|\, |z| < s\}$. □

From now on we will assume that our 4-manifold is given a Kähler metric that is flat nearby two points, p_1 and p_2.

We can *glue* a solution (A, ϕ) of (SW_f) to produce a solution (A_l, ϕ_l) of the following set of equations on $(M\#_l(S^1 \times S^3), g_l)$

$$D_{A_l}\phi_l = \mu(A_l, \phi_l) = \mu_l$$
$$F^+_{A_l} - q(\phi_l) = \nu(A_l, \phi_l) = \nu_l,$$

where $(\mu_l, \nu_l) \in \mathcal{S}(\mathfrak{c}) \times \Omega^2_+(M\#_l(S^1 \times S^3); i\mathbb{R})$. It is not difficult to see that

$$(\mu_l, \nu_l) \in L^2_1(M, g)$$
$$\lim_{l\to\infty} ||(\mu_l, \nu_l)||_{2,1} = 0,$$

Definition 14. We will denote by $\mathcal{M}_\theta(\mathfrak{c}_{0,1}) \subset \mathcal{M}(\mathfrak{c}_{0,1})$ the solution subspace of the SW-equations satisfying the extra condition

$$\mathrm{hol}_\gamma(A) = \theta,$$

and by $SW_\theta(\mathfrak{c}_{0,1})$ the cobordism invariant associated to this moduli space (counting solutions with appropriate sign). Note that the condition $\mathrm{hol}_\gamma(A) = \theta$ reduces the dimension of the moduli space by one.

Proposition 25. *Let $(M\#_l(S^1 \times S^3), g_l)$ be the connected sum of M with $S^1 \times S^3$ with a neck of length l. For every $\theta \in S^1$ and for every $l \gg 0$, there exists some generic perturbation $\eta_l \in \Omega^2_+(M\#(S^1 \times S^3); i\mathbb{R})$ with $\mathrm{supp}\,\eta_l \subset T_l$ such that $SW^{-1}_{\theta,l}(0, \eta_l) \neq \emptyset$, where $SW_{\theta,l}(A, \phi) = (D_A\phi, F^+_A - q(\phi))$ and $\mathrm{hol}_\gamma(A) = \theta$.*

Proof. Observe that the condition of η_l having $\mathrm{supp}\,\eta_l \subset T_l$ is not much of a restriction at all, because the space of such 2-forms is open and the set of generic perturbations is dense (see [10]).

Suppose otherwise, there exists some $\theta \in S^1$ such that for every $l \gg 0$ we have $SW^{-1}_{\theta,l}(0, \eta_l) = \emptyset$. This would imply that $SW^{-1}_\infty(0, 0) = \emptyset$ since we have seen (see Corollary 22) that $(M\#_l(S^1 \times S^3), g_l) \to (M_\infty, g_\infty)$, but this is a contradiction because we have proven (see Corollary 22), that $SW^{-1}_\infty(0, 0) \neq \emptyset$. □

Definition 15. We will say that $(\tilde{A}_l, \tilde{\phi}_l)$ on M_∞, $\mathcal{C}^0$-extends a solution (A_l, ϕ_l)

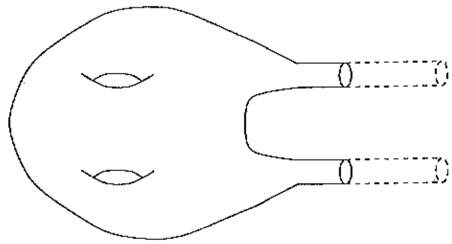

Figure 8.4: Manifold $M_l \subset M_\infty$ with a neck of length l

of $SW_{\theta,l}(A,\phi) = (0,\eta_l)$ on $M\#_l(S^1\times S^3)$ if

$$(\tilde{A}_l,\tilde{\phi}_l)|_{M_l} \equiv (A_l,\phi_l) \text{ and}$$
$$(\tilde{A}_l(t,x),\tilde{\phi}_l(t,x)) = (A_l(x), e^{-2\epsilon t}\phi_l(x))\text{for } (t,x)\in[l,\infty)\times S^3 \subset \mathbb{R}^+\times S^3.$$

Note that $(\tilde{A}_l,\tilde{\phi}_l)\in L^2_{0,\epsilon}(M_\infty, g_\infty)$ (see Fig. 8.4).

Remark. From now on we will fix a $U(1)$-connection A on $L_{\mathfrak{c}}$.

Lemma 26. *If for every $l \gg 0$ there exist two different irreducible solutions $[A_l^1,\phi_l^1]$ and $[A_l^2,\phi_l^2]$ of*

$$D_A\phi = 0$$
$$F_A^+ - q(\phi) = \eta_l$$

on $M\#_l(S^1\times S^3)$ for some generic perturbations η_l, then

$$(C_l,\psi_l) = (\|\tilde{\phi}_l^1 - \tilde{\phi}_l^2\|_{2,0,\epsilon}(\tilde{A}_l^1 - \tilde{A}_l^2), \frac{1}{\|\tilde{\phi}_l^1 - \tilde{\phi}_l^2\|_{2,0,\epsilon}}(\tilde{\phi}_l^1 - \tilde{\phi}_l^2))$$

satisfies

$$(C_l, \psi_l) \to (C, \psi) \in L^2_{1,\epsilon}(M_\infty, g_\infty)$$
$$\|\psi\|_{2,0,\epsilon} = 1,$$

where $(\tilde{A}^i_l, \tilde{\phi}^i_l)$ *C^0-extends* (A^i_l, ϕ^i_l) *to* (M_∞, g_∞) *for* $i = 1, 2$, *and* (A^i_l, ϕ^i_l) *are the unique representatives obtained by the gauge fixing condition* $\delta(A^i_l - A) = 0$.

Proof. Proposition 7 shows that $\phi^1_l \neq \phi^2_l$, so after C^0-extending these solutions we get $\|\tilde{\phi}^1_l - \tilde{\phi}^2_l\| \neq 0$. We will have on M_l,

$$\begin{aligned}
D_{\tilde{A}^1_l}\psi_l &= \frac{1}{\|\tilde{\phi}^1_l - \tilde{\phi}^2_l\|_{2,0,\epsilon}}(D_{\tilde{A}^1_l}\tilde{\phi}^1_l - D_{\tilde{A}^1_l}\tilde{\phi}^2_l) \\
&= -\frac{1}{\|\tilde{\phi}^1_l - \tilde{\phi}^2_l\|_{2,0,\epsilon}} D_{\tilde{A}^1_l}\tilde{\phi}^2_l \\
&= -\frac{1}{\|\tilde{\phi}^1_l - \tilde{\phi}^2_l\|_{2,0,\epsilon}} D_{\tilde{A}^2_l}\tilde{\phi}^2_l - \frac{1}{2}(\tilde{A}^1_l - \tilde{A}^2_l) \cdot \tilde{\phi}^2_l \\
&= -\frac{1}{2}(\tilde{A}^1_l - \tilde{A}^2_l) \cdot \tilde{\phi}^2_l.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
D^2_{\tilde{A}^1_l}\psi_l &= -\frac{1}{2}D_{\tilde{A}^1_l}((\tilde{A}^1_l - \tilde{A}^2_l) \cdot \tilde{\phi}^2_l) \\
&= -\frac{1}{2}(d(\tilde{A}^1_l - \tilde{A}^2_l) + \delta(\tilde{A}^1_l - \tilde{A}^2_l)) \cdot \tilde{\phi}^2_l + \frac{1}{2}(\tilde{A}^1_l - \tilde{A}^2_l) \cdot D_{\tilde{A}^1_l}\tilde{\phi}^2_l \\
&= -\frac{1}{2}d(\tilde{A}^1_l - \tilde{A}^2_l) \cdot \tilde{\phi}^2_l - \frac{1}{4}|(\tilde{A}^1_l - \tilde{A}^2_l)|^2\tilde{\phi}^2_l \\
&= -\frac{1}{2}(F^+_{\tilde{A}^1_l} - F^+_{\tilde{A}^2_l}) \cdot \tilde{\phi}^2_l - \frac{1}{4}|(\tilde{A}^1_l - \tilde{A}^2_l)|^2\tilde{\phi}^2_l.
\end{aligned}$$

By the Bochner formula,

$$\nabla^*_{\tilde{A}^1_l}\nabla_{\tilde{A}^1_l}\psi_l + \frac{s}{4}\psi_l + \frac{1}{2}F_{\tilde{A}^1_l}\cdot\psi_l = -\frac{1}{2}(F^+_{\tilde{A}^1_l} - F^+_{\tilde{A}^2_l})\cdot\tilde{\phi}^2_l - \frac{1}{4}\frac{1}{\|\tilde{\phi}^1_l - \tilde{\phi}^2_l\|^2_{2,0,\epsilon}}|C_l|^2\tilde{\phi}^2_l.$$

Since $(\tilde{A}^i_l, \tilde{\phi}^i_l)$, $i = 1, 2$ are solutions to (SW_f) on M_l, we have

$$F^+_{\tilde{A}^i_l} = e^{-f/2_l}q(\tilde{\phi}^i_l), \quad i = 1, 2.$$

We also have (since $(\tilde{A}^i_l, \tilde{\phi}^i_l)$ are $\mathcal{C}^0$-extensions of (A^i_l, ϕ^i_l), $i = 1, 2$),

$$|\tilde{\phi}^i_l(x)|^2 \leq -e^{f_l(x)}s(x)$$
$$|F^+_{\tilde{A}^i_l}(x)| \leq -e^{f_l(x)}s(x)/2,$$

on M_∞. Observe that even if we have $x_l \in T_l$ such that $e^{f_l(x_l)} \to \infty$ as $l \to \infty$, we would have $e^{f_l(x_l)}s(x_l) = 0$, since $s(x) = 0$ on T_l. Using these, it is not difficult to see that $(C_l, \psi_l) \in L^2_{1,\epsilon}(M_\infty, g_\infty)$. Moreover (C_l, ψ_l) are uniformly bounded in $L^2_{1,\epsilon}(M_\infty, g_\infty)$. These last estimates this are necessary in order to obtain the $L^2_{0,\epsilon}$ estimate on ψ. □

Lemma 27. *The same hypothesis as before. If $(\tilde{A}^i_l, \tilde{\phi}^i_l) \to (A_\infty, \phi_\infty)$ in the $L^2_{1,\epsilon}(M_\infty, g_\infty)$ topology, then we have*

$$\lim_{l\to\infty} \|D(SW_{\theta,l})_{(\tilde{A}^1_l,\tilde{\phi}^1_l)}(C_l, \psi_l)\|_{2,0,\epsilon} = 0$$

Proof. Observe that

$$D(SW_{\theta,l})_{(\tilde{A}_l^1,\tilde{\phi}_l^1)}(C_l,\psi_l) = \begin{pmatrix} d^+C_l - e^{-f_l}Dq_{\tilde{\phi}_l^1}(\psi_l) \\ \frac{1}{2}C_l\cdot\psi_l + D_{\tilde{A}_l^1}\psi_l \end{pmatrix}$$

Let us consider the first coordinate of $D(SW_{\theta,l})_{(\tilde{A}_l^1,\tilde{\phi}_l^1)}(C_l,\psi_l)$

$$\begin{aligned} &d^+C_l - e^{-f_l}Dq_{\tilde{\phi}_l^1}(\psi_l) \\ &\qquad = \|\tilde{\phi}_l^1 - \tilde{\phi}_l^2\|_{2,0,\epsilon}(F^+_{\tilde{A}_l^1} - F^+_{\tilde{A}_l^2}) - Dq_{\tilde{\phi}_l^1}(\psi_l). \end{aligned}$$

It is easy to see that the right-hand side of this equation vanishes as $l\to\infty$. Finally it is easy to see that the second coordinate of $D(SW_{\theta,l})_{(\tilde{A}_l^1,\tilde{\phi}_l^1)}(C_l,\psi_l)$ vanishes as $l\to\infty$. □

Proposition 28. *For every $\theta\in S^1$, $SW_\theta(\mathfrak{c}_{0,1})=1$.*

Proof. Assume that $SW_\theta(\mathfrak{c}_{0,1})\neq\pm1$. By Proposition 25 this implies for $l\gg 0$ there exist (at least) two different irreducible solutions (A_l^i,ϕ_l^i), $i=1,2$ on $(M\#_l(S^1\times S^3),g_l)$. By Lemma 26 and Lemma 27 we would have an element of $\ker DSW_\infty$ at (A_∞,ϕ_∞) the unique solution on (M_∞,g_∞), obtained in Corollary 22. But this is a contradiction since (A_∞,ϕ_∞) is a smooth point. The same kind of argument shows that $SW_\theta(\mathfrak{c}_{0,1})=1$ since $SW_\infty(\mathfrak{c})=1$. □

Chapter 9

Cohomology of $\mathcal{B}^*(\mathfrak{c})$

In this section we will build cohomology classes for $\mathcal{B}^*(\mathfrak{c})$ in order to detect B-classes (see Definition 12). To describe the cohomology of $\mathcal{B}^*(\mathfrak{c})$ we have to introduce the concept of *universal family of SW-connections* associated to a Spin^c structure $\mathfrak{c}$, parameterized by $\mathcal{B}^*(\mathfrak{c})$. A SW-connection is simply a pair (A, ϕ), where A is a $U(1)$-connection on $L_\mathfrak{c}$ and $0 \neq \phi \in S^+(\mathfrak{c})$.

A cohomology class $\beta \in H^i(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$ can be thought of as a homomorphism $\beta : H_i(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z}) \to \mathbb{Z}$, and the elements of $H_i(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$ can be thought of as homotopic classes of maps $f : T \to \mathcal{B}^*(\mathfrak{c})$, where T is a compact space. The maps $f : T \to \mathcal{B}^*(\mathfrak{c})$ are naturally interpreted in terms of families of SW-connections.

Definition 16. *A family of SW-connections in a bundle $L_\mathfrak{c} \to M$ parametrized by a space T* is a bundle $L \to T \times M$ with the property that each slice $L_t = L|_{\{t\}\times M}$ is isomorphic to $L_\mathfrak{c}$, together with a SW-connection $(A_\phi)_t = (A_t, \phi_t)$ in L_t, forming a family $A_\phi = \{(A_\phi)_t\}$.

Let $\pi_2 : \mathcal{C}^*(\mathfrak{c}) \times M \to M$ be the projection onto the second factor and let $\mathcal{L}_\mathfrak{c} \to \mathcal{C}^*(\mathfrak{c}) \times M$ be the pull-back line bundle, $\pi_2^* L_\mathfrak{c}$. Then $\mathcal{L}_\mathfrak{c}$ carries a tautological family of SW-connections A_ϕ, in which the SW-connection on the slice $\mathcal{L}_\mathfrak{c}|_{\{(A,\phi)\}}$ over $\{(A,\phi)\} \times M$ is $(\pi_2^*(A), \pi_2^*(\phi))$. The group $\mathcal{G}(\mathfrak{c})$ acts freely on $\mathcal{C}^*(\mathfrak{c}) \times M$ as well as on $\mathcal{L}_\mathfrak{c} = \mathcal{C}^*(\mathfrak{c}) \times L_\mathfrak{c}$, and there is therefore a quotient bundle

$$\mathbb{L}_\mathfrak{c} \to \mathcal{B}^*(\mathfrak{c}) \times M$$
$$\mathbb{L}_\mathfrak{c} = \mathcal{L}_\mathfrak{c}/\mathcal{G}(\mathfrak{c}).$$

The family of SW-connections A_ϕ is preserved by $\mathcal{G}(\mathfrak{c})$, so $\mathbb{L}_\mathfrak{c}$ carries an inherited family of SW-connections $\mathbb{A}_\phi$. This is the *universal family of SW-connections in* $L_\mathfrak{c} \to M$ *parameterized by* $\mathcal{B}^*(\mathfrak{c})$.

If a family of SW-connections is parameterized by a space T and carried by a bundle $L \to T \times M$, there is an associated map $f : T \to \mathcal{B}^*(\mathfrak{c})$ given by

$$f(t) = [A_t, \phi_t].$$

Conversely, given $f : T \to \mathcal{B}^*(\mathfrak{c})$ there is a corresponding pull-back family of connections carried by $(f \times I)^*\mathbb{L}_\mathfrak{c}$. These two constructions are inverses of one another: if f is determined by the above equation, then for each t there is a *unique* isomorphism ψ_t between the SW-connections in L_t and $(f \times I)^*(\mathbb{L}_\mathfrak{c})_t$, and as t varies these fit together to form an isomorphism $\psi : L \to (f \times I)^*\mathbb{L}_\mathfrak{c}$ between these two families. (The uniqueness of ψ_t results from the fact that $\mathcal{G}(\mathfrak{c})$ acts freely on $\mathcal{C}^*(\mathfrak{c})$). Thus:

Lemma 29. *The maps* $f : T \to \mathcal{B}^*(\mathfrak{c})$ *are in one-to-one correspondence with families of SW-connections on M parameterized by T, and this correspondence* *is obtained by pulling back from the universal family* $(\mathbb{L}_{\mathfrak{c}}, \mathbb{A}_{\phi})$.

Remark. Let $\{\gamma_i\}$ be fixed representatives for the generators of the free part of $H_1(M;\mathbb{Z})$. If $f_1, f_2 : T \to \mathcal{B}^*(\mathfrak{c})$ are homotopic, the corresponding bundles L_1 and L_2 are isomorphic, and the corresponding holonomy maps $h_1 : T \to (S^1)^{b_1}$ and $h_2 : T \to (S^1)^{b_1}$ are homotopic, where the holonomy map is defined as $h_i(t) = (\mathrm{hol}_{\gamma_1}(f_i(t)), \ldots, \mathrm{hol}_{\gamma_{b_1}}(f_i(t)))$.

There is a general construction which produces cohomology classes in $\mathcal{B}^*(\mathfrak{c})$, using the slant-product pairing

$$/ : H^{d-i}(\mathcal{B}^*(\mathfrak{c});\mathbb{Z}) \times H_i(M;\mathbb{Z}) \to H^i(\mathcal{B}^*(\mathfrak{c});\mathbb{Z}).$$

We have built over $\mathcal{B}^*(\mathfrak{c}) \times M$ a line bundle $\mathbb{L}_{\mathfrak{c}}$, so we can define a map

$$\mu : H_i(X;\mathbb{Z}) \to H^{2-i}(\mathcal{B}^*(\mathfrak{c});\mathbb{Z})$$

by

$$\mu(\alpha) = c_1(\mathbb{L}_{\mathfrak{c}})/\alpha.$$

If T is any $(2-i)$-cycle in $\mathcal{B}^*(\mathfrak{c})$, the class $\mu(\alpha)$ can be evaluated on T using the formula

$$\langle \mu(\alpha), T\rangle_{\mathcal{B}^*(\mathfrak{c})} = \langle c_1(\mathbb{L}_{\mathfrak{c}}), T \times \alpha\rangle_{\mathcal{B}^*(\mathfrak{c})\times M},$$

which expresses the fact that the slant product is the adjoint of the cross-product homomorphism. Next we will describe another way to build cohomology classes.

Definition 17. A closed curve $\gamma : S^1 \to M$ induces a *holonomy map*

$$\mathrm{hol}_\gamma : \mathcal{B}^*(\mathfrak{c}) \to S^1$$

defined as the holonomy of the SW-connections A_ϕ along γ. The pull-back of the canonical class $d\theta$ of S^1 defines a cohomology class on $H^1(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$ which we will call the *holonomy class along* γ.

Proposition 30. *The cohomology groups of* $\mathcal{B}^*(\mathfrak{c})$ *are generated by the image of the map* $\mu : H_i(X; \mathbb{Z}) \to H^{2-i}(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$. *Moreover, given* $\gamma \in H_1(M; \mathbb{Z})$, $\mu(\gamma)$ *is the holonomy class along* γ, $\mathrm{hol}_\gamma^*(d\theta)$.

Proof. First we will prove that if $\{\gamma_i\}$ are fixed representatives for the generators for the free part of $H_1(M; \mathbb{Z})$ then $\{\mu(\gamma_i)\}$ generates $H^1(\mathcal{B}^*(\mathfrak{c}); \mathbb{Z})$. It is enough to prove that for every i we can find $\beta_i : S^1 \to \mathcal{B}^*(\mathfrak{c})$ such that $\langle \mu(\gamma_i), \beta_i \rangle|_{\mathcal{B}^*(\mathfrak{c})} = 1$. Consider the line bundle $\gamma_i^* L_\mathfrak{c} \to S^1$, and observe that there is no obstruction to extend it to a line bundle $L \to S^1 \times S^1$ such that $\deg L = \langle c_1(L), S^1 \times S^1 \rangle = 1$. Let A_i be a $U(1)$-connection on L and consider the map

$$\mathrm{hol}_{\bullet \times S^1}(A_i) : S^1 \to S^1.$$

It is not difficult to see that $\deg L = \deg(\mathrm{hol}_{\bullet \times S^1}(A_i))$. After extending $A_i(t, \gamma_i)$ to a $U(1)$-connection on $L_\mathfrak{c} \to M$ for each t, we obtain (see remark below Lemma 29) our desired maps $\beta_i : S^1 \to \mathcal{B}^*(\mathfrak{c})$.

To prove the last statement we proceed as follows: let $\alpha : S^1 \to \mathcal{B}^*(\mathfrak{c})$,

$$
\begin{aligned}
\langle \mu(\gamma_i), \alpha \rangle_{\mathcal{B}^*(\mathfrak{c})} &= \langle c_1(\mathbb{L}_{\mathfrak{c}}), \alpha \times \gamma_i \rangle_{\mathcal{B}^*(\mathfrak{c}) \times M} \\
&= \langle c_1((\alpha \times \gamma_i)^*(\mathbb{L}_{\mathfrak{c}})), S^1 \times S^1 \rangle \\
&= \deg(\mathrm{hol}_{\bullet \times S^1}(A_i) : S^1 \to S^1) \\
&= \deg(\mathrm{hol}_{\gamma_i} \circ \alpha : S^1 \to S^1) \\
&= \langle \deg^*_{\beta_i}(d\theta), \alpha \rangle_{\mathcal{B}^*(\mathfrak{c})}.
\end{aligned}
$$

Finally we have to show that if $x \in M$ then $\mu(x)$ generates the cohomology of the $\mathbb{CP}^\infty$ factor. Since $Map(M, S^1)_o$ acts freely on $\mathcal{C}^*(\mathfrak{c})$, then it is easy to show that $\mathbb{L}_{\mathfrak{c}}|_{\mathcal{B}^*(\mathfrak{c})} \approx \mathcal{C}^*(\mathfrak{c})/\mathcal{G}_0(\mathfrak{c})$, where $\mathcal{G}_0(\mathfrak{c})$ is the kernel of the homomorphism $\mathcal{G}(\mathfrak{c}) \to S^1$ given by evaluating on the fiber over x. $\square$

Chapter 10

Applications

C. LeBrun [8] showed that under some mild conditions on M, $M\#k\,\overline{\mathbb{CP}^2}$ does not admit Einstein metrics. The precise statement is the following:

Theorem (C. LeBrun). *Let M be a smooth compact oriented 4-manifold with $2e + 3\sigma > 0$. Assume, moreover, that M has a non-trivial Seiberg-Witten invariant. If $k \geq \frac{25}{57}(2e + 3\sigma)$ then $M\#k\,\overline{\mathbb{CP}^2}$ does not admit an Einstein metric.*

Remark. The proof of this Theorem only requires that M has a Spin^c-structure $\mathfrak{c}$ that is a B-class.

Theorem 31. *Let $(M, \mathfrak{c})$ be a smooth compact Kähler surface with a* Spin^c*-structure $\mathfrak{c}$. There is a canonical* Spin^c *structure in the connected sum manifold $M\#(S^1 \times S^3)$ which we will denote by $\mathfrak{c}_{0,1}$. Moreover $d(\mathfrak{c}_{0,1}) = d(\mathfrak{c}) + 1$. If $\mathfrak{c}$ is a non-trivial SW-class for M then $\mathfrak{c}_{0,1}$ is a B-class for the connected sum $M\#(S^1 \times S^3)$.*

Proof. $SW_\theta(\mathfrak{c}_{0,1})$ is a cobordism invariant for every $\theta \in S^1$. Consider the smooth cobordism induced by the family of metrics g_l on $M\#(S^1 \times S^3)$ as

$l \to \infty$ and observe (Corollary 22) that $SW_\infty(\mathfrak{c}) = 1$. This shows that $\langle \mathrm{hol}^*_\gamma(d\theta), \mathcal{M}(\mathfrak{c}_{0,1})\rangle|_{\mathcal{B}^*(\mathfrak{c}_{0,1})} = 1$, where γ is a representative for the S^1 factor of the connected sum. This, the definition of a B-class and Proposition 30 complete the proof. □

Corollary 32. *Let $(M, \mathfrak{c})$ be a smooth compact oriented Kähler surface with a* Spin^c*-structure $\mathfrak{c}$. There is a canonical* Spin^c *structure in the connected sum $M\#2(S^1 \times S^3)$ which we will denote by $\mathfrak{c}_{0,2}$. Moreover $d(\mathfrak{c}_{0,2}) = d(\mathfrak{c}) + 2$. If $\mathfrak{c}$ is a non-trivial SW-class then $\mathfrak{c}_{0,2}$ is a B-class but has trivial Seiberg-Witten invariant.*

Proof. Theorem 31 shows that every time that we perform a connected sum with $S^1 \times S^3$ we *add a cycle* to the moduli space, that lies entirely in the $H^1(M\#(S^1 \times S^3); \mathbb{R})/H^1(M\#(S^1 \times S^3); \mathbb{Z})$ part of $\mathcal{B}^*(\mathfrak{c}_{0,1})$. □

Lemma 33. *Let $(M, \mathfrak{c})$ be a smooth compact oriented Kähler surface with a* Spin^c*-structure $\mathfrak{c}$ and $2e + 3\sigma > 0$. Assume that $\mathfrak{c}$ is a non-trivial SW-class. Let k, l be any two natural numbers. Then there is a B-class $\mathfrak{c}_{k,l}$ on $M_{k,l} = M\#k\,\overline{\mathbb{CP}^2}\#\,l(S^1 \times S^3)$ such that*

$$(c_1^+(\mathfrak{c}_{k,l}))^2 \geq (2e + 3\sigma)(M).$$

Proof. First observe that $M_{k,l} = (M\#k\,\overline{\mathbb{CP}^2})_{0,l}$. Since M is a Kähler surface, we know that $M\#k\,\overline{\mathbb{CP}^2}$ is also a Kähler surface, and its associated Spin^c structure $\mathfrak{c}_{k,0}$ satisfies $c_1(\mathfrak{c}_{k,0}) = c_1(\mathfrak{c}) + \sum_{j=1}^k E_j$, where $E_1, \ldots, E_k$ are generators for the pull-backs to $M\#k\,\overline{\mathbb{CP}^2}$ of the k copies of $H^2(\mathbb{CP}^2,\mathbb{Z})$ so that

$$c_1^+(\mathfrak{c})\cdot E_j \geq 0, \quad j=1,\dots,k.$$

Let $c_1(\mathfrak{c}_{k,l})$ be the first Chern class of $(\mathfrak{c}_{k,0})_{0,l}$ which is a B-class by Theorem 31, and notice that $c_1(\mathfrak{c}_{k,l}) = c_1(\mathfrak{c}_{k,0})$. One then has

$$\begin{aligned}
(c_1^+(\mathfrak{c}_{k,l}))^2 &= (c_1^+(\mathfrak{c}_{k,0}))^2 \\
&= \left(c_1^+(\mathfrak{c}) + \sum_{j=1}^{k} E_j^+\right)^2 \\
&= (c_1^+(\mathfrak{c}))^2 + 2\sum_{j=1}^{k} c_1^+(\mathfrak{c}_{0,l})\cdot E_j^+ + (\sum_{j=1}^{k} E_j^+)^2 \\
&\geq (c_1^+(\mathfrak{c}))^2 \\
&\geq (c_1(\mathfrak{c}))^2 \\
&= (2e+3\sigma)(M).
\end{aligned}$$

□

LeBrun's Theorem can be generalized in the following way:

Theorem 34. *Let $(M,\mathfrak{c})$ be a smooth compact oriented Kähler surface with a* Spin^c*-structure $\mathfrak{c}$ and $2e+3\sigma>0$. Assume that $\mathfrak{c}$ is a B-class. If $k+4l \geq \frac{25}{57}(2e+3\sigma)$ then $M_{k,l} = M\#k\,\overline{\mathbb{CP}^2}\#\,l(S^1\times S^3)$ does not admit an Einstein metric.*

Proof. The proof is the same as the one given by C. LeBrun [8], we reproduce here for completeness. For any Einstein metric g on M, C. LeBrun showed that

$$(2e+3\sigma)(M_{k,l}) = \frac{1}{4\pi^2}\int_{M_{k,l}} \left(2|W_+|^2 + \frac{s^2}{24}\right) d\mu > \frac{32}{57}(c_1^+(\mathfrak{c}))^2$$

for any B-class $\mathfrak{c}$ on $M_{k,l}$.

By Lemma 33, $M_{k,l}$ has a B-class $\mathfrak{c}_{k,l}$ with $(c_1^+(\mathfrak{c}_{k,l}))^2 \geq (2e+3\sigma)(M)$. Thus

$$(2e+3\sigma)(M) - k - 4l = (2e+3\sigma)(M_{k,l}) > \frac{32}{57}(2e+3\sigma)(M).$$

Assuming that $M_{k,l}$ admits an Einstein metric we get that

$$k + 4l < \frac{25}{57}(2e+3\sigma),$$

which contradicts the hypothesis. The result follows. □

There exists two well known topological obstructions to the existence of Einstein metrics on a differentiable compact oriented 4-manifold M.

The first one is Thorpe's inequality, that comes from the Gauss-Bonnet-Chern formula for the Euler characteristic $e(M)$ of M and from the Hirzebruch formula for the signature $\sigma(M)$ of M (see [2]), which allow us to express these two topological invariants in terms of the irreducible components of the curvature under the action of $SO(4)$. It can be stated in the following way

Theorem (N. Hitchin, J. Thorpe). *Let M be a compact oriented manifold of dimension 4. If $e(M) < \frac{3}{2}|\sigma(M)|$ then M does not admit any Einstein metric. Moreover, if $e(M) = \frac{3}{2}|\sigma(M)$ then M admits no Einstein metric unless it is either flat, or a $K3$ surface, or an Enriques surface, or the quotient of an*

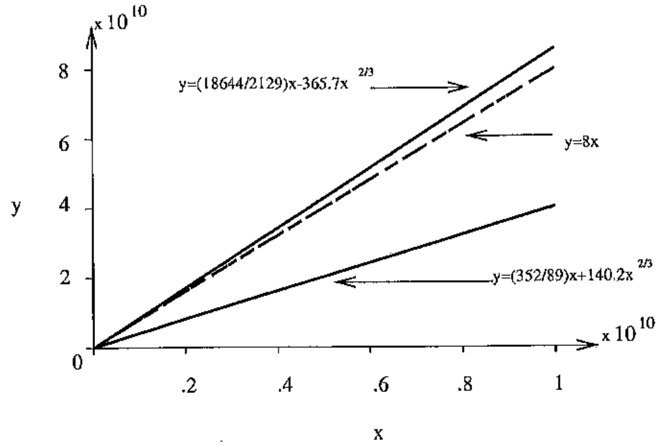

Figure 10.1: Chen's region

Enriques surface by a free antiholomorphic involution.

This theorem implies a previous result of M. Berger who proved that there exists no compact Einstein 4-manifold with a negative Euler characteristic.

On the other hand, combining the Gauss-Bonnet-Chern formula for the Euler characteristic with Gromov's estimation of simplicial volume $\|M\|$ of a Riemannian manifold M (see [4]), M. Gromov obtained the following obstruction

Theorem (M. Gromov). *Let M be a compact manifold of dimension 4. If $e(M) < \frac{1}{2592\pi^2}\|M\|$ then M does not admit any Einstein metric.*

A. Sambusetti (see [12]) found a topological obstruction to the existence of Einstein metrics on compact 4-manifolds which admit a non-zero degree map onto some compact real or complex hyperbolic 4-manifold. As a consequence, by connected sums, he produces infinitely many non-homeomorphic 4-manifolds which admit no Einstein metrics. This fact is not a consequence of Hitchin-Thorpe's or Gromov's obstruction theorems. A. Sambusetti also proves that any Euler characteristic and signature can be simultaneously realized by these non-homeomorphic manifolds admitting no Einstein metrics.

Definition 18. We say that a pair $(m, n) \in \mathbb{Z}^2$ is *admissible* if there exists a smooth compact oriented 4-manifold with Euler characteristic m and signature n. In fact a necessary and sufficient condition for $(m, n) \in \mathbb{Z}^2$ to be an admissible pair is that $m \equiv n \mod 2$.

Z. Chen [3] proved the following theorem:

Theorem (Z. Chen). *Let x, y be integers satisfying*

$$\frac{352}{89}x + 140.2x^{2/3} < y < \frac{18644}{2129}x - 365.7x^{2/3},$$
$$x > C,$$

where C is a large constant. There exists a simply connected minimal surface M of general type with $c_1^2(M) = y$, $\chi(M) = x$. Furthermore, M can be represented by a surface admitting a hyperelliptic fibration.

Remark. Recall that $\chi(M)$ denotes the Euler-Poincaré characteristic of the invertible sheaf $\mathcal{O}_M$. Using Noether's formula we have that

$$\chi(M) = \frac{c_1^2(M) + e(M)}{12} = \frac{e(M) + \sigma(M)}{4}.$$

If M is not a complex surface $e(M) + \sigma(M)$ is not necessarily a multiple of 4 but it is always an even number.

In our last result, the manifolds in Chen's theorem play the same rôle as that of the hyperbolic manifolds in Sambusetti's construction.

Theorem 35. *For each admissible pair (m, n) there exist an infinite number of non-homeomorphic compact oriented 4-manifolds which have Euler characteristic m and signature n, with free fundamental group and which do not admit Einstein metric.*

Proof. Let (m_0, n_0) be an admissible pair and consider the pair of integers $(x_0', y_0) = \left(\frac{m_0+n_0}{2}, 2m_0 + 3n_0\right)$. It is always possible to find (infinitely many) positive integers k and l such that

$$(x, y) = \left(\frac{x_0' + l}{2}, y_0 + k\right) \in \mathcal{Z}$$

$$4l + \frac{32}{57}k \geq \frac{25}{57}y_0$$

where $\mathcal{Z}$ denotes the set of $(x, y) \in \mathbb{Z}^2$ that satisfy the conditions of Chen's Theorem. The reason for this last statement is that the region $\mathcal{Z}_{\mathbb{R}}$ determine by $(x, y) \in \mathbb{R}^2$ such that

$$\frac{352}{89}x + 140.2x^{2/3} < y < \frac{18644}{2129}x - 365.7x^{2/3},$$

$$x > C,$$

where C is a large constant, is open, connected and not bounded (see Fig. 10.1).

If we denote by M the simply connected Kähler surface with $c_1^2 = y$ and $\chi = x$, then $M_{k,l} = M \# k\,\overline{\mathbb{CP}^2} \# l(S^1 \times S^3)$, is a manifold that realizes the pair

(m_0, n_0) and does not admit any Einstein metric. This last statement is a consequence of Theorem 34. $\square$

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