Gromov Invariants of Symplectic Fibrations

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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We study the Gromov invariants of the total space of a symplectic fibration $\pi : W \to M$, where (M, ω) is a symplectic 4-manifold and the fiber is equal to S^2. We find a relation between the Gromov invariatns of W and those of M, for the homology classes $\hat{A}$ such that $\pi(\hat{A}) \neq 0$. As an application we construct infinitelly many symplectic structures on W for $M = E(n)$, the simply connected minimal elliptic surface.

To my parents

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Chapter 1

Introduction

In this thesis we study the Gromov invariants of the total space of a symplectic fibration $\pi : W \to M$, considering in particular how the Gromov invariants of W and M are related.

We shall work with almost complex structures on W which are in certain sense *adapted* to the fibration. Such almost complex structures will be called *fibered* (see definition 2.3 below), and the set of all such structures will be denoted by $\mathcal{J}_{fib}$.

Assuming that the structure group of the fibration $\pi : W \to M$ is a compact group, we prove that the set of regular fibered almost complex structures on W is dense in $\mathcal{J}_{fib}$ (see Proposition 3.7 below). This regularity result will be essential for the proof of our main theorem. This concerns the special case when the fiber $F = S^2$ and dim $M = 4$ (see Theorem 4.2), and implies that if there are non-zero Gromov invariants on M, then we can find non-zero Gromov invariants on W (see Corollary 4.4).

We can apply our results to the construction of infinitely many symplectic structures on some of the 6-manifolds we mentioned. Namely, let $M = E(n)$ be the simply connected minimal elliptic surface with fiber T and first Chern class $c_1(E(n)) = (2-n)PD(T)$, and let $\pi : W \to E(n)$ be any S^2-bundle. Recall that two symplectic forms are said to be deformation equivalent if they can be joined by a path of symplectic forms.

Theorem 1.1 *W has infinitely many deformation classes of symplectic structures in the same cohomology class $[\Omega] \in H^2(W, \mathbb{R})$.*

This theorem is a generalization of a result of Ionel and Parker in [6], where they prove the same statement for $E(n) \times S^2$, and is related to the stability conjecture of Donaldson, that states that homeomorphic symplectic 4-manifolds when multiplied with S^2 should yield diffeomorphic but not deformation equivalent symplectic manifolds.

Chapter 2

Preliminaries

In this section we study properties of the metrics on a symplectic fibration that are associated to fibered almost complex structures. First, we recall the definition of a symplectic fibration and the conditions under which it supports a symplectic form. Then we define the class of *fibered* almost complex structures.

Definition 2.1 *A locally trivial fibration $\pi : W \to M$ with fiber F is said to be a* symplectic fibration, *if (F, σ) is a symplectic manifold and the structure group of the fibration can be reduced to a subgroup G of the group of symplectomorphisms of the fiber $Symp(F, \sigma)$.*

We shall denote by $F_x = \pi^{-1}(x)$ the inverse image of $x \in M$, and by σ_x the induced symplectic structure on the fiber. We are going to assume that G is compact.

For every point $y \in W$ denote by $Vert_y = \ker\, d\pi(y) = T_y F_{\pi(y)}$ the vertical tangent space. A *connection* Γ on W is a field of horizontal subspaces $Hor_y \subseteq T_y W$, for every $y \in W$, such that $TW \cong Vert \oplus Hor$, i.e. there exists a bundle map $\iota : \pi^*TM \longrightarrow TW$, such that $d\pi \circ \iota = id_{\pi^*TM}$, which implies that the short exact sequence of bundles

$$0 \longrightarrow Vert \longrightarrow TW \longrightarrow \pi^*TM \longrightarrow 0$$

splits.

We say that a two form $\tau \in \Omega^2(W)$ is *compatible with the fibration* if $\tau|_{F_x} = \sigma_x$ for every $x \in M$. Every compatible 2-form τ, gives rise to a connection Γ_τ, in the following way: for $y \in W$,

$$Hor_y = V_y^\tau = \{w \in T_yW \mid \tau(w,v) = 0, \text{ for every } v \in Vert\}.$$

Conversely, for every connection Γ on W there exists a two form τ, such that $Hor = Vert^\tau$. This form, however, is not unique, τ and τ' generate the same distribution if $Vert$ is in the kernel of $\tau - \tau'$, i.e., if $\iota(v)(\tau - \tau') = 0$ for every $v \in Vert$. Notice that it is not hard to construct a compatible 2-form on any symplectic fibration (see [12]).

Given a connection Γ, every path $\beta : [0,1] \to M$ determines a diffeomorphism $\Psi_\beta : F_{\beta(0)} \to F_{\beta(1)}$ sending a point $y_0 \in F_{\beta(0)}$ to the end point $y_1 = \overline{\beta}(1) \in F_{\beta(1)}$ of the unique horizontal lift $\overline{\beta}$ of β at y_0. Ψ_β is called the *holonomy* of β, or *parallel transport* along β. The connection Γ is called *symplectic* if Ψ_β is a symplectomorphism $(F_{\beta(0)}, \sigma_{\beta(0)}) \to (F_{\beta(1)}, \sigma_{\beta(1)})$, for every β.

If τ is a 2-form compatible with the fibration, then the connection Γ_τ, is symplectic if and only if τ is *vertically closed*, that is, if $d\tau(\eta_1, \eta_2, \cdot) = 0$ for every $\eta_1, \eta_2 \in Vert$ ([12]). Such a form will be called a *connection 2-form* on W. In particular, if τ is closed then Γ_τ is symplectic. It is not hard to prove that every symplectic connection Γ has the form Γ_τ, for some vertically closed 2-form τ on W (see [12]). However, τ need not be closed. The next theorem characterizes the fibrations for which τ may be taken to be closed. Guillemin, Lerman and Sternberg in [4] constructed a closed 2-form τ using the holonomy of a symplectic connection when the fiber is simply connected. McDuff and Salamon [12] generalized their construction whenever this holonomy is Hamiltonian around every contractible loop in the base.

Theorem 2.2 *[4],[12] Let $\pi : W \to M$ be a symplectic fibration and Γ be a symplectic connection on W. Then the following are equivalent*

(i) There exist a closed connection 2-form $\tau \in \Omega^2(W)$, such that $\Gamma = \Gamma_\tau$.

(ii) The holonomy of Γ around any contractible loop in M is Hamiltonian.

As a consequence, we see that if the base manifold (M, ω) is symplectic, then W carries a family of compatible symplectic structures given by $\Omega_\kappa = \tau + \kappa\pi^*\omega$, for κ sufficiently big. For this reason we shall just look at fibrations that support closed connection 2-forms τ and *Hor* will be $Vert^\tau$.

The next step is to consider almost complex structures on W tamed by Ω, it will be convenient to make use of almost complex structures adapted to the fibration in the following way,

Definition 2.3 *An almost complex structure J in W is called* fibered *if for some symplectic connection Γ on W, we have the following properties*

(i) J is Ω-tamed,

(ii) Hor and Vert are J-invariant,

(iii) π *is* (J, J_M)*-holomorphic, for some almost complex structure* J_M *on* M *tamed by* ω.

This set is non-empty and can be constructed as follows. *Vert*$\to W$ is a symplectic vector bundle. Hence it has a complex structure J_F ([12]). Define J to be the pull-back of J_M to the horizontal distribution *Hor*, and to be J_F on *Vert.*

Remark 2.4 *Assume that* J_M *and* J_F *are fixed, and let* $Hor = \iota(\pi^*TM)$ *and* $Hor' = \iota'(\pi^*TM)$ *be two horizontal distributions, and* J *and* J' *the corresponding fibered almost complex structures on* W. *Then* $J|_{Vert} = J_F = J'|_{Vert}$. *We would like to know the difference between* J *and* J' *on vectors belonging to* Hor. *Since both structures are fibered,* $J\iota = \iota J_M$ *and* $J'\iota' = \iota' J_M$. *Let* $X \in \pi^*TM$. *Then* $\iota(X) = \iota'(X) + (\iota - \iota')(X)$ *and*

$$J'\iota(X) = \iota'(J_M X) + J_F(\iota - \iota')(X) = \iota(J_M X) + \theta(X) = J\iota(X) + \theta(X),$$

where $\theta(X) = J_F(\iota - \iota') - (\iota - \iota')J_M$ *is a* (J_M, J_F)*-anti-linear homomorphism from* $(\pi^*TM, J_M) \to (Vert, J_F)$.

□

Notice that with such a J, if $u : (\Sigma, j) \to (W, J)$ is (j, J)-holomorphic, then $v := \pi \circ u$ is (j, J_M)-holomorphic. Assume that v is not constant, then if v is somewhere injective, then u has to be somewhere injective as well. We want to study the space of J-holomorphic maps in W, therefore we need to examine the Cauchy-Riemann operator (see Section 3 below), and for that we need an appropriate metric on W. Namely, choose a G-invariant metric g_F on F compatible with J_F and the metric $g_M(.,.) = \omega(J_M.,.)$ on M. Define the metric g on W to be equal to π^*g_M on Hor and to g_F on $Vert$, which makes Hor and $Vert$ perpendicular. Since g_F is G-invariant, g is a well-defined Riemannian metric on W. Notice in addition that J is g-invariant. Furthermore, with this metric π is a Riemannian submersion, that is, $d\pi(y)|_{Hor_y} : (Hor_y, \pi^*g_M) \to (T_{\pi(y)}M, g_M)$ is an isometry for every $y \in W$.

If $X \in TM$, $\overline{X} \in Hor \subseteq TW$ will denote its horizontal lift. Let ∇ be the Levi-Civita connection of g_M and $\overline{\nabla}$ the Levi-Civita connection of g (See [1]).

Lemma 2.5 *For every X, $Y \in TM$,*

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v, \tag{2.1}$$

and $[\overline{X}, \overline{Y}]^v(p)$ only depends on $\overline{X}(p)$ and $\overline{Y}(p)$, for $p \in W$.

Proof. It is enough to prove that $\langle \overline{h}, \overline{\nabla}_{\overline{X}}\overline{Y}\rangle = \langle h, \nabla_X Y\rangle$ for every $h \in TM$, and that $\langle v, \overline{\nabla}_{\overline{X}}\overline{Y}\rangle = \frac{1}{2}\langle v, [\overline{X}, \overline{Y}]\rangle$, for every $v \in Vert$. First observe that $\overline{X}\langle \overline{Y}, \overline{Z}\rangle = X\langle Y, Z\rangle$ and that $d\pi[\overline{X}, v] = 0$.

Let $h \in TM$ and $\overline{h}$ its horizontal lift, then by definition of the Levi-Civita connection in terms of the metric ([1], p. 55)

$$\begin{aligned}\langle \overline{h}, \overline{\nabla}_{\overline{X}}\overline{Y}\rangle &= \frac{1}{2}(\overline{Y}\langle \overline{X}, \overline{h}\rangle + \overline{X}\langle \overline{h}, \overline{Y}\rangle - \overline{h}\langle \overline{Y}, \overline{X}\rangle \\ &- \langle [\overline{Y}, \overline{h}], \overline{X}\rangle - \langle [\overline{X}, \overline{h}], \overline{Y}\rangle - \langle [\overline{Y}, \overline{X}], \overline{h}\rangle) \\ &= \frac{1}{2}(Y\langle X, h\rangle + X\langle h, Y\rangle - h\langle Y, X\rangle \\ &- \langle [Y, h], X\rangle - \langle [X, h], Y\rangle - \langle [Y, X], h\rangle) \\ &= \langle h, \nabla_X Y\rangle.\end{aligned}$$

Similarly, for $v \in Vert$,

$$\langle v, \overline{\nabla}_{\overline{X}}\overline{Y}\rangle = \frac{1}{2}\langle [\overline{X}, \overline{Y}], v\rangle.$$

To see that $[\overline{X}, \overline{Y}]^v$ is a tensor, let $f, g : W \to \mathbb{R}$,

$$[f\overline{X}, g\overline{Y}] = fg[\overline{X}, \overline{Y}] + f\overline{X}(g)\overline{Y} - g\overline{Y}(f)\overline{X},$$

by taking the vertical part of both sides we get that

$$[f\overline{X}, g\overline{Y}]^v = fg[\overline{X}, \overline{Y}]^v.$$

□

If F is a submanifold of W, then for any X, $Y \in TF$, $\overline{\nabla}_X Y = \nabla^F_X Y + B(X,Y)$, where $\nabla^F_X Y$ is the tangential component of $\overline{\nabla}_X Y$, and $B(X,Y)$ its normal component (called the Second Fundamental form of F). We say that

F is *totally geodesic* if $\overline{\nabla}_X Y = \nabla^F_X Y$, for every $X, Y \in TF$. This means that if γ is a geodesic in F with respect to $g|_F$, then γ is a geodesic in W with respect to g (see [1]).

The next lemma is taken from [16] and we include the proof for the convenience of the reader.

Lemma 2.6 *[16] The fibers of π are totally geodesic with respect to g.*

Proof. Let $\alpha : [0,1] \to F$ be a curve parameterized by arc length. We want to prove that $\overline{\nabla}_{\dot\alpha}\dot\alpha$ is a vertical vector. Let $h(0)$ be a horizontal vector at $\alpha(0)$. Let $\beta : [0,1] \to M$ be a curve in M with initial velocity $\dot\beta(0) = \pi_* h(0)$, then for every $s \in [0,1]$ we have the maps $\Psi_{\beta(s)} : F_{\beta(0)} \to F_{\beta(s)}$ or parallel transport.

Define $\sigma(t,s) = \Psi_{\beta(s)}(\alpha(t))$. By construction, $h(t) = \frac{\partial}{\partial s}\sigma(t,0)$ is a horizontal vector field along α. Define $f(s) = \int_0^1 \| \frac{\partial}{\partial t}\sigma(t,s) \|^2 \, dt$, the energy of the curve $\sigma(t,s)$ for fixed s. Since g_F is a G-invariant metric, then $\Psi_{\beta(s)}$ (or parallel transport) is an isometry for every s, and therefore $f(s)$ is a constant function in s, i.e. $\frac{\partial}{\partial s} f(0) = 0$.

On the other hand, we have that

$$\begin{aligned}
\frac{\partial}{\partial s} f(0) &= \frac{\partial}{\partial s}\int_0^1 \langle \frac{\partial}{\partial t}\sigma(t,s), \frac{\partial}{\partial t}\sigma(t,s)\rangle dt \\
&= 2\int_0^1 \langle \overline{\nabla}_s \frac{\partial}{\partial t}\sigma(t,s), \frac{\partial}{\partial t}\sigma(t,s)\rangle dt \\
&= 2\int_0^1 \langle \overline{\nabla}_t \frac{\partial}{\partial s}\sigma(t,s), \frac{\partial}{\partial t}\sigma(t,s)\rangle dt \\
&= 2(\int_0^1 \frac{d}{dt}\langle \frac{\partial}{\partial s}\sigma(t,s), \frac{\partial}{\partial t}\sigma(t,s)\rangle dt - \int_0^1 \langle \frac{\partial}{\partial s}\sigma(t,s), \overline{\nabla}_t \frac{\partial}{\partial t}\sigma(t,s)\rangle dt \\
&= 2\int_0^1 \langle h(t), \overline{\nabla}_{\dot\alpha}\dot\alpha\rangle dt.
\end{aligned}$$

Then there exists $t_1 \in [0,1]$ such that $\langle h(t_1), \overline{\nabla}_{\dot{\alpha}}\dot{\alpha}(t_1)\rangle = 0$. Defining $\alpha_1(t) = \alpha(\frac{t}{2})$, for $t \in [0,1]$, and applying the same procedure, we can find $t_2 \in [0, \frac{1}{2}]$ such that $\langle h(t_2), \overline{\nabla}_{\dot{\alpha}}\dot{\alpha}(t_2)\rangle = 0$. This procedure gives a sequence $\{t_i\}$ such that

- t_i converges to 0,
- $\langle h(t_i), \overline{\nabla}_{\dot{\alpha}}\dot{\alpha}(t_i)\rangle = 0$.

Therefore $\langle h(0), \overline{\nabla}_{\dot{\alpha}}\dot{\alpha}(0)\rangle = 0$. Since $h(0)$ is an arbitrary horizontal vector, this shows that $\overline{\nabla}_{\dot{\alpha}}\dot{\alpha}(0)$ is a vertical vector.

□

Lemma 2.7 *Parallel transport along vertical trajectories preserves the horizontal distribution.*

Proof. Let $x(t) \in F_{y_0}$, and $h(0)$ a horizontal vector at $x(0)$, then there exists a unique vector field h along x such that $\overline{\nabla}_{\dot{x}}h = 0$. We claim that h is horizontal. To see this, let Y be a vertical vector field along x, which is in addition parallel, i.e. such that $\overline{\nabla}_{\dot{x}}Y = \nabla^F_{\dot{x}}Y = 0$. Then

$$\begin{aligned}\frac{d}{dt}\langle h, Y\rangle &= \langle \overline{\nabla}_{\dot{x}}h, Y\rangle + \langle h, \overline{\nabla}_{\dot{x}}Y\rangle \\ &= \langle h, \overline{\nabla}_{\dot{x}}Y\rangle = 0.\end{aligned}$$

Therefore $\langle h, Y\rangle$ is constant, and since $\langle h(0), Y(0)\rangle = 0$, $\langle h, Y\rangle = 0$ along x. Since we can choose a basis of vertical vector fields that in are also parallel for the vertical vector space *Vert*, h has to be perpendicular to *Vert*.

□

Corollary 2.8 *Let $h \in TM$ and $\overline{h}$ its horizontal lift to TW. Let v be a vertical vector field. Then*

1. $\overline{\nabla}_v \overline{h}$ *is horizontal.*

2. $[v, \overline{h}](p)$ *depends on* $\overline{h}(p)$ *and the extension of v in a neighborhood of p.*

Proof. The first statement is immediate from previous lemma. The second statement follows from the fact that $\overline{\nabla}$ is torsion free, thus $(\overline{\nabla}_{\overline{h}} v)^v = -[v, h]$.

□

Chapter 3

Regularity

To study the space of J-holomorphic curves in W, we need to study the Cauchy-Riemann operator. In section 3.1 we summarize some basic facts about the moduli space of J-holomorphic curves. In section 3.2, we adapt the previous constructions to our fibered setting, and we establish the the existence of a long exact sequence in cohomology, induced by the short exact sequence

$$0 \to Vert \to TW \to \pi^*TM \to 0,$$

see Corollary 3.5. The main result in section 3.3 is Proposition 3.3, that states that the set of regular fibered almost complex structures in W is dense in $\mathcal{J}_{fib}$.

3.1 Set Up

Let (X, J) be any almost complex manifold, Σ be a closed Riemann surface of genus g, and $\mathcal{J}(\Sigma)$ the space of almost complex structures on Σ. For $g > 1$, denote by $\mathcal{T}_g$ the Teichmüller space of Σ, which parameterizes $\mathcal{J}(\Sigma)$. Thus

$\dim_{\mathbb{R}} \mathcal{T}_g = 6g - 6$ and there exists a smooth mapping

$$j : \mathcal{T}_g \to \mathcal{J}(\Sigma)$$
$$\tau \mapsto j(\tau).$$

A map $u : (\Sigma, j(\tau)) \to (X, J)$ is called a $(J, j(\tau))$-holomorphic map, if

$$\overline{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j(\tau)) = 0.$$

For any homology class $A \in H_2(X, \mathbb{Z})$, we shall denote by $\mathcal{M}_g(A, X, J)$ the moduli space of pairs (u, τ), where u is a $(J, j(\tau))$-holomorphic, somewhere injective map that represent the homology class A, and $\tau \in \mathcal{T}_g$. If we denote by *Maps* the set of somewhere injective maps $f : \Sigma \to X$ that represent the homology class A, then we can think of $\overline{\partial}_J$ as a section of a bundle $\mathcal{E} \to Maps \times \mathcal{T}_g$, where the fiber at (u, τ) is the space $\mathcal{E}_{(u,\tau)} = \Omega^{0,1}(u^*TX)$ of smooth J-anti-linear 1-forms on Σ with values in u^*TX. Then $\mathcal{M}_g(A, X, J) = \overline{\partial}_J^{-1}(0)$.

In order for $\mathcal{M}_g(A, X, J)$ to be a manifold, $\overline{\partial}_J$ needs to be transversal to the zero section, this means that the image of the linearization of $\overline{\partial}_J$ at $(u, \tau) \in \mathcal{M}_g(A, X, J)$,

$$d\overline{\partial}_J : T_{(u,\tau)}(Maps \times \mathcal{T}_g) \longrightarrow T_{((u,\tau),0)}\mathcal{E} = T_{(u,\tau)}(Maps \times \mathcal{T}_g) \oplus \mathcal{E}_{(u,\tau)},$$

is complementary to the tangent space of the zero section $T_{(u,\tau)}(Maps \times \mathcal{T}_g)$, i.e., when we project $d\overline{\partial}_J(u)$ onto its vertical part in $\mathcal{E}_{(u,\tau)}$ we get a map of maximal rank,

$$\mathcal{L}\overline{\partial}_J(u, \tau) : C^\infty(u^*TX) \oplus T_\tau \mathcal{T}_g \longrightarrow \Omega^{0,1}(u^*TX).$$

This map is given by

$$\mathcal{L}\overline{\partial}_J(u,\tau)(\xi,s) = Du(\xi) + \frac{1}{2}J \circ du \circ (j_*(s)),$$

where

$$Du : C^\infty(u^*TX) \to \Omega^{0,1}(u^*TX)$$

is the linearization of the Cauchy-Riemann equation at u ([13], [14]).

3.2 Cauchy-Riemann Operator on Symplectic Fibrations

We want to study the space of J-holomorphic curves in W for classes $\hat{A}$ such that $\pi_*(\hat{A}) \neq 0$, and relate it with the space of J_M-holomorphic curves in M in class $A := \pi_*(\hat{A})$.

Let $u : \Sigma \to W$ be an embedded J-holomorphic curve in class $\hat{A}$, such that $\pi_*\hat{A} = A \neq 0$ and $v = \pi \circ u : \Sigma \to M$ is an embedding. Recall that the short exact sequence of vector bundles

$$0 \to Vert \to TW \to \pi^*TM \to 0$$

splits. Then the complex bundle (u^*TW, J) splits as the direct sum

$$(u^*\mathit{Vert}, J_F) \oplus (v^*TM, J_M),$$

and consequently

$$\Omega^{0,1}(u^*TW, J) = \Omega^{0,1}(u^*\mathit{Vert}, J_F) \oplus \Omega^{0,1}(v^*TM, J_M).$$

Theorem 3.1 *Let $u : \Sigma \to W$ be as above, then the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\infty(u^* Vert) & \longrightarrow & C^\infty(u^*TW) & \longrightarrow & C^\infty(v^*TM) & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle D_F u}\downarrow & & {\scriptstyle Du}\downarrow & & {\scriptstyle Dv}\downarrow & & \\ 0 & \longrightarrow & \Omega^{0,1}(u^* Vert) & \longrightarrow & \Omega^{0,1}(u^*TW) & \longrightarrow & \Omega^{0,1}(v^*TM) & \longrightarrow & 0, \end{array}$$

where Dv is the linearization of the Cauchy-Riemann equation at v and $D_F u = Du|_{C^\infty(u^* Vert)}$.

Proposition 3.2 below proves the commutativity of the second square, and Proposition 3.4 proves the commutativity of the first square.

Proposition 3.2 *Let $\xi_1 \in C^\infty(v^*TM)$, $\overline{\xi}_1$ be its horizontal lift to TW and $X \in T\Sigma$. Then $\pi_* Du(\overline{\xi}_1)(X) = Dv(\xi_1)(X)$.*

Proof. It is not hard to prove that for every $X \in T\Sigma$ and $\xi_1 \in C^\infty(v^*TM)$, there exist vector fields $\tilde{\xi_1}$, $\tilde{X}$ and $\tilde{jX}$ in M that extend ξ_1, $dv(X)$ and $dv(jX)$, respectively, in a neighborhood of $v(\Sigma) \subseteq M$, with the property that $[\tilde{\xi_1}, \tilde{X}] = [\tilde{\xi_1}, \tilde{jX}] = 0$ (see [11]). By slight abuse of notation we write ξ_1, $dv(X)$ and $dv(jX)$ for the extensions to M. Then a formula for the operator

$$Dv : C^\infty(v^*TM) \to \Omega^{0,1}(v^*TM),$$

is given by

$$\begin{aligned} Dv(\xi_1)(X) &= \frac{1}{2}(\nabla_{\xi_1} dv(X) + \nabla_{\xi_1}(J dv(jX))) \\ &= \frac{1}{2}(\nabla_{dv(X)}\xi_1 + J_M \nabla_{dv(jX)}\xi_1 + (\nabla_{\xi_1} J_M)(dv(jX)), \end{aligned}$$

whose horizontal lift to TW is

$$\overline{Dv(\xi_1)(X)} = \frac{1}{2}(\overline{\nabla_{dv(X)}\xi_1 + J_M\nabla_{dv(jX)}\xi_1 + (\nabla_{\xi_1}J_M)(dv(jX))}).$$

On the other hand,

$$Du(\overline{\xi}_1)(X) = \frac{1}{2}(\overline{\nabla}_{\overline{\xi_1}}du(X) + \overline{\nabla}_{\overline{\xi_1}}(Jdu(jX))).$$

Define $X^v = du(X) - \overline{dv(X)}$, then

$$Du(\overline{\xi}_1)(X) = \frac{1}{2}(\overline{\nabla}_{\overline{\xi_1}}X^v + \overline{\nabla}_{\overline{\xi_1}}\overline{dv(X)} + \overline{\nabla}_{\overline{\xi_1}}(J(jX)^v) + \overline{\nabla}_{\overline{\xi_1}}(J\overline{dv(jX)})).$$

Since $\overline{\nabla}$ is torsion-free,

$$\begin{aligned} Du(\overline{\xi}_1)(X) = \frac{1}{2}(&\overline{\nabla}_{X^v}\overline{\xi}_1 + \overline{\nabla}_{J(jX)^v}\overline{\xi}_1 + [\overline{\xi}_1, X^v] + [\overline{\xi}_1, J(jX)^v] \\ &+ \overline{\nabla}_{\overline{dv(X)}}\overline{\xi}_1 + J\overline{\nabla}_{\overline{dv(jX)}}\overline{\xi}_1 + (\overline{\nabla}_{\overline{\xi}_1}J)(\overline{dv(jX)}) \\ &+ [\overline{\xi}_1, \overline{dv(X)}] + J[\overline{\xi}_1, \overline{dv(jX)}]). \end{aligned}$$

Now we use Lemma 2.5 (2.1) and the fact that $\overline{\nabla}_{X^v}\overline{\xi}_1 + \overline{\nabla}_{J(jX)^v}\overline{\xi}_1 = 0$,

$$\begin{aligned} Du(\overline{\xi}_1)(X) = \frac{1}{2}(&\overline{\nabla_{dv(X)}\xi_1} + \overline{J_M\nabla_{dv(jX)}\xi_1} + (\overline{\nabla}_{\overline{\xi}_1}J)(\overline{dv(jX)}) \\ &+ [\overline{\xi}_1, X^v] + [\overline{\xi}_1, J(jX)^v] + [\overline{\xi}_1, \overline{dv(X)}] + J[\overline{\xi}_1, \overline{dv(jX)}] \\ &+ \frac{1}{2}[\overline{dv(X)}, \overline{\xi}_1]^v + \frac{1}{2}J[\overline{dv(jX)}, \overline{\xi}_1]^v), \end{aligned}$$

and also

$$(\overline{\nabla}_{\overline{\xi}_1}J)(\overline{dv(jX)}) = \overline{(\nabla_{\xi_1}J)(dv(jX))} + \frac{1}{2}[\overline{\xi}_1, \overline{J_M dv(jX)}]^v - \frac{1}{2}J[\overline{\xi}_1, \overline{dv(jX)}]^v.$$

Then

$$
\begin{aligned}
Du(\overline{\xi}_1)(X) &= \frac{1}{2}(\overline{\nabla_{dv(X)}\xi_1 + J_M\nabla_{dv(jX)}\xi_1 + (\nabla_{\xi_1}J)(dv(jX))} \\
&+ \frac{1}{2}[\overline{\xi}_1, \overline{J_M dv(jX)}]^v - \frac{1}{2}J[\overline{\xi}_1, \overline{dv(jX)}]^v \\
&+ [\overline{\xi}_1, X^v] + [\overline{\xi}_1, J(jX)^v] + [\overline{\xi}_1, \overline{dv(X)}] + J[\overline{\xi}_1, \overline{dv(jX)}] \\
&+ \frac{1}{2}[\overline{dv(X)}, \overline{\xi}_1]^v + \frac{1}{2}J[\overline{dv(jX)}, \overline{\xi}_1]^v) \\
&= \overline{Dv(\xi_1)(X)} + \frac{1}{2}[\overline{\xi}_1, X^v] + \frac{1}{2}J[\overline{\xi}_1, (jX)^v],
\end{aligned}
$$

the last equality follows from the fact that we assumed that $[\xi_1, dv(X)] = 0 = [\xi_1, dv(jX)]$.

□

Remark 3.3 $[\overline{\xi}_1, X^v]$ *depends on* $\overline{\xi}_1(u(z))$ *and the extension of* X^v *along* $\overline{\xi}_1$. *It is, therefore, an operator of order* 0. *Analogously,* $[\overline{\xi}_1, J(jX)^v]$ *is an operator of order* 0.

Proposition 3.4 *Let* $\xi_2 \in C^\infty(u^*V)$ *and* $X \in T\Sigma$. *Then* $Du(\xi_2)(X) \in C^\infty(u^*Vert)$.

Proof.

$$Du\xi_2(X) = \frac{1}{2}(\nabla_{\xi_2}du(X) + \nabla_{\xi_2}(Jdu(jX))),$$

as before, let $X^v = du(X) - \overline{dv(X)}$,

$$Du\xi_2(X) = \frac{1}{2}(\overline{\nabla}_{\xi_2}X^v + \nabla_{\xi_2}(J(jX)^v) + \overline{\nabla}_{\xi_2}\overline{dv(X)} + \overline{\nabla}_{\xi_2}(J\overline{dv(jX)})),$$

using that $\overline{\nabla}$ is torsion free, then

$$
\begin{aligned}
Du\xi_2(X) &= \frac{1}{2}(\overline{\nabla}_{X^v}\xi_2 + J\overline{\nabla}_{(jX)^v}\xi_2 + (\overline{\nabla}_{\xi_2}J)(jX)^v + \overline{\nabla}_{\overline{dv(X)}}\xi_2 \\
&\quad + \nabla_{J\overline{dv(jX)}}\xi_2 + [\xi_2, X^v] + J[\xi_2, (jX)^v] + [\xi_2, \overline{dv(X)}] + [\xi_2, J\overline{dv(jX)}]) \\
&= \frac{1}{2}(\overline{\nabla}_{X^v}\xi_2 + J\overline{\nabla}_{(jX)^v}\xi_2 + (\overline{\nabla}_{\xi_2}J)(jX)^v + [\xi_2, X^v] + J[\xi_2, (jX)^v]).
\end{aligned}
$$

By Lemma 2.7, $\overline{\nabla}_{X^v}\xi_2$, $\overline{\nabla}_{(jX)^v}\xi_2$ and $(\overline{\nabla}_{\xi_2}J)(jX)^v$ are vertical vectors. Therefore $Du\xi_2(X) \in C^\infty(u^*V)$.

□

Therefore $Du(\xi) = D_M u(\overline{\xi}_1) + D_F u(\xi_2)$, where

$$D_M u = Du|_{C^\infty(v^*TM)} : C^\infty(v^*TM) \longrightarrow \Omega^{0,1}(u^*TW)$$

and

$$D_F u = Du|_{C^\infty(u^*\mathit{Vert})} : C^\infty(u^*\mathit{Vert}) \longrightarrow \Omega^{0,1}(u^*\mathit{Vert}).$$

By Proposition 3.2, $D_M u = D^h_M u + D^v_M u$, where $D^h_M u(\overline{\xi}_1) = \overline{Dv(\xi_1)}$ and $D^v_M u : C^\infty(v^*TM) \to C^\infty(u^*\mathit{Vert})$ is compact of order 0.

Theorem 3.1 implies the following

Corollary 3.5 *Let $u : \Sigma \to W$ be a J-holomorphic map representing the class $\hat{A}$ such that $\pi_*(\hat{A}) \neq 0$, then we have the following long exact sequence*

$$0 \to H^0(\Sigma, u^*\mathit{Vert}) \to H^0(\Sigma, u^*TW) \to H^0(\Sigma, v^*TM) \xrightarrow{\delta}$$

$$H^1(\Sigma, u^*\mathit{Vert}) \to H^1(\Sigma, u^*TW) \to H^1(\Sigma, v^*TM) \to 0,$$

where $\delta = D^v_M u$.

□

In this language, $H^1(\Sigma, u^*TW) = 0$ is equivalent to the surjectivity of $\mathcal{L}\overline{\partial}_J(u,\tau) = Du + \frac{1}{2}J \circ du \circ j_*$. Then the statement $\mathcal{L}\overline{\partial}_J(u,\tau)$ is surjective if and only if $Dv + \frac{1}{2}J_M \circ du \circ j_*$ and $D_M^v u + D_F u + \frac{1}{2}(J \circ du \circ j_*)^v$ are surjective is equivalent to $H^1(\Sigma, u^*TW) = 0$ if and only if $H^1(\Sigma, v^*TM)$=0 and δ is onto. For example, if $c_1(\mathit{Vert}) \cdot \hat{A} \geq 2g - 1$, then D_F is automatically surjective ([5]), i.e. $H^1(\Sigma, u^*\mathit{Vert}) = 0$. Then $\mathcal{L}\overline{\partial}_J(u,\tau)$ is surjective if and only if $Dv + \frac{1}{2}J_M \circ du \circ j_*$ is.

3.3 Regularity of Fibered Almost Complex Structures

As above, we denote by $\mathcal{M}_g(\hat{A}, W, J)$ the moduli space of pairs (u, τ), where u is a somewhere injective $(j(\tau), J)$-holomorphic map $\Sigma \to W$, such that $[u(\Sigma)] = \hat{A}$,and $\tau \in \mathcal{T}_g$, and by $\mathcal{M}_g(A, M, J_M)$ the corresponding moduli space of J_M-holomorphic maps in M representing the class A, and $\tau \in \mathcal{T}_g$. We say that a class $\hat{A}$ is simple, if is not a multiple of any other class.

Definition 3.6 *Let $(u,\tau) \in \mathcal{M}_g(\hat{A}, W, J)$, we say that (u, τ, J) is* regular *if $\mathcal{L}\overline{\partial}_J(u,\tau)$ is onto.*

Let

$$\mathcal{J}_{reg}(\hat{A}) = \{J | \mathcal{L}\overline{\partial}_J(u,\tau) \text{ is onto for every } (u,\tau) \in \mathcal{M}_g(\hat{A}, W, J)\},$$

$$\mathcal{J}_{reg} = \cap_{\hat{A}} \mathcal{J}_{reg}(\hat{A}),$$

and denote by $\mathcal{M}_g(\hat{A}, W, \mathcal{J}_{fib})$ the set

$$\{(u, \tau, J) \in Maps \times \mathcal{T}_g \times \mathcal{J}_{fib} \mid u \text{ is } J\text{-}holomorphic\}.$$

Proposition 3.7 *Let $\hat{A} \in H_2(W, \mathbb{Z})$ be a simple class such that $\pi_* \hat{A} \neq 0$, then $\mathcal{M}_g(\hat{A}, W, \mathcal{J}_{fib})$ is a smooth Banach manifold. Furthermore, the regular fibered almost complex structures on W form a set of the second category in $\mathcal{J}_{\{\}\lfloor}$.*

Proof. This proof is a modification of the one of Proposition 3.4.1 in [13]. Let $J \in \mathcal{J}_{fib}$. The tangent space $T_J \mathcal{J}_{fib}$ consists of sections of the bundle $End(TW, J)$, whose fiber at any point $p \in W$ is the space of linear maps $Y : T_pW \to T_pW$ of the form $Y = Y_1 + Y_2 + \theta$, where $Y_1 : (\pi^*TM)_p \to (\pi^*TM)_p$, $Y_2 : Vert_p \to Vert_p$, $\theta : (\pi^*TM)_p \to Vert_p$, and $JY + YJ = 0$ (this happens if and only if, $J_iY_i + Y_iJ_i = 0$, for $i = 1, 2$ and θ is a (J_M, J_F)-anti-linear homomorphism, see Remark 2.4).

As in Section 3.1, let $\mathcal{E} \longrightarrow Maps \times \mathcal{T}_g \times \mathcal{J}$ be the bundle whose fiber at (u, τ, J) is the space $\mathcal{E}_{(u,\tau,J)} = \Omega^{0,1}(u^*TW)$. Then we can define a section $\mathcal{F}$ of this bundle by $\mathcal{F}(u, \tau, J) = \overline{\partial}_J(u)$. We need to prove that

$$D\mathcal{F}(u, \tau, J) : C^\infty(u^*TW) \times T_\tau \mathcal{T}_g \times End(TW, J) \longrightarrow \Omega^{0,1}(u^*TW)$$

$$D\mathcal{F}(u, \tau, J)(\xi, s, Y) = Du(\xi) + \frac{1}{2} J \circ du \circ j_*(s) + \frac{1}{2} Y(u) \circ du \circ j(\tau)$$

is onto for all $J \in \mathcal{J}_{fib}$ and for all $(u, \tau) \in \mathcal{M}(\hat{A}, W, J)$, where $D\mathcal{F}$ is the linearization of $\mathcal{F}$ composed with the projection onto $\mathcal{E}_{(u,\tau,J)}$. Since Du is

Fredholm we only need to prove that the image of $D\mathcal{F}(u,\tau,J)$ is dense, i.e. that for every $\eta \in \Omega^{0,1}(u^*TW)$ there exist $Y \in End(TW,J)$ and $z \in S^2$ such that $\langle Y(u(z)) \circ du(z) \circ j, \eta(z)\rangle \neq 0$.

Let $z \in S^2$ be such that $du(z) \neq 0$ and $u^{-1}u(z) = \{z\}$. Such a point exists since we are considering somewhere injective maps.

Observe that if Du is not surjective, then there exists $\eta \in \Omega^{0,1}(u^*TW)$ such that $\langle Du(\xi), \eta\rangle = 0$. Since $\eta = \eta^v + \eta^h$, we can consider each part separately. If $\eta \in \Omega^{0,1}(u^*\pi^*TM)$, it is easy to find $Y_1(u(z)) : (\pi^*TM)_{u(z)} \to (\pi^*TM)_{u(z)}$ for which $\langle Y_1(u(z)) \circ \overline{dv_z} \circ j, \eta(z)\rangle \neq 0$. If $\eta \in \Omega^{0,1}(u^*Vert)$,

$$\langle Y(u) \circ du \circ j, \eta\rangle = \langle Y_2(u) \circ (du - \overline{dv}) \circ j, \eta\rangle + \langle \theta \circ \overline{dv} \circ j, \eta\rangle.$$

Then we have two cases. First, suppose that $du(X) - \overline{dv(X)} \neq 0$, then it is easy to see that there exists $Y_2 : Vert_{u(z)} \to Vert_{u(z)}$, such that $\langle Y_2(u(z)) \circ (du_z - \overline{dv_z}) \circ j, \eta(z)\rangle \neq 0$. Second, if for all z's as above, $du_z(X) - \overline{dv_z(X)} = 0$ for every $X \in T\Sigma$, that is, if du is tangent to the horizontal distribution. Then we can perturb the horizontal distribution to destroy the tangency at some point, this is equivalent to find $\theta : (\pi^*TM)_{u(z)} \to Vert_{u(z)}$ such that $\langle \theta \circ \overline{dv_z} \circ j, \eta(u(z))\rangle \neq 0$ (see Remark 2.4).

As in [13], Theorem 3.1.2(ii), we can consider the projection

$$\mathcal{M}_g(\hat{A}, W, \mathcal{J}_{fib}) \longrightarrow \mathcal{J}_{fib}.$$

A regular value J of this projection is an almost complex structure such that $\mathcal{L}\overline{\partial}_J(u,\tau)$ is onto for every $(u,\tau) \in \mathcal{M}_g(\hat{A}, W, J)$. By the Sard-Smale theorem, the set of regular and fibered almost complex structures is of the second category (i.e. countable intersection of open and dense sets).

□

Now let J_M be a generic almost complex structure on M, $v : \Sigma \to M$ be a J_M holomorphic curve, and $C = v(\Sigma)$. Let $X_C = \pi^{-1}(C) \subset W$.

Definition 3.8 *Let J_M and C be as above. The set of $J \in \mathcal{J}_{fib}$ such that $\pi_* J = J_M \pi_*$ will be denoted by $\mathcal{J}_{fib,J_M}$. We say that $J \in \mathcal{J}_{fib,J_M} \cap \mathcal{J}_{reg}$ is* super regular *for C, if $J|_{X_C}$ is also regular.*

Then we have the following

Lemma 3.9 *Given J_M regular and C as above, the set of super regular almost complex structures for C in $\mathcal{J}_{fib,J_M}$ is of the second category in $\mathcal{J}_{fib,J_M}$.*

Proof. First of all, $\mathcal{J}_{fib,J_M} \cap \mathcal{J}_{reg}$ is a set of the second category. To see this one can apply the same argument as in the proof of Proposition 3.7. In fact, if J_M is regular, then $D^h_M u + \frac{1}{2}(J \circ du \circ j_*)^h = Dv + \frac{1}{2} J_M \circ dv \circ j_*$ is surjective and then we can cover $\Omega^{0,1}(u^* Vert)$ with variations in J of the form $Y = Y_2 + \theta$ (see Proposition 3.7).

Second, $X_C := \pi^{-1}(C) \subset W$ is a symplectic fibration over C with fiber (F, σ). As above, the set of regular fibered almost complex structures on X_C is a set of the second category. Note also that we have a map

$$\mathcal{J}^W_{fib,J_M} \longrightarrow \mathcal{J}^{X_C}_{fib}$$

given by

$$J \mapsto J|_{X_C}.$$

This map is onto, since we can extend any symplectic connection on X_C to a symplectic connection on W, which implies that we can extend any fibered almost complex structures on X_C to a fibered almost complex structure on W (see Remark 2.4). Therefore the set of super regular almost complex structures for C is the interesection of $\mathcal{J}_{fib,J_M} \cap \mathcal{J}_{reg}$ with the inverse image of $\mathcal{J}_{fib}^{X_C} \cap \mathcal{J}_{reg}^{X_C}$.

□

Notice, that this lemma does not imply that there exist $J \in \mathcal{J}_{fib} \cap \mathcal{J}_{reg}$ such that for every J_M-holomorphic curve C, $J|_{X_C}$ is regular. In fact, it is possible for a $J \in \mathcal{J}_{fib} \cap \mathcal{J}_{reg}$ to admit a J-holomorphic curve $\hat{C} \subseteq W$ such that $\hat{C}$ represents a class in X_C that would not be represented for generic almost complex structure in X_C (see Example 2 at the end of Chapter 4).

If J is generic then $\mathcal{M}_g(\hat{A}, W, J)$ is a smooth manifold of dimension

$$2(n+1-3)(1-g) + 2c_1(W) \cdot \hat{A} + \dim G_g$$

and

$$\dim \mathcal{M}_g(A, M, J_M) = 2(n-3)(1-g) + 2c_1(M) \cdot A + \dim G_g,$$

where G_g is the reparameterization group. Thus $G_0 = PSL(2, \mathbb{C})$, G_1 is the extension of $SL(2, \mathbb{Z})$ by the torus T^2, and for $g \geq 2$, G_g is the mapping class group. There is an obvious map $pr : \mathcal{M}_g(\hat{A}, W, J) \longrightarrow \mathcal{M}_g(A, M, J_M)$, given by $pr(u) = \pi \circ u$.

Proposition 3.10 *Assume A is a simple class. If $J \in \mathcal{J}_{reg} \cap \mathcal{J}_{fib}$ then $J_M \in \mathcal{J}_{reg}(A)$. Furthermore, If $c_1(Vert) \cdot \hat{A} = g - 1$ then pr is orientation preserving.*

Proof. The first statement is an obvious consequence of Corollary 3.5. The hypothesis $c_1(Vert) \cdot \hat{A} = g - 1$ implies that

$$\dim \mathcal{M}_g(\hat{A}, W, J) = \dim \mathcal{M}_g(A, M, J_M).$$

Recall that an orientation of $\mathcal{M}_g(\hat{A}, W, J)$ is just a nowhere vanishing section of the complex line bundle $\det(Du) = \bigwedge^{max} \text{Ker } Du$ (up to multiplication by positive functions). We let

$$Du^J = \frac{1}{2}(Du - JDuJ),$$

that is a J-linear operator between Banach spaces ([14]). Given the decomposition of Du into its vertical and horizontal parts, $Du^J = (D^h_M u)^J + (D^v_M u)^J + D_F u^J$. Moreover, $Du = Du^J + Z_u$, where Z_u is a 0-th order term. Let

$$Du^t = (D^h_M u)^J + D_F u^J + t(D^v_M u)^J + tZ_u.$$

Then $\det(Du^t)$ $=\det(Du^0)$ for every t. On the other hand, $\text{Ker}D^h_M u$ and $\text{Ker}D_F u$ are complex vector spaces and their complex structure induce the complex structure on $\text{Ker}(Du)$. Therefore the canonical orientations of $\det(D^h_M u)$ and $\det(D_F u)$ induce the canonical orientation of $\det(Du)$.

□

Chapter 4

Gromov Invariants

In this section we prove the main theorem of this paper, Theorem 4.2. We say that $A \in H_2(M, \mathbb{Z})$ is a simple class if it is not a multiple of any other class. Theorem 4.2 implies that if there are non-zero Gromov invariants on M in the simple class A, then we can find non-zero Gromov invariants on W in the class $\hat{A}$, provided that $\pi_* \hat{A} = A$ and $c_1(Vert) \cdot \hat{A} \geq g - 1$. We shall use the language of [13] throughout this section.

4.1 Main Theorem

Let us recall the definition of Gromov invariants for a symplectic manifold (X, ω) of dimension $2n$. Let J be an almost complex structure on X tamed by ω, and let $A \in H_2(X, \mathbb{Z})$ be a simple class. (X, ω, J) is called *weakly monotone* if for every spherical homology class $B \in H_2(X, \mathbb{Z})$

$$\omega(B) > 0 \text{ and } c_1(X) \cdot B \geq 3 - n \text{ imply } c_1(X) \cdot B \geq 0.$$

The Gromov invariants of (X, ω) are a collection of homomorphisms

$$\Phi_{A,g} : H_d(X^p, \mathbb{Z})/Tor \to \mathbb{Z}$$

defined as follows. First of all, observe that the reparameterization group G_g acts on $\mathcal{M}_g(A, X, J) \times \Sigma^p$, by $\phi \cdot (u, z_1, \ldots, z_p) = (u \circ \phi^{-1}, \phi(z_1), \ldots, \phi(z_p))$, $\phi \in G_g$. Since $u \circ \phi^{-1}$ has the same image as u, let us divide by this action. Let

$$ev_{p,X} : \mathcal{M}_g(A, X, J) \times_{G_g} (\Sigma)^p \longrightarrow X^p$$

be the evaluation map given by,

$$ev_{p,X}([u, z_1, \ldots, z_p]) = (u(z_1), \ldots, u(z_p)).$$

If (X, ω) is weakly monotone and A is a simple class, then

$$ev_{p,X}(\mathcal{M}_g(A, X, J) \times_{G_g} (\Sigma)^p)$$

can be compactified by adding spaces of *cusp curves* of codimension less or equal to 2. In this case $ev_{p,X}$ defines a pseudo-cycle of dimension

$$2(n-3)(1-g) + 2c_1(X) \cdot A + 2p,$$

and therefore a homology class in X^p. Let $\alpha \in H_d(X^p, \mathbb{Z})/Tor$ be represented by a pseudo-cycle $f : D \to X^p$ transverse to $ev_{p,X}$. Then we set

$$\Phi_{A,g}(\alpha) := \begin{cases} ev_{p,X} \cdot f & \text{if } d = 2np \text{ - } \dim \mathcal{M}(A, X, J) \times_G (\Sigma)^p, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

If $\alpha = \alpha_1 \times \cdots \times \alpha_p$, with $\alpha_i \in H_*(X, \mathbb{Z})$, then we write $\Phi_A(\alpha_1, \cdots, \alpha_p)$ for $\Phi_A(\alpha)$.

Given the fact that we can compactify $ev_{p,X}(\mathcal{M}_g(A, X, J) \times_{G_g} (\Sigma)^p)$ with spaces of codimension greater or equal to 2, then one can prove that $\Phi_{A,g}(\alpha)$ is independent of the choice of generic J and of the pseudo-cycle representing α. Furthermore, $\Phi_{A,g}(\alpha)$ is an invariant of the deformation class of ω. If A is not a simple class, i.e. if $A = mB$ with $m > 1$ and $c_1(X) \cdot B > 0$, then multiply covered curves representing B form a space of codimension ≥ 2 in $ev_{p,X}(\mathcal{M}_g(A, X, J) \times_{G_g} (\Sigma)^p)$ and so do not contribute to $\Phi_{A,g}$. However, if $c_1(X) \cdot B = 0$, the moduli spaces $\mathcal{M}_g(A, X, J)$ and $\mathcal{M}_g(B, X, J)$ have the same virtual dimension and multiple covers of B may contribute to $\Phi_{A,g}$. In dimension 4, the classes B with this property are the ones that can be represented by embedded curves of genus $g = 1$ and with self intersection $B \cdot B = 0$ ([9], [13]).

The next Proposition is taken from [14], and expresses some basic properties of these invariants.

Proposition 4.1 *(Ruan-Tian) Let (X, ω) be a weakly monotone symplectic manifold, if A is a simple class, then*

1. $\Phi_{A,g}(\alpha_1, \cdots, \alpha_p) = 0$, if dim $\alpha_p = 2n$.

2. $\Phi_{A,g}(\alpha_1, \cdots, \alpha_p) = (\alpha_p \cdot A)\Phi_{A,g}(\alpha_1, \cdots, \alpha_{p-1})$, if dim $\alpha_p = 2n - 2$.

□

In what follows, M will denote a symplectic 4-manifold and $F = S^2$, therefore W will be a 6-dimensional manifold. From the dimensional condition

(4.1), if $\hat{A} \in H_2(W, \mathbb{Z})$ is a simple class, then $\Phi_{\hat{A},g}(\hat{\alpha}_1, \ldots, \hat{\alpha}_p) \neq 0$ only if

$$d = \sum_{i=1}^{p} \dim\, \hat{\alpha}_i = 4p - 2c_1(W) \cdot \hat{A}. \tag{4.2}$$

Similarly, if A is a simple class in M, then $\Phi_{A,g}(\alpha_1, \ldots, \alpha_p) \neq 0$ only if

$$d = \sum_{i=1}^{p} \dim\, \alpha_i = 2p - 2(g-1) - 2c_1(M) \cdot A. \tag{4.3}$$

Notice that, by Proposition 4.1, the relevant Gromov invariants of M occur when $\alpha_i = [pt] \in H_0(M, \mathbb{Z})$, $i = 1, \ldots p$. Let $\{x_1, \ldots x_p\}$ be a set of generic points in M, then according with (4.3),

$$\Phi_{A,g}(x_1, \ldots, x_p) \neq 0 \text{ only if } p = (g-1) + c_1(M) \cdot A,$$

that is, when $ev_{p,M}$ is a pseudo-cycle of the same dimension as M^p.

Let $p_1 = c_1(M) \cdot A + (g-1)$ and $p_2 = c_1(Vert) \cdot \hat{A} - (g-1)$. Notice that if $\hat{A}$ is a class in W, such that $\pi_*(\hat{A}) = A$, then $c_1(W) \cdot \hat{A} = c_1(M) \cdot A + c_1(Vert) \cdot \hat{A}$, thus $\dim\, \mathcal{M}(\hat{A}, W, J)/G_g = 2c_1(W) \cdot \hat{A} = 2p_1 + 2p_2$. Observe that to have some relation between the invariants in M and the invariants in W there is an obvious numerical constraint: from (4.2) we see that

$$p = \frac{1}{4}\left(\sum_{i=1}^{p} \dim\, \hat{\alpha}_i + 2p_1 + 2p_2\right)$$

and from (4.3)

$$p = \frac{1}{2}\left(\sum_{i=1}^{p} \dim\, \pi_* \hat{\alpha}_i + 2p_1\right).$$

Therefore

$$\sum_{i=1}^{p} \text{dim } \hat{\alpha}_i - 2\text{dim } \pi_*\hat{\alpha}_i = 2p_1 - 2p_2. \tag{4.4}$$

Let $B_j \in H_2(M, \mathbb{Z})$ and $\hat{B}_j \in H_2(W, \mathbb{Z})$, for $j = 1, \ldots n$ be such that $\pi_*(\hat{B}_j) = B_j$, for every j. Take m generic points y_i in W, for $i = 1, \ldots m$, l generic fibers F_k, for $k = 1, \ldots, l$, and let $x_i = \pi(y_i)$, for $i = 1, \ldots m$ and $x_{m+k} = \pi(F_k)$, for $k = 1, \ldots, l$.

Theorem 4.2 *Let $A \in H_2(M, \mathbb{Z})$ be a simple class, and let $\hat{A} \in H_2(W, \mathbb{Z})$ be a class such that $\pi_*\hat{A} = A$ and $c_1(Vert) \cdot \hat{A} \geq g - 1$. Then*

$$\Phi_{\hat{A},g}(y_1, \ldots y_m, F_1, \cdots, F_l, \hat{B}_1, \cdots, \hat{B}_n) = 2^g \Phi_{A,g}(x_1, \ldots, x_{m+l}, B_1, \cdots, B_n), \tag{4.5}$$

provided that $2m + n + l = c_1(W) \cdot \hat{A}$ and $m + l \geq p_1$. If $m + l > p_1$, then both sides are equal to 0.

Proof. We choose n pseudo-cycles (D_i, f_i) to represent $\hat{B}_i$, where $f_j : D_j \to W$, $[f_j(D_j)] = \hat{B}_j$, and l pseudo-cycles (E_k, g_k) to represent F, where $g_k : E_k \to W$, $[g_k(E_k)] = F$ and $\pi(g_k(E_k)) = pt$. Then we can represent

$$\Pi pt \times \Pi \hat{B}_j \times \Pi F_k$$

by

$$(D = \Pi y_i \times \Pi D_j \times \Pi E_k, R = \Pi f_j \times \Pi g_k),$$

and

$$\Pi pt \times \Pi B_j$$

by

$$(\Pi x_i \times \Pi B_j, r = \pi^p \circ R).$$

Recall that there is a map $pr : \mathcal{M}(\hat{A}, J, W) \to \mathcal{M}(A, J_M, M)$. We can extend it to a map $\mathcal{M}(\hat{A}, J, W) \times_G \Sigma^p \longrightarrow \mathcal{M}(A, J_M, M) \times_G \Sigma^p$ and obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(\hat{A}, J, W) \times_G \Sigma^p & \xrightarrow{ev_{p,W}} & W^p \\ {\scriptstyle pr}\big\downarrow & & \big\downarrow{\scriptstyle \pi^p} \\ \mathcal{M}(A, J_M, M) \times_G \Sigma^p & \xrightarrow{ev_{p,M}} & M^p. \end{array}$$

We can apply Lemma 4.3 below with $e = ev_{p,W}$ and $f = R$. Then we can assume that $ev_{p,W}$ and R are transverse, as well as $ev_{p,M}$ and $r = \pi^p \circ R$, and that in M intersection with $B_1, \ldots, B_n$ occurs away from $B_i \cap B_j$. .

If $m + l > c_1(M) \cdot A + (g-1) = p_1$, then we can perturb r and R such that

$$ev_{p,M}(\mathcal{M}(A, J_M, M) \times_G \Sigma^p) \cap h(D) = \emptyset.$$

Therefore both sides of (4.5) are equal to 0.

For the rest of the proof we shall assume that $p_1 = c_1(M) \cdot A + (g-1) = m + l$. Notice that by hypothesis, $c_1(W) \cdot \hat{A} = p_1 + p_2 = 2m + n + l$. Then $p_2 = m + l$. In this case

$$ev_{p,M}(\mathcal{M}(A, M, J_M)) \cap r(D)$$

consists of a finite collection of points,

$$[v, z_1, \cdots, z_p] \in \mathcal{M}(A, M, J_M) \times_G \Sigma^p$$

such that $v(z_i) = x_i$, $i = 1, \dots m$, $v(z_{m+j}) \in \pi(f_j(D_j))$, $j = 1, \dots, n$, and $v(z_{m+n+k}) \in \pi(g_k(E_k))$, $k = 1, \dots, l$. Since these intersections occur away from $B_i \cap B_j$, then we can assume that all points $z_1, \dots z_p$ are distinct. Let $C = v(\Sigma)$ and $X_C = \pi^{-1}(C)$. By Lemma 3.9 we can assume that J is super regular for these C's. If $u : \Sigma \to W$ is a J-holomorphic map, with $u(\Sigma) = \hat{C}$ such that C is its projection onto M, then $\hat{C}$ lies in the ruled surface $\pi^{-1}(C) = X_C$. It follows that $\hat{C}$ is a representative of the section class $\hat{A}$. Therefore, roughly speaking, to prove the theorem it is enough to count the number of J-holomorphic curves representing $\hat{A}$ in X_C.

The inverse image of $[v, z_1, \cdots, z_p]$ under pr is the set of points $[u, w_1, \cdots, w_p] \in \mathcal{M}_g(\hat{A}, W, J) \times_G \Sigma^p$, such that $\pi \circ u \circ \phi = v$ and $\phi(w_i) = z_i$, for some $\phi \in G$, for every $i = 1, \dots p$. Up to reparameterization, this is the set of sections of the bundle $X_C \to C$ that represent the class $\hat{A}$. We know that its virtual dimension is

$$2\hat{A} \cdot \hat{A} + 2(1-g) = 2c_1(Vert) \cdot \hat{A} + 2(1-g) = 2p_2.$$

Therefore

$$pr^{-1}(\{v\} \times \Sigma^p) = \mathcal{M}(\hat{A}, X_C, J|_{X_C}) \times_{G_g} \Sigma^p$$

has dimension $2p_2 + 2p$.

On the other hand,

$$(\pi^p)^{-1}(C^p) = X_C^p \subseteq W^p,$$

and so

$$ev_{p,W}|_{pr^{-1}(\{v\} \times \Sigma^p)} : \mathcal{M}(\hat{A}, X_C, J|_{X_C}) \times_G \Sigma^p \to X_C^p.$$

Observe that, $X_C \cap f_j(D_j)$ is a set of points, $j = 1, \ldots, n$, and $X_C \cap g_k(E_k) = g_k(E_k)$, $k = 1, \ldots, l$. Hence

$$R(D) \cap X_C^p = \Pi y_i \times \Pi(X_C \cap f_j(D_j)) \times \Pi(X_C \cap g_k(E_k)),$$

has dimension $2l$. Therefore, $ev_{p,W}(pr^{-1}(\{v\} \times \Sigma^p))$ and $R|_{R^{-1}(R(D)\cap X_C^p)}$ are pseudo-cycles of complementary dimension in X_C^p and

$$ev_{p,W}(pr^{-1}(\{v\} \times \Sigma^p)) \cap (R(D) \cap X_C^p)$$

consist of finitely many points.

By Proposition 6.7 in [9], if $d(\hat{A}) = 2(1-g) + 2\hat{A} \cdot \hat{A} \geq 0$, then $Gr(\hat{A}) = 2^g$; that is, for generic p_2 points in X_C there are holomorphic curves such that $\sum_u \nu(u) = 2^g$, where $\nu(u) = 1$ if $ev_{p_2, X_{C*}}(u)$ preserves orientation and $\nu(u) = -1$ if $ev_{p_2, X_{C*}}(u)$ reverses orientation. Any of these curves belong to a section class of X_C and by the choice of J the fibers are also J-holomorphic, then by positivity of intersections, they will intersect any representative of F transversally in one point. Hence $ev_{p,W}(pr^{-1}(\{v\} \times \Sigma^p)) \cdot (R(D) \cap X_A^p) = 2^g$.

The proof of Proposition 3.10 implies that the canonical orientation of $\mathcal{M}_g(\hat{A}, W, J)$ is determined by the canonical orientation of $\det(D_F u)$ and the canonical orientation of $\mathcal{M}_g(A, M, J_M)$. Therefore we have that

$$ev_{p,W} \cdot R = 2^g ev_{p,M} \cdot r.$$

□

To complete the proof of Theorem 4.2, we need to prove the following lemma. Let $\pi : X \to Z$ be a locally trivial fiber bundle. Define $\mathit{Diff}(X, \pi) \subseteq$

$\mathit{Diff}(X)$ to be the set of diffeomorphisms of X that descend to Z, i.e., $\phi \in \mathit{Diff}(X,\pi)$ if there exists $\varphi \in \mathit{Diff}(Z)$, such that $\pi \circ \phi = \varphi \circ \pi$.

Lemma 4.3 *Let $e : U \to X$ and $f : V \to X$ two pseudo-cycles of complementary dimensions. Then there exists a set of the second category in $\mathit{Diff}(X,\pi)$ such that e and $\pi \circ f$, and $\pi \circ e$ and $\pi \circ \phi \circ f$ are transverse, for every ϕ in this set.*

Proof. This is proved using standard arguments in differential topology (see [13] Lemma 7.1.2). For $\phi \in \mathit{Diff}(X,\pi)$, $T_\phi \mathit{Diff}(X,\pi)$ consists of vector fields $v \in TX$ such that $d\pi \circ v = w \circ \pi$ for some vector field $w \in TZ$.

Consider the following diagram

$$\begin{array}{ccc} U \times V \times \mathit{Diff}(X,\pi) & \xrightarrow{F} & X \times X \\ \downarrow & & \downarrow{\scriptstyle \pi^2} \\ U \times V \times \mathit{Diff}(Z) & \xrightarrow{G} & Z \times Z, \end{array}$$

where $F(u,v,\phi) = (e(u), \phi(f(v)))$, and $G(u,v,\psi) = (\pi(e(u)), \psi(\pi(f(v))))$. It is easy to see that F is transverse to Δ_X and G is transverse to Δ_Z. Then $F^{-1}(\Delta_X)$ is a submanifold of $U \times V \times \mathit{Diff}(X,\pi)$, and $G^{-1}(\Delta_Z)$ a submanifold of $U \times V \times \mathit{Diff}(Z)$. Consider the projections $F^{-1}(\Delta_X) \to \mathit{Diff}(X,\pi)$ and $G^{-1}(\Delta_Z) \to \mathit{Diff}(Z)$, then the regular values of this projections R_X, R_Z, respectively, are sets of the second category in $\mathit{Diff}(X,\pi)$ and in $\mathit{Diff}(Z)$, resp. Note that $pr : \mathit{Diff}(X,\pi) \to \mathit{Diff}(Z)$ is a submersion, therefore $pr^{-1}(R_Z)$ is a set of the second category in $\mathit{Diff}(X,\pi)$, and so is $pr^{-1}(R_Z) \cap R_X$.

□

Corollary 4.4 *Assume A is a simple class such that $\Phi_{A,g}(x_1, \dots, x_{p_1}) \neq 0$. Then there exist m and $\hat{A}$ such that $\pi_* \hat{A} = A$, and*

$$\Phi_{\hat{A},g}(y_1, \dots, y_m, F_1, \dots, F_l, \hat{B}_1, \dots \hat{B}_n) \neq 0,$$

for some l and n determined by m and $\hat{A}$.

Proof. Let $\hat{A}$ be any class in W with $\pi_* \hat{A} = A$ such that $c_1(Vert) \cdot \hat{A} \geq g - 1$, i.e. with $p_2 \geq 0$. Then we can choose suitable integers l, m, n with $p_1 = m + l$ and $p_2 = m + n$ by taking any m, $0 \leq m \leq min\{p_1, p_2\}$, and then defining $l = p_1 - m$, $n = p_2 - m$. Note also that the possible values of p_2 are constrained by the bundle.

□

4.2 Examples

1. Let $M = \mathbb{C}P^2$, and $W = \mathbb{P}(E \oplus \mathbb{C})$ the projectivization of the complex vector bundle $E \oplus \mathbb{C}$, where E is a line bundle over M with first Chern class $c_1(E) = 3\rho$. This bundle has two obvious sections $Z_+ = \mathbb{P}(0 \oplus \mathbb{C})$ and $Z_- = \mathbb{P}(E \oplus 0)$. Let L be the class of the line in $\mathbb{C}P^2$, and $\hat{L}_3$ be the class on W that is represented by a line in Z_+, then $c_1(Vert) \cdot \hat{L}_3 = 3$. In this case, $p_1 = c_1(M) \cdot L + (g-1) = 2$, and $p_2 = c_1(Vert) \cdot \hat{L}_3 + 1 - g = 4$. If $m = 0$, then we need two fibers F_1, F_2, and 4 two dimensional homology classes in W, say $B_i = \hat{L}_3$, $i = 1, \dots, 4$.

$$\Phi_{\hat{L}_3,0}(\hat{L}_3, \hat{L}_3, \hat{L}_3, \hat{L}_3, F_1, F_2).$$

If $m = 1$, then we need one fiber F and 3 two homology classes,

$$\Phi_{\hat{L}_3,0}(pt, \hat{L}_3, \hat{L}_3, \hat{L}_3, F).$$

Finally, if $m = 2$, then we just need 2 two homology classes,

$$\Phi_{\hat{L}_3,0}(pt, pt, \hat{L}_3, \hat{L}_3).$$

In all these cases, the hypothesis of theorem 4.2 are satisfied and therefore all the invariants are equal to

$$\Phi_{L,0}(pt, pt) = 1.$$

Nevertheless, it is not hard to find examples where the equality in 4.5 does not hold. Consider the following case,

$$\Phi_{\hat{L}_3,0}(pt, pt, \hat{L}_3, \hat{L}_3, Z_+) = 3\Phi_{\hat{L}_3,0}(pt, pt, \hat{L}_3, \hat{L}_3) = 3,$$

since $Z_+ \cdot \hat{L}_3 = 3$. But

$$\Phi_{L,0}(pt, pt, L, L, \pi(Z_+)) = 0,$$

because $\pi(Z_+) = M$.

□

2. If we are in the situation when $c_1(\mathit{Vert}) \cdot \hat{A} < g - 1$ and $c_1(TW) \cdot \hat{A} \geq 0$, then we could still have holomorphic curves in W representing the class $\hat{A}$. Notice that these curves would have to be non-regular on X_C. In [10], Le and Ono consider the following case, let $M = S^2 \times S^2$, and $W = \mathbb{P}(E)$, where E is a complex vector bundle of rank 2 over M with characteristic classes $c_1(E) = 0$

and $c_2(E) = \sigma_1\sigma_2$, where $[\sigma_i] \in H^2(S^2, \mathbb{Z})$ is a generator, for $i = 1, 2$. Let $A = [S^2 \times pt] \in H_2(M, \mathbb{Z})$. Since E is trivial over $S^2 \times pt$, there exist a class $\hat{A}_{-2} \in H_2(W, \mathbb{Z})$ such that $c_1(Vert) \cdot \hat{A}_{-2} = -2$. Observe that $c_1(W)(\hat{A}_{-2}) = 0$, so $\mathcal{M}_g(\hat{A}_{-2}, W, J)$ consists of isolated curves for generic J. They prove that if $X_B = \pi^{-1}(pt \times S^2)$ then the LHS of (4.5) is

$$\Phi_{\hat{A}_{-2},0}(X_B) = 1,$$

while the RHS of (4.5),

$$\Phi_{A,0}([pt \times S^2]) = 0,$$

since equality in (4.3) does not hold.

□

Chapter 5

Classification of S^2-bundles over 4-manifolds.

In this section we recall the classification, up to fiberwise homeomorphism, of S^2-bundles over 4-manifolds. All the results are taken from [2].

Let M be an oriented compact 4-manifold. Orientable S^2-bundles over M are in 1-1 correspondence with homotopy classes $[M, BDiff^+(S^2)]$ of mappings of M into the classifying space $BDiff^+(S^2) \cong BSO(3)$ (the last equality follows from the fact that $Diff^+(S^2)$ deformation retracts onto $SO(3)$). Therefore every (orientable) S^2-bundle can be seen as the sphere bundle of a rank 3 (orientable) vector bundle, or, alternatively, as a principal $SO(3)$-bundle.

Let $B \to M$ be a principal $SO(3)$-bundle and $W = B \times_{SO(3)} S^2$ its associated S^2-bundle. We define $w_2(W) \in H^2(M, \mathbb{Z}_2)$ to be equal to $w_2(B)$, the Second Stiefel-Whitney class of the principal bundle B, and $p_1(W) \in H^4(M, \mathbb{Z})$ to be equal to $p_1(B)$, the first Pontrjagin class of B.

Let B_1 and B_2 be two principal $SO(3)$-bundles over M, and $h_i : M \to BSO(3)$ be a classifying mapping for B_i, $i = 1, 2$. If $w_2(B_1) = w_2(B_2) = w_2$, then $B_1|_{M^{(3)}}$ is equivalent to $B_2|_{M^{(3)}}$, where $M^{(3)}$ is the three skeleton of M, therefore we can assume that $h_1|_{M^{(3)}} = h_2|_{M^{(3)}}$. In this case, we can define the *difference cocycle* as follows: without lost of generality we can assume that the cell decomposition of M has a unique 4-cell, e. Then $h_i|_e : e \to BSO(3)$ with the property that $h_1|_{\partial e} = h_2|_{\partial e}$, then $h_1 - h_2$ can be considered as a map from $S^4 = e \cup_{\partial e} e \to BSO(3)$ and its homotopy class $[h_1 - h_2]$ as an element of $\pi_4(BSO(3)) \cong \pi_3(SO(3)) \cong \mathbb{Z}$. Define the 4-cochain $\Delta(h_1, h_2) : \mathbb{Z}\{e\} \to \pi_4(BSO(3))$ by $\Delta(h_1, h_2)(ne) := n[(h_1 - h_2)]$. It is a closed cochain and it is called *difference cocycle.* Its cohomology class $d(h_1, h_2) = [\Delta(h_1, h_2)] \in H^4(M, \pi_4(BSO(3))) \cong H^4(M, \mathbb{Z})$.

Then we have the following two theorems

Theorem 5.1 *(Dold-Whitney) If B_1 and B_2 are as above, then*

$$p_1(B_1) - p_2(B_2) = -4d(h_1, h_2).$$

It is not hard to see that for any $w_2 \in H^2(M, \mathbb{Z}_2)$, and $d \in H^4(M, \mathbb{Z})$, there exist $h_1, h_2 : M \to BSO(3)$, such that the corresponding bundles have w_2 as its second Stiefel-Whitney class, and $d(h_1, h_2) = d$. Therefore not every pair $(w, p) \in H^2(M, \mathbb{Z}_2) \times H^4(M, \mathbb{Z})$ can be realized as the characteristic classes of an S^2-bundle over M.

Theorem 5.2 *(Dold-Whitney) If B_1 and B_2 are as above, they are equivalent if and only if there exist a cohomology class $x \in H^1(M, \mathbb{Z}_2)$ such that $d(h_1, h_2) = \beta x \cup \beta x + \beta(x \cup w_2)$, where β is the Bockstein homomorphism $\beta : H^1(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z})$, associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$.*

The next theorem characterizes the S^2-bundles whose structure groups can be reduced to $S^1 = SO(2)$. Recall that these bundles are exactly those that admit sections.

Theorem 5.3 *(Massey) Let W be an S^2-bundle over M. Then W admits a section if and only if there exist $\gamma \in H^2(M, \mathbb{Z})$ such that*

1. $w_2(W) \equiv \gamma \; mod(2)$,

2. $p_1(W) = \gamma \cup \gamma$.

An important class of examples comes from the projectivization of rank two complex vector bundles $\mathbb{P}(E)$; the relation between the characteristic classes of E, and those of $\mathbb{P}(E)$ as an S^2-bundle is the following:

$$w_2(\mathbb{P}(E)) \equiv c_1(E) \bmod(2)$$
$$p_1(\mathbb{P}(E)) = c_1^2(E) - 4c_2(E).$$

Conversely, if $W \to M$ is a given S^2-bundle, it is equivalent to $\mathbb{P}(E)$, for some complex vector bundle $E \to M$, if there exist $\gamma \in H^2(M, \mathbb{Z})$, such that $\gamma \equiv w_2(W) \bmod(2)$.

For example, if $H^3(M, \mathbb{Z})$ is torsion free, then $\beta \equiv 0$, where β is the Bockstein homomorphism

$$\cdots \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{Z}_2) \xrightarrow{\beta} H^3(M, \mathbb{Z}) \longrightarrow \cdots,$$

and therefore, any S^2-bundle will be the projectivization of a complex bundle.

Chapter 6

Non-deformation equivalent symplectic forms.

Let $M = E(n)$ be the simply connected minimal elliptic surface with fiber class T, and first Chern class $c_1(E(n)) = (2-n)PD(T)$. Let W be an S^2-bundle over $E(n)$. Since $H_3(E(n), \mathbb{Z})$ is torsion free, then W is the projectivization of a rank 2 complex vector bundle L (see section 5, [2]).

Let η be the first Chern class of the tautological line bundle in $\mathbb{P}(L)$. It is a class that restricts to the generator of the cohomology of the fiber, therefore there exist a symplectic form Ω_κ on W that represents the class $\eta + \kappa\pi^*[\omega]$, for $\kappa >> 0$, that is compatible with the fibration. We fix one such κ and write $\Omega = \Omega_\kappa$. Notice, in addition, that $c_1(TW) = \pi^* c_1(TM) + 2\eta - \pi^* c_1(L)$.

Our main result in this section is the following

Theorem 6.1 *$W_{E(n)}$ has infinitely many deformation classes of symplectic structures in the same cohomology class $[\Omega] \in H^2(W_{E(n)}, \mathbb{R})$.*

Its proof uses the construction of Fintushel and Stern in [3] that gives rise to an infinite family of smooth symplectic manifolds homeomorphic but not diffeomorphic to $E(n)$. Let $K \subset S^3$ be a fibered knot, i.e. there exists a fibration S^3-$K \to S^1$ with fiber a Riemann surface $X_0 \subset S^3$ whose boundary is equal to K in S^3. We say that K has *genus* g if the genus of X_0 is equal to g. If we perform a 0-surgery on S^3 along K, then we get a fibration $Z_K \to S^1$ with fiber equal to $X = X_0 \cup_{\partial X = S^1} D^2$, a closed Riemann surface of genus g. The meridional loop m to $K \subset S^3$ defines a section of the bundle $Z_K \to S^1$. One can prove that Z_K has the same homology as $S^2 \times S^1$.

Now define $X_K = Z_K \times S^1$. It projects onto $T^2 = S^1 \times S^1$ with fiber equal to X, and has the homology as $S^2 \times T^2$. X_K has a well defined section class $S \in H_2(X_K, \mathbb{Z})$, namely the section represented by the embedded torus $m \times S^1$. Furthermore, we can construct a symplectic form on X_K, such that the fiber X and the section class S are represented by symplectic submanifolds.

The fiber class of $E(n)$ is a symplectically embedded torus T with trivial normal bundle. Since the normal bundle of S is also trivial, then we can construct the fiber sum of $E(n)$ with X_K along T and S, $E(n,K) = E(n)\#_{T=S}X_K$. By theorem 5.2 of [6],

$$Gr^T_{E(n,K)} = Gr^T_{E(n)} - (\text{trace} f_{K*}) = 2 - n - (\text{trace}(f_K)_*),$$

where $f_K : X \to X$ is the monodromy of the knot K, and f_{K*} is the induced homomorphism on $H_1(X, \mathbb{Z})$. In [6], the authors use the invariants defined by Ruan and Tian in [15], where they use the moduli space $\mathcal{M}_{A,g,d}(A)$ of connected, perturbed holomorphic maps representing A with genus g and d marked points. To define the invariants in Section 4 we used a different moduli space and therefore a different compactification. Nevertheless, Theorem 4.5 in

[7] asserts that both invariants are equal.

By Proposition 5.6 in [6], there exists a homeomorphism $\phi : E(n, K) \to E(n, K')$ that preserves the class T, the periods of ω, and the Stiefel-Whitney class w_2. If K and K' have the same genus, then $\phi^* c_1(E(n, K')) = c_1(E(n, K))$. It also preserves the first Pontrjagin class.

Up to equivalence of S^2-bundles, that is fiberwise homeomorphisms, we can specify two cohomology classes $c_1(L) \in H_2(E(n), \mathbb{Z})$ and $c_2(L) \in H_4(E(n), \mathbb{Z})$, and consider the corresponding S^2-bundle $\pi_K : W_K \to E(n, K)$. The following theorem is going to be useful in the proof of Lemma 6.3

Theorem 6.2 *[8] If X and Y are closed, simply connected 6-manifolds with torsion free homology, then there exist a diffeomorphism, $\varphi : X \to Y$, if and only if there exists an isomorphism $\Upsilon : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ which preserves the cup product structure, w_2 and p_1. Moreover, φ is orientation preserving and $\varphi^* = \Upsilon$.*

□

Lemma 6.3 *If K and K' are two fibered knots, then there exist a diffeomorphism $W_K \to W_{K'}$ that preserves $[\Omega]$. If K and K' have the same genus, then the diffeomorphism also preserves the homotopy class of J.*

Proof. et $\phi : E(n, K) \to E(n, K')$ be the homeomorphism mentioned above. As S^2 bundles W_K and $\phi^* W_{K'}$ are isomorphic, since they have the same characteristic classes. Therefore there exist a bundle homeomorphism $\psi : W_K \to W_{K'}$

covering ϕ such that $\psi^* w_2(W_{K'}) = w_2(W_K)$, $\psi^* p_1(W_{K'}) = p_1(W_K)$, $\psi^*[\Omega_{K'}] = [\Omega_K]$, and if $g = g'$, $\psi^* c_1(W_{K'}) = c_1(W_K)$.

Then we can use Theorem 6.2 to get a diffeomorphism $\varphi : W_K \to W_{K'}$ that induces the same map in cohomology as ψ.

By Wall's Theorem, [17], if W is any almost complex 6-manifold, there is just one homotopy class of almost complex structures for each $c_1 \in H^2(W, \mathbb{R})$ whose mod 2 reduction is $w_2(W)$. Therefore the homotopy class of J is preserved if $g = g'$.

□

To complete the proof of Theorem 6.1, let $B \in H_2(E(n,K), \mathbb{Z})$ such that $B \cdot T \neq 0$, and let $\hat{B} \in H_2(W_K, \mathbb{Z})$ be a class in W_K such that $(\pi_K)_* \hat{B} = B$. In this case $p_1 = m + l = 0$ and $p_2 = c_1(Vert) \cdot \hat{T}$. Then choose a class $\hat{T}$ with the property that $c_1(Vert) \cdot \hat{T} \geq 0$. With such a choice, we need $p_2 = c_1(Vert) \cdot \hat{T}$ copies of $\hat{B}$ to get a non-zero Gromov invariant in W_K. By Theorem 4.2 we have,

$$\Phi^{W_K}_{\hat{T}_+}(\underbrace{\hat{B}, \ldots, \hat{B}}_{p_2}) = 2\Phi^{E(n,K)}_{T}(\underbrace{B, \ldots, B}_{p_2})$$
$$= 2(T \cdot B)^{p_2} \Phi^{E(n,K)}_{T}$$
$$= 2(T \cdot B)^{p_2}(2 - n - \text{trace}(f_K)_*).$$

Therefore, if $\text{trace}(f_K)_* \neq \text{trace}(f_{K'})_*$ then $\Phi_{\hat{T}}(W_K) \neq \Phi_{\hat{T}}(W_{K'})$.

□

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