A Dimensional Reduction of the Seiberg-Witten Equations and Geometric Quantization

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Abstract of the Dissertation

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In this thesis we dimenisonally reduce the Seiberg-Witten equations and study the resulting moduli spaces. We consider two reductions, one with a so-called Higgs field and one without a Higgs field. The second reduction is not new and gives us the the so-called vortex equations. The moduli space of the vortex equations is shown to carry symplectic and almost complex structures. We consider several moduli spaces arising out of the first reduction and show that some of them have hyperkähler structures. Finally, we construct a prequantum line bundle over one of these moduli spaces.

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To Sanyashi and other children like him, living in the streets of Calcutta.

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Chapter 1

Introduction and the main equations

Inspired by Hitchin's work on the dimensional reduction of the Self-Dual-Yang-Mills equations over a Riemann surface [6], we study the analogous question for the Seiberg-Witten equations. Though the main context of Seiberg-Witten theory is in four dimensions, the 2-dimensional reduction also seems worth exploring. Apart from the rich structure embedded in the various moduli spaces, it also admits the possibility of geometric quantization, which is a prerequisite to a $(2+1)$ topological field theory [2], [17], [3].

In chapter 1, we describe the reduction from Seiberg-Witten equations in $\mathbb{R}^4$ to $\mathbb{R}^2$ which are then defined over a compact Riemann surface M. On $\mathbb{R}^4$ the Seiberg-Witten equations do not have nice L^2-solutions. But when reduced to M we expect to have interesting solutions. We obtain three equations (1.1) – (1.3) as well as two vortex equations ($1v$) and ($2v$). The vortex equations on $\mathbb{C}$, which appear in the usual (without Higgs field) reduction of the Seiberg-Witten equations, [10], [16] play an important role in Taubes' proof of the equivalence of the Seiberg-Witten invariants and Gromov invariants for a certain class of 4-manifolds [16]. A new feature which appears in our reduction to (1.1) – (1.3) is the so-called Higgs field Φ.

In section 2.1 we describe the moduli space $\mathcal{M}_v$ of solutions to the vortex equations and compute its dimension. We find that under certain conditions it is a symplectic, almost complex manifold. In section 2.2, we realize equations (1.1) and (1.2) as moment maps on the configuration space with respect to the action of the gauge group. Realization of equation (1.1) as a moment map is not new and can also be found in [15]. Realization of equation (1.2) is new but motivated by analogous structure on the dimensionally reduced Yang-Mills equations in [6]. In section 2.2 we also exhibit a hyperkähler structure on the moduli space $\mathcal{M}$ of solution to (1.1) and (1.2), analogous to [6]. Note that this hyperkähler structure does not descend to $\mathcal{N}$, the moduli space of solutions to (1.1) – (1.3). In section 2.3 we describe Σ_ψ, the moduli space of solutions to equations (1.1) and (1.2) for a fixed appropriate class of spinors. We find that it is a hyperkähler manifold of finite dimension. We also compute the dimension of Σ_ψ.

In chapter 3 we present a brief survey of the determinant line bundle of families of elliptic operators. We follow [12] and [4]. In [12], Quillen constructs the Hermitian metric on the determinant line bundle for the family of Cauchy-Riemann operators on a vector bundle over a Riemann surface. Quillen computes the curvature form induced by the metric which turns out to be the standard Kähler form on the affine space of all Cauchy-Riemann operators. We present a brief survey of his construction.

In chapter 4 we give a brief survey of geometric quantization. The first step involves the construction of a prequantum line bundle $\mathcal{L}$ over a symplectic manifold such that its curvature form coincides with the symplectic form. The Hilbert space of states is given by the square integrable sections of $\mathcal{L}$. The second step in geometric quantization involves finding a polarization of the sympectic form, such that polarized sections of $\mathcal{L}$ give a finite dimensional Hilbert space. The second step works well when the manifold in question is compact, but it might work as well for manifolds with finite volume. In the same chapter we construct a prequantum bundle for Σ_Ψ. This construction is motivated by a similar construction for moduli space of flat connections over a Riemann surface which is used in Chern-Simons gauge theory [3]. The second step in the geometric quantization is beyond the scope of this thesis.

1.1 Dimensional Reduction of the Seiberg-Witten equations

In this section we dimensionally reduce the Seiberg - Witten equations on $\mathbb{R}^4$ to $\mathbb{R}^2$ and patch them up over a compact Riemann surface M.

The Seiberg-Witten equations on $\mathbb{R}^4$

This is a brief description of the Seiberg-Witten equations on $\mathbb{R}^4$, [15], [1], [8] .

Identify $\mathbb{R}^4$ (coordinates x_0, x_1, x_2, x_3) with the quaternions $\mathbb{H}$. Fix the constant spin structure $\Gamma : \mathbb{H} = T_x\mathbb{H} \to \mathbb{C}^{4\times 4}$, given by

$$\Gamma(\zeta) = \begin{bmatrix} 0 & \gamma(\zeta) \\ \gamma(\zeta)^* & 0 \end{bmatrix},$$

where

$$\gamma(\zeta) = \begin{bmatrix} \zeta_0 + i\zeta_1 & -\zeta_2 - i\zeta_3 \\ \zeta_2 - i\zeta_3 & \zeta_0 - i\zeta_1 \end{bmatrix}.$$

Thus $\gamma(e_0) = Id$, $\gamma(e_1) = I$, $\gamma(e_2) = J$, $\gamma(e_3) = K$ with

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$

so that $IJ = K$, $JK = I$, $KI = J$ and $I^2 = J^2 = K^2 = -Id$.

(Note: our choice of I , J and K are different from [15], but it does not make any difference).

Recall that $Spin^c(\mathbb{R}^4) = (Spin(\mathbb{R}^4) \times S^1)/\mathbb{Z}_2$. Since $Spin(\mathbb{R}^4)$ is a double cover of $SO(4)$, a *spin*^{c} - connection involves a connection ω on T $\mathbb{H}$

and a connection

$$A = i\sum_{j=1}^{4} A_j dx_j \in \Omega^1(\mathbb{H}, i\mathbb{R})$$

on the characteristic line bundle $\mathbb{H} \times \mathbb{C}$ which arises from the S^1 factor (see [15], [8], [1] for more details). We set $\omega = 0$, which is equivalent to choosing the connection on the trivial tangent bundle to be d. The curvature 2-form of the connection A is given by

$$F(A) = dA \in \Omega^2(\mathbb{H}, i\mathbb{R}).$$

Consider the covariant derivative acting on $\Psi \in C^\infty(\mathbb{H}, \mathbb{C}^2)$ (the positive spinor on $\mathbb{R}^4$) induced by the connection A on $\mathbb{H} \times \mathbb{C}$

$$\nabla_j \Psi = (\frac{\partial}{\partial x_j} + A_j)\Psi.$$

Then according to [15], the Seiberg-Witten equations for (A, Ψ) on $\mathbb{R}^4$ are equivalent to:

$$(SW1) \qquad \nabla_0 \Psi = I\nabla_1 \Psi + J\nabla_2 \Psi + K\nabla_3 \Psi,$$

and $(SW2)$

$$\begin{aligned}
F_{12} + F_{34} &= \frac{1}{2}\Psi^* I \Psi \\
&= \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2) \doteq \frac{1}{2}\eta_1, \\
F_{13} + F_{42} &= \frac{1}{2}\Psi^* J \Psi \\
&= i(Im\psi_1\bar{\psi}_2) \doteq \frac{1}{2}\eta_2, \\
F_{14} + F_{23} &= \frac{1}{2}\Psi^* K \Psi \\
&= -i(Re\psi_1\bar{\psi}_2) \doteq \frac{1}{2}\eta_3,
\end{aligned}$$

where $\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$.

Reduction

We closely follow the method used in [6] for the reduction of the Self-Dual-Yang-Mills equations. This seems to give the most general form of the reduced equations which contain the so-called Higgs field.

Namely, impose the condition that none of the A_i's and Ψ in $(SW1)$ and $(SW2)$ depend on x_3 and x_4, i.e. $A_i = A_i(x_1, x_2)$, $\Psi = \Psi(x_1, x_2)$ and set $\phi_1 = -2iA_3$ and $\phi_2 = -2iA_4$. The $(SW2)$ equations reduce to the following system on $\mathbb{R}^2$

$$F_{12} = \frac{1}{2}\eta_1,$$
$$\frac{1}{2}(\frac{\partial\phi_1}{\partial x_1} - \frac{\partial\phi_2}{\partial x_2}) = -\frac{1}{2}\eta_2,$$
$$\frac{1}{2}(\frac{\partial\phi_1}{\partial x_2} + \frac{\partial\phi_2}{\partial x_1}) = -\frac{1}{2}\eta_3.$$

Introducing complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ the last two equations can be rewritten as

$$\frac{\partial(\phi_1 + i\phi_2)}{\partial\bar{z}} = -\frac{1}{2}(\eta_2 + i\eta_3) = -\psi_1\bar{\psi}_2,$$

where $\frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$. Setting $\phi_1 + i\phi_2 = \phi$, we can rewrite the reduction of $(SW2)$ as the following two equations,

(1) $$F(A) = \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2)dz \wedge d\bar{z},$$

(2) $$2\bar{\partial}\Phi = -(\psi_1\bar{\psi}_2)dz \wedge d\bar{z},$$

where $\Phi = \phi dz - \bar{\phi} d\bar{z} \in \Omega^1(\mathbb{R}^2, i\mathbb{R})$ and $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^2, \mathbb{C})$ are spinors on $\mathbb{R}^2$.

Next consider equation $(SW1)$:

$$\nabla_1\psi - I\nabla_2\psi - J\nabla_3\psi - K\nabla_4\psi = 0$$

which is rewritten as

$$\begin{bmatrix} \frac{\partial}{\partial x_1} + iA_1 - i\frac{\partial}{\partial x_2} + A_2 & \frac{\partial}{\partial x_3} + iA_3 + i\frac{\partial}{\partial x_4} - A_4 \\ -\frac{\partial}{\partial x_3} - iA_3 + i\frac{\partial}{\partial x_4} - A_4 & \frac{\partial}{\partial x_1} + iA_1 + i\frac{\partial}{\partial x_2} - A_2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0,$$

and reduces to the following system on $\mathbb{R}^2$

$$\begin{bmatrix} 2\frac{\partial}{\partial z} + i(A_1 - iA_2) & -\phi \\ \bar{\phi} & 2\frac{\partial}{\partial \bar{z}} + i(A_1 + iA_2) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0.$$

Introducing $A^{1,0} = \frac{i}{2}(A_1 - iA_2)dz$ and $A^{0,1} = \frac{i}{2}(A_1 + iA_2)d\bar{z}$ where $A = i(A_1 dx + A_2 dy) = A^{1,0} + A^{0,1}$ (and $A^{0,1} = -A^{\bar{0},1}$) we can finally write it as

(3) $$\begin{bmatrix} -\frac{1}{2}\bar{\phi} d\bar{z} & -(\bar{\partial} + A^{0,1}) \\ (\partial + A^{1,0}) & -\frac{1}{2}\phi dz \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0.$$

We call equations $(1)-(3)$ as the dimensionally reduced Seiberg-Witten equations over $\mathbb{C}$.

1.1.1 The Dimensionally Reduced Equations on a Riemann surface

Let M be a compact Riemann surface of genus $g > 1$ with a conformal metric $ds^2 = hdz \otimes d\bar{z}$ and let ω be its Kähler form. Let K denote the canonical line bundle over M. For a fixed choice of $K^{\frac{1}{2}}$ we let $\psi_1, \psi_2 \in \Gamma(M, L)$ where $L = \bar{K}^{\frac{1}{2}}$. The metric ds^2 induces a Hermitian metric H on $\bar{K}^{\frac{1}{2}}$ and one can define a norm $|\psi|_H \in C^\infty(M)$ and an inner product $< \psi_1, \psi_2 >_H \in C^\infty(M)$ of the sections of L.

We denote by A a unitary connection on the holomorphic line bundle $K^{-\frac{1}{2}}$ over M. Since the connection A is unitary, it defines a connection on the anti-holomorphic line bundle $L = \bar{K}^{\frac{1}{2}}$ as well. Finally let $\Phi = \phi dz - \bar{\phi} d\bar{z} \in \Omega^1(M, i\mathbb{R})$. Note that $\Phi = \Phi^{1,0} + \Phi^{0,1}$ where $\Phi^{1,0} \in \Omega^{1,0}(M, \mathbb{C})$ and $\bar{\Phi}^{1,0} = -\Phi^{0,1}$.

Now we can rewrite the equations $(1) - (3)$ in an invariant form on M as follows:

$$F(A) = i\frac{(|\psi_1|_H^2 - |\psi_2|_H^2)}{2}\omega, \tag{1.1}$$

$$2\bar{\partial}\Phi = - < \psi_1, \psi_2 >_H \omega, \tag{1.2}$$

$$\begin{bmatrix} -\frac{1}{2}\bar{\phi} d\bar{z} & -\nabla''_A \\ \nabla'_A & -\frac{1}{2}\phi dz \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0. \tag{1.3}$$

Here $\nabla_A = \nabla''_A + \nabla'_A$, where $\nabla'_A : \Gamma(M, L) \to \Omega^{1,0}(M, L)$, and $\nabla''_A : \Gamma(M, L) \to \Omega^{0,1}(M, L)$, is a decomposition of the connection ∇_A induced by the decomposition of $T^*(M) = T^{*\prime}(M) \oplus T^{*\prime\prime}(M)$.

Gauge group action

There is an action of the gauge group $G = Map(M, S^1)$ on (A, Ψ, Φ),

$$(A, \Psi, \Phi) \to (A + u^{-1}du, u^{-1}\Psi, \Phi) \tag{1.4}$$

which leaves the solution space invariant. This action is free since $c_1(L) = 1 - g \neq 0$. This is because for a fixed point necessarily $\Psi \equiv 0$, which would imply, by equation (1.1), that $c_1(L) = 0$.

1.1.2 Vortex Equations

There is the usual reduction without a Higgs field which gives rise to the Vortex Equations:

$$F(A) = i\frac{(|\psi_1|_H^2 - |\psi_2|_H^2)}{2}\omega, \tag{1v}$$

and

$$\begin{bmatrix} 0 & -\nabla''_A \\ \nabla'_A & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0. \tag{2v}$$

Here we consider two cases. In case (a) $\psi_1, \psi_2 \in \Gamma(M, L)$ and A is a unitary connection on L.

In the case (*b*) $\psi_1 \in \Gamma(M,L)$ but $\psi_2 \in \Omega^{1,0}(M,L)$, so that $\nabla'_A : \Gamma(M,L) \to \Omega^{1,0}(M,L)$ as before, but $\nabla''_A : \Omega^{1,0}(M,L) \to \Omega^{1,1}(M,L)$.

Since A is unitary, it is easy to see that $(\nabla'_A)^\dagger = - * (\nabla''_A)$, where $\dagger$ stands for the adjoint operator and $* : \Omega^{1,1}(M,L) \to \Gamma(M,L)$ is the Hodge-star operator. This is because for $\Psi_1 \in \Gamma(M,L)$ and $\Psi_2 \in \Omega^{1,0}(M,L)$, ∇_A satisfies the equation

$$d < \Psi_1, \Psi_2 >_H = < \nabla_A \Psi_1, \Psi_2 >_H + < \Psi_1, \nabla_A \psi_2 >_H$$

where all the terms in the equation belong to $\Omega^{1,1}(M)$. Identifying types, one obtains,

$$\bar{\partial} < \Psi_1, \Psi_2 >_H = < \nabla'_A \Psi_1, \Psi_2 >_H + < \Psi_1, \nabla''_A \psi_2 >_H .$$

Integrating on M, one obtains that result.

This enables us to interpret $\begin{bmatrix} 0 & -\nabla''_A \\ \nabla'_A & 0 \end{bmatrix}$ as a Dirac operator.

x

Chapter 2

The Moduli Spaces

2.1 The moduli space of the Vortex Equations

The Vortex Equations in the last section can be rewritten as follows:

$$(1v) \qquad \mu \doteq F(A) - \frac{i}{2}(|\psi_1|^2_H - |\psi_2|^2_H)\omega = 0$$

$$(2v) \qquad \mathcal{D}_A \Psi \doteq \begin{bmatrix} 0 & -\nabla''_A \\ \nabla'_A & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0.$$

Recall that we have two cases.

Case (a): We let $\psi_1, \psi_2 \in \Gamma(M, L)$ and A be a unitary connection on L. Let $\tilde{S}$ denote the solution space. The gauge group G acts as $(A, \Psi) \to$

$(A+u^{-1}du, u^{-1}\Psi)$ and leaves $\tilde{\mathcal{S}}$ invariant. It acts freely on $\tilde{\mathcal{S}}$ since $g \neq 1$. We denote by $\tilde{\mathcal{M}}_v \doteq \tilde{\mathcal{S}}/G$ the moduli space of solutions to the vortex equations for case (a).

Case (b): We let $\psi_1 \in \Gamma(M, L), \psi_2 \in \Omega^{1,0}(M, L)$. We denote by $\mathcal{S}$ the corresponding solution space and by $\mathcal{M}_v \doteq \mathcal{S}/G$ the moduli space of solutions.

In this thesis we make an assumption that the moduli spaces $\mathcal{M}_v$ and $\tilde{\mathcal{M}}_v$ are non-empty. The proof of this result is beyond the scope of this thesis and we plan to address this question later. In [11] it has been shown that solutions to the vortex equations arising from a slightly different dimensional reduction do exist.

Proposition 2.1.1. *The moduli space $\mathcal{M}_v$ has virtual dimension $2g-2$.*

Note: The tangent space is given by linearization of equations $(1v)$ and $(2v)$, and can be seen to be zero set of a Fredholm operator. Its virtual dimension is the index of this operator.

Proof: The techniques used in this proof are standard [8], [6]. Consider the tangent space $T_p\, \mathcal{S}$ at a point $p = (A, \Psi) \in \mathcal{S}$, which is defined by the linearization of equations $(1v)$ and $(2v)$. Let $X = (\alpha, \beta) \in T_p\, \mathcal{S}$, where $\alpha \in \Omega^1(M i\mathbb{R})$ and $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \Gamma(M, S)$, where $S = L \oplus (L \otimes K)$, satisfy the equations

$$(1L) \qquad d\alpha = \frac{i}{2\sqrt{h}}[(\psi_1\bar{\beta}_1 + \beta_1\bar{\psi}_1) - \frac{1}{h}(\psi_2\bar{\beta}_2 + \beta_2\bar{\psi}_2)]\omega$$

$$(2L) \qquad \begin{bmatrix} 0 & -\nabla''_A \\ \nabla'_A & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha^{0,1} \\ \alpha^{1,0} & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0.$$

Taking into account the quotient by the gauge group G, we arrive at the following sequence $\mathcal{C}$

$$0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{d_1} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(M, S) \xrightarrow{d_2} \Omega^2(M, i\mathbb{R}) \oplus \Gamma(M, S') \to 0,$$

where recall that $S = L \oplus (L \otimes K)$, and $S' = (L \otimes \bar{K}) \oplus (L \otimes K)$, and $d_1 f = (df, -f\Psi)$,

$$d_2(\alpha, \beta_1, \beta_2) \doteq \quad ((d\alpha - \frac{i}{2\sqrt{h}}(\psi_1\bar{\beta}_1 + \beta_1\bar{\psi}_1) - \frac{1}{h}(\psi_2\bar{\beta}_2 + \beta_2\bar{\psi}_2)]\omega),$$
$$\mathcal{D}_A \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} -\alpha^{0,1}\psi_2 \\ \alpha^{1,0}\psi_1 \end{bmatrix}).$$

Clearly, dim $H^1(\mathcal{C}) =$ dim $T_p\ \mathcal{M}_v$. The sequence $\mathcal{C}$ is a complex, since

$$d_2(df, -f\Psi, 0) =$$
$$(d^2 f + \frac{i}{2\sqrt{h}}[(f\psi_1\bar{\psi}_1 + \psi_1\bar{f}\bar{\psi}_1) - \frac{1}{h}(f\psi_2\bar{\psi}_2 + \psi_2\bar{f}\bar{\psi}_2)]\omega,$$
$$\mathcal{D}_A \begin{bmatrix} -f\psi_1 \\ -f\psi_2 \end{bmatrix} + \begin{bmatrix} \bar{\partial} f\psi_2 \\ -\partial f\psi_1 \end{bmatrix}) = 0,$$

where we have used equation $(2v)$ and the fact that $f = -\bar{f}$.

Clearly, $H^0(\mathcal{C}) = 0$, because if $f \in ker(d_1)$, then $df = 0$ and $f\Psi = 0$, which implies $f = 0$ since $\Psi = 0$ would contradict, by equation $(1v)$, that $c_1(L) \neq 0$.

Therefore the virtual $dim T_p\mathcal{M}_v = -ind(\mathcal{C})$.

To calculate the index of $\mathcal{C}$, we consider the family of complexes $(\mathcal{C}^t, d^t)$, $0 \le t \le 1$, where $d_1^t = (df, -tf\Psi)$ and

$$d_2^t(\alpha, \beta_1, \beta_2) = ((d\alpha - \frac{it}{2\sqrt{h}}[(\psi_1\bar{\beta}_1 + \beta_1\bar{\psi}_1) - \frac{1}{h}(\psi_2\bar{\beta}_2 + \beta_2\bar{\psi}_2)]\omega), \mathcal{D}_A \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + t \begin{bmatrix} -\alpha^{0,1}\psi_2 \\ \alpha^{1,0}\psi_1 \end{bmatrix}).$$

Clearly, $ind(\mathcal{C}^t)$ does not depend on t. The complex $\mathcal{C}_0$ is

$$0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{d_1'} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(M, S) \xrightarrow{d_2'} \Omega^2(M, i\mathbb{R}) \oplus \Gamma(M, S') \to 0,$$

where $d_1'f = (df, 0)$ and $d_2'(\alpha, \beta) = (d\alpha, \mathcal{D}_A\beta)$, and decomposes into a direct sum of two complexes

$$(a) \qquad 0 \to \Omega^0(X, i\mathbb{R}) \xrightarrow{d} \Omega^1(M, i\mathbb{R}) \xrightarrow{d} \Omega^2(M, i\mathbb{R}) \to 0,$$

and

$$(b) \qquad 0 \to \Gamma(M, S) \xrightarrow{\mathcal{D}_A} \Gamma(M, S') \to 0.$$

The index of the complex (a) is $2 - 2g$.

The index of complex (b) is 0 since the operator $\mathcal{D}_A = \begin{bmatrix} 0 & -\nabla_A'' \\ \nabla_A' & 0 \end{bmatrix}$ is self-adjoint with $(\nabla_A'')^\dagger = -\nabla_A'$, so that $ind(\nabla_A') = -ind(\nabla_A'')$. (See section (1.1.2) case (b) for a proof).

Since the index of direct sum of complexes is the sum of the indices, the result follows. □

Note: This last part of the proof does not immediately work for $\tilde{\mathcal{M}}_v$.

2.1.1 Symplectic and almost complex structures

In the next theorem we describe symplectic and complex structures on $\mathcal{M}_v$. The results are valid for $\tilde{\mathcal{M}}_v$ as well.

Let $\mathcal{A}$ be the affine space over $\Omega^1(M, i\mathbb{R})$ of unitary connections on L and $S = L \oplus L$ in case (a) and $S = L \oplus (L \otimes K)$ in case (b), and set $\mathcal{B} = \mathcal{A} \times \Gamma(M, S)$.

First, we construct an obvious symplectic structure on the affine space $\mathcal{B} = \mathcal{A} \times \Gamma(M, S)$, which becomes *degenerate* when restricted to $\mathcal{S}$, the solution space to equations $(1v)$ and $(2v)$. However, the leaves of the characteristic foliation are the gauge orbits so that taking the quotient by the gauge group produces symplectic structures on the moduli spaces $\mathcal{M}_v$ and $\tilde{\mathcal{M}}_v$.

Second, we show that the standard complex structure on $\mathcal{B}$ descends to an almost complex structure on $\mathcal{M}_v$.

Specifically, let $p = (A, \Psi) \in \mathcal{B}$, $X = (\alpha_1, \beta)$, $Y = (\alpha_2, \eta) \in T_p\mathcal{B}$. Let

$$g(X, Y) = \int_M *\alpha_1 \wedge \alpha_2 + \int_M Re < \beta, \eta >_H \omega, \tag{2.1}$$

be a Riemannian metric on $\mathcal{B}$ and let

$$\Omega(X, Y) = -\int_M \alpha_1 \wedge \alpha_2 + \int_M Re < I\beta, \eta >_H \omega \tag{2.2}$$

be a symplectic form on $\mathcal{B}$. Let $\mathcal{I}_v = \begin{bmatrix} * & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} : T_p\mathcal{B} \to T_p\mathcal{B}$, where

$\mathcal{I}_v(\alpha,\beta) = (*\alpha, I\beta)$ be a complex structure on $\mathcal{B}$. Here $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $* : \Omega^1 \to \Omega^1$ is the Hodge star operator on M. Then, the symplectic form Ω is compatible with the complex structure $\mathcal{I}_v$ and with the metric g,

$$g(\mathcal{I}_v X, Y) = \Omega(X, Y).$$

Moreover, we have the following

Proposition 2.1.2. *The metric g, the symplectic form Ω and the complex structure $\mathcal{I}_v$ are invariant under the gauge group action.*

Proof: Let $p = (A, \Psi) \in \mathcal{B}$ and $u \in G$, where $u \cdot p = (A + u^{-1}du, u^{-1}\Psi)$. Then $u_* : T_p\mathcal{B} \to T_{u\cdot p}\mathcal{B}$ is given by the mapping (Id, u^{-1}) and it is now easy to check that g and Ω are invariant and $\mathcal{I}_v$ commutes with u_*.

Proposition 2.1.3. *The first vortex equation can be realised as a moment map $\mu = 0$ with respect to the action of the gauge group and the symplectic form Ω.*

Note The realization of the first vortex equation as a moment map is not new (see [15]).

Proof: Let $\zeta \in \Omega(M, i\mathbb{R})$ - Lie algebra of the gauge group; it generates a vector field X_ζ on $\mathcal{B}$ as follows :

$X_\zeta(A, \Psi) = (d\zeta, -\zeta\Psi) \in T_p\mathcal{B}$, $p = (A, \Psi) \in \mathcal{B}$.

We show next that X_ζ is Hamiltonian. Namely, define $H_\zeta : \mathcal{B} \to \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta \cdot (F_A - i\frac{(|\psi_1|_H^2 - |\psi_2|_H^2)}{2}\omega).$$

Then, we need to show that

$$dH_\zeta = \Omega \lrcorner X_\zeta.$$

Let $X = (\alpha, \beta) \in T_p\mathcal{B}$. We have

$$\begin{aligned} dH_\zeta(X) &= \int_M \zeta d\alpha - i\int_M \zeta Re(\psi_1\bar{\beta}_1 - \frac{1}{h}\psi_2\bar{\beta}_2)\frac{1}{\sqrt{h}}\omega \\ &= \int_M (-d\zeta)\wedge\alpha - \int_M Re < I\zeta \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} >_H \omega \\ &= \Omega(X_\zeta, X), \end{aligned}$$

where we use that $\bar{\zeta} = -\zeta$.

Thus we can define the moment map $\mu : \mathcal{B} \to \Omega^2(M, i\mathbb{R}) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A, \Psi) \doteq (F(A) - i\frac{(|\psi_1|_h^2 - |\psi_2|_h^2)}{2}\omega).$$

Thus equation $(1v)$ is $\mu = 0$.

□.

Lemma 2.1.4. *Let $X \in T_p\mathcal{S}$, $p \in \mathcal{S}$. Then $\mathcal{I}_v X \in T_p\mathcal{S}$ if and only if X is orthogonal to the gauge orbit $O_p = G \cdot p$.*

Proof Let $X_\zeta \in T_p O_p$, where $\zeta \in \Omega^0(M, i\mathbb{R})$. Then we have

$g(X, X_\zeta) = -\Omega(\mathcal{I}_v X, X_\zeta) = -\int_M \zeta \cdot d\mu(\mathcal{I}_v X)$, and therefore $\mathcal{I}_v X$ satisfies the equation $(1L)$ iff $d\mu(\mathcal{I}_v X) = 0$, i.e., iff $g(X, X_\zeta) = 0$ for all ζ.

Second, it is easy to check that $\mathcal{I}_v X$ satisfies equation $(2L)$ whenever X does.

□

Theorem 2.1.5. *$\mathcal{M}_v$ has a natural symplectic structure and an almost complex structure compatible with the symplectic form Ω and the metric g.*

Proof

First we show that the almost complex structure descends to $\mathcal{M}_v$. Then using this and the symplectic quotient construction we will show that Ω gives a symplectic structure on $\mathcal{M}_v$.

(a) To show that $\mathcal{I}_v$ descends as an almost complex structure we use standard technique in [6]. Let $\tau : \mathcal{S} \to \mathcal{S}/G = \mathcal{M}_v$ be the projection map and set $[p] = \tau(p)$. Then we can naturally identify $T_{[p]}\mathcal{M}_v$ with the quotient space $T_p\mathcal{S}/T_pO_p$, where $O_p = G \cdot p$ is the gauge orbit. Using the metric g on $\mathcal{S}$ we can realize $T_{[p]}\mathcal{M}_v$ as a subspace in $T_p\mathcal{S}$ orthogonal to T_pO_p. Then by lemma 2.1.4, this subspace is invariant under $\mathcal{I}_v$. Thus $I_{[p]} = \mathcal{I}_v|_{T_p(O_p)^\perp}$, gives the desired almost complex structure. This construction does not depend on the choice of p since $\mathcal{I}_v$ is G-invariant.

(b) The symplectic structure Ω descends to $\mu^{-1}(0)/G$, (by proposition 2.1.3 and by the symplectic quotient construction, since the leaves of the characteristic foliation are the gauge orbits). Now, as a 2-form Ω descends to $\mathcal{M}_v$, due to proposition 2.1.2 so does the metric g. To show that equation (2v) does not give rise to new degeneracy of Ω, we use the fact that both the metric g and $\mathcal{I}_v$ descend to $\mathcal{M}$. Since they are compatible with Ω, the latter is symplectic.

2.2 Hyperkähler Quotient

In this section we show that equations (1.1) and (1.2) can be realised as moment maps of the action of the gauge group G on the "configuration" space:

$$\mathcal{E} = \mathcal{A} \times \Gamma(M, \mathcal{S}) \times \mathcal{H}$$

where $S = L \oplus L$ and $\mathcal{H} = \Omega^1(M, i\mathbb{R})$.

Note: Realization of the first equation as a moment map is not new and can be found in [15]. The moment for the second equation and the hyperkähler quotient construction are new. We have been motivated by similar constructions for the dimensional reduction of the Self-Dual Yang-Mills equations in [6].

2.2.1 First equation as a moment map

Just as before, define a symplectic form on $\mathcal{E}$ as follows:

$$\begin{aligned}\Omega(X,Y) &= -\int_M \alpha^1 \wedge \alpha^2 + \int_M Re < I\beta^1, \beta^2 >_H \omega \\ &\quad + \int_M \gamma^1 \wedge \gamma^2,\end{aligned}$$

where $X = (\alpha^1, \beta^1, \gamma^1)$ and $Y = (\alpha^2, \beta^2, \gamma^2) \in T_p\mathcal{E}$, $p = (A, \Psi, \Phi)$ and

$$I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Clearly, the gauge group G acts on $\mathcal{E}$ by symplectomorphisms.

Let $\zeta \in \Omega(M, i\mathbb{R})$; as before it generates a vector field on $\mathcal{E}$ as follows :

$$X_\zeta(A, \Psi, \Phi) = (d\zeta, -\zeta\Psi, 0) \in T_p\mathcal{E}.$$

Proposition 2.2.1. *The vector field X_ζ is Hamiltonian.*

Proof: The proof is exactly the same as proposition 2.1.3.

Define $H_\zeta : \mathcal{E} \to \mathbb{C}$ as before

$$H_\zeta(A, \Psi, \Phi) = \int_M \zeta \cdot (F_A - i\frac{(|\psi_1|_H^2 - |\psi_2|_H^2)}{2}\omega).$$

Then,

$$\begin{aligned} dH_\zeta(X) \quad &= -\int_M (d\zeta) \wedge \alpha - \int_M Re < I\zeta \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} >_H \omega \\ &= \Omega(X_\zeta, X), \end{aligned}$$

exactly as before, where $X = (\alpha, \beta, \gamma) \in T_p\mathcal{E}$, (and we use that ζ is purely imaginary).

Thus we can define the moment map $\mu : \mathcal{E} \to \Omega^2(X, i\mathbb{R}) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) of the action to be

$$\mu(A, \Psi, \Phi) = (F(A) - i\frac{(|\psi_1|_H^2 - |\psi_2|_H^2)}{2}\omega),$$

so that equation(1.1) can be written as $\mu = 0$.

$\square$.

2.2.2 Second equation as a moment map

To realise the equation (1.2) as a moment map we need to define another symplectic form $\mathcal{Q}$ on $\mathcal{E}$, which is complex-valued,

$$\begin{aligned}\mathcal{Q}(X,Y) = \quad & -2\int_M \alpha_1^{0,1}\wedge\gamma_2^{1,0} + 2\int_M \alpha_2^{0,1}\wedge\gamma_1^{1,0} \\ & -\int_M (\beta_1^1\bar{\beta}_2^2 - \bar{\beta}_1^2\beta_2^1)\frac{1}{\sqrt{h}}\omega\end{aligned}$$

where $X=(\alpha_1,\beta_1,\gamma_1)$, $Y=(\alpha_2,\beta_2,\gamma_2)\in T_p\mathcal{E}$.

Proposition 2.2.2. *The vector field X_ζ is Hamiltonian with respect to the symplectic form $\mathcal{Q}$.*

Proof : Define the Hamiltonian to be

$$H_\zeta(A,\psi,\phi)=\int_M \zeta(2\bar{\partial}\Phi+\psi_1\bar{\psi}_2\frac{1}{\sqrt{h}}\omega),$$

where $\zeta\in\Omega(M,i\mathbb{R})$.

We need to show that $dH_\zeta=\mathcal{Q}\lrcorner X_\zeta$. Indeed for $X=(\alpha,\beta_2,\gamma)\in T_p\mathcal{E}$,

$$\begin{aligned}dH_\zeta(X) \quad &= \int_M \zeta(2\bar{\partial}\gamma+(\beta_2^1\bar{\psi}_2+\psi_1\bar{\beta}_2^2)\frac{1}{\sqrt{h}}\omega) \\ &= 2\int_M -\bar{\partial}\zeta\wedge\gamma+\int_M(\zeta\beta_2^1\bar{\psi}_2+\zeta\psi_1\bar{\beta}_2^2)\frac{1}{\sqrt{h}}\omega \\ &= 2\int_M -\bar{\partial}\zeta\wedge\gamma+\int_M(-\beta_2^1\bar{\zeta}\bar{\psi}_2+(\zeta\psi_1)\bar{\beta}_2^2)\frac{1}{\sqrt{h}}\omega \\ &= \mathcal{Q}(X_\zeta,X).\end{aligned}$$

where $X_\zeta=(d\zeta,-\zeta\Psi,0)$. Thus we can define the moment map of the action with respect to the form $\mathcal{Q}$ to be :

$$\mu_{\mathcal{Q}}=2\bar{\partial}\Phi+<\psi_1,\psi_2>_H\omega.$$

Thus equation (1.2) is precisely $\mu_Q = 0$.

$\square$.

2.2.3 Hyperkähler quotient

Proposition 2.2.3. *The configuration space $\mathcal{E}$ has a Riemannian metric g and three complex structures $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ which satisfy $\mathcal{I}\mathcal{J} = \mathcal{K}$, $\mathcal{J}\mathcal{K} = \mathcal{I}$, and $\mathcal{K}\mathcal{I} = \mathcal{J}$, and three symplectic structures*

$$\begin{aligned}\omega_1(X,Y) &= g(\mathcal{I}X,Y),\\ \omega_2(X,Y) &= g(\mathcal{J}X,Y),\\ \omega_3(X,Y) &= g(\mathcal{K}X,Y)\end{aligned}$$

such that $\omega_1 = \Omega$, and $\omega_2 + i\omega_3 = \mathcal{Q}$.

Proof We define the Riemannian metric g as follows:

$$\begin{aligned}g(X,Y) = &\int_M *\alpha_1 \wedge \alpha_2\\ &+ \int_M Re < \beta_1, \beta_2 >_H \omega + \int_M *\gamma_1 \wedge \gamma_2,\end{aligned}$$

where $X = (\alpha_1, \beta_1 = \begin{bmatrix} \beta_1^1 \\ \beta_1^2 \end{bmatrix}, \gamma_1)$, $Y = (\alpha_2, \beta_2 = \begin{bmatrix} \beta_2^1 \\ \beta_2^2 \end{bmatrix}, \gamma_2) \in T_p\mathcal{E}$.

Next let $\mathcal{I} = \begin{bmatrix} * & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -* \end{bmatrix}$, $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $\mathcal{J} = \begin{bmatrix} 0 & 0 & * \\ 0 & J & 0 \\ * & 0 & 0 \end{bmatrix}$, $J =$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\mathcal{K} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & K & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $* : \Omega^1(M) \to \Omega^1(M)$ is the Hodge-star operator. One verifies that $\mathcal{I}\mathcal{J} = \mathcal{K}$, $\mathcal{J}\mathcal{K} = \mathcal{I}$ and $\mathcal{K}\mathcal{I} = \mathcal{J}$. Then, clearly,

$$\begin{aligned} g(\mathcal{I}X, Y) \quad &= -\int_M \alpha_1 \wedge \alpha_2 + \int_M Re < I\beta_1, \beta_2 >_H \omega + \int_M \gamma_1 \wedge \gamma_2 \\ &= \Omega(X, Y). \end{aligned}$$

Finally,

$$\begin{aligned} &\omega_2(X, Y) = g(\mathcal{J}X, Y) \\ &= \int_M -\gamma_1 \wedge \alpha_2 - \int_M \alpha_1 \wedge \gamma_2 + \int_M Re(\beta_1^2 \bar{\beta}_2^1 - \beta_1^1 \bar{\beta}_2^2) \frac{1}{\sqrt{h}} \omega \end{aligned}$$

and

$$\begin{aligned} &\omega_3(X, Y) = g(\mathcal{K}X, Y) \\ &= \int_M - * \gamma_1 \wedge \alpha_2 + \int_M Re < K\beta_1, \beta_2 >_H + \int_M *\alpha_1 \wedge \gamma_2 \\ &= \int_M - * \gamma_1 \wedge \alpha_2 + \int_M Re(i\beta_1^2 \bar{\beta}_2^1 + i\beta_1^1 \bar{\beta}_2^2) \frac{1}{\sqrt{h}} \omega \\ &+ \int_M *\alpha_1 \wedge \gamma_2 \end{aligned}$$

so that indeed

$$\begin{aligned}
(\omega_2 + i\omega_3)(X,Y) &= \int_M (-\gamma_1 - i*\gamma_1)\wedge\alpha_2 \\
&+ \int_M [Re(\beta_1^2\bar{\beta}_2^1 - \beta_1^1\bar{\beta}_2^2) \\
&+ iRe(i\beta_1^2\bar{\beta}_2^1 + i\beta_1^1\bar{\beta}_2^2)]\frac{1}{\sqrt{h}}\omega \\
&\quad + \int(-\alpha_1 + i*\alpha_1)\wedge\gamma_2 \\
&= -2\int_M (\gamma_1)^{1,0}\wedge\alpha_2^{1,0} - \int_M (\beta_1^1\bar{\beta}_2^2 - \bar{\beta}_1^2\beta_2^1)\frac{1}{\sqrt{h}}\omega \\
&\quad -2\int_M (\alpha_1)^{0,1}\wedge\gamma_2^{1,0} \\
&= \mathcal{Q}(X,Y).
\end{aligned}$$

□.

Let $\tilde{\mathcal{S}} = \mu^{-1}(0)\cap\mu_{\mathcal{Q}}^{-1}(0)\subset\mathcal{E}$ be the solution space to the equations (1.1) and (1.2), and denote by $\mathcal{M} = \tilde{\mathcal{S}}/G$ the corresponding moduli space.

We assume $\mathcal{M}$ to be non-empty and to be a smooth manifold.

Theorem 2.2.4. *Let M be a compact Riemann surface of $g > 1$. Let $\mathcal{M}$ be the moduli space of solutions to equations* (1.1) *and* (1.2)*. We assume it to be a smooth manifold (possibly infinite dimensional). Then the Riemannian metric g induced by the metric on $\mathcal{E}$ is hyperkählerian, and $\mathcal{M}$ is hyperkähler.*

Proof: Since $\mathcal{I},\mathcal{J},\mathcal{K}$, g and Ω , $\mathcal{Q}$ are G-invariant, and $\mathcal{M}$ comes from a symplectic reduction, it follows that the symplectic forms ω_i, $i = 1,2,3$, descend to $\mathcal{M}$ as symplectic forms . Also, from the proof of theorem 2.1.5

and proposition 2.2.3 it follows that $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are well defined almost complex structures on $\mathcal{M}$. To show that they are integrable, we use the following lemma by Hitchin (see [6]).

Lemma 2.2.5. *Let g be an almost hyperkähler metric, with 2-forms ω_1, ω_2, ω_3 corresponding to almost complex structures $\mathcal{I}$,$\mathcal{J}$ and $\mathcal{K}$. Then g is hyperkähler if each ω_i is closed.*

2.3 The moduli space $\Sigma_{[\Psi]}$

The moduli space $\mathcal{M}$ of solutions to the first two equations (1.1) and (1.2) will be in general cumbersome to handle though it is hyperkähler, since the equations have only zero-th order terms in the spinors.

We define a new moduli space $\Sigma_{[\Psi]}$ as follows. Choose an appropriate Ψ such that $< \psi_1, \psi_2 >_H \omega$ is ∂-exact and let $\mathcal{W} \doteq \mathcal{A} \times \{G \cdot \Psi\} \times \mathcal{H} \subset \mathcal{E}$. Let $\mathcal{S}_1 \doteq \mathcal{W} \cap \tilde{S}$, where $\tilde{S}$ is the solution space to equations (1.1) and (1.2) on $\mathcal{E}$.

Define $\Sigma_{[\Psi]} \doteq \mathcal{S}_1/G$. Any point $p \in \Sigma_{[\Psi]}$ is given by $p = ([(A, \Psi, \Phi])$ where Ψ is now fixed, $[\cdot,\cdot,\cdot]$ denotes the gauge equivalence class and (A, Ψ, Φ) satisfy equations (1.1) and (1.2).

Note: $\mathcal{S}_1$ essentially consists of $(A, \Phi) \in \mathcal{A} \times \mathcal{H}$ such that $dA = \frac{i}{2} f_1 \omega$ and $2\bar{\partial}\Phi = f_2\omega$, where $f_1 = |\psi_1|^2_H - |\psi_2|^2_H$ and $f_2 = - < \psi_1, \psi_2 >_H \in C^\infty(M)$. We will see that if $(A, \Phi) \in \Sigma_{[\Psi]}$, then if one changes $A \to A' = A + \alpha$ such that $d\alpha = 0$, α unique upto exact forms, and $\Phi^{0,1} \to \Phi'^{0,1} =$

$\Phi^{0,1} + \eta^{0,1}$ such that $\bar{\partial}\eta^{0,1} = 0$ then, the $(A', \Phi') \in \Sigma_{[\Psi]}$. Thus $\Sigma_{[\Psi]}$ is an affine space.

Theorem 2.3.1. *Let M be a Riemann surface of genus $g > 1$. Fix the equivalence class of Ψ such that ψ_1 and ψ_2 are each not identically zero and such that $< \psi_1, \psi_2 >_H \omega$ is $\bar{\partial}$-exact. Assume that $\Sigma_{[\psi]}$ is a smooth non-empty manifold. Then, $\Sigma_{[\Psi]}$ is hyperkähler of dimension $4g$.*

Proof: On $\mathcal{W}$ one defines the same symplectic forms Ω, ω_2 and ω_3 as in the previous section. On $\Sigma_{[\Psi]}$ these forms restrict to

$$\Omega_{[\Psi]} = -\int_M \alpha^1 \wedge \alpha^2 + \int_M \gamma^1 \wedge \gamma^2,$$

$$\omega_{2[\Psi]} = -\int_M \gamma_1 \wedge \alpha_2 - \int_M \alpha_1 \wedge \gamma_2$$

$$\omega_{3[\Psi]} = -\int_M *\gamma_1 \wedge \alpha_2 + \int_M *\alpha_1 \wedge \gamma_2$$

which are, by arguments same as in the previous section, hyperkählerian with respect to the complex structures $\mathcal{I}_1 = \begin{bmatrix} * & 0 \\ 0 & -* \end{bmatrix}$, $\mathcal{J}_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $\mathcal{K}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and to the Riemannian metric

$$g(X, Y) = \int_M *\alpha_1 \wedge \alpha_2 + \int_M *\gamma_1 \wedge \gamma_2$$

where $X = (\alpha_1, \gamma_1)$ and $Y = (\alpha_2, \gamma_2) \in T_p\Sigma_{[\Psi]}$.

To calculate the dimension of $\Sigma_{[\Psi]}$, we linearize equations (1.1) and (1.2) to obtain:

$$(I) \qquad d\alpha = \frac{i}{2}(\beta_1\bar{\psi}_1 + \psi_1\bar{\beta}_1 - \beta_2\bar{\psi}_2 - \psi_2\bar{\beta}_2)\frac{1}{\sqrt{h}}\omega,$$

$$(II) \qquad \bar{\partial}\eta^{1,0} = -\frac{1}{2}(\psi_1\bar{\beta}_2 + \beta_1\bar{\psi}_2)\frac{1}{\sqrt{h}}\omega,$$

where $(\alpha, \beta, \eta) \in T_p\mathcal{W}$ and $\eta = \eta^{1,0} + \eta^{0,1}$. Now we note that $\beta = -\zeta\Psi \in T_\Psi\{G \cdot \Psi\}$ so that these two equations simplify to

$$(I) \quad d\alpha = 0,$$

$$(II) \quad \bar{\partial}\eta^{1,0} = 0.$$

Taking into account the gauge group action, we get

dim $\{\alpha \in \Omega^1(M, i\mathbb{R}) | d\alpha = 0\}/\{\alpha = df\} = 2g$. Also, dim $\{\eta \in \Omega^{1,0}(M, \mathbb{C}) | \bar{\partial}\eta = 0\} = 2g$. Thus the dim $T_p\Sigma_{[\Psi]} = 4g$.

□.

Chapter 3

Determinant Line Bundles

Here we give a survey of the determinant line bundle construction for the families of Cauchy-Riemann operators, following the papers [12], [4], [14].

3.0.1 Determinants of Cauchy-Riemann operators

Let M be a compact Riemann surface and E be a C^∞ vector bundle over M and let $\Omega^{p,q}(E)$ be the space of differential (p,q)-forms on M with values in E.

A Cauchy-Riemann operator in E is a linear mapping $D : \Omega^0(M, E) \to \Omega^{0,1}(M, E)$ that satisfies the property $D(fs) = \bar{\partial} f \otimes s + fDs$, where $f \in C^\infty(M)$ and $s \in \Omega^0(M, E)$. Locally, it looks like $D = \bar{\partial} + \alpha(z)$, where $\alpha(z)$ is a local $(0,1)$-form with values in $End(E)$.

Let $\mathcal{A}$ be the space of all Cauchy-Riemann operators in E. It is an affine space over the vector space $\mathcal{B} = \Omega^{0,1}(End(E))$.

To motivate the construction of the determinant line bundle we first consider the finite-dimensional case. Let $T : V^0 \to V^1$ be a linear map between two complex vector spaces of the same finite dimension. Then T induces a map from $\wedge^{top} V^0$ to $\wedge^{top} V^1$ and determines an element σ_T of the line $\wedge^{top}(V^0)^* \otimes \wedge^{top}(V^1)$, where $*$ stands for the dual vector space. Upon choosing a generator for this line, σ_T can be identified with a function $det(T)$ which is holomorphic in T and is non-zero exactly when the operator T is invertible.

In the infinite-dimensional case of the Cauchy-Riemann operators the above line is replaced by $\mathcal{L}_D = \wedge^{top}(KerD)^* \otimes \wedge^{top}(CokerD)$ where $KerD$ and $CokerD$ are finite-dimensional vector spaces. The family of $\mathcal{L}_D$ forms a holomorphic line bundle $\mathcal{L}$ over $\mathcal{A}$, called the determinant line bundle.

In the next section, following [12] and [4], we give a rigorous definition of $\mathcal{L}$. We also describe a Hermitian metric on $\mathcal{L}$, known as the Quillen metric, using the zeta-function determinant of the Laplacian.This is the idea of "analytic torsion" [14]. The inner product and the holomorphic structure determine a connection on $\mathcal{L}$, whose curvature remarkably turns out to be the standard Kähler form on $\mathcal{A}$. This is the part we would be interested for the purposes of geometric quantization of our moduli spaces.

3.1 Explicit Construction

Let E be a Hermitian vector bundle over M. For every $A \in \mathcal{A}$ let $D_A : \Gamma(M, E) \to \Omega^{0,1}(M, E)$ be the corresponding Cauchy-Riemann operator and let $D_A^\dagger : \Omega^{0,1}(M, E) \to \Gamma(M, E)$ be its adjoint.

Let $\Delta_A = D_A^\dagger D_A$ be the Laplacian of D_A. Then Δ_A has real, discrete spectrum. For $a > 0$, set $U^a = \{A \in \mathcal{A} | a \notin \text{Spec}\,(\Delta_A)\}$ and denote by $K_+^a(A)$, $A \in \mathcal{A}$, the direct sum of eigenspaces of Δ_A of eigenvalues $< a$. Then from general elliptic theory it follows that over the open set U^a, $K_+^a(A)$ form a smooth finite dimensional subbundle K_+^a of $\Gamma(M, E)$ (i.e. its dimension does not jump).

Similarly, denote by $K_-^a(A)$ the direct sum of eigenspaces of the operator $D_A D_A^\dagger$ of eigenvalues $< a$. Over U^a they form a finite dimensional subbundle K_-^a of $\Omega^{0,1}(M, E)$.

Set $\lambda^a = (\wedge^{top} K_+^a)^* \otimes \wedge^{top} K_-^a$. This is a smooth line bundle over U^a. Now recall that the determinant line bundle λ has the fiber over $A \in U^a$ $\lambda_A = \wedge^{top}(Ker D_A)^* \otimes \wedge^{top}(Coker D_A)$. The following exact sequence

$$0 \to Ker D_A \to K_+^a(A) \xrightarrow{D_A} K_-^a(A) \to Ker D_A^* \to 0$$

allows us to identify λ with λ^a over U^a.

Transition functions

For $a, b \notin Spec\Delta_A$, $0 < a < b$, define $K_+^{(a,b)}(A)$ and $K_-^{(a,b)}(A)$ as the direct sum of eigenspaces of $D_A^\dagger D_A$ and $D_A D_A^\dagger$ corresponding to eigenvalues μ with $a < \mu < b$. The vector spaces $K_+^{(a,b)}(A)$ and $K_-^{(a,b)}(A)$ form smooth subbundles $K_+^{(a,b)}$ and $K_-^{(a,b)}$ of $\Gamma(M,E)$ and $\Omega^{0,1}(M,E)$ over $U^a \cap U^b$. Define

$$\lambda^{(a,b)} = (\wedge^{top} K_+^{(a,b)})^* \otimes \wedge^{top} K_-^{(a,b)}$$

and let $D_A^{(a,b)}$ be the restriction of D_A to $K_+^{(a,b)}$, so that $D_A : K_+^{(a,b)} \to K_-^{(a,b)}$.

Clearly, over $U^a \cap U^b$,

$$\lambda^b = \lambda^a \otimes \lambda^{(a,b)}$$

and the identification of λ^a and λ^b over $U^a \cap U^b$ is given by the mapping

$$s \in \lambda^a \to s \otimes detD_A^{(a,b)} \in \lambda^b.$$

Clearly, the line bundle λ over $\mathcal{A}$ is holomorphic.

Quillen's metric on λ

As subbundles of $\Gamma(M,E)$ and $\Omega^{0,1}(M,E)$, the bundles K_+^a and K_-^a over U^a are Hermitian. The bundles $\lambda^a, \lambda^{(a,b)}$ are then naturally endowed with metrics $|\ |^a$ and $|\ |^{(a,b)}$.

Over $U^a \cap U^b$, the vector spaces $K_+^a(A)$ and $K_-^a(A)$ are orthogonal to $K_+^{(a,b)}(A)$ and $K_-^{(a,b)}(A)$ so that for $s \in \lambda^a$,

$$|s \otimes detD_A^{(a,b)}|^b = |s|^a |detD_A^{(a,b)}|^{(a,b)}.$$

When identifying λ with λ^a or λ^b , the metrics $|\ |^a$ and $|\ |^b$ are related to each other by

$$|\ |^b = |\ |^a |\mathrm{det}D_A^{(a,b)}|^{(a,b)}.$$

and in general do not define a metric on λ.

To correct this discrepancy, one uses a zeta function regularized determinant of Δ_A (see [12]).

Let $\zeta(s)$ be the zeta-function of the elliptic operator Δ_A. It is a meromorphic function of s which is defined to be $\sum \eta^{-s}$ for $Re\ s > 1$, where η runs over all non-zero eigenvalues of Δ_A. The zeta-function $\zeta(s)$ is regular at $s = 0$ and depends smoothly on A.

Similarly, let $\zeta(s)_{>a} = \sum \eta^{-s}$ for $Res > 1$, where the sum is taken over all eigenvalues η of Δ_A such that $\eta > a$, and for $0 < a < b$, also define $\zeta^{(a,b)}(s)$ as $\zeta^a(s) = \zeta^{(a,b)}(s) + \zeta^b(s)$. One has the relation

$$|detD_A^{(a,b)}|^{(a,b)} = exp\{-\frac{1}{2}\frac{\partial \zeta^{(a,b)}}{\partial s}(0)\},$$

which suggests the following definition of the metric $\|\cdot\|^a$ on λ^a

$$\|l\|^a = |l|^a exp\{-\frac{1}{2}\frac{\partial \zeta^a}{\partial s}(0)\},$$

where $l \in \Gamma(U^a, \lambda^a)$.

Theorem 3.1.1. *Under the canonical identification of the determinant line bundle λ with λ^a over U^a , the metrics $\|\cdot\|^a$ patch into a smooth metric $\|\cdot\|$ on λ over $\mathcal{A}$, called Quillen's metric.*

The proof follows easily from the definitions.

Quillen curvature

Quillen's remarkable discovery is the computation of the curvature form of the canonical unitary connection ∇_Q in the determinant line bundle λ.

Namely, recall that the affine space $\mathcal{A}$ is an infinite-dimensional Kähler manifold; for every $A \in \mathcal{A}$, $T'_A(\mathcal{A}) = \Omega^{0,1}(M, EndE)$ and the corresponding Kähler form is given by

$$\Omega(X,Y) = \frac{i}{2}\int_M Tr(X \wedge *Y),$$

where $X, Y \in \Omega^{0,1}(M, EndE)$, and $* : \Omega^{0,1}(M, EndE) \to \Omega^{1,0}(M, EndE)$ is the Hodge-star operator and $Tr : \Gamma(E, EndE) \to C^\infty(M)$ is the trace map.

Then one has the following theorem of Quillen

Theorem 3.1.2.

$$F(\nabla_Q) = \frac{2i}{\pi}\Omega.$$

For a proof see [12].

Chapter 4

Prequantization

4.1 Survey of Geometric Quantization

Let (M, ω) be a symplectic manifold.

Step 1: Prequantization. This is a construction of a Hilbert space $\mathcal{H}$ and a correspondence between classical observables - functions on M - and operators on $\mathcal{H}$ such that the Poisson bracket of the functions corresponds to the commutator of the operators.

If ω is integral in $H^2(M)$, a natural way to construct such a Hilbert space would be to define $\mathcal{H}$ as the space of square integrable sections of a line bundle L which has a connection with curvature given by $i\omega$ [17]. Then to each $f \in C^\infty(M)$ one assigns an operator O_f acting on a section s of L defined by $O_f : s \to i\nabla_{X_f} s + fs$, where $X_f = (\omega)^{-1}(-df)$ is the Hamiltonian vector field.

Proposition 4.1.1. *[17] This satisfies the following conditions :*

1. *The mapping $f \to O_f$ is linear.*
2. *If $f = 1$, then $O_f = Id$, where Id is the identity operator.*
3. *If $\{f_1, f_2\} = f_3$, then $[O_{f_1}, O_{f_2}] = -iO_{f_3}$.*

Step 2: Geometric Quantization

Assuming that M is compact and the symplectic form ω has a real or Kähler polarization, one defines a subspace $\mathcal{H}_0 \subset \mathcal{H}$ consisting of the sections which are parallel along the polarization. If M is compact the Hilbert space $\mathcal{H}_0$ is finite dimensional. This construction is useful in the context of topological field theories [3].

Note: This second step will be beyond the scope of this thesis.

4.1.1 Examples

1. Geometric Quantization of the Cotangent Bundle [17].

Let $M = T^*Q$ of an n dimensional manifold Q. The canonical symplectic form on M is given by

$$\omega = d\theta,$$

where θ is the canonical 1-form on M, the so-called Liouville form.

The prequantum line bundle L over M is a trivial line bundle $M \times \mathbb{C}$ with the connection $\nabla = d - i\theta$. For instance, when $Q = \mathbb{R}^n$ with coordinate functions q^i, coordinate functions for M will be $p_i, q^i, i = 1, .., n$. They correspond to the following operators on $\mathcal{H} = L^2(M)$:

$Q_{p_i} = -i\frac{\partial}{\partial q^i}$ and $Q_{q^i} = i\frac{\partial}{\partial p_i} + q^i$, $i = 1, ..., n$.

For the geometric quantization we consider L^2-functions which are constant along the fibres of $M = T^*Q$, i.e. $\mathcal{H}_0 = L^2(Q)$. Thus we arrive at canonical quantization in quantum mechanics:

$\hat{p}_i = -i\frac{\partial}{\partial q^i}$ and $\hat{q}_i = q_i$, $i = 1, ..., n$.

2. Geometric quantization of the moduli space of flat connections in the context of Chern-Simons Theory (A sketch).

The moduli space of flat connections of a principal G-bundle on a Riemann surface has been quantized in the following way [3], [4], [7], [12], [19]. Consider the determinant line bundle $\mathcal{L} = det(Ker\bar{\partial}_A)^* \otimes det(Coker\bar{\partial}_A)$. It carries the Quillen metric such that the canonical unitary connection has a curvature form which coincides with the natural Kähler form on the moduli space of flat vector bundles over M of a given rank. This provides a geometric quantization of the moduli space.

4.2 Prequantization of $\Sigma_{[\psi]}$

Step 1 : Definition of the prequantum line bundle $\mathcal{L}$ over $\mathcal{W}$.

(a) Over the affine space $\mathcal{A} \times \mathcal{H}$ consider the family of Cauchy-Riemann operators $D_{(A,\Phi)} : \Gamma(M, L) \to \Omega^{0,1}(M, L)$, given by $D_{(A,\Phi)} = \nabla''_A + \Phi^{0,1}$ where $\nabla_A = \nabla'_A + \nabla''_A$ and $\Phi^{0,1}$ is the $(0, 1)$-part of $\Phi \in \mathcal{H}$.

We consider the determinant line bundle $\mathcal{R}$ of the family of these operators, which is holomorphic with respect to the complex structure given by $\mathcal{I}_2 = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ on the affine space $\mathcal{A} \times \mathcal{H}$, where $*$ is the Hodge star operator.

Then, it follows from theorem 3.1.2 that the Quillen curvature 2-form of $\mathcal{R}$ is given by

$$F(1) \qquad \mathcal{F}_{\mathcal{R}}(X,Y) = -\frac{1}{\pi}\int_M \tau^{0,1} \wedge \kappa^{1,0},$$

where $X = (\alpha_1, \gamma_1), Y = (\alpha_2, \gamma_2) \in T_{(A,\Phi)}(\mathcal{A} \times \mathcal{H})$, and $\tau = \alpha_1 + \gamma_1$, $\kappa = \alpha_2 + \gamma_2 \in \Omega^1(M, i\mathbb{R})$.

(b) Next consider the antiholomorphic bundle $\tilde{\mathcal{R}}$ which is the determinant line bundle of the adjoint $D^{\dagger}_{(A,\Phi)} = \nabla'_A + \Phi^{1,0}$ of $D_{(A,\Phi)}$, where $D^{\dagger}_{(A,\Phi)} : \Omega^{0,1}(M,L) \to \Omega^{1,1}(M,L)$.

The determinant line bundle of the family $D^{\dagger}_{(A,\Phi)}$ is anti-holomorphic with respect to the complex structure $\mathcal{I}_2$ and the Quillen curvature is given by

$$F(2) \qquad \mathcal{F}_{\tilde{\mathcal{R}}}(X,Y) = -\frac{1}{\pi}\int_M \tau^{1,0} \wedge \kappa^{0,1}.$$

(c) Similarly as in (a) we construct the determinant line bundle $\mathcal{T}$ for the the family of Cauchy-Riemann operators $C_{(A,\Phi)} = \nabla''_A - \Phi^{0,1}$. The Quillen curvature is given by

$$F(3) \qquad \mathcal{F}_{\mathcal{T}}(X,Y) = -\frac{1}{\pi}\int_M \tau^{0,1} \wedge \kappa^{1,0},$$

where $\tau = \alpha_1 - \gamma_1$, $\kappa = \alpha_2 - \gamma_2 \in \Omega^1(M, i\mathbb{R})$.

(d) Similarly as in (b) we construct the determinant line bundle $\mathcal{T}^\dagger$ of the operator $C^\dagger_{(A,\Phi)}$ and find its Quillen curvature to be

$$F(4) \qquad \mathcal{F}_{\tilde{\mathcal{T}}}(X,Y) = -\frac{1}{\pi}\int_M \tau^{1,0} \wedge \kappa^{0,1},$$

where $\tau = \alpha_1 - \gamma_1$, $\kappa = \alpha_2 - \gamma_2$, $\in \Omega^1(M, i\mathbb{R})$.

Set

$$\mathcal{L} \doteq \mathcal{R} \otimes \tilde{\mathcal{R}} \otimes \mathcal{T} \otimes \tilde{\mathcal{T}}.$$

This is a well defined line bundle over $\mathcal{A} \times \mathcal{H}$.

We now compute the curvature of $\mathcal{L}$ which is given by $F(1)$ $+F(2)$ $+F(3)$ $+F(4)$. One easily calculates this curvature to be

$$\Omega'(X,Y) = -\frac{2}{\pi}\int_M (\alpha_1 \wedge \alpha_2 + \gamma_1 \wedge \gamma_2),$$

where $X = (\alpha_1, \gamma_1), Y = (\alpha_2, \gamma_2) \in T_p(\mathcal{A} \times \mathcal{H})$.

Step 2: Definition of the prequantum line bundle over $\Sigma_{[\Psi]}$.

We show that $\mathcal{R}$ descends to the quotient by the gauge group.

Let $u \in G$ act on $\mathcal{A} \times \mathcal{H}$. Then it is easy to check that if $(A, \Phi) \to (A + u^{-1}du, \Phi) = (A \cdot u, \Phi)$, then $D_{(A\cdot u,\Phi)}u^{-1}s = u^{-1}D_{(A,\Phi)}s$, where $s \in \Gamma(M, L)$ and $D^\dagger_{(A\cdot u,\Phi)}u^{-1}t = u^{-1}D^\dagger_{(A,\Phi)}t$, where $t \in \Omega^{0,1}(M, L)$. Thus for every $u \in G$ there is an isomorphism of the eigenspaces of the operators $\Delta = D^\dagger_{(A,\Phi)}D_{(A,\Phi)}$ and $\Delta_u = D^\dagger_{(A\cdot u,\Phi)}D_{(A\cdot u,\Phi)}$ given by $s \to u^{-1}s$.

It follows from this that there is a smooth isomorphism of the fibers of $\mathcal{R}$ over (A, Φ) and $(A \cdot u, \Phi)$. Namely, over $(A, \Phi) \in U^a$ the fiber of $\mathcal{R}$ is $(\wedge^{top}K^a_+)^* \otimes (\wedge^{top}K^a_-)$. By the previous remark, the fiber of $\mathcal{R}$ over

$(A \cdot u, \Phi) \in U^a$ is then $(\wedge^{top} K^a_{u+})^* \otimes (\wedge^{top} K^a_{u-})$ where K^a_{u+} is the direct sum of eigenspaces of eigenvalues $< a$ of Δ_u. These fibers are isomorphic by the canonical isomorphism of $K^a_\pm$ and $K^a_{u\pm}$ which takes $s \in K^a_\pm \to u^{-1}s \in K^a_{u\pm}$.

Now the gauge group action on $\Gamma(M, L)$ induces the action $s \to u^{-1}s$, where $s \in K^a_\pm$. Thus we can identify the fibers in canonical way when we take quotient by the gauge group. Thus $\mathcal{R}$ is well defined over the quotient space.

Same result holds for $\tilde{\mathcal{R}}$, $\mathcal{T}$ and $\tilde{\mathcal{T}}$ and we have the following proposition:

Proposition 4.2.1. *The line bundle $\mathcal{L}$ is well defined over $(\mathcal{A} \times G \cdot \Psi \times \mathcal{H})/G$. When restricted to $\Sigma_{[\Psi]} \subset (\mathcal{A} \times G \cdot \Psi \times \mathcal{H})/G$ it has the following curvature 2-form*

$$F(X, Y) = -\frac{2}{\pi} \int_M \alpha \wedge \eta - \frac{2}{\pi} \int_M \gamma \wedge \delta,$$

where $X = (\alpha, \gamma) \in T_p\Sigma_{[\Psi]}$ and $Y = (\eta, \delta) \in T_p\Sigma_{[\Psi]}$.

Step 3 The curvature form is a symplectic form.

Theorem 4.2.2. *The form $\Omega' = \frac{\pi}{2}F$ is a symplectic form on $\Sigma_{[\Psi]}$ compatible with the almost complex structure $\mathcal{I}' = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and the metric g*

$$g(X, Y) = \int_M *\alpha \wedge \eta + \int_M *\gamma \wedge \delta.$$

Proof: The proof goes along the same lines as the proof of theorem 2.1.5. On the affine space $\mathcal{W}$ define a new symplectic form

$$\Omega_2(X, Y) = -\int_M \alpha \wedge \eta + \int_M Re < I\beta_1, \beta_2 >_H - \int_M \gamma \wedge \delta,$$

where $X = (\alpha, \beta_1, \gamma)$, $Y = (\eta, \beta_2, \delta) \in T_p\mathcal{W}$. Let $\mathcal{I}_2 = \begin{bmatrix} * & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & * \end{bmatrix}$ be a complex structure on $\mathcal{W}$, where $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. The symplectic form Ω_2 is compatible with $\mathcal{I}_2$ and the following metric on $\mathcal{W}$

$$g(X,Y) = \int_M *\alpha \wedge \eta + \int_M Re < \beta_1, \beta_2 >_H + \int_M *\gamma \wedge \delta.$$

The two equations we are considering are, for an appropriate gauge-equivalence class of Ψ,

$$F(A) = \frac{i}{2}(|\psi_1|_H^2 - |\psi_2|_H^2)\omega, \tag{1.1}$$

$$\bar{\partial}\Phi = -\frac{1}{2} < \psi_1, \psi_2 >_H \omega. \tag{1.2}$$

Once again, it is easy to see that equation (1.1) can be realised as a moment map $\mu_{\Omega_2} = \mu = 0$, where μ is as in proposition 2.1.3 and is now considered as a moment map with respect to Ω_2.

Now on $\mathcal{W}$, the linearization of (1.1) and (1.2) is

$$d\alpha = 0, \tag{1l}$$

$$\bar{\partial}\gamma = 0, \tag{2l}$$

where $(\alpha, \beta) = (-\zeta\Psi, \gamma) \in T_p\mathcal{W}$ and $\zeta \in \Omega^0(M, i\mathbb{R})$.

It is easy to check that if $X \in T_p\mathcal{W}$ satisfies equation $(2l)$ then $\mathcal{I}_2 X$ satisfies it as well. Now the proof of theorem 2.1.5 gives the result for $\Sigma_{[\Psi]}$ with Ω_2 descending to a symplectic form Ω'.

$\square$.

Thus, we have the following

Theorem 4.2.3. *$\mathcal{L}$ is a prequantum line bundle over $\Sigma_{[\Psi]}$.*

Note: Since $\Sigma_{[\Psi]}$ is an affine space, this construction by itself maynot be useful. But we have been able to show a similar construction on $\tilde{\mathcal{M}}_v$ which we donot include here.

4.3 Concluding Remarks

We are planning to prove that the moduli spaces are non-empty , smooth and $\mathcal{M}_v$, $\tilde{\mathcal{M}}_v$ and $\mathcal{N}$ are of finite volume. We also want to address the question of what topology we would like to impose on these moduli spaces.

We have constructed the prequantum bundle on $\tilde{\mathcal{M}}_v$ by incorporating the spinors in the determinant line bundle construction and are planning to extend this construction to $\mathcal{N}$, the moduli space of solutions to $(1.1) - (1.3)$. Next, we wish to adapt the second step of geometric quantization for manifolds of finite volume and thus hope to complete the geometric quantization of the moduli spaces.

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