Hardy Modules

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Abstract of the Dissertation
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The Hardy space is a module over the ring of bounded analytic functions. In this thesis, we will study the submodules and the quotient modules of the Hardy space. Special attentions are given to the compressions of the shift operators to those submodules and quotient modules.

To my mother and father.

Contents

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Introduction

Operator theory has been greatly enriched after the introduction of the Hilbert spaces of analytic functions. On one hand, the analytic function theory makes it possible to reformulate and solve many classical operator theoretical problems; on the other hand, it opens many new field of studies in which algebra, geometry and topology also play fundamental roles. A very illustrative example is the study of the *Hardy space* over the unit disk, the results of which have found essential applications not only in operator theory itself such as to the invariant subspace problem, the BDF theory and the theory of hyponormal operators, but also in index theory, group representation theory and statistics.

In recent years, many attempts have been made to explore the multi-variate analogue of this study. Discovered by Ronald G. Douglas, many of these attempts can be systemized if the language of module is adopted and a groundwork was laid down in [**DP**]. This module language not only properly emphasized the key problems in the multi-variate operator theory but also made clear its connections with algebraic geometry and commutative algebra. But operator theory has its own infinite dimensional nature which distinguish itself from algebra and geometry. This thesis will focus on some operator theoretical problems in the *Hardy modules* even though certain algebriac aspect will also be touched upon.

In Chapter 1 we first study some properties of the submodule of the Hardy space over the bidisk and give an estimate of a paritular numerical invariant of the submodule. Then we will study a specific kind of submodules defined by linear equations and prove that under certain conditions these submodules are all similar to the Hardy space itself.

In some respect the compression operators on the *quotient module* are the multi-variate analogies of the Nagy-Foias models for completely nonunitary contractions. In Chapter 2 we will first show how the spectral properties of the compression operators are related to the zero varieties and then prove a so called *rigidity* phenomenon for quotient modules. Results in this chapter are the joint work with Ronald G. Douglas.

In Chapter 3 we will study the commutation problems of the compression operators and show that the cross commutators of certain compressions are Hilbert-Schmidt. The results can be viewed as the two-variate analogue of the classical Berger-Shaw theorem for single hyponormal operators.

We conclude this thesis with a brief review of some results and their prospect of applications .

Chapter 1

Preliminaries

In this chapter we will introduce the notations that we will use later on and list some theorems related to the studies in this thesis.

Hardy space We let $\mathbf{C}^n$ denote the cartesian product of n copies of the complex field. The points of $\mathbf{C}^n$ are thus ordered n-tuples $z = (z_1, z_2, ..., z_n)$ and z^s stands for the n-tuple $(z_1^{s_1}, z_2^{s_2}, ..., z_n^{s_n})$ for any multi-index $s = (s_1, s_2, ..., s_n) \in \mathbf{Z}_+^n$, where $\mathbf{Z}_+$ is the set of nonnegative integers. In the case $n = 2$, we shall use z, w instead of z_1, z_2 to denote the coordinate functions. The ring of polynomials of $z = (z_1, z_2, ..., z_n)$ will be denote by $\mathcal{R}$ though sometimes the standard notation $\mathbf{C}[z_1, z_2, ..., z_n]$ is also used to avoid possible confussions. The ideal generated by polynomials $p_1, p_2, ..., p_n$ is denoted by $(p_1, p_2, ..., p_n)$.

D^n will be the unit polydisk in $\mathbf{C}^n$ with distinguished boundary T^n, where T is the unit circle. The closure of polynomials over D^n under the supremum norm will be denoted by $A(D^n)$ and called the polydisk algebra. The Hardy space $H^2(D^n)$ is the Hilbert space of holomorphic functions over D^n which satisfy the inequality

$$sup_{0\leq r<1} \int_{T^n} |f(rz)|^2 dm < \infty,$$

where dm is the normalized Lebesgue measure on T^n. The norm $\|f\|$ of a function $f \in H^2(D^n)$ is defined by

$$\|f\|^2 := sup_{0\leq r<1} \int_{T^n} |f(rz)|^2 dm.$$

The inner product induced by this norm will be denoted by $< \cdot, \cdot >$.

By Fatou's theorem, every function in $H^2(D^n)$ has nontangential limit at almost every point of T^n. If we let $\hat{f}$ denote the boundary function of $f \in H^2(D^n)$, then

$$\hat{f} \in H^2(T^n, dm) := \overline{span}\{z^s : z \in T^n \; and \; s \in \mathbf{Z}_+^n\},$$

where the closure is taken in $L^2(T^n, dm)$. And it is also well known that each function in $H^2(T^n, dm)$ has an unique analytic extension to D^n which belongs in $H^2(D^n)$. For convenience, we identify $H^2(D^n)$ with $H^2(T^n, dm)$ and will use f to denote its boundary value $\hat{f}$ as well.

For any bounded function ϕ on $A(D^n)$, we define a so called *Toeplitz* operator T_ϕ mapping $H^2(D^n)$ to itself such that

$$T_\phi(f) = P(\phi f),$$

where P is the orthogonal projection from $L^2(T^n, dm)$ to $H^2(D^n)$. The study of Toeplitz operators has been very active in the last thirty years and many nice results were abtained. Good references for this subject are [**Do**2] and [**Up**].

$H^\infty(D^n)$ is the space of all bounded holomorphic functions in D^n with

$$\|f\|_\infty = sup|f(z)|, \quad z \in D^n,$$

and it is easily seen that $H^\infty(D^n)$ is a Banach algebra with pointwise multiplication and addition. The collection of invertible elements in algebra $H^\infty(D^n)$ is denoted by $[H^\infty(D^n)]^{-1}$.

It is well known that the space $H^2(D^n)$ is an $A(D^n)$ -module with action defined by the pointwise multiplication by $A(D^n)$ functions. For any $h \in H^2(D^n)$, we let

$$[h] := \overline{A(D^n)h}^{H^2}$$

be the submodule generated by the function h. A function $h \in H^2(D^n)$ is called *Helson outer*(denoted outer(H)) if $[h]$ is equal to $H^2(D^n)$ and is called *inner* if $|h(z)|$ is equal to 1 almost everywhere on T^n. It is easy to see that when h is inner,

$$[h] = hH^2(D^n).$$

Compressions of the shift operators If M is a proper submodule of $H^2(D^n)$ and we let

$$p : H^2(D^n) \longrightarrow M, \quad q : H^2(D^n) \longrightarrow H^2(D^n) \ominus M$$

be the orthogonal projections, then one checks that the map $S : A(D^n) \longrightarrow \mathcal{B}(H^2(D^n) \ominus M)$ defined by

$$S_f g := qfg, \quad f \in A(D^n), \ \ g \in H^2(D^n) \ominus M,$$

is a homomorphism which turns $H^2(D^n) \ominus M$ into a quotient $A(D^n)$-module, where $\mathcal{B}(H)$ stands for the collection of all the bounded linear operators acting on the Hilbert space H. One sees that the operators $S_{z_1}, S_{z_2}, ..., S_{z_n}$ are compressions of the Toeplitz operators $T_{z_1}, T_{z_2}, ..., T_{z_n}$ onto $H^2(D^n) \ominus M$. The restrictions of $T_{z_1}, T_{z_2}, ..., T_{z_n}$ to the submodule M will be denoted by $R_{z_1}, R_{z_2}, ..., R_{z_n}$. For convenience we denote S_{z_j} simply by S_j and R_{z_j} by R_j, $j = 1, 2, ..., n$. The main purpose of this thesis is to study the relation between the operator theoretical properties of S_{z_j}'s and R_{z_j}'s and the properties of the submodule M.

Some related results The studies of the submodules of $H^2(D^n)$ were carried out by many authors. Here we only list some results that are pertinent to our studies.

Theorem 1.0.1 *([Ge]) If h is a polynomial in the polynomial ring $\mathbb{C}[z_1, z_2, ..., z_n]$, then*

$$[h] = H^2(D^n)$$

if and only if h has no zero in D^n.

This theorem will be used very often to exclude some trivial cases in the proof of many of the results in the thesis.

If H_1 and H_2 are two $A(D^n)$ modules, then H_1 is said to be *unitarily equivalent(similar)* to H_2 if there is a unitary(invertible) module map from H_1 to H_2. A bounded module map T from H_1 to H_2 is called *quasi-affine* if it is one to one and has dense range. H_1 and H_2 are said to be *quasi-similar* if there are quasi-affine module maps from H_1 to H_2 and from H_2 to H_1. One sees that similarity implies quasi-similarity.

Theorem 1.0.2 *([**ACD**]) A submodule M is unitarily equivalent to $H^2(D^n)$ if and only if*

$$M = \phi H^2(D^n)$$

for some inner function ϕ.

Theorem 1.0.3 *([**DF**]) If H_1 and H_2 are two submodules of $H^2(D^n)$ then $H^2(D^n) \ominus H_1$ is unitarily equivalent to $H^2(D^n) \ominus H_2$ if and only if $H_1 = H_2$.*

The phenomenon described in this theorem is called the *rigidity* phenomenon. We will say more about the rigidity phenomenon in Chapter 3.

Two bounded operators A, B are said to *doubly commute* if A commutes with B and its adjoint B^*.

Theorem 1.0.4 *([**Ma**]) If M is a submodule in $H^2(D^2)$, then R_z doubly commutes with R_w on M if and only if M is unitarily equivalent to $H^2(D^2)$, i.e.*

$$M = \phi H^2(D^2)$$

for some inner function ϕ by Theorem 1.0.2.

In Chapter 2, we will study the conditions on M under which S_z doubly commutes with S_w on $H^2(D^n) \ominus M$.

Theorem 1.0.5 *([**CMY**]) If M is a submodule of $H^2(D^2)$ that is generated by homogenuous polynomials, then $R_z^* R_w - R_w R_z^*$ is Hilbert-Schmidt.*

In Chapter 5 we will generalize this result to all the submodules generated by polynomials. Our result settle a question raised by Curto. A corresponding study for the cross commutator $S_z^*S_w - S_wS_z^*$ will also be carried out there.

Chapter 2

Submodules

In this chapter, we will mainly study the elementary properties of the submodules of $H^2(D^2)$ even though some of the results still hold for $H^2(D^n)$.

We first give an estimate of the dimension of the quotient $M \ominus IM$, where M is any submodule of $H^2(D^2)$ and $I \subset \mathcal{R}$ is any ideal. Then will study the conditions on M under which S_z doubly commutes with S_w on $H^2(D^2) \ominus M$. Section 2.3 is devoted to the proof of a theorem which identifies a specific kind of submodules in $H^2(D^2) \oplus H^2(D^2)$.

2.1 Proper Submodule

It is well known that the collection of functions in $H^2(D^2)$ which vanish at the origin (0, 0) is a closed proper subset of $H^2(D^2)$. It is also not hard to believe that $zM + wM$ should not be dense in M for any submodule M since both z and w vanish at the origin. We begin this section by giving a proof of this elementary fact.

Let $E_k = span\{z^k,\ z^{k-1}w,\ ...,\ zw^{k-1},\ w^k\}$ and P_k be the orthogonal projection from $H^2(D^2)$ to E_k. Then

$$\sum_{k=0}^{\infty} P_k = I.$$

For $f \in H^2(D^2)$, we define

$$ord(f) = min\{k : P_k f \neq 0\}.$$

Lemma 2.1.1 *If $f \in H^2(D^2)$, then there is a positive constant ϵ such that $g \in H^2(D^2)$ and $\|f - g\| < \epsilon$ imply $ord(f) \geq ord(g)$.*

Proof. If $ord(f) = m$, then

$$f = \sum_{k=m}^{\infty} P_k f$$

and

$$\|f\|^2 = \sum_{k=m}^{\infty} \|P_k f\|^2.$$

Choose $\epsilon = \|P_m f\|$, then for every $g \in H^2(D^2)$ with $\|f - g\|^2 < \epsilon^2$,

$$\epsilon^2 > \|f - g\|^2 = \sum_{k=0}^{m-1} \|P_k g\|^2 + \|P_m f - P_m g\|^2 + \sum_{k=m+1}^{\infty} \|P_k f - P_k g\|^2.$$

Therefore,

$$\|P_m f - P_m g\| < \epsilon$$

and hence

$$\begin{aligned} \|P_m g\| &\geq \|P_m f\| - \|P_m f - P_m g\| \\ &= \epsilon - \|P_m f - P_m g\| > 0 \end{aligned}$$

This implies

$$ord(g) \leq m.$$

□

For any subset $S \subset H^2(D^2)$, we denote its closure in $H^2(D^2)$ by $\overline{S}$ and define

$$ord(S) := min\{ord(f) : f \in S\}.$$

The following corollary is a direct consequence of Lemma 2.1.1 and its proof.

Corollary 2.1.2 *a.* $ord(S) = ord(\overline{S})$. *b.* $P_k(S) = P_k(\overline{S})$ $\forall k \geq 0$.

For any submodule $M \subset H^2(D^2)$, we let

$$E_k(M) := P_k(M).$$

It is not hard to see that $E_k(M)$ is a subspace of E_k. One checks also that for each $f \in H^2(D^2)$,

$$zP_kf = P_{k+1}zf, \quad wP_kf = P_{k+1}wf.$$

These observations and Corollary 2.1.2(b) yield the following

Lemma 2.1.3 *For any submodule $M \subset H^2(D^2)$,*

$$E_{k+1}(zM + wM) = zE_k(M) + wE_k(M) \subset E_{k+1}(M), \quad k \geq 0.$$

Corollary 2.1.4 *For any submodule $M \subset H^2(D^2)$,*

$$ord(\overline{zM + wM}) = ord(M) + 1.$$

In particular $\overline{zM + wM}$ is proper in M.

Proof. From the definition, $ord(M) = min\{k : E_k(M) \neq \{0\}\}$. Lemma 2.1.3 implies that $E_k(M) \neq \{0\}$ if and only if $E_{k+1}(zM + wM) \neq \{0\}$ and hence the corollary follows from Corollary 2.1.2(a). □

Similar arguments will show that $\overline{(z-\alpha)M + (w-\beta)M}$ is proper in M for any $(\alpha, \beta) \in D^2$, hence the following

Corollary 2.1.5 *If $I \subset \mathcal{R}$ is an ideal whose zero variety $V(I)$ intersects D^2 nontrivially, then $\overline{IM}$ is proper in M.*

Proof. Select any point, say (α, β), in $V(I) \cap D^2$. Then I is contained in the ideal generated by $z - \alpha$ and $w - \beta$(denoted by $(z-\alpha,\ w-\beta)$). Hence

$$\overline{IM} \subset \overline{(z-\alpha,\ w-\beta)M} = \overline{(z-\alpha)M + (w-\beta)M}.$$

□

The following theorem gives an estimate of the dimension of $M \ominus IM$.

Theorem 2.1.6 *If $I \subset \mathcal{R} = \mathbf{C}[z, w]$ is an ideal and $M \subset H^2(D^2)$ is a submodule, then*

$$dim(M \ominus IM) \leq dim(\mathcal{R}/I)rank(M).$$

Proof. We assume $dim(\mathcal{R}/I) = m_1 < \infty$ with a basis $\{v_1,\ v_2,\ ...,\ v_{m_1}\}$ for $\mathcal{R}/I$ and $rank(M) = m_2 < \infty$ with a generating set $\{e_1,\ e_2,\ ...,\ e_{m_2}\}$ for M.

If $\phi \in M \ominus IM$, then there is a sequence of polynomials $\{f_j^n : n \geq 0,\ j = 1,\ 2,\ 3, ...,\ m_2\}$ such that

$$\lim_{n\to\infty} \sum_{j=1}^{m_2} f_j^n e_j = \phi$$

in $H^2(D^2)$. For each f_j^n, we write

$$f_j^n = f_{j,I}^n + r_j^n$$

with $f_{j,I}^n \in I$ and $r_j^n \in \mathcal{R}/I$. If we let $P : M \longrightarrow M \ominus IM$ be the orthogonal projection, then

$$\begin{aligned}\phi = P\phi &= P(\lim_{n\to\infty}\sum_{j=1}^{m_2} f_j^n e_j)\\ &= \lim_{n\to\infty}\sum_{j=1}^{m_2} P(f_{j,I}^n e_j + r_j^n e_j)\\ &= \lim_{n\to\infty}\sum_{j=1}^{m_2} P(r_j^n e_j).\end{aligned}$$

Since $\{v_1,\ v_2,\ ...,\ v_{m_1}\}$ is a basis for R/I, we can write

$$r_j^n = \sum_{i=1}^{m_1} c_{j,i}^n v_i,$$

where $c_{j,i}^n,\ n \geq 0,\ 1 \leq i \leq m_1,\ 1 \leq j \leq m_2$ are constants. Then,

$$\phi = \lim_{n\to\infty}\sum_{j=1}^{m_2}\sum_{i=1}^{m_1} c_{j,i}^n P(v_i e_j).$$

and hence $\phi \in span\{P(v_i e_j),\ 1 \leq i \leq m_1,\ 1 \leq j \leq m_2\}$. Therefore,

$$dim(M \ominus IM) \leq m_1 m_2 = dim(\mathcal{R}/I)rank(M).$$

□

Corollary 2.1.7 *If $M \subset H^2(D^2)$ is a submodule, then*

$$dim(M \ominus (zM + wM)) \leq rank(M).$$

Proof. Let $I = (z,\ w) \subset \mathcal{R}$, then $dim(\mathcal{R}/I) = 1$ and $IM = zM + wM$. The corollary thus follows directly from Theorem 2.1.6. □

Corollary 2.1.8 *If* $I_1,\ I_2,\ ...,\ I_k$ *are ideals in* $\mathcal{R}$ *and we set*

$$\hat{I}_j = I_1 I_2 \cdots I_{j-1} I_{j+1} \cdots I_k, \quad J = \hat{I}_1 + \hat{I}_2 + \cdots + \hat{I}_k,$$

then

$$dim(\cap_{j=1}^k [I_j] \ominus [\prod_{j=1}^k I_j]) \leq dim(\mathcal{R}/J) rank(\cap_{j=1}^k [I_j]).$$

Proof. We denote $\cap_{j=1}^k [I_j]$ by N. For any $\phi \in N$, there is a sequence of polynomials $\{p_j^n : 1 \leq j \leq k,\ n \geq 0\}$ such that $\{p_j^n :\ \ n \geq 0\} \subset I_j$ and

$$\lim_{n\to\infty} p_j^n = \phi$$

for each $1 \leq j \leq k$. If $f_j \in \hat{I}_j$, $j = 1,\ 2,\ ...,\ k$, then

$$(\sum_{j=1}^k f_j)\phi = \sum_{j=1}^k \lim_{n\to\infty} f_j p_j^n.$$

But for each j, $f_j p_j^n \in \hat{I}_i I_j = I_1 I_2 \cdots I_k$, so

$$(\sum_{j=1}^k f_j)\phi \in [\prod_{j=1}^k I_j].$$

This shows $JN \subset [\prod_{j=1}^k I_j]$ and hencs

$$dim(N \ominus [\prod_{j=1}^k I_j]) \leq dim(N \ominus JN).$$

The corollary then follows from Theorem 2.1.6. □

The equality in Corollary 2.1.8 holds in some cases.

Example. If $I_1 = I_2 = (z,\ w)$, then $J = (z,\ w)$ and hence $dim(\mathcal{R}/J) = 1$. It is also easy to see that $[I_1 I_2] = [(z^2,\ zw,\ w^2)]$ and one checks that

$$[(z,\ w)] \ominus [(z^2,\ zw,\ w^2)] = span\{z,\ w\}.$$

Therefore,

$$dim([(z,\ w)] \ominus [(z^2,\ zw,\ w^2)]) = 2 = dim(\mathcal{R}/J)rank([(z,\ w)]).$$

By Corollary 2.1.8, if $J = \mathcal{R}$ then $\cap_{j=1}^{k}[I_j] = [\prod_{j=1}^{k} I_j]$. We can improve the result a little bit. For simplicity we state the improved result for $k = 2$.

Corollary 2.1.9 *If $I_1,\ I_2$ are ideals of $\mathcal{R}$ such that $(I_1+I_2)\cap(H^\infty(D^2))^{-1} \neq \emptyset$, then*

$$[I_1] \cap [I_2] = [I_1 I_2].$$

Proof. In the proof of Corollary 2.1.8, we see that $(I_1+I_2)([I_1]\cap[I_2]) \subset [I_1 I_2]$. If $(I_1 + I_2) \cap (H^\infty(D^2))^{-1} \neq \emptyset$ then

$$[I_1] \cap [I_2] = (I_1 + I_2)([I_1] \cap [I_2]) \subset [I_1 I_2] \subset [I_1] \cap [I_2].$$

Hence,

$$[I_1] \cap [I_2] = [I_1 I_2].$$

□

Corollary 2.1.7 can be used to prove the existence of a submodule of $H^2(D^2)$ with infinite rank. A concrete example was constructed in **[Ru]**.

Corollary 2.1.10 *There is a submodule $M \subset H^2(D^2)$ with $rank(M) = \infty$.*

Proof. It is well known that the Bergman space $L_a^2(D)$ and the quotient space $H^2(D^2) \ominus [z-w]$ are unitarily equivalent $A(D)$ modules(see [**Ru**]) and a subspace $K \subset H^2(D^2) \ominus [z-w]$ is invariant for S_z if and only if $K = H \ominus [z-w]$ for some submodule H that contains $[z-w]$. So by Corollary 5.5[**BFP**] and Proposition 10.1[**BFP**] there are submodules M and N with

$$[z-w] \subset N \subset M \subset H^2(D^2)$$

such that

$$dim(M \ominus N) = \infty,$$

and the compression $p_\perp z = 0$ on $M \ominus N$, where $p_\perp$ is the orthogonal projection from $H^2(D^2)$ onto $M \ominus N$. This means that if $f \in M \ominus N$ then $zf \perp (M \ominus N)$. But $p_\perp w - p_\perp z = p_\perp(z-w) = 0$, so we also have $wf \perp (M \ominus N)$. This implies that

$$M \ominus N \subset M \ominus \overline{(zM + wM)},$$

and hence,

$$\infty = dim(M \ominus N) \leq dim(M \ominus \overline{(zM + wM)}).$$

The corollary then follows from Corollary 2.1.7. $\square$

2.2 Conditions for $[S_z,\ S_w^*] = 0$

We recall from Chapter 1 that given a submodule M, $R_z,\ R_w$ are restrictions of the Toeplitz operattors $T_z,\ T_w$ on M and $S_z,\ S_w$ are the compressions of $T_z,\ T_w$ to the quotient space $H^2(D^2) \ominus M$.

Theorem 1.0.4 says that R_z, R_w doubly commute on some submodule M if and only if $M = \phi H^2(D^2)$ for some inner function ϕ. In this section we will study the necessary conditions for S_z to doubly commute with S_w. Some related questions will also be studied.

Proposition 2.2.1 *If M is a submodule such that the commutator $[S_z,\ S_w^*] = 0$ on $K = H^2(D^2) \ominus M$, then M contains a function in one variable.*

Proof. By **[Ya]**(also see Proposition 4.3.1),

$$[S_z,\ S_w^*] = -(I-p)\overline{w}pz, \tag{2.1}$$

where p is the orthogonal projection from $H^2(D^2)$ onto M. So for $f,\ g$ in K,

$$\begin{aligned} 0 = - < [S_z,\ S_w^*]f,\ g > &= < (I-p)\overline{w}pzf,\ g > \\ &= < pzf,\ pwg > . \end{aligned}$$

One also checks that for every $h \in M$,

$$< pzf,\ zh >=< zf,\ zh >=< f,\ h >= 0,$$

i.e. pz maps K into $M \ominus zM$ and similarly pw maps K into $M \ominus wM$. By Corollary 2.1.5,

$$(M \ominus zM) \cap (M \ominus wM) = M \ominus \overline{zM + wM} \neq \{0\}.$$

So either $pz(K)$ is not dense in $M \ominus zM$ or $pw(K)$ is not dense in $M \ominus wM$. We assume $pz(K)$ is not dense in $M \ominus zM$; then there is a $\phi \in M \ominus zM$ such that

$$< zf,\ \phi >=< pzf,\ \phi >= 0,$$

for any $f \in K$. Therefore ϕ is orthogonal to both zM and zK and hence is orthogonal to $zM \oplus zK = z(M \oplus K) = zH^2(D^2)$. So ϕ is a function in w only.

□

Corollary 2.2.2 *M is a submodule such that $K = H^2(D^2) \ominus M$ is invariant for multiplication by z if and only if*

$$M = \phi H^2(D^2)$$

for some inner function ϕ depending on w only.

Proof. If K is invariant for z then by the proof of Proposition 2.2.1 every function in $M \ominus zM$ depends only on w, and hence $M \ominus zM$ is also invariant for the multiplication by w. By Beurling's Theorem,

$$M \ominus zM = \phi H^2(D)$$

for some inner function ϕ depending on w only. Hence,

$$M = \oplus_{i=0}^{\infty} z^i (M \ominus zM) = \phi \oplus_{i=0}^{\infty} z^i H^2(D) = \phi H^2(D^2).$$

Conversely, if $M = \phi H^2(D^2)$ for some inner function ϕ depending only on w and f is any function in $K = H^2(D^2) \ominus M$, then obviously

$$< zf,\ \phi w^j >= 0$$

for $j \geq 0$. For any $i \geq 1$ and $j \geq 0$,

$$< zf,\ \phi z^i w^j >=< f,\ \phi z^{i-1} w^j >= 0.$$

In conclusion, $zf \in K$ and hence $K = H^2(D^2) \ominus M$ is invariant under the multiplication by z.

□

Corollary 2.2.3 *If $M = [h]$ for some function h that is holomorphic in a neighborhood of $\overline{D^2}$ with*

$$\overline{Z(h)} \cap \overline{D^2} = \overline{Z(h) \cap D^2},$$

then $[S_z,\ S_w^*] = 0$ on $K = H^2(D^2) \ominus M$ implies that either $\sigma(S_z) \cap D$ or $\sigma(S_w) \cap D$ is discrete.

Proof. If $[S_z,\ S_w^*] = 0$ on K then there is a function, say ϕ, in M depending only on one variable, say w. If $\{w_j :\ 1 \le j \le N\}$ are the zeros of ϕ in $D(N$ could be ∞.), then

$$Z(h) \subset D \times \{w_j :\ 1 \le j \le N\}.$$

By [**DYa**](see also Theorem 3.1.5),

$$\sigma(S_w) = \pi_2(\overline{Z(h) \cap D^2}) \subset \overline{\{w_j :\ 1 \le j \le N\}}.$$

□

Corollary 2.2.4 *If h is a polynomial in $\mathcal{R}$, then $[S_z,\ S_w^*] = 0$ on $H^2(D^2) \ominus [h]$ if and only if*

$$[h] = GH^2(D^2)$$

with G an inner function depending only on one variable.

Proof. First of all if $Z(h) \cap D^2 = \emptyset$ then $[h] = H^2(D^2)$ by Theorem 1.0.1. So we now assume $Z(h) \cap D^2 \neq \emptyset$.

If $[S_z,\ S_w^*] = 0$ on $H^2(D^2) \ominus [h]$, then by Proposition 2.2.1 $[h]$ contains a function, say $\phi(w)$, of only one variable. Suppose $\{w_j :\ 1 \leq j \leq N \leq \infty\}$ are the distinct zeros of ϕ in D, then

$$Z(h) \cap D^2 \subset \cup_{j=0}^{N} D \times \{w_j\}.$$

We assume $\{w_j :\ 1 \leq j \leq k\}$ are all the zeros of ϕ such that

$$Z(h) \cap D \times \{w_j\} \neq \emptyset.$$

Since h can't have isolated zeros, so

$$h(z, w_j) = 0,\ \forall z \in D,\ 1 \leq j \leq k.$$

But since h is a polynomial, we must have

$$(w - w_j) \mid h(z, w),\ 1 \leq j \leq k < \infty.$$

If for each j, we let

$$n_j := max\{n :\ (w - w_j)^n \mid h(z, w)\},$$

then

$$h(z, w) = \prod_{j=1}^{k} (w - w_j)^{n_j} p(z, w),$$

for some polynomial p. From the construction above,

$$Z(p) \cap D^2 = \emptyset$$

and hence is outer(H) by Theorem 1.0.1. If we let

$$G(w) := \prod_{j=1}^{k} (\frac{w - w_j}{1 - \overline{w_j} w})^{n_j},$$

then

$$[h] = (\prod_{j=1}^{k} (w - w_j)^{n_j}) H^2(D^2) = GH^2(D^2).$$

Coversely, if $[h] = GH^2(D^2)$ with G an inner function depending only on one variable, say w, then $H^2(D^2) \ominus [h]$ is invariant under the multiplication by z and hence

$$[S_z,\ S_w^*] = -(I - p)\overline{w} p z = 0$$

by Equality (2.1).

□

Corollary 2.2.5 *$H^2(D^2)$ can not be decomposed as an orthogonal direct sum of two proper submodules.*

Proof. If M and $K = H^2(D^2) \ominus M$ are both submodules then by Corollary 2.2.2,

$$M = \phi_1 H^2(D^2) = \phi_2 H^2(D^2),$$

for some inner functions $\phi_1,\ \phi_2$ in different variables. This is possible only when ϕ_1 and ϕ_2 are both scalars, hence $M = H^2(D^2)$. □

Actually no two submodules can even have positive angle. It is a easy consequence of the following lemma.

Lemma 2.2.6 *If $M \subset H^2(D^2)$ is a nontrivial submodule, then the minimal unitary dilation of M is $L^2(T^2,\ dm)$, where dm is the normalized Lebesgue measure on the torus T^2.*

Proof. We let

$$\hat{M} := \overline{\{z^i w^j f : \ f \in M, \ i, \ j : \ integers\}},$$

where the closure is taken in $L^2(T^2, \ dm)$. Then $\hat{M}$ is an closed subspace of $L^2(T^2, \ dm)$ jointly invariant for the multiplications by $z, \ w$ and $\overline{z}, \ \overline{w}$. Then by Lemma 3 in [**GM**],

$$\hat{M} = 1_E L^2,$$

for some measurable subset $E \subset T^2$. But $\hat{M}$ contains M and it is well known that nonzero functions in $H^2(D^2)$ can't vanish on a subset of T^2 with positive measure. So $E = T^2$ and

$$\hat{M} = L^2.$$

□

Corollary 2.2.7 *No two submodules of $H^2(D^2)$ can have positive angle.*

Proof. If $M, \ N$ are two nontrivial submodules, then it suffices to show that

$$sup\{| < f, \ g > | \ : \ f \in M, \ g \in N, \ \|f\| = \|g\| = 1\} = 1.$$

Let $f \in M, \ g \in N$ be any two nonzero functions, then by Lemma 2.2.6,

$$[\hat{f}] = [\hat{g}] = L^2.$$

So for any small positive number ϵ we can find polynomials p_1 and p_2 in four variables $z, \ w, \ \overline{z}, \ \overline{w}$ such that

$$\|p_1 f\| = \|p_2 g\| = 1$$

and

$$\|1 - p_1 f\| \leq \epsilon, \quad \|1 - p_2 g\| \leq \epsilon.$$

Then,

$$\begin{aligned} |< p_1 f,\ p_2 g >| &= |< 1 + p_1 f - 1,\ 1 + p_2 g - 1 >| \\ &\geq 1 - \|p_1 f - 1\| - \|p_2 g - 1\| - \|p_1 f - 1\|\|p_2 g - 1\| \\ &\geq 1 - 2\epsilon - \epsilon^2 \end{aligned}$$

We now choose a sufficiently large integer K such that $z^K w^K p_1,\ z^K w^K p_2$ are polynomials in $z,\ w$ only. Then $z^K w^K p_1 f \in M$ and $z^K w^K p_2 g \in N$ and

$$|< z^K w^K p_1 f,\ z^K w^K p_2 g >| = |< p_1 f,\ p_2 g >| \geq 1 - 2\epsilon - \epsilon^2.$$

□

2.3 Equation $p_1 f_1 + p_2 f_2 = 0$

If $p_1,\ p_2$ are two polynomials in $\mathcal{R}$, we consider the equation

$$p_1 f_1 + p_2 f_2 = 0. \tag{2.2}$$

The solutions of Equation (2.2) in $H^2(D^2) \oplus H^2(D^2)$ will be denoted by $Ker(p_1, p_2)$. If we equip $H^2(D^2) \oplus H^2(D^2)$ with a module action of $A(D^2)$ defined by

$$g \cdot (f_1, f_2) = (gf_1, gf_2),$$

then $Ker(p_1, p_2)$ is a submodule of $H^2(D^2) \oplus H^2(D^2)$. Since $Ker(p_1, p_2) = Ker(qp_1, qp_2)$ for any $q \in H^\infty(D^2)$, we will always assume that the greatest common divisor of $p_1,\ p_2$(denoted by $GCD(p_1,p_2)$) is 1. In this section will show that $Ker(p_1,p_2)$ is similar to $H^2(D^2)$ if $p_1,\ p_2$ have no common zero on the boundary of D^2.

Theorem 2.3.1 *If $p_1,\ p_2$ are two polynomials that have no common zeros on the boundary of D^2, then $Ker(p_1,p_2)$ is similar to $H^2(D^2)$ as an $A(D^2)$ module.*

Proof. If $GCD(p_1,p_2)=1$ then the quotient $\mathcal{R}/(p_1,p_2)$ is finite dimensional by [**Ya**](also see Lemma 4.7.1); hence p_1,p_2 have a finite number of common zeros. We denote these zeros by

$$\{(z_i,w_i):\ 1\le i\le l\}\subset \mathbf{C}^2.$$

Without loss of generality, we assume $(z_i,w_i)\in D^2$, for $1\le i\le k$ and $(z_i,w_i)\in \mathbf{C}^2\setminus D^2$, for $k+1\le i\le l$, and set

$$m(z,w)=\prod_{i=1}^{k}(z-z_i)(w-w_i),\ r(z,w)=\prod_{i=k+1}^{l}(z-z_i)(w-w_i).$$

Then by Nullstellensatz, a suitable power of $m(z,w)r(z,w)$ is in the ideal $(p_1,\ p_2)$. Without loss of generality(which will be clear from the proof) we assume $m(z,w)r(z,w)\in(p_1,\ p_2)$. So there are two polynomials $q_1,\ q_2$ such that

$$m(z,w)r(z,w)=q_1p_1+q_2p_2.$$

By the assumption, $r(z,w)$ has no zero in $\overline{D^2}$, so $r^{-1}\in H^\infty(D^2)$ and

$$m=r^{-1}q_1p_1+r^{-1}q_2p_2.$$

We let $\phi_i = r^{-1}q_i, \ i = 1, \ 2.$

If $(f_1, f_2) \in Ker(p_1, p_2)$, then

$$\begin{aligned} mf_1 + p_2(\phi_1 f_2 - \phi_2 f_1) &= (m - \phi_2 p_2) f_1 + \phi_1 p_2 f_2 \\ &= \phi_1 p_1 f_1 + \phi_1 p_2 f_2, \end{aligned}$$

hence

$$mf_1 + p_2(\phi_1 f_2 - \phi_2 f_1) = \phi_1(p_1 f_1 + p_2 f_2) = 0. \tag{2.3}$$

If we let

$$T = \begin{pmatrix} 1 & 0 \\ -\phi_2 & \phi_1 \end{pmatrix},$$

then by Equation (2.3) , T is a bounded module map from $H^2(D^2) \oplus H^2(D^2)$ to $H^2(D^2) \oplus H^2(D^2)$ which maps $Ker(p_1, \ p_2)$ into $Ker(m, \ p_2)$.

It is easy to check that T is injective; in what follows we will show that T also maps $Ker(p_1, \ p_2)$ onto $Ker(m, \ p_2)$.

First of all, if $GCD(m, \ p_2) = s(z, w)$ then

$$Ker(m, \ p_2) = Ker(m/s, \ p_2/s),$$

so again we assume that $GCD(m, \ p_2) = 1$.

If $(h_1, \ h_2) \in Ker(m, \ p_2)$ then

$$mh_1 = -p_2 h_2.$$

But since $GCD(m, \ p_2) = 1$, $\frac{h_2}{m}$ must be holomorphic. If we set

$$\psi_m = \prod_{i=1}^{k} (1 - \overline{z_i} z)^{-1} (1 - \overline{w_i} w)^{-1},$$

then $\psi_m \in H^\infty(D^2)$ and $m\psi_m$ is an inner function. Therefore,

$$\begin{aligned}\int_{T^2} |\frac{h_2}{m}|^2 d\mu &= \int_{T^2} |\frac{h_2}{m\psi_m}|^2 |\psi_m|^2 d\mu \\ &= \int_{T^2} |h_2|^2 |\psi_m|^2 d\mu \\ &\le \|\psi_m\|_\infty^2 \|h_2\|^2 < \infty,\end{aligned}$$

i.e. $\frac{h_2}{m} \in H^2(D^2)$.

If we set

$$f_1 = h_1, \quad f_2 = p_1 h_2/m,$$

then

$$\begin{aligned}p_1 f_1 + p_2 f_2 &= p_1 h_1 + p_2 p_1 h_2/m \\ &= p_1(\frac{mh_1 + p_2 h_2}{m}) = 0,\end{aligned}$$

i.e. $(f_1,\ f_2) \in Ker(p_1,\ p_2)$. Furthermore,

$$\begin{aligned}T\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} f_1 \\ -\phi_2 f_1 + \phi_1 f_2 \end{pmatrix} \\ &= \begin{pmatrix} h_1 \\ \frac{-\phi_2 m h_1 + \phi_1 p_1 h_2}{m} \end{pmatrix} \\ &= \begin{pmatrix} h_1 \\ \frac{(\phi_2 p_2 + \phi_1 p_1) h_2}{m} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.\end{aligned}$$

This shows that T maps $Ker(p_1,\ p_2)$ onto $Ker(m,\ p_2)$; hence $Ker(p_1,\ p_2)$ and $Ker(m,\ p_2)$ are similar modules.

Moreover from the discussions which lead to Equation (2.3),

$$Ker(m,\ p_2) = \{(-p_2h/m,\ h) :\ h \in mH^2(D^2)\},$$

and hence is similar to $mH^2(D^2) = m\psi_m H^2(D^2)$. But $m\psi_m H^2(D^2)$ is unitarily equivalent to $H^2(D^2)$ because $m\psi_m$ is inner. $\square$

Chapter 3

Quotient Modules

We recall from Chapter 1 that the quotient $H^2(D^n) \ominus [h]$ is an $A(D^n)$ module and the coordinate functions z_1, z_2, ..., z_n act on $H^2(D^n) \ominus [h]$ as bounded linear operators. In this chapter, we first make a study of the spectral properties of these operators and reveal how these properties are related to the function h. Then we will have a look at the analytic continuation problem. At last, we will show a rigidity phenomenon of quotient Hardy modules.

Let us also recall that $H^\infty(D^n)$ is the space of all bounded holomorphic functions in D^n with

$$\|f\|_\infty = sup|f(z)|, \quad z \in D^n.$$

The collection of invertible elements in algebra $H^\infty(D^n)$ is denoted by $[H^\infty(D^n)]^{-1}$. The H^∞ spaces over other domains are similarly defined. Suppose Ω is any open set that contains $\overline{D^n}$. For any natural number j less than or equal to n and any $\mu \in \overline{D}$, we set

$$S^j_\mu := \{z \in \Omega | z_j = \mu\},$$

which is called the *slice* of Ω at $z_j = \mu$. In many places of this chapter, we will assume some functions to be holomorphic in a neighborhood of $\overline{D^n}$. The slice sets will be useful in the discussions there. For any h holomorphic in a domain, $Z(h)$ will denotes the set of zeros of h in that domain.

In Section 3.1, we make a study of the spectra of the operators S_1, S_2, ..., S_n as well as the joint spectrum of the n-tuple $(S_1, S_2, ..., S_n)$. Section 3.2 is devoted to the study of some functional properties of certain functions in $H^2(D^n)$. In Section 3.3, we establish a rigidity phenomenon of quotient modules.

3.1 Spectra

We recall from Chapter 1 that S_1, S_2, ..., S_n denote the compressions of the Toeplitz operators T_{z_1}, T_{z_2}, ..., T_{z_n} onto some quotient module, say $H^2(D^2) \ominus [h]$. Cowen and Rubel made a study of the joint spectrum of the tuple $(S_1, S_2, ..., S_n)$ and showed a close relation to the zero set of h. In this section we will show that under some conditions the spectrum of S_j is exactly the projection of the zero set to the jth coordinate.

We proceed by proving the following

Lemma 3.1.1 *If h is holomorphic in a neighborhood of $\overline{D^n}$ and $h(\lambda, z') \in [H^\infty(S^1_\lambda)]^{-1}$, then $\lambda \in \rho(S_1)$, the resolvent set of S_1.*

Proof. Consider the function

$$F(z_1, z') = \frac{1 - h(z_1, z')h^{-1}(\lambda, z')}{z_1 - \lambda}.$$

By the Weierstrass Preparation Theorem[**Kr**, Thm. 6.4.5], the numerator of F has $z_1 - \lambda$ as a factor, and hence F is a bounded holomorphic function over D^n. So S_F is a bounded operator on $H^2(D^n) \ominus [h]$.

For every $f \in H^2(D^n) \ominus [h]$,

$$\begin{aligned}(S_1 - \lambda)S_F f &= q(z_1 - \lambda)Ff \\ &= q(1 - h(\cdot,\cdot)h^{-1}(\lambda,\cdot))f \\ &= qf - qh(\cdot,\cdot)h^{-1}(\lambda,\cdot)f \\ &= qf = f,\end{aligned}$$

where q is the orthogonal projection from $H^2(D^n)$ to $H^2(D^n)\ominus[h]$. This shows that

$$(S_1 - \lambda)S_F = I.$$

Since S_1 commutes with S_F, we also have that

$$S_F(S_1 - \lambda) = I$$

i.e. $\lambda \in \rho(S_1)$. □

Similar statements are true for the operators S_2, ..., S_n with the corresponding assumptions on h.

In essence, if λ is not the j-th coordinate of any of the zeros of h in $\overline{D^n}$, then λ is in the resolvent set of S_j. It is actually possible to give a complete description of the spectra of these compression operators when $Z(h)$ satisfies certain conditions, but it is convenient to have a look at their *joint spectrum* first.

Let us first give the definition of the joint spectrum. A good reference of this subject is Chapter III in [**Hö**].

Suppose $\mathcal{B}$ is a commutative Banach algebra with unit e, and

$$a = (a_1,\ a_2,\ \ldots,\ a_n)$$

is a tuple of elements in $\mathcal{B}$. We say that a is *non-singular* if there are elements $b_1,\ b_2,\ \ldots,\ b_n \in \mathcal{B}$ with

$$\sum_{i=1}^{n} a_i b_i = e.$$

The tuple a is called *singular* if it is not non-singular. The *joint spectrum* of the tuple a is defined as

$$\sigma(a) := \{z \in \mathbf{C}^n :\ a - ze \textit{ is singular.}\}$$

Here $a - ze$ denotes the tuple $(a_1 - z_1,\ a_2 - z_2,\ \ldots,\ a_n - z_n)$.

Now we state two more lemmas which are special cases of the results in [**CR**]. The proofs here are only slightly different.

Lemma 3.1.2 *Suppose h is holomorphic in a neighborhood of $\overline{D}^n$ and*

$$\overline{Z(h)} \cap \overline{D^n}$$

is not empty, then the joint spectrum

$$\sigma(S_1, S_2, \ldots, S_n) \subset \overline{Z(h)} \cap \overline{D^n}.$$

Proof. Without loss of generality we may assume that h is holomorphic in a pseudoconvex neighborhood U of $\overline{D}^n$. Then, for any $\lambda = (\lambda_1,\ \lambda_2, \ldots,\ \lambda_n) \in$

$\overline{D}^n \setminus \overline{Z(h)} \cap \overline{D^n}$, we can write h as

$$h(z_1, z_2, ..., z_n) = h(\lambda_1, \lambda_2, ..., \lambda_n) + \sum_{j=1}^{n}(z_j - \lambda_j)g_j$$

for some functions g_j that are also holomorphic in U [**Kr**, Thm.7.2.9]. since

$$h(z)h^{-1}(\lambda) = 1 + h^{-1}(\lambda)\sum_{j=1}^{n}(z_j - \lambda_j)g_j,$$

it follows for any $f \in H^2(D^n) \ominus [h]$ that

$$\begin{aligned} -h^{-1}(\lambda)\sum_{j=1}^{n}(S_j - \lambda_j)S_{g_j}f &= -q(h^{-1}(\lambda)\sum_{j=1}^{n}(z_j - \lambda_j)g_j)f \\ &= q(1 - h(\cdot)h^{-1}(\lambda))f \\ &= qf = f. \end{aligned}$$

This implies that

$$\sum_{j=1}^{n}(S_j - \lambda_j)S_{g_j} = I,$$

and hence $\lambda \in \rho(S_1,\ S_2, ...,\ S_n)$ for any $\lambda \in \overline{D}^n \setminus \overline{Z(h)} \cap \overline{D^n}$, or equivalently

$$\sigma(S_1, S_2, ..., S_n) \subset \overline{Z(h)} \cap \overline{D^n}.$$

□

Here we note that $\overline{Z(h)} \cap \overline{D^n}$ not empty doesn't imply that $[h]$ is proper. For example, $[z+w+2]$ is equal to $H^2(D^2)$ [**Ge**]. In Lemma 3.1.2 we excluded the trivial case $[h] = H^2(D^2)$. In case $Z(h) \cap D^n$ is not empty, we have an inclusion in the other direction.

Lemma 3.1.3 *If h is holomorphic in a neighborhood of $\overline{D}^n$ and $Z(h) \cap D^n$ is not empty, then*

$$Z(h) \cap D^n \subset \sigma(S_1, S_2, ..., S_n).$$

Proof. Suppose $\lambda = (\lambda_1,\ \lambda_2, \ldots,\ \lambda_n) \in Z(h) \cap D^n$. It is easy to see that

$$\overline{\sum_{j=1}^{n}(S_j - \lambda_j)(H^2(D^n) \ominus [h]) + [h]} \subset \overline{\sum_{j=1}^{n}[z_j - \lambda_j] + [h]},$$

but λ is a common zero of the functions $z_1 - \lambda_1,\ z_2 - \lambda_2, \ldots,\ z_n - \lambda_n$ and h, so $\overline{\sum_{j=1}^{n}[z_j - \lambda_j] + [h]}$ is a proper subset of $H^2(D^n)$. This implies that

$$\sum_{j=1}^{n}(S_j - \lambda_j)(H^2(D^n) \ominus [h]) \neq H^2(D^n) \ominus [h]$$

i.e. $\lambda \in \sigma(S_1,\ S_2, \ldots,\ S_n)$. □

Using module resolution and tensor product one can prove the inclusion in this lemma for other submodules. But the statement here is good enough for our purpose.

Combining Lemmas 3.1.2 and 3.1.3 , we have the following

Theorem 3.1.4 *If h is holomorphic in a neighborhood of $\overline{D}^n$ such that*

$$\overline{Z(h)} \cap \overline{D^n} = \overline{Z(h) \cap D^n}, \tag{3.1}$$

then

$$\sigma(S_1,\ S_2, \ldots,\ S_n) = \overline{Z(h) \cap D^n}.$$

For $j = 1,\ 2,\ 3,\ \ldots,\ n$ and any $z \in \overline{D^n}$, we let

$$\pi_j z := z_j$$

be the projection to the j-th coordinate of z. It is well known that,

$$\pi_j \sigma(S_1,\ S_2, \ldots,\ S_n) \subset \sigma(S_j).$$

Combining Lemma 3.1.1 and the above theorem, we have

Theorem 3.1.5 *If h is holomorphic in a neighborhood of $\overline{D}^n$ that satisfies Condition (3.1) in Theorem 3.1.4, then for $j = 1, 2, \ldots, n$,*

$$\sigma(S_j) = \pi_j(\overline{Z(h) \cap D^n}).$$

Proof. It suffices to show that

$$\sigma(S_j) \subset \pi_j(\overline{Z(h) \cap D^n}).$$

In fact, if μ is inside the complement of $\pi_j(\overline{Z(h) \cap D^n})$, then fixing $z_j = \mu$, h doesn't vanish on the closure of S^j_μ. Lemma 3.1.1 then concludes that $\mu \in \rho(S_j)$. □

Theorem 3.1.5 will be used in Section 3.2 to make a study of the analytic continuation problem.

3.2 Analytic continuation

In [**AC**], Ahern and Clark made a study of the analytic continuation of functions in certain quotient Hardy modules. In this section, we are going to use a result from their work and the results obtained in Section 3.1 to study the analytic continuation problem. Again we find that the zero set plays an important role.

Corollary 3.2.1 *If h is holomorphic in a neighborhood of $\overline{D}^n$ and h is in $[H^\infty(S^j_\lambda)]^{-1}$ setting $z_j = \lambda$ with $|\lambda| = 1$, then every function in $H^2(D^n) \ominus [h]$ has an analytic continuation to a neighborhood of $D^{j-1} \times \{\lambda\} \times D^{n-j}$.*

Proof. We prove the assertion for $j = 1$.

Every function of $H^2(D^n) \ominus [h]$ has the property that

$$f(\lambda_1, \lambda_2, ..., \lambda_n) = < f, (I - \overline{\lambda_1} S_1)^{-1}(I - \overline{\lambda_2} S_2)^{-1} \cdots (I - \overline{\lambda_n} S_n)^{-1} q1 >,$$

where q is the projection from $H^2(D^n)$ onto $H^2(D^n) \ominus [h]$([**AC**]). If we replace λ_1 by λ, then we have

$$f(\lambda, \lambda_2, ..., \lambda_n) = \overline{\lambda} < f, (\lambda I - S_1)^{-1}(I - \overline{\lambda_2} S_2)^{-1} \cdots (I - \overline{\lambda_n} S_n)^{-1} q1 > .$$

But from Lemma 3.1.1, if λ is in the resolvent set of S_1, then the right side extends analytically in the first variable to a fixed neighborhood of λ and the corollary follows. □

In essence, this corollary means that if a $\lambda \in T$ is not the j-th coordinate of any of the zeros of h in $\overline{A(D^n)}$, then every function of $H^2(D^n) \ominus [h]$ has an analytic continuation to a fixed neighborhood of $D^{j-1} \times \{\lambda\} \times D^{n-j}$.

Example : If $h(z, w) = z - \mu w$ for some nonzero $\mu \in D$, then $h(\lambda, w)$ is holomorphic in a neighborhood of $\overline{D}$ for every $\lambda \in T$ and $h(\lambda, \cdot)$ is invertible in $H^\infty(D)$. Then by the corollary, all the functions of $H^2(D^2) \ominus [z - \mu w]$ are analytic in a neighborhood of the unit disk in the first variable.

When $n = 1$, it is well known that every function of $H^2(D)$, the Hardy space over the unit disk, has an inner-outer(H) factorization. But that is far from the case even when $n = 2$. Functions like $z_1 + z_2$ don't even factorize as the product of two $H^1(D^2)$ functions([**Ru**, pp 63]). (Here we alert the reader that the notion of outer function considered in [**Ru**] is not the same as that used here even though they are the same when $n = 1$. We refer the reader to [**Ru**] for a detailed discussion.) Equipped with Corollary 3.2.1 and a theorem from [**AC**], we find a simple way to determine that certain functions have no inner-outer(H) factorization.

Theorem 3.2.2 *Suppose h is holomorphic in a neighborhood of $\overline{D^n}$ satisfying the condition 3.1 in theorem 3.1.4 such that*

1. $Z(h) \cap D^n$ is not a subset of a countable union of slices of D^n, and

2. there is an integer $j \leq n$ such that $\pi_j\overline{(Z(h) \cap D^n)}$ doesn't contain the unit circle T.

Then h has no inner-outer(H) factorization.

We note that condition 1 demands in particular that $Z(h) \cap D^n$ is not empty. The proof uses the following theorem of P.Ahern and D.Clark[**AC**].

Theorem 3.2.3 *Suppose $M = gH^2(D^n)$ where g is inner, $\lambda = (\lambda_1, \lambda_2, ...\lambda_n) \in \partial D^n$ with $|\lambda_j| = 1$, and there is a neighborhood B of λ such that every function in $H^2(D^n) \ominus M$ has an analytic continuation into B. Then g is a function of z_j alone. In particular , if more than one of the λ_i has modulus 1, g is a constant and $M^{\perp} = 0$.*

We now come to the proof of Theorem 3.2.2.

Proof. Suppose h is a function with the properties mentioned in the theorem. If h has the inner-outer(H) factorization GF, then

$$Z(h) \cap D^n = Z(G) \cap D^n$$

and $[h] = GH^2(D^n)$.

If $\mu \in T \setminus \pi_j\overline{(Z(h) \cap D^n)}$, then $\mu \in \rho(S_j)$ from Theorem 3.1.5. Let $B(\mu) \subset \rho(S_j)$ be a neighborhood of μ. Corollary 3.2.1 shows that each f of $H^2(D^n) \ominus GH^2(D^n)$ has an analytic continuation into $D^{j-1} \times B(\mu) \times D^{n-j}$. Then the theorem of Ahern and Clark implies that G depends on z_j only. Therefore $Z(h) \cap D^n$ must be a subset of a countable union of slices of D^n which contradicts the assumption. □

In view of the results in [**ACD**], Theorem 3.2.2 enables one to contruct many examples of submodules that are not equivalent to $H^2(D^n)$.

3.3 Rigidity

Submodules with *thin* zero sets exhibit the so called *rigidity* phenomenon [**DY**][**Pa**]. Things are much different when the zero sets are hyper-surfaces. For example, it is well known that M is equivalent to gM for any submodule if g is inner. But the zero sets of M and gM can be quite different. In this section, we prove a theorem which shows that for quotient modules this is by far not the case.

Let us first recall some definitions. If H_1 and H_2 are two $A(D^n)$ modules, then H_1 is said to be *unitarily equivalent(similar)* to H_2 if there is a unitary(invertible) module map from H_1 to H_2. A bounded module map T from H_1 to H_2 is called *quasi-affine* if it is one to one and has dense range. H_1 and H_2 are said to be *quasi-similar* if there are quasi-affine module maps from H_1 to H_2 and from H_2 to H_1. One sees that similarity implies quasi-similarity.

In [**DF**], R. G. Douglas and C. Foias have shown that if H_1 and H_2 are two submodules of $H^2(D^n)$ then $H^2(D^n) \ominus H_1$ is unitarily equivalent to $H^2(D^n) \ominus H_2$ if and only if $H_1 = H_2$. In [**DC**], R. G. Douglas and Xiaoman Chen were able to prove that if either J_1 or J_2 is principal in $\mathcal{R}$, then $H^2(\Omega) \ominus [J_1]$ is quasi-similar to $H^2(\Omega) \ominus [J_2]$ if and only if $J_1 = J_2$, where Ω can any bounded domain. They proved the result through a detailed analysis of the zero varieties of the two ideals. We refer the reader to [**FS**] and [**Wo**] for definitions of Hardy spaces over general domains.

Recently we discovered a direct approach which generalize the results in [**DC**]. This approach was also suggested by Keren Yan in a less general context some years ago. We state our result in the polydisk case.

Proposition 3.3.1 *If M_1 and M_2 are submodules of $H^2(D^n)$ such that there is a quasi-affine module map from $H^2(D^n) \ominus M_1$ to $H^2(D^n) \ominus M_2$, then every bounded function in M_1 is also contained in M_2.*

Proof. Suppose

$$p_1 : H^2(D^n) \longrightarrow M_1, \quad q_1 : H^2(D^n) \longrightarrow H^2(D^n) \ominus M_1,$$

$$p_2 : H^2(D^n) \longrightarrow M_2, \quad q_2 : H^2(D^n) \longrightarrow H^2(D^n) \ominus M_2,$$

are projections and let the operator $T : H^2(D^n) \ominus M_1 \longrightarrow H^2(D^n) \ominus M_2$ be the quasi-affine module map. Then

$$q_2 f T = T q_1 f, \quad \text{for any } f \in A(D^n).$$

As multiplication operators acting on $H^2(D^n)$, $H^\infty(D^n)$ is the *weak* operator closure of $A(D^n)$, so the equality

$$q_2 f T = T q_1 f$$

holds for every $f \in H^\infty(D^n)$. In particular, for any bounded function $g \in M_1$, it follows

$$q_2 g T = T q_1 g = 0,$$

and hence the operator $q_2 g = 0$ since T has dense range. If we choose $q_2 1 \in H^2(D^n) \ominus M_2$, then

$$0 = q_2 g(q_2 1) = q_2(g).$$

This shows that $g \in M_2$. □

This proposition leads to the following theorem which shows the rigidity phenomenon of quotient Hardy modules.

Theorem 3.3.2 *If M_1 and M_2 are submodules both generated by bounded holomorphic functions , then $H^2(D^n) \ominus M_1$ and $H^2(D^n) \ominus M_2$ are quasi-similar $A(D^n)$ modules if and only if $M_1 = M_2$.*

Proof. Sufficiency is obvious.

From the above theorem every bounded function of M_1 also belongs in M_2. Since M_1 is generated by bounded functions and M_2 is a closed submodule, we have that

$$M_1 \subset M_2.$$

Similarly we also have the inclusion

$$M_2 \subset M_1,$$

and hence

$$M_1 = M_2.$$

□

Here we note that the above theorem in the case of the Hardy space over the unit disk is also implied by the Livsic-Moeller theorem[**Ni**]. We also point out that the proofs work for other $A(D^n)$ modules, such as the weighted Hardy modules and even the Bergman modules.

We end this chapter by a conjecture suggested by Theorem 3.3.2.

Conjecture. *If M_1 and M_2 are submodules of $H^2(D^n)$, then $H^2(D^n) \ominus M_1$ is similar to $H^2(D^n) \ominus M_2$ if and only if $M_1 = M_2$.*

Chapter 4

Cross Commutators

Let us recall that if $(h) \subset A(D^2)$ is the principal ideal generated by a polynomial h, then its closure $[h](\subset H^2(D^2))$ and the quotient $H^2(D^2) \ominus [h]$ are both $A(D^2)$ modules. As in Chapter 1, we let $R_z,\ R_w$ be the actions of the coordinate functions z and w on $[h]$, and let $S_z,\ S_w$ be the actions of z and w on $H^2(D^2) \ominus [h]$. In this chapter, we will show that R_z and R_w, as well as S_z and S_w, essentially doubly commute. Moreover, both $[R_w^*,\ R_z]$ and $[S_w^*,\ S_z]$ are actually Hilbert-Schmidt.

4.1 General Questions

The Berger-Shaw theorem says that the self-commutator of a multicyclic hyponormal operator is trace class[**BS**]. It is interesting to study the multivariate analogue of this theorem. In [**DY1**], the authors reformulated the theorem in an algebraic language and showed that if the spectrum of a finite rank hyponormal module is contained in an algebraic curve then the module is *reductive* which means that the module actions are all essentially normal. They also gave examples showing that it is generally not the case if the spectrum of the module is of higher dimension. However, many examples show that the *cross* commutators don't seem to have a close relation with the spectra of modules but are generally 'small'. This suggests that the following general questions may have positive answers.

Questions: *Suppose $T_1,\ T_2$ are two doubly commuting operators acting on a separable Hilbert space H and $R_1,\ R_2$ are the restrictions of them to a jointly invariant subspace that is finitely generated by $T_1,\ T_2$.*

1. Is the cross commutator $[R_1^*,\ R_2]$ in some Schatten p-class?

2. Is the product $[R_1^*,\ R_1][R_2^*,\ R_2]$ also small?

3. What about the compressions of $T_1,\ T_2$ to the orthogonal complement of M?

A special case of the first question was studied by Curto, Muhly and Yan in [**CMY**]. The second question was raised by R. Douglas. The third one appears naturally from the study of essentially reductive quotient modules. Note that when $T_1 = T_2$ the first two questions are answered positively by the Berger-Shaw theorem.

In this chapter we will make a study of these questions in the case $H = H^2(D^2)$, the Hardy space over the bidisk, and $T_1,\ T_2$ are the multiplications by the two coordinate functions z and w. Then a closed subspace of $H^2(D^2)$ is jointly invariant for T_1 and T_2 if and only if it is an $A(D^2)$ submodule. We will have a look at the third question first because it turns out to be the easiest.

The answer to the second question is a consequence of the answer to the first one. Some related questions will also be studied in this chapter.

We now do some preparations.

We let E', E be two separable Hilbert spaces of infinite dimension and $\{\delta_j' : j \geq 0\}$, $\{\delta_j : j \geq 0\}$ are orthonormal bases for E' and E respectively. We let $H^2(E)$ denote the E-valued Hardy space, i.e.

$$H^2(E) := \{\sum_{j=0}^{\infty} z^j x_j : |z| = 1, \sum_{j=0}^{\infty} \|x_j\|_E^2 < \infty\}.$$

It is well known that every function in $H^2(E)$ has an analytic continuation to the whole unit disc D. For our convenience, we will not distinquish the functions of $H^2(E)$ from their extensions to D. We let T_z be the Toeplitz operator on $H^2(E)$ such that for any $f \in H^2(E)$,

$$T_z f(z) = z f(z).$$

One sees that T_z is a shift operator of infinite multiplicity.

A $\mathcal{B}(E', E)$-valued analytic function $\theta(z)$ on D is called *left-inner(inner)* if its boundary values on the unit circle T are almost everywhere isometries(unitaries) from E' into E. Therefore, multiplication by a left-inner θ defines an isometry from $H^2(E')$ into $H^2(E)$.

A closed subspace $M \subset H^2(E)$ is called invariant if

$$T_z M \subset M.$$

The Lax-Halmos theorem([Ni]) gives a complete description of invariant subspaces in terms of left-inner functions.

Theorem 4.1.1 (Lax-Halmos) *M is a nontrivial invariant subspace of $H^2(E)$ if and only if there is a closed subspace $E' \subset E$ and a $\mathcal{B}(E', E)$-valued left-inner function θ such that*

$$M = \theta H^2(E'). \tag{4.1}$$

The representation is unique in the sense that

$$\theta H^2(E') = \theta' H^2(E'') \Longleftrightarrow \theta = \theta' V,$$

where V is a unitary from E' onto E''.

In order to make a study of the Hardy modules over the bidisc, we identify the space E with another copy of the Hardy space. Then $H^2(E) = H^2(D) \otimes E$ will be identified with $H^2(D) \otimes H^2(D) = H^2(D^2)$. We do this in the following way.

Let u be the unitary map from E to $H^2(D)$ such that

$$u\delta_j = w^j, \quad j \geq 0.$$

Then $U = I \otimes u$ is a unitary from $H^2(D) \otimes E$ to $H^2(D) \otimes H^2(D)$ such that

$$U(z^i \delta_j) = z^i w^j, \quad i, \ j \geq 0.$$

It is not hard to see that $M \subset H^2(E)$ is invariant if and only if $UM \subset H^2(D^2)$ is invariant under multiplication by the coordinate function z. This identification enables us to use the Lax-Halmos theorem to study certain properties of sub-Hardy modules over the bidisk which we will do in Section 4.2. Throughout this chapter, we will let $d|z|$ denote the normalized Lebesgue measure on the unit circle T and $d|z|d|w|$ be the product measure on the torus T^2.

4.2 Hilbert-Schmidt operators

In this section we prove two technical lemmas and an important corollary.

Suppose θ is left inner with values in $B(E',\ E)$ and δ is any fixed element of E . We now define an operator N from $\theta E'$ to the Hardy space $H^2(D)$ over the unit disc as the following:

$$N(\theta(z)\sum_{j=0}^{\infty}\alpha_j\delta_j^{'}) :=< \theta(z)\sum_{j=0}^{\infty}\alpha_j\delta_j^{'}, \delta >_E, \tag{4.2}$$

where $\sum_{j=0}^{\infty}\alpha_j\delta_j^{'}$ is any element in E'.

Lemma 4.2.1 *N is Hilbert-Schmidt and*

$$tr(N^*N) = \int_T \|\theta^*(z)\delta\|_{E'}^2 d|z|. \tag{4.3}$$

Proof. Since θ is left inner, $\{\theta\delta_j : j \geq 0\}$ is an orthonormal basis for $\theta E'$. To prove the lemma, it suffices to show that $\sum_{j=0}^{\infty} < N^*N\theta\delta_j^{'}, \theta\delta_j^{'} >_{\theta E'}$is finite. In fact ,

$$\begin{aligned}
\sum_{j=0}^{\infty} < N^*N\theta\delta_j^{'}, \theta\delta_j^{'} >_{\theta E'} &= \sum_{j=0}^{\infty} < N\theta\delta_j^{'}, N\theta\delta_j^{'} >_{H^2} \\
&= \sum_{j=0}^{\infty}\int_T | < \theta(z)\delta_j^{'}, \delta >_E |^2 d|z| \\
&= \sum_{j=0}^{\infty}\int_T | < \delta_j^{'}, \theta^*(z)\delta >_{E'} |^2 d|z| \\
&= \int_T \sum_{j=0}^{\infty} | < \delta_j^{'}, \theta^*(z)\delta >_{E'} |^2 d|z| \\
&= \int_T \|\theta^*(z)\delta\|_{E'}^2 d|z|.
\end{aligned}$$

□

So in general

$$tr(N^*N) \leq \|\delta\|^2,$$

and the equality holds when θ is inner.

Back to the $H^2(D^2)$ case, this lemma has an important corollary. Let us first introduce some operators.

For any bounded function f we let $T_f := Pf$ be the Toeplitz operator on $H^2(D^2)$, where P is the projection from $L^2(T^2)$ to $H^2(D^2)$. For every non-negative integer j and $\lambda \in D$, we let operators N_j and N_λ from $H^2(D^2)$ to $H^2(D)$ be such that for any $f(z,w) = \sum_{k=0}^{\infty} f_k(z)w^k \in H^2(D^2)$

$$N_j f(z) = f_j(z), \quad N_\lambda f(z) = f(z,\ \lambda).$$

Then one verifies that N_j is a contraction for each j and $\|N_\lambda\| = (1-|\lambda|^2)^{-1/2}$. Furthermore,

$$\sum_{k=0}^{\infty} T_{w^k} N_k = I \text{ on } H^2(D^2), N_\lambda = \sum_{k=0}^{\infty} \lambda^k N_k. \tag{4.4}$$

In what follows we will be mainly interested in the restrictions of $N_k,\ N_\lambda$ to certain subspaces and will use the same notations to denote these restrictions.

Corollary 4.2.2 *For any $A(D^2)$-submodule $M \subset H^2(D^2)$, N_j and N_λ are Hilbert-Schmidt operators restricting on $M \ominus zM$ for each $j \geq 0$ and $\lambda \in D$, and*

$$\begin{aligned} tr(N_j^* N_j) &\leq 1, \\ \|p_\perp \frac{1}{1-\bar{\lambda} w}\|^2 \leq tr(N_\lambda^* N_\lambda) &\leq (1-|\lambda|^2)^{-1}, \end{aligned}$$

where $p_\perp$ is the projection from $H^2(D^2)$ onto $M \ominus zM$.

Proof. Because M is invariant under the multiplication by z, U^*M is invariant under T_z, where U is defined in the last paragraph of Section 4.1, and hence

$$U^*M = \theta H^2(E')$$

for some Hilbert space E' and a left inner function θ. Then

$$U^*(M \ominus zM) = \theta H^2(E') \ominus z\theta H^2(E') = \theta(H^2(E') \ominus zH^2(E')) = \theta E'.$$

Let us first deal with the operator N_λ.

In Lemma 4.2.1, if we choose $\delta = \sum_{j=0}^{\infty} \overline{\lambda^j}\delta_j \in E$, then for any $f(z,w) = \sum_{j=0}^{\infty} f_j(z)w^j$ inside $M \ominus zM$, $U^*f = \sum_{j=0}^{\infty} f_j(z)\delta_j$ is in $\theta E'$, and

$$\begin{aligned} NU^*f(z) &= N(\sum_{j=0}^{\infty} f_j(z)\delta_j) = < \sum_{j=0}^{\infty} f_j(z)\delta_j, \delta > \\ &= \sum_{j=0}^{\infty} f_j(z)\lambda^j = N_\lambda f(z). \end{aligned}$$

So $N_\lambda = NU^*$, hence is Hilbert-Schmidt by Lemma 4.2.1, and

$$tr(N_\lambda^* N_\lambda) = tr(U^*N^*NU) = tr(N^*N).$$

The inequality

$$tr(N_\lambda^* N_\lambda) \leq (1 - |\lambda|^2)^{-1}$$

comes from the remarks following the proof of Lemma 4.2.1. We now show the inequality

$$\|p_\perp \frac{1}{1 - \overline{\lambda} w}\|^2 \leq tr(N_\lambda^* N_\lambda).$$

If $\{g_0,\ g_1,\ g_2,\ \ldots\}$ is an orthonormal basis for $M \ominus zM$. Then

$$N_\lambda g_k(z) = g_k(z,\lambda) = \int_T \frac{g_k(z,w)}{1 - \lambda \overline{w}} d|w|,$$

and therefore

$$\begin{aligned} tr(N_\lambda^* N_\lambda) &= \sum_{k=0}^{\infty} \int_T | \int_T \frac{g_k(z,w)}{1-\lambda\overline{w}} d|w| \, |^2 d|z| \\ &\geq \sum_{k=0}^{\infty} | \int_T \int_T \frac{g_k(z,w)}{1-\lambda\overline{w}} d|w| d|z| \, |^2 \\ &= \sum_{k=0}^{\infty} | < g_k, (1-\overline{\lambda}w)^{-1} > |^2 \\ &= \|p_\perp \frac{1}{1-\overline{\lambda}w}\|^2. \end{aligned}$$

For operators N_j, $j = 0, 1, 2, \ldots$, we choose δ to be δ_j, $j = 0, 1, 2, \ldots$ correspondingly in Lemma 4.2.1. Similar calculations will establish the assertion and the inequalities.

□

If $\mathcal{L}^2$ denotes the collection of all the Hilbert-Schmidt operators acting on some Hilbert space K, then for any a, b of $\mathcal{L}^2$,

$$< a, b > := trace(b^* a)$$

defines an inner product which turns $(\mathcal{L}^2, < \cdot, \cdot >)$ into a Hilbert space. If $|\cdot|$ is the norm induced from this inner product, then

$$|xay| \leq \|x\| \|y\| |a|, \tag{4.5}$$

for any $a \in \mathcal{L}^2$ and bounded operators x and y[**GK**,pp79], where $\|\cdot\|$ is the operator norm.

Lemma 4.2.3 *Suppose A, B are two contractions such that [A,B]=AB-BA is Hilbert-Schmidt and* $f(z) = \sum_{j=0}^{\infty} c_j z^j$ *is any holomorphic function over the unit disc such that* $\sum_{j=0}^{\infty} j|c_j|$ *converges, then* $[f(A), B]$ *is also Hilbert-Schmidt.*

Proof. We observe that for any positive interger n,

$$\begin{aligned} & [A^n, B] \\ = \ & A^nB - BA^n \\ = \ & A^nB - A^{n-1}BA + A^{n-1}BA - BA^n \\ = \ & A^{n-1}[A,B] + [A^{n-1}, B]A \\ = \ & \vdots \\ = \ & A^{n-1}[A,B] + A^{n-2}[A,B]A + \cdots + A[A,B]A^{n-2} + [A,B]A^{n-1}, \end{aligned}$$

hence

$$|[A^n, B]| \le n|[A,B]|$$

by inequality (1-5). If we let $f_n(z) = \sum_{j=0}^{n} c_j z^j$ then $[f_n(A), B]$ is in $\mathcal{L}^2$ and

$$\begin{aligned} |[f_n(A), B] - [f(A), B]| &= |[\sum_{j=n+1}^{\infty} c_j A^j, B]| \\ &\le \sum_{j=n+1}^{\infty} |c_j||[A^j, B]| \\ &\le \sum_{j=n+1}^{\infty} j|c_j||[A,B]| \end{aligned}$$

From the assumption on f,

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} j|c_j||[A,B]| = 0,$$

hence $[f(A), B]$ is also in $\mathcal{L}^2$, i.e. Hilbert-Schmidt. □

Corollary 4.2.2 is crucial for the rest of the sections and Lemma 4.2.3 will enable us to get around some technical difficulties.

4.3 Decomposition of Cross Commutators

In this section we will define the compression operators and decompose their cross commutators. We begin by recalling some notations from Chapter 1.

For any $h \in H^2(D^2)$, we let

$$[h] := \overline{A(D^2)h}^{H^2}$$

denote the submodule generated by h. Here we note that h is called *inner* if

$$|h(z,w)| = 1 \quad a.e. \quad on\ T^2.$$

It is not hard to see that

$$[h] = hH^2(D^2)$$

when h is inner. h is called *outer* in the sense of Helson(H) if

$$[h] = H^2(D^2).$$

Given any submodule M, we can decompose $H^2(D^2)$ as

$$H^2(D^2) = (H^2(D^2) \ominus M) \oplus M,$$

and let

$$\begin{aligned} p &: H^2(D^2) \longrightarrow M, \\ q &: H^2(D^2) \longrightarrow H^2(D^2) \ominus M \end{aligned}$$

be the projections. For any $f \in H^\infty(D^2)$, we let S_f and R_f be the compressions of the operator T_f to $H^2(D^2) \ominus M$ and M respectively, i.e.

$$S_f = qfq, \quad R_f = pfp.$$

In sections 4.4 and 4.5 we will prove that when $M = [h]$ with h a polynomial, the cross commutators $[S_w^*,\ S_z]$ and $[R_w^*,\ R_z]$ are both Hilbert-Schmidt. To avoid the technical difficulties, we prove the assertion for the operators $[S_{\varphi_\lambda}^*,\ S_z]$ and $[R_{\varphi_\lambda}^*,\ R_z]$ first, where $\varphi_\lambda(w) = \frac{w-\lambda}{1-\bar{\lambda}w}$ with some $\lambda \in D$ such that $h(z,\lambda) \neq 0$ for all $z \in T$, and then apply Lemma 4.2.3.

First we need to have a better understanding of the two cross commutators $[S_w^*,\ S_z]$ and $[R_w^*,\ R_z]$. In view of the decomposition

$$H^2(D^2) = (H^2(D^2) \ominus M) \oplus M,$$

we can decompose the Toeplitz operators on $H^2(D^2)$ correspondingly.

If we regard φ_λ as a multiplication operator on $H^2(D^2)$, then

$$T_{\varphi_\lambda} = \begin{pmatrix} q\varphi_\lambda q & 0 \\ p\varphi_\lambda q & p\varphi_\lambda p \end{pmatrix},$$

$$T_z = \begin{pmatrix} qzq & 0 \\ pzq & pzp \end{pmatrix},$$

and

$$T_{\varphi_\lambda}^* T_z - T_z T_{\varphi_\lambda}^*$$
$$= \begin{pmatrix} q\overline{\varphi}_\lambda qzq + q\overline{\varphi}_\lambda pzq - qzq\overline{\varphi}_\lambda q & q\overline{\varphi}_\lambda pzp - qzq\overline{\varphi}_\lambda p \\ p\overline{\varphi}_\lambda pzq - pzq\overline{\varphi}_\lambda q & p\overline{\varphi}_\lambda pzp - pzq\overline{\varphi}_\lambda p - pzp\overline{\varphi}_\lambda p \end{pmatrix}.$$

It is well known that T_z doubly commutes with T_w on $H^2(D^2)$(see Theorem 1.0.4). Because φ_λ is a function of w only, it is then not hard to verify that

$$T^*_{\varphi_\lambda}T_z - T_zT^*_{\varphi_\lambda} = 0,$$

so we have that

$$q\overline{\varphi}_\lambda qzq + q\overline{\varphi}_\lambda pzq - qzq\overline{\varphi}_\lambda q = 0,$$

and

$$p\overline{\varphi}_\lambda pzp - pzq\overline{\varphi}_\lambda p - pzp\overline{\varphi}_\lambda p = 0,$$

i.e.

$$\begin{aligned} q\overline{\varphi}_\lambda qzq - qzq\overline{\varphi}_\lambda q &= -q\overline{\varphi}_\lambda pzq, \\ p\overline{\varphi}_\lambda pzp - pzp\overline{\varphi}_\lambda p &= pzq\overline{\varphi}_\lambda p. \end{aligned}$$

Thus we have a following

Proposition 4.3.1

$$S^*_{\varphi_\lambda}S_z - S_zS^*_{\varphi_\lambda} = -q\overline{\varphi}_\lambda pzq, \tag{4.6}$$

$$R^*_{\varphi_\lambda}R_z - R_zR^*_{\varphi_\lambda} = pzq\overline{\varphi}_\lambda p. \tag{4.7}$$

4.4 Essential commutativity of S_w^* and S_z

In this section we will prove the essential commutativity of S_w^* and S_z on $H^2(D^2) \ominus [h]$ when h is a polynomial. As we noted in the last section, we first prove the assertion for $S_{\varphi_\lambda}^*$ and S_z.

We first observe that for any $f \in H^2(D^2) \ominus [h]$ and any $g \in [h]$,

$$< pzf, zg >_{H^2} = < zf, zg >_{H^2} = < f, g >_{H^2} = 0.$$

So pz actually maps $H^2(D^2) \ominus [h]$ into $[h] \ominus z[h]$. Therefore, $S_{\varphi_\lambda}^* S_z - S_z S_{\varphi_\lambda}^*$ can be decomposed as

$$H^2(D^2) \ominus [h] \xrightarrow{-pz} [h] \ominus z[h] \xrightarrow{q\overline{\varphi}_\lambda} H^2(D^2) \ominus [h]. \tag{4.8}$$

This observation has an interesting corollary when h is inner.

Corollary 4.4.1 *If h is inner, then $S_w^* S_z - S_z S_w^*$ is at most of rank 1 on $H^2(D^2) \ominus [h]$.*

Proof. First we note that when $\lambda = 0$, $\varphi_\lambda(w) = w$. If h is inner, then

$$[h] = hH^2(D^2),$$

and $\{w^n h | n = 0, 1, 2, ...\}$ is an orthonormal basis for $[h] \ominus z[h]$. For any function $f(z, w) = \sum_{j=0}^{\infty} c_j w^j h$ inside $[h] \ominus z[h]$,

$$q\overline{w}f = q\overline{w}c_0 h + q(\sum_{j=1}^{\infty} c_j w^{j-1} h) = c_0 q\overline{w}h.$$

This shows that $q\overline{w}$ is at most of rank one and hence $S_w^* S_z - S_z S_w^* = -q\overline{w}pz$ is at most of rank one. □

This corollary enables us to give an operator theoretical proof of an interesting fact first noticed by W. Rudin in a slightly different context[**Ru**,pp123].

Corollary 4.4.2 *$h(z,w) = z - w$ has no inner-outer(H) factorization.*

Proof. As before, we let S_z, S_w be the compressions of T_z, T_w to $H^2(D^2) \ominus [h]$ and set

$$e_n = \frac{1}{\sqrt{n+1}}(z^n + z^{n-1}w + \cdots + zw^{n-1} + w^n), \ n = 0, 1, 2, \ldots.$$

One verifies that $\{e_n | n = 0, 1, 2, ...\}$ is an orthonormal basis for $H^2(D^2) \ominus [z - w]$. Experts will know that $H^2(D^2) \ominus [z - w]$ is actually the Bergman space over the unit disk. One then easily checks that

$$\begin{aligned} S_z &= S_w, \\ S_w e_n &= \frac{\sqrt{n+1}}{\sqrt{n+2}} e_{n+1}, \\ S_w^* e_n &= \frac{\sqrt{n}}{\sqrt{n+1}} e_{n-1}, \quad n \geq 1, \end{aligned}$$

Therefore,

$$[S_w^*, S_w] e_n = \frac{1}{n(n+1)}, \ n = 0, 1, 2, \ldots.$$

If $z - w$ had an inner-outer factorization, then $[z - w] = gH^2(D^2)$ for some inner function g and

$$[S_w^*, S_w] = [S_w^*, S_z]$$

would be at most a rank one operator which conflicts with the above computation. □

Similar methods can be used to show that functions like $z - \mu w^n$, for $|\mu| < 1$ and n a nonnegative integer, have no inner-outer(H) factorization.

We now come to the main theorem of this section.

Theorem 4.4.3 *If $h \in H^\infty(D^2)$ and there is a fixed $\lambda \in D$ and a positive constant L such that*

$$L \leq |h(z, \lambda)| \tag{4.9}$$

for almost every $z \in T$ then $S_w^* S_z - S_z S_w^*$ on $H^2(D^2) \ominus [h]$ is Hilbert-Schmidt.

Proof. We first show that $S_{\varphi_\lambda}^* S_z - S_z S_{\varphi_\lambda}^*$ is Hilbert-Schmidt. By (4.7), it will be sufficient to show that

$$q\overline{\varphi}_\lambda : [h] \ominus z[h] \longrightarrow H^2(D^2) \ominus [h]$$

is Hilbert-Schmidt.

Let us recall that the operator N_λ from $[h] \ominus z[h]$ to $H^2(D)$ is defined by

$$N_\lambda g = g(\cdot, \lambda),$$

and it is Hilbert-Schmidt by Corollary 4.2.2. Suppose

$$hf_0,\ hf_1,\ hf_2,\ \ldots$$

is an orthonormal basis for $[h] \ominus z[h]$.

We first show that $h(z,w)f_k(z,\lambda) \in [h]$ for every k. In fact,

$$\begin{aligned}\int_T |f_k(z,\lambda)|^2 d|z| &\leq L^{-2} \int_T |h(z,\lambda) f_k(z,\lambda)|^2 d|z| \\ &= L^{-2} \|N_\lambda(hf_k)\|^2 < \infty,\end{aligned}$$

i.e. $f_k(z,\lambda) \in H^2(D)$ and hence $h(z,w)f_k(z,\lambda) \in [h]$ since h is bounded. Furthermore,

$$\begin{aligned}\|h(\cdot,\ \cdot)f_k(\cdot,\lambda)\|^2 &\leq \|h\|_\infty^2 \|f_k(\cdot,\lambda)\|^2 &(4.10)\\ &\leq \|h\|_\infty^2 L^{-2} \|N_\lambda(hf_k)\|^2. &(4.11)\end{aligned}$$

Next, we observe that

$$q\overline{\varphi}_\lambda h f_k = q\overline{\varphi}_\lambda h(f_k - f_k(\cdot,\lambda)) + q\overline{\varphi}_\lambda h f_k(\cdot,\lambda). \tag{4.12}$$

Since $f_k(z,w) - f_k(z,\lambda)$ vanishes at $w = \lambda$ for every $z \in D$, it has $\varphi_\lambda(w)$ as a factor, and hence

$$q\overline{\varphi}_\lambda h(f_k - f_k(\cdot,\lambda)) = 0. \tag{4.13}$$

Combining (4.10), (4.11) and (4.12),

$$\begin{aligned}\sum_{k=0}^\infty \|q\overline{\varphi}_\lambda h f_k\|_{H^2(D^2)}^2 &= \sum_{k=0}^\infty \|q\overline{\varphi}_\lambda h(f_k - f_k(\cdot,\lambda)) + q\overline{\varphi}_\lambda h f_k(\cdot,\lambda)\|_{H^2(D^2)}^2 \\ &= \sum_{k=0}^\infty \|q\overline{\varphi}_\lambda h f_k(\cdot,\lambda)\|_{H^2(D^2)}^2 \\ &\leq \sum_{k=0}^\infty \|h(\cdot,\cdot) f_k(\cdot,\lambda)\|_{H^2(D^2)}^2 \\ &\leq \|h\|_\infty^2 L^{-2} \sum_{k=0}^\infty \|h(\cdot,\lambda) f_k(\cdot,\lambda)\|_{H^2(D)}^2 \\ &= \|h\|_\infty^2 L^{-2} tr(N_\lambda^* N_\lambda).\end{aligned}$$

This shows that $q\overline{\varphi}_\lambda$, and hence $[S_{\varphi_\lambda}^*, S_z]$, is Hilbert-Schmidt.

Assuming $\hat{\varphi}_\lambda(w) = \overline{\varphi_\lambda(\overline{w})}$, one verifies that

$$S^*_{\varphi_\lambda} = \hat{\varphi}_\lambda(S^*_w).$$

The fact that

$$\hat{\varphi}_\lambda(\hat{\varphi}_\lambda(w)) = w$$

and an application of Lemma 4.2.3 with $f = \hat{\varphi}_\lambda$ then imply that $[S^*_w, S_z]$ is Hilbert-Schmidt. $\square$

In Theorem 4.4.3, if h is continuous to the boundary of $D \times D$, then the inequality (4.8) will hold once there is a $\lambda \in D$ such that $h(z,\lambda)$ has no zero on T. This idea leads to the assertion that $S^*_w S_z - S_z S^*_w$ is Hilbert-Schmidt on $H^2(D^2) \ominus [h]$ for any polynomial h in two complex variables. But we need to recall some knowlege from complex analysis before we can prove it.

Suppose G is a bounded open set in the complex plane $\mathbf{C}$. We let $A(G)$ denote the collection of all the functions that are holomorphic on G and are continuous to the boundary of G and $Z(f)$ denote the zeros of f.

To make a study of zero sets of polynomials, we need a classical theorem in several complex variables.

Theorem 4.4.4 *Let*

$$h(z,w) = z^n + a_1(w)z^{n-1} + \cdots + a_n(w)$$

be a pseudopolynomial without multiple factors, where the $a_j(w)$'s are all in $A(G)$. Further let

$$D_h := \{w \in G | \Delta_h(w) = 0\},$$

where $\Delta_h(w)$ *is the discriminant of* h. *Then for any* $w_0 \in G - D_h$ *there exists an open neighborhood of* $U(w_0) \subset G - D_h$ *and holomorphic functions* $f_1,\ f_2,\ \ldots,\ f_n$ *on* U *with* $f_i(w) \neq f_j(w)$ *for* $i \neq j$ *and* $w \in U$, *such that*

$$h(z,w) = (z - f_1(w))(z - f_2(w)) \cdots (z - f_n(w))$$

for all $w \in U$ *and all complex number* z.

This theorem is taken from [**GF**], but similar theorems can be found in other standard books on several complex variables.

This theorem reveals some information on the zero sets of polynomials which we state as

Corollary 4.4.5 *For any polynomial* $p(z,w)$ *not having* $z - \lambda$ *with* $|\lambda| = 1$ *as a factor, the set*

$$Y_p = \{w \in \mathbf{C} | p(z,w) = 0 \ for\ some\ z \in T\}$$

has no interior.

Proof. We first assume that p is irreducible and write $p(z,w)$ as

$$p(z,w) = a_0(w)z^n + a_1(w)z^{n-1} + \cdots + a_n(z)$$

with $a_j(w)$ polynomials of one variable and $a_0(w)$ not identically zero. Then on $\mathbf{C} \setminus Z(a_0)$, we have

$$p(z,w) = a_0(w)\left(z^n + \frac{a_1(w)}{a_0(w)} z^{n-1} + \cdots + \frac{a_n(w)}{a_0(w)}\right).$$

Let Δ_p be the discriminant(See [**GF**] for the definition.) of p. If p is irreducible, Δ_p is not identically zero, and so neither is the discriminant of

$$q(z,w) = z^n + \frac{a_1(w)}{a_0(w)} z^{n-1} + \cdots + \frac{a_n(w)}{a_0(w)}.$$

This implies that the pseudopolynomial $q(z,w)$ has no multiple factor either.

We now prove the corollary for the irreducible polynomial p. We do it by showing that given any open disk $B \subset \mathbf{C}$, there is a $w \in B$ which is not in Y_p.

Given any small open disc B and a point w_0 in $B \setminus \{Z(\Delta_p) \cup Z(a_0)\}$, the above theorem shows the existence of an open neighborhood $U \subset B$ of w_0 and holomorphic functions f_1, f_2, ..., f_n on U with $f_i(w) \neq f_j(w)$ for $i \neq j$ and $w \in U$ such that

$$p(z,w) = a_0(w)(z - f_1(w))(z - f_2(w)) \cdots (z - f_n(w)), \qquad (4.14)$$

for all $z \in \mathbf{C}$. Then $f_1(w)$ can not be a constant λ of modulus 1 because p doesn't have factors of the form $z - \lambda$ from the assumption. So we can choose a smaller open disc $B_1 \subset U$ such that $f_1(B_1) \cap T$ is empty. Carrying the same argument out for f_2 on B_1, we have an open disc $B_2 \subset B_1$ such that $f_2(B_2) \cap T$ is empty. Continuing this procedure, we have discs B_1, B_2, ..., B_n such that $B_j \subset B_{j-1}$ for $j = 2, 3, ..., n$. Then for any $w \in B_n$, $p(z,w)$ will have no zero on T and hence w is not in Y_p.

If p is an arbitary polynomial not having $z - \lambda$ with $|\lambda| = 1$ as a factor, we factorize p into a product of irreducible polynomials as

$$p(z,w) = p_1^{d1} p_2^{d2} \cdots p_m^{dm}.$$

If we let

$$Y_j = \{w \in \mathbf{C} | p_j(z,w) = 0 \ for \ some \ z \in T\},$$

then $Y_p \subset \cup_{j=1}^m Y_j$, hence it has no interior. □

We feel it may be interesting to have a closer look at the set Y_p, but that is not the purpose of this paper. The result in Corollary 4.4.5 is good enough for us to state

Theorem 4.4.6 *For any polynomial h, $S_w^* S_z - S_z S_w^*$ is Hilbert-Schmidt on $H^2(D^2) \ominus [h]$.*

Proof. Suppose h is any polynomial. If h is of the form $(z - \lambda)g$ for some polynomial g and some λ of modulus 1, then $[h] = [g]$ because $z - \lambda$ is outer(H). So without loss of generality, we assume that h doesn't have this kind of factor. Then from the above corollary, $h(z, \mu)$ has no zeros on T for any $\mu \in D - Y_h$. Theorem 4.4.3 and the observations immediately after it then imply that $[S_w^*, S_z]$ is Hilbert-Schmidt. □

We recall that in Chapter 1 we defined an operator S_f by

$$S_f x := qfx$$

for any function $f \in A(D^2)$ and any $x \in H^2(D^2) \ominus [h]$, where q is the projection from $H^2(D^2)$ onto $H^2(D^2) \ominus [h]$ and this turns $H^2(D^2) \ominus [h]$ into a Hilbert $A(D^2)$ quotient module. The module is called *essentially reductive* if S_f is essentially normal for every $f \in A(D^2)$. It is easy to see that $H^2(D^2) \ominus [h]$ is essentially reductive if and only if both $[S_z^*, S_z]$ and $[S_w^*, S_w]$

are compact. Currently we don't know how to characterize those functions h for which $H^2(D^2)\ominus[h]$ is essentially reductive, even though some partial results are available. [**Do**] and [**DP**] are good references on this topic. However, if we consider $H^2(D^2)\ominus[h]$ as a module over the the subalgebra $A(D) \subset A(D^2)$, Theorem 4.4.6 yields the following

Corollary 4.4.7 *Assume h is a polynomial. If there is a $g \in A(D)$ and a $f \in [h] \cap H^\infty(D^2)$, such that*

$$z = g(w) + f(z, w),$$

then $H^2(D^2)\ominus[h]$ is an essentially reductive module over $A(D)$ with the action defined by

$$f \cdot x := f(S_z)x$$

for all $f \in A(D)$ and all $x \in H^2(D^2)\ominus[h]$.

Proof. It suffices to show that S_z is essentially normal. From the assumption on f, S_f is equal to 0. Since $z - g(w) = f(z, w)$, we have that

$$S_z = S_g = g(S_w).$$

Suppose $\{p_n\}$ is a sequence of polynomials which converges to g in supremum norm, then from Lemma 4.2.3, $[S_z^*, p_n(S_w)]$ is compact for each n and it is also not hard to see that $[S_z^*, p_n(S_w)]$ converges to $[S_z^*, g(S_w)]$ in the operator norm, and hence $[S_z^*, S_z] = [S_z^*, g(S_w)]$ is compact. □

This corollary shows in particular that $H^2(D^2)\ominus[h]$ is essentially reductive over $A(D^2)$ when h is linear.

4.5 Essential commutativity of R_w^* and R_z

In Section 4.4 we proved that the module actions of the two coodinate functions z, w on the quotient module $H^2(D^2) \ominus [h]$ essentially doubly commute when h is a polynomial. It is then natural to ask if there is a similar phenomenon in the case of submodules. A result due to Curto, Muhly and Yan[**CMY**] answered the question affirmatively in a special case and Curto asked if it is true for any polynomially generated submodule[**Cu**]. Since $\mathbf{C}[z, w]$ is Noetherian, one only needs to look at the submodules generated by a finite number of polynomials. In this section we will answer Curto's question partially and a complete answer will be given in Section 4.7.

One might think that the submodule case should be easier to deal with than the quotient module case because z, w act as isometries on submodules. But it turns out that the submodule case is more subtle and needs a finer analysis.

Let us now get down to the details.

Suppose M is a submodule and R_w and R_z are the module actions by coordinate functions z and w. It is obvious R_w and R_z are commuting isometries. In [**CMY**], Curto, Muhly and Yan made a study of the essential commutativity of operators R_w^*, R_z in the case that M is generated by a finite number of homogeneous polynomials. They were actually able to show that $[R_w^*, R_z]$ is Hilbert-Schmidt. In this section we will show that this is also true when M is generated by an arbitrary polynomial. The same result for the case that M is generated by a finite number of polynomials is a corollary of this result and will be treated in Section 4.6.

We suppose h is a polynomial that doesn't have a factor $z - \mu$ with $|\mu| = 1$. Then from Corollary 4.4.5 there is a $\lambda \in D$ such that $h(z, \lambda)$ is bounded away from 0 on T. As in Section 4.4, we will see that this is crucial in the development of the proofs.

For a bounded analytic function $f(z, w)$ over the unit bidisk, we recall that R_f is the restriction of the Toeplitz operator T_f onto $[h]$ and by Proposition 4.3.1,

$$R^*_{\varphi_\lambda} R_z - R_z R^*_{\varphi_\lambda} = pzq\overline{\varphi_\lambda}p.$$

We let

$$p_1 : H^2(D^2) \longrightarrow \varphi_\lambda[h], \quad q_1 : H^2(D^2) \longrightarrow [h] \ominus \varphi_\lambda[h]$$

be the projections, then $p = p_1 + q_1$. It is not hard to see that

$$(R^*_{\varphi_\lambda} R_z - R_z R^*_{\varphi_\lambda})p_1 = pzq\overline{\varphi_\lambda}p_1 = 0.$$

Moreover, by the remarks preceding Proposition 4.3.1,

$$T_z T_{\overline{\varphi_\lambda}} = T_z T^*_{\varphi_\lambda} = T^*_{\varphi_\lambda} T_z = T_{\overline{\varphi_\lambda}} T_z,$$

and hence,

$$\begin{aligned} R^*_{\varphi_\lambda} R_z - R_z R^*_{\varphi_\lambda} &= pzq\overline{\varphi_\lambda}(p_1 + q_1) \\ &= pzq\overline{\varphi_\lambda}q_1 \\ &= pz(P - p)\overline{\varphi_\lambda}q_1 \\ &= pT_z T_{\overline{\varphi_\lambda}} q_1 - pzp\overline{\varphi_\lambda}q_1 \\ &= pT_{\overline{\varphi_\lambda}} T_z q_1 - pzp\overline{\varphi_\lambda}q_1 \\ &= p\overline{\varphi_\lambda}zq_1 - pzp\overline{\varphi_\lambda}q_1, \end{aligned}$$

where P is the projection from $L^2(T^2)$ to $H^2(D^2)$. For any $f \in [h] \ominus \varphi_\lambda[h]$ and $g \in [h]$,

$$< p\overline{\varphi_\lambda} f, g >=< f, \varphi_\lambda g >= 0,$$

i.e.

$$p\overline{\varphi_\lambda} q_1 = 0. \tag{4.15}$$

So we have that

$$R^*_{\varphi_\lambda} R_z - R_z R^*_{\varphi_\lambda} = p\overline{\varphi_\lambda} z q_1.$$

Furthermore, (4.14) also implies that

$$\begin{aligned} p\overline{\varphi_\lambda} z q_1 &= p\overline{\varphi_\lambda}(p_1 + q_1) z q_1 \\ &= p\overline{\varphi_\lambda} p_1 z q_1 + p\overline{\varphi_\lambda} q_1 z q_1 = p\overline{\varphi_\lambda} p_1 z q_1. \end{aligned}$$

Since $p\overline{\varphi_\lambda}$ acts on $\varphi_\lambda[h]$ as an isometry, the above observations then yield

Proposition 4.5.1 *$[R^*_{\varphi_\lambda}, R_z]$ is Hilbert-Schmidt on $[h]$ if and only if $p_1 z q_1$ is Hilbert-Schmidt and*

$$tr([R^*_{\varphi_\lambda}, R_z]^*[R^*_{\varphi_\lambda}, R_z]) = tr((p_1 z q_1)^*(p_1 z q_1)).$$

We further observe that, for any $f \in [h] \ominus \varphi_\lambda[h]$ and $g \in \varphi_\lambda[h]$,

$$< p_1 z f, z g >=< f, g >= 0.$$

So the range of operator $p_1 z q_1$ is a subspace of $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$. If we let $p_\perp$ be the projection from $\varphi_\lambda[h]$ onto $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$ then

$$p_1 z q_1 = p_\perp z q_1. \tag{4.16}$$

We will prove that $p_\perp z q_1$ is Hilbert-Schmidt after some preparation.

Suppose

$$h = \sum_{j=0}^{m} a_j(z) w^j$$

is a polynomial and that

$$|h(z,\lambda)| \geq \epsilon,$$

for some fixed positive ϵ and all $z \in T$. Assume $\mathcal{H}$ to be the L^2-closure of $span\{h(z,w)z^j | j \geq 0\}$, then $\mathcal{H} \subset [h]$ and we have the following

Lemma 4.5.2 $\mathcal{H} = \{h(z,w)f(z) : f \in H^2(D)\} = hH^2(D)$.

Proof. It is not hard to check that $hH^2(D) \subset \mathcal{H}$.

For the other direction, we assume hf is any function in $\mathcal{H}$ and need to show that $f \in H^2(D)$. In fact, if $p_n(z),\ n \geq 1$ is a sequence of polynomials such that $h(z,w)p_n(z),\ n \geq 1$, converges to $h(z,w)f(z,w)$ in $L^2(T^2)$, then $h(z,\lambda)p_n(z),\ n \geq 1$, converges to $h(z,\lambda)f(z,\lambda)$ in $L^2(T)$ by the boundedness of N_λ. Our assumption on h then implies that $p_n(z),\ n \geq 1$, converges to $f(z,\lambda)$ in $L^2(T)$, and in particular, $f(z,\lambda) \in H^2(D)$. This in turn implies that $h(z,w)p_n(z),\ n \geq 1$, converges to $h(z,w)f(z,\lambda)$ in $L^2(T^2)$ since h is a bounded function. Hence by the uniqueness of the limit,

$$h(z,w)f(z,w) = h(z,w)f(z,\lambda),$$

and therefore

$$f(z,w) = f(z,\lambda).$$

□

It is interesting to see from this lemma and Corollary 4.4.5 that $hH^2(D)$ is actually closed in $H^2(D^2)$ for any polynomial h not having a factor $z-\mu$ with $|\mu|=1$.

Lemma 4.5.3 *The operator $V:[h]\longrightarrow \mathcal{H}$ defined by*

$$V(hf)=h(z,w)f(z,\lambda)$$

is bounded.

Proof. First of all $h(z,\lambda)f(z,\lambda) = N_\lambda(hf)$ is in $H^2(D)$ and hence so is $f(z,\lambda)$ since $|h(z,\lambda)|\geq\epsilon$ on T. So V is indeed a map from $[h]$ to $\mathcal{H}$.

Next we choose a number M sufficiently large such that

$$\int_T |h(z,w)|^2 d|w| \leq M\epsilon^2 \leq M|h(z,\lambda)|^2$$

for all $z\in T$. Then for any $h(z,w)f(z,w)\in[h]$,

$$\begin{aligned}\|V(hf)\|^2 &= \int_{T^2}|h(z,w)f(z,\lambda)|^2d|z|d|w|\\ &= \int_T\left(\int_T|h(z,w)|^2d|w|\right)|f(z,\lambda)|^2d|z|\\ &\leq M\int_T|h(z,\lambda)f(z,\lambda)|^2d|z|\\ &\leq M(1-|\lambda|^2)^{-1}\|hf\|^2\end{aligned}$$

□

This lemma enables us to reduce the problem further.

For any $h(z,w)f(z,w)\in[h]\ominus\varphi_\lambda[h]$,

$$p_\perp zhf = p_\perp zV(hf)+p_\perp z(hf-Vhf).$$

But

$$zh(z,w)f(z,w) - zV(hf)(z,w) = zh(z,w)(f(z,w) - f(z,\lambda)),$$

and since $f(z,w) - f(z,\lambda)$ vanishes at $w = \lambda$ for every z, it has φ_λ as a factor; hence $z(hf - V(hf)) \in z\varphi_\lambda[h]$. Therefore by the definition of $p_\perp$,

$$p_\perp zhf = p_\perp zV(hf) + p_\perp z\varphi_\lambda hg = p_\perp zV(hf). \tag{4.17}$$

To prove that $p_\perp zq_1$ is Hilbert-Schmidt, it then suffices to show that $p_\perp z$ restricted on $\mathcal{H}$ is Hilbert-Schmidt. Before proving it, we make another observation and state a lemma.

Since $h(z,w)$ is a polynomial and

$$\int_T |h(z,w)|^2 d|w| = \sum_{k=0}^{m} |a_k(z)|^2,$$

the Riesz-Fejér theorem implies that there is a polynomial $Q(z)$ such that

$$|Q(z)|^2 = \int_T |h(z,w)|^2 d|w|$$

on T. If Q vanishes at some $\mu \in T$, then $a_k(\mu) = 0$ for each k, and hence h has a factor $(z - \mu)$. But this contradicts our assumption on h. So we can find a positive constant, say η, such that

$$|Q(z)| \geq \eta, \tag{4.18}$$

for all $z \in T$.

Suppose $\{h(z,w)f_n(z) | n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$, then

$$\begin{aligned} \delta_{i,j} &= \int_{T^2} h(z,w)f_i(z)\overline{h(z,w)f_j(z)}d|z|d|w| \\ &= \int_T \left(\int_T |h(z,w)|^2 d|w|\right) f_i(z)\overline{f_j(z)}d|z| \\ &= \int_T Q(z)f_i(z)\overline{Q(z)f_j(z)}d|z|. \end{aligned}$$

So $\{Q(z)f_k(z)|k \geq 0\}$ is orthonormal in $H^2(D)$, but of course it may not be complete.

Lemma 4.5.4 *The linear operator* $J : \overline{span\{Qf_k|k \geq 0\}} \longrightarrow H^2(D)$ *defined by*

$$J(Qf_k) = f_k, \qquad k \geq 0,$$

is bounded.

Proof. By Inequality (4.17), for any function $Qf \in \overline{span\{Qf_k|k \geq 0\}}$,

$$\int_T |f(z)|^2 d|z| \leq \eta^{-2} \int_T |Q(z)f(z)|^2 d|z|.$$

□

Now we are in the position to prove

Proposition 4.5.5 $p_\perp z$ *restricted to* $\mathcal{H}$ *is Hilbert-Schmidt.*

Proof. Assume $\{g_k|k \geq 0\} \subset [h] \ominus z[h]$ is an orthonormal basis and, as above, $\{h(z,w)f_n(z)|n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$. Since φ_λ is inner, $\{\varphi_\lambda(w)g_k(z,w)|k \geq 0\}$ is an orthonormal basis for $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$. Therefore, by the first identity of (4.4) and the expression of h,

$$\begin{aligned} p_\perp z h f_n &= \sum_{k=0}^{\infty} < zhf_n, \varphi_\lambda g_k > \varphi_\lambda g_k \\ &= \sum_{k=0}^{\infty} < \sum_{i=0}^{m} z a_i w^i f_n, \varphi_\lambda \sum_{j=0}^{\infty} T_{w^j} N_j g_k > \varphi_\lambda g_k. \end{aligned}$$

Note that a_i's and f_n are functions of z only, so $\sum_{i=0}^{m} za_i w^i f_n$ is orthogonal to $\sum_{j=m+1}^{\infty} w^j \varphi_\lambda N_j g_k$ because the latter has the factor w^{m+1}. It then follows that

$$\begin{aligned} p_\perp zhf_n &= \sum_{k=0}^{\infty} < \sum_{i=0}^{m} za_i w^i f_n, \varphi_\lambda \sum_{j=0}^{\infty} T_{w^j} N_j g_k > \varphi_\lambda g_k \\ &= \sum_{k=0}^{\infty} < \sum_{i=0}^{m} za_i w^i f_n, \sum_{j=0}^{m} \varphi_\lambda w^j N_j g_k > \varphi_\lambda g_k \\ &= \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \varphi_\lambda g_k \left(\int_T za_i(z) f_n(z) \overline{N_j g_k(z)} d|z| \right) \left(\int_T w^i \overline{\varphi_\lambda(w) w^j} d|w| \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i,j=0}^{m} c_{ij} < f_n, T^*_{za_i} N_j g_k >_{H^2(D)} \right) \varphi_\lambda g_k, \end{aligned}$$

where

$$c_{ij} = \int_T w^i \overline{\varphi_\lambda(w) w^j} d|w|.$$

If $c := max\{|c_{ij}| \ : 0 \le i, j \le m\}$, then the Cauchy inequality yields

$$\begin{aligned} \|p_\perp zhf_n\|^2 &= \sum_{k=0}^{\infty} \left| \sum_{i,j=0}^{m} c_{ij} < f_n, T^*_{za_i} N_j g_k >_{H^2(D)} \right|^2 \\ &\le (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} | < f_n, T^*_{za_i} N_j g_k >_{H^2(D)} |^2 \\ &= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} | < J(Qf_n), T^*_{za_i} N_j g_k >_{H^2(D)} |^2 \\ &= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} | < Qf_n, J^* T^*_{za_i} N_j g_k >_{H^2(D)} |^2, \end{aligned}$$

where J is the operator defined in Lemma 4.5.4. Therefore, by the fact that $\{Qf_n : n \ge 0\}$ is orthogonal in $H^2(D)$ and the fact that N_j is Hilbert-Schmidt on $[h] \ominus z[h]$ for each j,

$$\begin{aligned} \sum_{n=0}^{\infty} \|p_\perp zhf_n\|^2 &\le (mc)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} | < Qf_n, J^* T^*_{za_i} N_j g_k >_{H^2(D)} |^2 \\ &= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \sum_{n=0}^{\infty} | < Qf_n, J^* T^*_{za_i} N_j g_k >_{H^2(D)} |^2 \end{aligned}$$

$$
\begin{aligned}
&\leq (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \|J^* T^*_{za_i} N_j g_k\|^2_{H^2(D)} \\
&= (mc)^2 \sum_{i,j=0}^{m} \|J^* T^*_{za_i}\|^2 \sum_{k=0}^{\infty} \|N_j g_k\|^2_{H^2(D)} \\
&= (mc)^2 \sum_{i,j=0}^{m} \|J^* T^*_{za_i}\|^2 tr(N_j^* N_j) < \infty.
\end{aligned}
$$

□

Theorem 4.5.6 *$[R_w^*, R_z]$ is Hilbert-Schmidt on $[h]$ for any polynomial h.*

Proof. If $h = (z - \lambda)h_1$ for some polynomial h_1 and $\lambda \in T$, then $[h] = [h_1]$. If h_1 is a nonzero constant then $[h_1] = H^2(D^2)$ and hence

$$R_w = T_w, \quad R_z = T_z.$$

Therefore $[R_w^*, R_z] = 0$. So without loss of generality, we may assume h doesn't have a factor $z - \lambda$ for some $\lambda \in T$. Propositions 4.5.1, 4.5.5 and Equality (4.15) together imply that $[R^*_{\varphi_\lambda}, R_z]$ is Hilbert-Schmidt. An argument similar to that in the end of the proof of Theorem 4.4.3 establishes our assertion. □

4.6 Operator $[R_z^*, \ R_z][R_w^*, \ R_w]$ on $[h]$

In this section we are going use the result of Section 4.5 to prove the following

Theorem 4.6.1 *The operator $[R_z^*, \ R_z][R_w^*, \ R_w]$ is Hilbert-Schmidt on $[h]$ when h is a polynomial.*

Proof. For the same reason as in the proof of Theorem 4.5.6, we assume that h doesn't have a factor $z-\mu$ for $\mu \in T$. Then by Corollary 4.4.5, $h(z,\lambda)$ is bounded away from zero on T for some $\lambda \in D$. To make our computations clearer, we assume that $h(z,0)$ is bounded away from 0 on T. Then one sees that for any $hf \in [h]$, $h(f - f(\cdot,\ 0))$ is a function in $w[h]$. Therefore,

$$\begin{aligned}
[R_w^*,\ R_w]hf &= hf - R_w R_w^* hf \\
&= hf - R_w R_w^* h(f - f(\cdot,\ 0) + f(\cdot,\ 0)) \\
&= hf - h(f - f(\cdot,\ 0)) - R_w R_w^* hf(\cdot,\ 0) \\
&= hf(\cdot,\ 0) - R_w R_w^* hf(\cdot,\ 0),
\end{aligned}$$

hence

$$[R_w^*,\ R_w]hf = [R_w^*,\ R_w]h(\cdot,\ \cdot)f(\cdot,\ 0). \tag{4.19}$$

Similarly,

$$\begin{aligned}
[R_z^*,\ R_z]hf(\cdot,\ 0) &= hf(\cdot,\ 0) - R_z R_z^* hf(\cdot,\ 0) \\
&= hf(\cdot,\ 0) - R_z R_z^* h(f(\cdot,\ 0) - f(0,\ 0) + f(0,\ 0)) \\
&= hf(\cdot,\ 0) - h(f(\cdot,\ 0) - f(0,\ 0)) - R_z R_z^* hf(0,\ 0) \\
&= hf(0,\ 0) - f(0,\ 0) R_z R_z^* h,
\end{aligned}$$

hence

$$[R_z^*,\ R_z]hf(\cdot,\ 0) = f(0,0)[R_z^*,\ R_z]h. \tag{4.20}$$

By the essential commutativity of R_z^* and R_w(Theorem 4.5.6), and Equality (4.18), Equality (4.19),

$$[R_z^*,\ R_z][R_w^*,\ R_w]hf$$

$$\begin{aligned} &= [R_z^*,\ R_z][R_w^*,\ R_w]h(\cdot,\ \cdot)f(\cdot,\ 0) \\ &= [R_w^*,\ R_w][R_z^*,\ R_z]h(\cdot,\ \cdot)f(\cdot,\ 0) + Khf(\cdot,\ 0) \\ &= f(0,\ 0)[R_w^*,\ R_w][R_z^*,\ R_z]h + Khf(\cdot,\ 0), \end{aligned}$$

where K a Hilbert-Schmidt operator from Theorem 4.5.6. If we let $A,\ B$ be operators from $[h]$ to itself such that for any $hf \in [h]$

$$Ahf = f(0,\ 0)h; \qquad Bhf = h(\cdot,\ \cdot)f(\cdot,\ 0),$$

then the above computation shows that

$$[R_z^*,\ R_z][R_w^*,\ R_w] = [R_w^*,\ R_w][R_z^*,\ R_z]A + KB.$$

We observe that A is a rank one operator with kernel $\overline{z[h] + w[h]}$ and

$$dim([h] \ominus (\overline{z[h] + w[h]})) = 1$$

by Corollary 2.1.4 and Corollary 2.1.7, hence A is a bounded. Thus to prove that $[R_z^*,\ R_z][R_w^*,\ R_w]$ is Hilbert-Schmidt, it suffices to check that B is bounded, but this is clear from our assumption on h and Lemma 4.5.3.

If $h(z, \lambda)$ is bounded away from zero on T for some non-zero $\lambda \in D$, then similar computations will show that $[R_z^*,\ R_z][R_{\varphi_\lambda}^*,\ R_{\varphi_\lambda}]$ is Hilbert-Schmidt. Then applying Lemma 4.2.3 twice will establish the assertion. □

One sees that the proof of Theorem 4.6.1 depends heavily on the fact that $R_z,\ R_w$ are isometries. A corresponding study for the product $[S_z^*,\ S_z][S_w^*,\ S_w]$ is thus expected to be harder and we plan to return to that at a later time.

4.7 An Improvement

In this section we will generalize the major theorems obtained so far to the case when $[h]$ is replaced by submodules generated by a finite number of polynomials. Here we need a fact from commutative algebra which we state in a form that fits into our work. Readers may find more information in [**Ke**]. We thank Professor C. Sah for showing us his proof of the following statement.

Lemma 4.7.1 *Suppose* $p_1,\ p_2,\ \ldots,\ p_k$ *are polynomials in* $C[z,w]$ *such that the greatest common divisor* $GCD(p_1,\ p_2,\ \ldots,\ p_k) = 1$, *then the quotient*

$$C[z,w]/(p_1,\ p_2,\ \ldots,\ p_k)$$

is finite dimensional.

Proof. First of all, $\mathcal{R}$ is a Unique Factorization Domain(UFD) of Krull dimension 2.

We denote the ideal $(p_1,\ p_2,\ \ldots,\ p_k)$ by I and suppose

$$I = \cap_{s=1}^{n} I_s$$

is the irredundant primary representation of I. If we let $J_s = \sqrt{I_s}$ be the radical of I_s, $s = 1,\ 2,\ \ldots,\ n$, then each J_s is prime and it is either maximal or minimal since the Krull dimension of $\mathcal{R}$ is 2. In an UFD, every minimal prime ideal is principal [**ZS**, p 238]. Since $GCD(p_1,\ p_2,\ \ldots,\ p_k) = 1$, the associated prime ideals $J_1,\ J_2,\ \ldots,\ J_s$ must all be maximal and hence each J_s must have the form $(z - z_s, w - w_s)$ with $(z_s, w_s) \in \mathbf{C}^2$, $s = 1,\ 2,\ \ldots,\ n$, mutually different.

Therefore, we can choose an integer, say m, sufficiently large such that

$$J_s^m = (z - z_s, w - w_s)^m \subset I_s$$

for each s. Then,

$$\cap_{s=1}^n J_s^m \subset \cap_{s=1}^n I_s = I,$$

and therefore,

$$dim(\mathcal{R}/I) \leq dim(\mathcal{R}/(\cap_{s=1}^n J_s^m)).$$

By the Nullstellensatz, one easily checks that

$$J_i^m + J_j^m = \mathcal{R}, \quad i \neq j.$$

The Chinese Remainder Theorem then implies that

$$\mathcal{R}/(\cap_{s=1}^n J_s^m) = \prod_{s=1}^n \mathcal{R}/J_s^m,$$

and hence

$$dim(\mathcal{R}/I) \leq \prod_{s=1}^n dim(\mathcal{R}/J_s^m) = (\frac{m(m+1)}{2})^n$$

□

It would be interesting to generalize this lemma to polynomial rings of higher Krull dimensions.

If $h_1,\ h_2,\ ...,\ h_k$ are polynomials and we set

$$G = GCD(h_1,\ h_2,\ ...,\ h_k) \quad and \quad f_j = h_j/G, \tag{4.21}$$

$j = 1,\ 2, ...,\ k$, then

$$GCD(f_1,\ f_2,\ ...,\ f_k) = 1.$$

If $\{e_1,\ e_2,\ ...,\ e_m\}$ is a basis for

$$C[z,w]/(f_1,\ f_2,\ ...,\ f_k),$$

then for any polynomial $g(z,w)$,

$$g(z,w) = \sum_{i=1}^{m} c_i e_i(z,w) + r(z,w)$$

with $r \in (f_1,\ f_2,\ ...,\ f_k)$ and some constants $c_i,\ i = 1,\ 2,\ ...,\ m$. Therefore,

$$G(z,w)g(z,w) = \sum_{i=1}^{m} c_i G(z,w)e_i(z,w) + G(z,w)r(z,w). \tag{4.22}$$

It is easy to see that $G(z,w)r(z,w) \in (h_1,\ h_2,\ ...,\ h_k)$ and hence $(G)/(h_1,\ h_2,\ ...,\ h_k)$ is also finite dimensional.

Corollary 4.7.2 *If M is a submodule of $H^2(D^2)$ generated by a finite number of polynomials, then*

(a) $[S_z^*,\ S_w]$ is Hilbert-Schmidt on $H^2(D^2) \ominus M$;

(b) $[R_z^*,\ R_w]$ is Hilbert-Schmidt on M;

(c) $[R_z^*,\ R_z][R_w^*,\ R_w]$ is Hilbert-Schmidt on M.

Proof. Suppose $h_1,\ h_2,\ ...,\ h_k$ are polynomials and $M = [h_1,\ h_2,\ ...,\ h_k]$ is the closed submodule generated by $h_1,\ h_2,\ ...,\ h_k$. We assume $G,\ f_i,\ i = 1,\ 2, ...,\ k$, and $e_j,\ j = 1,\ 2, ...,\ m$ to be as in (4.20) and (4.21). Consider the space

$$\mathcal{K} := span\{e_j : j = 1,\ ,2, ...,\ m\} + M.$$

It is closed because $span\{e_j : j = 1,\ ,2, ...,\ m\}$ is finite dimensional. For any polynomial g, Identity (4.21) implies that $Gg \in \mathcal{K}$, and hence $[G] \subset \mathcal{K}$. The inclusion

$$[G] \ominus M \subset \mathcal{K} \ominus M$$

then forces $[G] \ominus M$ to be finite dimensional. We let

$$p_G : H^2(D^2) \longrightarrow [G], \quad q_G : H^2(D^2) \longrightarrow H^2(D^2) \ominus [G],$$

$$p_M : H^2(D^2) \longrightarrow M, \quad q_M : H^2(D^2) \longrightarrow H^2(D^2) \ominus M,$$

$$p_\perp : H^2(D^2) \longrightarrow [G] \ominus M,$$

be the projections. Then $p_\perp$ is of finite rank and

$$p_G = p_M + p_\perp, \quad q_G = q_M - p_\perp.$$

One verifies that

$$\begin{aligned} p_G z p_G &= p_M z p_M + p_M z p_\perp + p_\perp z p_M + p_\perp z p_\perp, \\ q_G z q_G &= q_M z q_M - q_M z p_\perp - p_\perp z q_M + p_\perp z p_\perp, \end{aligned}$$

and consequently, $p_G z p_G - p_M z p_M$ and $q_G z q_G - q_M z q_M$ are of finite rank. Similarly, $q_G w q_G - q_M w q_M$ and $q_G w q_G - q_M w q_M$ are also of finite rank. The assertion in this corollary then follows easily from Theorem 4.4.6, 4.5.6 and 4.6.1. □

We conclude this chapter by a conjecture suggested by Corollary 4.7.2.

Conjecture. *The assertions in Corollary 4.7.2 still hold if M is replaced by any finitely generated submodule.*

Concluding remarks

1. As we mentioned in the beginning of Section 2.1, the collection of functions in $H^2(D^2)$ which vanish at the origin $(0,\ 0)$ is a closed proper subset of $H^2(D^2)$ and it well known that this subspace has the form $zH^2(D^2) + wH^2(D^2)$. In particular, $zH^2(D^2) + wH^2(D^2)$ is a closed subspace of $H^2(D^2)$. If M is an arbitrary submodule of finite rank, then Theorem 2.1.7 says that $dim(M \ominus (zM + wM))$ is finite and we wonder if $zM + wM$ is still closed. If this is true then many studies on submodules will be simplified.

2. In Section 2.3, we considered the solution space of the equation $p_1 f_1 + p_2 f_2 = 0$. But the techniques there don't generalize to more general equations like

$$p_1 f_1 + p_2 f_2 + \cdots + p_k f_k = 0, \quad k > 2.$$

The case that $p_1,\ p_2$ have common zeros on the boundary seems much harder. We feel the singular measures may play some roles in this case.

3. Theorem 3.2.2 says that if a function h has an inner-outer(H) factorization, then its zero set can't be arbitrary. Corollary 4.4.5 seems to give us a useful picture of those zeros of polynomials that lie on the boundary. It therefore may be possible to characterize those polynomials which have inner-outer(H) factorizations.

4. By the classical Berger-Shaw theorem, the traces of the selfcommutators of hyponormal operators are related to the Lebesgue measure of the spectra. Now we see from Theorem 4.4.6 that if h is a polynomial, then the cross commutator $S_zS_w^*-S_w^*S_z$ is Hilbert-Schmidt on $H^2(D^2)\ominus[h]$. And by Theorem 3.1.4 the joint spectrum $\sigma(S_z,\ S_w)$ is the closure of $Z(h)$ in $\overline{D^2}$. One is then led naturally to think about the possible relations between the Hilbert-Schmidt norm of $S_zS_w^* - S_w^*S_z$ and the *mass* of $Z(h)$. Some examples were calculated, but the relationship is still very far from clear.

5. The proof of Proposition 4.5.5 actually provides an estimate of the Hilbert-Schmidt norm of $R_zR_w^* - R_w^*R_z$ in terms of the coefficients of the polynomial h. But this estimate doesn't seem to give any interesting implications since $R_zR_w^* - R_w^*R_z$ can be zero on very nontrivial submodules. Then what is the Hilbert-Schmidt norm of $R_zR_w^* - R_w^*R_z$ related to? There isn't even a guess.

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