Upper Bounds on the Length of the Shortest Closed Geodesic on Simply Connected Manifolds

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The subject of the thesis is upper bounds on the length of the shortest closed geodesic on simply connected manifolds. We will give two explicit estimates which hold for manifolds with nontrivial second homology group. The first estimate will depend on the diameter and on the (possibly negative) lower bound on the sectional curvature; the second estimate will depend on the volume and the upper bound on the sectional curvature and holds under some additional assumptions about the homotopy type of the manifold.

The technique that we develop in order to obtain the first result will also enable us to estimate homotopy distance between any two closed curves

on compact simply connected manifolds with sectional curvature bounded below and diameter bounded above. More precisely, let c be a constant such that any curve of length L can be contracted to a point through the curves of length less than or equal to cL. There exists a homotopy connecting any two closed curves such that the length of the trajectory of the points during this homotopy has an explicit upper bound in terms of the lower bound of the curvature, the upper bound of the diameter and c.

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Chapter 1

History of the Problem and Techniques

1.1 Introduction

The study of closed geodesics was originated by Darboux, Hadamard, Poincare and Birkhoff at the end of 19th century. It was continued among others, by Klingenberg, Lyusternik and Fet, Gromoll and Meyer, Cheeger, Sullivan and Vigue-Poirrier. Among the many different subjects they studied we will distinguish three major questions:

1. Does there exist at least one closed geodesic on any compact Riemannian manifold?

2. Are there infinitely many closed geodesics on any compact Riemannian manifold?

3. How is the length of the shortest closed geodesic connected with the other parameters of a Riemannian manifold, such as volume, diameter and curvature? (There we can distinguish two subquestions: a) finding lower bounds on the length of the shortest closed geodesic and b) finding upper bounds on the length of the shortest closed geodesic, preferably, in terms of the volume or the diameter.)

The first question was positively answered by Lyusternik and Fet in 1952. We will discuss their result in section 1.4. The second question is still open, but significant contribution to its solution was made by Gromoll and Meyer, who had shown that there always exist infinitely many distinct periodic geodesics on compact manifold under some topological condition, namely unboundedness of the sequence of Betti numbers of the free loop space on a manifold, and Sullivan and Vigue-Poirrier, who had shown that this condition is satisfied if the cohomology ring of the manifold or any of its covers is not generated by one element (see [10, 20]).

In 1970 J. Cheeger found a lower bound on the length of the shortest closed geodesic on the Riemannian manifold with sectional curvature bounded from below, diameter bounded from above, and volume bounded below by a positive constant (see [5]).

This result is particularly important due to the following Klingenberg's Lemma that establishes the connection between the injectivity radius of a manifold and the length of the shortest closed geodesic.

Lemma 1.1.1. *(Klingenberg) Let M be a Riemannian manifold, and for* *any $p \in M$ let $q \in M$ be a point that realizes the distance from p to its cut locus $C_M(p)$. Then one of the following statements has to be satisfied:*

(1) q is conjugate to p along some minimizing geodesic γ connecting p and q.

(2) there exist exactly two minimizing geodesics γ and σ from p to q and $\gamma'(l) = -\sigma'(l)$, where $l = d(p,q)$.

(see [9]).

In particular, if p is a point such that $d(p, C_M(p))$ is the smallest and p and q are not conjugate along a minimizing geodesic then reversing roles of p and q we see that p and q lie on a smooth closed geodesic. Further recall that if M is a compact Riemannian manifold with the sectional curvature bounded from above by H then the distance between any pair of conjugate points is $\geq \pi/\sqrt{H}$. Combining these facts together we see that either there exists a closed geodesic γ in M, such that $inj(M) = \frac{1}{2}l(\gamma)$, where $inj(M)$ denotes the injectivity radius of a manifold M or $inj(M) \geq \frac{\pi}{\sqrt{H}}$. Now Cheeger's estimate yields a lower bound on the injectivity radius of any manifold with curvature, diameter and volume bounded from above, and consequently helps to establish the finitness of the number of diffeomorphism types of n-dimensional manifolds with bounded $|K|$, diameter and volume bounded from below.

Our interests lie in finding upper bounds on the length of the shortest closed geodesic. The only simply connected manifold for which explicit bounds of such nature were known prior to our work is S^2 (Croke, 1989).

Croke has shown that $l(M) \leq 31\sqrt{A}$ and $l(M) \leq 9d$, where A is the area and d is the diameter of a manifold M diffeomorphic to the 2-dimensional sphere. However, we are not aware of any explicit upper bounds for $l(M)$ valid for any metric on a simply connected manifold of dimension ≥ 3.

For nonsimply connected compact manifolds the bound $l(M) \leq 2d$ is almost obvious and well-known. For the class of essential manifolds Gromov (see [12]) found an upper bound $l(M) \leq c(n)vol(M)^{1/n}$. (Recall that a closed manifold M is essential if there exists a map $f : M \longmapsto K(\Pi, 1)$, such that $f_*([M]) \neq 0$, where $K(\Pi, 1)$ is aspherical space with the fundamental goup Π. In particular, 1-essential manifolds are not simply connected.) That result is a generalization of previous work by Loewner, Pu and Berger, among others.

In the thesis we have found some upper bounds for $l(M)$ in terms of either a lower bound of sectional curvature and upper bound of the diameter of M, where M is any compact simply connected manifold such that $H_2(M) \neq \{0\}$ or in terms of the upper bound of the sectional curvature and a lower bound of the volume of M, if M is a simply connected manifold satisfying some topological constraints, described later. The exact statement of our results will be given in Chapter 2.

The techniques we used in our work include Morse theory for the space of all closed curves on a manifold, some ideas of Gromov from [12] based on the obstruction theory, and a modification of the techniques developed by Gromov in order to prove his theorem on the curvature, diameter and Betti numbers, (see [11]).

In this chapter we will give a more detailed description of previous results in this area, in particular, the ones that are going to be relevant for our work, and describe some of the techniques that were used in the thesis.

In section 1.2 we will define the Birkhoff Curve Shortening Process, that allows one to obtain closed geodesics, and that is repeatedly used in our work. In section 1.3 we will describe the previous results for nonsimply connected manifolds, in particular, the existence of a closed geodesic on compact nonsimply connected manifold, and Gromov's estimate for 1-essential manifolds. In section 1.4 we will talk about the results of Croke, and, finally, in section 1.5 we will discuss Gromov's paper [11], which will be important for our work despite it seeming irrelevance to the study of closed geodesics.

1.2 Birkhoff Curve Shortening Process

At this point we would like to discuss the following technique that is often helpful in "finding" a closed geodesic. This technique was introduced by Birkhoff; we will closely follow the description of it in [8].

Birkhoff Curve Shortening Process Let Λ^E be the space of all piecewise differentiable closed curves parametrized proportionally to its arclength of energy bounded from above by E, where the energy of a piecewise differentiable curve $\gamma(t)$ defined on the interval $[a, b]$ is equal to $\int_a^b |\frac{d\gamma}{dt}|^2 dt$. Note that for curves that are parametrized by a parameter proportional to arc length on the interval $[0, 1]$ the energy of a curve is equalt to the square of its length. Given a curve $\alpha \in \Lambda^E$, B.C.S.P. (Birkhoff Curve Shortening Process) will provide us with a way of assigning to it another curve of smaller energy (length). We will proceed as follows: choose N large enough so that $\sqrt{E}/N$ is smaller then the injectivity radius of a manifold. Let α_1 be the unique piecewise geodesic closed curve such that it agrees with α at the points i/N for all integers $0 \leq i \leq N$, and such that $\alpha_1|_{[i/N,(i+1)/N]}$ is a minimizing geodesic parametrized proportionally to its arclength.

We note that α and α_1 can be joined with the following homotopy, $s \in [0,1]$, $i = \{0, 1..., N\}$

$$\alpha_s(\frac{i}{N} + t) = \begin{cases} \gamma_i^s(t) & \text{if } 0 \leq t \leq \frac{s}{N} \\ \alpha(\frac{i}{N} + t) & \text{if } \frac{s}{N} \leq t \leq \frac{1}{N}, \end{cases}$$

where γ_i^s is the minimizing geodesic connecting $\alpha(i/N)$ and $\alpha(i/N + 2N)$ parametrized proportionally to arclength. Since length (γ_i^s) is less than radius of injectivity for all i, s, it is unique.

Next we define α_2 to be the unique piecewise geodesic closed curve with α_2 agreeing with α_1 at the points $(2i+1)/(2N)$, which is also parametrized proportional to its arclength on each interval $[(2i+1)/(2N), (2i+3)/(2N)]$. We then note that α_1 and α_2 can also be joined by the similar homotopy.

In other words, our procedure consists of breaking our original curve α into N segments so that the length of each segment is less than the injectivity radius of a manifold, and then substituting for each segment a minimizing geodesic. We, thus, obtain a piecewise geodesic curve of N segments. Then we repeat the procedure, by dividing each segment in half and connecting "the middles" of those segments with minimal geodesics.

Now we can similarly define a sequence $\alpha_i, i = 2, 3, ...$, where α_{i+1} is obtained from α_i by dividing each geodesic segment of α_i in half and connecting "the middles" of those segments with minimal geodesics. Birkhoff had shown that this sequence converges to a (possibly trivial) closed geodesic in the same free homotopy class as the original closed curve α.

1.3 Upper bounds on the length of a shortest closed geodesic on nonsimply connected manifolds

It is not surprising that the results on the closed geodesics, beginning with the existence of the closed geodesic, are much more readily obtainable in the case of nonsimply connected compact manifolds. The existence of a closed geodesic on nonsimply connected manifold was proven by Cartan. Only a little additional work is required to show that the length of the shortest closed geodesic is bounded from above by two times the diameter of a manifold. The upper bound in terms of the volume of a manifold, as we said before, was obtained by Gromov for nonsimply connected compact manifold, under certain topological restrictions.

Even though, the case of simply connected manifolds is rather different, both of those results can be generalized for simply connected manifolds assuming additional restriction on the curvature.

By the free homotopy classes of closed curves on M we will mean homotopy classes of continuous maps $S^1 = [0,1]/\{0,1\} \longrightarrow M$.

Theorem 1.3.1. *(Cartan) Let M be compact nonsimply connected manifold. There exists a closed geodesic in the every nonconstant free homotopy class Γ, of closed curves on M,(cf. [4], Theorem 4.12).*

Proof. Let m be the infimum of the lengths of piecewise differentiable curves that belong to Γ. Consider a sequence $\{\gamma_i\}, \gamma_i \in \Gamma$ of piecewise differentiable curves, such that $length(\gamma_i)$ approaches m as i approaches infinity. Without loss of generality we can assume that they are broken geodesics defined on the interval $[0,1]$ parametrized proportional to their arclengths. By Ascoli-Arzela Theorem, this sequence converges uniformly to a continuos closed curve $\gamma_0 : [0,1] \longrightarrow M$.

Now we construct a new curve γ and show that it is a closed geodesic. The curve γ is defined as follows: we use the partion $0 = t_0 < t_1 < \ldots < t_j = 1$ to subdivide the unit interval $[0,1]$, so that $\gamma_0|_{[t_{i-1},t_i]}$ is contained in a totally normal neighborhood. Then we define γ to be a unique piecewise geodesic curve that agrees with γ_0 at the points $\gamma_0(t_i)$, where i is the integer between 0 and k. In order to show that γ is a geodesic it is sufficient to show that the length of γ is exactly m, (otherwise we could shorten it and obtain a contradiction). To show that the length of γ is m we assume that it is strictly greater then m then let $\epsilon = \frac{l(\gamma)-m}{2k+1}$. There exists integer j such that $l(\gamma_j) - m < \epsilon$ and $d(\gamma_j(t), \gamma_0(t)) < \epsilon$ for all $t \in [0,1]$. Denote $\gamma_j^i = \gamma_j|_{[t_{i-1},t_i]}$, and $\gamma|_{[t_{i-1},t_i]}$ by γ^i. $\sum_1^k(l(\gamma_j^i)+2\epsilon) = l(\gamma_j)+2\epsilon k < l(\gamma) = \sum_1^k l(\gamma^i)$. Therefore, there exists i such that $1 \leq i \leq k$ and $l(\gamma_j^i) + 2\epsilon < l(\gamma^i)$, which contradicts the fact that γ^i is minimizing. □

The following simple fact is well-known.

Proposition 1.3.2. *The length of the shortest closed geodesic on a nonsimply connected compact Riemannian manifold is always less then or equal to twice the diameter of a manifold.*

Proof. Consider homotopically nontrivial closed curve $\gamma(t)$ parametrized proportionally to its arclength. Use the partition $0 = t_0 < t_1 < ... < t_k = 1$ to subdivide the interval $[0,1]$ so that the length$(\gamma|_{[t_i,t_{i+1}]}) < \delta$, for any positive δ. Now let $p \in M$ be any point $p \notin \gamma(t)$. Consider closed curves T_i formed by $\sigma_i \cup \gamma|_{[t_i,t_{i+1}]} \cup -\sigma_{i+1}$, where $\sigma_i's$ are minimizing geodesics that connect p with $\gamma(t_i)$. At least one of the above curves is homotopically nontrivial.

Now let us use the above procedure repeatedly for $\delta = \frac{1}{n}$, where n is a positive integer, thus obtaining a sequence $\{T_{\frac{1}{n}}\}$ of homotopically nontrivial curves, such that the length of a curve $T_{\frac{1}{n}}$ is less than or equal to $2d + \frac{1}{n}$, where d is a diameter of a manifold. Therefore, the length $l(T_{\frac{1}{n}})$ converges to $2d$. We parametrize these curves proportionally to their arclength. By Ascoli-Arzela Theorem there exists a subsequence of the sequence which converges to T_0, which will also be homotopically nontrivial. Now let us construct a new curve T as follows: use a partition $0 = t_0 < t_1 < ... < t_j = 1$ to subdivide the unit interval $[0,1]$ so that $T_0|_{[t_i,t_{i+1}]}$ is contained in a totally normal neighborhood, and then let T be the closed piecewise geodesic curve that agrees with T_0 at the points $T_0(t_i)$ and minimizes distances between $T_0(t_i)$ and $T_0(t_{i+1})$. T is homotopic to T_0, thus it is also homotopically nontrivial. Moreover, the length of T is less than or equal to $2d$. We can now apply B.C.S.P. and since T is homotopically nontrivial we must obtain a closed geodesic of length $\leq 2d$. □

Another natural question is whether it is possible to find an upper bound for the length of the shortest closed geodesic in terms of the volume of a manifold. The major contribution to this problem was done by Gromov in [12], who had established this result for the following class of manifolds.

Definition 1.3.3. (1-essential manifold) 1-essential manifold is a compact manifold that admits a map $f : M \longrightarrow K(\Pi, 1)$, such that $f_*[M] \neq 0$, where $[M]$ is the fundamental homology class of M, f_* is the induced homomorphism, and $K(\Pi, 1)$ denotes the aspherical space with the fundamental group Π, i.e. the space K that has the following properties: $\pi_1(K) = \Pi$, and $\pi_n(K) = 0$, for $n \neq 1$.

In [12] Gromov had proven the following theorem:

Theorem 1.3.4. *Let M be a compact, 1-essential Riemannian manifold of dimension n. Then $l(M) \leq const_n(vol)^{\frac{1}{n}}$, where $l(M)$ is a length of a shortest closed geodesic on M, $vol(M)$ is a volume of M, and $const_n <$ $6(n+1)n^n\sqrt{(n+1)!}$*

The theorem was proved by establishing the following two inequalities:

1. Fill Rad $M \leq const'_n (vol M)^{\frac{1}{n}}$,

2. $l(M) \leq 6$Fill Rad M, where $const'_n < (n+1)n^n\sqrt{(n+1)!}$, and Fill Rad M is a filling radius of a manifold, defined below.

Definition 1.3.5. (Filling Radius of n-dimensional Manifold M Topologically Imbedded into X) Filling radius, denoted by Fill Rad $(M \subset X)$, where X is an arbitrary metric space, is the infimum of $\epsilon > 0$, such that M bounds in the ϵ-neighborhood $N_\epsilon(M)$, i.e. homomorphism $H_n(M) \longrightarrow H_n(N_\epsilon(M))$ induced by the inclusion map vanishes, where n is the dimension of a manifold M.

Definition 1.3.6. (Filling Radius of an abstract manifold M) Filling radius, denoted Fill Rad M, of an abstract manifold M is Fill Rad $(M \subset X)$, where $X = L^\infty(M)$, and the embedding of M into X is a function that to each point p of M assigns a distance function $p \longmapsto f_p = d(p,q)$.

We will present a proof of the second inequality (Lemma 1.2.B in [12]). Similar ideas are used in a proof of one of our results.

Lemma 1.3.7. *Let M be a compact, 1-essential manifold of dimension n, then $l(M) \leq 6$Fill Rad M.*

Proof. M is 1-essential, thus there exists $f : M \longrightarrow K(\Pi, 1)$, such that $f_*[M] \neq \{0\}$. For any positive δ, M bounds in its Fill Rad $(M) + \delta$-neighborhood in $L^\infty(M)$. Let $M = \partial W$, where W is a compact $(n+1)$-dimensional polyhedron in $N_{FillRad(M)+\delta}(M)$. We will try to extend a map to W in the following way: first, by extending f to 0-skeleton of W and then to 1-skeleton. Had we been able to extend this map to 2-skeleton we would have been able to extend it to the whole of W, but that shouldn't be possible, since M is 1-essential. (Here we somewhat oversimplify by assuming that W can be triangulated. In case it cannot, we can still approximate it by a simplicial complex.)

Extending to 0-skeleton: Subdivide W, so that all simplices σ have diameter $diam(\sigma) \leq \delta$ for some $\delta > 0$. Map vertices $w_i \in W$ to vertices of triangulation $m_i \in M$ for which $d(w_i, m_i) \leq d(w_i, M) + \delta$. Suppose m_i, m_j come from vertices w_i, w_j of some simplex in W. Then $d(m_i, m_j) \leq d(m_i, w_i) + d(w_i, w_j) + d(w_j, m_j) \leq 2\text{Fill Rad } M + 5\delta$. Thus, m_i, m_j can be joined by a geodesic of length less then or equal to $2\text{Fill Rad } M + 5\delta$.

Extending to 1-skeleton: Now send the 1-simplices $[w_i, w_j] \subset W \setminus M$ to the above geodesics joining m_i and m_j. (In addition, we can assume all 1-simplices in M to be already short). We can see that the boundary of each 2-simplex in W is sent to a curve of length $\leq 6\text{Fill Rad } M + 15\delta$ in M (and then to $K(\Pi, 1)$ by f). As it was stated before, it is impossible to extend f to the 2-skeleton of W. Thus, one of those curves must be homotopically nontrivial, and of length $\leq 6\text{Fill Rad } M + 15\delta$, which shows that there is a closed geodesic of length that is less then 6 Fill Rad $M + 15\delta$ and letting δ go to zero we obtain a closed geodesic of length that is less than 6 Fill Rad M. □

1.4 The length of the shortest closed geodesic on simply connected manifolds

As we have said in the introduction the proof of the existence of a closed geodesic on compact simply connected manifold was given by Lyusternik and Fet. Their proof uses Morse theory on the space of closed curves, (see [1, 17] for more details).

Theorem 1.4.1. *Let M be compact simply connected Riemannian manifold. Then there exists at least one closed geodesic on M.*

In order to briefly state the proof of this theorem we will need several facts:

1. Let $\Lambda M = \mathrm{Map}(S^1, M)$ denote the space of continuous maps from S^1 to M and let ΩM be the space of fixed point loops. Then

$$\pi_q(\Lambda M) = \pi_q(M) \oplus \pi_q(\Omega M);$$

   and

$$\pi_{q+1}(M) \simeq \pi_q(\Omega M).$$

   Therefore, there exists $i > 0$ such that $\pi_i(\Lambda M) \neq \{0\}$.

2. Given $c > 0$ let $\Lambda^c M$ denote the closed subset $E^{-1}([0, c])$, where E is the energy defined on piecewise differentiable curves.

Let $P_N M$ be the set of all geodesic polygons consisting of N segments. For any fixed m there exists N_m such that $\pi_k(P_N M) = \pi_k(\Lambda^c M)$ for all $k \leq m, N \geq N_m$.

3. Let us also recall that closed geodesics are critical points of the energy function (or equivalently of the length function) on $P_N M$.

Proof. Consider the smallest i such that $\pi_i(\Lambda M) \neq \{0\}$, (or, equivalently $\pi_{i+1}(M) \neq 0$ and $\pi_i(M) = 0$) and $\pi_i(M) = 0$. Let $\nu \in \pi_i(\Lambda M)$ and $\nu \neq 0$. It is easy to see that ν can be represented by a continuous map of S^i into the space $\Lambda^* M$ made of piecewise differentiable closed curves. Let $c = \sup_{x \in S^i}(E(\nu(x)))$. Consider Λ^{2c}. For any fixed m there exists N_m such that for all $k \leq i, N \geq N_m, \pi_k(P_N^{2c} M) = \pi_k(\Lambda^{2c} M)$. Moreover, ν can be deformed into $P_N^{2c} M$ without increase of energy in the process of homotopy. Thus, $\pi_i(P_N^{2c} M) \neq 0$.

Suppose $P_N^{2c} M$ has no closed geodesics. Then the energy function E on $P_N^{2c} M$ has no critical points on $P_N^{2c} M$ other than constant paths. Let us define a vector field X on $P_N^{2c} M$ by the formula $- < X, Y >= dE(Y)$. The vector field X does not vanish on $P_N^{2c} M \setminus M$. Therefore we can deform $P_N^{2c} M$ into the tubular neighborhood of M, which can be retracted to M. But that would mean that $\pi_i(P_N^{2c} M) = \pi_i(M)$, which is a contradiction, since $\pi_i(M) = 0$. □

Suppose now, that we want to estimate the length of the closed geodesic on compact simply connected manifold. In light of the above proof, we can see that if we would actually construct a nontrivial element $h(q,t) \in \pi_i(\Lambda M)$ such that the length of each curve $h(q,*)$ is bounded from above by some constant, that would imply the existence of the closed geodesic, which length is going to be bounded by the same constant. That is exactly what Croke has done in [8] and what we are going to do later on.

Theorem 1.4.2. *(Croke) For any metric on S^2 the length of the shortest closed geodesic $l(S^2)$ satisfies the following two inequalities:*

1. $l(S^2) \leq 9D$
2. $l(S^2) \leq 31\sqrt{A}$, *where D is the diameter and A is the area of a manifold.*

The proof of (1) uses two lemmas (Lemma 1.4.5 and Berger's lemma, which we will not prove here). But first, we need the following definition:

Definition 1.4.3. (Convexity to a Domain) Let γ be a simple (no self intersections) closed curve on a surface M, which divides M into two components. Let Δ be one of those components. Then we will say that γ is convex to Δ if there is an $\epsilon > 0$ such that for all $x, y \in \gamma$, with $d(x,y) < \epsilon$ the minimizing geodesic τ from x to y satisfies $\tau \in \overline{\Delta}$.

Lemma 1.4.4. *(Berger) Let M be a compact Riemannian manifold and $x, y \in M$ be such that $d(x,y) = D$, where D is a diameter of M. Then for all $V \in T_xM$ there exists a minimizing geodesic γ from $x = \gamma(0)$ to y with $\langle\gamma'(0), V\rangle \geq 0$, (cf. [15], Lemma 1.1).*

Lemma 1.4.5. *(Croke) Let α_1 and α_2 be two piecewise smooth curves from x to y such that $\alpha_1 \cup -\alpha_2$ forms a simple closed curve which is convex to an open disc Δ. Assume that for every $z \in \Delta$, $d_{\overline{\Delta}}(x,z) \leq D$, where $d_{\overline{\Delta}}$ represents distance as measured in Δ, and D is some real number. Then either there is a nontrivial closed geodesic lying in $\overline{\Delta}$ of length less than or equal to $L = l(\alpha_1 \cup -\alpha_2)$ or α_1 is homotopic to α_2 through curves from x to y lying in $\overline{\Delta}$ of length less than or equal to $3L + 2D$.*

Proof of Theorem 1.4.2(1). Let us choose two points $x, y \in M$ that are within distance D from each other, where D is the diameter of a manifold and apply Berger's lemma. Then for every $V \in T_xM$ there exists a minimal geodesic σ that connects x and y, such that $\langle V, \sigma'(0)\rangle \geq 0$. Let $U = \{\sigma_i\}_{i=1}^n$ be a finite set of minimal geodesics from x to y satisfying the same property. We can assume that $n > 2$. (Since if n=2 then $\sigma_1 \cup -\sigma_2$ is a closed geodesic of length $2D$.) Let us order this set, so that σ_{i+1} is a curve immediately to the right of σ_i. Note that $\sigma_i \cap \sigma_j = \{x, y\}$ for $i \neq j$, and that we can order $\sigma_i's$ so that $\angle(\sigma_i'(0), \sigma_{i+1}'(0)) \leq \pi$ and similarly $\angle(-\sigma_i'(D), -\sigma_{i+1}'(D)) \leq \pi$. (That is $\sigma_i \cup -\sigma_{i+1}$ is a simple closed curve which is convex to the domain Δ_i lying between them.)

We can now apply the above Lemma to curves $\sigma_i \cup -\sigma_{i+1}$. We see that either there is a closed geodesic of length $\leq 2D$ or $-\sigma_i$ is homotopic to $-\sigma_{i+1}$ through curves of length $\leq 8D$ lying in $\overline{\Delta_i}$. We now describe a homotopy from the point curve $\{x\}$ to the point curve $\{y\}$:

$$\{x\} \sim (\sigma_1 \cup -\sigma_1) \sim (\sigma_1 \cup -\sigma_2) \sim ... \sim (\sigma_1 \cup -\sigma_n) \sim (\sigma_1 \cup -\sigma_1) \sim \{x\},$$

where the homotopy from $-\sigma_i$ to $-\sigma_{i+1}$ is through the curves in $\overline{\Delta_i}$. We have, thus, constructed a closed curve in the space of closed curves, such that the length of each of those cureves is bounded from above by $9D$. It is not difficult to see that this curve is not contractible. Therefore, the length of the shortest closed geodesic is $\leq 9D$. $\square$

The proof of (2) follows from (1) and the following corollary to the coarea formula:

$$\int_a^b L(S(x,t))dt \leq A,$$

where $S(x,t)$ are circles of radius t centered at x.

Under certain assumptions this formula will allow one to partition a sphere into 3 or 4 domains, such that the length of the boundary of each domain is $\leq const.\sqrt{A}$, where *const.* is an absolute constant. We then apply Birkhoff curve shortening process to the boundary of each domain. If each of the boundary converges to a point, we can then construct a homotopy in the same style as a homotopy in Lemma 1.4.5, which will be a nontrivial closed curve in the space of closed curves. (See [8] for details.)

1.5 Diameter, curvature and critical points of the distance function

Now we would like to discuss the results that are seemingly unrelated to closed geodesics. They are mostly based on the paper by Gromov [11] in which he gives an estimate of the sum of Betti numbers in terms of the diameter and the curvature of a manifold obtaining the following inequality that holds for a manifold M with curvature $K \geq -H, H > 0$, $\sum b^i(M^n) \leq c(n)^{\sqrt{H}d+1}$.

One of the elements of Gromov's proof is the theory of critical points of distance function.

Grove and Shiohama observed in [15] that it is possible to define critical point of a distance function $\rho_p(x) = d(x,p)$ on a complete Riemannian manifold M^n, so that the following Isotopy Lemma holds (see [6] for more details).

Definition 1.5.1. (Critical Points of Distance Function) The point y will be called a critical point of ρ_p if for all $V \in T_pM$ there is a minimal geodesic γ from p to y, such that the angle $\angle(V, \gamma'(0)) \leq \frac{\pi}{2}$.

Lemma 1.5.2. *(Isotopy Lemma) Let* $r_1 < r_2 \leq \infty$, *and let* $B_{r_i}(p) = \{x/d(x,p) < r_i\}$ *be a metric ball on a complete Riemannian manifold* M^n. *Suppose that* $\overline{B_{r_2}(p)}/B_{r_1}(p)$ *is free of critical points of* ρ_p, *then*

1. *this region is homeomorphic to* $\partial B_{r_1}(p) \times [r_1, r_2]$.

2. $\partial B_{r_1}(p)$ *is a topological submanifold.*

Proof. If x is noncritical with respect to p, then there exists $V \in T_xM$ such that $\angle(V, \gamma'(0)) < \frac{\pi}{2}$ for all minimal geodesics γ from x to p. V can be extended to a vector field V_x on a neighborhood U_x of x, satisfying the same condition, i.e. $\angle(\gamma'(0), V_x(y)) < \frac{\pi}{2}$, where $y \in U_x$ and γ is a minimal geodesic from p to y. Take a finite (or locally finite, in case $r_2 = \infty$) open cover $\{U_x\}$ of $\overline{B_{r_2}(p)}/B_{r_1}(p)$ and consider a smooth partition of unity $\{f_i\}$, subordinate to that cover. Let $V = \sum f_i V_{x_i}$. Normalize V, so that $|V| = 1$. By the first variation formula, the integral curve Ψ of V satisfies the following:

$$\rho_p(\Psi(t_2)) - \rho_p(\Psi(t_1)) \leq (t_1 - t_2)\cos(\frac{\pi}{2} - \epsilon)$$

for small ϵ. The first statement follows.

Now let $q \in \partial B_{r_1}(p)$, and let σ be a minimal geodesic from q to p and let S be a piece of the totally geodesic hypersurface at q, normal to σ. Then for $z \in S$ sufficiently close to q, each integral curve through z intersects $\partial B_{r_1}(p)$ in exactly one point, $z' \in \partial B_{r_1}(p)$. Thus, the map $z \longmapsto z'$ gives us a local chart for $\partial B_{r_1}(p)$ at q. □

The proof of Gromov is based on the following ideas.

1. First idea is that a certain sequence of critical points has a bounded number of elements. The upper bound is given in terms of the diameter and a lower bound on the curvature of the manifold.

2. Second idea is that to each metric ball one can assign a number, which we will call a rank of a ball. Correspoinding to each metric ball we can then construct a sequence of critical points, as in (1), such that the number of elements in the sequence equals to the rank of the ball, and

thus obtain and inequality that bounds the rank of the whole manifold regarded as a metric ball in terms of its curvature and diameter.

3. One can then show that the sum of Betti numbers is bounded above by the rank of a manifold. The way it is done is the following. Let $b^i(A,B)$ denote a rank of the map $f^* : H^i(B) \longrightarrow H^i(A)$, which is the map induced by $f : A \longrightarrow B$. Then one introduces the notion of the content of a ball, $B_r(p)$ denoted $cont(r,p)$, which is equal to $\sum b^i(r,p)$, where $b^i(r,p)$ is defined to be $b^i(B_r(p), B_{5r}(p))$. Note that if $r > diam(M)$ then $b^i(r,p) = b^i(M)$, thus, in this case, $cont(r,p) = \sum b^i(M)$. Therefore, the sum of Betti numbers can be estimated from above by estimating the content of a ball of radius greater than diameter of the manifold. The generalized Mayer-Vietoris sequence allows one to estimate the content of a ball in terms of contents of smaler balls, which in turn allows one to establish connection between the content of a ball and its rank.

Now we would like to describe some details of Gromov's proof relevant to our work.

Definition 1.5.3. (Compression) We will say that a metric ball $B_r(p)$ compresses to a metric ball $B_s(q)$ and write $B_r(p) \longmapsto B_s(q)$ if the following conditions are satisfied:

1. $5s + d(p,q) \leq 5r$.

2. There exists a homotopy $F_\tau : B_r(p) \longrightarrow B_s(q)$ with F_0 being the identity and $F_1 : B_r(p) \subset B_s(q)$.

Definition 1.5.4. (Rank) Rank(r,p):=0, if $B_r(p) \longmapsto B_s(q)$, with $B_s(q)$ contractible.

Rank (r,p):=j, if rank(r,p) is not $\leq j-1$ and if $B_r(p) \longmapsto B_s(q)$, such that for all $q' \in B_s(q)$ and $s' \leq s/10$, we have $rank(s', q') \leq j-1$.

Informally speaking, rank of a ball can be described as the number of steps it takes to get from this ball to the contractible ball, where each step consists of moving a ball of a radius r to a ball of a smaller radius s and then subdividing the ball into balls of radius $s/10$ as described in the definition.

Definition 1.5.5. (Incompressible Ball) A ball, $B_r(p)$, will be called incompressible if $B_r(p) \longmapsto B_s(q)$ implies $s \geq r/2$.

We remark that any ball can be compressed either to a contractible ball or to an incompressible ball.

Definition 1.5.6. (Modified Rank)

1. $rank'(r,p) := 0$ if $B_r(p) \longmapsto B_s(q)$, with $B_s(q)$ contractible.

2. $rank'(r,p) := j$ if $rank'(r,p) \neq j-1$ and $B_r(p) \longmapsto B_s(q)$ such that $B_s(q)$ is incompressible and for all $q' \in B_s(q)$ and $s' \leq s/10$, we have $rank(q', s') \leq j-1$.

Note that rank(r,p) $\leq rank'(r,p)$.

Lemma 1.5.7. *Let q_1 be critical with respect to p and q_2 satisfy $d(p, q_2) \geq \nu d(p, q_1)$ for some $\nu > 1$. Let γ_1, γ_2, be minimal geodesics from p to q_1, q_2 respectively and let $\theta = \angle(\gamma_1'(0), \gamma_2'(0))$. Then*

1. *If $K_M \geq 0, \theta \geq \cos^{-1}(\frac{1}{\nu})$.*

2. *If $K_M \geq H, (H < 0)$ and $d(p, q_2) \leq d$, then*

$$\theta \geq \cos^{-1}(\frac{\tanh(\frac{\sqrt{H}d}{\nu})}{\tanh(\sqrt{H}d)})$$

Proof. The proof is an application of Toponogov's theorem together with the law of cosines in the first case and hyperbolic law of cosines in the second case. (See [6, 11] for more details.) □

The Corollary to this Lemma will allow us to estimate the number of critical points to p in the sequence $q_1, q_2, ..., q_N$, where $d(p, q_{i+1}) \geq \nu d(p, q_i)$, $\nu > 1$.

Corollary 1.5.8. *Let $q_1, ...q_N$ be a sequence of critical points of p, with $d(p, q_{i+1}) \geq \nu d(p, q_i)$. Then $N \leq \frac{\int_0^\pi (\sin s)^{n-2} ds}{\int_0^{\theta/2} (\sin s)^{n-2} ds}$, where θ is a function of the dimension of a manifold and ν, when curvature of the manifold is bounded from below by 0, and θ is a function of the dimension of a manifold, ν and the diameter when curvature is bounded from below by -1.*

Proof. Consider minimal geodesics γ_i that connect p and q_i. Consider also the set of the unit tangent vectors $\{\gamma_i'(0)\}$ that can be viewed as a subset of the unit sphere in the tangent space of a manifold M at p. Let θ_i be the angle between p and q_i. Then the balls of radius $\theta_i/2$ about the $\gamma_i'(0)$ are mutually disjoint. Thus, the number of points in the sequence is $\leq \frac{volS^{n-1}}{\min volB(p,\theta_i/2)}$, where $B(p, \theta_i/2)$ denote balls in S^{n-1}. Take $\theta = \min \theta_i$ then the result will follow. □

We will also need the following lemma that will allow us to establish a connection between rank of a metric ball and critical points.

Lemma 1.5.9. *Let M^n be a complete Riemannian manifold, and let $B_r(p) \subset M^n$ be a metric ball on M^n. Assume $5s + d(p, y) \leq 5r$; $d(p, y) \leq 2r$. Then if $B_r(p)$ does not compress to $B_s(y)$, there exists a critical point, x, of y, such that $s \leq d(x, y) \leq r + d(p, y)$.*

Proof. The proof is an application of the Isotopy Lemma. □

We will now give an informal proof of the key Proposition.

Proposition 1.5.10. *Let M^n be a Riemannian manifold with diameter d and curvature $K \geq H, H \leq 0$ then for any $p \in M$ and any r $rank(r,p) \leq C$, where C is a function of a dimension of manifold if $H = 0$, and C is a function of the dimension of a manifold, diameter and the lower bound H on curvature, if H is negative.*

Proof. We will show that $rank'(r, p)$ is bounded by C, and since $rank(r, p) \leq rank'(r, p)$ the result will follow. Given a ball $B_r(p)$ of $rank'(r, p) = j$ we construct a sequence $\{x_1, .., x_j\}$ of points critical to a point y using the following strategy:

1. First, we construct a sequence of incompressible balls $B_{r_i}(p_i)$ with decreasing radii. Without loss of generality we assume that $B_r(p)$ to be incompressible and let $B_{r_j}(p_j) = B_r(p)$. By definition of $rank'(r,p)$ there exists a point $\tilde{p}_{j-1} \in B_r(p)$ and $\tilde{r}_{j-1} \leq r_j/10$ such that $rank'(\tilde{r}_{j-1}, \tilde{p}_{j-1}) = j-1$. That ball compresses to an incompressible ball $B_{r_{j-1}}(p_{j-1})$ of $rank'(r_{j-1}, p_{j-1}) = j-1$. If we continue in the above fasion we obtain our sequence.

2. We let $y = p_0$. Since $B_{r_i}(p_i)$ is incompressible, in particular, it does not compress to $B_{r_i/2}(y)$. One can also check that $B_{r_i}(p_i)$ and $B_{r_i/2}(y)$ satisfy the conditions of Lemma 1.5.9 Thus, there exists a critical point x_i as in Lemma 1.5.9

3. One can check that a sequence of points we obtain this way satisfies the condition $d(y, x_{i+1}) \geq \nu d(y, x_i)$ with $\nu = 5/4$.

4. We then apply Corollary 1.5.8 that gives us a bound on the number of elements in the sequence, which is exactly the same as $rank'(r,p) = j$.

□

Chapter 2

New Bounds for the Length of the Shortest Closed Geodesic on a Compact Simply Connected Manifold

2.1 Introduction

In the thesis we will prove two theorems relating the length of the shortest closed geodesic on simply connected Riemannian manifold either to the diameter or to the volume of the manifold. That work was motivated by the paper [12] of Gromov, in which he asks whether it is always possible to find a constant c(n), such that the length of the shortest closed geodesic is bounded from above by c(n)$vol(M)^{1/n}$,where n is the dimension of the manifold. (See also [3, 13] for the related topics.) Gromov himself had solved this problem for essential manifolds in [12]. (In particular, all compact surfaces with the exception of a sphere are essential, and so are all the manifolds that admit Riemannian metric of nonpositive sectional curvature.) Combined with the result of Croke [8] it finished the problem for the compact surfaces. The only known to us results for the simply connected manifolds of higher dimension are those of Ballman, Thorbergsson and Ziller [2], who, in particular, have investigated the case of spheres endowed with a $\frac{1}{4}$-pinched metric of positive sectional curvature, and the results of Croke [8] and Treibergs [18] for convex hypersurfaces.

In the thesis we will find two upper bounds on the length of the shortest closed geodesic on a simply connected Riemannian manifold with nontrivial second homology group. Our first estimate will be in terms of the lower bound on the sectional curvature and an upper bound on the diameter of the manifold.

Theorem 2.1.1. *Let Ψ be a class of simply connected compact Riemannian manifolds with non-trivial second homology group and sectional curvature $K \geq -1$, diameter $\leq D$. Then the length of the shortest closed geodesic $\gamma(t)$ on any manifold $M^n \in \Psi$ is bounded from above by*

$$f(D,n) = e^{e^{c(n)(D+1)}},$$

where $c(n) = 250(n+1)$.

Corollary 2.1.2. *Let Ψ' be a class of simply connected compact Riemannian manifolds with non-trivial second homology group and sectional curvature $K \geq 0$, diameter $\leq D$. Then the length of the shortest closed geodesic $\gamma(t)$ on any manifold $M^n \in \Psi'$ is bounded from above by $e^{e^{2c(n)}} D$.*

Proof. follows from the Theorem 2.1.1 by a rescaling argument. □

Our second result requires an additional assumption on the topology of the manifold. Namely, the manifolds in question should be 2-essential, some of the examples of which are manifolds that are homotopically equivalent to Kähler manifolds, in particular $\mathbf{C}P^n$.

Theorem 2.1.3. *Let Υ be a class of 2-essential simply connected compact Riemannian manifolds with sectional curvature $K \leq 1$ and volume $\leq V$. Then the length of the shortest closed geodesic $\gamma(t)$ on any manifold $M^n \in \Upsilon$ is bounded from above by*

$$g(V,n) = c_1(n) + (c_2(n)V)^{c_3(n)(V^{1/n}+1)},$$

where $c_1(n) = 10^3(n+1)n\sqrt[n]{n!}$, $c_2(n) = 10^3(n+1)n(n!)^2$, $c_3(n) = 10^5(n+1)n\sqrt[n]{n!}$.

Let us now recall a definition of 2-essential manifolds:

Definition 2.1.4. We will say that a compact and orientable manifold M of dimension n is 2-essential if there exists $f : M^n \to \mathbf{C}P^\infty$ such that $f_*([M]) \neq \{0\}$, where $[M]$ is the fundamental homology class of M.

Let ΛM denote a free loop space over M, and let $i > 0$ be such that $\pi_i(\Lambda M) \neq 0$, but $\pi_i(M) = 0$. Let us once again recall the result of Lyusternik and Fet, and the idea that in order to estimate the length of the closed geodesic on compact simply connected manifold it is enough to construct a nontrivial element $h(q,t) \in \pi_i(\Lambda M)$, such that the length of each curve $h(q,*)$ is bounded from above by some constant, which would imply the existence of the closed geodesic with length bounded from above by the same constant.

When manifolds has a nontrivial second homology group $\pi_2(M) \neq 0$ and $\pi_1(M) = 0$. It follows that

$$\pi_1(\Lambda M) = \pi_1(M) \oplus \pi_1(\Omega M) \simeq \pi_1(\Omega M) \simeq \pi_2(M) \neq 0.$$

Therefore, the problem can be reduced to constructing a nontrivial element $H_\tau(t)$ of $\pi_1(\Lambda M)$, such that for all τ length $H_\tau \leq$ f(D,n) in the first case or g(V,n) in the second case. We will be using the notions of the width of the homotopy and homotopy distance introduced in [19] as follows:

Definition 2.1.5. (Width of the Homotopy) Let $F_\tau(t)$ be a homotopy that connects two closed curves parametrized by $t \in [0,1]$ on a Riemannian manifold M. We say that W_F is the width of the homotopy $F_\tau(t)$ if $W_F = \max_{t\in[0,1]}$ length of the curve $F_\tau(t)$. That is W_F is the maximal length of the trajectory described by a point of one of the original closed curves during the homotopy. More generally if X, Y are metric spaces and $F : X \times [0,1] \longrightarrow Y$ is a homotopy then W_F is defined as $\sup_{x\in X}$length $F(x,*)$.

Definition 2.1.6. (Homotopy Distance) Let $\alpha_1(t), \alpha_2(t)$ be two curves then homotopy distance $d_H(\alpha_1, \alpha_2)$=$\inf_H W_H$, where H is any homotopy between α_1 and α_2.

The constructions of $H_\tau(t)$ in proofs of Theorems 2.1.1 and 2.1.3 are somewhat different. We will summarize the proof of Theorem 2.1.1 and then indicate the points where the proof of Theorem 2.1.3 will deviate from the proof of Theorem 2.1.1.

There are several essential ideas in this proof that we want to emphasize.

1. In order to construct a nontrivial element $H_\tau(t)$ of $\pi_1(\Lambda M)$ such that length $H_\tau \leq f(D, n)$ we will have to learn how to construct a homotopy of any closed curve $\gamma(t)$ of length $\leq 3d$ to a point, and that homotopy has to have some special properties. What we have in mind is the following: there are two parameters of the homotopy that we need to control at the same time, i.e. the length of curves in the homotopy and homotopy width (by controling we will mean providing an upper bound). We will, actually be satisfied if we have only "partial" control of the width, that is at least two selected points on $\gamma(t)$ do not "travel for too long" until they reach p, i.e. $H_\tau(t_i)$ is bounded, where $i = 1, 2$. The attempt at using Birkhoff curve shortening process fails for the reason that even though we have absolute control of the length of the curves in the homotopy, each point on $\gamma(t)$ can travel a long distance till it gets to p.

2. Such a homotopy as in the previous paragraph can be constructed by first producing a different homotopy from which we demand only that its width should be bounded. The assumption that $l(M) > 3d$ will be used, where $l(M)$ denotes the length of the shortest closed geodesic. We can then construct a new homotopy based on the previous one that will satisfy the necessary conditions, (see Figure 11.)

3. In order to accomplish Step 2 we will need to use notions very similar to those used by Gromov in [11], i.e. rank and compressibility, that will be substituted by effective rank and effective compressibility. These notions will be defined in section 2.3. We will be able to show that effective rank is bounded through the curvature and diameter of the manifold.

4. The result of the various estimates will be that (assuming $l(M) > 3d$) for every curve $\gamma(t)$ of length $\leq 3d$ there exists a homotopy of this curve to a point, such that the length of curves in the homotopy is bounded by $e^{e^{200(n+1)(d+1)}}$, and that we can insure that for some two points on the curve, the distance they travel is bounded by the same function. At that point we will be able to construct a nontrivial 2-cycle with some special properties. The argument that we will use in order to come up with it is the following: we will triangulate M, so that the diameter of each simplex is less than injectivity radius of M, we will then consider $H_2(M)$ and select cycle σ composed of simplices of the triangulation

and representing a nontrivial element in $H_2(M)$, next we will attempt to "fill" σ using the following procedure. First we will pick any point p in M. Then we will join the point p with all the vertices of σ by minimal geodesics. After that we will consider all the closed curves composed of two geodesic segments that join p and vertices v_1 and v_2 of σ and an edge of σ that joins v_1 and v_2. We will then use the "nice" homotopy to connect those closed curves with some points. Thus, we will obtain some cycles of m of a specific shape, at least one of which should be nontrivial, in order for σ not to bound. Finally, we will construct a nontrivial element of $\pi_1(\Lambda M)$ with the desired properties.

Let us now somewhat extrapolate on Step 3 since most of the thesis will be dedicated to it. That work was mainly inspired by [11]. We will establish the connection between effective rank of the ball and the homotopy distance between any closed curve inside that ball and some point. That is, we will find an upper bound on the homotopy distance between a curve and a point in terms of the effective rank. The proof of this uses the induction procedure on the effective rank of a ball containing the curve, that we will denote as $rank'_a$. For the curve $\gamma(t)$ that lies inside the ball, which radius is less than an injectivity radius there exists an obvious homotopy of bounded width to the center of the ball. It is only slightly harder to construct a homotopy of bounded width for the curve inside the ball of $rank'_a = 0$, (but perhaps, with radius greater then the injectivity radius.) The above will be the base of induction. Now let us roughly describe how we can construct a homotopy of bounded width for the curve that lies inside the ball $B_r(p)$ of $rank'_a = 1$, since that will make the induction step clear.

We will begin by showing that there exists a finite sequence of closed piecewise geodesic curves that starts at our given curve and ends with some constant curve, such that two consecutive curves in the sequence are sufficiently close to each other. We will then construct a homotopy between two neighboring curves by reducing the problem to finding a homotopy of a closed curve that lies totally inside the ball $B_{r/10}(p')$, which is a subset of our original ball. That will be accomplished by first "bringing" the curve to a ball of $rank'_a = 0$ and then by homotoping it to the point, (see Figure 1 below, and also Figures 3-6.)

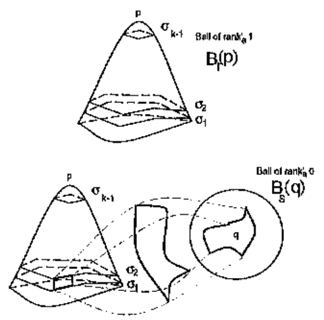

Figure 2.1: A homotopy between two neighboring curves

Note that the procedure that we develop in Steps 2 and 3 in order to construct a homotopy of width bounded in terms of the lower bound on the sectional curvature and the diameter can also be used to prove the following theorem.

Theorem 2.1.7. *Let M^n be compact Riemannian manifold with the sectional curvature $K \geq -1$ and diameter $\leq D$. Suppose also that for any closed curve $\omega(t)$ there exists a homotopy $H_\tau(t)$ of this curve to a point such that the length of the curves $H_{\tau_*}(t)$ in the homotopy is bounded from above by $c \cdot length(\omega(t))$ for all τ_*. Then there exists a homotopy $\Xi_\tau(t)$ of $\omega(t)$ to a point such that*

$$W_\Xi \leq e^{e^{\xi \cdot (n+1)(D+1)c}},$$

where ξ is an absolute constant that can be explicitly calculated.

To prove Theorem 2.1.3 we will first assume that injectivity radius of M is not too small, otherwise Klingenberg's lemma together with Berger injectivity radius estimate will give an estimate on the length of the shortest closed geodesic and we would be done. But now, having the lower bound on the injectivity radius it will be much simpler to construct the required homotopy with special properties that we discussed above. We can then use some obstruction theory used by Gromov in [12] to construct a homotopy nontrivial map of the boundary of 3-simplex into M with the following special properties: denote the images of the vertices a, b, c, d, images of 1-simplices will be geodesics joining a, b, c and d, and its faces will be formed by surfaces generated by the homotopies with the "nice" properties described in Step 1 and proceed as in the first case.

Sections 2.2–2.5 will be dedicated to estimating the homotopy distance between closed p.w.g. curve and a point for class Ψ. In section 2.2 we will define the notions of a-effective compressibility and a-effective rank. In section 2.3 we will prove the lemma that will be essential in estimating a-effective rank. It will establish that the number of elements in the sequence of ϵ-almost critical points is finite for some ϵ under some conditions. In section 2.4 we will establish the connection between the effective rank and the homotopy distance, and in section 2.5 we will show that $rank'_a$ is bounded from above by $e^{3(n+1)(d+1)}$ for some a. Combining results of sections 2.4 and 2.5 we will obtain Theorem 2.1.7.

In section 2.6 we will construct the required homotopy for any $M \in \Psi$. We will deal with class Υ in section 2.7. Finally in section 2.8 we will prove Theorem 2.1.1 and use some ideas from the paper [12] of Gromov to finish proof of Theorem 2.1.3. In sections 2.2, 2.3, and 2.5 we will closely follow the proof of the main theorem from [11] of Gromov as it was done in [6].

2.2 Basic Definitions

Definition 2.2.1. (a-Effective Compressibility) Let a be a positive number. We will say that $B_r(p)$ a-effectively compresses to $B_s(q)$ and write $B_r(p) \longmapsto_a B_s(q)$ if the following conditions are satisfied:

1. 5s+d(p,q)$\leq$5r.
2. There exists a homotopy $F_\tau : B_r(p) \to B_s(q)$ with F_0 being the identity and $F_1 : B_r(p) \subset B_s(q)$.
3. $W_F \leq ar$.

Definition 2.2.2. (a-Effective Rank)

1. $rank_a(r,p)$:=0, if $B_r(p) \longmapsto_a B_s(q)$ with $B_s(q)$ a -effectively contractible.

2. $rank_a$ (r,p):=j if $rank_a$ (r,p) is not $\leq$ j-1 and if $B_r(p) \longmapsto_a B_s(q)$ such that for all $q' \in B_s(q)$ with $s' \leq$ s/10, we have $rank_a(s', q')$ $\leq$j-1.

Definition 2.2.3. (a-Effectively Incompressible Ball) A ball $B_r(p)$ is called a-effectively incompressible if $B_r(p) \longmapsto_a B_s(q)$ implies that $s > r/2$.

Lemma 2.2.4. *Any ball $B_r(p)$ can be 3a-effectively compressed either to an a-effectively contractible ball or to a ball that is incompressible a-effectively.*

Proof. Suppose $B_r(p) \longmapsto_a B_{s_1}(q_1)$. Then there are three possibilities:

1. $B_{s_1}(q_1)$ is a-contractible.

2. $B_{s_1}(q_1)$ is incompressible a-effectively.

3. $B_{s_1}(q_1)$ is compressible a-effectively, but not a-effectively contractible.

In case of 1 or 2 we are done, since a-effective compressibility implies $3a$-effective compressibility. In the third case, $B_{s_1} \longmapsto_a B_{s_2}$ such that $s_2 \leq s_1/2$ by definition of a-effective compressibility. Once again we have three possibilities for $B_{s_2}(q_2)$. It can be either a-effectively contractible, a-effectively incompressible, or a-effectively compressible. Consider the last case and obtain $B_{s_2} \longmapsto_a B_{s_3}$ such that $s_3 \leq s_2/2$, and so on. The above process will have to terminate by our arriving either at a-effectively compressible ball, or the ball that is incompressible a-effectively. We, thus,obtain a sequence:$F^1, F^2, F^3, ..., F^n$ of homotopies such that $W_{F^1} \leq ar, W_{F^2} \leq as_1 \leq ar, W_{F^3} \leq as_2 \leq as_1/2 \leq ar/2, ..., W_{F^n} \leq (ar)/2^{(n-1)}$. Thus, we can get to a-effectively contractible or a-effectively incompressible ball applying one homotopy after the other and the width W_F of final homotopy will be $\leq ar + ar + ar/2 + \cdots + ar/2^{(n-1)} \leq 3ar$. □

2.3 Modified Gromov's Lemma

In this section we will prove a slightly generalized version of the well-known Gromov lemma about sequence of critical points, (see [6, 11] for the proof of the original lemma).

Definition 2.3.1. (ϵ-Almost Critical Point) We will say that a point q on a manifold M is ϵ-almost critical with respect to p, if for all vectors v in the tangent space M_q, there exists a minimal geodesic γ from q to p with the absolute value of the angle $\angle v, \gamma'(0) \leq \pi/2 + \epsilon$.

Lemma 2.3.2. *(Modified Gromov's Lemma) Let q_1 be ϵ-almost critical point with respect to p and let q_2 satisfy $d(p, q_2) \geq \nu d(p, q_1)$ for some $\nu > 1$. Let γ_1, γ_2 be minimal geodesics from p to q_1, q_2 respectively, and let θ be the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$. If sectional curvature K_M of the manifold M is bounded below by -1 and $d(p, q_2) \leq d$ then*

$$\cos\theta \leq \frac{\tanh\frac{d}{\nu}}{\tanh d}(\sin\epsilon + 1) + \sin\epsilon.$$

Proof. Let $a = d(p, q_1), b = d(q_1, q_2), c = d(p, q_2)$. Also let γ_3 be a minimal geodesic from q_1 to q_2. Since q_2 is ϵ-almost critical point to p, there exists a minimal geodesic $\sigma(t)$ from q_2 to p such that the angle $\angle\sigma'(0), \gamma_3'(0) \leq \pi/2+\epsilon$, (see Figure 2.) We will apply the Toponogov comparison theorem twice to hinges $\sigma(t), \gamma_3(t)$ and $\gamma_1(t), \gamma_2(t)$, which in combination with hyperbolic law of cosines will yield inequalities (1) and (2) respectively.

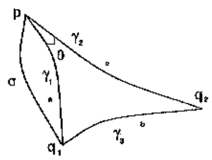

Figure 2.2: Two hinges

$$\cosh c \leq \cosh a \cosh b - \sinh a \sinh b \cos(\pi/2 + \epsilon) \tag{2.1}$$

$$\cosh b \leq \cosh a \cosh c - \sinh a \sinh c \cos\theta. \tag{2.2}$$

Let us substitute the inequality (2.2) into (2.12) to obtain:

$$\cosh c \leq \cosh a(\cosh a \cosh c - \sinh a \sinh c \cos\theta) + \sinh a \sinh b \sin\epsilon.$$

Now, let us use the triangle inequality to see that

$$\cosh c \leq \cosh^2 a \cosh c - \cosh a \sinh a \sinh c \cos\theta + \sinh a \sinh(a + c) \sin\epsilon.$$

Therefore, using hyperbolic functions identities we obtain:

$$0 \leq \sinh^2 a(\sin\epsilon + 1) - \cosh a \sinh a \tanh c(\cos\theta - \sin\epsilon);$$

$$\cosh a \tanh c(\cos\theta - \sin\epsilon) \leq \sinh a(\sin\epsilon + 1)$$

$$\cos\theta \leq \frac{\tanh a}{\tanh c}(\sin\epsilon + 1) + \sin\epsilon \leq \frac{\tanh(d/\nu)}{\tanh d}(\sin\epsilon + 1) + \sin\epsilon.$$

□

It is clear that unless the expression on the right is strictly less than 1, our lemma will not provide any additional information. Thus, we need to find such an ϵ that

$$\frac{\tanh(d/\nu)}{\tanh d}(\sin\epsilon + 1) + \sin\epsilon = x < 1.$$

We will use Lemma 2.3.2. in the situation when ν=5/4. In this case let

$$c_d = \frac{\tanh(4d/5)}{\tanh d},$$

and let

$$x = \frac{c_d + 1}{2}.$$

It is clear that both c_d and x are strictly less then 1. Take ϵ such that

$$\sin\epsilon = \frac{1 - c_d}{2(c_d + 1)}.$$

After some calculation we see that

$$\frac{1}{\sin\epsilon} = \frac{2(e^{9d/5} - e^{-9d/5})}{e^{d/5} - e^{-d/5}} \leq 18e^{8d/5}.$$

Lemma 2.3.2 implies that $\cos\theta \leq (c_d + 1)/2$. After some calculation we see

$$(c_d + 1)/2 \leq 1 - \frac{e^{-8d/5}}{2}.$$

and hence

$$\theta \geq e^{-4d/5}.$$

Corollary 2.3.3. *Let $q_1, q_2, ..., q_N$ be a sequence of ϵ-almost critical points of p, where* $\sin\epsilon$ *satisfies the above condition. Suppose also that $d(p, q_{i+1}) \geq (5/4)d(p, q_i)$. Then $N \leq (n-1)\pi^{n-1}e^{4(n-1)d/5}$.*

Proof. Consider minimal geodesics γ_i that join p and q_i. Next consider the set of the unit tangent vectors $\{\gamma_i'(0)\}$ that can be viewed as a subset of the unit sphere in the tangent space of M at p. Let θ_i be the angle between p, q_i. Then the balls of radius $\theta_i/2$ about the $\gamma_i'(0)$ are mutually disjoint. Thus the number of points in the sequence N $\leq \frac{volS^{n-1}}{\min volB(p,\theta_i/2)}$, where $B(p, \theta_i/2)$ denote balls in S^{n-1} and

$$\frac{volS^{n-1}}{\min volB(p, \theta_i/2)} = \frac{\int_0^\pi (\sin s)^{n-2} ds}{\int_0^{\theta/2} (\sin s)^{n-2} ds},$$

for $\theta = \min\theta_i$. Since $\sin s \geq 2s/\pi$, on the interval $(0, \pi/2)$ we estimate:

$$N \leq \frac{\pi^{n-1}(n-1)}{\theta^{n-1}}.$$

Now substitute the lower bound for θ and obtain the result. □

2.4 Homotopy Distance and a-Effective Rank of a Ball

Throughout this section we will be working with simply connected compact manifolds. In addition we will also assume that the manifold M doesn't have any geodesics of length $\leq 3d$, where d is the diameter of the manifold. In that case we will examine how homotopy distance between a curve $\alpha(t) \in B_r(p)$ of a-effective rank m and a center of a ball, depends on the rank m and a diameter d of a manifold M. (Note that the only reason we need an assumption $l(M) > 3d$ is to establish the connectedness of the space of all parametrized curves of length bounded from above by $3d$. Indeed, in that case any curve of length $\leq 3d$ can be contracted to a point without any length increase. However, we can weaken our hypothesis by assuming that any curve of length $L \leq 3d$ can be contracted to a point through the curves of length $\leq cL$ for some constant c in order to show that in that case a curve of any length can be contracted to a point in such a way that the width of the homotopy will be bounded in terms of the rank of the ball in which that curve lies and the constant c.)

Definition 2.4.1. (Modified a-Effective Rank)

1. $rank'_a(p,r) := 0$ if $B_r(p) \longmapsto_{3a} B_s(q)$ where $B_s(q)$ is a-contractible.

2. $rank'_a(p,r) := j$ if $rank'_a(p,r) \neq j-1$ and $B_r(p) \longmapsto_{3a} B_s(q)$ such that $B_s(q)$ is a-incompressible and for all $q' \in B_s(q)$ with $s' \leq s/10$, we have

$rank'_a(q', s') \leq$ j-1.

Lemma 2.4.2. *Let $\gamma(\tau) \in B_r(p)$ with $rank'_a(p,r) = 0$ be a closed curve. There exists a homotopy F_τ of $\gamma(t)$ to a point with $W_{F_\tau} \leq 4ar \leq 4ad$.*

Proof. Since $rank'_a(p,r) = 0$ there exists an a-effectively contractible $B_s(q)$ such that $B_r(p) \longmapsto_a B_s(q)$, implying the existence of a homotopy F^2_τ with $W_{F^2_\tau} \leq 3ar$, such that $F^2_0(\gamma(t)) = \gamma(t)$ and $F^2_1(\gamma(t)) \subset B_s(q)$. But a-effective contractibility of $B_s(q)$ implies the existence of a homotopy F^1_τ such that

1. $W_{F^1_\tau} \leq ar$;

2. $F^1_0(F^2_1(\gamma(t))) = F^2_1(\gamma(t))$;

3. $F^1_1(F^2_1(\gamma(t))) = q'$,

where $q' \in B_s(q)$.

Take the composition of the above homotopies and obtain F_τ with $W_{F_\tau} \leq 4ar$. □

We are now ready to show that for every closed curve inside the ball of a'-effective rank m the homotopy distance between the curve and any point on M is bounded by the function that depends exponentially on the a'-effective rank of the ball and the diameter of the manifold, where $a' = \frac{1}{\sin\epsilon}$, and where $\epsilon = \frac{1-c_d}{2(1+c_d)}$ as it was defined in section 2.2. (Thus, $\frac{1}{\sin\epsilon} \leq 18e^{8d/5}$). More precisely, we will show that there exists a homotopy F^m of that curve to a point, such that $W_{F^m} \leq e^{A(d+1)(m+1)}$, where $A = 2 \cdot 10^5(n+1)$, n is the dimension of the manifold.

Our proof will be by induction on the rank of the ball and will be done in five steps.

Let us first note that the above statement is true, when $m = 0$. Since for any closed curve inside that ball $B_r(p)$ of rank 0, there exists homotopy F^0, such that $W_{F^0} \leq 4a'r$,(by Lemma 2.4.2), thus, $W_{F^0} \leq \frac{4r}{\sin\epsilon} \leq e^{A(d+1)}$.

Let us assume now that the above statment is true for the curve lying in the ball of a'-effective rank m, that is there exists a point q_m and homotopy F^m, such that $W_{F^m} \leq f(m) = e^{A(d+1)(m+1)}$. We want to show that for any closed curve lying inside that ball of a'-effective rank $m+1$ there exists a homotopy F^{m+1} to a point q_{m+1}, such that $W_{F^{m+1}} \leq e^{A(d+1)(m+2)}$. The homotopy F^{m+1} will be a product of several homotopies. We will proceed as follows.

Step 1 Given α_1 we will show that there exists a homotopy that we will call h^1 that connects our curve α_1 with the curve α_2 inside a ball $B_s(q)$, such that the width of h^1 will be bounded by $3a'r(\leq 3a'd)$ and for every $q' \in B_s(q)$ and $s' \leq s/10$ the ball $B_{s'}(q')$ has rank m.

Step 2 We will use our induction assumption to show that for any two curves $\alpha_1(t), \alpha_2(t) \subset B_s(q)$ such that $d(\alpha_1(t), \alpha_2(t)) \leq s/20$ there exists a homotopy h^2 with the width $\leq 4f(m)+4s/10(\leq 4f(m)+4d/10.)$ (see Figures 3-6).

Step 3 Using our assumptions that there is no closed geodesic of length $\leq 3d$, (thus, no closed geodesic of length $\leq 3s$) we can show that any curve of length bounded from above by $3s$ in the ball $B_s(q)$ can be homotoped to a point q by a homotopy h^3 with the width $\leq (960^n e^{d(n-1)})^{1441} W_{h^2}$.

Step 4 For any curve α_2, regardless of its length, in a ball $B_s(q)$ there exists a homotopy h^4 to a point q such that the width of the homotopy is bounded by $2s + 4W_{h^3} (\leq 2d + 4W_{h^3})$.

Step 5 Take the composition of h^1 and h^4 to get the required homotopy and estimate its width.

We will now proceed with the proofs.

Step 1 Immediately follows from the definition of the a-effective rank of the ball.

Step 2 will be the result of

Lemma 2.4.3. *Let $\alpha_1(t), \alpha_2(t)$ be two closed curves in a ball $B_s(q)$ with the distance $d(\alpha_1(t), \alpha_2(t)) \leq s/20$ for all t. Suppose also that $B_s(q)$ has the property that for every q', s' all balls $B_{s'}(q') \subset B_s(q)$ have ranks $\leq m$. Then there exists a homotopy h^2 between those two curves with $W_{h^2} \leq 4f(m) + 4s/10$ $(\leq 4f(m) + 4d/10)$.*

Proof. Take $\alpha_1(t), \alpha_2(t)$, such that d $(\alpha_1(t), \alpha_2(t)) < s/20$. W.L.O.G. we can assume that α_1:[0,1] $\longrightarrow M$, α_2:[0,1] $\longrightarrow M$ are broken geodesics. We will partition the interval [0,1] into segments, such that each quadrangle with vertices $\alpha_1(t_i)$, $\alpha_1(t_{i+1})$, $\alpha_2(t_{i+1})$, $\alpha_2(t_i)$ and edges $\alpha_j|_{[t_i,t_{i+1}]}$,j=1,2, $\sigma_j(s)$, where $\sigma_j(s)$ is minimal geodesic joining $\alpha_1(t_i)$ and $\alpha_2(t_i)$, lies inside a metric ball of radius s/10. This can be done by requiring that the length of the curve $(\alpha_j|_{[i,i+1]}) \leq s/30$.

We will describe the homotopy, by providing the description of the images of the curve $\alpha_1(t)$ under the homotopy. Let $\alpha_1^i = \alpha_1|_{[t_i,t_{i+1}]}$ and $\alpha_2^i = \alpha_2|_{[t_i,t_{i+1}]}$. Then we claim:

1. $\alpha_1(t)$ is homotopic to the curve $\gamma_1 = \bigcup_{i=1} \alpha_1^i \cup \sigma_1 \cup -\sigma_1$, (see Figure 3.) Moreover, $W_{g^1} \leq s/10$, where g^1 is the homotopy between α_1 and γ_1.

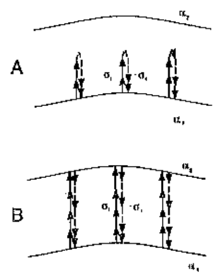

Figure 2.3: Homotopy of Lemma 2.4.3.

2. The curve $\gamma_1(t)$ will be homotopic to the curve $\gamma_2(t) = \bigcup_{i=1} \alpha_1^i \cup \sigma_i \cup -\alpha_2^i \cup \alpha_2^i \cup -\sigma_i$ with $W_{g^2} \leq (3s)/10$, (see Figure 4.)
3. The curve $\gamma_2(t)$ will be homotopic to the curve $\gamma_3(t)$, where $\gamma_3(t) = \bigcup_{i=1} \alpha_2^i \cup F_\tau^m(\alpha_2(t_{i+1}))|_{[0,\tau_*]} \cup F_\tau^m(t) \cup -F_\tau^m(\alpha_2(t_{i+1}))$ and $W_{g^3} \leq 2f(m)$, (see Figure 5.)

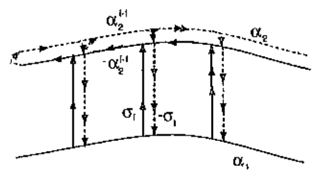

Figure 2.4: Homotopy of Lemma 2.4.3 (second stage).

4. The curve $\gamma_3(t)$ is homotopic to the curve $\gamma_4(t)$, where $\gamma_4(t) = \bigcup_{i=1} \alpha_2^i \cup F_\tau^m(\alpha_2(t_{i+1})) \cup -F_\tau^m(\alpha_2(t_{i+1}))$ and $W_{g^4} \leq 2f(m)$, (see Figure 6).

Finally, we observe that $\gamma_4(t)$ is homotopic to $\alpha_2(t)$ and notice that $W_{h^2} \leq 4f(m) + 4s/10$ as required. □

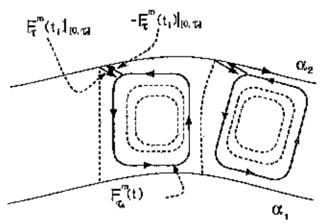

Figure 2.5: Homotopy of Lemma 2.4.3 (third stage).

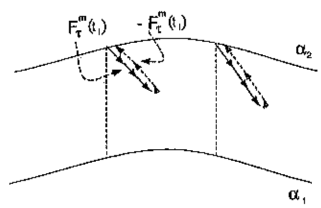

Figure 2.6: Homotopy of Lemma 2.4.3(final stage).

Step 3 will require proofs of several lemmas.

Lemma 2.4.4. *Let $\Phi(B_r(p))$ be the space of piecewise differentiable closed curves of length $\leq L$ parametrized proportionally to their arclength in $B_r(p)$, a ball of radius r in a manifold M and let N be the upper bound on the number of elements in some cover $\{B_{\epsilon/24}(p_i)\}$ of the ball $B_r(p)$ for some ϵ. There exists an $\epsilon/4$-net on the δ-neighborhood $N_\delta(\Phi(B_r(p)))$ with the number of elements*

$$Q \leq N^{24L/\epsilon+1},$$

where δ is some positive number and the length of every closed curve in this $\epsilon/4$-net is $\leq 3L$.

Proof. Given the cover $\{B_{\epsilon/24}(p_i)\}$ of $B_r(p)$ we will construct an $\epsilon/4$-net in the δ-neighborhood of $\Phi(B_r(p))$ as follows. We will consider the set I of $\gamma_i(t)$, where each $\gamma_i(t)$ will be a curve composed of the geodesic segments, that join points p_k and p_j if and only if $d(p_k, p_j) \leq \epsilon/8$ with the additional condition that length $\gamma_i \leq 3L$. The number of such curves will be $\leq N^{24L/\epsilon+1}$. If we will impose the additional condition of all curves being closed then our set J of such curves will be a subset of I and the number of elements of J will also be $\leq N^{24L/\epsilon+1}$. We claim that J is our $\epsilon/4$-net.

For let $\gamma(t) \subset B_r(p)$ be any curve parametrized proportionally to its arclength, in particular, $\gamma : [0, l] \to B_r(p), l \leq L$. Let us partition [0,l] into segments $[t_j, t_{j+1}]$ such that $t_{j+1} - t_j = \epsilon/24$. For each t_j we will select p_j such that $\gamma(t_j) \in B_{\epsilon/24}(p_j)$. Note that $d(p_{j+1}, p_j) \leq \epsilon/8$ by triangle inequality. We will then construct a curve $\sigma(t)$ by joining centers of the balls by minimal geodesics. Note: $\sigma(t)$ will not be parametrized proportionally by its arclength, but $d(\sigma(t), \gamma(t)) < \epsilon/8$. □

Lemma 2.4.5. *Let $B_r(p) \subset M^n$, where M is as in the Lemma 2.4.4 and the length of the shortest closed geodesic $l(M) > L$. Let $\alpha(t) \subset \Phi(B_r(p))$, where $B_r(p) \subset M$. There exists a finite sequence $\{\sigma_i\}_{i=1}^k$ of closed broken geodesics, such that $d(\sigma_1(t), \alpha(t)) \leq \epsilon/4$ for all t; $\sigma_k = p$; length(σ_i) $\leq 3L, d(\sigma_i, \sigma_{i+1}) < \epsilon$ and the number k of elements in this sequence is $\leq Q$, where Q is as in the Lemma 2.4.4.*

Proof. Let $\alpha(t)$ be any curve parametrized proportionally to its arclength of length $\leq L$. There exists a path P_τ in a space of all curves of length bounded from above by L and parametrized proportionally to the arclength connecting $\alpha(t)$ and a constant curve p. (Just use Birkhoff curve shortening process.) By the above Lemma we can construct an $\epsilon/4$-net in the δ-neighborhood of $\Phi(B_r(p))$, that is the space of curves of length $\leq L$ and parametrized proportionally to their arclength. Now consider the sequence: $P_1, ..., P_k$ such that $d(P_i, P_{i+1}) < \epsilon/4, P_1 = \alpha_1, P_k = p$. We know that for all p_i there exists $\sigma_i \in N_\delta(\Phi(B_r(p)))$ such that $d(P_i, \sigma_i) < \epsilon/4$. Thus we obtain sequence σ_i such that $\sigma_k = p$ and $d(\sigma_i, \sigma_{i+1}) \leq \epsilon$. But we still have to estimate the number of elements in that sequence. So far, the way our sequence was constructed it is possible that there exist i, j such that $i \neq j$, but $\sigma_i = \sigma_j$. To avoid that we can delete all the subsequences between all the repetitive elements from the sequence without changing the essential properties of the sequqnce, so our new sequqnce obtained in this way will be nonrepetitive and the number of elements will be less than or equal to Q. □

Let us now apply these Lemmas to the compact manifold M with the sectional curvature $K \geq -1$ and where $L = 3r$. We estimate the number of points in the $\epsilon/24$ net on $B_r(p)$ using Bishop-Gromov volume comparison theorem, (see [9]). Consider the maximal number of pairwise disjoint balls in $B_r(p)$ of radius $\epsilon/48$. Then the set of balls $\{B_{\epsilon/24}(p_i)\}_1^N$ will cover $B_r(p)$, where N can be estimated to be

$$N \leq \frac{vol(B_r(p))}{vol(B_{\epsilon/48}(p_i))} \leq \frac{vol(B'_r(p'))}{vol(B'_{\epsilon/48}(p'_i))},$$

where $B'_r(p'), B'_{\epsilon/48}(p'_i)$ are balls of radii $r, \epsilon/48$respectively on a manifold of constant curvature -1. Calculation shows that

$$N \leq \frac{\int_0^r \sinh^{n-1} t dt}{\int_0^{\epsilon/48} \sinh^{n-1} t dt}.$$

Since the balls $\{B_{\epsilon/6}(p_i)\}$ cover $B_r(p)$, the set $\{p_i\}$ will be $\epsilon/24$-net. Let $\epsilon = r/20$. Then $Q \leq (960e^{d(n-1)})^{1441}$.

We are now ready to complete Step 3. Let $\gamma(t) \in B_r(p)$ be a curve of length $\leq 3r$ and let $\epsilon = r/20$. There exists a sequence of p.w. geodesics parametrized proportionally to their arclength of length $\leq 3r$ and having properties 1-3. Thus, we can obtain the required homotopy by taking a composition of homotopies between the consecutive curves. It is easy to see that the upper bound on the width of the final homotopy is what has been required.

Step 4 will require the lemma below.

Lemma 2.4.6. *Let $\gamma(t) \in B_r(p)$ be a p.w. geodesic curve of any length in $B_r(p)$ parametrized proportionally to its arclength. Then there is a homotopy*

$H_\tau(t)$ of $\gamma(t)$ to a point such that $W_{H_\tau} \leq 2d + 4W_{h^3}$, where $h^3_\tau(t)$ is the homotopy connecting a curve of length $\leq 3r$ to a point, (see Step 3).

Proof. Let $\gamma(t)$ have length l. Partition the interval [0,l] into the subintervals $[t_i, t_{i+1}]$ such that $t_{i+1} - t_i \leq r$. Also let σ_i be minimal geodesic joining the points: $\gamma(t_i)$ and the center of the ball. Let $\gamma^i = \gamma|[t_i, t_{i+1}]$ then we claim that $\gamma(t)$ is homotopic to the curve $\gamma^1 = \bigcup \gamma^{i-1} \cup \sigma_i \cup -\sigma_i$ and $W_{h^1} \leq 2r(\leq 2d)$, where h^1 is the homotopy, (see Figures 7 and 8)

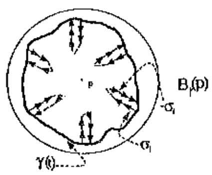

Figure 2.7: Homotopy of Lemma 2.4.6

Let T^i be a geodesic triangle with vertices at $\gamma(t_i), \gamma(t_{i+1}), p$ and edges: $\sigma_i, \gamma^i, \sigma_{i+1}$. By Step 3, for each T_i there exists a homotopy that conects it to some point p_i. Let us call this homotopy $H^{m+1,i}$. Consider a curve $H^{m+1,i}_\tau(p)$ joining p and p_i, and denote $H^{m+1,i}_\tau|_{[0,t_*]}$ by $H^{m+1,i}_*$. Then we claim that γ^1 is homotopic to $\gamma^2 = \bigcup H^{m+1,i}_*(p) \cup H^{m+1,i}_{\tau_*}(T_i) \cup -H^{m+1,i}_*(p)$ with the width of the homotopy bounded by $2d + 2W_{H^{m+1,i}_\tau}$, (see Figure 9.)

We can see that γ^2 is homotopic to $\gamma^3 = \bigcup H^i \cup -H^i$, which is homotopic to a point, (see Figure 10.)

□

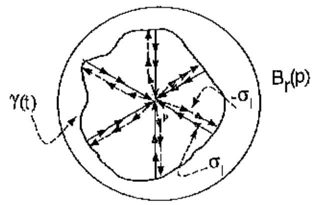

Figure 2.8: Homotopy of Lemma 2.4.6 (second stage).

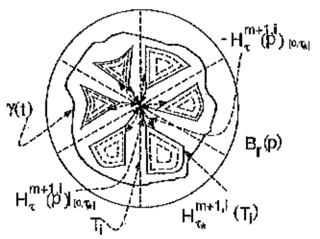

Figure 2.9: Homotopy of Lemma 2.4.6 (third stage).

Step 5 We combine the results of Steps 1-4 and get the desired result:

$$W_{F^{m+1}} \leq e^{A(d+1)(m+2)},$$

where $A = 2 \cdot 10^5(n+1)$.

As it was stated at the beginning of the section, we can now prove the next lemma by following Steps 1-5.

Lemma 2.4.7. *Let M^n be a compact manifold with the sectional curvature $K \geq -1$ and diameter d. Suppose that for any curve $\omega(t)$ there exists a homotopy connecting this curve with a point, such that the length of curves in the homotopy is bounded from above by length$(\omega) \cdot c$. Then there exists a (possibly different) homotopy connecting the above curve to a point, such*

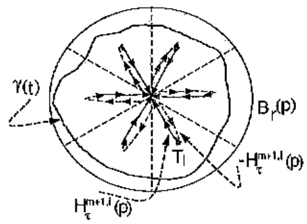

Figure 2.10: Homotopy of Lemma 2.4.6 (final stage).

that the width of this homotopy is bounded from above by

$$e^{\rho(c,n)(d+1)(m+2)},$$

where m is the modified a-effective rank of the ball in which the curve lies, and where $\rho(c,n) = const. \cdot (n+1) \cdot c$ and const. can be estimated.

2.5 An Upper Bound on the Effective Rank

In this section we will establish an upper bound for the $\frac{1}{\sin \epsilon}$-effective rank of the ball.

Once again the proof of the following proposition is a modification of the similar proof in [11]. We will need to use an effective version of the Isotopy Lemma [6] that was proved by Grove and Peterson in [14], (Theorem 1.6, page 199).

Lemma 2.5.1. *Let $B_{r_1}(p)$ and $B_{r_2}(p)$ be two metric balls on a manifold M with $r_2 < r_1$. Suppose that there is no ϵ-almost critical points to p on the complement of $B_{r_2}(p)$ in the $B_{r_1}(p)$. Then there exists a homotopy that* *deforms $B_{r_1}(p)$, so that it lies inside $B_{r_2}(p)$ and the width of this homotopy is bounded from above by $\frac{r_1}{\sin\epsilon}$.*

Proposition 2.5.2. *Let M be a compact manifold of diameter d such that its curvature $K \geq -1$. Then modified a'-effective rank of any ball in M, where $a' = 1/\sin\epsilon$, $\epsilon = \frac{1-c_d}{2(1+c_d)}$ as in section 2.2, will be bounded by $(n-1)\pi^{n-1}e^{4(n-1)d/5}$.*

In order to prove this proposition, we will first have to prove the following

Lemma 2.5.3. *Let $B_r(p)$ be a ball of radius r in a complete Riemannian manifold M. Assume $5s + d(p,y) \leq 5r; d(p,y) \leq 2r$. Then if $B_r(p)$ doesn't $\frac{1}{\sin\epsilon}$-effectively compress to $B_s(y)$ there exists an ϵ-almost critical point x of y with $s \leq d(x,y) \leq r + d(p,y)$.*

Proof. Let us assume that there are no ϵ-almost critical points in the complement of the $B_s(y)$ in $B_{d(p,y)+r}(y)$. Then the bigger ball can be $\frac{1}{\sin\epsilon}$-effectively deformed into a smaller one (this follows from the effective version of the Isotopy Lemma), but $B_r(p) \subset B_{d(p,y)+r}(y) \subset B_{5r}(p)$,which is a contradiction. Therefore, there exists and ϵ-almost critical point x in the complement of $B_s(y)$ in $B_{d(p,y)+r}(y)$. □

The next Lemma will be a slight modification of the Lemma in [11], where the term "$\frac{1}{\sin\epsilon}$-effectively" will be added in appropriate places (see also [6] for the proof of the original Lemma).

Lemma 2.5.4. *Let M^n be Riemannian manifold and let $rank'_{\frac{1}{\sin\epsilon}}(r,p) = j$. Then there exists $y \in B_{5r}(p)$ and $x_j, ...x_1 \in B_{5r}(p)$ such that for all $i \le j$, x_i is ϵ-almost critical with respect to y and $d(x_i, y) \ge (5/4)d(x_{i-1}, y)$.*

Proof. We begin by considering the metric ball $B_r(p)$ of $rank'_{\frac{1}{\sin\epsilon}}(r,p) = j$. By the definition of the $\frac{1}{\sin\epsilon}$-effective $rank'$ $B_r(p) \longmapsto_{\frac{3}{\sin\epsilon}} B_{r_j}(p_j)$, such that the following conditions are satisfied:

1. $B_{r_j}(p_j)$ is $\frac{1}{\sin\epsilon}$-effectively incompressible;
2. there exists $p'_{j-1} \in B_{r_j}(p_j)$ and $r'_{j-1} \le r_j/10$ such that

$$rank'_{\frac{1}{\sin\epsilon}}(r'_{j-1}, p'_{j-1}) = j - 1.$$

Similarly, the ball $B_{r'_{j-1}}(p'_{j-1}) \longmapsto_{\frac{3}{\sin\epsilon}} B_{r_{j-1}}(p_{j-1})$ such that

1. $B_{r_{j-1}}(p_{j-1})$ is $\frac{1}{\sin\epsilon}$-effectively incompressible;
2. there exists $p'_{j-2} \in B_{r_{j-1}}(p_{j-1})$ and $r'_{j-2} \le r_{j-1}/10$, such that

$$rank'_{\frac{1}{\sin\epsilon}}(r'_{j-2}, p'_{j-2}) = j - 2.$$

etc.

Note: $B_{3r_j/2}(p) \supset B_{5r_{j_1}}(p_{j_1})$.

By proceeding in the above fashion we obtain the sequence of balls $B_{r_i}(p_i), i = 0, 1, ..., j.$ such that for $1 \le i \le j B_{r_i}(p_i)$ is $\frac{1}{\sin\epsilon}$-incompressible and $B_{3r_i/2} \supset B_{5r_{i-1}}(p_{i-1}), r_{i-1} \le r_i/10$. Let $y = p_0$. Then $y \in B_{3r_i/2}(p_i)$ for all $1 \le i \le j$. In particular, $d(p_i, y) + 5r_i/2 \le 4r_i < 5r_i$ and $d(p_i, y) \le$

$3r_i/2 < 2r_i$. Since $B_{r_i}(p_i)$ is $\frac{1}{\sin\epsilon}$-incompressible it doesn't $\frac{1}{\sin\epsilon}$-effectively compress to $B_{r_i/2}(y)$. Therefore, by Lemma 2.5.1 there exists an ϵ-almost critical point x_i with

$$r_i/2 \leq d(x_i, y) \leq r_i + 2 \cdot (3/2)r_i = 4r_i.$$

Then $d(x_i, y) \geq r_i/2 \geq 5r_{i-1} \geq (5/4) \cdot 4r_{i-1} \geq 5d(x_{i-1}, y)/4$. □

Corollary 2.5.5. *$rank'_{\frac{1}{\sin\epsilon}}(r,p) \leq (n-1)\pi^{n-1}e^{4(n-1)d/5} \leq e^{3(d+1)(n-1)}$, which proves Proposition 2.5.2.*

Corollary 2.5.6. *Let M be compact manifold with $K \geq -1$ and $d \leq D$. Assume also that there is no closed geodesics of length $\leq 3d$ on M. Then for any closed curve $\alpha(t)$ there exists a point $p \in M$ and a homotopy H_τ contracting $\alpha(t)$ to the point such that*

$$W_{H_\tau} \leq e^{e^{B(d+1)}},$$

where $B = 100(n+1)$.

The slight change in the hypothesis will also lead to the

Proof of Theorem 2.1.7. Indeed, let us assume that for any closed curve $\omega(t) \subset M^n$ there exists a homotopy connecting this curve with a point, such that the length of curves in the homotopy is bounded from above by $length(\omega) \cdot c$ (instead of the assumption $l(M) > 3d$), where M^n is a compact manifold with $K \geq -1$ and $d \leq D$. Then there exists a homotopy connecting the above curve to a point, such that the width of this homotopy is bounded from above by

$$e^{\rho(c,n)(D+1)(m+2)},$$

where m is the modified a-effective rank of the ball in which the curve lies, and where $\rho(c,n) = const. \cdot (n+1)c$, (see Lemma 2.4.7). Now substitute the estimate for the modified a-effective rank for m from Corollary 2.5.5. and we will obtain the result. □

We are now ready to construct a homotopy with the following properties:

1. the length of the curves in the homotopy is bounded;
2. there exists a point p for which the length of the curve $H_\tau(p)$ is bounded.

2.6 Construction of a Homotopy with Curves of Bounded Length

In this section we will prove the following proposition:

Proposition 2.6.1. *Let $M^n \subset \Psi$, and $\gamma(t)$ a closed curve in M^n. Then there exists a homotopy $H_\tau^{new}(t)$ satisfying the following properties:*

1. $H_0^{new}(t) = \gamma(t)$;

2. $H_1^{new}(t) = p$, *where* $p \in M^n$;

3. $\sup_{\tau_*} length H_{\tau_*}^{new}(t) \leq length(\gamma) + e^{e^{\Lambda(d+1)}}$, *where* $\Lambda = 200(n+1)$;

4. *length* $H_\tau^{new}(t_i) \leq e^{e^{B(d+1)}}$, *where* $\gamma(t_i), i = 1, 2$ *are two selected points on the curve* $\gamma(t)$, *B=100(n+1).*

Proof. Let us begin by observing that for any two curves $\alpha_1(t), \alpha_2(t)$ such that $d(\alpha_1(t), \alpha_2(t)) < inj/3$ for any t a function that for any $t \in (0,1)$ assigns the minimal geodesic that joins $\alpha_1(t)$ and $\alpha_2(t)$ is continuous.

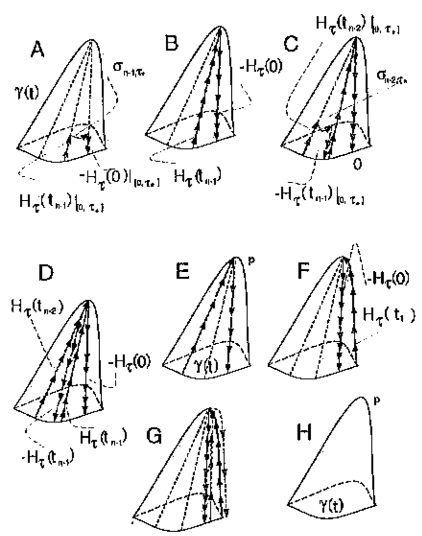

Figure 2.11: Homotopy of Proposition 2.6.1.

Let $\gamma(t)$ be a closed curve as above, and let $H_\tau(t)$ be the homotopy from Corollory 2.5.6. For any τ we will partition the unit interval into n subintervals $[t_i, t_{i+1}], t_0 = t_n$ so that $d(H_\tau(t_i), H_\tau(t_{i+1})) \leq inj/3$. That partition is possible to achieve because of the continuity of H_τ. Let $\sigma_{i,\tau}(s)$

be the minimal geodesic that joins points $H_\tau(t_i)$ and $H_\tau(t_{i+1})$. We are now ready to describe our new homotopy, (see also Figure 11.) We claim:

1. $\gamma(t)$ is homotopic to the curve $\lambda_1 = \gamma|_{[0,t_{n-1}]} \cup H_\tau(t_{n-1})|_{[0,\tau_*]} \cup \sigma_{(n-1),\tau_*} \cup -H_\tau(0)|_{[0,\tau_*]}$. Moreover, the length of curves in the homotopy is bounded by $l(\gamma) + 4W_H$, (see Figure 11A.)

2. λ_1 is homotopic to $\lambda_2 = \gamma|[0, t_{n-1}] \cup H_\tau(t_{n-1}) \cup -H_\tau(0)$, (see Figure 11B.)

3. λ_2 is homotopic to $\lambda_3 = \gamma|_{[0,t_{n-2}]} \cup H_\tau(t_{n-2}) \cup -H_\tau(t_{n-1}) \cup H_\tau(t_{n-1}) \cup -H_\tau(0)$. The length of curves in the homotopy is bounded by $l(\gamma) + 8W_H \leq l(\gamma) + e^{e^{200(n+1)(d+1)}}$, (see Figures 11C and D.)

4. λ_3 is homotopic to $\lambda_4 = \gamma|[0, t_{n-2}] \cup H_\tau(t_{n-2}) \cup -H_\tau(0)$, (see Figure 11E.)

5. λ_4 is homotopic to $\lambda_5 = \gamma|[0, t_1] \cup H_\tau(t_1) \cup -H_\tau(0)$, (see Figure 11F.)

6. λ_5 is homotopic to $\lambda_6 = H_\tau(t_1) \cup -H_\tau(t_1) \cup -H_\tau(0) \cup H_\tau(0)$ which is homotopic to a point p, (see Figures 11G and H.) Note also that for points $\{t_i\}, l(H_\tau^{new}(t_i)) \leq 4W_{H_\tau} \leq e^{e^{200(n+1)(d+1)}}$ and that we can partition the unit interval in such a way that some selected points on the curve γ are among $\gamma(t_i)$.

$\square$

In section 2.7 we will establish a similar result for a compact 2-essential manifolds with $k \leq 1$ and $vol(M) \leq V$, where k is the sectional curvature.

2.7 Manifolds with bounded from above curvature

Our result will be based on two inequalities:

1. Croke's isoperimetric inequality: $vol(B) \geq const._n r^n$, where B(r,p) is any metric ball of radius r centered at $p \in M$ and $r < inj(M)/2$. We can take $const._n = \frac{2^{n-1}}{(n!)^2}$, (see [7, 4]).

2. Berger's inequality: $vol(M) \geq c_n(inj(M)/\pi)^n$ where M is compact manifold of dimension n, and c_n can be estimated to be $\frac{\pi^n}{n!}$, (see [4]).

We will also need the following corollary to Klingenberg's lemma:

Lemma 2.7.1. *Let M^n be compact manifold with the sectional curvature bounded from above, i.e. $K \leq 1$. Then $inj(M) \geq min(\pi, l(M)/2)$, where l(M) is the length of the shortest simple closed geodesic, (see [9].)*

Suppose $l(M) = 2inj(M)$. Then we are done, because by Berger's inequality $l(M) = 2inj(M) \leq c_n' vol(M)^{1/n}$. Therefore, from now on we will assume that $l(M) > 2inj(M) > 2\pi$. Our approach will be similar to that of Section 2.3, but instead of first constructing a homotopy of bounded width we will right away construct a homotopy similar to the one of Section 2.6. We will show that it is possible to construct such a homotopy if the distance between two curves $d(\alpha_1(t), \alpha_2(t)) \leq \pi/9$. Then we will construct a sequence of curves $\{\sigma_i(t)\}_{i=1}^m$ such that

1. $\sigma_1(t) = \alpha(t)$;

2. $\sigma_m = p$, where p is a point on a manifold.

3. $d(\sigma_i, \sigma_{i+1}) \leq \pi/9$ and

4. $m \leq (\frac{(n!)^2 vol(M)}{20^{n-1}})^{216L/\pi+1}$

Lemma 2.7.2. *Let $\Phi_L(M)$ be the space of piecewise differentiable closed curves of length $\leq L$ parametrized proportionally to their arclength. There exists $\pi/36$-net on the δ-neighborhood $N_\delta(\Phi_L(M))$ of such curves and the number of elements in $\pi/36$-net will be $\leq (\frac{(n!)^2 vol(M)}{20^{n-1}})^{216L/\pi+1}$.*

Proof. First, we will have to construct $\pi/216$-net on M and estimate the number of elements in it. It will be done using Croke's inequality. Let us consider the maximal number of pairwise disjoint balls in M of radius $\pi/432$. The number of such balls will be $\leq \frac{vol(M)}{\max vol(B_{\pi/432}(p_i))} \leq \frac{vol(M)}{\frac{2^{n-1}432^n}{(n!)^2\pi^n}} = const._n' vol(M)$. The set $(B_{\pi/216}(p_i))_{i=1}^m$ is a cover of M, thus the set of points $\{p_i\}$ will be $\pi/216$-net on M. We will construct the required net on $N_\delta(\Phi_L(M)$ by joining p_i and p_j with minimal geodesics if and only if $d(p_i, p_j) \leq \pi/72$ and considering its subset consisting of closed curves. The number of elements in that set can be estimated to be

$$\leq [const._n' vol(M)]^{216L/\pi+1} = (\frac{(n!)^2\pi^n vol(M)}{2^{n-1}432^n})^{216L/\pi+1}$$

$$\leq (\frac{(n!)^2 vol(M)}{20^{n-1}})^{216L/\pi+1}.$$

Let us now apply Lemma 2.4.4, substituting $\pi/9$ for ϵ. □

Lemma 2.7.3. *Let M^n be compact simply-connected manifold with $K \geq 1$, $vol(M) \leq V$, and $l(M) > \pi$. Let $\gamma(t) \in \Phi_\pi(M)$. There exists a finite sequence σ_i of broken geodesics such that $d(\sigma_1(t), \gamma(t)) \leq \pi/36$; $\sigma_n = p$; length$(\sigma_i) \leq 3\pi$; $d(\sigma_i, \sigma_{i+1}) < \pi/9$ and the number n of elements in this sequence is $\leq N$, where $N = (const.'_n vol(M))^{216L/\pi+1}$.*

Proof. as in Lemma 2.4.5. □

We are now ready to construct a homotopy with the required properties.

Lemma 2.7.4. *Let M^n be as in Lemma 2.7.3, and let $\gamma(t) \subset M$ be any closed curve. Then there exists a homotopy $H_\tau(t)$ of a curve to a point p, such that*

1. $\sup_\tau\{$ *length* $H_\tau(t)/t \in [0,1]\} \leq 2L + \pi$;
2. *length* $H_\tau(t_i), \tau \in [0,1] \leq \frac{\pi(const.'_n vol(M))^{216L/\pi+1}}{9} \leq (const.'_n vol(M))^{216L/\pi+1}$ *for at least two selected points $t_i, i = 1, 2$.*

Proof. First we will show that any two curves $\alpha_1(t), \alpha_2(t)$ such that

$$d(\alpha_1(t), \alpha_2(t)) \leq \pi/9$$

can be connected by a homotopy $H'_\tau(t)$ for which

1. $\sup_\tau\{lengthH_\tau(t)/t \in [0,1]\} \leq 2L + \pi$.
2. $lengthH_\tau(t_i), t \in [0,1] \leq \pi/9$.

Then we will apply Lemma 2.7.3 to get a desired homotopy.

Let us subdivide the interval [0,1] into subintervals $[t_i, t_{i+1}]$,so that the figure with vertices: $\alpha_1(t_i)$, $\alpha_(t_i)$, $\alpha_2(t_{i+1})$, and $\alpha_1(t_{i+1})$ and edges $\alpha_j|_{[t_i,t_{i+1}]}$ and $\sigma_i(s)$, where $\sigma_i(s)$ is a minimal geodesic joining $\alpha_1(t_i)$ and $\alpha_2(t_i)$ lies inside the ball of radius $\pi/3$. That is possible to achieve by demanding that max length $\alpha_j|_{[t_i,t_{i+1}]} \leq \pi/9$. The function that for each s assigns the minimal geodesic $\beta_s^i(t)$ that joins points $\sigma_i(s)$ and $\sigma_{i+1}(s)$ is continuous. W.L.O.G. assume that $\alpha_j|_{[t_i,t_{i+1}]}$ is a geodesic. Let us call it α_j^i. Then we claim:

1. α_1 is homotopic to $\gamma_1 = \alpha_1|_{[0,t_{n-1}]} \cup \sigma_{t_{n-1}}(s)|_{[0,s_*]} \cup \beta_{s_*}^{n-1} \cup -\sigma_0(s)$. Moreover, the length of curves in the homotopy is bounded by $4\pi/9 + l(\alpha_1)$.

2. γ_1 is homotopic to $\gamma_2 = \alpha_1|_{[0,t_{n-1}]} \cup \sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$. Length of curves in the homotopy bounded by $2\pi/9 + l(\alpha_1) + l(\alpha_2)$.

3. γ_2 is homotopic to $\gamma_3 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}}|_{[0,s_*]} \cup \beta_{s_*}^{n-2} \cup -\sigma_{t_{n-1}}|_{[0,s_*]} \cup \sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$. Length of curves in the homotopy is $\leq 2\pi/3 + l(\alpha_1) + l(\alpha_2)$.

4. γ_3 is homotopic to $\gamma_4 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}} \cup \alpha_2|_{[t_{n-2},t_{n-1}]} \cup -\sigma_{t_{n-1}}\sigma_{t_{n-1}} \cup \alpha_2|_{[t_{n-1},0]} \cup -\sigma_0$.

5. γ_4 is homotopic to $\gamma_5 = \alpha_1|_{[0,t_{n-2}]} \cup \sigma_{t_{n-2}} \cup \alpha_2|_{[t_{n-2},0]} \cup -\sigma_0$.

6. γ_5 is homotopic to $\gamma_6 = \alpha_1|_{[0,t_1]} \cup \sigma_{t_1} \cup \alpha_2|_{[t_1,0]} \cup -\sigma_0$.

7. γ_6 is homotopic to $\gamma_7 = \sigma_0 \cup \alpha_2 \cup -\sigma_0$, which is homotopic to α_2. Note that the maximal length of curves in the resulting homotopy is bounded from above by $\pi + 2L$. Note also that for all t_i length $H(t_i) \leq \pi/9$, (see Figure 12.)

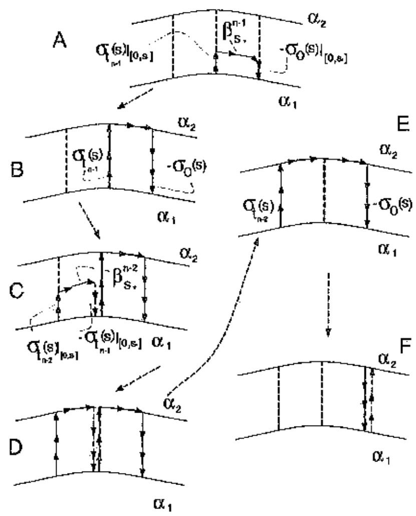

Figure 2.12: Homotopy of Lemma 2.7.4.

Now let us apply Lemma 2.7.3, i.e. take a sequence of broken geodesics σ_i. We know that σ_i is homotopic to σ_{i+1}, where the homotopy H^i has the desired properties. Take the composition of those homotopies to obtain $H_\tau(t)$. It is clear that it will satisfy (1) and (2) of the Lemma. $\square$

2.8 Construction of a closed path in the space of closed curves

Before we attempt to prove our main theorems we will need the following two definitions from [12]:

Definition 2.8.1. (Filling Radius of n-dimensional Manifold M Topologically Imbedded into X) Filling radius, denoted by Fill Rad(M $\subset X$), where X is an arbitrary metric space, is the infimum of $\epsilon > 0$, such that M bounds in the ϵ-neighborhood $N_\epsilon(M)$, i.e. homomorphism $H_n(M) \to H_n(N_\epsilon(M))$ induced by the inclusion map vanishes.

Definition 2.8.2. (Filling Radius of an Abstract Manifold) Filling radius Fill Rad M of an abstract manifold M is Fill Rad $(M \subset X)$, where $X = L^\infty(M)$, i.e. the Banach space of bounded Borel functions f on M, and the embedding of M into X that to each point p of M assigns a distance function $p \mapsto f_p = d(p, q)$.

The idea of filling radius will be used in as much as we will need the following result proved by Gromov in [12].

Theorem 2.8.3. *Let M be a closed connected Riemannian manifold of dimension n. Then Fill Rad* $M \leq (n+1)n\sqrt[n]{n!}(volM)^{1/n}$.

We are now ready to prove the following proposition:

Proposition 2.8.4. *Let M^n be compact 2-essential manifold of dimension n, with the property that for every closed curve $\gamma(t)$ of length $\leq 6FillRadM$, there exists a homotopy $H_\tau(t)$ of that curve to a point p, such that length of curves in the homotopy is bounded by L_1 and for two selected points on $\gamma(t) : \gamma(t_i), i = 1, 2$ the length $H_\tau(t_i) \leq L_2$. Then there exists a closed path in the space $\Lambda(M^n)$ of closed curves on a manifold with the property that the length of each curve is bounded by $f(L_1, L_2) = 3L_1 + 6L_2$ and such that this path represents a non-trivial element of $\pi_1(\Lambda M^n)$.*

Proof. The proof will be done in two steps: The idea of Step 1 is to obtain a nontrivial element of $H_2(M)$ with some special properties described below. This will be done the following way: we will consider the filling of M that we will call W, and then try to extand the map $f : M \to \mathbf{C}P^\infty$ to W. We will then obtain that element as an obstruction. This part of the proof will be a modification of Lemma 1.2 B and the Proposition of Gromov in [12] (page 136). In Step 2 we will construct the path.

Step 1 Let W be a filling of M. Since M is 2-essential there exists a function $f : M \to \mathbf{C}P^\infty$ such that $f_*[M] \neq 0$. We will try to extend f to W. Let us proceed as follows: first, extend f to 0-skeleton of W, then to 1-skeleton of W, etc. This process will have to be interrupted at the 4th stage since we know that f cannot be extended to W.

Extending to 0-skeleton Subdivide W, so that all simplices have diam(σ) $\leq \delta$. Send vertices $w_i \in W$ to vertices of triangulation $m_i \in M$ for which $d(w_i, m_i) \leq d(w_i, M) + \delta <$ Fill Rad M $+\delta$. Suppose m_i, m_j come from vertices w_i, w_j of some simplex in W. Then $d(m_i, m_j) \leq d(m_i, w_i) + d(w_i, m_j) \leq d(m_i, w_i) + d(w_i, w_j) + d(w_j, m_j) \leq$ 2Fill Rad M$+3\delta$. Thus, m_i, m_j can be joined by geodesic of length $<$ 2Fill Rad M $+3\delta$.

Extending to 1-skeleton Send the 1-simplices $[w_i, w_j] \subset W/M$ to the above geodesics joining m_i and m_j. (In addition, we assume all 1-simplices in M to be already short.) So we can see that the boundary of each 2-simplex in W is sent to a curve of length $<$ 6Fill Rad M $+9\delta$, (note, that it is also $\leq 3d$).

Extending to 2-skeleton let σ be a 2-simplex of W. Consider its boundary $\partial\sigma$ and the image of $\partial\sigma$ under f. It will be a closed curve consisting of broken geodesics. Let us call it $\gamma(t)$. By our hypothesis we know that there exists a special homotopy $H_\tau(t)$ of that curve to a p oint. We will then map σ to the surface determined by this homotopy.

We have thus succeeded in extending our map to the 2-skeleton. Extending the map to 3-skeleton would have been equivalent to extending it to the whole of W, but that is impossible, since it would contradict $f_*[M^n] \neq 0$. Therefore, there exists a 3-simplex in W, such that the image of its boundary ω represents a nontrivial element of $H_2(M)$.

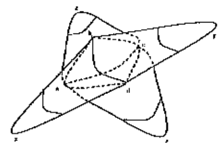

Figure 2.13: Nontrivial element of $H_2(M)$ from Proposition 2.8.4.

Step 2 Consider ω. Let's denote its vertices a,b,c,d. (Here, we will call vertices of ω the images of vertices a^*, b^*, c^*, d^* of Ω.) Let x,y,z,s be images of faces of Ω. (They are obtained by contracting closed curves [a,b,d], [b,d,c], [a,b,c], [d,a,c] in M by the homotopies from the hypothesis of this proposition, which we will call H^x_τ, H^y_τ, H^z_τ, H^s_τ respectively.) Let us examine the face x of ω. We will claim that

1. a, regarded as a constant curve, is homotopic to $[a,b]\cup[b,a]$, (see Figure 14.A.) and the length of curves in this homotopy $\leq L_1$. .

2. $[a,b]\cup[b,a]$ is homotopic to $[a,b]\cup H^x_\tau(b)\cup -H^x_\tau(a)$ and the length of curves in the homotopy is $\leq 2L_1+2L_2$. Figure 14.B. and C.

3. $[a,b]\cup H^x_\tau(b)\cup -H^x_\tau(a)$ is homotopic to $[a,b]\cup H^x_\tau|_{[0,\tau_*]}(b)\cup H^x_{\tau_*}([a,b])\cup -H^x_\tau|_{[0,\tau_*]}(a)$. Figure 14.D. The length of curves in the homotopy is bounded from above by $2L_1+2L_2$.

4. $[a,b]\cup H^x_\tau|_{[0,\tau_*]}(b)\cup H^x_{\tau_*}(t)\cup -H^x_\tau|_{[0,\tau_*]}(a)$ is homotopic to $[a,b]\cup[b,d]\cup[d,a]$, and the length of the curves in the homotopy is . $\leq 2L_1+2L_2$. Figure 14E.

By the same type of constructions we have

5. $[a,b]\cup[b,d]\cup[d,a]$ homotopic to $[a,z]\cup[z,b]\cup[b,y]\cup[y,d]\cup[d,s]\cup[s,a]$ homotopic to $[a,c]\cup[c,b]\cup[b,c]\cup[c,d]\cup[d,c]\cup[c,a]$ homotopic to $[a,c]\cup[c,a]$ homotopic to a, where the length of the curves in the homotopy is $\leq 3L_1+6L_2$. See Figure 14F and G.

□

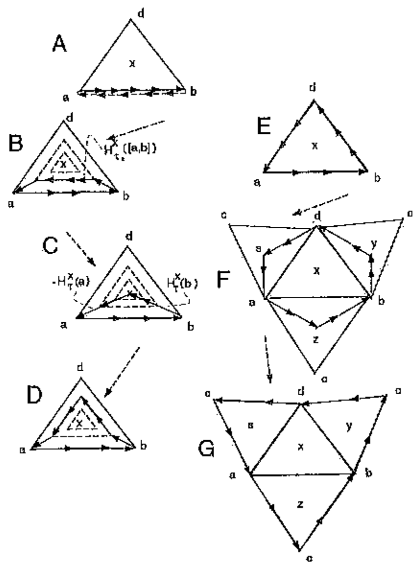

Figure 2.14: Nontrivial element of $H_1(\Lambda M)$.

Now we can prove the main theorems. The proof of Theorem 2.1.1 will be a combination of application of results of Sections 2.2–2.6 and Step 2 of Proposition 2.8.4.

Proof of Theorem 2.1.1. Let us triangulate M^n so that diameter of any simplex τ is less than the injectivity radius of M^n. Select any cycle σ composed of the simplices of the triangulation and representing a nontrivial element in $H_2(M^n)$. Since any simplex τ is located in the ball with radius smaller then the injectivity radius, we can assume that 2-faces of simplices of σ are surfaces generated by contracting the boundary to the center of that ball. (Note that the width of the homotopy in that case is $\leq d$.) Pick a point $p \in M^n$. Now join p with all vertices of σ by minimal geodesics. Now consider all the closed curves that are formed by three segments: $[p, v_1]$,$[p, v_2]$, $[v_1, v_2]$ where $[p, v_1]$, $[p, v_2]$ are minimal geodesics joning point p with vertices v_1, v_2, and $[v_1, v_2]$ is an edge of σ. The length of such closed curves will be $\leq 3d$. So we can contract each of them to the point using the homotopy of Proposition 2.6.1. We thus obtain 2-cycles, at least one of which should represent a nontrivial element of $H_2(M)$ in order for σ not to bound. This cycle has the shape of the one in figure 13, i.e. its vertices are p, v_1, v_2, v_3, where $v_1, v_2, v_3 \in \sigma$, its edges are curves of length $\leq d$, and its faces are surfaces generated by the homotopies of Proposition 2.6.1 (except for the face that lies in σ, which is generated by the homotopy that is even nicer since it lies inside the ball of radius that is less than the injectivity radius of M^n.) We can now denote this cycle ω and follow Step 2 of Proposition 2.8.4 where $L_1 \leq e^{e^{200(n+2)(d+1)}}$ and $L_2 \leq e^{e^{100(n+2)(d+1)}}$, where L_1 and L_2 are as in Proposition 2.8.4. Therefore, by Step 2 of Proposition 2.8.4 there exists a closed nontrivial curve in the space of all closed curves such that the length of each curve is bounded by $3L_1 + 6L_2 \leq e^{e^{250(n+2)(d+1)}} \leq e^{e^{250(n+2)(D+1)}}$. □

Proof of Theorem 2.1.3. If $l(M) = 2\text{inj(M)}$ then $l(M) \leq 2\sqrt[n]{n!}vol(M)^{1/n} \leq \sqrt[n]{n!}V^{1/n}$ and we are done. Otherwise, the proof is similar to that of Theorem 2.1.1 and is an application of Lemma 2.7.4, Theorem 2.8.3, and Proposition 2.8.4. Note that in this case

$$L_1 = 12FillRadM + \pi,$$

$$L_2 = (\frac{(n!)^2 vol(M)}{20^{n-1}})^{216FillRadM+1}.$$

Obvious substitutions and calculations imply the result. □

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