On the filling volume of Riemannian manifolds

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Abstract of the Dissertation
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In this paper, the filling volume of M. Gromov for the circle of length 2π is found to be equal to 2π and uniquely realized by the round, unit hemisphere. An application is a geometric inequality of isoperimetric type. This in turn is used to find upper estimates for both Cheeger's constant and the first eigenvalue of the Laplacian assuming an upper curvature bound. The geometric inequality and both upper estimates are realized in the case of a sphere of constant curvature. A more general problem of finding metrics of minimal volume that satisfy a lower bound on the distance function restricted to the boundary is posed, and partial results characterizing these metrics are given.

To my parents.

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Chapter 1

Introduction

M. Gromov [17] showed that there is a lower bound for the volume of a Riemannian manifold, $\mathrm{Vol}(M,g) \geq c(n)\mathrm{Sys}^n(M,g)$, where $c(n)$ is a constant depending only on n, the dimension of M, and $\mathrm{Sys}(M,g)$,the systole of (M,g), is the length of the shortest non-contractable curve (given topological assumptions). To do this, he defined two Riemannian invariants: the filling radius and the filling volume (see [4] for a summary). M. Katz [20] has calculated the filling radius for $\mathbb{RP}^n$ and S^n with constant curvature metrics as well as finding lower bounds for $\mathbb{CP}^n$, $\mathbb{HP}^n$ and the Cayley plane with their standard metrics. In this paper we show that the filling volume of S^1 with length 2π is realized by the hemisphere of constant unit curvature (Theorem 3.1), as conjectured by M. Gromov. The problem of finding the filling volume of S^1 is one of minimizing the area of an orientable surface with boundary S^1 while keeping the distances between points on the boundary from falling below those of the standard metric on S^1. R. Michel [23] found that fixing the distance on the boundary of a surface leads to rigidity of the metric. This result does not use the hypothesis of non-positive curvature unlike other results of rigidity obtained by fixing distance on the boundary. Another case of rigidity of the metric given conditions on the boundary was established by V. Bangert in [1], where he showed that if the length of all non-constant geodesics with end points in the boundary of a compact n-manifold is equal to π, then the manifold is isometric to a unit hemisphere.

Using the filling volume of the circle, we prove that

$$\mathrm{Area}(M, g) - \mathrm{Area}(B_r(p), g) \geq \frac{8r^2}{\pi}, \tag{1.1}$$

for a complete, orientable Riemannian surface (M, g) and $B_r(p)$, a metric ball at $p \in M$ of radius $r \leq \mathrm{conv}(p)$, the convexity radius at p. This inequality is used to find an upper bound for Cheeger's isoperimetric constant h on a compact manifold, given an upper curvature bound (Corollary 4.2). P. Buser [7] used Cheeger's constant to give an upper bound of the first non-zero eigenvalue of a complete Riemannian manifold assuming a lower curvature bound. We employ (1.1) to find an upper estimate for the first non-zero eigenvalue of the Laplacian of a complete, orientable Riemannian manifold of finite volume, in terms of convexity radius and an upper curvature bound. This gives a sharp estimate of the kind obtained by M. Berger [3] and C.B. Croke [11] with the added assumption of an upper curvature bound. The philosophy of this approach is similar to that used by P. Li and S.-T. Yau [22] to find sharp bounds for the first eigenvalue of the Laplacian for compact surfaces. However, the result is very different in nature. Other upper estimates for λ_1 involve either a lower bound on curvature (sectional, Ricci or mean curvature) [5, 6, 8, 9, 10, 14, 15, 21], or on the genus [2, 8, 12, 18, 19, 25, 26].

The strategy for finding the filling volume of the circle is to reduce consideration to the simply connected case, which was solved by M. Gromov [17]. A slightly different version of this result is needed, and the statement and proof are found in Chapter 1 (Proposition 2.8). The constructions used to arrive at the simply connected case and to prove uniqueness are introduced as lemmata in Chapter 2. The third chapter contains the statement and proof of the main result. Applications including a sharp, geometric inequality of isoperimetric type, and eigenvalue estimates are given in Chapter 4. Finally, in Chapter 5 we motivate and pose a more general problem of finding minimal volumes for a manifold with boundary with a lower bound on the (extrinsic) distance function restricted to the boundary. Partial results are given on the characterization of metrics which realize minimal volume. They are in terms of the number of geodesics through each point in the interior that realize the lower bound on distances between points in the boundary. The notation used is listed in Appendix A.

Chapter 2

Basic Constructions

Definition 2.1 *Let (V, h) be a compact Riemannian n-manifold. We say that (M, g, φ)* fills *(V, h) (or (M, g, φ) is a* filling *of (V, h)) iff (M, g) is a Riemannian $(n+1)$-manifold with boundary complete as a metric space of finite volume, $\varphi : V \to \partial M$ is a diffeomorphism, and g is such that the extrinsic distance, $\mathrm{d}_g(\varphi(x), \varphi(y)) \geq \mathrm{d}_h(x, y)$ for all $x, y \in V$. Also, if V is orientable, then M is required to be orientable.*

In [17], Gromov defines the filling volume of a compact Riemannian manifold and shows that it is equivalent to the following.

Definition 2.2 *The* filling volume, *of a compact Riemannian manifold (V, h), denoted by* FillVol(V, h), *is the infemum of volumes of all (M, g, φ) which fill (V, h).*

In fact, Gromov shows (Proposition 2.2.A in [17]) that the infemum is the same if we restrict to any given M and φ when the dimension of V is at least two. From Theorem 3.1 and a simple construction, it follows that this result also holds in dimension one i.e. when $V = S^1$ (see Remark 3.2).

We will use the following terminology. If (X, d) is a length space, and $\gamma : [a, b] \to M$ is a rectifiable curve, then we call γ a segment if $L(\gamma) = \mathrm{d}(\gamma(a), \gamma(b))$. We say that γ is a geodesic if $(X, \mathrm{d}) = (M, \mathrm{d}_g)$ for some Riemannian manifold (M, g) (possibly with boundary) and γ is a critical point of the energy functional

$$\mathrm{E}(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_g^2 dt.$$

We will be concerned with certain segments between points in the boundary of a filling. In a slightly more general setting we make the following

Definition 2.3 *Let* (X, d_X) *be a length space such that for some continuous, symmetric function* $\rho : \partial X \times \partial X \to \mathbb{R}$ $\mathrm{d}_X(p, q) \geq \rho(p, q)$ *for all* $p, q \in \partial X$. *A rectifiable curve* $\gamma : [0, 1] \to X$ *is called* taut *with respect to* ρ *(or* ρ-taut*) iff* $\gamma(0) = p$ *and* $\gamma(1) = q$ *for some* $p, q \in \partial X$ *and the length of* γ *is equal to* $\rho(p, q)$.

If (M, g, φ) fills (V, h), then the distance between any pair of points on the boundary of M is bounded below by the distance of their preimage under φ. For this filling is to realize the filling volume of (V, h), there must be sufficiently many taut curves. The following lemma shows that there is at least one such curve through each point in the interior of M.

Lemma 2.4 *If* (M, g, φ) *fills* (V, h) *and there exists* $U \subset M$ *an open set such that for all* $p \in U$ *there is no* $\gamma : [0, 1] \to M$ *through* p *which is* φ*-taut, then there is a metric* $\tilde{g}$ *on* M *with* $\mathrm{Vol}(M, \tilde{g}) < \mathrm{Vol}(M, g)$*, such that* $(M, \tilde{g}, \varphi)$ *fills*

(V, h). Thus, if the filling is of minimal volume. there must be at least one φ-taut curve through each point in the interior of M.

(Proof)

Without loss of generality, we may assume that $U \subset M \backslash \partial M$. Take $p \in U$, $r > 0$ and A compact so that $B_r(p) \subset A \subset U$. Let $\psi : M \longrightarrow [\eta, 0]$ be a smooth function with support $\operatorname{supp} \psi \subset B_r(p)$ and $\inf \psi = \eta < 0$. Define a new metric on M, $\tilde{g} = e^{2\psi} g$. We clearly have that $\mathrm{Vol}(M, \tilde{g}) < \mathrm{Vol}(M, g)$. Also, $\tilde{g} = g$ on $M \backslash B_r(p)$. Set

$$R = \inf\{\mathrm{d}_g(\varphi(x), q) + \mathrm{d}_g(q, \varphi(y)) - \mathrm{d}_h(x, y) \mid q \in A \text{ and } x, y \in V\}.$$

Now, A and V are compact, $A \subset U$ and there are no taut curves through U. Therefore $R > 0$ by the continuity of the distance function. We are free to choose $r \leq R/2$.

Take any smooth curve $\alpha : [0, 1] \longrightarrow M$ with $\alpha(0) = \varphi(x), \alpha(1) = \varphi(y)$ for $x, y \in V$. If α does not enter $B_r(p)$, then clearly $L_{\tilde{g}}(\alpha) = L_g(\alpha)$. Otherwise, let

$$a = \inf\{t \in [0, 1] \mid \alpha(t) \in B_r(p)\} \quad \text{and} \quad b = \sup\{t \in [0, 1] \mid \alpha(t) \in B_r(p)\}.$$

Since $\tilde{g} = g$ on $M \backslash B_r(p)$ and φ is distance nondecreasing, we have that

$$\begin{aligned} L_{\tilde{g}}(\alpha) &\geq L_g(\alpha|_{[0,a]}) + L_g(\alpha|_{[b,1]}) \\ &\geq \mathrm{d}_g(\alpha(0), \alpha(a)) + \mathrm{d}_g(\alpha(a), \alpha(1)) - \mathrm{d}_g(\alpha(a), \alpha(b)) \\ &\geq \mathrm{d}_h(\varphi(x), \varphi(y)) + R - 2r\,. \end{aligned}$$

Thus, distances between points in $\varphi(V)$ are bounded below as desired.

Now to conclude the proof we will show the set of points p without φ-taut curve passing through them is open. Suppose that there is a sequence $\{p_j\}_{j\in\mathbb{N}}$ converging to p and a sequence of pairs $\{(x'_j, y'_j)\}_{j\in\mathbb{N}} \subset V \times V$ with

$$\mathrm{d}_g(x_j, p_j) + \mathrm{d}_g(p_j, y_j) \leq \mathrm{d}_h(x'_j, y'_j),$$

where $x_j = \varphi(x'_j)$ and $y_j = \varphi(y'_j)$. By the triangle inequality,

$$\mathrm{d}_g(x_j, p_j) + \mathrm{d}_g(p_j, y_j) = \mathrm{d}_h(x'_j, y'_j).$$

As V is compact, so is $V \times V$ and so we may pass to a subsequence of pairs $\{(x'_j, y'_j)\}_{j\in\mathcal{I}}$ convergent to $(x', y') \in V \times V$. By the triangle inequality, we have that

$$\begin{aligned}
\mathrm{d}_g(p, x) &\leq \mathrm{d}_g(p, p_j) + \mathrm{d}_g(p_j, x_j) + \mathrm{d}_g(x_j, x) \\
\mathrm{d}_g(p, y) &\leq \mathrm{d}_g(p, p_j) + \mathrm{d}_g(p_j, y_j) + \mathrm{d}_g(y_j, y),
\end{aligned}$$

where $x = \varphi(x')$ and $y = \varphi(y')$. Therefore

$$\begin{aligned}
\mathrm{d}_g(p, x) + \mathrm{d}_g(p, y) &\leq 2\mathrm{d}_g(p, p_j) + \mathrm{d}_g(x_j, x) + \mathrm{d}_g(y_j, y) + \mathrm{d}_h(x'_j, y'_j) \\
&\leq 2\mathrm{d}_g(p, p_j) + \mathrm{d}_g(x_j, x) + \mathrm{d}_g(y_j, y) \\
&\quad + \mathrm{d}_h(x', y') + \mathrm{d}_h(x'_j, x') + \mathrm{d}_h(y'_j, y') \\
&\leq 2\mathrm{d}_g(p, p_j) + 2\mathrm{d}_g(x_j, x)) + 2\mathrm{d}_g(y_j, y) + \mathrm{d}_h(x', y').
\end{aligned}$$

Now, for any $\epsilon > 0$, there exists an N so that for all $j > N$

$$\max\{\mathrm{d}_g(p, p_j), \mathrm{d}_g(x_j, x), \mathrm{d}_g(y_j, y)\} < \epsilon/6\,,$$

so we have that for any $\epsilon > 0$

$$\mathrm{d}_g(p, x) + \mathrm{d}_g(p, y) \leq \mathrm{d}_h(x', y') + \epsilon\,.$$

Therefore,

$$\begin{aligned} d_g(p,x) + d_g(p,y) &\le d_h(x',y') \\ \Rightarrow d_g(p,x) + d_g(p,y) &\ge d_g(x,y) \ge d_h(x',y') \end{aligned}$$

so, by the triangle inequality,

$$d_g(p,x) + d_g(p,y) = d_h(x'.y') .$$

This implies the existence of a taut curve through p, a contradiction.

□

Note that the change of metric in Lemma 2.4 is a conformal one. Consider a conformal change of metric $\tilde{g} = f^2 g$ on an n-dimensional manifold M. Then for any C^1 curve $\gamma : [a, b] \to M$,

$$L_{\tilde{g}}(\gamma) = \int_a^b \sqrt{f^2 g(\dot{\gamma}, \dot{\gamma})} = \int_a^b f \|\dot{\gamma}\|_g .$$

Also, $\det \tilde{g} = f^{2n} \det g$, so

$$\mathrm{Vol}(M, \tilde{g}) = \int_M \sqrt{\det \tilde{g}} = \int_M f^n \sqrt{\det g} = \|f\|_{L^n}^n .$$

The following lemma shows that in each class of conformally equivalent metrics there is a weak solution to the minimal filling problem. Suppose that (M^n, g, φ) $(n \ge 2)$ is fills (V, h) and denote

$$\mathcal{P} = \left\{ \gamma \in C^1([0,1], M) \mid \gamma(0), \gamma(1) \in \partial M \right\} .$$

Let $\mathcal{F}$ be the set of all conformal factors f for which $(M, f^2 g, \varphi)$ fills (V, h):

$$\mathcal{F} = \left\{ f \in C^\infty(M, \mathbb{R}^+) \mid \forall \gamma \in \mathcal{P}, \int_0^1 f(\gamma) \|\dot{\gamma}\| dt \ge d_V(\varphi^{-1}(\gamma(0)), \varphi^{-1}(\gamma(1)) \right\} .$$

Lemma 2.5 *If there exists a function f_0 in the L^n-closure of $\mathcal{F}$ such that*

$$||f_0||_{L^n} = \inf_{f \in \mathcal{F}} ||f||_{L^n}.$$

then this function is unique and so each class of conformally equivalent metrics on M has at most one weak solution to the problem of finding the metric of minimal volume which fills (V, h) via φ.

This is a special case of Lemma 5.3 which is proved in Chapter 5.

Next we state a technical lemma used to show the compactness of certain sets of curves.

Lemma 2.6 *Let (M, g) be a smooth, compact Riemannian manifold with smooth (possibly empty) boundary and $\{\alpha_k : [0,1] \to M\}_{k \in \mathbb{N}}$ a sequence of piecewise C^1-curves in M. If*

$$\sup_{k \in \mathbb{N}} \sup_{t \in [0,1]} ||\dot{\alpha}_k(t)|| < \infty.$$

then there exists a uniformly convergent subsequence. In particular, the distance between any given pair of points in M is realized by a segment.

(Proof)

Since the speed of these curves is bounded above, they form an equicontinuous set and M is compact, so $\{\alpha_k(t)\}$ has compact closure for all $t \in [0,1]$. The Arzelà-Ascoli Theorem [13] then applies to show that $\{\alpha_k\}$ has compact closure in the compact-open topology on $C^0([0,1], M)$ and is equicontinuous. Therefore we can pass to a subsequence $\{\alpha_k\}$ which converges to a continuous curve α. $k > N$.

To show the existence of a segment, we take a sequence of C^1 curves joining a pair of points whose lengths converge monotonically to the distance. There must then be a continuous curve joining these two points whose length is equal to the distance. Because the curve is locally minimal for the length functional, it must be a g-geodesic in the interior of M and C^1. Also, if it stays in ∂M for time of non-zero measure, it must be a geodesic in the induced metric on ∂M. Therefore, it is C^1 and piecewise C^∞.

□

The next lemma is key in proving the main result. Applying it a finite number of times to a filling of $(S^1, 2pi)$ by a compact manifold results in a filling of $(S^1, 2\pi)$ by a disc of smaller area. Let

$$\mathcal{A} = \{\alpha \in \Omega(M) \mid \alpha \text{ is simple and not contractable in } W = M/\sim\},$$

where $x \sim y$ iff $x, y \in \partial M$, and $\Omega(M)$ is the set of all piecewise C^1 loops in M.

Lemma 2.7 *Let (M, g, φ) fill $(S^1, 2\pi)$, with M compact, and assume that there exits $\alpha \in \mathcal{A}$ which does not intersect ∂M and realizes the infimum of the lengths of curves in $\mathcal{A}$. Then there is a manifold of strictly smaller area than (M, g) which fills $(S^1, 2\pi)$.*

(Proof)

Parametrize $\alpha : \mathbb{R} \to M$ to have constant speed, period one and so that $\alpha|_{[0,1)}$ is injective. It follows that $\alpha|_{[\sigma, \sigma+\frac{1}{2}]}$ is a geodesic and minimizing for any σ and furthermore that there are exactly two minimal connections from $\alpha(\sigma)$

to $\alpha(\sigma+\frac{1}{2})$, specifically $\alpha|_{[\sigma,\sigma+\frac{1}{2}]}$ and $\alpha|_{[\sigma+\frac{1}{2},\sigma+1]}$. To see this, suppose that there exists a minimal curve $\gamma:[\sigma,\tau]\to M$ from $\alpha(\sigma)$ to $\alpha(\tau)$, with $\sigma<\tau\leq\sigma+\frac{1}{2}$. Without loss of generality we may assume that $\gamma|_{(\sigma,\tau)}$ does not intersect α. Let $\beta_1=\alpha|_{[\sigma,\tau]}*\bar{\gamma}$ and $\beta_2=\gamma*\alpha|_{[\tau,\sigma+1]}$, where $\bar{\gamma}(t)=\gamma(\tau+\sigma-t)$. Note that β_j are simple. If either β_j is not contractable in W (defined above), then there is a C^1 curve homotopic to β_j of strictly shorter length. To find such a curve, let T be such that

$$p=\beta_j(T)=\alpha(\tau)=\gamma(\tau).$$

Since both α and γ are geodesics, they meet transversally at p. Hence, by strong local convexity of Riemannian manifolds, for $\epsilon>0$ sufficiently small there is a unique, minimal geodesic from $\beta_j(T-\epsilon)$ to $\beta_j(T+\epsilon)$ which is shorter than the path between these two points along β_j. This gives a new curve $\tilde{\beta}_j$ which is simple and homotopic to β_j for ϵ sufficiently small. Therefore $\tilde{\beta}_j\in\mathcal{A}$ and,

$$L_g(\tilde{\beta}_j)<L_g(\beta_j)=L_g(\alpha).$$

This contradicts the minimality of the length of α in $\mathcal{A}$. If both β_1 and β_2 are contractable in W, then so is α, contradicting $\alpha\in\mathcal{A}$. Therefore, α is a simple, periodic geodesic.

Consider $N=M\backslash\alpha$ a manifold with boundary $\partial N=\partial M\amalg\alpha_1\amalg\alpha_2$, where α_1,α_2 are disjoint components of ∂N and are diffeomorphic to α (see figure). We have a natural map $f:N\to M$ which identifies α_1 and α_2 with α, i.e. $f(\alpha_1(t))=f(\alpha_2(t))=\alpha(t)$. This map is a local diffeomorphism on $N\backslash(\alpha_1\cup\alpha_2)$ and is surjective. Let $\varphi_0:S^1\to N$ be the unique map such that $f\circ\varphi_0=\varphi$

and let $g_0 = f^*g$. Then $\mathrm{Area}(N, g_0) = \mathrm{Area}(M, g)$. Also, φ_0 is distance non-decreasing because the set of possible curves from $\varphi_0(S^1)$ to $\varphi_0(S^1)$ can be seen as a subset of those from ∂M to ∂M. Since α is minimal in $\mathcal{A}$, and in particular is a geodesic, the only taut curves through points on α are transversal to it with exactly one point of intersection.

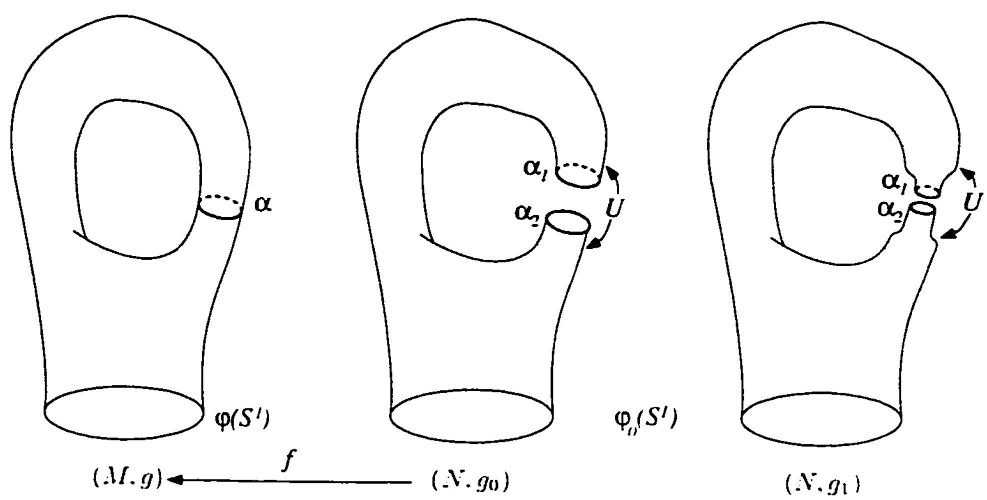

Figure 2.1: Construction of (N, g_0) and (N, g_1)

Therefore, no point on α_j can have a φ_0-taut g_0-segment passing through it. Whence there exists a neighbourhood U of $\alpha_1 \cup \alpha_2$ such that for all $p \in U$ there is no taut g_0-segment passing through p.

Take Fermi coordinates given by the distance to $\alpha_1 \cup \alpha_2$ on a subset of U. Let ξ_2 be the coordinate field chosen so that $f_*\xi_2|_{\alpha_j} = \dot{\alpha}$ for $j = 1, 2$. Take ξ_1 normal to ξ_2 chosen to agree with the orientation on N induced by f. The frame $\{\xi_1, \xi_2\}$ is everywhere g_0-orthogonal. Define a new metric on N which stretches g_0 along ξ_1 and shrinks it along ξ_2 (see figure). In particular, let

$$\langle X, Y\rangle_{g_1} = (1 + \eta\psi)\langle X, Y\rangle_{g_0} - 2\eta\psi\langle X, \xi_2\rangle_{g_0}\langle Y, \xi_2\rangle_{g_0},$$

where $0<\eta<1$ and $\psi = \tilde{\psi} \circ f$, $\tilde{\psi}: M \to [0,1]$ is a smooth cutoff function with $\operatorname{supp} \tilde{\psi} \subset B_\delta(\alpha)$. We take $\tilde{\psi}$ to be constant on level sets of the g-distance to α so that $\tilde{\psi}(\alpha(t)) = 1$ for all t. If we choose $\delta > 0$ sufficiently small this is well-defined. We will show that in this new metric. α_j are both shorter, further from ∂M, and are still geodesics so that α is minimal in $\mathcal{A}$ in the metric $\tilde{g}$ on M defined by

$$\langle X, Y\rangle_{\tilde{g}} = (1+\eta\tilde{\psi})\langle X, Y\rangle_{g_0} - 2\eta\tilde{\psi}\langle X, f_*\xi_2\rangle_{g_0}\langle Y, f_* xi_2\rangle_{g_0}.$$

Note that $g_1 = f * \tilde{g}$. For $\delta > 0$ sufficiently small the support of ψ does not intersect $\varphi_0(S^1)$ and any curve realizing the g_0-distance from a point in $N \setminus \operatorname{supp}\psi$ to α_j is tangent to ξ_1 whenever it is inside the support of ψ. Also, $\{\xi_j\}$ is g_1-orthogonal, and ψ is constant on level sets of the g_0-distance to $\alpha_1 \cup \alpha_2$. Therefore any curve realizing the g_1-distance from a point in $N \setminus \operatorname{supp}\psi$ to α_j is also tangent to ξ_1 when it is inside the support of ψ, so

$$\mathrm{d}_{g_1}(\alpha_j, N \setminus \operatorname{supp}\psi) > \mathrm{d}_{g_0}(\alpha_j, N \setminus \operatorname{supp}\psi).$$

Also, g_0 and g_1 are equal outside of $\operatorname{supp}\psi$ therefore,

$$\mathrm{d}_{g_1}(\alpha_j, \varphi_0(S^1)) > \mathrm{d}_{g_0}(\alpha_j, \varphi_0(S^1)). \tag{2.1}$$

In the metric $\tilde{g}$, α will be a periodic geodesic and be minimal in $\mathcal{A}$ as follows. For any other piecewise C^1-curve $\beta: [0,1] \to M$ in $\mathcal{A}$,

$$\begin{aligned} L_{\tilde{g}}(\beta) &= \int_0^1 \sqrt{(1+\eta\tilde{\psi})\|\dot{\beta}(t)\|_g^2 - 2\eta\tilde{\psi}\langle\dot{\beta}(t), \xi_2\rangle_{g_0}^2}\,dt \\ &\geq \int_0^1 \sqrt{(1-\tilde{\psi})\|\dot{\beta}(t)\|_g^2}\,dt \end{aligned}$$

$$\geq \int_0^1 \sqrt{1-\eta}\|\dot{\beta}(t)\|_g dt$$
$$\geq \sqrt{1-\eta}L_g(\alpha)$$

since α is minimal in $\mathcal{A}$. Now ξ_2 is tangent to α and $\tilde{\psi}\circ\alpha \equiv \eta$, so

$$L\tilde{g}(\alpha) = \sqrt{1-\eta}L_g(\alpha).$$

and it follows that

$$L_{\tilde{g}}(\beta) \geq \sqrt{1-\eta}\, L_g(\alpha) = L_{\tilde{g}}(\alpha).$$

Therefore in the metric $\tilde{g}$, α is minimal in $\mathcal{A}$ and so, in the metric g_1, α_j are geodesics. For $j=1,2$,

$$L_{g_1}(\alpha_j) = \sqrt{1-\eta}\, L_{g_0}(\alpha_j) < L_{g_0}(\alpha_j). \tag{2.2}$$

Now we will show that if η is chosen sufficiently small,

$$\varphi_0 : (S^1, 2\pi) \to (N, g_1)$$

is distance non-decreasing. Define a sequence of metrics $\tilde{g}_k$ on N by the formula

$$\langle X, Y\rangle_{\tilde{g}_k} = (1+\tilde{\psi}_k)\langle X, Y\rangle_{g_0} - 2\tilde{\psi}_k\langle X, \xi_2\rangle_{g_0}\langle Y, \xi_2\rangle_{g_0}.$$

where $\tilde{\psi}_k = 2^{-k}\psi$, so $\tilde{g}_k \to g_0$ in C^∞. Since $\{\xi_j\}$ is g_0-orthonormal,

$$\det(\tilde{g}_k) = (1+\tilde{\psi}_k)(1-\tilde{\psi}_k)\det(g_0) = (1-\tilde{\psi}_k^2)\det(g_0) \leq \det(g_0),$$

with strict inequality inside $\operatorname{supp}\psi$. Therefore,

$$\operatorname{Area}(N, \tilde{g}_k) < \operatorname{Area}(N, g_0). \tag{2.3}$$

Suppose now that for all k, $\varphi_0 : (S^1, 2\pi) \to (N, \tilde{g}_k)$ fails to be distance non-decreasing. Then, there exists a sequence of pairs $\{(x'_k, y'_k)\} \subset S^1 \times S^1$ such that if $x_k = \varphi_0(x'_k)$ and $y_k = \varphi_0(y'_k)$,

$$\mathrm{d}_{\tilde{g}_k}(x_k, y_k) < \mathrm{d}_{S^1}(x'_k, y'_k) \le \mathrm{d}_{g_0}(x_k, y_k). \tag{2.4}$$

Since $(N, \tilde{g}_k)$ is a compact manifold with boundary, we can apply Lemma 2.6 to obtain $\gamma_k : [0,1] \to N$ a $\tilde{g}_k$-segment from x_k to y_k. We have that

$$L_{\tilde{g}_k}(\gamma_k) = \mathrm{d}_{\tilde{g}_k}(x_k, y_k) < \mathrm{d}_{S^1}(x'_k, y'_k) \le \mathrm{d}_{g_0}(x_k, y_k) \le L_{g_0}(\gamma_k),$$

and so γ_k must enter $\operatorname{supp}\psi$. Therefore, γ_k enters $N \backslash \partial N$ and is locally $\tilde{g}_k$-minimizing there so if $\gamma_k(t) \in N \backslash \partial N$, then γ_k is a $\tilde{g}_k$-geodesic on $(t-\epsilon, t+\epsilon)$ for some $\epsilon > 0$. Also, γ_k is C^1, piecewise smooth and $\{\gamma_k\}$ have bounded speed in the g_0 metric. Therefore, we can apply Lemma 2.6 once more to get a uniformly convergent subsequence $\gamma_k \to \gamma$. In particular we have that $\gamma_k(0) = x_k \to \gamma(0) = x$, and $\gamma_k(1) = y_k \to \gamma(1) = y$. From the triangle inequality,

$$\mathrm{d}_{g_0}(x, y) < \epsilon + \mathrm{d}_{g_0}(x_k, y_k),$$

for any $\epsilon > 0$ as long as k is sufficiently large. Therefore,

$$\mathrm{d}_{g_0}(x, y) \le \liminf_k \mathrm{d}_{g_0}(x_k, y_k) \le \liminf_k L_{g_0}(\gamma_k).$$

Also by the triangle inequality and by (2.4),

$$\mathrm{d}_{g_0}(x, y) > \mathrm{d}_{\tilde{g}_k}(x_k, y_k) - \epsilon$$

for k sufficiently large. This is true for all $\epsilon > 0$, so we have that

$$\mathrm{d}_{g_0}(x, y) \ge \limsup_k \mathrm{d}_{\tilde{g}_k}(x_k, y_k) = \limsup_k L_{g_0}(\gamma_k)$$

because $\tilde{g}_k \to g_0$ uniformly. Therefore,

$$\limsup_k L_{g_0}(\gamma_k) = \liminf_k L_{g_0}(\gamma_k) = \mathrm{d}_{g_0}(x, y)$$

and also,

$$\lim_{k\to\infty} L_{g_0}(\gamma_k) = L_{g_0}(\gamma)$$

again, because $\gamma_k \to \gamma$ uniformly. Therefore,

$$\mathrm{d}_{g_0}(x, y) = L_{g_0}(\gamma)$$

i.e. γ is a g_0-segment, so it is C^1 and piecewise smooth. Also, it is a g_0-geodesic inside $N \setminus \partial N$. Now, every γ_k intersects the compact set $\operatorname{supp}\psi \subset U$. Therefore, so does γ.

Furthermore, (2.4) along with continuity of the distance function and the fact that,

$$\lim_{k\to\infty} \mathrm{d}_{\tilde{g}_k}(x_k, y_k) = \lim_{k\to\infty} L_{\tilde{g}_k}(\gamma_k) = L_{g_0}(\gamma) = \lim_{k\to\infty} \mathrm{d}_{g_0}(x_k, y_k),$$

imply that $L_{g_0}(\gamma) = \mathrm{d}_{S^1}(x', y')$ where $\varphi_0(x') = x$ and $\varphi_0(y') = y$, so γ is φ_0-taut. This is a contradiction because γ enters U. Therefore, we can choose k sufficiently large so that if $g_1 = \tilde{g}_k$, $\varphi_0 : (S^1, 2\pi) \to (N, g_1)$ is distance non-decreasing.

We proceed inductively and obtain a sequence of metrics $\{g_k\}$ on N, where g_{k+1} stretches g_k along ξ_1 and shrinks along ξ_2 inside $B_\delta(\alpha_1 \cup \alpha_2)$. In particular, we have

$$\langle X, Y\rangle_{g_k} = (1 + \eta\psi_k)\langle X, Y\rangle_{g_0} - 2\eta\psi_k\langle X, \xi_2\rangle_{g_0}\langle Y, \xi_2\rangle_{g_0},$$

where $\{\psi_k\} \subset C^\infty(N, [0, 1))$ is a sequence of functions with the following properties:

(i) They are constant on level sets of the distance to $\alpha_1 \cup \alpha_2$ attaining their suprema on $\alpha_1 \cup \alpha_2$.

(ii) $\operatorname{supp} \psi_k \subset B_\delta(\alpha_1 \cup \alpha_2)$

(iii) $\psi_k(x)$ is non-decreasing with k for all $x \in N$

(iv) $\forall \delta > 0\ \exists\, k_\delta$ such that $\psi_i(x) = \psi_j(x)$ for all $i, j \geq k_\delta$ and for all x with $\mathrm{d}_{g_0}(x, \alpha_1 \cup \alpha_2) > \delta$.

(v) φ_0 is distance non-decreasing into (N, g_k).

We have that $\mathrm{d}_{g_k}(\varphi_0(S^1), \alpha_j)$, $L_{g_k}(\alpha_j)$, and $\mathrm{Area}(N, g_k)$ are all strictly decreasing with k, by (2.1), (2.2), and (2.3), respectively. For all these metrics $\{\alpha_j\}$ are geodesics in $\mathcal{A}$. If $\lim_{k\to\infty} L_{g_k}(\alpha_j) > 0$, then since $L_{g_{k+1}}(\alpha_j) = \sqrt{1 - \eta_k} L_{g_k}(\alpha_j)$, we have that ψ_k converges to a continuous function ψ_∞ which by (iv) is smooth on $N \backslash (\alpha_1 \cup \alpha_2)$. For $\delta > 0$, take a smooth approximation $\tilde{\psi}_\delta \geq \psi_\infty$ constant on level sets of the distance to $\alpha_1 \cup \alpha_2$ with $\sup |\psi_\infty - \tilde{\psi}_\delta| \leq \delta$, so that $\tilde{\psi}_\delta \circ f^{-1}$ is well-defined and smooth, and $\tilde{\psi}_\delta(x) = \psi_\infty(x)$ for all x with either $\mathrm{d}_{g_0}(x, \alpha_1 \cup \alpha_2) > \delta$, or $x \in \alpha_1 \cup \alpha_2$. Define a metric $\tilde{g}_\delta$ by,

$$\langle X, Y \rangle_{\tilde{g}_\delta} = e^{2\lambda} \left((1 + \tilde{\psi}_\delta) \langle X, Y \rangle_{g_0} - 2\tilde{\psi}_\delta \langle X, \xi_2 \rangle_{g_0} \langle Y, \xi_2 \rangle_{g_0} \right),$$

where $\lambda \in C^\infty(N)$ is a non-negative function constant on level sets of the distance to $\alpha_1 \cup \alpha_2$, with $\operatorname{supp} \lambda = \operatorname{supp} \psi_\infty \backslash (\alpha_1 \cup \alpha_2)$, $\lambda \circ f^{-1}$ well-defined and smooth and $\|f\|_{L^2}$ sufficiently small so that $\mathrm{Area}(N, e^{2\lambda} g_\infty) < \mathrm{Area}(M, g)$.

Thus, $L_{\tilde{g}_\delta}(\alpha_j) = L_{g_\infty}$ and $\lim_{\delta\to 0} \|v\|_{\tilde{g}_\delta} \geq \|v\|_{g_\infty}$ with strict inequality for v based at a point in the support of λ. The conformal factor $e^{2\lambda}$ will serve to stretch out lengths of curves so that for $\delta > 0$ sufficiently small, $\varphi_0 : (S_1, 2\pi) \to (N, \tilde{g}_\delta)$ will be distance non-decreasing, yet $\text{Area}(N, \tilde{g}_\delta) < \text{Area}(M, g)$. Also, α_j will be geodesics, minimal in $\mathcal{A}$, and will not intersect $\varphi_0(S^1)$. We can proceed to apply a change of metric once more so that for any $l > 0$, we have $L_{\bar{g}}(\alpha_j) < l$ for some metric $\bar{g}$ such that $\varphi_0 : (S^1, 2\pi) \to (N, \bar{g})$ is distance nondecreasing.

Since α_j are totally geodesic, we may glue in two discs, each with area less than twice that of the round hemisphere of radius l so that the map given by inclusion composed with φ_0 is distance non-decreasing. The area of the two discs can be made arbitrarily small by taking l sufficiently small, so that the resulting manifold $(\tilde{N}, \tilde{g})$ will have $\text{Area}(\tilde{N}, \tilde{g}) < \text{Area}(M, g)$. Therefore, $(\tilde{N}, \tilde{g})$ is a Riemannian manifold, complete as a metric space with strictly smaller area than (M, g) and $\varphi_1 : S^1 \to N$, the map induced by φ_0 is distance non-decreasing, as desired. □

Proposition 2.8 (M. Gromov [17] pg. 59) *Let (M, g) be a Riemannian manifold with boundary, diffeomorphic to a 2-disc such that (M, g, φ) fills $(S^1, 2\pi)$. Then,*

$$\text{Area}(M) \geq 2\pi,$$

with equality if and only if (M, g) is isometric to the hemisphere of constant curvature one.

(Proof)

Let $N = M/\sim$ where $x_1 \sim x_2$ if and only if $x_j = \varphi(x'_j)$ with $x'_j \in S^1$ and $d_{S^1}(x'_1, x'_2) = \pi$. If N is isomorphic to $(\mathbb{R}P^2$ with the canonical metric, we are done. Otherwise, take a sequence of smooth metrics $\{g_k\}$ on M so that the induced metric g'_k on N is smooth, and so that $g_k \to g$ smoothly on compact subsets of $M \backslash \partial M$ with $\|v\|_{g_k} \geq \|v\|_g$ for all $v \in TM$. Then we have,

$$\text{Area}(N, g'_k) = \text{Area}(M, g_k) \to \text{Area}(M, g),$$

and since any non-contractable curve in N must come from a curve in M which hits some antipodal pair, the systole

$$\text{Sys}(N, g'_k) \geq \text{Sys}(N, g') = \inf\{d_g(x, y) | x \sim y\} \geq \pi.$$

Then by the isosystolic inequality of P. M. Pu [24],

$$\text{Area}(M, g_k) = \text{Area}(N, g'_k) \geq \frac{2}{\pi} \text{Sys}(N, g'_k)^2 \geq 2\pi.$$

Since all metrics on the disc are conformally equivalent, there is by Lemma 2.5 below, a unique metric of minimal area which fills S^1 via φ. This must be the metric of constant curvature one. Thus equality is only attained in this case and there is strict inequality if we take k sufficiently large. Therefore, in the limit as $k \to \infty$, we have the desired result. $\square$

Chapter 3

Filling volume of the circle

In this section, all surfaces are assumed to be orientable. The proof is accomplished by direct construction. It is shown, using Lemma 2.4, that given a filling of S^1 by a non-compact manifold there is a filling by a compact one with strictly smaller area. Then, given a filling that is compact, but not simply connected, a manifold of strictly smaller area is constructed using Lemma 2.7. This construction, if repeated finitely many times provides a simply connected filling (i.e. a disc) with strictly smaller area. M. Gromov's proposition (Proposition 2.8) is then applied to show that the object of minimal area is in fact the round hemisphere.

Theorem 3.1 *If (M, g, φ) fills $(S^1, \Phi 2\pi)$ then,* $\mathrm{Area}(M, g) \geq 2\pi$*, with equality if and only if (M, g) is isomorphic to the hemisphere of constant curvature one.*

(Proof)

Suppose first that M is not compact. Let

$$\tilde{A} = \{p \in M \mid \mathrm{d}_g(p, \partial M) > \pi/2\}.$$

By Sard's theorem, there exists $r > \pi/2$ a regular value of the distance to ∂M, and

$$A = \{p \in M \mid \mathrm{d}_g(p, \partial M \geq r\}.$$

We have that

$$\tilde{A} \supset A = \bigcup_{j=1}^{k} A_j,$$

a disjoint union of connected components with ∂A diffeomorphic to $k' \geq k$ copies of S^1. For $\epsilon > 0$, take $\psi_\epsilon : M \to (0, 1]$ a smooth function in $L^1(M)$ such that $\psi_\epsilon(x) \equiv 1$ for $x \in M \setminus A$ and $\psi_\epsilon(x) \leq \epsilon$ for $x \in A_\delta$, where

$$A_\delta = \{x \in A \mid \mathrm{d}_g(x, M \setminus A) \geq \delta\}$$

for $\delta > 0$ chosen sufficiently small so that ∂A_δ is diffeomorphic to a disjoint union of k' copies of S^1. Let

$$g_\epsilon = \psi_\epsilon^2 g.$$

We have that (M, g_ϵ, φ) fills $(S^1, 2\pi)$ because any curve from ∂M to ∂M entering $\{x \mid g_\epsilon(x) \neq g(x)\}$ must have length at least π. Also, it is clear that

$$\mathrm{Area}(M, g_\epsilon) < \mathrm{Area}(M. g)$$

with

$$\lim_{\epsilon \to 0} \mathrm{Area}(M, g_\epsilon) \leq \mathrm{Area}(M \setminus A_\delta, g) < \mathrm{Area}(M, g),$$

and

$$\lim_{\epsilon \to 0} L_{g_\epsilon}(\partial A_j) = 0.$$

For $\epsilon > 0$ sufficiently small, $M \setminus A$ can be capped off with discs of area small enough to make a new compact manifold (M', g') with

$$\mathrm{Area}(M', g') < \mathrm{Area}(M. g)$$

which fills $(S^1, 2\pi)$ via φ' the map induced by φ.

Now we assume without loss of generality that (M, g) is compact. If M is simply connected, then apply Proposition 2.8. Otherwise, consider the following set of curves

$$\mathcal{A} = \{\alpha \in \Omega(M) \mid \alpha \text{ is simple and not contractable in } W = M/\sim\},$$

where $x \sim y$ iff $x, y \in \partial M$, and $\Omega(M)$ is the set of all piecewise C^1 loops in M. If M is not simply connected, then $\mathcal{A}$ is not empty as it contains the generator of the free homotopy group of any handle. Now apply Lemma 2.6 to obtain $\alpha : [0,1] \to M$ realizing the minimal length of all curves in $\mathcal{A}$. Assuming that α does not intersect ∂M, then it must be a simple curve as follows.

There is a sequence $\{\alpha_j\}_{j\in\mathbf{N}} \subset \mathcal{A}$ uniformly convergent to α. Suppose that α is not simple and has a self-intersection inside $M \backslash \partial M$. We may choose a parametrization of α so that it has constant speed and,

$$\alpha(0) = \alpha(1) = \alpha(\tau) = p \notin \partial M$$

with $0 < \tau < 1$. Since $\{\alpha_j\}_{j\in\mathbf{N}}$ are all simple, and curves locally separate sets in dimension two, α is not only self-intersecting at p, but also self-tangent i.e. $\dot\alpha(0) = \pm\dot\alpha(\tau)$. By uniqueness of geodesics, this contradicts the minimality of α. Therefore, α cannot have any self-intersections in the interior of M.

If α does not intersect ∂M, we may apply Lemma 2.7 along α to get a manifold (N_1, g_1) and a distance non-decreasing map $\varphi_1 : (S^1, 2\pi) \to (N_1, g_1)$ (i.e. a filling of $(S^1, 2\pi)$) with

$$\text{Area}(N_1, g_1) < \text{Area}(M, g).$$

If every curve realizing the g-minimal length in $\mathcal{A}$ intersects ∂M, fix one such curve α with a parametrization of constant speed such that $\alpha(0)=\alpha(1)\notin\partial M$ and let

$$\mathcal{J}_\delta=\{t \mid \mathrm{d}_g(\alpha(t),\partial M) < \delta\}$$

Since α must be C^1, $\mathcal{J}_\delta$ is, by continuity of the distance to ∂M, a disjoint union of open, non-empty sub-intervals of $[0,1]$. Take $\alpha_\delta : [0,1] \to M$ such that $\alpha_\delta(t) = \alpha(t)$ for all $t \notin \mathcal{J}_\delta$, and so that $\mathrm{d}_g(\alpha_\delta(t), \partial M) = \delta$ for all $t \in \mathcal{J}_\delta$. For $\delta > 0$ sufficiently small, α_δ can be parametrized to be piecewise smooth and so that $\|\dot\alpha_\delta(t)\| = \|\dot\alpha(t)\| = c$, if $t\notin\mathcal{J}_\delta$, and $\|\dot\alpha_\delta(t)\| = c_\delta$ otherwise, where $c_\delta > 0$ is a constant. Also, $\mathrm{d}_g(\alpha_\delta), \partial M) \geq \delta$, and

$$\alpha = \lim_{\delta\to 0^+} \alpha_\delta,$$

converges uniformly. Therefore, $\alpha_\delta \in \mathcal{A}$ for $\delta > 0$ sufficiently small and furthermore,

$$L_g(\alpha) = \lim_{\delta\to 0^+} L_g(\alpha_\delta).$$

In particular,

$$|L_g(\alpha) - L_g(\alpha_\delta)| \leq A\delta, \tag{3.1}$$

for some $A > 0$ as follows. Let $X(t)$ be a C^1, piecewise smooth unit vector field along α with $\langle X, \dot\alpha\rangle_{g_0} \equiv 0$. For all $\delta > 0$ sufficiently small,

$$\alpha_\delta(t) = \exp_{\alpha(t)}\left(f_\delta(t)X(t)\right)$$

where f_δ is a Lipschitz and piecewise smooth function. By construction, $f_\delta(t) = 0$ for all $t \notin \mathcal{J}_\delta$, and $|f_\delta(t)| \leq \delta$ for all $t \in [0,1]$. Therefore, for some $A > 0$ we have that

$$\|\dot{\alpha}_\delta(t)\|_g - \|\dot{\alpha}(t)\|_g \leq A\delta$$

for almost every t and for $\delta > 0$ sufficiently small and for $t \notin \partial\mathcal{J}_\delta$ (i.e. almost everywhere). Equation (3.1) follows after integrating.

Define $\psi_\delta : M \to [0, \eta]$ by

$$\psi_\delta(x) = \eta f\left(\frac{1}{\delta}\, \mathrm{d}_g(x, \partial M)\right)$$

where f is a smooth, non-increasing function with $f(t) = 1$ for $t \in [0, 1/3]$, and $f(t) = 0$ for $t \geq 1/2$. Then ψ_δ is smooth, has support in $B_{\delta/2}(\partial M)$, and $\psi_\delta = \eta$ for all $x \in B_{\delta/3}(\partial M)$, where $\eta = \sup \psi_\delta$ will be chosen shortly. Let

$$g_\delta = e^{2\psi_\delta} g.$$

This conformal change of metric will serve to push the minimal curves in $\mathcal{A}$ off of ∂M with an easily estimated (and small) increase in area. Note that,

$$L_{g_\delta}(\alpha_\delta) = L_g(\alpha_\delta). \tag{3.2}$$

Equation (3.1) implies that

$$L_g(\alpha) + A\delta \geq L_g(\alpha_\delta) \tag{3.3}$$

Choose η so that

$$A < \frac{2}{3} e^\eta < 2A.$$

By Lemma 2.6, there exists $\beta \in \mathcal{A}$ such that

$$L_{g_\delta}(\beta) = \inf\{L_{g_\delta}(\gamma) \mid \gamma \in \mathcal{A}\}. \tag{3.4}$$

Suppose that $\beta \cap \partial M$ is not empty. Then (3.2), (3.3), the choice of η, and the g-minimality of α give that,

$$\begin{aligned} L_{g_\delta}(\beta) &\geq L_g(\beta) + 2\left(e^{\eta}\frac{\delta}{3}\right) \\ &> L_g(\alpha) + A\delta \\ &\geq L_g(\alpha_\delta) \\ &= L_{g_\delta}(\alpha_\delta) \end{aligned}$$

This is a contradiction of (3.4), so β does not intersect ∂M. Denote by β_δ a g_δ-minimal curve in $\mathcal{A}$. It is a closed, g_δ-geodesic not freely homotopic to ∂M which does not intersect ∂M and therefore (by an argument analogous to the one above for a g-minimal curve of $\mathcal{A}$ which does not intersect ∂M) is simple.

Clearly, $\text{Area}(M, g_\delta) \geq \text{Area}(M, g)$. Also, by choice of η and δ,

$$\begin{aligned} \text{Area}(M, g_\delta) &= \int_M e^{2\psi_\delta} d\mu_g \\ &\leq \text{Area}(M, g) + e^{2\eta}\text{Area}(B_\delta(\partial M), g) \\ &\leq \text{Area}(M, g) + C\delta e^{2\eta} \\ &\leq \text{Area}(M, g) + 9CA^2\delta \end{aligned}$$

for some constant $C > 0$. Thus, $\text{Area}(M, g_\delta)$ can be made arbitrarily close to $\text{Area}(M, g)$.

Furthermore, because for all $v \in TM$, $\|v\|_{g_\delta} \geq \|v\|_g$, we have that (M, g_δ, φ) fills $(S^1, 2\pi)$. Apply Lemma 2.7 along β_δ to obtain a manifold with boundary (N^δ, g'_δ) which fills $(S^1, 2\pi)$ so that

$$\text{Area}(N^\delta, g'_\delta) < \text{Area}(M, g_\delta).$$

For any given $\epsilon > 0$,

$$\text{Area}(M, g) > \text{Area}(M, g_\delta) - \epsilon, \tag{3.5}$$

if we take $\delta > 0$ sufficiently small. We may make an a priori lower estimate $\iota > 0$ so that for all $\delta > 0$ sufficiently small,

$$\text{Area}(M, g_\delta) > \text{Area}(N^\delta, g'_\delta) + \iota. \tag{3.6}$$

Using this estimate we choose δ sufficiently small so that (3.5) holds for $\epsilon < \iota$. Therefore, if we choose $(M_1, g_1) = (N^\delta, g'_\delta)$ for $\delta > 0$ sufficiently small, (3.5) and (3.6) give that $\text{Area}(M_1, g_1) < \text{Area}(M, g)$.

If the above argument is applied once for each handle in M, the resulting manifold (M_m, g_m) is simply connected. If $\text{Area}(M, g) \leq 2\pi$ and M is not simply connected, then $\text{Area}(M_m, g_m) < \text{Area}(M, g) \leq 2\pi$. This contradicts Proposition 2.8. It follows that $\text{Area}(M, g) \geq 2\pi$ with equality if and only if (M, g) is isomorphic to the hemisphere of constant curvature one.

The proof of the existence of the positive lower estimate (3.6) is obtained by cutting out β_δ to obtain a manifold with boundary as before, but this time shrinking only in a small compact set near the boundary. It goes as follows: By the compactness of both M and $[0, 1]$, and the fact that

$$L_g(\beta_\delta) \leq L_{g_\delta}(\beta_\delta) \leq L_{g_\delta}(\alpha) \leq L_g(\alpha),$$

we can apply Lemma 2.6 and can take a sequence

$$\{\beta_j = \beta_{\delta_j}\}_{j \in \mathbb{N}} \subset \{\beta_\delta\}_{\delta > 0}$$

which converges uniformly to β, such that $\delta_j \to 0$ monotonically. Since $\{\beta_j\} \subset \mathcal{A}$ and converge uniformly, β is not trivial in W. Thus there exists $\tau \in [0, 1]$ such that $\beta(\tau), \beta_j(\tau) \notin \partial M$ for all $j \in \mathbb{N}$. For any curve γ, we have by construction that,

$$L_g(\gamma) \leq L_{g_j}(\gamma) \leq L_{g_k}(\gamma). \tag{3.7}$$

for $j \geq k$ (we denote $g_j = g_{\delta_j}$). In particular,

$$L_{g_j}(\beta_j) \leq L_{g_j}(\beta_k) \leq L_{g_k}(\beta_k),$$

by the g_j-minimality of β_j in $\mathcal{A}$. Furthermore,

$$\begin{aligned} \lim_{j\to\infty} L_{g_j}(\beta_j) &\leq \lim_{j\to\infty} L_{g_j}(\alpha_{\delta_j}) \\ &= \lim_{j\to\infty} L_g(\alpha_{\delta_j}) \\ &= L_g(\alpha), \end{aligned}$$

by (3.1). Also, (3.7) and the g-minimality of α in $\mathcal{A}$ imply that for all j,

$$L_{g_j}(\beta_j) \geq L_g(\beta_j) \geq L_g(\alpha).$$

Therefore, $\lim_{j\to\infty} L_{g_j}(\beta_j) = \lim_{j\to\infty} L_g(\beta_j) = L_g(\alpha)$, and the uniform convergence $\beta_j \to \beta$ gives that, $L_g(\beta) = L_g(\alpha)$. Thus, β is a g-minimal element of $\mathcal{A}$ so it is C^1, piecewise smooth and is a g-geodesic outside ∂M. If β has self-intersections, they must be in ∂M. By assumption, β intersects ∂M.

As in the proof of Lemma 2.7, construct manifolds,

$$N_j = M \backslash \beta_j,$$

with boundary

$$\partial N_j = \partial M \amalg \check{\beta}_j \amalg \hat{\beta}_j.$$

where ∂M, $\check{\beta}_j$ and $\hat{\beta}_j$ are disjoint components of ∂N_j. Let $f_j : N_j \to M$ be the map identifying $\check{\beta}_j$ and $\hat{\beta}_j$ with $\beta_j \subset M$. Let $\tilde{g}_j = f_j^* g_j$ and $\bar{g}_j = f_j^* g$ be metrics on N_j. Let $\varphi_j : S^1 \to N_j$ be such that $\varphi = f_j \circ \varphi_j$. Without loss of generality, we may assume that

$$\mathrm{d}_{\tilde{g}_j}(\check{\beta}_j, \partial M) \leq \mathrm{d}_{\tilde{g}_j}(\hat{\beta}_j, \partial M).$$

We will shrink in a neighbourhood of a point near $\hat{\beta}_j$ to obtain the desired estimate. The proof of Lemma 2.7 shows that since (M, g_j, φ) is a filling of $(S^1, 2\pi)$, φ_j is distance non-decreasing. Also, there are no φ_j-taut $\tilde{g}_j$-segments through any point on $\hat{\beta}_j$. By (3.7) this holds for $\bar{g}_j$ as well. In particular, it holds for τ such that $\beta_j(\tau)$ does not converge to a point in ∂M. Furthermore, there is a neighbourhood of $\hat{\beta}_j(\tau)$ which no φ_j-taut $\bar{g}_j$-segment enters. Thus we can take $\rho_j > 0$ sufficiently small so that for all $p \in U_j = \{q \in N_j \,|\, \mathrm{d}_{\bar{g}_j}(q, \hat{\beta}_j(\tau))\}$ there is no φ_j-taut $\bar{g}_j$-segment through p. Now, suppose that $\rho_\infty = \inf\{\rho_j\} = 0$. If γ is φ_j-taut in $\bar{g}_j$ for some j, then $f_j \circ \gamma$ is φ-taut in g_j and

$$L_{\bar{g}_j}(\gamma) = L_g(f_j(\gamma)) \leq L_{g_j}(f_j(\gamma)) = \mathrm{d}_{S^1}(x_j, y_j).$$

The inequality is strict because $f_j \circ \gamma$ enters $\operatorname{supp} \psi_j = B_{\delta_j/2}(\partial M)$. Since (M, g, φ) fills S^1, this is impossible, therefore, $\rho_\infty > 0$.

Construct a metric space

$$V = (M \backslash \beta) \amalg \left[\varphi(S^1) \cap \beta\right] \amalg \check{\beta} \amalg \hat{\beta}$$

as a disjoint union. The boundary of V is a union $\partial V = (\partial M \backslash \beta) \amalg \check{\beta} \amalg \hat{\beta}$ where $(\partial M \backslash \beta) \amalg \check{\beta}$ and $\hat{\beta}$ are the connected components of the boundary. Let $f : V \to M$ be the natural map with

$$f(\check{\beta}(t)) = f(\hat{\beta}(t)) = \beta(t)$$

for all $t \in [0, 1]$. The distance on V is that of the length space induced by f and is denoted by d_V. The map $\Phi : S^1 \to V$ such that $\varphi = f \circ \Phi$ is distance non-decreasing. There is an open neighbourhood U of $\hat{\beta}(\tau)$, diffeomorphic to $\mathbb{R}^2_+ \cap B_1(0)$ which is a Riemannian manifold with boundary $\partial U = \hat{\beta} \cap U$, and metric $h = f^*|_U g$. Since β does not lie entirely inside ∂M and is a geodesic in $M \backslash \partial M$, $\hat{\beta}(t)$ is a geodesic of the Riemannian manifold (U, h) for all t near τ. Therefore, there can be no Φ-taut curve $\gamma : [0, 1] \to V$ through $\hat{\beta}(\tau)$. This implies that there is a neighbourhood

$$U' = \{q \in U \mid \mathrm{d}_h(q, \hat{\beta}(\tau)) < \rho'\}$$

with no Φ-taut curves through U'. Let

$$A_j = \{q \in M \mid \mathrm{d}_{g_j}(q, \beta_j(\tau)) \leq \rho',\ \mathrm{d}_{g_j}(q, \partial M) \geq \mathrm{d}_g(\beta_j(\tau), \partial M)\}.$$

Take

$$\rho = \frac{1}{2} \min\{\rho', \rho_\infty, \mathrm{inj}(A_j, g_j)\} \tag{3.8}$$

where the minimum, taken over all j, is non-zero because the A_j are compact and all are a bounded distance from ∂M. Let

$$r_j = \inf \Big\{ \mathrm{d}_{\bar{g}_j}(\varphi_j(x), q) + \mathrm{d}_{\bar{g}_j}(q, \varphi_j(y)) - \mathrm{d}_{S^1}(x, y) \Big| \mathrm{d}_{\bar{g}_j}(q, \hat{\beta}(\tau)) \leq \rho,\ x, y \in S^1 \Big\}. \tag{3.9}$$

We claim that

$$r = \liminf r_j > 0. \tag{3.10}$$

Suppose that $r = 0$. Then for all j there exists a C^1-curve $\tilde{\gamma}_j : [0,1] \to N_j$ parametrized to have constant speed with

$$\begin{aligned} \tilde{\gamma}_j(0) &= \varphi_j(x_j), \\ \tilde{\gamma}_j(1) &= \varphi_j(y_j), \\ \text{and } \tilde{\gamma}_j(s_j) &= q_j, \end{aligned}$$

for $x_j, y_j \in S^1$, $\mathrm{d}_{\tilde{g}_j}(q_j, \beta(\tau)) \leq \rho$, $s_j \in (0,1)$, $\tilde{\gamma}_j$ a $\tilde{g}_j$-segment and

$$\lim_{j\to\infty} \left[L_{\tilde{g}_j}(\tilde{\gamma}_j) - \mathrm{d}_{S^1}(x_j, y_j) \right] = 0.$$

If we let $\gamma_j = f_j \circ \tilde{\gamma}_j : [0,1] \to M$, then

$$\lim_{j\to\infty} \left[L_{g_j}(\gamma_j) - \mathrm{d}_{S^1}(x_j, y_j) \right] = 0.$$

By compactness of M, S^1, and $[0,1]$, and the fact that

$$\limsup L_g(\gamma_j) \leq \mathrm{diam}(S^1) + \epsilon$$

for some $\epsilon > 0$, applying Lemma 2.6, we may pass to a subsequence $\{\gamma_j\}_{j\in\mathcal{N}}$ which converges uniformly to γ with, $\gamma(0) = \varphi(x)$, $\gamma(0) = \varphi(y)$, and $\gamma(s) = q$, where $x_j \to x$, $y_j \to y \in S^1$, $s_j \to s \in [0,1]$, and $q_j \to q \in M$ with $\mathrm{d}_g(q, \beta(\tau)) \leq \rho$. Then, since $g_j \to g$ in measure, $\|v\|_{g_j} \to \|v\|_g$ for $v \in T_pM$ where $p \in M \setminus \partial M$ and $\|v\|_{g_j}$ is bounded for $p \in \partial M$,

$$\begin{aligned} \lim_{j\to\infty} L_{g_j}(\gamma_j) &= \lim_{j\to\infty} L_{g_j}(\gamma) \\ &= L_g(\gamma). \end{aligned}$$

Therefore, $L_g(\gamma) = \mathrm{d}_{S^1}(x, y)$. Since $\gamma_j \to \gamma$ uniformly and γ_j lifts to a continuous curve $\tilde{\gamma}_j$ in N_j, (i.e. $\gamma_j = f_j \circ \tilde{\gamma}_j$), γ must lift to a continuous curve $\tilde{\gamma}$ in V with $L_V(\tilde{\gamma}) = L_g(\gamma) = \mathrm{d}_{S^1}(x, y)$ because γ cannot be transversal to β when no γ_j intersects β. This contradicts the choice of ρ in (3.8), showing that $r > 0$.

Now we consider the sequence of Riemannian manifolds with boundary

$$\left\{(N_j, \tilde{g}_j, \varphi_j(\partial M) \amalg \tilde{\beta}_j \amalg \hat{\beta}_j\right\}_{j \in \mathbb{N}}$$

and obtain new metrics $\tilde{g}_j'$ on N_j as in Lemma 2.7. but only changing $\tilde{g}_j$ inside $\{q \in N_j \mid \mathrm{d}_{\tilde{g}_j}(q, \hat{\beta}_j(\tau)) < R\}$ with

$$R = \frac{1}{2} \min\{\rho, r/2\}.$$

As in the proof of Lemma 2.7, we have that $\varphi_j : S^1 \to (N_j, \tilde{g}_j')$ is distance non-decreasing and

$$\mathrm{Area}(N_j, \tilde{g}_j') < \mathrm{Area}(N_j, \tilde{g}_j).$$

Let

$$\iota_j = \mathrm{Area}(N_j, \tilde{g}_j) - \mathrm{Area}(N_j, \tilde{g}_j').$$

Since $\beta_j(\tau) \to \beta(\tau)$, and $r > 0$, we can choose $\tilde{g}_j'$ so that

$$\iota = \frac{1}{2} \liminf(\iota_j) > 0.$$

Take j sufficiently large so that

$$\mathrm{Area}(M, g_j) < \mathrm{Area}(M, g) + \iota.$$

This gives the desired a priori estimate and proves the theorem. □

Remark 3.2

If we take the round hemisphere of unit radius, embed $(S^1, 2\pi)$ as the equator, and stretch the metric slightly at the dome so that there is a compact set K of non-zero measure with $d(K, \partial M) > \pi/2$, then we can attach any number of handles or spikes to K, so that the total area is arbitrarily close to 2π. This construction shows that for any manifold M with boundary diffeomorphic to S^1, there is a sequence $\{g_k\}$ of smooth Riemannian metrics on M and a smooth, distance non-decreasing map $\varphi: (S^1, 2\pi) \to (M, g_k)$ such that (M, g_k) is complete as a metric space and

$$\lim_{k \to \infty} \text{Area}(M, g_k) = 2\pi.$$

Therefore, the infimum of the areas of (M, g, φ) which fill $(S^1, 2\pi)$ is the same regardless of the topology of M. This extends a result Gromov proved for dimensions two and higher (Proposition 2.2.A in [17]).

Chapter 4

Applications

Theorem 4.1 *If (M, g) is any orientable, Riemannian 2-manifold, $p \in M$ and $r \leq \operatorname{conv}(p)$ then,*

$$\operatorname{Area}(M, g) - \operatorname{Area}(B_r(p), g) \geq \frac{8r^2}{\pi}$$

with equality for all p if and only if (M, g) is isometric to a sphere of constant curvature κ and $r = \pi/2\sqrt{\kappa}$.

(Proof)

If $r < \operatorname{conv}_p$, then $B_r(p)$ is strictly convex and the length of its boundary is at least $4r$. Fix a diffeomorphism $\varphi : S^1 \to \partial B_r(p)$ with constant speed $||\dot{\varphi}||_g = L_g(\partial B_r(p))$. If $N = M \setminus B_r(p)$, then (N, g, φ) fills the circle of length $4r$ as follows. Fix $x, y \in S^1$ and apply Lemma 2.6 to obtain $\beta : [0, 1] \to N$ realizing the distance between $\varphi(x)$ and $\varphi(y)$ in N. If the image of β is not in ∂N, then there exist $z_1, z_2 \in S^1$ and $0 < t_1 < t_2 < 1$ with $\varphi(z_j) = \beta(t_j)$ so that $\beta|_{(t_1,t_2)} \subset N \setminus \partial N$. By the strict convexity of $B_r(p) \subset M$, $\beta|_{(t_1,t_2)}$ is not minimal in M and therefore not minimal in N. Whence there is a curve in N

of strictly shorter length from $\beta(t_1)$ to $\beta(t_2)$. This contradicts the minimality of β. Therefore, all minimal curves in N between points on ∂N lie in ∂N and so (N, g, φ) must be a filling of $(S^1, 4r)$. It follows that

$$\begin{aligned}\text{Area}(M, g) - \text{Area}(B_r(p), g) &= \text{Area}(N, g) \\ &\geq \text{FillVol}(S^1, 4r) \\ &= \text{FillVol}(S^1, 2\pi)(2r/\pi)^2.\end{aligned}$$

By continuity the inequality holds for $r = \text{conv}(p)$. We have equality if and only if $M \backslash B_r(p)$ is isomorphic to the hemisphere of constant curvature $(\pi/2r)^2$. If equality holds for every $p \in M$, then M must be a sphere of constant curvature.

□

Using the Bishop-Gunther volume comparison theorem [16], we immediately have the following

Corollary 4.2 *If (M, g) is a compact, orientable 2-manifold, $r \leq \text{conv}(p)$ for some $p \in M$, and the curvature is bounded above by κ on $B_r(p)$, then Cheeger's constant*

$$h \leq \frac{L(\partial B)}{\min\{\nu(r, \kappa), 8r^2/\pi\}}$$

where $\nu(r, \kappa)$ is the volume of the ball of radius r in the simply connected surface of constant curvature κ.

Theorem 4.3 *If M is a compact Riemannian surface, $r = \text{conv}(M)$, and the curvature of (M, g) is bounded above by κ, then the first non-zero eigenvalue* *of the Laplacian, λ_1, satisfies*

$$\lambda_1 \leq \frac{\pi^2\,[\mathrm{Area}(M) - \tau(\kappa,r)] - 8\pi r^2}{4r^2\tau(\kappa,r)}.$$

$$\text{where}\quad \tau(\kappa,r) = \int_{B(r)} \cos^2\left(\pi \mathrm{d}_\kappa(x,q)/2r\right) d\mu_\kappa(x)$$

$$= \frac{\pi}{\pi^2 - \kappa r^2}\begin{cases} \pi^2(1-\cos(r\sqrt{\kappa}))\kappa^{-1} - 2r^2 & \kappa > 0 \\ (\pi^2-4)r^2/2 & \kappa = 0 \\ \pi^2(1-\cosh(r\sqrt{-\kappa}))\kappa^{-1} - 2r^2 & \kappa < 0 \end{cases}$$

and $B(r)$ is any ball of radius r in the simply connected space of constant curvature κ with canonical measure $d\mu_\kappa$.

Equality is satisfied if M is a sphere of constant curvature κ.

Remark 4.4

In the case that M is complete but not necessarily compact, if one can find two points p_j and two radii r_j with $B_{r_j}(p_j)$ convex and disjoint such that $\psi(p_1,r_1) = \psi(p_2,r_2)$ (see (4.1) below for a definition of ψ), then the estimate (4.3) below is possibly sharper than the one in the theorem. In particular, if (M,g) has a nontrivial isometry group, or even has disjoint open subsets isometric to one another, then one can choose convex, isometric metric balls and apply (4.3).

(Proof of Theorem 4.3)

Let p_1, p_2 be any distinct points in M and

$$\psi(p,r) = \int_{B_r(p)} F\left(\mathrm{d}(x,p), r\right) d\mu_g(x), \tag{4.1}$$

where $F(t,r) = \cos(\pi t/2r)$. Take $r_j \leq \mathrm{conv}(p_j)$ such that $B_{r_1}(p_1)$ and $B_{r_2}(p_2)$ are disjoint. Without loss of generality, we can assume that $\psi(p_1, r_1) \leq \psi(p_2, r_2)$. Now, for $r < \mathrm{inj}(p)$, $\psi(p,r)$ is continuous and strictly decreasing as a function of r, and $\psi(p,r) \to 0$ as $r \to 0$. Therefore, we can take $r_2 > 0$ sufficiently small so that $\psi(p_1, r_1) = \psi(p_2, r_2)$. As in [3, 11], it is natural to use a test function that approximates an eigenfunction (with lowest non-zero eigenvalue) of the Laplacian on spheres of constant curvature. Let

$$f(x) = F\left(\mathrm{d}(x,p_1), r_1\right) \chi(p_1, r_1) - F\left(\mathrm{d}(x,p_2), r_2\right) \chi(p_2, r_2),$$

where $\chi(p_j, r_j)$ is the characteristic function of $B_{r_j}(p_j)$. We have that $\int_M f = 0$ and

$$\begin{aligned} \|df\|^2(x) &= \frac{\pi^2}{4r_1^2}\left[1 - F^2(\mathrm{d}(x,p_1), r_1)\right] \chi(p_1, r_1) \\ &+ \frac{\pi^2}{4r_2^2}\left[1 - F^2(\mathrm{d}(x,p_2), r_2)\right] \chi(p_2, r_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_M \|df\|^2 &\leq \frac{\pi^2}{4r_1^2}\left[\mathrm{Area}(B_{r_1}(p_1)) - \int_{B_{r_1}(p_1)} F^2(\mathrm{d}(x,p_1), r_1) d\mu_g(x)\right] \\ &+ \frac{\pi^2}{4r_2^2}\left[\mathrm{Area}(B_{r_2}(p_2)) - \int_{B_{r_2}(p_2)} F^2(\mathrm{d}(x,p_2), r_2) d\mu_g(x)\right] \\ &\leq \frac{\pi^2}{4}\left[\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\mathrm{Area}(M) - \frac{16}{\pi} - \frac{1}{r_1^2}\tau(\kappa, r_1) - \frac{1}{r_2^2}\tau(\kappa, r_2)\right]. \end{aligned}$$

The last inequality uses (Theorem 4.1) as well as the following fact. If we let d_κ be the distance function on Q^2_κ (the simply connected, two dimensional space of constant curvature κ), then, using the comparison of the volume form as expressed in normal coordinates (Lemma 4.5) we have that,

$$\int_{B_r(p)} F^2(\mathrm{d}(x,p),r)d\mu_g(x) \geq \int_{B(r,\kappa)} F^2(\mathrm{d}_\kappa(x,p'),r)d\mu_g(x) = \tau(\kappa,r), \qquad (4.2)$$

where $B(r,\kappa) = B_r(p') \subset Q^2_\kappa$ and $p' \in Q^2_\kappa$. Furthermore,

$$\begin{aligned}\int_M f^2 &= \int_{B_{r_1}(p_1)} F^2(\mathrm{d}(x,p_1),r_1)d\mu_g(x) + \int_{B_{r_2}(p_2)} F^2(\mathrm{d}(x,p_2),r_2)d\mu_g(x)\\ &\geq \tau(\kappa,r_1)+\tau(\kappa,r_2),\end{aligned}$$

also by (4.2). Since f is Lipshitz and has compact support, it is in $H^1_0(M)$, and by the minimax principle we have that

$$\lambda_1 \leq \frac{\pi^2(r_1^2+r_2^2)\mathrm{Area}(M) - 16\pi r_1^2 r_2^2 - \pi^2 r_2^2\tau(\kappa,r_1) - \pi^2 r_1^2\tau(\kappa,r_2)}{4r_1^2r_2^2[\tau(\kappa,r_1)+\tau(\kappa,r_2)]}. \qquad (4.3)$$

Let $r < \mathrm{conv}(M)$ and $p,q \in M$ be points satisfying $d(p,q) \geq 2r$ such that $\psi(p,r) = \psi(q,r)$. If two such points exist, then we can apply (4.3) with $r_1 = r_2 = r$ and the result follows immediately.

Now we show the existence of such $p,q \in M$. Let $V_p = M \setminus B_{2r}(p)$. Since $r < \mathrm{conv}(M)$, it follows that $2r < \mathrm{inj}(M)$ and therefore V_p has non-empty interior for all $p \in M$. Suppose that for all p, $\psi(p,r) \notin \psi(V_p,r)$. Consider the sets

$$\begin{aligned} U_+ &= \{p \in M \mid \psi(p,r) > \psi(V_p,r)\},\\ U_- &= \{p \in M \mid \psi(p,r) < \psi(V_p,r)\},\\ U_0 &= \{p \in M \mid \exists q,q' \in V_p \text{ s.t. } \psi(q,r) < \psi(p,r) < \psi(q',r)\}.\end{aligned}$$

These sets are disjoint and their union is M. They are also open, as follows. Fix $p \in M$ and let $\{p_j\}$ be a sequence converging to p. Take q in the interior of V_p. Then for some $\delta > 0$, $\mathrm{d}(p,q) = 2r + 2\delta$, and for all j sufficiently large, $\mathrm{d}(p_j,q) > 2r+\delta$. Also, if $\psi(q,r) > \psi(p,r)$, and we take j large enough, then by the continuity of ψ, $\psi(q,r) > \psi(p_j,r)$. Similarly, if $\psi(q,r) < \psi(p,r)$, then $\psi(q,r) < \psi(p_j,r)$ for j sufficiently large. It follows that the sets U_+, U_- and U_0 are open, which contradicts the connectedness of M.

Therefore, the estimate holds for all $r < \mathrm{conv}(M)$. By continuity the result then holds for $r = \mathrm{conv}(M)$. One can easily see that the first eigenvalue for the sphere of constant curvature κ, $\lambda_1 = 2\kappa$, realizes equality. □

Theorem 4.3 appeals to a volume comparison lemma. The proof follows that of the Bishop-Gunther comparison theorem given in [16]. Let (M^n, g) be a Riemannian manifold with sectional curvature bounded above by κ. Consider spherical normal coordinates at $p \in M$. Denote by

$$J(u,t)dx_1 \wedge \cdots \wedge dx_n = (\exp_p^*)_{tu} d\mu_g$$

the volume form in these coordinates and by $\tilde{J}$ that for $\tilde{p} \in Q^n_\kappa$, the simply connected, n-dimensional space of constant curvature κ.

Lemma 4.5 *With the notation given above,*

$$J(u,t) \geq \tilde{J}(\tilde{u},t).$$

for all unit vectors $u \in T_pM$, $\tilde{u} \in T_{\tilde{p}}Q^n_\kappa$ *and all* $t < \min\{\mathrm{inj}(p), \mathrm{inj}(\tilde{p})\}$.

(Proof)

Fix $p \in M$ and take $r < \min\{\text{inj}(p), \text{inj}(\tilde{p})\}$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and a point $q = \exp_p(ru)$ with $u = e_1$ a unit vector. Let $\alpha(t) = \exp_p(tu)$, a normal geodesic from p to q, and $\{E_1, \ldots, E_n\}$ a parallel frame along α with $E_j(0) = e_j$. Let $\tilde{\alpha}$, and $\{\tilde{E}_1, \ldots, \tilde{E}_n\}$ be given by similar constructions in Q^n_κ. Fix Jacobi fields $\{Y_1, \ldots, Y_n\}$ along α with $Y_j = y_jE_j$, $y_j(0) = 0$, and $y_j(r) = 1$. Then we have that

$$(exp_p)_*(tu)(e_j) = \frac{1}{t}Y_j(t)$$

for $j = 2, \ldots, n$, and

$$(\exp_p)_*(tu)(e_1) = \alpha'(t).$$

Therefore, in these coordinates,

$$\begin{aligned} J(u,t) &= \sqrt{\det \left\langle \frac{1}{t|y_i'(0)|}Y_i(t), \frac{1}{t|y_j'(0)|}Y_j(t) \right\rangle_g} \\ &= \frac{1}{ct^{(n-1)}}\sqrt{\det \langle Y_i(t), Y_j(t)\rangle_g}, \end{aligned}$$

where $c = |y_2'(0) \cdots y_n'(0)|$. Let $\mathcal{D} = \det\langle Y_i, Y_j\rangle_g = c^2t^{2n-2}J^2$.

$$\begin{aligned} \frac{d\mathcal{D}}{dt} &= c^2(2n-2)t^{2n-3}J^2 + c^2t^{2n-2}2J\frac{dJ}{dt} \\ &= c^2t^{2n-2}J^2\left[\frac{2n-2}{t} + \frac{2J'}{J}\right] \\ &= \mathcal{D}\left[\frac{2n-2}{t} + \frac{2J'}{J}\right], \end{aligned}$$

therefore, since at $t = r$, $\mathcal{D}(r) = 1$,

$$\mathcal{D}'(r) = \frac{2n-2}{r} + \frac{2J'(r)}{J(r)}. \tag{4.4}$$

The same calculation can be repeated in Q^n_κ and so

$$\tilde{\mathcal{D}}'(r) = \frac{2n-2}{r} + \frac{2\tilde{J}'(r)}{\tilde{J}(r)}. \tag{4.5}$$

Furthermore, at $t = r$, $\langle Y_i(r), Y_j(r)\rangle_g = I_{n-1}$ so,

$$\begin{aligned}
\frac{d\mathcal{D}}{dt}(r) &= \det\langle Y_i(r), Y_j(r)\rangle_g \operatorname{trace}\left[\langle Y_i(r), Y_j(r)\rangle_g^{-1} \frac{d}{dt}\langle Y_i(r), Y_j(r)\rangle_g\right] \\
&= \operatorname{trace}\left[\frac{d}{dt}\langle Y_i(r), Y_j(r)\rangle_g\right] \\
&= 2\sum_{j=2}^{n} \langle Y_j(r), Y_j'(r)\rangle_g \\
&= \sum_{j=2}^{n} \int_0^r \langle Y_j', Y_j'\rangle_g - R(c', Y_j, c', Y_j) \\
&= \sum_{j=2}^{n} I(Y_j, Y_j) \qquad (4.6)
\end{aligned}$$

Also, let $\{\tilde{Y}_1, \ldots, \tilde{Y}_n\}$ be the Jacobi fields along $\tilde{\alpha}$ with $\tilde{Y}_j(r) = \tilde{E}_j(r)$ and $\tilde{Y}_j(0) = 0$. Then a similar calculation shows that

$$\frac{d\tilde{\mathcal{D}}}{dt}(r) = \sum_{j=2}^{n} I(\tilde{Y}_j, \tilde{Y}_j) \qquad (4.7)$$

Now if we consider the vector fields $X_j(t) = y_j(t)\tilde{E}_j(t)$ along $\tilde{\alpha}$, we see that for $j = 1, \ldots, n$

$$\begin{aligned}
I(Y_j, Y_j) &= \int_0^r \langle Y_j', Y_j'\rangle_g - R(\alpha', Y_j, \alpha', Y_j) \\
&= \int_0^r \langle y_j' E_j, y_j' E_j\rangle_g - y_j^2 R(\alpha', E_j, \alpha', E_j) \\
&\geq \int_0^r (y_j')^2 - y_j^2 \kappa \\
&= \int_0^r \langle y_j' \tilde{E}_j, y_j' \tilde{E}_j\rangle_\kappa - y_j^2 \kappa \\
&= I(X_j, X_j) \\
&\geq I(\tilde{Y}_j, \tilde{Y}_j) \qquad (4.8)
\end{aligned}$$

by the index theorem (see [16] Lemma 3.103) since $X_j(0) = \tilde{Y}_j(0) = 0$, $X_j(r) = \tilde{Y}_j(r) = \tilde{E}_j(r)$ and $\tilde{Y}_j$ is a Jacobi field along $\tilde{\alpha}$. Finally, we apply

(4.8),(4.4),(4.4),(4.6), (4.7) and obtain

$$\frac{J'(r)}{J(r)} \geq \frac{\bar{J}'(r)}{\bar{J}(r)}. \tag{4.9}$$

Since this is true for all $r < \min\{\mathrm{inj}(p),\ \mathrm{inj}(\bar{p})\}$. we may integrate 4.9 and obtain the desired result.

□

Finally, we have the following result which is a corollary of Lemma 2.4.

Corollary 4.6 *Let (V, h) be a compact, Riemannian n-manifold. Then unless (V, h) is null cobordant, there does not exist a smooth filling (M, g, φ) of (V, h) with*

$$\mathrm{Vol}(M, g) = \mathrm{FillVol}(V, h).$$

(Proof)

If (M, g) is complete but not compact, there exists an open set $U \subset M$ with

$$\mathrm{d}_g(U, \partial M) > \frac{1}{2}\mathrm{diam}(V, h).$$

Apply Lemma 2.4 to this set U. This implies that no such (M, g) can minimally fill (V, h).

□

Chapter 5

Generalized filling problems

The problem of finding a minimal filling can be generalized to the following question. Given a manifold X with boundary and a continuous function

$$\rho: \partial X \times \partial X \to \mathbb{R}$$

what is the minimal volume of all smooth metrics g on X such that

$$\mathrm{d}_g(x, y) \geq \rho(x, y) \qquad \text{for all } x, y \in \partial X,$$

and how can one characterize the minimizing metric.

One motivation for asking such a question is to consider (M, g, φ), a filling of (V, h) a compact Riemannian n-manifold, and take $U \subset M$ open with smooth boundary. Define a function ρ on $\partial U \times \partial U$ by

$$\rho(p, q) = \sup\{\mathrm{d}_h(x, y) - \mathrm{d}_g(x, p) - \mathrm{d}_g(y, q) \mid x, y \in \partial M\}$$

The constraint that (M, g, φ) be a filling for (V, h) induces a more general filling problem on $X = \bar{U}$.

Claim 5.1 *For any metric $\tilde{g}$ on M such that*

$$\begin{aligned} \tilde{g}|_{M\setminus U} &= g|_{M\setminus U} \\ \text{and } \mathrm{d}_{\tilde{g}}(p,q) &\geq \rho(p,q) \end{aligned}$$

for all $p,q \in \partial U$, $(M,\tilde{g})$ also fills V.

(Proof)

Take any $\alpha : [0,1] \to M$ piecewise smooth with $\alpha(0) = x, \alpha(1) = y \in \partial M$. Let

$$\begin{aligned} t_0 &= \inf\{t|\alpha(t) \in U\}. \\ t_1 &= \sup\{t|\alpha(t) \in U\}. \end{aligned}$$

and $p{=}\alpha(t_0)$, $q{=}\alpha(t_1) \in \partial U$. We have that

$$\begin{aligned} L_{\tilde{g}}(\alpha) &= L_g(\alpha|_{[0,t_0]}) + L_{\tilde{g}}(\alpha|_{[t_0,t_1]}) + L_g(\alpha|_{[t_1,1]}) \\ &\geq \mathrm{d}_g(x,p) + \mathrm{d}_{\tilde{g}}(p,q) + \mathrm{d}_g(q,y) \\ &\geq \mathrm{d}_h(x,y) - \rho(p,q) + \mathrm{d}_{\tilde{g}}(p,q) \\ &\geq \mathrm{d}_h(x,y), \end{aligned}$$

as desired.

□

Claim 5.2 *If we take ρ as above, then for any $p, p', q, q' \in \partial U$,*

1. $\rho(p,q) = \rho(q,p)$
2. $\rho(p,q) \leq \mathrm{d}_g(p,q)$

and 3. $|\rho(p,q) - \rho(p',q')| \leq \mathrm{d}_g(p,p') + \mathrm{d}_g(q,q')$

(Proof)

1. This is clear from the definition of ρ.

2. By the triangle inequality,

$$d_g(x,p) + d_g(p,q) + d_g(q,y) \geq d_g(x,y) \geq d_h(x,y).$$

   Therefore, for every $x, y \in \partial M$,

$$d_g(p,q) \geq d_h(x,y) - d_g(x,p) - d_g(y,q)$$

   as claimed.

3. Since ∂M is compact, we can choose $x, y \in \partial M$ such that

$$\rho(p,q) = d_h(x,y) - d_g(x,p) - d_g(y,q)$$

   Then,

$$\begin{aligned}
\rho(p',q') - \rho(p,q) &\geq d_h(x,y) - d_g(x,p') - d_g(y,q') \\
&\quad - d_h(x,y) + d_g(x,p) + d_g(y,q) \\
&\geq -d_g(x,p') + d_g(x,p) - d_g(y,q') + d_g(y,q) \\
&\geq -d_g(p,p') - d_g(q,q')
\end{aligned}$$

   Similarly,

$$\rho(p,q) - \rho(p',q') \geq -\left[d_g(p,p') + d_g(q,q')\right].$$

   Also,

$$\begin{aligned}
\rho(p,q) - \rho(p',q') &\leq d_h(x,y) - d_g(x,p') - d_g(y,q') \\
&\quad - d_h(x,y) + d_g(x,p) + d_g(y,q) \\
&\leq d_g(p,p') + d_g(q,q')
\end{aligned}$$

and the claim follows.

□

Now we return to the more general question. Let M be an $(n+1)$-dimensional manifold-with-boundary, ρ a symmetric, continuous function on $\partial M \times \partial M$, and we consider metrics g on M such that

$$\mathrm{d}_g(p,q) \geq \rho(p,q).$$

What is the minimal volume of such metrics? As before, we will denote

$$\mathcal{P} = \left\{ \gamma \in \mathcal{C}^1([0,1], M) \;\middle|\; \gamma(0), \gamma(1) \in \partial M \right\}.$$

Define

$$\mathcal{F}_g = \left\{ f \in \mathcal{C}^\infty(M, \mathbb{R}^+) \mid \forall \gamma \in \mathcal{P}, \int_0^1 f(\gamma) \|\dot{\gamma}\| dt \geq \rho(\gamma(0), \gamma(1)) \right\}$$

Lemma 5.3 *If there exists a function $f_0 \in \mathcal{F}_g$ such that*

$$\|f_0\|_{L^n} = \inf_{f \in \mathcal{F}_g} \|f\|_{L^n}$$

then this function is unique and so each conjugacy class of M has at most one weak solution to the minimal volume problem.

(Proof)

First we will show that $\mathcal{F}_g$ as defined above is convex.

Let $f_0, f_1 \in \mathcal{F}_g$, and $f_t = t f_1 + (1-t) f_0$. Clearly f_t is $\mathcal{C}^\infty$ and positive. Also, if $\gamma \in \mathcal{C}^1([0,1], M)$ with $\gamma(0), \gamma(1) \in \partial M$, then,

$$\begin{aligned} \int_0^1 f_t(\gamma)\|\dot{\gamma}\| dt &= t \int_0^1 f_1(\gamma)\|\dot{\gamma}\| dt + (1-t) \int_0^1 f_0(\gamma)\|\dot{\gamma}\| dt \\ &\geq t\rho(\gamma(0), \gamma(1)) + (1-t)\rho(\gamma(0), \gamma(1)) \\ &= \rho(\gamma(0), \gamma(1)), \end{aligned}$$

so $\mathcal{F}_g$ is convex.

Now take two sequences $\{f_j\}, \{\tilde{f}_j\} \subset \mathcal{F}$ which have decreasing L^n-norm. Let

$$f_j^t = tf_j + (1-t)\tilde{f}_j.$$

We have that $f_j \to f$ and $\tilde{f}_j \to \tilde{f}$ in L^n. Suppose that both sequences are minimizing, i.e.

$$||f||_{L^n} = ||\tilde{f}||_{L^n} = \inf_{F \in \mathcal{F}} ||F||_{L^n} = a.$$

Given $\epsilon > 0$, we have that for j sufficiently large

$$||f_j^t||_{L^n} \leq t||f_j||_{L^n} + (1-t)||\tilde{f}_j||_{L^n} \leq a + \epsilon.$$

Also, because $f_j^t \in \mathcal{F}$, $||f_j^t||_{L^n} \geq a$. Therefore, in the limit as $j \to \infty$, for all $0 \leq t \leq 1$,

$$\lim_{j \to \infty} ||f_j^t||_{L^n}^n = \int_M (tf + (1-t)\tilde{f})^n d\text{Vol}_g = a^n,$$

which implies that $f = \tilde{f}$ a.e. □

The following lemma uses a non-conformal change of metric to show that either at each point in the interior of M every tangent vector is tangent to a taut curve, or else there is a metric of smaller volume which satisfies the condition on the distance between points of the boundary. This is a strengthening of Lemma 2.4.

Theorem 5.4 *If there exists $p \in M \setminus \partial M$ and $v_0 \in T_pM$ a unit vector such that there are no taut curves through p tangent to v_0, then there exists a metric $\tilde{g}$ under which M is complete as a metric space with*

$$\text{Vol}(M, \tilde{g}) < \text{Vol}(M, g)$$

for which

$$\mathrm{d}_{\tilde{g}}(x, y) \geq \rho(x, y)$$

is distance non-decreasing.

(Proof)

The proof is by direct construction of the new metric. Note that we may assume that (M, g) is compact as otherwise, there is a point with no minimal geodesics through it that go from ∂M to ∂M.

We will denote by $\mathcal{T} \subset \mathcal{C}^1([0,1], M)$ the subset of all $\mathcal{C}^1$ curves with constant speed whose endpoints are in ∂M.

If such a v exists, then there exists an open set $U \subset M \setminus \partial M$ with a unit coordinate field ξ on U such that for any $q \in U$ there is some $\theta \in (0, 1)$ such that if $\gamma \in \mathcal{T}$, $\gamma(\tau) = q$ and,

$$|\langle \dot{\gamma}(\tau), \xi \rangle_{g_0}| > \theta \|\dot{\gamma}\|_g$$

then,

$$L_g(\gamma) > \rho(\gamma(0), \gamma(1))$$

Take $K \subset U$ compact with nonempty interior. Then there exists $\theta_0 \in (0, 1)$ such that for any $\gamma \in \mathcal{T}$ with $\gamma(\tau) \in K$ and $|\langle \dot{\gamma}(\tau), \xi \rangle_{\tilde{g}}| \geq \theta_0 \|\gamma\|_g$ then $L_g(\gamma) > \rho(\gamma(0), \gamma(1))$. By the compactness of K, if

$$R = \inf \left\{ L_g(\gamma) - \rho(\gamma(0), \gamma(1)) \;\middle|\; \gamma \in \mathcal{T}, \exists \tau \in (0,1) \text{ s.t. } |\langle \dot{\gamma}(\tau), \xi \rangle_{g_0}| > \theta_0 \|\dot{\gamma}\|_g \right\} \tag{5.1}$$

then $R > 0$. Construct a new metric $\tilde{g}$, defined by

$$\langle X, Y \rangle_{\tilde{g}} = (1 + a\psi) \langle X, Y \rangle_{g_0} - b\psi \langle X, \xi \rangle_{g_0} \langle Y, \xi \rangle_{g_0} \tag{5.2}$$

Here $\psi \in C^\infty(M, [0,1])$ is a cutoff function with $\operatorname{supp}\psi = B_\epsilon(q) \subset K$ (ϵ will be determined shortly), and a, b are positive constants satisfying

$$b < 1 + a \tag{5.3}$$

$$b < a/\theta_0^2 \tag{5.4}$$

$$b > 1 + a - (1+a)^{1-n} \tag{5.5}$$

$$b > a. \tag{5.6}$$

The existence of such an a, b is proved as follows. For $a > 0$, $0 < (1+a)^{1-n} < 1$. Therefore $1 + a - (1+a)^{1-n} > a$ and so (5.6) follows from (5.5). We want to show that (5.4) and (5.5) are compatible (as the others clearly are). This is possible if we have

$$1 + a - (1+a)^{1-n} < a/\theta_0^2$$

which is certainly true if

$$1 + a \leq a/\theta_0^2.$$

This holds for a sufficiently large because $0 < \theta_0 < 1$. We then take

$$1 + a - (1+a)^{1-n} < b < 1 + a \leq a/\theta_0^2$$

and all desired conditions on a and b will be satisfied.

It is easily seen that equations (5.3)-(5.6) are equivalent to

$$1 + a - b > 0 \tag{5.7}$$

$$(1+a)^{n-1}(1 + a - b) < 1 \tag{5.8}$$

$$1 > a/b > \theta_0^2 \tag{5.9}$$

Let $\theta = \sqrt{a/b}$ so that $1 > \theta > \theta_0$. The new metric $\tilde{g}$ is positive definite by (5.7). Condition (5.8) implies that $\det \tilde{g}(x) < \det g(x)$ at points x where $\psi(x) = 1$. Fix

$$0 < \epsilon < \frac{1}{2}\sqrt{\frac{1+a-b}{1+a}}\, R.$$

so that $\operatorname{supp}\psi \cap \partial M$ is empty and $\operatorname{supp}\psi \subset K$. Take ψ so that $\psi^{-1}(1)$ has sufficiently large measure in $\operatorname{supp}\psi$ to insure $\mathrm{Vol}(M, \tilde{g}) < \mathrm{Vol}(M, g)$. We take condition (5.9) so that $\tilde{g}$ shrinks in directions close to ξ so that distances between points in the boundary under $\tilde{g}$ will still be bounded below by ρ, as follows.

Take any $\gamma \in \mathcal{T}$ which realizes the $\tilde{g}$-distance between its endpoints. We will show that

$$L_{\tilde{g}}(\gamma) \geq \rho(\gamma(0), \gamma(1)). \tag{5.10}$$

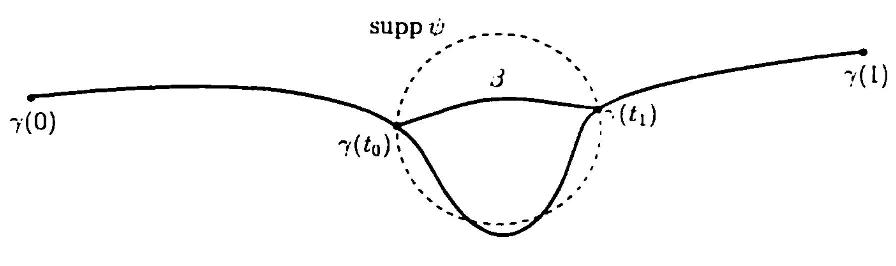

Figure 5.1: A curve γ realizing the $\tilde{g}$-distance between two points in ∂M

Let

$$\begin{aligned} t_0 &= \inf\{t \in [0,1] \mid \gamma(t) \in \operatorname{supp}\psi\} \\ \text{and} \quad t_1 &= \sup\{t \in [0,1] \mid \gamma(t) \in \operatorname{supp}\psi\}. \end{aligned}$$

Since g and $\tilde{g}$ agree on $M \setminus \operatorname{supp} \psi$, (5.10) follows unless $0 < t_0 < t_1 < 1$ as γ would not even enter $\operatorname{supp} \psi$. Also, if for all t such that $\gamma(t) \in K$.

$$|\langle \dot{\gamma}(t), \xi \rangle_{g_0}| \leq \theta,$$

then it follows from (5.2) that $\|\dot{\gamma}(t)\|_{\tilde{g}} \geq \|\dot{\gamma}(t)\|_g$ for all t, and so in this case (5.10) also holds. Finally, if for some t with $\gamma(t) \in K$ we have that

$$|\langle \dot{\gamma}(t), \xi \rangle_{g_0}| > \theta,$$

Then (5.9) and (5.1) together give that

$$L_g(\gamma) \geq \rho(\gamma(0), \gamma(1)) + R \tag{5.11}$$

Furthermore. if β is a curve realizing the g-distance between $\gamma(t_0)$ and $\gamma(t_1)$, then (5.2) and the fact that γ realizes the $\tilde{g}$ distance between $\gamma(t_0)$ and $\gamma(t_1)$, give that

$$\begin{aligned} L_g(\gamma|_{[t_0,t_1]}) &\leq \frac{1}{\sqrt{1+a-b}} L_{\tilde{g}}(\gamma|_{[t_0,t_1]}) \\ &\leq \frac{1}{\sqrt{1+a-b}} L_{\tilde{g}}(\beta) \\ &\leq \frac{\sqrt{1+a}}{\sqrt{1+a-b}} L_g(\beta) \\ &\leq \sqrt{\frac{1+a}{1+a-b}}\, 2\epsilon \\ &< R \end{aligned} \tag{5.12}$$

Therefore by (5.1) and (5.11),

$$\begin{aligned} L_{\tilde{g}}(\gamma) &= L_{\tilde{g}}(\gamma|_{[0,t_0]}) + L_{\tilde{g}}(\gamma|_{[t_0,t_1]}) + L_{\tilde{g}}(\gamma|_{[t_1,1]}) \\ &> L_g(\gamma|_{[0,t_0]}) + L_g(\gamma|_{[t_1,1]}) \end{aligned}$$

$$
\begin{aligned}
&= L_g(\gamma) - L_g(\gamma|_{[t_0,t_1]}) \\
&\geq \rho(\gamma(0),\gamma(1)) + R - L_g(\gamma|_{[t_0,t_1]}),
\end{aligned}
$$

and (5.10) follows immediately from (5.12).

□

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Appendix A

Notation

$B_r(p)$	open metric ball of radius r at p
$B_r(A)$	$\bigcup_{p\in A} B_r(p)$
$L_g(\alpha)$	length of the curve α in the metric g
$\mathrm{d}_g(p,q)$	distance between points p and q in the metric g
	The subscript g will be omitted when the meaning is clear.
$\mathrm{d}_g(A,B)$	distance between sets A and B in the metric g
	i.e. $\inf\{\mathrm{d}_g(a,b) \mid a \in A, b \in B\}$
$\operatorname{supp}\psi$	support of function ψ
$\bar{U}$	closure of the set U

$\alpha * \beta$	usual product of curves α and β
$\mathrm{Sys}(M, g)$	systole of Riemannian manifold (M, g) i.e. the infimum of lengths of all non-contractable curve in M.
$d\mu_g$	Riemannian measure of metric g
$\mathrm{Vol}(A)$	volume of a set A in this measure
$\mathrm{Area}(A)$	area of a set A in the measure μ_g for a surface
(S^n, can)	sphere of dimension n with the standard metric
(S^1, l)	circle of length l
$\mathrm{diam}(V, g)$	diameter of the set V in the metric g i.e. $\sup\{\mathrm{d}_g(p, q) \mid p, q \in V\}$
$\mathrm{inj}(p, g)$	injectivity radius at p in the metric g
$\mathrm{conv}(p, g)$	convexity radius (strict convexity) at p in the metric g
$\mathrm{conv}(g)$	convexity radius of the metric g on a compact manifold