Positive paths and length minimizing geodesics in Hofer's geometry

A Dissertation Presented

by

Jennifer Robin Slimowitz

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

May 1998

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation

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A compactly supported time dependent Hamiltonian function defined on a symplectic manifold (M, ω) induces a vector field on M. The flow of this vector field is a path starting at the identity in the group $Ham^c(M)$ of compactly supported Hamiltonian diffeomorphisms of M.

Given a flow in $Ham^c(M)$ which is a stable geodesic, there is a point in M, fixed by each element of the flow, around which the linearization of the flow is a positive path in $\mathrm{Sp}(2n)$. In the first part of this dissertation, we examine the way the set of positive loops sits inside the set of all loops in $\mathrm{Sp}(2n)$. For $n = 1$ and

$n = 2$, we show that if two positive loops in $\mathrm{Sp}(2n)$ are homotopic, they are in fact homotopic through positive loops.

In the second part of this dissertation, we consider length minimizing geodesics in the space $Ham^c(M)$. We show that rotation in one homogeneous coordinate through π radians in $\mathbf{CP^2}$ and $\widetilde{\mathbf{CP^2}}$ is a length minimizing path between its endpoints. More generally, our main theorem states that if M has dimension four, any path in $Ham^c(M)$ which is generated by an autonomous Hamiltonian and which has no non-constant closed trajectory is length minimizing among all homotopic paths. To prove this theorem, we provide an upper bound for the Hofer-Zenhder capacity for manifolds of the type $M \times D^2$ where M has dimension four.

To my family

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Acknowledgements

I thank my advisor, Dusa McDuff, for allowing me to write this dissertation under her supervision and introducing me to symplectic geometry. I learned so much by giving talks and going to conferences, and I could not have done this without her backing me up. Most of all, I wish to acknowledge her guidance in solving many mathematical problems, her sharing of her insight with me, and her helpful comments on many previous drafts of this thesis.

Certainly, I would have never eneterd graduate school in mathematics without the encouragement and support of my mentors from college. I thank Joel Cohen, Mike Reed, and Bill Pardon for instilling in me enough self confidence to embark on this journey and for providing advice all the way through.

Finally, I owe a great deal to the friends from Stony Brook for teaching me most of what I know about analysis and geometry and for fun times at the beach.

Chapter 1

Introduction

Symplectic geometry is the study of a $2n$ dimensional manifold M endowed with a closed, non-degenerate 2-form ω. Using ω, a compactly supported time dependent Hamiltonian function H_t on M induces a vector field whose flow ϕ_t^H for $0 \leq t \leq 1$ is a path in the group $Ham^c(M)$ of compactly supported Hamiltonian symplectomorphisms of M. Conversely, every path in $Ham^c(M)$ arises in this way as the flow of some Hamiltonian function. $Ham^c(M)$ is an infinite dimensional manifold about which very little is known. However, Hofer has constructed a bi-invariant norm on $Ham^c(M)$ under which the size of a diffeomorphism ϕ is the infimum of the lengths of all of the paths from the identity to ϕ. The length of a path in $Ham^c(M)$ is defined to be the integral of the total variation of its generating Hamiltonian function [HZ94]. More precisely, given a path ϕ_t^H for $0 \leq t \leq 1$ generated by the Hamiltonian H_t, the length of ϕ_t^H is the length $L(H_t)$ of H_t, where

$$L(H_t) = \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) dt.$$

Although the Hofer norm is simply defined, in practice it is very hard to calculate. One case in which we have a possibility to calculate the Hofer norm of ϕ is when there is a natural path from the identity to ϕ, e.g. a path induced by a circle action. Here, the Hamiltonian which generates the path is just the moment map of the action. All of the specific paths which we will examine in this dissertation arise from circle actions.

A geodesic in $Ham^c(M)$ is a path in this space for which sufficiently short segments are locally length minimizing. These geodesics give rise in a natural way to the notion of positive paths in the linear symplectic group $\mathrm{Sp}(2n)$. This thesis has two parts: one examining positive paths in $\mathrm{Sp}(2n)$ and one dealing with length minimizing paths in $\mathrm{Ham}^c(M)$. First, we describe how the set of positive loops based at the identity in $\mathrm{Sp}(2)$ and $\mathrm{Sp}(4)$ sits inside of the set of all loops. Second, for M of dimension 4, we show that a path in $Ham^c(M)$ starting at the identity which induces no non-constant closed trajectories of points in M is length minimizing among all homotopic paths. This includes the special cases of rotation by π radians in one homogeneous coordinate in complex projective space $\mathbf{CP^2}$ and in $\mathbf{CP^2}$ blown up at one point. We also provide an upper bound for the Hofer-Zehnder capacity for symplectic manifolds of the type $M \times D^2$ where M has dimension four.

1.1 Positive Paths

A positive path in the group of real symplectic matrices $\mathrm{Sp}(2n)$ is a smooth path $A(t)$ whose derivative A' satisfies

$$A'(t) = JP_tA(t)$$

where P_t is a positive definite symmetric matrix (dependent on t) and J is the standard complex structure. It is easy to see that positive paths are exactly those generated by negative definite time dependent quadratic Hamiltonians on $\mathbf{R^{2n}}$. The simplest example of a positive path is the counter clockwise rotation $A(t) = e^{Jkt}$ where k is any positive integer; here $P_t = kI$.

The relationship between positive paths and geodesics under the Hofer norm motivates Lalonde and McDuff's paper [LM97]. A compactly supported Hamiltonian $H_t : \mathbf{R^{2n}} \to \mathbf{R}$ generates a flow ϕ_t^H which is a geodesic under the Hofer norm if and only if around each $t_0 \in [0,1]$ there exists a short interval I such that there exist two points x and X in $\mathbf{R^{2n}}$ so that x is a minimum and X is a maximum of H_t for all $t \in I$. Around x, the linearized flow A_t of ϕ_t^H is a positive path in $\mathrm{Sp}(2n)$, and it is called short if 1 is not an eigenvalue of the A_t for any $t \in [0,1]$. A geodesic is said to be stable if it is a local minimum of the length functional. In [LM95a] and [LM95b], it is shown that if one x and one X can be chosen for H_t for the entire interval $[0,1]$ and the linearizations of the flow at x and X are short, then ϕ_t^H is a stable geodesic. Lalonde and McDuff study the linearized flow and positive paths in general in order to obtain topological information about stable geodesics in [LM97]. Their work further develops Krein's theory by analyzing short positive paths whose eigenvalues lie off of the unit circle. They show that any short positive path may be extended to a short positive path whose endpoint is diagonalizable with eigenvalues on S^1, and also that the space of short positive paths which end at such a matrix is path connected [LM97].

In addition, they define the positive fundamental group $\pi_{1,pos}(\mathrm{Sp}(2n))$ to be the semigroup generated by positive loops with base point at the identity, where two loops are considered equivalent if one can be deformed to the other via a family of positive loops. Recall that $\pi_1(\mathrm{Sp}(2n)) = \mathbf{Z}$ since the symplectic linear group deformation retracts onto the unitary group. If γ is a positive loop in $\mathrm{Sp}(2n)$, then the homotopy class of γ is actually in the subset $\mathbf{N}$ of $\mathbf{Z}$. In [LM97], Lalonde and McDuff pose the natural question: "Is the map

$$\pi_{1,pos}(\mathrm{Sp}(2n)) \to \pi_1(\mathrm{Sp}(2n))$$

which sends the positive homotopy class of a positive loop to its ordinary homotopy class injective?" Here, we prove the following two theorems:

Theorem 3.2.3 *Suppose $A_t, B_t \in \mathrm{Sp}(2)$ are two positive loops based at I. Then, A_t and B_t are homotopic if and only if they are homotopic through positive loops. Thus, the natural map from*

$$\pi_{1,pos}(\mathrm{Sp}(2)) \to \pi_1(\mathrm{Sp}(2))$$

is injective and onto $\mathbf{N}$.

Theorem 4.1.1 *Let $A_t, B_t : [0, 2\pi] \to \mathrm{Sp}(4)$ be positive loops in $\mathrm{Sp}(4)$ with base point I. Then A_t and B_t are homotopic if and only if they are homotopic through positive loops. Thus, the natural map*

$$\pi_{1,pos}(\mathrm{Sp}(4)) \to \pi_1(\mathrm{Sp}(4))$$

is injective and onto $\mathbf{N} - \{1\}$.

In Chapter 2, we examine the behavior of positive loops in Sp($2n$) by looking at the projection of these loops in the stratified space of symplectic conjugacy classes. In Chapters 3 and 4, we characterize these projections and construct homotopies between them, and then lift the results to Sp($2n$) by means of a lifting lemma. The main difficulty in the four dimensional case is to show that any positive loop is positively homotopic to a loop whose eigenvalues lie in $S^1 \cup \mathbf{R}$. Lalonde and McDuff look at generic paths and those meeting isolated codimension two singularities; here we will occassionally need to look at paths which cross singularities of higher codimension. The technical lemmas are proven in Chapter 5.

The theorems about positive paths that we prove here are only for dimensions 2 and 4. As the dimension increases, the space of symplectic conjugacy classes gets more complicated. This is the impediment to proving analogous results for larger matrix groups.

1.2 Length Minimizing Geodesics

A length minimizing geodesic ϕ_t for $0 \leq t \leq 1$ in $Ham^c(M)$ is a path which is an absolute minimum of the length functional among all paths from ϕ_0 to ϕ_1. Classification of length minimizing geodesics is the logical extension of the work done on general geodesics in $Ham^c(M)$ by Bialy-Polterovich in [BP94] and Lalonde-McDuff in [LM95a]. Since general geodesics are short only in a local sense, it makes sense next to consider those paths which are globally length minimizing.

Lalonde and McDuff give the first non-trivial example of a globally length minimizing path when they show that rotation through π radians on S^2 is a length minimizing geodesic in $Ham(S^2)$ [LM95a]. This leads us to ask whether rotation of $\mathbf{CP^2}$ through π radians is length minimizing in $Ham(\mathbf{CP^2})$. In fact, by following the procedure of embedding balls as outlined in [LM95a], we prove in Chapter 7 that rotation on $\mathbf{CP^2}$ is indeed a length minimizing geodesic.

Theorem 7.2.1 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\mathbf{CP^2})$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

It is easy to verify that the flow from Theorem 7.2.1 is generated by the Hamiltonian function P where

$$P[z_0 : z_1 : z_2] = \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

Note that since P is independent of time,

$$L(P) = \max_{x \in \mathbf{CP^2}} P(x) - \min_{x \in \mathbf{CP^2}} P(x) = \frac{\pi}{2}.$$

The next natural path to examine is rotation on the symplectic blow up $\widetilde{\mathbf{CP^2}}$ of $\mathbf{CP^2}$. We define $(\widetilde{\mathbf{CP^2}}, \tau_\lambda)$ to be the manifold obtained by removing from $\mathbf{CP^2}$ an open 4-ball of radius λ and collapsing its boundary S^3 along the fibers of the Hopf map. If we think of $\mathbf{CP^2}$ as a 4-ball of radius 1 with the boundary S^3 collapsed along the fibers of the Hopf map, then $(\widetilde{\mathbf{CP^2}}, \tau_\lambda)$ is an annulus $\{(w_0, w_1) \mid (1-\lambda^2) \leq |w_0|^2 + |w_1|^2 \leq 1\}$ with both boundaries collapsed along the Hopf fibers.

We define $(\widetilde{\mathbf{CP^2}}_0, \tau_\lambda)$ to be the symplectic blow up of $\mathbf{CP^2}$ by a ball of radius λ centered at the point $[1:0:0]$, equipped with the Fubini-Study symplectic form. Similarly, we define $(\widetilde{\mathbf{CP^2}}_1, \tau_\lambda)$ to be the corresponding blow up at the point $[0:1:0]$. It is not hard to see that rotation on $\mathbf{CP^2}$ in the first homogeneous coordinate is also well defined on both $\widetilde{\mathbf{CP^2}}_0$ and $\widetilde{\mathbf{CP^2}}_1$. It is only necessary to verify that the rotation keeps the set of removed points for either blow up invariant and that the rotation is well defined under the equivalence imposed on the boundary. It is important to realize, however, that the rotation in the first homogeneous coordinate is qualitatively different in the two different blow ups; in $\widetilde{\mathbf{CP^2}}_0$ each point on the exceptional divisor is fixed, whereas in $\widetilde{\mathbf{CP^2}}_1$, the points on the exceptional divisor rotate.

Note that the function P is well defined on $\widetilde{\mathbf{CP^2}}_0$ and $\widetilde{\mathbf{CP^2}}_1$. When blowing up, we collapse the boundary of the ball of radius λ along orbits of an S^1 action, and P (defined on $\mathbf{CP^2}$) is invariant under this action. Therefore, P, defined appropriately, is the Hamiltonian function which generates rotation in the first homogeneous coordinate on all three manifolds $\mathbf{CP^2}$, $\widetilde{\mathbf{CP^2}}_0$, and $\widetilde{\mathbf{CP^2}}_1$. Hence, we will use P to denote this Hamiltonian function on each of them, and it will be clear from context which domain we are considering.

Although P is well defined both on $\widetilde{\mathbf{CP^2}}_0$ and $\widetilde{\mathbf{CP^2}}_1$, its length is not the same on both manifolds. P applied to $\widetilde{\mathbf{CP^2}}_0$ has $L(P) = \frac{\pi}{2}(1-\lambda^2)$ whereas P

applied to $\widetilde{\mathbf{CP^2}}_1$ has $L(P) = \frac{\pi}{2}$. Because $L(P)$ decreases when blowing up from $\mathbf{CP^2}$ to $\widetilde{\mathbf{CP^2}}_0$, it is easy to generalize Theorem 7.2.1 to $\widetilde{\mathbf{CP^2}}_0$. On the other hand, since $L(P)$ does not decrease when it is applied to $\widetilde{\mathbf{CP^2}}_1$, Theorem 7.2.1 does not immediately generalize to $\widetilde{\mathbf{CP^2}}_1$. Thus, we must develop alternative techniques to prove the following theorem.

Theorem 7.4.5 *The path ϕ_t^P for $0 \le t \le 1$ in $Ham(\widetilde{\mathbf{CP^2}}_1)$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Theorem 7.4.5 is a direct consequence of the next result. Before stating it, we need some preparation. We say a path $\phi_t \in Ham^c(M)$ which starts from the identity has no non-constant closed trajectory in time less than 1 if

$$\phi_{t_0}(x_0) = x_0 \text{ for some } t_0 \in (0,1], x_0 \in M \Rightarrow \phi_t(x_0) = x_0 \ \forall t \in [0,1].$$

We let $S^2(a)$ denote the sphere equipped with a symplectic form σ which satisfies $\int_{S^2} \sigma = a$.

Theorem 7.4.4 *Let (M, ω) be a symplectic manifold of dimension four. Let ϕ_t for $0 \le t \le 1$ be a path in $Ham^c(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0 is the identity diffeomorphism and ϕ_t has no non-constant closed trajectory in time less than 1. Then, ϕ_t for $0 \le t \le 1$ is length minimizing among all homotopic paths between the identity and ϕ_1.*

This theorem generalizes Hofer's parallel result for $\mathbf{R^{2n}}$. He proves that the flow of an autonomous Hamiltonian in $\mathbf{R}^{2n}$ which admits no non-constant closed trajectory in time less than 1 is a length minmizing path in Section 5.7 of [HZ94].

For the proof of Theorems 7.4.5 and 7.4.4, we follow the criteria for length minimizing geodesics from [LM95a], described in Chapter 6, using capacities and quasi-cylinders. In order to complete the proofs, we need to use the Hofer-Zehnder capacity c_{HZ} on a large class of manifolds, including $\widetilde{\mathbf{CP}^2}$. However, in [HV92], Hofer and Viterbo only prove that c_{HZ} may be used on manifolds M which are weakly exact, and $\widetilde{\mathbf{CP}^2}$ is not of this type. Hence, in Chapter 8, we go back to the original proof in [HV92] and modify it using the theory of J-holomorphic curves to show that we can use c_{HZ} on all M if M has dimension four or less. The main statement we prove is the following:

Theorem 7.4.3 *Suppose that the manifold (M, ω) is a symplectic manifold of dimension four. Then,*

$$c_{HZ}(M \times D^2(a), \omega \oplus \sigma) \leq a.$$

Remark 1.2.1 *Theorem 7.4.4 as it is now stated has limited scope. The restriction to manifolds of dimension four is required in order to deal with multiply covered curves on $M \times S^2$ at the end of Section 8.2. However, recent advances by Fukaya-Ono, Li-Tian, Liu-Tian, Ruan, and Siebert in the theory of $J-$holomorphic curves will most likely allow us to generalize to other dimensions. In particular, Liu and Tian have just proved that the Weinstein* *conjecture holds for all manifolds using the difficult theory of Gromov-Witten invariants [LT97]. It seems as if their methods, properly applied, might indeed imply that Theorem 7.4.3 and hence Theorem 7.4.4 will hold for all manifolds.*

In other related work, Polterovich examines a rotation similar to ϕ_t^P on the monotone manifold $(\widetilde{\mathbf{CP}^2}_0, \tau_{1/\sqrt{3}})$, the blow up of $\mathbf{CP^2}$ obtained by removing a ball of radius $\frac{1}{\sqrt{3}}$ centered at the point $[1:0:0]$. He works with the coarse Hofer norm, a variant of the original Hofer norm, and examines the path ψ_t where

$$\psi_t[z_0 : z_1 : z_2] = [e^{2\pi i t} z_0 : z_1 : z_2].$$

Note that ψ_t is just ϕ_t^P traversed at double speed. He shows that the loop formed by ψ_t for $0 \leq t \leq 1$ is a length minimizing representative of its homotopy class in $Ham(\widetilde{\mathbf{CP}^2}_0)$ with respect to the coarse Hofer norm [Pol96]. In Chapter 9, we show that ψ_t is a length minimizing representative of its homotopy class with respect to the regular Hofer norm, as well.

Theorem 9.0.8 *The loop ψ_t for $0 \leq t \leq 1$ is length minimizing in its homotopy class in $Ham^c(\widetilde{\mathbf{CP}^2}_0)$. In fact, it is length minimizing in its homotopy class when considered as a loop in $\mathbf{CP^2}$ or $\widetilde{\mathbf{CP}^2}_1$, as well.*

Chapter 2

The Behavior of Positive Paths

If A_t is a path in $\mathrm{Sp}(2n)$, we can look at the ways the eigenvalues of the matrices along the path change with respect to time. In this chapter, we describe the restrictions that positivity places on a path in terms of the movement of its eigenvalues. In addition, we prove a lifting lemma which will allow us to lift a positive homotopy from the space of symplectic conjugacy classes to $\mathrm{Sp}(2n)$.

2.1 The Splitting Number

A useful tool for describing the movement of eigenvalues along a positive path is the splitting number. The notion of splitting number arises from Krein theory, described in [Eke86] and [Eke89], and is explained further in Lalonde and McDuff [LM97]. They define the non-degenerate Hermitian symmetric form β on $\mathbf{C}^{2n}$ by $\beta(v, w) = -i\overline{w}^T Jv$ where J is the standard $2n \times 2n$ block matrix with the identity in the lower left box and minus the identity in the upper right box. They prove the

Lemma 2.1.1 *If $A \in \mathrm{Sp}(2n)$ has eigenvector v with eigenvalue $\lambda \in S^1$ of multiplicity 1, then $\beta(v,v) \in \mathbf{R} - \{0\}$.*

Hence, for any simple eigenvalue $\lambda \in S^1$ we may define the splitting number $\sigma(\lambda) = \pm 1$ where $\beta(v,v) \in \sigma(\lambda)\mathbf{R}^+$. Using properties of β, we can check that $\sigma(\lambda) = -\sigma(\overline{\lambda})$. As an illustration, when $n = 1$, the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues i and $-i$ corresponding to the eigenvectors

$$v_i = \begin{pmatrix} 1 \\ -i \end{pmatrix} \text{ and } v_{-i} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Computing, we find that $\beta(v_i, v_i) = 2$ so $\sigma(i) = 1$ and, similarly $\sigma(-i) = -1$.

In a more general setting, if $\lambda \in S^1$ has multiplicity > 1, we set $\sigma(\lambda)$ to be equal to the signature of β on the corresponding eigenspace. It is a straightforward calculation to see that the symplectic conjugacy class of a diagonalizable element in $\mathrm{Sp}(2n)$ with all of its eigenvalues on the circle is determined by its spectrum and corresponding splitting numbers. Hence, for each pair of conjugate eigenvalues $\{\lambda, \overline{\lambda}\} \in S^1$, there exist two symplectic conjugacy classes in $\mathrm{Sp}(2)$: one where λ has positive splitting number (and $\overline{\lambda}$ has negative splitting number) and one where λ has negative splitting number (and $\overline{\lambda}$ has positive splitting number). Note that there is no corresponding notion for real eigenvalues or the eigenvalues on S^1 of a non-diagonalizable matrix.

A natural question to ask is, "What restrictions does positivity impose upon movement of eigenvalues?" Krein's lemma states that under a positive flow, simple eigenvalues on S^1 with $+1$ splitting number move counter clockwise while those with -1 must move clockwise [Eke89]. In [LM97], Lalonde and McDuff show that when a positive path has a pair of eigenvalues that enter S^1, they must do so at a matrix which has a 2×2 Jordan block symplectically conjugate to

$$N_\lambda^+ = \begin{pmatrix} \lambda & -\lambda \\ 0 & \lambda \end{pmatrix}$$

where λ represents the eigenvalue on S^1. Similarly, when a pair leaves S^1, it does so at a matrix with a Jordan block symplectically conjugate to

$$N_\lambda^- = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}.$$

These restrictions are, in fact, the only ones dictated on generic paths by the positivity condition, leaving us with the following statement:

Lemma 2.1.2 *A positive path in Conj may move freely between conjugacy classes when its eigenvalues are away from S^1. On S^1, the eigenvalues move according to splitting number by Krein's lemma, and when entering and leaving S^1, they behave according to the above results of Lalonde and McDuff.*

For example, there are 4 open regions in Sp(4) whose union is dense:

(i) $\mathcal{O}_C$, consisting of all matrices with 4 distinct eigenvalues of the form $\{\lambda, \overline{\lambda}, \frac{1}{\lambda}, \frac{1}{\overline{\lambda}}\} \in \mathbf{C} - (\mathbf{R} \cup \mathbf{S^1})$;

(ii) $\mathcal{O}_{\mathcal{U}}$, consisting of all matrices with eigenvalues on $S^1 - \{1, -1\}$ where each eigenvalue has multiplicity 1 or multiplicity 2 with non-zero splitting numbers;

(iii) $\mathcal{O}_{\mathcal{R}}$, consisting of all matrices whose eigenvalues have multiplicity 1 and lie on $\mathbf{R} - \{0, 1, -1\}$

(iv) $\mathcal{O}_{\mathcal{U},\mathcal{R}}$, consisting of all matrices with 4 distinct eigenvalues, one pair on $S^1 - \{1, -1\}$ and the other on $\mathbf{R} - \{0, 1, -1\}$.

We will describe the other higher codimension regions later. Lemma 2.1.2 tells us that positive paths may move freely in $\mathcal{O}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{R}}$, but their behavior is restricted when in and when entering or leaving $\mathcal{O}_{\mathcal{U}}$ and $\mathcal{O}_{\mathcal{U},\mathcal{R}}$.

2.2 A Lifting Lemma

The following basic facts about positive paths from [LM97] will be very useful:

Lemma 2.2.1 **(i)** *The set of positive paths is open in the C^1 topology.*

(ii) *Any piecewise positive path may be C^0 approximated by a positive path.*

We now begin the discussion of homotopy and develop the tools necessary to prove the injectivity of map from $\pi_{1,pos}(\mathrm{Sp}(2n)) \to \pi_1(\mathrm{Sp}(2n))$. Given a homotopy whose endpoints are positive paths, we need to produce a homotopy between those two endpoints where each path in the homotopy is a positive path. We will consider the projection of the original homotopy to *Conj*, the space of symplectic conjugacy classes. Let π denote this projection:

$$\pi(A) = \cup_X \{XAX^{-1} : X \in \mathrm{Sp}(2n)\} \in Conj.$$

After altering the projection of the homotopy in *Conj* in a specific way to make each path in it positive, we lift it to Sp(2) or Sp(4).

Now we will state some useful definitions and two propositions which will enable us to execute the lifting.

Definition 2.2.2 *A point in* $\mathrm{Sp}(2n)$ *is called a* **generic point** *if all of its eigenvalues have multiplicity 1. A path in* $\mathrm{Sp}(2n)$ *is called a* **generic path** *if all of its points are generic or lie on the codimension 1 boundary part of a generic region, and the codimension 1 boundary points are isolated. These definitions also hold for points and paths in Conj.*

Definition 2.2.3 *A path* a_t *in Conj is called* **positive** *if there exists a positive path* $A_t \in \mathrm{Sp}(2n)$ *such that* $\pi(A_t) = a_t$. *A homotopy* $H(s,t) \in \mathrm{Sp}(2n)$ *is called* **positive** *if for every* s_0, $H(s_0,t)$ *is a positive path. A homotopy* $h(s,t) \in Conj(\mathrm{Sp}(2n))$ *is called* **positive** *if it is made up of positive paths in Conj, i.e. for every* s_0, *there is a positive path* $H(s_0,t) \in \mathrm{Sp}(2n)$ *such that* $\pi(H(s_0,t)) = h(s_0,t)$.

Proposition 2.2.4 *Let* $A_t \in \mathrm{Sp}(2n)$ *be a generic positive path joining two generic points* A_0 *and* A_1. *Then the set of positive paths in* $\mathrm{Sp}(2n)$ *which lift* $\pi(A_t) \in Conj$ *is path connected.*

Proof: Here is the idea of Lalonde and McDuff's proof from [LM97]. Suppose B_t and C_t are two paths which lift $\pi(A_t)$. We may assume that B_t crosses codimension 1 strata at finitely many times t_i. Note that each fiber of $\pi : \mathrm{Sp}(2n) \to \mathit{Conj}$ is path connected since $\mathrm{Sp}(2n)$ is. Hence, using Lemma 2.2.1, we may homotop C_t around those times to $X_i B_t X_i^{-1}$ for t close to t_i for some symplectic matrix X_i. Let ξ_B be the vector field tangent to to the curve B_t and define the vector field $\xi_C = X_i \xi_B X_i^{-1}$ over neighborhoods of each C_{t_i}. If we extend ξ_C appropriately and take the convex combination vector fields $s\xi_B + (1-s)\xi_C$, these new positive vector fields have integral curves which also project to $\pi(A_t)$. Thus, the family of integral curves as s varies from 0 to 1 gives a path between B_t and C_t within the lifts of $\pi(A_t)$. □

Certainly, if A_t and B_t are positively homotopic paths in $\mathrm{Sp}(2n)$, then $\pi(A_t)$ and $\pi(B_t)$ are positively homotopic in *Conj*. The next proposition shows that when each path in the homotopy is generic, the converse is also true.

Proposition 2.2.5 *Let $h(s,t)$ be a positive homotopy of generic loops based at the identity in Conj*$(\mathrm{Sp}(2n))$ *where* $h : [0,1] \times [0, 2\pi] \to \mathit{Conj}$ *and*

$$h(0,t) = a_t \quad h(1,t) = b_t.$$

Also, let $A_t, B_t : [0, 2\pi] \to \mathrm{Sp}(2n)$ *be any two positive generic loops based at* I *so that* $\pi(A_t) = a_t$ *and* $\pi(B_t) = b_t$. *Then, there exists a positive homotopy* $H(s,t) : [0,1] \times [0,2\pi] \to \mathrm{Sp}(2n)$ *such that* $H(0,t) = A_t$ *and* $H(1,t) = B_t$.

Proof: The proof of this proposition mimics that of the previous one, only here we must introduce parameters. After dealing with the technicalities of locally lifting h around each codimension 1 point as in Proposition 2.2.4, we are left with a finite sequence of $H^i(s,t) : [s_i, s_{i+1}] \times [0, 2\pi] \to \mathrm{Sp}(2n)$, homotopies defined on some partition $[s_i, s_{i+1}]$ of $[0,1]$. Here, $\pi(H^i(s,t)) = h(s,t)$, and each loop $H^i(s_i,t) : [0,2\pi] \to \mathrm{Sp}(2n)$ is a generic positive path. Using Proposition 2.2.4, for each i, we glue $H^i(s_{i+1},t)$ to $H^{i+1}(s_{i+1},t)$ via a family of positive loops, all of which project to $h(s_{i+1},t)$ in *Conj*, and let H be the resultant homotopy. At the end of this paper, we give the full details concerning the lifting of some specific homotopies in Sp(4). □

Hence, to prove the injectivity of the map from $\pi_{1,pos} \to \pi_1$, we need only construct a positive homotopy of generic paths in *Conj* between the projections of the two given endpoints. This is exactly what will happen in Sp(2). It turns out, however, that the homotopy we construct in *Conj*(Sp(4)) may have some paths which are non-generic and go through points of codimension two and higher. We will deal with this by finding specific lifts of the homotopy in neighborhoods of these points to Sp(4). We then join these lifts to the given homotopy using Proposition 2.2.4.

Chapter 3

The Positive Fundamental Group of Sp(2)

We begin this chapter by describing the space $\mathcal{C}onj(\mathrm{Sp}(2))$. Then, we will prove Theorem 3.2.3 and some related facts about positive paths.

3.1 The Structure of $\mathcal{C}onj(\mathrm{Sp}(2))$

Here is a review of the structure of the stratified space of symplectic conjugacy classes of Sp(2) as described in [LM97], along with some additional details.

A generic matrix in Sp(2) has two distinct eigenvalues and belongs to one of the following regions:

(i) $\mathcal{O}_{\mathcal{U}}$, consisting of all matrices with eigenvalues $\{\lambda, \overline{\lambda}\} \in S^1$

(ii) $\mathcal{O}_{\mathcal{R}}$, consisting of all matrices with real eigenvalues $\{\lambda, \frac{1}{\lambda}\}$ where $|\lambda| \geq 1$.

We will divide each of $\mathcal{O}_{\mathcal{R}}$ and $\mathcal{O}_{\mathcal{U}}$ naturally into two parts: $\mathcal{O}_{\mathcal{R}}^+$ and $\mathcal{O}_{\mathcal{R}}^-$ for positive or negative eigenvalues and $\mathcal{O}_{\mathcal{U}}^+$ and $\mathcal{O}_{\mathcal{U}}^-$ based on the sign of the imaginary part of the eigenvalue with positive splitting number.

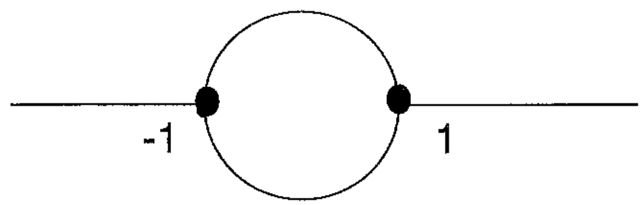

Figure 3.1: $Conj(\mathrm{Sp}(2))$

We see that the non-generic matrices are the identity matrix I and $-I$ and the non-diagonalizable matrices with a double eigenvalue of 1 or -1. The space of symplectic conjugacy classes of Sp(2) (remember this requires similarity by a symplectic matrix) can be described by the set $S^1 \cup (1,\infty) \cup (-\infty,-1)$ in the plane with the points 1 and -1 tripled, as depicted in Figure 3.1.

This can be seen as follows: identify $A \in \mathcal{O}_{\mathcal{R}}$ with its eigenvalue λ whose absolute value is greater than 1. Clearly, all such matrices are conjugate. For $A \in \mathcal{O}_{\mathcal{U}}$, we can distinguish between the two eigenvalues $\{\lambda, \overline{\lambda}\}$ by the notion of splitting number as described above. Associating A to its eigenvalue with positive splitting number produces a well-defined equivalence class, accounting for each element in S^1. I and $-I$ each comprise their own equivalence class; associate I with 1 and $-I$ with -1. If A is non-diagonalizable with double eigenvalue -1, then A is conjugate to either

$$N_{-1}^{+} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

or

$$N_{-1}^{-} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

In either case, we send A to -1, and we have 3 conjugacy classes at -1: $-I, N_{-1}^{+}, N_{-1}^{-}$.

Similarly, if A is nondiagonalizable with double eigenvalue 1, then A is conjugate to either N_1^+ or N_1^- , and we have three conjugacy classes at 1: I, N_1^+, N_1^-. The space of $A \in \mathrm{Sp}(2)$ which project to either $N_{-1}^{+}, N_{-1}^{-}, N_1^+$, or N_1^- is of codimension 1. By Lemma 2.1.2 , we know that positive paths in *Conj* enter S^1 via N_{-1}^{+} and N_1^+ and leave via N_{-1}^{-} and N_1^-.

3.2 The main result for Sp(2)

Definition 3.2.1 *A* **simple** *path $\gamma(t)$ in Conj*(Sp(2)) *has at most one local minimum and no local maxima each time it enters $\pi(\mathcal{O}_{\mathcal{R}}^{-})$, and has at most one local maximum and no local minima each time it enters $\pi(\mathcal{O}_{\mathcal{R}}^{+})$.*

Lemma 3.2.2 *If γ_t is a simple path along the real axis in Conj with bounded eigenvalues, it is positive.*

Proof: By Lemma 2.2.1, it suffices to show that for all $\alpha, \beta \in \mathbf{R}^{+} - (0, 1]$ where $\alpha \neq \beta$, there is a positive path $A_t \in \mathrm{Sp}(2)$ such that the function $t \to \pi(A_t)$ is an embedding of $[0, 1]$ onto $[\alpha, \beta]$ sending 0 to α and 1 to β. Consider the path $e^{Jt}B$ where

$$B = \begin{pmatrix} \beta & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$$

the projection of $e^{Jt}B$ to the real axis in *Conj* depends only on the trace of the matrix, since we can recover the eigenvalues from the trace and the determinant which we know is 1. So, by examining the movement of the trace of $e^{Jt}B$, we can determine the flow of $\pi(e^{Jt}B)$ on the real axis. We know that $\pi(e^{Jt}B)$ must travel counter clockwise along the circle by Krein's Lemma, so once we figure out what the trace is doing, we will get the trajectory of the path in all of *Conj*. The derivative of the trace of $e^{Jt}B$ at time t is

$$-(\beta + \frac{1}{\beta}) \sin t$$

which is negative for $0 < t < \pi$, zero for $t = \pi$, and positive for $\pi < t < 2\pi$. Note that at $t = \pi$, $e^{Jt}B = -B$. Hence, $\pi(e^{Jt}B)$ finishes by coming off the circle through N_1^- and travelling up the real axis, past α, to β. We can let A_t be the reparametrization of the last portion of $e^{Jt}B$ which projects to $[\alpha, \beta]$. Similarly, if $\alpha > \beta$, we will let A_t be the first part of $e^{Jt}C$ where

$$C = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}.$$

□

Theorem 3.2.3 *Suppose* $A_t, B_t \in \mathrm{Sp}(2)$ *are two positive loops based at* I*. Then,* A_t *and* B_t *are homotopic if and only if they are homotopic through positive loops. Thus, the natural map from*

$$\pi_{1,pos}(\mathrm{Sp}(2)) \to \pi_1(\mathrm{Sp}(2))$$

is injective and onto $\mathbf{N}$.

Proof: Certainly, if A_t and B_t are homotopic through positive loops, then they are homotopic.

Conversely, if A_t and B_t are homotopic, then the homotopy descends to a homotopy of the projections of the paths $\pi(A_t)$ and $\pi(B_t)$ in *Conj*. Thus, the two projections of the paths travel around S^1 the same number of times; this homotopy invariant is the Maslov index. We can assume the paths only go though I at times 0 and 1 and are generic away from these points, as positivity is an open condition. We will show that $\pi(A_t)$ and $\pi(B_t)$ are both homotopic through positive paths in *Conj* to a standard path γ_t with appropriate Maslov index and, thus, that they are homotopic through positive loops in *Conj* to each other. Since any piecewise positive path may be C^0 approximated by a positive path, it will suffice to do the homotopy in pieces, first considering the parts of the paths on the circle, and then considering the parts on the real axis.

Let γ_t be a loop at I in *Conj* which goes around S^1 the same number of times as $\pi(A_t)$ and $\pi(B_t)$. Choose γ_t so that it is a simple path. (Thus, γ_t doesn't swivel back and forth more than once along the real axis each time it leaves the circle.)

Lemma 2.1.2 tell us that by reparametrizing $\pi(A_t)$, we can make it equal to γ_t for the times when γ_t takes values on the circle. The new parametrization is positively homotopic to $\pi(A_t)$, so we need only search for a homotopy from $\pi(A_t)$ to γ_t when these paths take values on the real line.

From Lemma 3.2.2, we know that the portion of each simple loop in *Conj* on the real line is positive. If $\pi(A_t)$ is simple, it can be easily homotoped through positive paths to γ_t just by "stretching" through simple and therefore positive paths. If $\pi(A_t)$ is not simple, then we can slightly perturb it to have finitely many local maxima and minima. Then, we can consider each "bump" as a simple path, and flatten each one individually by passing through simple and therefore positive paths. After smoothing out all the bumps in this manner, we are left with one simple piecewise positive path in *Conj* positively homotopic to $\pi(A_t)$. We can estimate this path closely by a simple postive path positively homotopic to $\pi(A_t)$, and by moving through simple paths, homotop it to γ_t.

Hence, $\pi(A_t)$ is homotopic through positive loops to γ_t. In the same way, $\pi(B_t)$ is also homotopic through positive loops to γ_t, and so $\pi(A_t)$ and $\pi(B_t)$ are positively homotopic in *Conj*.

All of these homotopies are through generic paths; hence by Proposition 2.2.5, we can lift this homotopy to Sp(2), and the proof is complete. □

Corollary 3.2.4 *Let* $A_t : [0, 2\pi] \to \mathrm{Sp}(2)$ *be a positive loop. Then* A_t *is positively homotopic to* $e^{Jkt}K$ *where* k *is the Maslov index of* A_t *and* $A_0 = K$.

Proof: Since the Maslov index completely dictates the homotopy class of a loop, A_t is homotopic to $e^{Jkt}K$. Hence by Theorem 3.2.3, A_t is positively homotopic to $e^{Jkt}K$. □

3.3 Other results for paths in Sp(2)

Here are a few interesting remarks concerning positive paths in Sp(2):

Remark 3.3.1 *At a point $A \in \mathrm{Sp}(2)$, the intersection of the positive cone and the tangent vectors pointing in the direction of the conjugacy class of A is*

$$\{JPA \cap (MA - AM)\}$$

where P is positive definite symmetric and $M \in \mathbf{sp}(2)$. If

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, |\lambda| > 1$$

then this intersection is

$$\left\{ \begin{pmatrix} 0 & -z\lambda \\ \frac{x}{\lambda} & 0 \end{pmatrix} \middle| x, z \in \mathbf{R}^+ \right\}.$$

The intersection of the positive cone and tangent vectors pointing within the conjugacy class at BAB^{-1} for $B \in \mathrm{Sp}(2)$ is

$$\left\{ B \begin{pmatrix} 0 & -z\lambda \\ \frac{x}{\lambda} & 0 \end{pmatrix} B^{-1} \middle| x, z > 0 \right\}.$$

Hence, if

$$M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for $b, c > 0$, then the path $\gamma(t) = e^{Mt} A e^{-Mt}$ is a positive path staying in the conjugacy class of A.

Theorem 3.3.2 *Given any two elements in the same conjugacy class in $\mathcal{O}_{\mathcal{R}}^{\pm} \in \mathrm{Sp}(2)$, there exists a positive path within the conjugacy class from one to the other.*

Proof: This is a direct result of Lobry's Theorem which may be stated as follows: Let M be a smooth, connected, paracompact manifold of dimension n with a set of complete vector fields $\{X^i | i \in I\}$ for some index set I. Consider the smallest family of vector fields containing the X^i which is closed under Jacobi bracket. At each point of M, the values of the elements of this family are vectors in the tangent space to M which generate a linear subspace S. If $dim(S) = n$ for all points in M, the positive orbit of a point under the vector fields X^i is the whole manifold. (See Lobry [Lob74], Sussman [Sus87], and Grasse and Sussman [GS90].)

In our specific case, M is the conjugacy class of an element in $\mathcal{O}^{\pm}_{\mathcal{R}}$, a smooth, connected 2 dimensional paracompact manifold. Let A represent the diagonal element of this conjugacy class with eigenvalues λ and $\frac{1}{\lambda}$. Our index set $I = \mathbf{R}^+ \times \mathbf{R}^+$ and our vector fields at BAB^{-1} will be the positive vectors in $T_{BAB^{-1}}M$:

$$X^{x,z}_{BAB^{-1}} = B \begin{pmatrix} 0 & -z\lambda \\ \frac{x}{\lambda} & 0 \end{pmatrix} B^{-1}.$$

At each point in M the dimension of the subspace spanned by the $X^{x,z}$ is 2. Closure under Jacobi bracket would only add more vector fields and hence increase the dimension of the subspace which is spanned, so the $dim(S) \geq 2$ at all points in the manifold. But, $dim(S) \leq dim(T_{BAB^{-1}}M) = 2$, so $dim(S) = 2$. Lobry's theorem applies, and we can move within the conjugacy class positively from any one element to any other. □

Remark 3.3.3 *There exist positive paths $\gamma^{\pm}(t) \in \mathrm{Sp}(2)$ such that*

$$\lim_{t\to\infty} trace(\gamma^{\pm}(t)) = \pm\infty.$$

Proof: It suffices to find γ^+, as then we could just set $-\gamma^+ = \gamma^-$. Take the path ${\gamma^+}_0 = e^{JPt}A_0$ where

$$A_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \frac{1}{\lambda_0} \end{pmatrix}, \ \lambda_0 > 1 \ , \ P = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Note that ${\gamma^+}_0(0) = A_0$. If we take the derivative of the trace of ${\gamma^+}_0$, we find that

$$\frac{d}{dt} tr(e^{JPt}A_0) = \lambda_0(\cos t - \sin t) - \frac{1}{\lambda_0}(\sin t + \cos t)$$

which is positive for $t < \tan^{-1}\left(\frac{\lambda_0^2-1}{\lambda_0^2+1}\right)$ and zero for $t = \tan^{-1}\left(\frac{\lambda_0^2-1}{\lambda_0^2+1}\right)$. At this local maximum, the trace of ${\gamma^+}_0$ is $2\lambda_0^3 + \frac{2}{\lambda_0} > 2\lambda_0^3$.

The idea for creating a positive path whose trace goes to ∞ involves gluing together successive paths of the type ${\gamma^+}_0$ using Lemma 2.2.1. We start at some diagonal matrix A_0 as above and let the first leg of γ^+ be ${\gamma^+}_0$ until time $t_0 = \tan^{-1}\left(\frac{\lambda_0^2-1}{\lambda_0^2+1}\right)$. By Theorem 3.3.2, there exists a positive path in the conjugacy class between ${\gamma^+}_0(t_0)$ and the diagonal element representing this conjugacy class, say A_1. We can glue this path and ${\gamma^+}_0$ together to get a positive path from A_0 ending at the diagonal element A_1 with $tr(A_1) > \lambda_0^3$.

We let the second leg of γ^+ be ${\gamma_1}^+(t) = e^{JPt}A_1$, or actually some reparametrization of this path to obtain the part where trace increases past λ_0^9 followed by a positive path in the conjugacy class to the diagonal element A_2 with $tr(A_2) > \lambda_0^9$. Continue in this manner gluing paths together, using e^{JPt} to increase the trace followed by a path to the diagonal element of the conjugacy class. We can see that the resultant path will have trace tending to ∞, as with each step the trace not only increases, but it grows in a polynomial fashion. $\square$

Chapter 4

Positive paths in Sp(4)

The goal of this chapter is to prove Theorem 4.1.1. First, we must carefully analyze the space *Conj* (Sp(4)).

4.1 The topology of *Conj* (Sp(4))

Conj ($Sp(4)$) is substantially more complicated than *Conj* ($Sp(2)$); here we briefly recall its topology as described in [LM97]. Remember, we have the splitting number we can associate to simple eigenvalues on the circle which gives us a notion of directionality, but we have no corresponding idea for other eigenvalues.

Generic regions:

(i) $\mathcal{O}_C$, consisting of all matrices with 4 distinct eigenvalues in $\mathbf{C} - (\mathbf{R} \cup \mathbf{S^1})$; one conjugacy class for each quadruple;

(ii) $\mathcal{O}_U$, consisting of all matrices with eigenvalues on S^1 where each eigenvalue has multiplicity 1 (or multiplicity 2 with non-zero splitting numbers); four (or two) conjugacy classes for each quadruple corresponding to the possible splitting numbers;

(iii) $\mathcal{O}_{\mathcal{R}}$, consisting of all matrices whose eigenvalues have multiplicity 1 and lie on $\mathbf{R} - \{0, 1, -1\}$; one conjugacy class for each quadruple;

(iv) $\mathcal{O}_{\mathcal{U},\mathcal{R}}$, consisting of all matrices with 4 distinct eigenvalues, one pair on $S^1 - \{1, -1\}$ and the other on $\mathbf{R} - \{0, 1, -1\}$; two conjugacy classes for each quadruple corresponding to the possible splitting numbers of the pair on S^1.

Codimension 1 boundaries of these regions:

(v) $\mathcal{B}_{\mathcal{U}}$, consisting of all non-diagonalizable matrices whose spectrum consists of a pair of conjugate points in $S^1 - \{1, -1\}$ each of multiplicity 2 and splitting number 0; two conjugacy classes for each quadruple: $\mathcal{B}_{\mathcal{U}}^-$ containing those matrices from which positive paths enter $\mathcal{O}_{\mathcal{C}}$ and $\mathcal{B}_{\mathcal{U}}^+$ containing those matrices from which positive paths enter $\mathcal{O}_{\mathcal{U}}$;

(vi) $\mathcal{B}_{\mathcal{R}}$, consisting of all non-diagonalizable matrices whose spectrum is a pair of distinct points $\lambda, 1/\lambda \in \mathbf{R} - \{0, 1, -1\}$ each of multiplicity 2; one conjugacy class for each quadruple;

(vii) $\mathcal{B}_{\mathcal{U},1}$, consisting of all non-diagonalizable matrices with spectrum $\{\lambda, \overline{\lambda}, \pm 1, \pm 1\}$ with $\lambda \in S^1 - \{1, -1\}$; two conjugacy classes for each quadruple, corresponding to N_1^- (call this one $\mathcal{B}_{\mathcal{U},1}^-$) and N_1^+ (call this one $\mathcal{B}_{\mathcal{U},1}^+$);

(viii) $\mathcal{B}_{\mathcal{R},1}$, consisting of all non-diagonalizable matrices with spectrum $\{\lambda, 1/\lambda, \pm 1, \pm 1\}$ with $\lambda \in \mathbf{R} - \{0, 1, -1\}$; two conjugacy classes for each quadruple, corresponding to N_1^- (call this one $\mathcal{B}_{\mathcal{R},1}^-$) and N_1^+ (call this one $\mathcal{B}_{\mathcal{R},1}^+$).

It is useful to remember that generic positive paths move from $\mathcal{O}_{\mathcal{U}}$ to $\mathcal{O}_{\mathcal{C}}$ through $\mathcal{B}_{\mathcal{U}}^-$ and move back into $\mathcal{O}_{\mathcal{U}}$ through $\mathcal{B}_{\mathcal{U}}^+$. Postive paths from $\mathcal{O}_{\mathcal{R}}$ to $\mathcal{O}_{\mathcal{C}}$ and from $\mathcal{O}_{\mathcal{C}}$ to $\mathcal{O}_{\mathcal{R}}$ pass through $\mathcal{B}_{\mathcal{R}}$. Positive paths going from $\mathcal{O}_{\mathcal{U}}$ to $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ pass through $\mathcal{B}_{\mathcal{U},1}^-$ and they return to $\mathcal{O}_{\mathcal{U}}$ through $\mathcal{B}_{\mathcal{U},1}^+$. Finally, positive paths moving from $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ to $\mathcal{O}_{\mathcal{R}}$ pass through $\mathcal{B}_{\mathcal{R},1}^-$, and those returning to $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ pass through $\mathcal{B}_{\mathcal{R},1}^+$.

In addition, there are two important strata of higher codimension:

(ix) $\mathcal{B}_{\mathcal{R},\mathcal{D}}$, consisting of all diagonalizable matrices with two real eigenvalues each of multiplicity two; 1 conjugacy class for each quadruple;

(x) $\mathcal{B}_{\mathcal{U},\mathcal{D}}$, consisting of all diagonalizable matrices with a conjugate pair of eigenvalues on S^1, each of multiplicity two with 0 splitting number; 1 conjugacy class for each quadruple.

We now state the main theorem of this chapter:

Theorem 4.1.1 *Let* $A_t, B_t : [0, 2\pi] \to \mathrm{Sp}(4)$ *be positive loops in* $\mathrm{Sp}(4)$ *with base point* I. *Then* A_t *and* B_t *are homotopic if and only if they are homotopic through positive loops. Hence, the natural map*

$$\pi_{1,pos}(\mathrm{Sp}(4)) \to \pi_1(\mathrm{Sp}(4))$$

is injective and onto $\mathbf{N} - \{1\}$.

Certainly, if two loops are homotopic through positive loops, then they are homotopic.

The proof of the converse will come in several steps. By Proposition 2.2.5, it will be sufficient to produce the positive homotopy of generic loops in *Conj* which can be lifted to Sp(4). We will carefully examine the stratification of *Conj* to determine the behavior of a generic positive path. The idea is to first show that $\pi(A_t)$ and $\pi(B_t)$ can be positively homotoped out of $\pi(\mathcal{O}_\mathcal{C})$, leaving two loops in *Conj* postively homotopic to $\pi(A_t)$ and $\pi(B_t)$ which are entirely contained in

$$\mathcal{S} = \pi(\mathcal{O}_\mathcal{U}) \cup \pi(\mathcal{O}_\mathcal{R}) \cup \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \cup \pi(\mathcal{B}_{\mathcal{U},1}) \cup \pi(\mathcal{B}_{\mathcal{R},1}) \cup \pi(\mathcal{B}_{\mathcal{U},\mathcal{D}}).$$

$\mathcal{S}$ is the set of all open strata with eigenvalues in $S^1 \cup \mathbf{R}$ along with some boundary components to make it a connected set. Then, we view these paths as residing in $Conj(\mathrm{Sp}(2)) \times Conj(\mathrm{Sp}(2)) \subset Conj(\mathrm{Sp}(4))$, allowing us to use results about Sp(2). Finally, we show that two standard paths which have eigenvalues traversing the circle with different speeds but with the same number of total rotations are positively homotopic. Using these lemmas we produce the homotopy in *Conj*, and then lift it to Sp(4) to prove the theorem. We will postpone the technical proofs to the last section.

4.2 Pushing positive paths out of $\pi(\mathcal{O}_\mathcal{C})$

Lemma 4.2.1 *Let A_t be a positive generic loop with base point I. Then, $\pi(A_t)$ can be positively homotoped out of $\pi(\mathcal{O}_\mathcal{C})$ to a loop contained in $\mathcal{S}$.*

Proof:

We can slightly perturb any path so that it enters $\mathcal{O}_\mathcal{C}$ only a finite number of times, hence we assume that $\pi(A_t)$ enters $\pi(\mathcal{O}_\mathcal{C})$ only a finite number of times. Krein shows that the very beginning and end of positive loops based at the identity must be in $\mathcal{O}_\mathcal{U}$. More specifically, he shows that there exist positive ϵ and δ such that for all times t where $0 < t < \epsilon$ and $2\pi - \delta < t < 2\pi$ the path is in $\mathcal{O}_\mathcal{U}$ [Eke89]. Therefore we need to consider the different ways in which $\pi(A_t)$ can leave $\pi(\mathcal{O}_\mathcal{U})$, enter $\pi(\mathcal{O}_\mathcal{C})$, and return to $\pi(\mathcal{O}_\mathcal{U})$, and construct positive homotopies from each type to paths in $\mathcal{C}onj$ which remain in $\mathcal{S}$. Then, we can positively homotop each escape into $\pi(\mathcal{O}_\mathcal{C})$ back into $\mathcal{S}$ individually to result in a loop in $\mathcal{C}onj$ postively homotopic to $\pi(A_t)$ and entirely contained in $\mathcal{S}$.

First, notice that no positive path can travel directly from $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ to $\mathcal{O}_\mathcal{C}$ or $\mathcal{O}_\mathcal{C}$ to $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ without crossing a boundary component of codimension greater than one. Therefore, since A_t is generic, it cannot contain these transitions. Similarly, A_t cannot go directly from $\mathcal{O}_\mathcal{U}$ to $\mathcal{O}_\mathcal{R}$ or vice versa without crossing a higher codimensional boundary; to avoid this, it must pass through $\mathcal{O}_{\mathcal{U},\mathcal{R}}$ or $\mathcal{O}_\mathcal{C}$ at an intermediate time.

If $\pi(A_t)$ travels directly from $\pi(\mathcal{O}_\mathcal{C})$ to $\pi(\mathcal{O}_\mathcal{R})$ and back to $\pi(\mathcal{O}_\mathcal{C})$, Lalonde and McDuff show how it can be perturbed to stay only in $\pi(\mathcal{O}_\mathcal{C})$ [LM97].

Taking this into account, there are four distinct ways for $\pi(A_t)$ to leave

$\pi(\mathcal{O}_{\mathcal{U}})$, enter $\pi(\mathcal{O}_{\mathcal{C}})$, and return to $\pi(\mathcal{O}_{\mathcal{U}})$:

$$\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \tag{1}$$

$$\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \tag{2}$$

$$\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \tag{3}$$

$$\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \tag{4}$$

At each transition, the path crosses the appropriate codimension one boundary. Note that in each case, when the path is in $\pi(\mathcal{O}_{\mathcal{R}})$, it has either come directly from or will go directly into $\pi(\mathcal{O}_{\mathcal{C}})$. When passing between $\pi(\mathcal{O}_{\mathcal{R}})$ and $\pi(\mathcal{O}_{\mathcal{C}})$, both real eigenvalues of multiplicty two are positive, or both are negative. It is impossible to travel in real numbers from positive to negative without going through zero, and no symplectic matrix has 0 for an eigenvalue. Therefore, all four eigenvalues will remain positive or all will remain negative for the entire time that $\pi(A_t)$ is in $\pi(\mathcal{O}_{\mathcal{R}})$.

Any generic positive path in $\mathcal{C}onj$ can be broken up into finitely many sections which lie in $\mathcal{S}$ connected by parts of type (1), (2), (3), or (4). Note that in between each escape into $\pi(\mathcal{O}_{\mathcal{C}})$, while the path is in $\mathcal{S}$, there is a time when one pair of eigenvalues is $\{1, 1\}$ and a time where one pair is $\{-1, -1\}$. This is due to Lemma 2.1.2 and the fact that eigenvalues with positive and negative splitting number must meet on S^1 in order for the path to cross $\pi(\mathcal{B}_{\mathcal{U}}^-)$ and enter $\pi(\mathcal{O}_{\mathcal{C}})$. Hence, the different journeys into $\pi(\mathcal{O}_{\mathcal{C}})$ are separated by time and will not overlap at all. If we could show how to positively homotop any path of type (1), (2), (3), or (4) back into $\mathcal{S}$, we could start with the first diversion that occurs (with respect to time) of $\pi(A_t)$ into $\pi(\mathcal{O}_{\mathcal{C}})$, homotop it back into $\mathcal{S}$, continue in the same way one at a time with subsequent diversions, and eventually end up with a path contained entirely in $\mathcal{S}$ and positively homotopic to $\pi(A_t)$. Thus, the proof of Lemma 4.2.1 is now reduced to showing that any path of type (1), (2), (3), or (4) in *Conj* is positively homotopic to a positive path which lies in $\mathcal{S}$.

Note that (2) and (3) are opposites. If we can perturb case (3) properly , then we can also perturb case (2) in a similar manner. Thus, we will only work out the details for cases (1), (3), and (4).

Lemma 4.2.2 *Any path a_t of type (1) in Conj is positively homotopic to a positive path which lies in $\mathcal{S}$.*

Proof: Using Lemma 2.1.2 we can see that all paths of type (1) with the same endpoints in $\pi(\mathcal{O}_{\mathcal{U}})$ are homotopic. It is therefore sufficient to consider a model path of type (1) and produce the homotopy for this case. We assume the eigenvalues of a_t remain on one pair of conjugate rays in $\pi(\mathcal{O}_{\mathcal{C}})$, and that a_t simply goes out along these rays to a point where the norm of the largest eigenvalue is k and comes back. Denote the elements of $\pi(\mathcal{B}_{\mathcal{U}}^-)$ and $\pi(\mathcal{B}_{\mathcal{U}}^+)$ where a_t enters and leaves $\pi(\mathcal{O}_{\mathcal{U}})$ as $\pi(X^-)$ and $\pi(X^+)$, respectively.

We will find a continuous family of positive paths in Sp(4) which leave $\mathcal{O}_{\mathcal{U}}$ at X^-, go into $\mathcal{O}_{\mathcal{C}}$ along the appropriate ray, return along that ray, and re-enter $\mathcal{O}_{\mathcal{U}}$ at X^+. These paths should travel successively less far into $\mathcal{O}_{\mathcal{C}}$, with their limit not going into $\mathcal{O}_{\mathcal{C}}$ at all, but staying on $\mathcal{O}_{\mathcal{U}}$ and passing through $X \in \mathcal{B}_{\mathcal{U},\mathcal{D}}$. Then, the projection of these paths to *Conj* gives us the homotopy required by the lemma. Note that if we find this continuous family of positive paths for one X, we can do so for any other $Y \in \mathcal{B}_{\mathcal{U},\mathcal{D}}$ by multiplying by

$X^{-1}Y$. Hence, without loss of generality, we can assume that

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

which has eigenvalues $\{i, i, -i, -i\}$.

Consider the path in Sp(4)

$$\gamma_k(r) = e^{Jr} \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1/k \\ -k & 0 & 0 & 0 \\ 0 & -1/k & 0 & 0 \end{pmatrix}$$

as r varies in a neighborhood of 0. The eigenvalues of γ_k travel around the circle, leaving at X^- when r is such that

$$\cos^2 r = \frac{4k^2}{(1+k^2)^2}$$

to travel up the imaginary axis to the point $\{ki, -ki, i/k, -i/k\} = \pi(\gamma_k(0))$. Then, they move back down the imaginary axis to X^+, and re-enter the circle. All the while that γ_k is in $\mathcal{O}_C$, its eigenvalues stay on the imaginary axis. The family γ_k as we let $k \to 1$ is the continuous family of positive paths we need. Note that the last path in the homotopy will go through the non-generic stratum $\mathcal{B}_{\mathcal{U},\mathcal{D}}$. □

Case (4) requires us to consider exactly what part of $\pi(\mathcal{O}_\mathcal{R})$ $\pi(A_t)$ enters. First consider the case where both journeys into $\pi(\mathcal{O}_\mathcal{R})$ are in $\pi(\mathcal{O}_\mathcal{R}^+)$ or both are in $\pi(\mathcal{O}_{\mathcal{R}}^-)$. We know from Lemma 2.1.2 that movement in $\pi(\mathcal{O}_{\mathcal{C}})$ is unrestricted by the positivity condition; hence we can positively collapse the portion in $\pi(\mathcal{O}_{\mathcal{C}})$ back to either $\pi(\mathcal{O}_{\mathcal{R}}^+)$ or $\pi(\mathcal{O}_{\mathcal{R}}^-)$. If, instead, this part of $\pi(A_t)$ moves $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^+) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^-) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$ or its opposite, the analysis is more complicated. We will call these cases (4a) and come back to them later.

Now let us consider case (3). Assume without loss of generality that $\pi(A_t)$ enters $\pi(\mathcal{O}_{\mathcal{R}}^+)$ instead of $\pi(\mathcal{O}_{\mathcal{R}}^-)$. We can describe scenario (3) by graphing the motion of the eigenvalues in the complex plane as in Figure 4.1.

To begin, all four eigenvalues are on the circle, two conjugate pairs approaching the real axis. Then, the first pair passes through $\mathcal{N}_1^-$, enters the real axis and the path is in $\pi(\mathcal{O}_{\mathcal{U},\mathcal{R}})$. The second pair, still on the circle, migrates to the real axis also, eventually meets the first pair, and we have two real eigenvalues of multiplicity two. At that moment, which we assume to be $t = \frac{1}{2}$, the path breaks into $\pi(\mathcal{O}_{\mathcal{C}})$. Eventually the eigenvalues return to the circle as two plus/minus pairs, and continue rotating in the required direction.

Lemma 4.2.3 *Any path a_t of type (3) in Conj is positively homotopic to a positive path which is contained in $\mathcal{S}$.*

Proof: Using Lemma 2.1.2 it is easy to see that all generic paths of type (3) with the same end points in $\pi(\mathcal{O}_{\mathcal{U}})$ are positively homotopic. Therefore, it suffices to start with one path of this type and first show how to homotop it to a certain standard path b_t. b_t has the same first two configurations as a_t, but, instead of the second pair entering the real axis, the first pair re-enters

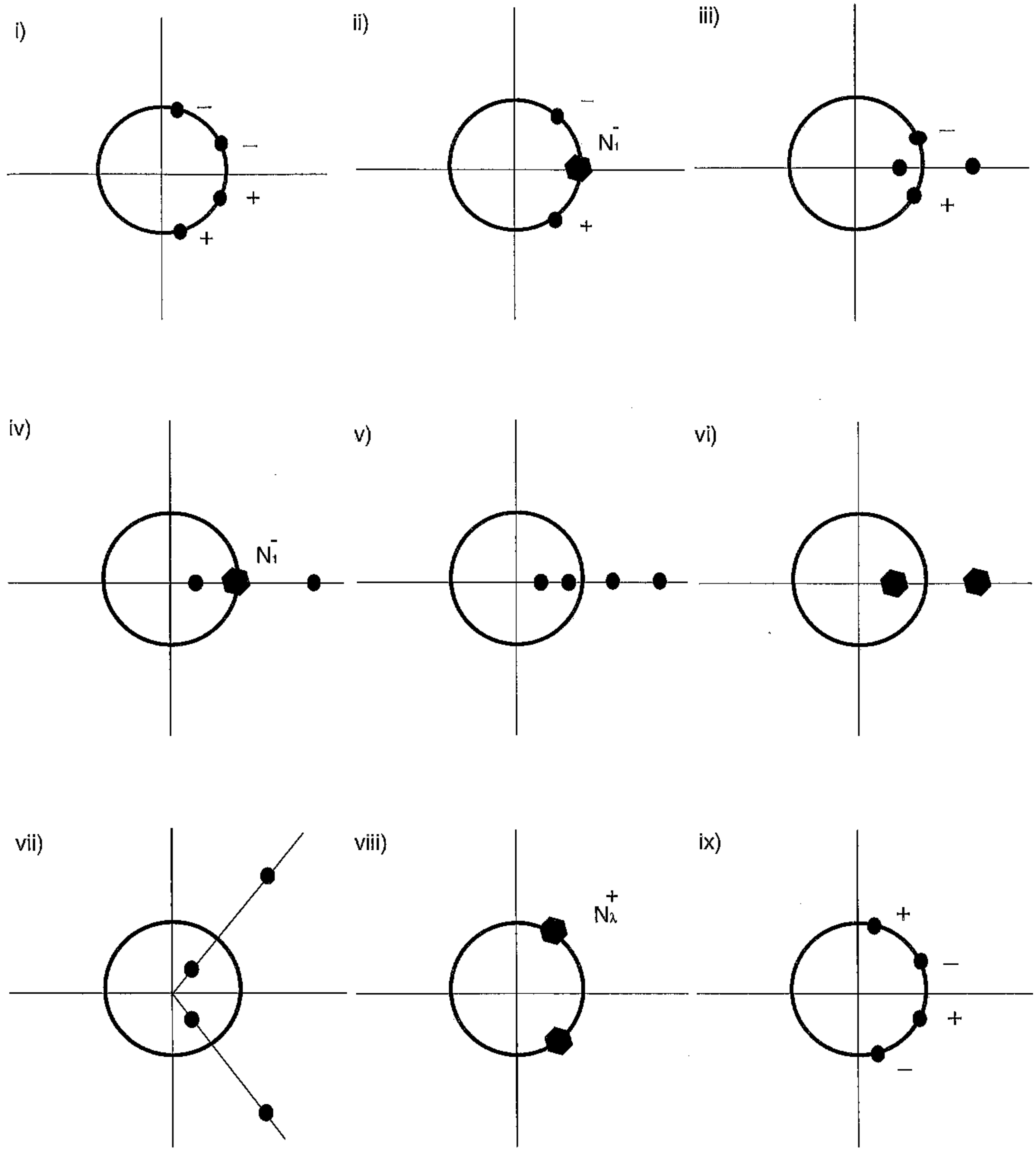

Figure 4.1: a_t in Case 3

the circle and the path is in $\pi(\mathcal{O}_{\mathcal{U}})$ again. Then, the positive eigenvalue from the first pair meets the eigenvalue with negative splitting number from the second pair, and vice versa, and the path escapes into $\pi(\mathcal{O}_{\mathcal{C}})$. Finally, this path returns directly to $\pi(\mathcal{O}_{\mathcal{U}})$. b_t is depicted in Figure 4.2.

The path b_t is of type (1), and we have already shown in Lemma 4.2.2 how to positively homotope such paths out of $\pi(\mathcal{O}_{\mathcal{C}})$. Therefore, if we can construct the positive homotopy from a_t to b_t, we will be done with case (3).

The family of positive paths a_t^s (where $a_t^0 = a_t$ and $a_t^1 = b_t$) we need to construct will all start in $\pi(\mathcal{O}_{\mathcal{U}})$ and then go into $\pi(\mathcal{O}_{\mathcal{U},\mathcal{R}})$. Here, s is the homotopy variable and t is the time variable. The first paths in the family will then enter $\pi(\mathcal{O}_{\mathcal{R}}^+)$ and break away into $\pi(\mathcal{O}_{\mathcal{C}})$ at time $t = \frac{1}{2}$, just as a_t does. The point at which the a^s enter $\pi(\mathcal{O}_{\mathcal{C}})$ will progressively get closer and closer to and eventually hit the class of some matrix with eigenvalues $\{1,1,1,1\}$ at $s = \frac{1}{2}$ in *Conj*. The paths subsequent to this will not enter $\pi(\mathcal{O}_{\mathcal{R}}^+)$, but rather will go back to $\pi(\mathcal{O}_{\mathcal{U}})$ from $\pi(\mathcal{O}_{\mathcal{U},\mathcal{R}})$. These paths will enter $\pi(\mathcal{O}_{\mathcal{C}})$ from $\pi(\mathcal{B}_{\mathcal{U}}^-)$ at time $t = \frac{1}{2}$ at points starting from the class of the matrix with 1 as a quadruple eigenvalue, and travel up the circle. Every path in the family will reach $\pi(\mathcal{O}_{\mathcal{C}})$ and travel back to $\pi(\mathcal{O}_{\mathcal{U}})$ ending at the same point as a_t and b_t.

Since movement in $\pi(\mathcal{O}_{\mathcal{C}})$ is not restricted under positivity, it suffices to find the family of positive paths a_s^t at the infinitesmal level. We will only construct the path $a_{\frac{1}{2}}^s$ and its forward and backward tangent vectors to a_s^t at $t = \frac{1}{2}$, since the rest of the construction is straightforward. We need to find two continuous vector fields along a continuous (not necessarily positive) path

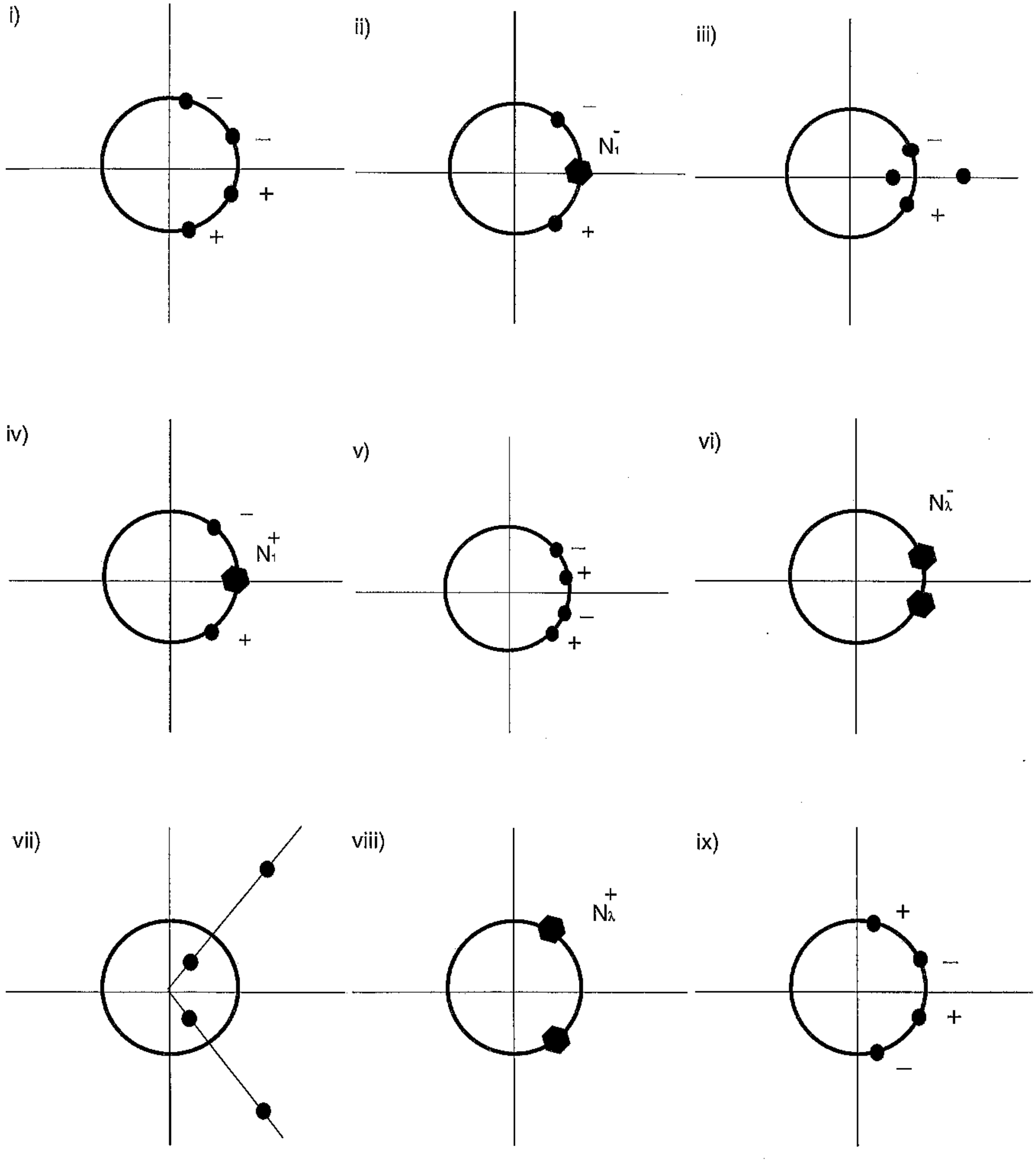

Figure 4.2: The standard path b_t

$q^s = a^s_{\frac{1}{2}} \in \pi(\mathcal{B}_\mathcal{R} \cup \mathcal{B}^-_\mathcal{U})$ (except when $s = \frac{1}{2}$) which goes from $q^0 \in \pi(\mathcal{B}_\mathcal{R})$, through some point with eigenvalues $\{1, 1, 1, 1\}$ at $s = \frac{1}{2}$, to $q^1 \in \pi(\mathcal{B}^-_\mathcal{U})$. Here, q^0 is the point in *Conj* where a_t enters $\pi(\mathcal{O}_\mathcal{C})$, and q^1 is the point in *Conj* where b_t enters $\pi(\mathcal{O}_\mathcal{C})$. We need to find one positive continuous vector field pointing into $\pi(\mathcal{O}_\mathcal{C})$ at every point along q^s, and one negative continuous vector field pointing into $\pi(\mathcal{O}^+_\mathcal{R})$ at the points on q^s with real eigenvalues and pointing into $\pi(\mathcal{O}_\mathcal{U})$ for all other points on q^s. We will explicitly find a lift Q^s of such a path and vector fields in Sp(4); their projections to *Conj* will satisfy the required properties. We set

$$\mathcal{N}_1^{-,-} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The proof of Lemma 4.2.3 is now reduced to the following:

Lemma 4.2.4 *There exists a path $Q^s : [0,1] \to \mathcal{B}_\mathcal{R} \cup \mathcal{B}^-_\mathcal{U} \cup \mathcal{N}_1^{-,-}$ where $Q^0 \in \mathcal{B}_\mathcal{R}$, $Q^{\frac{1}{2}} = \mathcal{N}_1^{-,-}$, and $Q^1 \in \mathcal{B}^-_\mathcal{U}$ satisfying two properties:*

(i) *There exists a (positive) vector field along Q^s pointing into $\mathcal{O}_\mathcal{C}$ of the form JPQ^s for a positive definite P.*

(ii) *There exists a negative vector field along Q^s pointing into $\mathcal{O}^+_\mathcal{R}$ when $Q^s \in \mathcal{B}_\mathcal{R}$ and pointing into $\mathcal{O}_\mathcal{U}$ elsewhere.*

Proof: The proof of this lemma will be deferred to Section 5. □

Now, the analysis for case (3) is finished. We leave case (2) to the reader because it is very similar to case (3), and we are left only with case (4a): $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^{+}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^{-}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$ or $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^{-}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}^{+}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$. This case is a combination of cases (2) and (3). Homotop the first part of the path from $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}})$ to $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}})$ exactly the same way as in case (3). Then, homotop the second part from $\pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$ to $\pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$ exactly the same way as in case (2). This leaves a type (1) path in *Conj* positively homotopic to $\pi(A_t)$ which travels $\pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{C}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U},\mathcal{R}}) \Rightarrow \pi(\mathcal{O}_{\mathcal{U}})$. Since type (1) cases have already been examined, the proof of Lemma 4.2.1 is now complete. □

4.3 Constructing the positive homotopy

In this section, we construct the homotopy necessary to prove Theorem 4.1.1.

Lemma 4.3.1 *If a_t is a positive loop in S based at I constructed by the methods of Lemma 4.2.1, then a_t is positively homotopic in Conj to*

$$\pi \begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J\ell t} \end{pmatrix}$$

for some positive integers k, ℓ.

Proof: Note that the only time a_t may go through a point with two eigenvalues of multiplicity two is when forced to go through $\mathcal{B}_{\mathcal{U},\mathcal{D}}$ as in Lemma 4.2.2. However, for each double pair of eigenvalues, there is only one conjugacy class in $\pi(\mathcal{B}_{\mathcal{U},\mathcal{D}})$ which we can write as the class of an element in block diagonal form. Hence, we can find positive loops $X_t, Y_t \in \mathrm{Sp}(2)$ such that

$$\pi \begin{pmatrix} X_t & 0 \\ 0 & Y_t \end{pmatrix} = a_t \in \mathcal{C}onj$$

where $X_0 = Y_0 = I$. The result now follows from Theorem 3.2.3 $\square$

The next lemma shows that the positive homotopy class of

$$\begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J\ell t} \end{pmatrix} \in \mathrm{Sp}(4)$$

where $k, \ell > 0$ depends only on the sum $k+\ell$, the same invariant as the regular homotopy class. Let $\sim_+$ mean positively homotopic.

Lemma 4.3.2 *Let $k, \ell \geq 1$, $n \geq 3$, $n > k$, $n > \ell$ where k, ℓ, n are all integers. Then,*

$$\begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J(n-k)t} \end{pmatrix} \sim_+ \begin{pmatrix} e^{J\ell t} & 0 \\ 0 & e^{J(n-\ell)t} \end{pmatrix}$$

where $J \in \mathrm{Sp}(2)$.

Proof: The straightforward but detailed proof of this lemma is deferred to Section 5.

Now we have all the tools to prove the main theorem.

Proof of Theorem 4.1.1:

Take A_t and B_t to be two homotopic positive loops in Sp(4) based at I. By Lemmas 4.2.1 , 4.3.1 , and 4.3.2 , $\pi(A_t)$ is positively homotopic to $\pi(B_t)$. Denote this homotopy in *Conj* by $h(s,t)$.

The final step in the proof will be to use $h(s,t)$ to produce a homotopy $H(s,t) \in \mathrm{Sp}(4)$ where $H(0,t) = a_t$ and $H(1,t) = b_t$. If all of the loops in $h(s,t)$ are generic in *Conj* except at the base point I, then by Proposition 2.2.5, $h(s,t)$ can be lifted to Sp(4) and the proof of the theorem is complete. Consider the case, then, when some loop in $h(s,t)$ is not generic; i.e. there exists some $s_0 \in [0,1]$ such that $h(s_0,t)$ passes through a boundary component of codimension greater than 1 or stays in a codimension 1 boundary stratum for more than one instant. Note that $\pi(A_t)$ and $\pi(B_t)$ are generic, so one of the steps in the construction of $h(s,t)$ above must have introduced this nongeneric behavior. There are three isolated ways in which this can happen:

(i) by the construction in Lemma 4.2.2 where a path goes through the stratum of diagonalizable elements with 2 pairs of double eigenvalues $\{\lambda, \lambda, \overline{\lambda}, \overline{\lambda}\}$ on the circle, $\pi(\mathcal{B}_{\mathcal{U},\mathcal{D}})$,

(ii) while being homotoped out of $\pi(\mathcal{O}_C)$, by the construction in case (2),(3) or (4a) of Lemma 4.2.1 where the paths are forced to go through $\pi(\mathcal{B}_{\mathcal{R},\mathcal{D}})$ or $N_1^{-,-}$,

(iii) in the proof of Lemma 4.3.1 where loops are forced to pass through I or $-I$.

The proof of Proposition 2.2.5 which allows us to lift a positive homotopy of generic loops fails if a loop is non-generic. To connect the $H^i(s_{i+1}, t)$ to $H^{i+1}(s_{i+1}, t)$ via positive loops using the Proposition 2.2.4 , we need to know that $h(s_{i+1}, t)$ is a generic loop in $\mathcal{C}onj$. However, the argument can be patched rather easily for the particular homotopy $h(s, t)$ constructed above. It is enough to show how to produce H locally around s_{i+1} when $h(s_{i+1}, t)$ has one diversion into $\pi(\mathcal{B}_{\mathcal{U},\mathcal{D}})$ or $\pi(\mathcal{B}_{\mathcal{R},\mathcal{D}})$ or $N_1^{-,-}$ as produced in Lemma 4.2.1 and when there are finitely many points at I or $-I$ as in Lemma 4.3.1. The final three lemmas complete our discussion.

Lemma 4.3.3 *If $h(s_{i+1}, t)$ is non generic because it enters $\pi(\mathcal{B}_{\mathcal{U},\mathcal{D}})$ at time $t = t_0$ as in Lemma 4.2.2, we can construct a local lifting of h.*

Proof: In the proof of Lemma 4.2.2 we actually constructed a lift of h for s, t in some interval $[s_{i+1} - \epsilon, s_{i+1} + \epsilon] \times [t_0 - \delta, t_0 + \delta]$. However, the paths at $s = s_{i+1} \pm \epsilon$ are not generic, as they still go through $\mathcal{B}_{\mathcal{U},\mathcal{D}}$ at time $t = t_0$. It is not hard to see that one can stil patch these different local lifts by the argument of Proposition 2.2.4. The important thing is that the fibres of π are always connected and there is only one non-generic point on each path. $\square$

Lemma 4.3.4 *If $h(s_{i+1}, t)$ is non generic because it enters $\pi(\mathcal{B}_{\mathcal{R},\mathcal{D}})$ or $N_1^{-,-}$ as in Lemma 4.2.1, we can construct a local lifting of h.*

Proof: In the proof of this Lemma 4.2.1, we actually constructed a lift $H(s, t)$ of $h(s, t)$ for $s \in [s_{i+1} - \delta, s_{i+1} + \delta]$ for some $\delta > 0$ such that $h(s_{i+1} - \delta, t)$ and $h(s_{i+1} + \delta, t)$ are generic loops in $\mathcal{C}onj$. We can relabel the s_i appopriately and apply the remainder of the proof of Proposition 2.2.5 to lift the entire homotopy. □

Lemma 4.3.5 *If $h(s_{i+1}, t)$ is non generic because it passes through I or $-I$ at times other than* 0 *and* 2π*, we can construct a local lifting of* h.

Proof: By compactness, there are finitely many such times, say $\{t_j\} |_{1\leq j\leq N}$. Then, for each interval $[t_j, t_{j+1}]$, $h(s_{i+1}, t)$ is is a positive generic path in *Conj* starting and ending at I or $-I$. Call this path $h_j(s_{i+1}, t)$. By Lemma 2.2.4, the space of positive lifts of $h_j(s_{i+1}, t)$ is path connected. Thus, we can connect $H^i_j(s_{i+1}, t)$ to $H^{i+1}_j(s_{i+1}, t)$ for each $1 \leq j \leq N$ independently, and arrive at a piecewise positive homotopy in Sp(4) between $H^i(s_{i+1}, t)$ and $H^{i+1}(s_{i+1}, t)$. Since piecewise positive paths can be approximated arbitrarily closely by positive paths, we can find a positive homotopy in Sp(4) between $H^i(s_{i+1}, t)$ and $H^{i+1}(s_{i+1}, t)$. As in the proof of Proposition 2.2.5, we patch together the $H^i(s_{i+1}, t)$ and $H^{i+1}(s_{i+1}, t)$ to obtain $H(s, t)$. □

Chapter 5

Technical Proofs for Positive Paths

This chapter contains the proofs of the technical lemmas needed in Chapter 4. We will restate the lemmas here for the convenience of the reader.

Lemma 4.2.4 *There exists a path* $\Delta^s : [0,1] \to \mathcal{B}_{\mathcal{R}} \cup \mathcal{B}_{\mathcal{U}}^- \cup \mathcal{N}_1^{-,-}$ *where* $\Delta^0 \in \mathcal{B}_{\mathcal{R}}$, $\Delta^{\frac{1}{2}} = \mathcal{N}_1^{-,-}$, *and* $\Delta^1 \in \mathcal{B}_{\mathcal{U}}^-$ *satisfying two properties:*

(i) *There exists a (positive) vector field along* Δ^s *pointing into* $\mathcal{O}_{\mathcal{C}}$ *of the form* $JP\Delta^s$ *for a positive definite* P.

(ii) *There exists a negative vector field along* Δ^s *pointing into* $\mathcal{O}_{\mathcal{R}}^+$ *when* $\Delta^s \in \mathcal{B}_{\mathcal{R}}$ *and pointing into* $\mathcal{O}_{\mathcal{U}}$ *elsewhere.*

Proof:

First, we need to construct the path Δ^s. The first part of Δ^s will travel within the boundary components from $\Delta^0 \in \mathcal{B}_{\mathcal{R}}$ where $\pi(\Delta^0) = \delta^0$ to $\Delta^\epsilon \in \mathcal{B}_{\mathcal{R},\mathcal{D}}$.

Suppose that $\delta^\epsilon = \pi(\Delta^\epsilon) \in \pi(\mathcal{B}_{\mathcal{R},\mathcal{D}})$ has eigenvalues $\lambda, \lambda, \frac{1}{\lambda}, \frac{1}{\lambda}$ and $\delta^1 \in \pi(\mathcal{B}_{\mathcal{U}}^-)$ has eigenvalues $a+bi, a+bi, a-bi, a-bi$ where $a^2+b^2=1$. Let

$\Delta^s : [0,1] \to \mathrm{Sp}(4)$ be the path defined as

$$\Delta^s = \begin{cases} \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} & if \quad \epsilon < s < \frac{1}{2} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & if \quad s = \frac{1}{2} \\ \begin{pmatrix} x & x & \sqrt{1-x^2} & \sqrt{1-x^2} \\ 0 & x & 0 & \sqrt{1-x^2} \\ -\sqrt{1-x^2} & -\sqrt{1-x^2} & x & x \\ 0 & -\sqrt{1-x^2} & 0 & x \end{pmatrix} & if \quad s > \frac{1}{2} \end{cases}$$

where $x = 2 - a + (2a-2)s$ and $\mu = \frac{\lambda - 1}{\epsilon - \frac{1}{2}}(s - \frac{1}{2}) + 1$. Then $\pi(\Delta^s) = \delta^s$ lies in the appropriate regions.

We will now look for a positive continuous vector field along Δ^s which points into $\mathcal{O}_\mathcal{C}$ at every point and project it to *Conj* to get the needed vector fields along δ^s. The original path A_t gives us one positive vector, say v_0, pointing into $\mathcal{O}_\mathcal{C}$ at Δ^0. We claim that $JP\Delta^s$ is a positive vector at Δ^s

pointing into $\mathcal{O}_{\mathcal{C}}$ for all $\epsilon < s \leq 1$, where

$$P = \begin{pmatrix} 10 & 0 & 0 & 1 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 1 & 0 & 0 & 12 \end{pmatrix}.$$

Since the positive cone is open and convex, join v_0 to $JP\Delta^{\epsilon}$ by a family of poisitive vectors pointing into $\mathcal{O}_{\mathcal{C}}$ along the path Δ^s for $0 < s < \epsilon$. Then, we can continue the vector field along Δ^s by letting the tangent vector at time s equal $JP\Delta^s$ for all $\epsilon \leq s \leq 1$. This vector field is certainly continuous and positive, we need only prove the claim that it points into $\mathcal{O}_{\mathcal{C}}$ for all time.

When $s > \frac{1}{2}$, $\Delta^s \in \mathcal{B}_{\mathcal{U}}^-$, and thus any positive vector points into $\mathcal{O}_{\mathcal{C}}$. Also, by construction, our positive vector field points into $\mathcal{O}_{\mathcal{C}}$ for $s < \varepsilon$. Hence, we need only consider $\varepsilon \leq s \leq \frac{1}{2}$. We check the direction of these vectors by examining the behavior of the symmetric functions of the eigenvalues of paths in their directions. For all matrices in $\mathcal{B}_{\mathcal{R},\mathcal{D}} \cup \mathcal{B}_{\mathcal{R}} \cup \mathcal{N}_1^{-,-}$, $\sigma_2 = \frac{\sigma_1^2}{4} + 2$ while, on the other hand, matrices in $\mathcal{O}_{\mathcal{C}}$ satisfy $\sigma_2 > \frac{\sigma_1^2}{4} + 2$ and those in $\mathcal{O}_{\mathcal{R}}$ satisfy $\sigma_2 < \frac{\sigma_1^2}{4} + 2$.

We look at the derivatives

$$\tfrac{d}{dr}|_{r=0}\sigma_1(e^{JPr}\Delta^s) = \sigma_1'(s)$$
$$\tfrac{d}{dr}|_{r=0}\sigma_2(e^{JPr}\Delta^s) = \sigma_2'(s)$$

Since $\sigma_2 = \frac{\sigma_1^2}{4} + 2$ for all points on Δ^s for $s \leq \frac{1}{2}$, if $\sigma_2'(s) > (\frac{\sigma_1(s)^2}{4} + 2)'$, then we know that $JP\Delta^s$ points into $\mathcal{O}_{\mathcal{C}}$. More generally, if

$$\frac{d^k}{dr^k}|_{r=0}(\sigma_2(s)) = \frac{d^k}{dr^k}|_{r=0}(\frac{\sigma_1^2(t)}{4} + 2)$$

for all $k \leq n$, and

$$\frac{d^n}{dr^n}|_{r=0}(\sigma_2(s)) > \frac{d^n}{dr^n}|_{r=0}(\frac{\sigma_1^2(s)}{4} + 2)$$

then $JP\Delta^s$ points into $\mathcal{O}_C$.

Let us consider specifically the point $\Delta^{\frac{1}{2}} = \mathcal{N}_1^{-,-}$. If we examine the symmetric functions of $e^{JQr}\mathcal{N}_1^{-,-}$ for general symmetric

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ q_2 & q_5 & q_6 & q_7 \\ q_3 & q_6 & q_8 & q_9 \\ q_4 & q_7 & q_9 & q_{10} \end{pmatrix}$$

we find that $\sigma_2' = (\frac{\sigma_1^2}{4} + 2)'$ for all Q. Going to the second derivative, $\sigma_2'' < (\frac{\sigma_1^2}{4} + 2)''$, except if $q_3 = 0$ and $q_1 = q_8$, in which case $\sigma_2'' = (\frac{\sigma_1^2}{4} + 2)''$. Imposing these two restrictions on Q and looking at the third derivatives, we find $\sigma_2''' > (\frac{\sigma_1^2}{4} + 2)'''$ if $q_1 > 0$ and $q_4 \neq q_6$. Hence, $e^{JQr}\mathcal{N}_1^{-,-}$ is a positive path pointing into $\mathcal{O}_C$, if Q is a positive definite matrix satisfying $q_1 > 0$, $q_3 = 0$, $q_4 \neq q_6$, and $q_1 = q_8$. Indeed, the aforementioned matrix P satisfies these conditions, and we can check that

$$\begin{aligned} \sigma_2' = (\tfrac{\sigma_1^2}{4} + 2)' &= 20 \\ \sigma_2'' = (\tfrac{\sigma_1^2}{4} + 2)'' &= -680 \\ \sigma_2''' &= -17560 \\ (\tfrac{\sigma_1^2}{4} + 2)''' &= -17600 \end{aligned}$$

for the path $e^{JPr}\mathcal{N}_1^{-,-}$, and hence this path does travel into $\mathcal{O}_C$.

Additionally, consider the path $e^{JPr}\mathcal{N}_{1+y}^{-,-}$ where

$$\mathcal{N}_{1+y}^{-,-} = \begin{pmatrix} 1+y & 1 & 0 & 0 \\ 0 & \frac{1}{1+y} & 0 & 0 \\ 0 & 0 & 1+y & 1 \\ 0 & 0 & 0 & \frac{1}{1+y} \end{pmatrix}.$$

This path satisfies

$$\sigma_2' = (\tfrac{\sigma_1^2}{4} + 2)'$$
$$\sigma_2'' > (\tfrac{\sigma_1^2}{4} + 2)''$$

for all $y > 0$.

The matrices in Δ^s for $\varepsilon < s < \frac{1}{2}$ are all of the form $\mathcal{N}_{1+y}^{-,-}$ for some $y > 0$. Therefore, the positive vector field which we have constructed on this portion of the path, $JP\mathcal{N}_{1+y}^{-,-}$ points into $\mathcal{O}_{\mathcal{C}}$ and the proof of the claim is completed.

Finally, we need to construct a negative (so the reverse flow would be positive) vector field along Δ^s which points into $\mathcal{O}_{\mathcal{R}}$ in the direction of decreasing trace for $s < \frac{1}{2}$ and into $\mathcal{O}_{\mathcal{U}}$ for $s \geq \frac{1}{2}$. For $s \geq \frac{1}{2}$, $\Delta^s \in \mathcal{B}_{\mathcal{U}}^- \cup \mathcal{N}_1^{-,-}$, and all negative vectors based at Δ^s will point into $\mathcal{O}_{\mathcal{U}}$. Therefore, if we find any negative continuous vector field along Δ^s for $s < \frac{1}{2}$, any negative continuous extension of it will provide us with vectors pointing into $\mathcal{O}_{\mathcal{U}}$ for the duration of Δ^s. We can pick such an extension to match the tangent vector of γ at the point Δ^1.

For $\epsilon \le s \le \frac{1}{2}$, on Δ^s, we have block matrices of the form

$$\begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix}.$$

It would be sufficient, then, to find a negative definite 2×2 matrix Q_2 such that

$$JQ_2 \begin{pmatrix} \mu & 1 \\ 0 & \frac{1}{\mu} \end{pmatrix}$$

points into $\mathcal{O}_{\mathcal{R}}$ in the direction of decreasing trace for all μ. Then, set Q_4 equal to the 4×4 block matrix with Q_2 in the upper left and lower right blocks, and vector field $JQ_4\Delta^s$ is a negative, continuous vector field pointing into $\mathcal{O}_{\mathcal{R}}$ in the direction of decreasing trace for $\epsilon \le s \le \frac{1}{2}$. For $s < \epsilon$, we can continuously perturb Q_4 so that $JQ_4\Delta^s$ is a negative vector field pointng into $\mathcal{O}_{\mathcal{R}}$ in the direction of decreasing trace which matches the given tangent vector to A_t at Δ^0. However, matrices Q_2 are plentiful; one can be chosen which can be slightly perturbed along Δ^s to match the tangent vector to A_t at Δ^0. □

Lemma 4.3.2 *Let $k, \ell \ge 1$, $n \ge 3$, $n > k$, $n > \ell$ where k, ℓ, n are all integers. Then,*

$$\begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J(n-k)t} \end{pmatrix} \sim_+ \begin{pmatrix} e^{J\ell t} & 0 \\ 0 & e^{J(n-\ell)t} \end{pmatrix}$$

where $J \in \mathrm{Sp}(2)$.

Proof :

The positive homotopy between the two paths is

$$H(\theta,t) = \begin{pmatrix} e^{Jt} & 0 \\ 0 & e^{J(1+n-k-\ell)t} \end{pmatrix} P_\theta \begin{pmatrix} e^{J(k-1)t} & 0 \\ 0 & e^{J(\ell-1)t} \end{pmatrix} P_\theta^{-1}$$

for $t \in [0, 2\pi]$ and $\theta \in [0, \frac{\pi}{2}]$ where

$$P_\theta = \begin{pmatrix} \cos(\theta)I & -\sin(\theta)I \\ \sin(\theta)I & \cos(\theta)I \end{pmatrix} \in \mathrm{Sp}(4).$$

Here, I represents the 2×2 identity matrix. $H(\theta, t)$ is certainly a homotopy, as it is the product of symplectic matrices for all time and hence always contained in $\mathrm{Sp}(4)$, and

$$H(0,t) = \begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J(n-k)t} \end{pmatrix}$$

$$H(\frac{\pi}{2},t) = \begin{pmatrix} e^{J\ell t} & 0 \\ 0 & e^{J(n-\ell)t} \end{pmatrix}.$$

We must check that this is a positive homotopy, i.e. $H(\theta, t)$ is a positive path for any fixed $\theta \in [0, \frac{\pi}{2}]$. Let R be the 4×4 matrix such that

$$\frac{d}{dt}|_{t=t_0} H(\theta,t) = JRH(\theta,t_0).$$

Certainly, R depends on both θ and t_0. $H(\theta, t)$ is positive if and only if R is a positive definite matrix for all θ and for all t_0. R must be symmetric since $JRH(\theta, t_0)$ is in the tangent space of $\mathrm{Sp}(4)$ at the point $H(\theta, t_0)$, thus it will be sufficient to prove that the eigenvalues of R are positive real.

Without loss of generality, assume that $k > \ell$ and $k, \ell \leq \frac{n}{2}$. The second assumption is justified because ,

$$\begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J\ell t} \end{pmatrix} \sim_+ \begin{pmatrix} e^{J\ell t} & 0 \\ 0 & e^{Jkt} \end{pmatrix}$$

under the positive homotopy

$$G(\theta, t) = P_\theta \begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J\ell t} \end{pmatrix} P_\theta^{-1}$$

for $\theta \in [0, \frac{\pi}{2}]$. $G(\theta, t)$ is positive for any fixed θ since it is the conjugate of a positive path, and

$$G(0, t) = \begin{pmatrix} e^{Jkt} & 0 \\ 0 & e^{J\ell t} \end{pmatrix}$$

$$G(\frac{\pi}{2}, t) = \begin{pmatrix} e^{J\ell t} & 0 \\ 0 & e^{Jkt} \end{pmatrix}.$$

We now compute R to determine its eigenvalues. Let J denote both the standard 2×2 and 4×4 matrix, its dimension will be clear by context. Let

$r = 1 + n - k - \ell$ to make computations easier.

$$\frac{d}{dt}H(\theta,t) = \begin{pmatrix} Je^{Jt} & 0 \\ 0 & rJe^{Jrt} \end{pmatrix} P_\theta \begin{pmatrix} e^{J(k-1)t} & 0 \\ 0 & e^{J(\ell-1)t} \end{pmatrix} P_\theta^{-1} +$$

$$\begin{pmatrix} e^{Jt} & 0 \\ 0 & e^{Jrt} \end{pmatrix} P_\theta \begin{pmatrix} (k-1)Je^{J(k-1)t} & 0 \\ 0 & (\ell-1)Je^{J(\ell-1)t} \end{pmatrix} P_\theta^{-1}$$

$$= J\left(\begin{pmatrix} I & 0 \\ 0 & rI \end{pmatrix} + J^{-1}\begin{pmatrix} e^{Jt} & 0 \\ 0 & e^{Jrt} \end{pmatrix} P_\theta \times \right.$$

$$\left.\begin{pmatrix} (k-1)Je^{J(k-1)t} & 0 \\ 0 & (\ell-1)Je^{J(\ell-1)t} \end{pmatrix} P_\theta^{-1} H(\theta,t)^{-1}\right) H(\theta,t).$$

Multiplying the terms in the parentheses gives

$$R = \begin{pmatrix} (1 + (k-1)\cos^2\theta + (l-1)\sin^2\theta)I & \cos\theta\sin\theta(k-\ell)e^{Jt(1-r)} \\ \cos\theta\sin\theta(k-\ell)e^{Jt(r-1)} & (r + (k-1)\sin^2\theta + (\ell-1)\cos^2\theta)I \end{pmatrix}.$$

R has two eigenvalues of multiplicity two which happen to be independent of t:

$$\lambda_1 = \frac{1}{2}(n + \sqrt{(k-\ell)^2 + 2\cos(2\theta)(k-\ell)(1-r) + (1-r)^2})$$
$$\lambda_2 = \frac{1}{2}(n - \sqrt{(k-\ell)^2 + 2\cos(2\theta)(k-\ell)(1-r) + (1-r)^2}).$$

Certainly, since n is positive, λ_1 is positive for all θ. To check that λ_2 is positive, we must show

$$\sqrt{(k-\ell)^2 + 2\cos(2\theta)(k-\ell)(1-r) + (1-r)^2} \;<\; n.$$

Recall the previously justified assumptions that $k > \ell$ and $k, \ell \leq \frac{n}{2}$. If $k = \ell = \frac{n}{2}$, then $r = 1$ and the left hand side of the inequality is 0 which is certainly less than n. If, on the other hand, either k or ℓ is less than $\frac{n}{2}$, then $(1-r)$ is negative while $(k-\ell)$ is positive. Hence,

$$\begin{aligned}
\sqrt{(k-\ell)^2 + 2\cos(2\theta)(k-\ell)(1-r) + (1-r)^2} &\leq \sqrt{(k-\ell)^2 - 2(k-\ell)(1-r) + (1-r)^2} \\
&= \sqrt{((k-\ell) - (1-r))^2} \\
&= k - \ell - 1 + r \\
&= n - 2\ell \\
&< n
\end{aligned}$$

and thus λ_2 is positive for all θ. Hence, R is a positive definite matrix, and $H(\theta, t)$ is a positive homotopy. □

Chapter 6

The space $Ham^c(M)$ and criteria for length minimizing paths

In this chapter, we review the relevant material about paths in $Ham^c(M)$. In addition, we describe the techniques of Lalonde and McDuff for showing a path is length minimizing.

6.1 Background

If (M, ω) is a $2n$ dimensional symplectic manifold, any compactly supported time dependent Hamiltonian function H_t on M induces a vector field X_H on M defined by the equation

$$-dH(v) = \omega(X_H, v) \text{ for all } v \in TM.$$

The flow ϕ_t^H of X_H for $0 \leq t \leq 1$ is a path starting at the identity in the group $Ham^c(M)$ of compactly supported Hamiltonian symplectomorphisms of M. Recall that any path $\phi_t \in Ham^c(M))$ is actually the time t flow of some Hamiltonian H_t, and the length $L(\phi_t)$ of the path ϕ_t is defined to be

$$L(\phi_t) = L(H_t) = \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) dt.$$

Given some element $\phi \in Ham^c(M)$, consider the set of paths

$$\{\gamma : [0,1] \to Ham^c(M) \,|\, \gamma(0) = \text{ identity and } \gamma(1) = \phi\}.$$

Hofer has constructed a norm on $Ham^c(M)$ which defines the size $\| \phi \|$ of $\phi \in Ham^c(M)$ to be the infimum of the lengths of all such paths γ.

Consider the Hamiltonian $P : \mathbf{CP^2} \to \mathbf{R}$ given in Chapter 1:

$$P[z_0 : z_1 : z_2] = \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

P induces the rotation ϕ_t^P in the first homogeneous coordinate in $\text{Ham}^c(\mathbf{CP^2})$ given by

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2].$$

If we restrict to the time interval $0 \leq t \leq 1$, the path ϕ_t^P travels from the identity to rotation by π radians. In this thesis, we will show that ϕ_t is in fact the shortest path between its two endpoints, i.e. that $\| \phi_1^P \| = L(P)$.

Lemma 6.1.1 *The Hamiltonian P defined on $\mathbf{CP^2}$ has $L(P) = \frac{\pi}{2}$.*

Proof: Since P is independent of time,

$$L(P) = \max_{x \in \mathbf{CP^2}} P(x) - \min_{x \in \mathbf{CP^2}} P(x) = \frac{\pi}{2}.$$

6.2 Sufficient conditions for a path to be length minimizing

We now briefly describe the theory that Lalonde and McDuff use to develop their criteria for length minimizing geodesics. In [LM95a], they first derive a geometric way of detecting that, for two Hamiltonians H_t and K_t on M, $L(H_t) \leq L(K_t)$. Then, they determine sufficient conditions involving symplectic capacities for these geometric requirements to be satisfied. To begin, we must make a few definitions and set some notation.

Suppose we have H, a compactly supported time dependent Hamiltonian function on the symplectic manifold (M^{2n}, ω). We may assume, by adding a constant, that

$$\min_{x \in M, t \in [0,1]} H_t(x) = 0.$$

We write for the graph of H

$$\Gamma_H = \{(x, H_t(x), t)\} \subset M \times \mathbf{R} \times [0,1].$$

Now, let

$$h_\infty = \max_{x \in M, t \in [0,1]} H_t(x)$$

and suppose $\ell(t) : [0,1] \to [-\delta, 0]$ is a function which is negative and close to zero. A thickening of the area under Γ_H is

$$R_H^-(\frac{\nu}{2}) = \{(x, s, t) \,\big|\, \ell(t) \leq s \leq H_t(x)\} \subset M \times [\ell(t), h_\infty] \times [0,1]$$

where $\int_0^1 -\ell(t)dt = \frac{\nu}{2}$. Similarly, we can define $R_H^+(\frac{\nu}{2})$ to be a slight thickening of the area above H:

$$R^+_H(\frac{\nu}{2}) = \{(x,s,t) \,\Big|\, H_t(x) \le s \le \mu_H(t)\} \subset M \times [0,\mu_H(t)] \times [0,1]$$

where $\mu_H(t)$ is a function dependent on H and t such that

$$\mu_H(t) \ge \max_{x\in M} H_t(x) \text{ and } \int_0^1 (\mu_H(t) - h_\infty)dt = \frac{\nu}{2}.$$

We define

$$R_H(\nu) = R^-_H(\frac{\nu}{2}) \cup R^+_H(\frac{\nu}{2}) \subset M \times \mathbf{R} \times [0,1].$$

For example, if we consider the P defined above on $\mathbf{CP^2}$,

$$\Gamma(P) = \left\{\left([z_0 : z_1 : z_2], \frac{\pi}{2}\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, t\right)\right\} \subset \mathbf{CP^2} \times [0,\frac{\pi}{2}] \times [0,1]$$

$$R^-_P(\frac{\nu}{2}) = \left\{([z_0 : z_1 : z_2], s, t) \,\middle|\, \ell(t) \le s \le \frac{\pi}{2}\frac{|z_1|^2.}{|z_0|^2+|z_1|^2+|z_2|^2}\right\}$$

$$\subset \mathbf{CP^2} \times [\ell(t), \frac{\pi}{2}] \times [0,1]$$

$$R^+_P(\frac{\nu}{2}) = \left\{([z_0 : z_1 : z_2], s, t) \,\middle|\, \frac{\pi}{2}\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2} \le s \le \mu_P(t)\right\}$$

$$\subset \mathbf{CP^2} \times [0, \mu_P(t)] \times [0,1]$$

and

$$R_P(\nu) = \{([z_0 : z_1 : z_2], s, t) \,|\, \ell(t) \le s \le \mu_P(t)\} \subset \mathbf{CP^2} \times \mathbf{R} \times [0,1].$$

Note that we can equip $R^-_H(\frac{\nu}{2})$, $R^+_H(\frac{\nu}{2})$, and $R_H(\nu)$ with the product symplectic form $\Omega = \omega \oplus ds \wedge dt$. We need the following definition from [LM95a] which describes manifolds such as $(R_H(\nu), \Omega)$:

Definition 6.2.1 *Let (M,ω) be a symplectic manifold and D a set diffeomorphic to a disc in $(\mathbf{R}^2, ds \wedge dt)$. Then, the manifold $Q = (M \times D, \Omega)$ endowed with the symplectic form Ω is called a* **quasi-cylinder** *if*

(i) *Ω restricts to ω on each fibre $M \times \{pt\}$;*

(ii) *Ω is the product $\omega \oplus (ds \wedge dt)$ near the boundary $M \times \partial D$, and, in the case where M is non-compact, outside a set of the form $X \times D$ for some compact subset X in M.*

Note that $(R_H(\nu), \Omega)$ is a quasi-cylinder symplectomorphic to $M \times D(L(H)+\nu)$ where $D(a)$ denotes the disk with area a. Since $\Omega = \omega \oplus ds \wedge dt$ everywhere, not just near the boundary, $R_H(\nu)$ is called a **split** quasi-cylinder. We define the area of a compact quasi-cylinder $(M \times D(a), \Omega)$ to be the number A such that

$$\text{vol } (M \times D(a), \Omega) = A \cdot \text{ vol } (M, \omega).$$

The area of $R_P(\nu)$, therefore, is $\nu + \frac{\pi}{2}$.

Now, suppose H_t and K_t are two Hamiltonians on M such that $\phi_1^H = \phi_1^K$ and the path ϕ_t^H for $0 \le t \le 1$ is homotopic (with fixed endpoints) to the path ϕ_t^K in $Ham^c(M)$. We may join Γ_K to Γ_H via the map

$$g(x,s,t) = (\phi_t^H \circ \phi_t^K(x), s - K(x) + H(\phi_t^H \circ \phi_t^{K^{-1}}(x)), t).$$

This map g extends to a symplectomorphism of $R_K^+(\frac{\nu}{2})$, and we define

$$(R_{H,K}(\nu), \Omega) = R_H^-(\frac{\nu}{2}) \cup R_K^+(\frac{\nu}{2}).$$

Because the loop $\phi_t^H \circ \phi_t^{K-1}$ is contractable in $Ham^c(M)$, Lalonde and McDuff are able to show that $(R_{H,K}(\nu), \Omega)$ is a quasi-cylinder diffeomorphic to

$$M \times \{s, t \in \mathrm{R}^2 \,|\, \lambda(t) \leq s \leq \mu_H(t)\} \cong M \times D(L(H) + \nu).$$

Note that $R_{H,K}(\nu)$ is not necessarily a split quasi-cylinder, and thus the area of $R_{H,K}(\nu)$ is not necessarily $L(H) + \nu$.

The key to the analysis in [LM95a] is the following lemma, whose proof we include for the convenience of the reader.

Lemma 6.2.2 *(Lalonde-McDuff,[LM95a], Part II, Lemma 2.1) Suppose that $L(K_t) < L(H_t) = A$. Then, for sufficiently small $\nu > 0$, at least one of the quasi-cylinders $(R_{H,K}(\nu), \Omega)$ and $(R_{K,H}(\nu), \Omega)$ has area $< A$.*

Proof: Choose $\nu > 0$ so that

$$L(K_t) + 2\nu < L(H_t),$$

and suppose first that M is compact. Evidently,

$$\begin{aligned} \mathrm{vol}(R_{H,K}(\nu)) + \mathrm{vol}(R_{K,H}(\nu)) &= \mathrm{vol}(R_H(\nu)) + \mathrm{vol}(R_K(\nu)) \\ &= (\mathrm{vol}M) \cdot (L(H_t) + L(K_t) + 2\nu) \\ &< 2(\mathrm{vol}M) \cdot L(H_t) \end{aligned}$$

where $R_H(\nu) = R_H^-(\frac{\nu}{2}) \cup R_H^+(\frac{\nu}{2})$. If M is non-compact, we may restrict to a large compact piece X of M and then take the volume. □

Lemma 6.2.2 tells us that if the area of both quasi-cylinders $(R_{H,K}(\nu), \Omega)$ and $R_{K,H}(\nu), \Omega)$ is greater than or equal to $L(H_t)$, then $L(H_t) \leq L(K_t)$. To develop their criteria for length minimizing paths, Lalonde and McDuff use the theory of symplectic capacities to estimate the area of quasi-cylinders. A symplectic capacity is a function from the set of symplectic manifolds to $\mathbf{R} \cup \{\infty\}$ satisfying certain properties; in particular, it is a symplectic invariant. For more information on symplectic capacities, see [HZ94]. Suppose we have chosen a particular capacity c and symplectic manifold (M, ω). We say the **capacity-area inequality** holds for c on M if

$$c(M \times B^2(r)) \leq \pi r^2$$

holds for all quasi-cylinders $M \times B^2(r)$ where $B^2(r)$ is the closed 2-ball of radius r. In the next section, we will give examples of manifolds and capacities that satisfy this condition. Although capacities are applied to symplectic manifolds, we may define the capacity of a Hamiltonian in the following way.

Definition 6.2.3 *The capacity* $\mathbf{c(H)}$ *of a Hamiltonian function* H_t *is defined as*

$$c(H) = \min\{\inf_{\nu>0} c(R_H^-(\frac{\nu}{2})), \inf_{\nu>0} c(R_H^+(\frac{\nu}{2}))\}.$$

Now, take a manifold M and a capacity c such that the capacity-area inequality holds for c on M, and suppose that we have a Hamiltonian $H_t : M \to \mathbf{R}$ for which

$$c(H) \geq L(H_t).$$

Then, for any Hamiltonian K_t generating a flow ϕ_t^K which is homotopic with fixed end points to ϕ_t^H (and thus has $\phi_1^K = \phi_1^H$), we can embed $R_H^-(\frac{\nu}{2})$ into

$R_{H,K}(\nu)$ and $R_H^+(\frac{\nu}{2})$ into $R_{K,H}(\nu)$. Thus, we know

$$L(H_t) \leq c(H) \leq c(R_H^-(\frac{\nu}{2})) \leq c(R_{H,K}(\nu))$$

$$L(H_t) \leq c(H) \leq c(R_H^+(\frac{\nu}{2})) \leq c(R_{K,H}(\nu)),$$

with the last inequality in both lines holding by the monotonicity property of capacities. Since capacity-area inequality holds, we know that the areas of both quasi-cylinders $R_{H,K}(\nu)$ and $R_{K,H}(\nu)$ must be greater than or equal to $L(H_t)$ and, hence by Lemma 6.2.2, that $L(K_t) \geq L(H_t)$. This proves the proposition from [LM95a]:

Proposition 6.2.4 *(Lalonde-McDuff, [LM95a], Part II, Proposition 2.2) Let M be any symplectic manifold and $H_{t\in[0,1]}$ a Hamiltonian generating an isotopy ϕ_t^H from the identity to ϕ_1^H. Suppose there exists a capacity c such that the following two conditions hold:*

(i) *$c(H) \geq L(H_t)$ and*

(ii) *there exists a class $\mathcal{S}$ of Hamiltonian isotopies homotopic rel endpoints to ϕ_t^H , $t \in [0,1]$, which is such that the capacity-area inequality holds (with respect to the given capacity c) for all quasi-cylinders $R_{H,K}(\nu)$ and $R_{K,H}(\nu)$ corresponding to Hamiltonians $K_t \in \mathcal{S}$.*

Then, the length of the path ϕ_t^H is minimal among all paths in $\mathcal{S}$.

Hence, to show that H_t generates a length minimizing geodesic ϕ_t^H for $t \in [0,1]$ among all paths homotopic rel endpoints, we need only produce a capacity c that satisfies the above conditions (i) and (ii). In fact, Lalonde and McDuff show that if the capacity-area inequality holds for all split quasi-cylinders of the form $M \times B^2(r)$, then it also holds for all $R_{H,K}$ in Proposition 4.4 of [LM95a]. Therefore, it will be enough to find a capacity that satisfies (i) and satisfies (ii) for all split quasi-cylinders, $M \times B^2(r)$. Our $\mathcal{S}$ will be the set of all Hamiltonians K_t where $\phi_1^K = \phi_1^H$ and ϕ_t^K is homotopic rel endpoints to ϕ_t^H.

Chapter 7

Rotation in $\mathbf{CP}^2$ and $\widetilde{\mathbf{CP}}^2$

In this chapter, we describe the two symplectic capacities we will use and explain their relevance to $\mathbf{CP^2}, \widetilde{\mathbf{CP^2}}_0$, and $\widetilde{\mathbf{CP^2}}_1$.

7.1 Capacities

The symplectic capacities we will work with in this paper are the Gromov capacity, c_G, and the Hofer-Zehnder capacity, c_{HZ}. We recall their definitions for the convenience of the reader.

Definition 7.1.1 *Let (N, ω) be a symplectic manifold of dimension $2n$.*

(i) *The* **Gromov capacity**

$$c_G(N, \omega) = \sup \left\{ \pi r^2 \;\middle|\; \begin{array}{l} \exists \textit{ a symplectic embedding} \\ \phi : (B^{2n}(r), \omega_0) \to (N^{2n}, \omega) \end{array} \right\}$$

where $(B^{2n}(r), \omega_0)$ is the open $2n$-dimensional ball with radius r endowed with the standard symplectic form.

(ii) *The* **Hofer-Zehnder capacity**

$$c_{HZ}(N,\omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}_{ad}(N,\omega)\}$$

where $\mathcal{H}_{ad}(N,\omega)$ *consists of all of the autonomous Hamiltonians on* N *satisfying the properties*

(a) *There exists a compact set* $\kappa \subset N \setminus \partial N$ *depending on* H *so that* $H \,|\, (N \setminus \kappa) = \max(H)$ *is constant.*

(b) *There is a nonempty open set* U *depending on* H *such that* $H \,|\, U = 0$.

(c) $0 \leq H(x) \leq \max(H)$ *for all* $x \in N$.

(d) *All* T*-periodic solutions of the Hamiltonian system* $\dot{x} = X_H(x)$ *on* N *with* $0 \leq T \leq 1$ *are constant.*

To check that the capacity-area inequality holds on split quasi-cylinders for either of these capacities is a non-trivial procedure. By using J-holomorphic curve techniques, Lalonde and McDuff show in [LM95a] that it holds for c_G on manifolds M, compact at ∞, which are of 4 dimensions or fewer or which are semi-monotone. Recently, they have shown that it holds for all M in [LM96] .

7.2 Rotation in $\mathbf{CP^2}$ is length minimizing

Hence, condition (ii) from Proposition 6.2.4 is satisfied for c_G on any manifold, and in particular on $\mathbf{CP^2}$. In the proof of the next theorem, we construct specific embeddings of 6-balls to show that rotation through π radians around the second coordinate in $\mathbf{CP^2}$ is length minimizing among all homotopic paths.

Theorem 7.2.1 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\mathbf{CP^2})$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Proof: To prove this theorem, we will use Gromov capacity c_G and the criteria from Proposition 6.2.4. Note that this criteria only tells us if ϕ_t^P will be length minimizing within its homotopy class. However, $\pi_1(Ham(\mathbf{CP^2})) = \mathbf{Z}_3$, generated by rotation through 2π radians in one coordinate [Gro85]. We can use the arguments of Chapter 9 to show that any non-null homotopic loop has length greater than or equal to π. By Lemma 6.1.1 the path ϕ_t^P has $L(\phi_t^P) = L(P) = \frac{\pi}{2}$. Hence, if the hypotheses from Proposition 6.2.4 are satisfied, ϕ_t^P will actually be length minimizing among non-homotopic paths as well as homotopic ones.

The Hamiltonian function $P : \mathbf{CP^2} \to \mathbf{R}$ which generates our path ϕ_t^P is

$$P([z_0 : z_1 : z_2]) = \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

By Lemma 6.1.1, $L(P) = \frac{\pi}{2}$. Hence, we need only show $c_G(P) = \frac{\pi}{2}$. Recall that the capacity of P is the minimum of the capacities of R_P^- and R_P^+, the regions below and above the graph

$$\Gamma_P = \{x, s, t \,|\, P(x) = s\}$$

of P in the six dimensional compact manifold

$$\mathbf{CP^2} \times [0, \frac{\pi}{2}] \times [0, 1]$$

endowed with the product symplectic form $\tau_0 \oplus ds \wedge dt$.

Since R_P^- and R_P^+ are quasi cylinders with area $L(P) = \frac{\pi}{2}$ and the capacity area inequality holds on $\mathbf{CP^2}$ for c_G, we know that

$$c_G(P) \leq L(P) = \frac{\pi}{2}.$$

To show that $c_G(P) \geq \frac{\pi}{2}$, we show both $c_G(R_P^-) \geq \frac{\pi}{2}$ and $c_G(R_P^+) \geq \frac{\pi}{2}$ by symplectically embedding a 6-ball of radius $1/\sqrt{2} - \epsilon$ into each region.

We explicitly construct a symplectic embedding of $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ into R_P^- and R_P^+. First, we consider

$$R_P^- = \left\{ [z_0 : z_1 : z_2], s, t \,\middle|\, 0 \leq t \leq 1, \ell(t) \leq s \leq \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right\}$$

$$\subset \mathbf{CP^2} \times [0, \frac{\pi}{2}] \times [0, 1]$$

where $\ell(t)$ is some negative function close to 0. In fact, we will embed $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ in the subset of R_P^- where $s \geq 0$. The embedding will be done in two steps: first, we embed a 4-ball in $\mathbf{CP^2}$ and then embed a 2-ball in the two extra graph dimensions.

To understand what is happening geometrically, we identify $\mathbf{CP^2}$ with its image under the moment map of the T^2 action

$$(\theta_0, \theta_1)([z_0 : z_1 : z_2]) = [e^{\pi i \theta_0} z_0 : e^{\pi i \theta_1} z_1 : z_2]$$

with $0 \leq \theta_0, \theta_1 \leq 1$. The moment map for this action $\rho : \mathbf{CP^2} \to \mathbf{R^2}$ is given by

$$\rho([z_0 : z_1 : z_2]) = \left(\frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)$$

and the image of $\mathbf{CP^2}$ under ρ is the right triangle pictured in Figure 7.1. The Hamiltonian P is projection onto the horizontal axis and its image is the interval $[0, \frac{\pi}{2}]$.

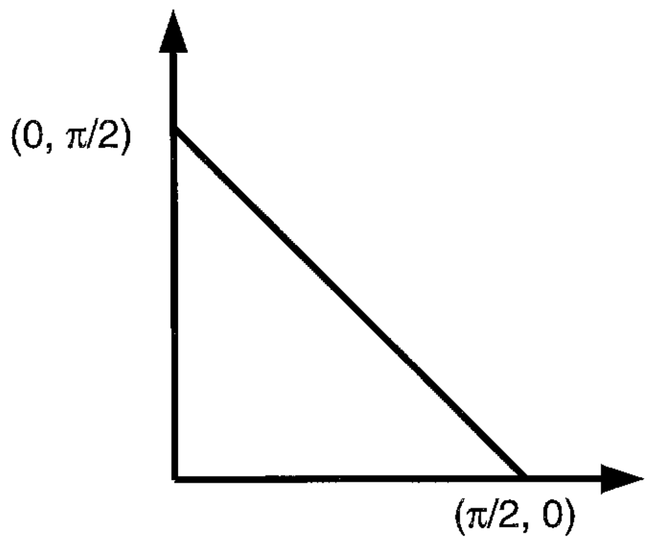

Figure 7.1: Image of $\mathbf{CP^2}$ under ρ

Let $i^- : \mathbf{C^2} \to \mathbf{CP^2}$ be the map

$$i^-(z_1, z_2) = [\sqrt{1 - |z_1|^2 - |z_2|^2} : z_1 : z_2].$$

Note that i^- restricted to $B^4(s) = \{(z_1, z_2) \,|\, |z_1|^2 + |z_2|^2 \leq s^2\}$ is a symplectic embedding for $s < 1$. The image of i^- composed with ρ is the shaded triangle in Figure 7.2.

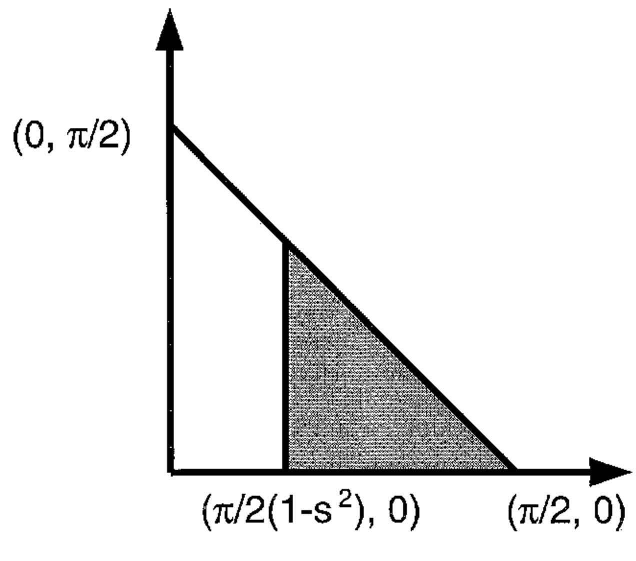

Figure 7.2: Image of $B^4(s)$ under $\rho \circ i^-$

Choose an $r \leq 1/\sqrt{2}$. For any $\epsilon > 0$, we can symplectically embed $B^2(r-\epsilon)$ (the closed 2-ball of radius $r-\epsilon$) into the smaller rectangle in Figure 7.3 because the area of the ball is $\pi(r-\epsilon)^2$ and the area of the rectangle is $(\frac{\pi}{\sqrt{2}}r)(\frac{2}{\sqrt{2}}r) = \pi r^2$. Denote this mapping by ψ_r^-. Let $R = \frac{1}{\sqrt{2}} - \epsilon$. It is possible to choose the family of maps ψ_r^- so that they fit together to form a smooth map ψ_R^- on $B^2(R)$ such that for $r < R$,

$$\psi_R^-|_{B^2(r)} = \psi_r^- .$$

In particular, this means the images of nested circles under ψ_r^- are disjoint and nested inside the larger rectangle in Figure 7.3.

We define the map $\Psi^- : B^6(\frac{1}{\sqrt{2}} - \epsilon) \to R_P^-$ by

$$\Psi^-(z_1, z_2, u, v) = (i^-(z_1, z_2), \psi_R^-(u, v))$$

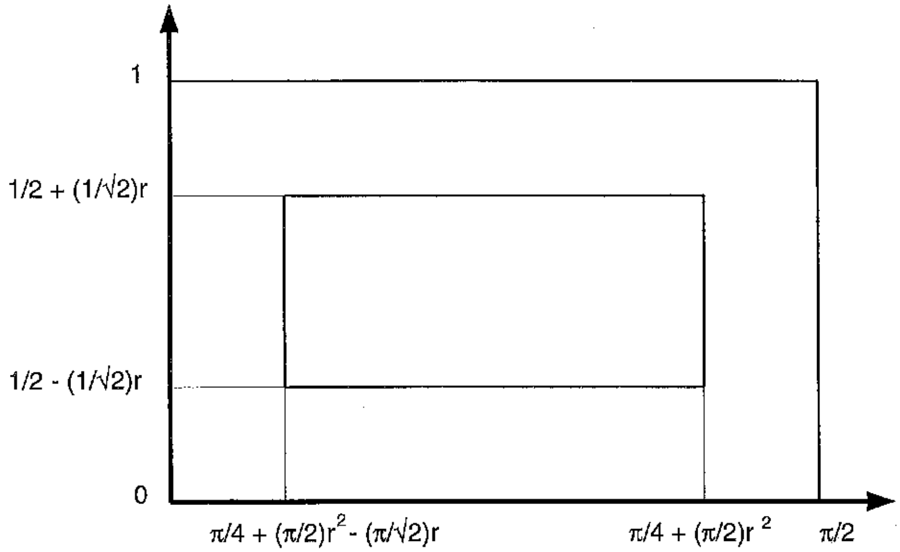

Figure 7.3: Image of $B^2(r)$ under ψ_r^-

where the domain coordinates lie in $\mathbf{C} \times \mathbf{C} \times \mathbf{R} \times \mathbf{R}$ and satisfy $|z_1|^2 + |z_2|^2 + u^2 + v^2 \leq (1/\sqrt{2} - \epsilon)^2$. Ψ^- will be the required embedding. We must show that Ψ^- is well defined, i.e. the image of Ψ^- does actually lie in R_P^-. Once this has been demonstrated, it is easy to see that Ψ^- is symplectic since it is the product of two symplectic maps into a symplectic manifold given the product symplectic structure.

Since the map i^- obviously is a well defined embedding, we must only check that for a given point $(z_1, z_2, u, v) \in B^6(\frac{1}{\sqrt{2}} - \epsilon)$, the image of $\psi_R^-(u, v)$ is contained in $[0, \frac{\pi}{2}(1 - |z_1|^2 - |z_2|^2)] \times [0, 1] \subset \mathbf{R^2}$. We let $u^2 + v^2 = r^2$ and use the fact that

$$\psi_R^-|_{B^2(r)} = \psi_r^-.$$

The height of the rectangle (the second coordinate of the image of ψ_r^-) covers the region

$$\left[\frac{1}{2} - \frac{1}{\sqrt{2}}r, \frac{1}{2} + \frac{1}{\sqrt{2}}r\right]$$

which is contained in the required interval $[0,1]$ for all $r \in [0, \frac{1}{\sqrt{2}}]$. For any given r, the width of the rectangle (the first coordinate of the image of ψ_r^-) covers the region

$$\left[\frac{\pi}{4} + \frac{\pi}{2}r^2 - \frac{\pi}{\sqrt{2}}r, \frac{\pi}{4} + \frac{\pi}{2}r^2\right].$$

As is required, the function $\frac{\pi}{4} + \frac{\pi}{2}r^2 - \frac{\pi}{\sqrt{2}}r$ is greater than zero and decreasing for all values of $r \in [0, \frac{1}{\sqrt{2}}]$. For the final check, we must examine the upper endpoint of the first coordinate of the image of ψ_r^-, $\frac{\pi}{4} + \frac{\pi}{2}r^2$, to ascertain that it is less than or equal to $\frac{\pi}{2}(1 - |z_1|^2 - |z_2|^2) = P \circ i^-(z_1, z_2)$ for u, v such that $(z_1, z_2, u, v) \in B^6(\frac{1}{\sqrt{2}} - \epsilon)$. This is a simple calculation hinged on the fact that P applied to the image under i^- of any 3-sphere is constant:

$$\begin{aligned}
\tfrac{\pi}{4} + \tfrac{\pi}{2}r^2 &= \tfrac{\pi}{4} + \tfrac{\pi}{2}(u^2 + v^2) \\
&\leq \tfrac{\pi}{4} + \tfrac{\pi}{2}(\tfrac{1}{2} - |z_1|^2 - |z_2|^2) \\
&= \tfrac{\pi}{2}(1 - |z_1|^2 - |z_2|^2).
\end{aligned}$$

Hence, the map Ψ^- is a well defined symplectic embedding of $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ into R_P^-.

In a similar manner we can define an embedding Ψ^+ of $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ into R_P^+, the region above Γ_P. We do this in two parts, as before, but now we want to center our ball in the $\mathbf{CP^2}$ portion away from $[1:0:0]$.

Let $i^+ : \mathbf{C^2} \to \mathbf{CP^2}$ be the map

$$i^+(z_1, z_2) = [z_1 : \sqrt{1 - |z_1|^2 - |z_2|^2} : z_2].$$

Note that i^+ restricted to $B^4(s) = \{z_1, z_2 \mid |z_1|^2 + |z_2|^2 \leq s^2\}$ is a symplectic embedding for $s < 1$. The image of i^+ composed with ρ is the shaded triangle in Figure 7.4.

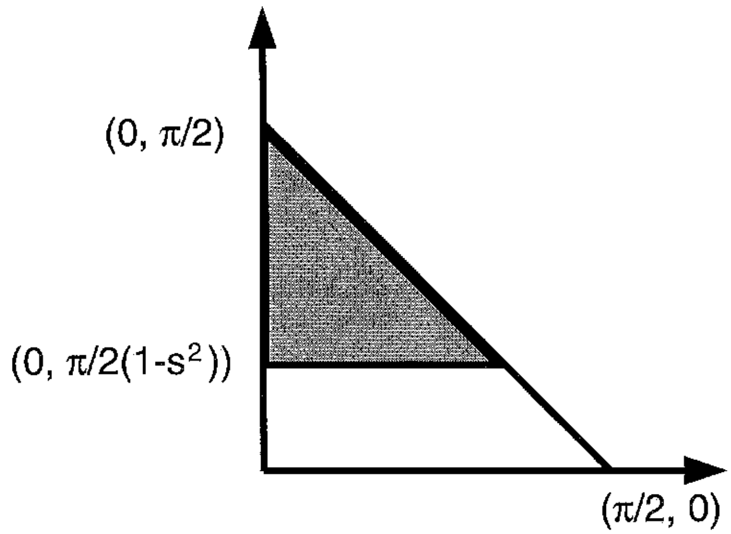

Figure 7.4: Image of $B^4(s)$ under $\rho \circ i^+$

The map i^+ will be the first part of Ψ^+.

Next, note that we can symplectically embed $B^2(r - \epsilon)$ into the smaller rectangle in Figure 7.5 because the area of the ball is $\pi(r - \epsilon)^2$ and the area of this rectangle is πr^2.

We denote this mapping by ψ_r^+. As in the previous set up, we may assume that for $r < R$, $\psi_R^+|_{B^2(r)} = \psi_r^+$. Then, we define $\Psi^+ : B^6(\frac{1}{\sqrt{2}} - \epsilon) \to R_P^+$ by

$$\Psi^+(z_1, z_2, u, v) = (i^+(z_1, z_2), \psi_R^+(u, v))$$

where the domain coordinates lie in $\mathbf{C} \times \mathbf{C} \times \mathbf{R} \times \mathbf{R}$ and satisfy $|z_1|^2 + |z_2|^2 + u^2 + v^2 \leq (1/\sqrt{2} - \epsilon)^2$. Just as we checked that Ψ is a well defined symplectic

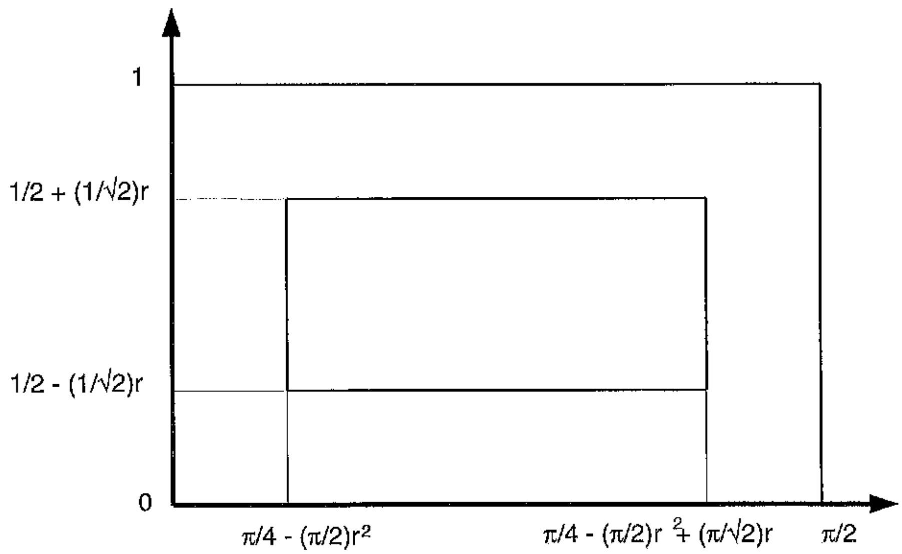

Figure 7.5: Image of $B^2(r)$ under ψ_r^+

embedding, we may verify that Ψ^+ is also a well defined symplectic embedding.

□

7.3 Blowing up $\mathbf{CP^2}$ at $[1:0:0]$

Here we begin our treatment of P applied to various blow ups of $\mathbf{CP^2}$. Note that P picks out the norm of the first homogeneous coordinate of a point. Hence, if we blow up at the point $[1:0:0]$ by removing a 4-ball of radius λ to get $\widetilde{\mathbf{CP^2}}_0$, the rotation ϕ_t^P for $0 \leq t \leq 1$ fixes the exceptional divisor. In addition, in moving from $\mathbf{CP^2}$ to $\widetilde{\mathbf{CP^2}}_0$, we have altered the domain of P in a consequential way.

Lemma 7.3.1 *The Hamiltonian P defined on $\widetilde{\mathbf{CP^2}}_0$ has $L(P) = \frac{\pi}{2}(1-\lambda^2)$.*

Proof: Written out in homogeneous coordinates,

$$\widetilde{\mathbf{CP^2}}_0 = \{[\sqrt{1-|z_1|^2-|z_2|^2} : z_1 : z_2] \,\big|\, \lambda^2 \le |z_1|^2 + |z_2|^2 \le 1\}$$

with the appropriate equivalence relation on the exceptional divisor. Hence, it is easy to see that

$$L(P) = \max_{x\in\widetilde{\mathbf{CP^2}}_0} P(x) - \min_{x\in\widetilde{\mathbf{CP^2}}_0} P(x) = \frac{\pi}{2}(1-\lambda^2) - 0 = \frac{\pi}{2}(1-\lambda^2).$$

□

The image of $\widetilde{\mathbf{CP^2}}_0$ under the map ρ is the quadrilateral in Figure 7.6. Since the map P is projection onto the horizontal axis, with this quadrilateral as its domain, P has image $[0, \frac{\pi}{2}(1-\lambda^2)]$. This verifies Lemma 7.3.1.

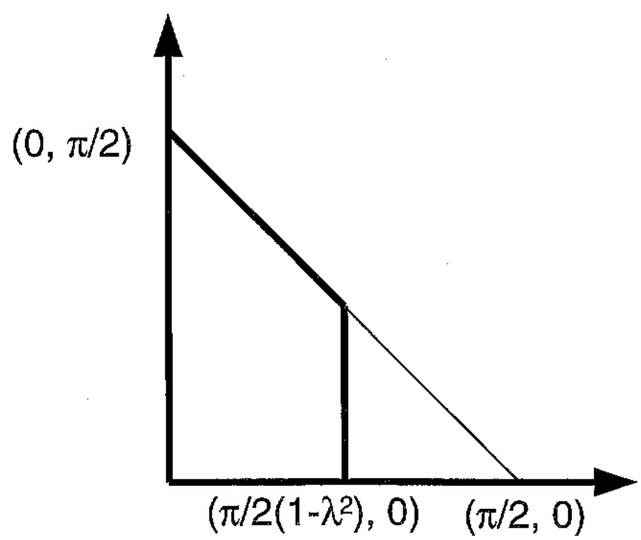

Figure 7.6: Image of $\widetilde{\mathbf{CP^2}}_0$ under ρ

Theorem 7.3.2 *The path ϕ_t^P for $0 \le t \le 1$ in $Ham(\widetilde{\mathbf{CP^2}}_0)$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) *and rotation by* π *radians in the first coordinate* (ϕ_1^P).

Proof:

By using the embeddings from the $\mathbf{CP^2}$ case adjusted appropriately, we can show that $c_G(P) = \frac{\pi}{2}(1-\lambda^2)$. By Proposition 6.2.4 and Lemma 7.3.1, this will tell us that ϕ_t^P is length minimizing in its homotopy class. The results of Chapter 9 can be applied to show that ϕ_t^P is actually globally length minimizing.

To show that $c_G(R_P^+) \geq \frac{\pi}{2}(1-\lambda^2)$ requires no additional work; we may use the embedding Ψ^+ from the $\mathbf{CP^2}$ case. However, to prove $c_G(R_P^-) \geq \frac{\pi}{2}(1-\lambda^2)$ takes some manipulation. We must produce a new embedding $\Upsilon^- : B^6(\sqrt{\frac{1-\lambda^2}{2}} - \epsilon) \to R_P^-$ because the old embedding, Ψ^-, has in its image some points that were removed under the blow up.

Consider the open shaded triangle in Figure 7.7 for some s where $s^2 \in [0, 1-\lambda^2]$. By Delzant's theorem, the preimage of this set under the map ρ is a symplectic submanifold. This preimage is equal to the set $U_s \subset \widetilde{\mathbf{CP^2}}_0$ where

$$U_s = \left\{ [z_0 : z_1 : z_2] \,\middle|\, |z_0|^2 = (1-\lambda^2-\tau^2), |z_1|^2 < \tau^2, 0 \leq \tau^2 < s^2 \right\}.$$

We will prove that there exists a symplectic embedding j_s^- of $B^4(s-\epsilon)$ into U_s. U_s is symplectomorphic to the set $V_s \subset \mathbf{R^4}$ where

$$V_s = \left\{ (z_0, B^2(\sqrt{1-\lambda^2-|z_0|^2})) \,\middle|\, z_0 \in \mathbf{C}, 1-\lambda^2-s^2 < |z_0|^2 < 1-\lambda^2 \right\}.$$

V_s is just a set of 2-balls fibered over an annulus. If we cut this annulus to make it a rectangle (this does not change the symplectic capacity), we arrive

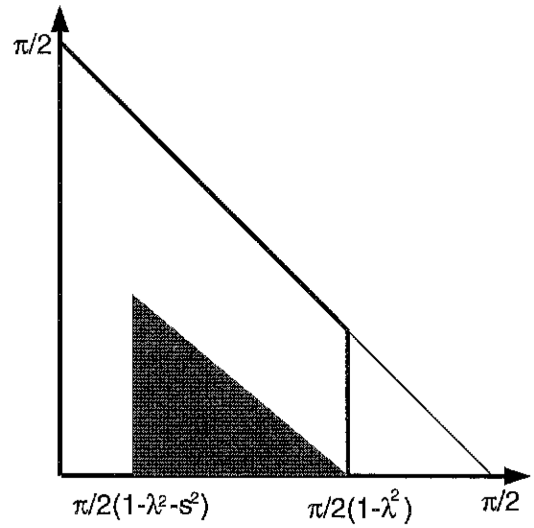

Figure 7.7: Image of $B^4(s)$ under $\rho \circ j^-$

at the set

$$T_s = \left\{(x, y, B^2(\sqrt{s^2 - y^2}))\,\middle|\, 0 \leq x < \pi s, 0 \leq y < s\right\} \subset \mathbf{R}^4.$$

T_s is a generalized trapezoid, that is it consists of balls fibered over a rectangle. It is not hard to show that the capacity of T_s is the same as the capacity of the more standard trapezoid

$$T^4(\pi s^2) = \left\{(x, y, B^2(\sqrt{s^2 - \frac{y}{\pi}})\,\middle|\, 0 \leq x < 1, 0 \leq y < \pi s^2\right\}.$$

In Lemma 3.6 of [LM95a], it is shown that the capacity of $T^4(\pi s^2)$ is equal to the capacity of $B^4(s)$. Hence,

$$c_G(U_s) = c_G(T_s) = c_G(T^4(\pi s^2)) = c_G(B^4(s)) = \pi s^2$$

and we can embed $B^4(s-\epsilon)$ into U_s for any $\epsilon > 0$. Call this embedding j_s^-. Consider the family of maps $j_s^- : B^4(s-\epsilon) \to U_s$ for all $0 \leq s \leq \sqrt{1-\lambda^2}$. Let

$$S = \sqrt{\frac{1-\lambda^2}{2}} - \epsilon.$$

Without loss of generality, we may assume that the family of maps j_s satisfies

$$j_S^-|_{B^4(s)} = j_s^-$$

for $s \leq S$, so that 3-spheres of constant radius appear as vertical lines in the moment map picture. To be precise, if $(w_o, w_1) \in \mathbf{C^2}$ and $|w_0|^2 + |w_1|^2 = s^2$, then $\rho \circ j_S^-(w_0, w_1)$ lies on the vertical line through the point $(\frac{\pi}{2}(1-\lambda^2-s^2), 0)$. Thus, P applied to the image of 3-spheres under j_S^- is constant.

Now, we have an embedding j_S^- from $B^4(\sqrt{\frac{1-\lambda^2}{2}}-\epsilon)$ into $\widetilde{\mathbf{CP^2}}_0$. Our next task is to work with the other two dimensions and construct Υ^-.

Fix an $r < \sqrt{\frac{1-\lambda^2}{2}}$. We can symplectically embed $B^2(r)$ into the smaller rectangle in Figure 7.8 because the area of the ball is πr^2 and the area of the rectangle is

$$\left(\frac{\pi}{2}(\sqrt{1-\lambda^2})r\right)\left(\frac{2}{\sqrt{2(1-\lambda^2)}}\right) = \pi r^2.$$

Denote this embedding by v_r^-. As before, we assume that for $r < S$, $v_S^-|_{B^2(r)} = v_r^-$, and define $\Upsilon^- : B^6(S) \to R_P^-$ by

$$\Upsilon^-(w_0, w_1, u, v) = \left(j^-(w_0, w_1), v_S^-(u, v)\right).$$

Using the fact that P is constant along the image under j^- of 3-spheres, it is routine to check that in fact Υ^- is well defined. $\square$

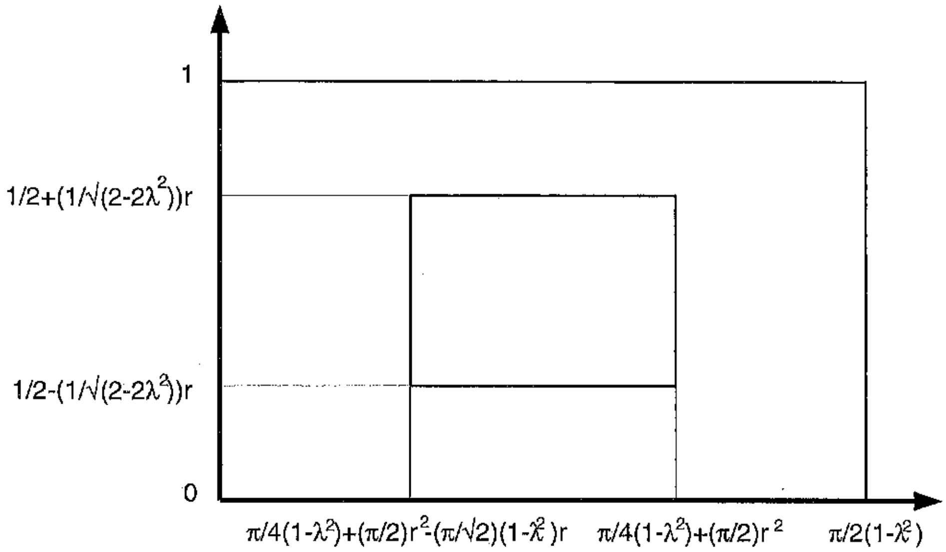

Figure 7.8: Image of $B^2(r)$ under υ_r^-

7.4 Blowing up $\mathbf{CP^2}$ at $[0:1:0]$

Now, suppose instead that we consider $\mathbf{CP^2}$ blown up at the point $[0:1:0]$ by removing a 4-ball of radius λ. We call this manifold $\widetilde{\mathbf{CP^2}}_1$.

Lemma 7.4.1 *The Hamiltonian P defined on $\widetilde{\mathbf{CP^2}}_1$ has $L(P) = \frac{\pi}{2}$.*

Proof: Written out in homogeneous coordinates,

$$\widetilde{\mathbf{CP^2}}_1 = \{[z_0 : \sqrt{1-|z_0|^2-|z_2|^2} : z_2] \,\big|\, \lambda^2 \leq |z_0|^2 + |z_2|^2 \leq 1\}$$

with the appropriate equivalence relation on the exceptional divisor. Hence, it is easy to see that

$$L(P) = \max_{x\in\widetilde{\mathbf{CP^2}}_1} P(x) - \min_{x\in\widetilde{\mathbf{CP^2}}_1} P(x) = \frac{\pi}{2}.$$

□

The image of ρ applied to $\widetilde{\mathbf{CP^2}}_1$ is the quadrilateral depicted in Figure 7.9. The map P defined on $\widetilde{\mathbf{CP^2}}_1$ is again projection onto the horizontal axis and has image $[0, \frac{\pi}{2}]$, verifying Lemma 7.4.1.

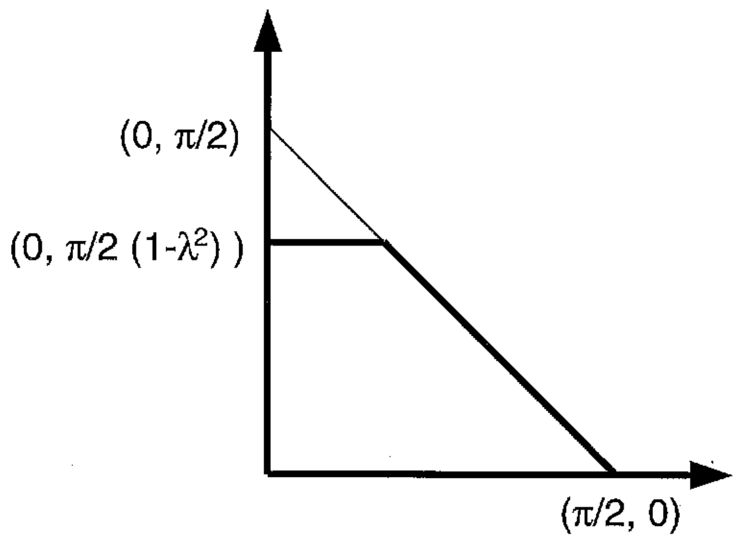

Figure 7.9: Image of $\widetilde{\mathbf{CP^2}}_1$ under ρ

In $\widetilde{\mathbf{CP^2}}_1$, the exceptional divisor is not fixed under the rotation ϕ_t^P. When we blow up $\mathbf{CP^2}$ in this way, Lemma 7.4.1 tells us that $L(P) = \frac{\pi}{2}$, i.e. the length of P has not decreased. However, the volume of the manifold has decreased, and there is not a straight forward way to embed large enough 6-balls to show that $c_G(P) = \frac{\pi}{2}$. It is not certain that $c_G(P) < \frac{\pi}{2}$, it is only that we cannot show equality by using the methods from the other cases.

These two blow ups, $\widetilde{\mathbf{CP^2}}_0$ and $\widetilde{\mathbf{CP^2}}_1$, are essentially the only two different types of blow ups of $\mathbf{CP^2}$ on which the rotation ϕ_t^P will be well defined. In order for the rotation to descend properly, we must blow up at a fixed point of the rotation in $\mathbf{CP^2}$. There is an isolated fixed point at $[1:0:0]$ and a fixed sphere consisting of the points of the form $[0:z_1:z_2] \subset \mathbf{CP^2}$. From blowing up at the isolated fixed point, we obtain $\widetilde{\mathbf{CP^2}}_0$, and from blowing up at a point on the fixed sphere we obtain $\widetilde{\mathbf{CP^2}}_1$.

Because $L(P)$ does not decrease when moving from $\mathbf{CP^2}$ to $(\widetilde{\mathbf{CP^2}}_1, \tau_\lambda)$, we cannot use the Gromov capacity to show that the rotation induced by P on $\widetilde{\mathbf{CP^2}}_1$ is length minimizing. The natural alternative is to use Hofer-Zehnder capacity, since c_{HZ}, unlike c_G, will not necessarily decrease after blow up. Thus we need to examine the conditions under which the capacity-area inequality holds for c_{HZ}. Recall that a symplectic manifold (M, ω) is **weakly exact** if ω restricted to $\pi_2(M)$ is zero. The following theorem from [HV92] is quoted as Theorem 1.17 in [LM95a]:

Theorem 7.4.2 *(Hofer-Viterbo) Suppose that (M, ω) is weakly exact. Then for all $a > 0$,*

$$c_{HZ}(M \times D^2(a), \omega \oplus \sigma) \leq a.$$

However, $\widetilde{\mathbf{CP^2}}$ is not weakly exact, as the Hurewicz homomorphism is an isomorphism between $H_2(\widetilde{\mathbf{CP^2}}, \mathbf{Z})$ and $\pi_2(\widetilde{\mathbf{CP^2}})$. In order to eventually apply Proposition 6.2.4 to $\widetilde{\mathbf{CP^2}}$ using c_{HZ}, in Chapter 8 we will go back to the original proof of Theorem 7.4.2 and show that the restriction that M is weakly exact can be changed to M has dimension 4. Hence, in Chapter 9 we arrive at

Theorem 7.4.3 *Suppose that (M,ω) is a symplectic manifold of dimension four. Then for all $a > 0$,*

$$c_{HZ}(M \times D^2(a), \omega \oplus \sigma) \leq a.$$

Theorem 7.4.3 enables us to prove the following main result.

Theorem 7.4.4 *Let (M,ω) be a symplectic manifold of dimension four. Let ϕ_t for $0 \leq t \leq 1$ be a path in $Ham^c(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0 is the identity diffeomorphism and ϕ_t has no non-constant closed trajectory in time less than 1. Then, ϕ_t for $0 \leq t \leq 1$ is length minimizing among all homotopic paths between the identity and ϕ_1.*

Finally, as a consequence of Theorem 7.4.4 we show

Theorem 7.4.5 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\widetilde{\mathbf{CP}^2}_1)$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Chapter 8

The capacity-area inequality for c_{HZ}

In the first section of this chapter, we analyze the proof of Theorem 7.4.2 which states sufficient conditions on M for c_{HZ} to satisfy the capacity-area inequality on M. Then, in the second section, we show that the weakly exact hypothesis in this theorem can be changed to dimension four.

8.1 Hofer and Viterbo's proof of Theorem 7.4.2

We now examine Hofer and Viterbo's proof of Theorem 7.4.2 to determine why they need the weakly exact condition [HV92]. Unfortunately, their notation is different from the notation in [LM95a], so we will first need to provide some sort of dictionary to explain the theorem as they have stated it.

Let $[S^2, V]$ be the set of homotopy classes of maps from S^2 to V. We apply ω to such a class $\alpha \in [S^2, V]$ by evaluating ω on the representative of α

in $H_2(V, \mathbf{Z})$. Define

$$m(M,\omega) = \inf\{\langle \omega, \alpha\rangle \,|\, \alpha \in [S^2, V],\ 0 < \ \langle \omega, \alpha\rangle\}.$$

Note that if M is weakly exact, $m(M,\omega) = \infty$. If for some particular class $\alpha \in H_2(V)$ we have $\langle \omega, \alpha\rangle = m(M,\omega)$, then α is called ω-minimal. The theorem of Hofer and Viterbo which is equivalent to Theorem 7.4.2 is

Theorem 8.1.1 *(Hofer-Viterbo, [HV92], Theorem 1.12) Let (M,ω) be a compact symplectic manifold and let σ be a volume form for S^2 such that $\int_{S^2} \sigma = a$ and*

$$0 < a \leq m(M,\omega).$$

Suppose $K : M \times S^2(a) \to \mathbf{R}$ is a smooth (time independent) Hamiltonian such that

$$K\,|_{\mathcal{U}(*)} = k_0 \textit{ and } K\,|_{\mathcal{U}(M\times\{\infty\})} = k_\infty$$

for suitable neighborhoods of $M \times \{\infty\}$ and some point $* \notin M \times \{\infty\}$. Suppose

$$k_0 < k_\infty \quad \textit{and} \quad k_0 \leq K \leq k_\infty.$$

Then, the Hamiltonian system $\dot{x} = X_K(x)$ on the symplectic manifold $(M \times S^2(a), \omega \oplus \sigma)$ possesses a non-constant T-periodic solution with

$$0 < (k_\infty - k_0)T < a.$$

The task now at hand is to see why Theorem 8.1.1 is equivalent to Theorem 7.4.2. Remember that

$$c_{HZ}(N,\omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}_{ad}(N,\omega)\}$$

where $\mathcal{H}_{ad}(N, \omega)$ consists of all of the autonomous Hamiltonians on N satisfying the properties:

(a) There exists a compact set $\kappa \subset N \setminus \partial N$ depending on H so that $H \,|\, (N \setminus \kappa) = \max(H)$ is constant.

(b) There is a nonempty open set U depending on H such that $H \,|\, U = 0$.

(c) $0 \leq H(x) \leq \max(H)$ for all $x \in N$.

(d) All T-periodic solutions of the Hamiltonian system $\dot{x} = X_H(x)$ on N with $0 \leq T \leq 1$ are constant.

Clearly, proving Theorem 7.4.2 is the same as showing that any properly normalized Hamiltonian K on $M \times D^2(a)$ with $\max(K) > a$ has a non-constant orbit with period $T \leq 1$. In Theorem 8.1.1, Hofer and Viterbo consider the completion $M \times S^2(a)$ of $M \times D^2(a)$. For simplicity, we will also denote the symplectic form on $S^2(a)$ by σ. The neighborhood $U(M \times \infty) \subset M \times S^2(a)$ corresponds to a neighborhood of $\partial(M \times D^2(a))$ in Theorem 7.4.2. The hypotheses concerning the values k_0 and k_∞ in Theorem 8.1.1 correspond to the conditions (a) (b), and (c) describing the requirements for K to be a member of $\mathcal{H}_{ad}$. The hypothesis $0 < a \leq m(M, \omega)$ in Theorem 8.1.1 is satisfied for all a if and only if M is weakly exact. Finally, the quantity $k_\infty - k_0$ corresponds to $\max(K)$. Hence, to show the equivalence of the two theorems we need to suppose in Theorem 8.1.1 that $k_\infty - k_0 \geq a$ and show that we get a closed non-constant orbit of period $T \leq 1$. In fact, the conclusion of Theorem 8.1.1

tells us exactly that we get a non-constant orbit of period

$$T < \frac{a}{k_\infty - k_0},$$

so that if $k_\infty - k_0 \geq a$ then $T \leq 1$.

We eventually want to prove Theorem 8.1.1 without the hypothesis $a \leq m(M, \omega)$. For the symplectic manifold $(M \times S^2, \omega \oplus \sigma)$, let $\mathcal{J}$ be the set of all compatible smooth almost complex structures J on $M \times S^2$. The original proof of Theorem 8.1.1 uses $J-$holomorphic curves with a split compatible almost complex structure $J \in \mathcal{J}$ on $M \times S^2$ that is regular for the class A in the sense of Theorem 3.1.2 of [MS94]. Hofer and Viterbo use a split J so that they can easily verify the condition $a < m(M, \omega)$ in certain settings. Since this condition is exactly the hypothesis we will remove, in this discussion we do not need to restrict ourselves to a split J. We will, however, need to impose more regularity conditions on J later.

After a J is fixed, the proof of Theorem 8.1.1 proceeds by determining the S^1-cobordism class of a certain moduli space of J-holomorphic spheres whose image is in $M \times S^2(a)$. This moduli space $\mathcal{H}(J)$ consists of the set of maps $u \in C^\infty(S^2, M \times S^2(a))$ that satisfy

$$[u] = [\{\text{pt}\} \times S^2(a)] = A \in H_2(M \times S^2(a), \mathbf{Z})$$

$$\int_D u^*\omega = \frac{1}{2}\langle \omega, A\rangle \text{ where } D = \{z \,\big|\, |z| \leq 1\}$$

$$u(0) = \{*\}, \;\; u(\infty) \in M \times \{\infty\}$$

$$\overline{\partial}_J u = 0.$$

Hofer and Viterbo show the S^1-cobordism class of $\mathcal{H}(J)$ is not zero and hence a related family $\mathcal{C}$ of perturbed J-holomorphic spheres is not compact. Specifically,

$$\mathcal{C} = \{(\lambda, u) \in [0, \infty) \times \mathcal{B} \,|\, \bar{\partial}_J u + \lambda k(u) = 0\}$$

where $k(u)$ is basically a scaling of the gradient of K and $\mathcal{B}$ is the set of maps $u \in H^{2,2}(S^2, M \times S^2(a))$ that satisfy

$$[u] = A \in H_2(M \times S^2(a), \mathbf{Z})$$

$$\int_D u^*\omega = \frac{1}{2}\langle \omega, A\rangle \text{ where } D = \{z \,\big|\, |z| \leq 1\}$$

$$u(0) = \{*\}, \;\; u(\infty) \in M \times \{\infty\}.$$

We can see that the map u for $(\lambda, u) \in \mathcal{C}$ is almost fixed. Since J is regular, the dimension of the moduli space of perturbed $J-$holomorphic spheres of class A is $2c_1(A) + \dim(M) + 2 = 6 + \dim(M)$ ([MS94], Theorem 3.12). However, $\mathcal{C}$ does not consist of all of these spheres; the restrictions placed upon the elements in $\mathcal{B}$ reduce the dimension of $\mathcal{C}$ greatly. The first normalization condition on the area imposes a loss of 1 dimension. The next restriction, fixing the image of $\{0\}$, imposes a loss of $\dim(M) + 2$ dimensions. Finally, restriction the image of $\{\infty\}$ results in a loss of 2 dimensions. Hence, the set of spheres we are considering in the second factor of $\mathcal{C}$ will have dimension $6 + \dim(M) - 1 - (\dim(M) + 2) - 2 = 1$. This degree of freedom corresponds to rotation by S^1 of S^2. Note, then, that $\mathcal{C}$ is a two dimensional space: one dimension for the λ coordinate and one dimension which corresponds to this S^1 rotation.

Hofer and Viterbo analyze the noncompactness of $\mathcal{C}$ and show that it cannot be due to a bubbling off of perturbed J-holomorphic curves. Since there are no bubbles, there are uniform bounds on the derivatives of the u. They view the u not as maps from the sphere, but rather as maps from the non-compact cylinder $S^1 \times \mathbf{R}$. Hence, $\mathcal{C}$ consists of maps with finite energy whose domain is an infinitely long cylinder. In the same manner as in Floer theory, Hofer and Viterbo show the noncompactness of $\mathcal{C}$ produces a sequence of maps that converge to a closed non-constant orbit x which is a solution of the equation $\dot{x} = X_K(x)$.

When we remove the restriction $a \leq m(M, \omega)$, each of the steps in the proof of Theorem 8.1.1 goes through with only minor adjustments, except for the proof of the statement that there are no bubbles. It turns out, however, that this difficulty can be overcome. In the next section, we give a new proof that shows when we remove the above area condition in the case where M has dimension 4, it is still true that generically, no sequence of elements in $\mathcal{C}$ converges to a bubble.

8.2 Noncompactness in $\mathcal{C}$ cannot be due to bubbling

We will show that for generic $J \in \mathcal{J}$, the space of bubbles which are limits of sequences of elements in $\mathcal{C}$ is empty. We first show that for generic J, the space of cusp curves which have two components in empty. There are two distinct cases we must consider. The first case is when the point where the derivitave blows up in S^2 is $\{0\}$ or $\{\infty\}$ and the second case is when the point where the derviative blows up lies elsewhere. We examine in detail the first case, as the second is similar but more simple.

We can represent the $\lambda k(u)$ perturbed component of the cusp curve by the class $A - Y$ and the J-holomorphic bubble by the class X. Let us for now assume that $X = Y$, and therefore that the homological sum of the two component classes is A. Note that this need not be the case: since we only consider simple cusp curves as limiting elements, we may have had to reduce a multiply covered curve and thus have lost some homology. We will discuss this later on and see that, since we assumed M has dimension four, it poses no obstacle.

We define the universal moduli spaces

$$\mu^\lambda(A-Y, \mathcal{J}) = \{(u, J) \,|\, u : S^2 \to M \times S^2(a), [\mathrm{Im}(u)] = A-Y,\ \bar{\partial}_J u + \lambda k(u) = 0\}$$

and

$$\mu(Y, \mathcal{J}) = \{(v, J) \,|\, v : S^2 \to M \times S^2(a), [\mathrm{Im}(v)] = Y,\ \bar{\partial}_J v = 0\}.$$

We will write $\mu^\lambda(A - Y, J)$ or $\mu(Y, J)$ when we wish to refer to the moduli space consisting of curves corresponding to a single J.

We must show that for a generic J, the subset of elements in $\mu^\lambda(A - Y, J) \times \mu(Y, J)$ which satisfy the restrictions imposed by C and which are bubbles is empty. Assume that the point where the derivative blows up to form the bubble is $\{0\}$; the case where it is $\{\infty\}$ is handled similarly. A picture of the cusp curve is shown in Figure 8.1.

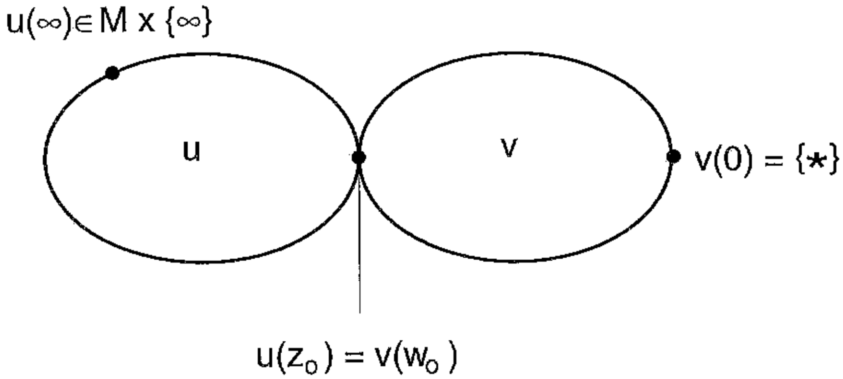

Figure 8.1: Bubbling at $\{0\}$

Let $\mathcal{U}^\lambda$ be the space

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times_{G_\infty} S^2 \times \mu(Y, J) \times_{G_0} S^2$$

The four dimensional reparametrization group G_∞ of S^2 that fixes $\{\infty\}$ acts on the first S^2 and the four dimensional reparametrization group G_0 that fixes $\{0\}$ acts on the second S^2. (Note that for different types of bubbles we will be able to quotient by different symmetry groups.) Define the space

$$\mathcal{U} = \{\lambda, \mathcal{U}^\lambda \,|\, \lambda \in [0, \lambda_\infty]\}$$

where λ_∞ is some constant that depends on A and M. We let $\mathcal{U}_J$ be the restriction of $\mathcal{U}$ to a particular $J \in \mathcal{J}$.

Consider the evaluation map ev where

$$ev : \mathcal{U} \to (M \times S^2)^4$$

by

$$ev(\lambda, J, u, z_0, v, w_0) = (u(\infty), u(z_0), v(w_0), v(0))\,.$$

Let

$$\mathcal{D} = ev^{-1}\left((\{\infty\} \times M), \Delta, \{*\}\right)$$

where Δ stands for the diagonal in $(M \times S^2) \times (M \times S^2)$. We let $\mathcal{D}_J$ be the restriction of $\mathcal{D}$ to a particular $J \in \mathcal{J}$. Our aim is to prove that for a generic J, $\mathcal{D}_J$ is empty. Note here that we have not included in $\mathcal{D}$ the area normalization condition

$$\int_{|z|\leq 1} u^*\omega = \frac{1}{2}\langle \omega, A\rangle$$

that is included in $\mathcal{C}$. We omit it to make the analysis more simple. Clearly, if $\mathcal{D}_J$ is empty when we do not require that u satisfy this condition, $\mathcal{D}_J$ will be empty if we do insist it satisfies this restriction.

Lemma 8.2.1 *There exists a set of complex structures $\mathcal{J}_0 \subset \mathcal{J}$ of second category such that for $J \in \mathcal{J}_0$, $\mathcal{D}_J$ is empty.*

Proof: First, we show that $\mathcal{D}$ is a manifold by proving that ev is transversal onto the set $(\{\infty\} \times M, \Delta, \{*\})$ in $(M \times S^2)^4$. Then, to calculate the codimension of $\mathcal{D}_J$, we apply the theory in the proof of Theorem 6.3.2 from [MS94]. These results tell us that since the projection map from $\mathcal{U}$ onto $\mathcal{J}$ is a Fredholm operator, there will be a set $\mathcal{J}_{reg}$ of second category in $\mathcal{J}$ such that for $J \in \mathcal{J}_{reg}$, the manifold $\mathcal{D}_J \subset \mathcal{U}_J$ has its expected codimension. The final step will be to show this codimension is greater than the dimension of $\mathcal{U}_J$, and hence $\mathcal{D}_J$ is empty.

Lemma 8.2.2 *The map ev is transversal onto $(\{\infty\} \times M, \Delta, \{*\})$.*

Proof: Choose some $(\lambda, J, u, v) \in [0, \lambda_\infty] \times \mathcal{J} \times \mu^\lambda(A-Y, J) \times \mu(Y, J)$. Note that u and v have no restrictions placed on them except for their homology class. First, suppose that if their images do touch in $M \times S^2$ as a limit of curves in $\mathcal{C}$, the point in $M \times S^2$ at which they meet is neither $\{*\}$ nor $u(\infty) \in \{\infty\} \times M$. Therefore, the preimages of the sets $(\{\infty\} \times M), \Delta,$ and $\{*\}$ are all separated in S^2. Since transversality is a local condition, it is enough to show that

$$\pi_1 \circ ev \text{ is transversal to } \{\infty\} \times M \subset M \times S^2$$

$$\pi_2 \times \pi_3 \circ ev \text{ is transversal to } \Delta \subset (M \times S^2)^2$$

and

$$\pi_4 \circ ev \text{ is transversal to } \{*\} \in M \times S^2$$

where π_i denotes projection onto the ith copy of $M \times S^2$ in the image of ev. By Lemma 6.1.2 in [MS94], since we have no restrictions on u and v, these maps are indeed transversal onto the necessary regions.

Now, suppose the point in $M \times S^2$ at which they meet is the image of $\{0\}$ or $\{\infty\}$ under u. Even though u would be restricted near the meeting point, v would not. Hence, since the normal bundle to Δ at the point (p, p) is spanned by the vectors $0 \times T_p(M \times S^2)$, we would still have transversality. □

Lemma 8.2.3 *There exists a set of second category $\mathcal{J}_{reg} \subset \mathcal{J}$, so that the codimension of $\mathcal{D}_J$ in $\mathcal{U}_J$ is $4n + 6$ for all $J \in \mathcal{J}_{reg}$.*

Proof: By Lemma 8.2.2, we can use the Fredholm theory from Theorem 6.3.2 of [MS94]. This tells us that for a set of almost complex structures $\mathcal{J}_{reg}$

of second category, $J \in \mathcal{J}_{reg}$ implies

$$\begin{aligned} \text{codimension of } \mathcal{D}_J &= \text{codimension of } ((\{\infty\} \times M), \Delta, \{*\}) \\ &= 2 + (2n+2) + (2n+2) \\ &= 4n+6. \end{aligned}$$

□

Finally, we calculate the dimension of $\mathcal{U}_J$.

Lemma 8.2.4 *There exists a set of second category $\mathcal{J}'_{reg} \subset \mathcal{J}$ so that for $J \in \mathcal{J}'_{reg}$, the dimension of $\mathcal{U}_J$ is $4n+5$.*

Proof: Theorem 3.1.2 from [MS94] states that for a set of second category $\mathcal{J}'_{reg}(B) \subset \mathcal{J}$, the dimension of the moduli space $\mu(B, J) = 2c_1(B) + 2n + 2$ where B is a 2-homology class in $M \times S^2$. If we let our class be $A - Y$ and Y, we see that for $J \in \mathcal{J}'_{reg}(A-Y) \cap \mathcal{J}'_{reg}(Y) = \mathcal{J}'_{reg}$,

$$\begin{aligned} \dim \mathcal{U}_J &= 1 + 2c_1(A-Y) + 2n + 2 + 2c_1(Y) + 2n + 2 + 2 - 4 + 2 - 4 \\ &= 2c_1(A) + 4n + 1 \\ &= 4n+5. \end{aligned}$$

□

For $J \in \mathcal{J}_{reg} \cap \mathcal{J}'_{reg} = \mathcal{J}_0$, $\mathcal{U}_J$ has dimension $4n+5$ and $\mathcal{D}_J$ has codimension $4n+6$. Hence, for these J, $\mathcal{D}_J$ will be empty. Note that $\mathcal{J}_0$ is of second category since it is the intersection of two second category sets . Thus, we have proven Lemma 8.2.1. □

Proposition 8.2.5 *Suppose (M, ω) is a symplectic manifold of dimension four. Then, there exists a set of second category of regular almost complex* *structures on $M \times S^2$ for which the space of bubbles which are limits of sequences of elements in C will be empty.*

Proof: Lemma 8.2.1 tells us that for generic J, the space of such bubbles that are cusp curves with two components, neither of which is multiply covered, where the bubble is formed by the derivative blowing up at $\{0\}$, is empty. To deal with other types of bubbling in a two component cusp curve is similar. We must be careful, though to quotient out by the appropriate symmetry groups.

To show that multiple bubbles would not occur, the argument from the proof of Lemma 8.2.1 can be modified. For each additional bubble, we would increase the number of homology classes used to form $\mathcal{U}_J$ by 1 and increase the number of S^2 used by 2. (See Theorem 6.3.2 from [MS94]). This adds $2n + 2 + 4 = 2n + 6$ to the dimension of $\mathcal{U}_J$, and we may reduce by the six dimensional reparametrization group $\mathrm{PSL}(2, \mathbf{C})$ to get $2n$ added dimensions. The transversality results would carry through. The codimension of $\mathcal{D}_J$ with one added bubble would increase by $2n+2$. Hence, again, the codimension of $\mathcal{D}_J$ would be greater than the dimension $\mathcal{U}_J$, so $\mathcal{D}_J$ will be empty.

Finally, we must deal with the possibility of multiply covered curves. Without loss of generality, assume that the cusp curve has two components: the $\lambda k(u)$ perturbed $J-$holomorphic component of class $A-Y$ and the $J-$holomorphic bubble component of class X. Suppose that X has been reduced from the multiply covered dX where $dX + Y = 0$ in homology for some positive integer d. Since M has dimension four, $M \times S^2$ has dimension six. Therefore, all classes representable by a $J-$holomorphic or perturbed $J-$holomorphic curve give a nonnegative integer when paired with the first Chern class. In particular, dX is representable so

$$c_1(X) = \frac{1}{d} \cdot c_1(dX) \geq 0.$$

This gives us

$$c_1(A) = c_1(A - Y) + d \cdot c_1(X) \geq c_1(A - Y) + c_1(X).$$

When we imitate the proof of Lemma 8.2.4, we see that the space we would consider as the domain of the evaluation map is

$$\mathcal{U}'_J = (\mu^\lambda(A - Y) \times_G S^2) \times (\mu(X) \times_{\mathrm{PSL}(2,\mathbf{C})} S^2).$$

We calculate

$$\begin{aligned} \dim \mathcal{U}'_J &= 2n + 2 + 2c_1(A - Y) + 2 - 4 + 2n + 2 + 2c_1(X) + 2 - 6 \\ &= -2 + 2c_1(A - Y + X) + 4n \\ &\leq -2 + 2c_1(A) + 4n \\ &= \dim \mathcal{U}_J. \end{aligned}$$

Note, however, that the codimension of $\mathcal{D}_J$ is the same. Hence, again, generically $\mathcal{D}_J$ will be empty. We could deal with the case when $A - Y$ has been reduced from a multiply covered component in a similar manner. □

Chapter 9

Proofs of main theorems about length minimizing paths

We will restate the theorems here as we prove them.

Theorem 7.4.3 *Suppose (M,ω) is a symplectic manifold of dimension four. Then,*

$$c_{HZ}(M \times D^2(a), \omega \oplus \sigma) \leq a.$$

Proof: Fix an almost complex structure $J \in \mathcal{J}$ on $M \times S^2$ so that Proposition 8.2.5 holds. Note that Proposition 8.2.5 implies that Theorem 8.1.1 and hence Theorem 7.4.2 hold for M, if M has dimension four. This completes the proof. $\square$

Theorem 7.4.4 *Let (M,ω) be a symplectic manifold of dimension four. Let ϕ_t for $0 \leq t \leq 1$ be a path in $Ham^c(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0 is the identity diffeomorphism and ϕ_t has no non-constant closed trajectory in time less than 1. Then, ϕ_t for $0 \leq t \leq 1$ is length minimizing among all homotopic paths between the identity and ϕ_1.*

Proof: Theorem 7.4.3 implies that the capacity-area inequality holds for c_{HZ} for all split quasi-cylinders. We can repeat the proof from Proposition 4.4 of [LM95a] to show that it holds for all quasi-cylinders. Thus, c_{HZ} satisfies condition (ii) of Theorem 6.2.4 for any Hamiltonian H on M if M has dimension four. Now, we choose an autonomous H that generates a flow ϕ_t^H which has no non-constant closed trajectories for $0 < t \leq 1$. In order to show that H generates a length minimizing geodesic among all homotopic paths, we must show that $c_{HZ}(H) \geq L(H)$ verifying condition (i) of Theorem 6.2.4. We now invoke Proposition 3.1 from [LM95a]:

Proposition 9.0.6 *(Lalonde-McDuff) Let M be any symplectic manifold and $H : M \to \mathbf{R}$ be any compactly supported Hamiltonian with no non-constant closed trajectory in time less than 1. Then*

$$c_{HZ}(H) \geq L(H).$$

Proof: We give here a sketch of the proof. Using H, we can construct a specific Hamiltonian $\overline{H}$ on $R_H^-(\frac{\nu}{2})$ and show that $\overline{H} \in \mathcal{H}_{ad}(R_H^-(\frac{\nu}{2}))$. Then, it is easy to show that $m(\overline{H}) \geq m(H) = L(H)$, so $c_{HZ}(R_H^-(\frac{\nu}{2})) \geq L(H)$ and hence $c_{HZ}(H) \geq L(H)$. □

It follows that ϕ_t^H for $0 \leq t \leq 1$ is a length minimizing geodesic among all paths homotopic with fixed endpoints from the identity to ϕ_1^H, and we are finished with the proof of Theorem 7.4.4. □

Theorem 7.4.5 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\widetilde{\mathbf{CP}^2}_1)$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Proof: We must show that the path induced by rotation is actually length minimizing among all paths. We define the functional $L : \pi_1(Ham^c(M)) \to \mathbf{R}$ by

$$L([\gamma]) = \inf_{\gamma \in [\gamma]} L(\gamma)$$

and let

$$r_1(M) = \inf(\{\text{Im } L : \pi_1(Ham^c(M) \to \mathbf{R}\} \cap (0, \infty))$$

if this set is not empty, and ∞ otherwise.

Now we use the result from [LM95a]:

Proposition 9.0.7 *(Lalonde-McDuff) Suppose we have a manifold M and a capacity c which satisfies condition (ii) of Proposition 6.2.4. The path ϕ_t^H is length minimizing amongst all paths with the same endpoints if $c(H) = L(H) \leq \frac{r_1}{2}$.*

For the P given above, we know $c_{HZ}(P) = L(P_t) = \frac{\pi}{2}$. Hence, we only need prove that $r_1(\widetilde{\mathbf{CP}^2}) \leq \pi$. In their preprint [AM97], Abreu and McDuff exactly calculate that $\pi_1(Ham^c(\widetilde{\mathbf{CP}^2}_0)) = \mathbf{Z}$. A generator for this group is the class of the loop ψ_t for $0 \leq t \leq 1$ where

$$\psi_t[z_0 : z_1 : z_2] = [e^{2\pi i t} z_0 : z_1 : z_2].$$

Certainly, $L(\psi) = \pi$. Note that the loop ψ_t is just the loop traced by ϕ_t^P traversed twice as fast. To show $r_1(\widetilde{\mathbf{CP}^2}_0) = \pi$, we must show $L([\psi]) = \pi$, i.e. ψ is length minimizing in its homotopy class.

Theorem 9.0.8 *The loop ψ_t for $0 \leq t \leq 1$ is length minimizing in its homotopy class in $Ham^c(\widetilde{\mathbf{CP^2}}_0)$. In fact, it is length minimizing in its homotopy class when considered as a loop in $\mathbf{CP^2}$ or $\widetilde{\mathbf{CP^2}}_1$, as well.*

Proof: We cannot use Theorem 7.4.4 to prove this directly, because ψ is a loop based at the identity and therefore has non-constant closed trajectories. However, Theorem 7.4.4 does tell us that $\psi|_{[0,t_0]}$ will be length minimizing in its homotopy class for any $t_0 < 1$ on $\mathbf{CP^2}$, $\widetilde{\mathbf{CP^2}}_0$, or $\widetilde{\mathbf{CP^2}}_1$. The following lemma, then, completes the proof of Theorem 9.0.8 and Theorem 7.4.5.

Lemma 9.0.9 *Suppose we have a path $\gamma : [0,1] \to Ham^c(M)$ and that for all $t_0 < 1$, the path $\gamma|_{[0,t_0]}$ is length minimizing in its homotopy class among paths with fixed endpoints γ_0 and γ_{t_0}. Then, γ_t for $0 \leq t \leq 1$ is length minimizing in its homotopy class among paths with fixed endpoints γ_0 and γ_1.*

Proof: Suppose there is another path, β, homotopic with fixed endpoints to γ, such that $L(\beta) < L(\gamma)$. Choose $\epsilon > 0$ such that

$$L(\beta) + \epsilon = L(\gamma).$$

Fix $t_0 \in (0,1)$ so that $L(\gamma|_{[0,t_0]})$ has length $L(\gamma) - \frac{\epsilon}{3} = L(\beta) + \frac{2\epsilon}{3}$. Then, since the composition loop $\beta^{-1} \circ \gamma$ is null homotopic, we have two homotopic paths from γ_0 to γ_{t_0}:

$$\gamma|_{[0,t_0]} \text{ and } \gamma^{-1}|_{[t_0,1]} \circ \beta$$

with lengths $L(\beta) + \frac{2\epsilon}{3}$ and $L(\beta) + \frac{\epsilon}{3}$ respectively. Since our hypothesis states that $\gamma|_{[0,t_0]}$ is length minimizing in its homotopy class, we have a contradiction. □

with fixed endpoints γ_0 and γ_{t_0}. Then, γ_t for $0 \leq t \leq 1$ is length minimizing in its homotopy class among paths with fixed endpoints γ_0 and γ_1.

Proof: Suppose there is another path, β, homotopic with fixed endpoints to γ, such that $L(\beta) < L(\gamma)$. Choose $\epsilon > 0$ such that

$$L(\beta) + \epsilon = L(\gamma).$$

Fix $t_0 \in (0,1)$ so that $L(\gamma|_{[0,t_0]})$ has length $L(\gamma) - \frac{\epsilon}{3} = L(\beta) + \frac{2\epsilon}{3}$. Then, since the composition loop $\beta^{-1} \circ \gamma$ is null homotopic, we have two homotopic paths from γ_0 to γ_{t_0}:

$$\gamma|_{[0,t_0]} \text{ and } \gamma^{-1}|_{[t_0,1]} \circ \beta$$

with lengths $L(\beta) + \frac{2\epsilon}{3}$ and $L(\beta) + \frac{\epsilon}{3}$ respectively. Since our hypothesis states that $\gamma|_{[0,t_0]}$ is length minimizing in its homotopy class, we have a contradiction.

□

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