A generalization of the Morse complex

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Abstract of the Dissertation
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Let $f : X \longrightarrow \mathbb{R}$ be a Morse-Bott function on a compact manifold, whose gradient-like flow φ_τ satisfies a generalization of the Smale condition and is 'tame' near the critical manifolds. We show that such a flow satisfies the finite volume condition of Harvey and Lawson [HL97b]. This implies that φ_τ gives rise to deformations of both the de Rham complex of differential forms on X and the complex of smooth singular chains transverse to the unstable manifolds of critical sets. We describe the structure of $\lim_{\tau \longrightarrow \infty} \varphi_\tau(T)$ for T in either of the two complexes. In particular, we show how the deformation of the singular chains yields an effectively computable model of the homology of X in terms of a generalized Morse complex $(\mathcal{M}, \partial_f)$. The chain groups of this complex can be identified with the (suitably shifted) groups of singular chains in the critical sets, and the differential is explicitly given in terms of the flow. Applications include computations of the homology of a fibration and G-equivariant homology for manifolds with an action of a compact Lie group G. The methods also give partial results about the ring structure of $H^*(X)$.

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Chapter 1

Introduction

It is a classical result of Morse theory that a finite dimensional compact manifold X has the homotopy type of a CW-complex with one cell for each critical point p of a Morse function f on X (where the dimension of the cell is equal to the index of the critical point p). In the late 1950's Smale introduced a simple transversality condition on the stable and unstable sets associated to a gradient-like flow for f, which can be used to substantially enhance this result. In fact, one can set up an algebraic chain complex $(\mathcal{M}, \partial_f)$ whose chain groups $\mathcal{M}_k$ are generated by the critical points of index k and whose boundary operator is given by counting the flow lines between critical points whose indices differ by one. Then it is a theorem that this chain complex computes the homology of X. Although this result is implicit in the work of Thom and Smale, it was only brought to broad attention in the early 1980's by Witten's well-known paper [Wit82].

In a recent paper [HL97b], Reese Harvey and Blaine Lawson introduce a new point of view to the subject. They observe that for Morse-Smale functions the total graph T in $X \times X$ of a 'canonically flat' gradient flow φ_T has finite volume and gives rise to an equation of currents

$$\partial T = \Delta - P \tag{1.1}$$

where $\Delta \subset X \times X$ is the diagonal and $P = \lim_{T \longrightarrow \infty} \text{graph}(\varphi_T)$. In fact one proves that

$$P = \sum_{p \in Cr(f)} [S_p] \times [U_p] \tag{1.2}$$

where S_p and U_p are the stable and unstable sets of the critical point p and $[Y]$ denotes the current of integration over the locally closed submanifold $Y \subset X$. Now one uses the fact that currents C in the product $X \times X$ give rise to operators $\mathbb{C} : \Omega^k(X) \longrightarrow \mathcal{D}_{n-k}(X)$ mapping differential forms on X to currents in such a way that the current equation (1.1) translates into a chain homotopy

$$\mathbb{T} \circ \mathbf{d} + \mathbf{d} \circ \mathbb{T} = \mathbb{I} - \mathbb{P}. \tag{1.3}$$

Here $\mathbb{T}$ and $\mathbb{P}$ are the operator associated to T and P, respectively, and $\mathbb{I}$ is the inclusion of twisted forms into currents associated to the diagonal Δ. This equation (1.3) has several important consequences. First, we see immediately that $\mathbb{P}$ commutes with exterior differentiation $\mathbf{d}$. Also, $\mathbb{P}$ induces a homology isomorphism, because $\mathbb{I}$ does. But since

$$\mathbb{P}(\beta) = \sum_{p \in Cr(f)} \left(\int_{U_p} \beta \right) [S_p], \tag{1.4}$$

we see that the image of $\mathbb{P}$ is the finitely generated complex $\mathcal{S}_f$ spanned over $\mathbb{R}$ by the stable manifolds of the critical points. The remarkable fact is that the gradient flow of a Morse-Smale function gives rise to an explicit deformation of the de Rham complex to the finite-dimensional complex $\mathcal{S}_f$. One easily checks that the boundary map in $\mathcal{S}_f$ is given by counting flow lines as mentioned before, and so it can be thought of as an explicit geometric realization of the algebraic complex $(\mathcal{M}, \partial_f)$. In fact the span of the stable manifolds over the integers forms an invariant subcomplex $\mathcal{S}_f^{\mathbb{Z}}$, which can be viewed as the deformation of the singular chains transverse to the unstable sets of the critical points. This complex $\mathcal{S}_F^{\mathbb{Z}}$ computes the integer homology of the underlying manifold X.

It is this explicit deformation which gives rise to interesting applications. For example, Reese Harvey and Blaine Lawson [HL97a] use an extension of these techniques to establish and compute canonical universal residue forms that arise in studying the relation of characteristic classes of two vector bundles with the singularities of a bundle map between them.

In this work, we extend the methods of Harvey and Lawson to the case of generalized Morse functions, where the critical set is a finite disjoint union of embedded submanifolds. These functions were first studied by Bott [Bot54], who proved the analogue of the classical Morse inequalities in this setting and applied them to obtain information about the topology of loop spaces. For certain 'tame' gradient-like flows which satisfy an appropriate version of the Smale transversality condition, we again prove that the total graph T of the flow has finite volume. Equation (1.1) holds as before and the products in (1.2) are now replaced by fibre products over the connected components of the critical sets:

$$P = \sum_{F_i \in \pi_0(Cr(f))} S_F \times_F U_F. \tag{1.5}$$

The associated operator $\mathbb{P}$ no longer has finitely generated image, as it is given by

$$\mathbb{P}(\beta) = \sum_{F_i \in \pi_0(Cr(f))} Res_F(\beta)[S_F], \tag{1.6}$$

where the smooth residue form $Res_F(\beta)$ is the pull-back to S_F of the fibre integral of β over U_F. However, $\mathbb{P}$ still acts on sufficiently transverse chains, and the geometric version of the main theorem (Theorem 4.4) asserts that the image complex $\mathcal{S}_f^{\mathbb{Z}}$ in this case is given by stable bundles of smooth chains in the critical sets. As before one may interpret these results as giving an explicit deformation of the de Rham complex (or more generally the complex of sufficiently transverse chains) to a geometric model $\mathcal{S}_f^{(\mathbb{Z})}$ of a Morse complex. In this model the boundary map is again explicitly given in terms of the gradient flow, so that it can be used to effectively compute the homology of X with $\mathbb{Z}$ or at least $\mathbb{Z}_2$ coefficients. By degenerating diagonals in higher products $X \times X \times ... \times X$ we obtain some partial information about the ring structure of $H^*(X)$.

The class of functions to which our results are applicable is quite general. It includes pull-backs of Morse-Smale functions from the base space of a fibration to the total space, as well as many invariant functions for the action of a compact Lie group G on X. In fact one can use the later to obtain results about the G-equivariant homology of X. But although a Morse function can always be made Morse-Smale by a change of metric, the same is not true for generalized Morse functions (cf. Example 2.3).

Some of these results for real coefficients were obtained by Austin and Braam [AB94] using methods from Floer theory. Our approach naturally yields results over $\mathbb{Z}$ and $\mathbb{Z}_2$, which are not at all immediate from their frame-work. We also expect the point of view of deformations to yield new applications.

Here is a brief outline of the body of this work. In Chapter 2 we set up the basic notation and definitions and provide examples of generalized Morse functions to which this theory applies. Chapter 3 contains the statement and proof of the basic structure theorem about the total graph T of the flow φ_T. In the course of the proof we find that the closure of the stable and unstable sets S_F and U_F of a critical manifold F are the image of smooth families of compact manifolds with corners under smooth maps. This observation allows us to define the concept of a stable bundle of a smooth chain in F in Chapter 4. Here the main theorems about the operator $\mathbb{P}$ on forms and on smooth chains transverse to the unstable bundles are proved. In Section 5.1 we give the calculation of the integer homology groups of $SO(n)$ as an example of how the methods work in practice. Section 5.2 explains how to compute G-equivariant homology from a function invariant under the action of some compact group G on X. Section 5.3 contains results about cup product. For completeness, the facts from geometric measure theory which are used throughout are collected in the Appendix. It also contains a brief section on Whitney stratifications, as well as a proof of the Morse Lemma for Morse-Bott functions.

Chapter 2

Mathematical Preliminaries

In this chapter we recall some standard definitions, mostly to introduce notation that will be used throughout the rest of this work. A few new concepts are also introduced.

In what follows, X will denote a smooth, compact, n-dimensional manifold without boundary. A Morse function $f : X \to \mathbb{R}$ is a smooth function on X whose critical points are non-degenerate in the sense that the Hessian is a non-degenerate bilinear form. A function $f : X \to \mathbb{R}$ is called a generalized Morse function (or Morse-Bott function) if its critical set $Cr(f)$ is a disjoint union of finitely many embedded submanifolds, and again the Hessian at a critical point p in the critical manifold F, which is now defined on T_pM/T_pF, is nondegenerate. The index λ_p of a critical point p is defined to be the index of the Hessian at p. Since this number only depends on the critical set F containing p, we will often denote it by λ_F. We also define the number $\lambda_F^* = n - n_F - \lambda_F$, where n_F is the dimension of the critical manifold F containing p. A vector field V on X is called a gradient-like vector field for the function f if its zero set coincides with the set $Cr(f)$ of critical points of f and $df(V) > 0$ on the complement of $Cr(f)$. Note that the gradient with respect to any metric is a gradient-like vector field in this sense. To any vector field we can associate its flow φ_τ, and for gradient-like fields the fixed point set $Fix(\varphi_\tau)$ of the flow will coincide with the critical set $Cr(f)$ of the function.

Definition 2.1 *A flow φ_τ on X is called* **tame** *if the fixed point set $Fix(\varphi_\tau)$ consists of a finite union of disjoint smooth submanifolds $\{F_i\}$ and each fixed point $p \in F$ has a coordinate neighborhood $(u, x, v) : \mathcal{U}_p \longrightarrow \mathbb{R}^{\lambda_F} \times \mathbb{R}^{n_F} \times \mathbb{R}^{\lambda_F^*}$ such that the flow in these coordinates is given by $\varphi_\tau(u, x, v) = (e^{-\tau}u, x, e^{\tau}v)$.*

In section 4 of the Appendix, we prove the following statement:

Theorem A.8 *Let $f : X \longrightarrow \mathbb{R}$ be a generalized Morse function , and let F be a connected component of its critical set. Then there exists a normal bundle with a splitting $N = N^+ \oplus N^-$ and a metric such that the function is given as $f(u, x, v) = f(F) + |v|^2 - |u|^2$. In particular, the gradient flow for this metric is locally of the form*

$$\varphi_\tau(u, x, v) = (e^{-\tau}u, x, e^{\tau}v).$$

For each $p \in Cr(f)$ we can define its stable set S_p and its unstable set U_p as usual by

$$S_p := \{x \in X : \lim_{\tau\to\infty} \varphi_\tau(x) = p\} \qquad \text{and}$$

$$U_p := \{x \in X : \lim_{\tau\to-\infty} \varphi_\tau(x) = p\}$$

For tame gradient-like flows of generalized Morse functions these sets are diffeomorphically embedded open disks of dimension λ_p and λ_p^*, respectively. We define the stable (resp. unstable) set of a critical manifold F to be the union of the stable (resp. unstable) sets of its points and denote it by S_F (resp. U_F). Recall that the gradient flow of a Morse function f on X is said to satisfy the **Smale condition** if for all critical points $p, q \in Cr(f)$ the stable manifold S_p is transverse to the unstable manifold U_q. In order to generalize the results of [HL97b] to the case of generalized Morse functions, we need an appropriate version of the Smale condition.

Definition 2.2 *A flow φ_τ is said to satisfy the* **generalized Smale condition** *if its fixed point set $Fix(\varphi_\tau)$ consists of a finite disjoint union of embedded submanifolds and for any two fixed points $p, q \in Fix(\varphi_\tau)$ we have that U_p is transverse to S_{F_q} and S_q is transverse to U_{F_p}, where F_x denotes the connected component of $Fix(\varphi_\tau)$ containing x.*

A generalized Morse function f is said to be generalized Morse-Smale if there is a gradient flow for f satisfying the generalized Smale condition.

We say $p \prec q$ if there is a (possibly piecewise) flow line from p to q. Similarly we have a relation $F \prec F'$ between critical manifolds.

For a generalized Morse-Smale function $F \prec F', F \neq F'$ implies the inequalities

$$\lambda_F < \lambda_{F'} \text{ and } \lambda_F^* > \lambda_{F'}^*$$

These are of course equivalent in the case $n_F = n_{F'} = 0$.

Remark 2.3 *Even though it is true that a (regular) Morse function can be made Morse-Smale by a change of metric, the same is* **not** *true for generalized Morse functions.*

For example, take the torus in $\mathbb{R}^3$ as pictured below with the height function. In coordinates $(\varphi, \psi) \in [0, 2\pi) \times [0, 2\pi)$ on the torus one could take it to be $h(\varphi, \psi) = (2 + cos2\varphi)(1 + cos\psi)$.

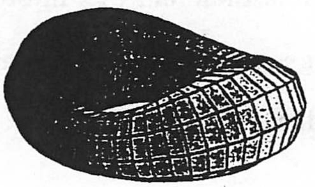

Figure 2.1: A certain embedding of the torus in $\mathbb{R}^3$

It has the 'bottom circle' $\{\psi = \pi\}$ as absolute minimum and on the 'top circle' it has two saddle points, $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$, and two maxima, $(0, 0)$ and $(\pi, 0)$. Notice that the fibres of the unstable bundle of the bottom circle have dimension 1, as do the stable manifolds of the saddle points. However, for **any** metric, there will be some fibre(s) that intersect(s) these stable manifolds in flow lines, i.e. with dimension 1, which contradicts transversality.

On the other hand, we do have the following important examples.

Example 2.4 *If f is a Morse-Smale function on X and Y is any compact manifold, then the pull-back of f to the product $X \times Y$ is a generalized Morse-Smale function in this sense. More generally, if $P \to X$ is a fibre bundle (and a Riemannian submersion) with compact fibre Y, and f is a Morse-Smale function on X, its pull-back to P is generalized Morse-Smale in our sense.*

To see that, e.g., U_p is transverse to S_F just note that they project to transverse objects on X and the tangent space to the fibre is contained in TS_F.

As an illustration one can take the function $f : S^3 \to \mathbb{R}$ given by $f(z_0, z_1) = |z_1|^2 - |z_0|^2$. Here the function is pulled back from S^2 via the Hopf fibration.

The same argument as for Example 2.4 shows that the set of generalized Morse-Smale functions is closed under pull-backs to the total space of a submersion. It is also not hard to see that if $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are generalized Morse-Smale, then so is $f + g : X \times Y \to \mathbb{R}$.

Example 2.5 *Another family of examples where the generalized Morse-Smale condition can often be easily verified arises in the context of compact Kähler manifolds with a $\mathbb{C}^*$-action with fixed points. Here one can construct an associated Morse function with critical set precisely the fixed point set of the action [Fra59].*

As an illustration, take the action of $\mathbb{C}^*$ on $\mathbb{P}^n$ given by

$$\tau \cdot [z_0 : \ldots : z_n] = [z_0 : \tau z_1 : \ldots : \tau z_n]$$

The fixed point set consists of the point $[1 : 0 : \ldots : 0]$ and the hyperplane $\{z_0 = 0\}$ and the associated function is

$$f([z]) = \frac{1}{|z|^2} \sum_{i=1}^{n} |z_i|^2$$

One of the central concepts of [HL97b], which will also play an important rôle here, is a finite volume flow.

Definition 2.6 *A flow φ_τ on X is called a* **finite volume flow** *if the pull-back by the total flow map $\Phi(x,\tau) = \varphi_\tau(x)$ of any (and therefore every) metric on X to $(X \setminus Fix(\varphi_\tau)) \times (0,\infty)$ has finite volume.*

For gradient-like flows this condition is equivalent to the requirement that the total graph of the flow

$$T = \{(x, \varphi_\tau(x)) \ : \ 0 \leq \tau < \infty\} \subset X \times X$$

has finite volume. Both conditions are independent of the choice of metric on X.

Chapter 3

The Geometry of the Gradient Flow

In this chapter f will be a generalized Morse function with tame gradient-like flow φ_τ. We will investigate the geometry of φ_τ assuming that it satisfies the generalized Smale condition. In particular, we want to prove that the sets S_F and U_F are images of certain families of compact manifolds with corners over F under well-behaved maps. This will enable us to prove an important structure theorem.

Theorem 3.1 *Let f be a generalized Morse function on X and let φ_τ be a tame gradient-like flow for f satisfying the generalized Smale condition. Then there is an equation of currents*

$$\partial T = \Delta - P \tag{3.1}$$

where $T = \{(x, \varphi_\tau(x)) \in X \times X \mid \tau \in (0, \infty)\}$ is the total graph of the flow, Δ is the diagonal and $P = \lim_{\tau \to \infty} graph(\varphi_\tau)$. The current P is given by

$$P = \sum_{F_i \in \pi_0(Cr(f))} [S_i \times_{F_i} U_i],$$

where $S_i \times_{F_i} U_i \subset X \times X$ *denotes the fibre product of the stable and unstable bundles of* F_i.

This theorem generalizes Theorem 6.2 of [HL97b], and some elements of the proof are very similar. The main technical tool is a lemma that describes what happens to the image of a smooth manifold with corners transverse to the stable manifolds as it 'flows through' a critical level (for precise statements see Lemma 3.3). Using this information we will deduce that φ_τ is a finite-volume flow, which then allows us to obtain the current equation (3.1). The final step is the identification of the current P.

We will often make statements about both stable and unstable sets of critical manifolds and points in them. Since the generalized Smale condition is symmetric with respect to time reversal, it is enough to prove the assertion just for one of them, which we choose to be the stable sets.

Without loss of generality we will assume that different connected critical manifolds correspond to different critical values. This is easy to achieve: if F_1 and F_2 happened to be two critical manifolds with the same value of f, we could add a tiny bump function to f which is supported in some small neighborhood of F_1, is constant in a smaller neighborhood of F_1, and has sufficiently small derivative so as to not create new critical points. In any case, this will change none of the essential features of f and is only done to simplify the language in the following presentation.

Our first aim is to prove that the closure of the stable bundle S_F over a critical set F is the image of a smooth manifold with corners $\widetilde{S_F}$ which again fibers over F, such that each fiber is also a smooth manifold with corners. Recall that a manifold with corners of dimension k is a Hausdorff topological space Y such that every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k \ : \ x_i \geq 0 \text{ for all } i = 1, ..., k\}$. It is called smooth if for any two such charts $\psi_1 : U_1 \longrightarrow \mathbb{R}^k_+$ and $\psi_2 : U_2 \longrightarrow \mathbb{R}^k_+$ the composition $\psi_1 \circ \psi_2^{-1} : \psi_2(U_1 \cap U_2) \longrightarrow \psi_1(U_1 \cap U_2)$ extends to a smooth map between open sets of $\mathbb{R}^k$. Y is naturally stratified by the sets $\{Y_r\}_{0 \leq r \leq k}$, where p belongs to Y_r if for some (and hence every) chart exactly r coordinates of p are 0. It is clear that the set Y_r is a smooth manifold of dimension $k - r$. Y_0 is the open and dense set of interior points of Y, Y_1 is the set of regular boundary points and the union of the other Y_i is the set of corners. A map σ from a manifold with corners Y to a manifold X is called smooth if for any (and hence every) coordinate chart $\psi : U \longrightarrow \mathbb{R}^k_+$ the composition $\sigma \circ \psi^{-1}$ extends to a smooth map of some neighborhood of $\psi(U)$ in $\mathbb{R}^k$. Such a map is called **completely transverse** to a submanifold L of the range if its restriction to all the sets Y_r is transverse in the usual sense that $\sigma_*(T_y Y_r) + T_{\sigma(y)} L = T_{\sigma(y)} X$.

Let $\rho : Y \longrightarrow X$ be a smooth map from a smooth manifold with corners Y to a smooth manifold X which is completely transverse to the submanifold $L \subset X$. We define the oriented blow-up $\widetilde{Y}$ of the manifold with corners Y along $\sigma^{-1}(L)$ as follows: First, for any smooth vector bundle $E \longrightarrow B$ we define the oriented blow-up of the zero section as $\widetilde{E} := S(E) \times [0, \infty)$ with the smooth projection map $\pi(b, \theta, r) = (b, r\theta)$, where b is a point in B and θ is a point in the fibre of the sphere bundle $S(E)$ of E over b. The sphere bundle $S(E)$ is defined using any metric on E, but it is easily seen that two different choices of metric give smoothly equivalent bundles. Then to obtain $\widetilde{Y}$, we construct the normal bundle to $\rho^{-1}(L)$ in Y in such a way that its fibre at $y \in Y_r$ is tangent to Y_r. The **oriented blow-up $\widetilde{Y}$ of Y along $\rho^{-1}(L)$** is then defined to be the oriented blow-up of the zero section in this normal bundle.

More generally, given a smooth manifold with corners Y and a smooth manifold Z, consider a smooth projection $\pi : Y \longrightarrow Z$ which is completely transverse to all points $z \in Z$. We will call such a creature a **smooth family of manifolds with corners** over Z. With this notation we have

Lemma 3.2 *Let $\pi : Y \longrightarrow Z$ be a smooth family of compact manifolds with corners over Z, and let $\rho : Y \longrightarrow X$ be a smooth map whose restriction to $\pi^{-1}(z)$ is completely transverse to $L \subset X$ for all $z \in Z$. Then we can construct an oriented blow-up $\widetilde{Y}$ along $\rho^{-1}(L)$ which again is a smooth family of compact manifolds with corners over Z.*

Proof: First assume $\pi : Y \longrightarrow Z$ is actually a smooth fibration such that ρ is transverse to L when restricted to any fibre. Then the codimension of $L' = \rho^{-1}(L)$ in each fibre of π is the same as the codimension of L' in Y. Let $V \subset TY$ be the vertical tangent bundle, whose fibre at the point $y \in \pi^{-1}(z)$ is $T_y(\pi^{-1}(z))$. Let $V_0 \subset V|_{L'}$ be the subbundle whose fibre at $y \in \pi^{-1}(z)$ is $T_y(\pi^{-1}(z) \cap L')$. Choosing a metric on Y, we define a normal bundle structure for L' as $\nu = V_0^{\perp} \subset V|_{L'}$. By construction it has the property that its restriction to any $L'_z = \pi^{-1}(z) \cap L'$ gives a normal bundle structure for L'_z as a submanifold of $\pi^{-1}(z)$.

In general, this procedure can be applied inductively to the strata Y_r of Y to obtain a normal bundle which is tangent to the fibres of π. Again this has the effect that the normal bundle of $\rho^{-1}(L)$ in each fibre is the restriction of the normal bundle of $\rho^{-1}(L)$ in Y. Now it is easy to see that the blow-up $\tilde{Y}$ of Y is fibered by the blow-ups of the fibres of π. □

Denote by $S_F(\varepsilon)$ the ε-sphere bundle in S_F. Then there exists a smooth projection σ from $S_F(\varepsilon) \times [f(F) - \varepsilon, f(F)]$ onto the ε-disk bundle of S_F. In fact σ can be extended to $S_F(\varepsilon) \times [c + \delta, f(F)]$, where $c < f(F)$ is the next critical value and $\delta > 0$ is arbitrary.

We will show how to alter $S_F(\varepsilon)$ by a finite sequence of oriented blow-ups (one for each critical value $c < f(F)$) such that at the final stage we can extend σ to a map of $\widetilde{S_F} = \widetilde{S_F(\varepsilon)} \times [\min f, f(F)]$ onto the closure of S_F.

In general, let us make the following

Inductive Assumption: There exist a manifold with corners Y and a critical value $c < f(F)$ with the following properties:

(1) $\pi : Y \longrightarrow F$ is a smooth family of compact manifolds with corners over F, i.e. π is completely transverse to all $p \in F$.

(2) For any $\delta > 0$ there exists a surjective smooth map $\sigma : Y \times [c + \delta, f(F)] \longrightarrow \overline{S_F} \cap f^{-1}([c+\delta, f(F)])$ [1], whose restriction to $\pi^{-1}(p) \times \{c+\delta\}$ for any $p \in F$ is completely transverse to all unstable bundles of critical

[1]We will see below that the smooth structure on $Y \times [c + \delta, f(F)]$ has to be adjusted slightly to make this map smooth.

manifolds. Furthermore, the restriction of the map σ to the subset Y_0 of interior points of its domain is a diffeomorphism onto its image.

This assumption is clearly satisfied by $Y = S_F(\varepsilon)$ for the first critical value $c < f(F)$.

Let us denote the restriction of σ to $Y \times \{c+\delta\}$ by $\sigma_{c+\delta}$. If F' is the critical manifold with critical value c, the intersection of $U_{F'}$ with $f^{-1}(c+\delta)$ is a smoothly embedded sphere bundle over F', so in particular it is a smooth submanifold of X. By part (2) of the assumption the oriented blow-up $\widetilde{Y}$ of Y along $\sigma_{c+\delta}^{-1}(U_{F'})$ exists, and by Lemma 3.2 we may assume that it satisfies part (1) of the assumption.

We want to prove that $\widetilde{Y}$ satisfies part (2) of the assumption for the next critical value $c' < c$. In particular, we want to show how to map $\widetilde{Y} \times [c-\delta, c+\delta]$ onto the closure of $S_F \cap f^{-1}([c-\delta, c+\delta])$. The further extension of the map as required by (2) is then easily constructed by using the product structure on $f^{-1}([c'+\delta, c-\delta])$ given by the flow. Thus the main technical assertion is contained in

Lemma 3.3 *Let $p : W \longrightarrow Z$ be a smooth family of compact k-dimensional manifolds with corners over some smooth manifold Z and let $\sigma_{c+\delta} : W \longrightarrow f^{-1}(c+\delta)$ be a smooth map whose restriction to each fibre $p^{-1}(z)$ is completely transverse to all unstable bundles of critical manifolds. Let F' be the critical manifold with critical value c and let $\pi : \widetilde{W} \longrightarrow W$ be the oriented blow-up of W along $\sigma_{c+\delta}^{-1}(U_{F'})$. Then there exists a family of maps $\widetilde{\sigma}_t$ of $\widetilde{W}$ into X, parametrized by $t \in [c-\delta, c+\delta]$ and having the following properties:*

(1) $\widetilde{\sigma}_{c+\delta}(\widetilde{w}) = \sigma_{c+\delta}(\pi(\widetilde{w}))$ *for all* $\widetilde{w} \in \widetilde{W}$*, i.e.* $\widetilde{\sigma}_{c+\delta}$ *is just the lift of* $\sigma_{c+\delta}$ *to* $\widetilde{W}$.

(2) The image of $\widetilde{\sigma}$ *parametrizes the closure of the backward time image of* $\sigma_{c+\delta}(W)$ *under the flow* φ_T *in* X *by the value of* f*, i.e.* $\widetilde{\sigma}_t(\widetilde{W})) \subset f^{-1}(t)$ *and* $x \in \widetilde{\sigma}(\widetilde{W} \times [c-\delta, c+\delta])$ *if and only if* $x \in \overline{S_F} \cap f^{-1}([c-\delta, c+\delta])$*. In particular,* $\widetilde{\sigma}_t(\widetilde{w}) \in f^{-1}(c-\delta)$ *if and only if* $t = c-\delta$.

(3) The restriction of the map $\widetilde{\sigma}_{c-\delta} : \widetilde{W} \longrightarrow f^{-1}(c-\delta)$ *to each fibre of* $\widetilde{p} : \widetilde{W} \longrightarrow Z$ *is completely transverse to all unstable bundles of critical manifolds.*

(4) There is a **canonical** *smooth structure on* $\widetilde{W} \times [c-\delta, c+\delta]$ *with respect to which the family* $\widetilde{\sigma}_t$ *becomes a smooth map* $\widetilde{\Sigma}$*. It agrees with the product structure* $\widetilde{W} \times [c-\delta, c+\delta]$ *away from* $\widetilde{W} \times \{0\}$ *and has the effect of introducing new corners in the domain along the preimage of* F'.

Before proving Lemma 3.3 we observe that, starting from $S_F(\varepsilon)$, we can perform a sequence of blow-ups, one for each critical level $c < f(F)$, to construct the promised set $\widetilde{S_F} \simeq \widetilde{S_F(\varepsilon)} \times [\min f, f(F)]$ which fibres over F and parametrizes the closure of S_F in the sense that the map is injective on an open dense subset. Here we write $\simeq$ to remind the reader of the fact that the smoothness structure has been altered at the pre-images of the various critical set $F' \prec F$. We summarize this result in the following theorem, which will play a central part in this and the following chapter.

Theorem 3.4 *The closure $\overline{S_F}$ of the stable manifold of any critical manifold F is the image of a smooth family of compact manifolds with corners $\widetilde{S_F} \longrightarrow F$ under a smooth map $\tilde{i}_{S_F}$ whose restriction to the interior points of $\widetilde{S_F}$ is a diffeomorphism onto an open dense set of S_F.*

Similarly, the closure $\overline{U_F}$ of the unstable manifold of any critical manifold F is the image of a smooth family of compact manifolds with corners $\widetilde{U_F} \longrightarrow F$ under a smooth map $\tilde{i}_{U_F}$ whose restriction to the interior points of $\widetilde{U_F}$ is a diffeomorphism onto an open dense set of U_F.

Note that the second assertion simply follows from the observation that the generalized Smale condition is invariant under time-reversal, which interchanges the rôles of stable and unstable manifolds.

Later in this chapter we will have occasion to make use of the following observation.

Remark 3.5 *Let K be a smooth compact submanifold of dimension k, contained in some level set of f, which is transverse to all stable (resp. unstable) manifolds of critical sets. Our inductive argument shows that its forward (resp. backward) time image under the flow is the image of a smooth compact manifold with corners of dimension $k+1$, and so in particular has finite volume.*

Let us now complete the proof of Theorem 3.4 by giving the proof for the key technical assertion that was post-poned before.

Proof: (of Lemma 3.3) We will first reduce the proof to more local considerations. As an initial step, observe that outside any neighborhood of the blow-up locus the extension can be constructed from the gradient flow, since on the complement of the critical set F' the gradient-like flow provides us with a canonical product structure.

Now let us cover F' with a finite number of coordinate charts U_α, such that a neighborhood of F' is a given as a union $\cup V'_\alpha$, where each V'_α is of the form

$$V'_\alpha \cong D^{\lambda_{F'}} \times U_\alpha \times D^{\lambda^*_{F'}} \tag{3.2}$$

and on it the gradient-like flow has the special form

$$\varphi_\tau(u, x, v) = (e^{-\tau}u, x, e^\tau v)$$

For some $\delta > 0$ sufficiently small, the union of these coordinate blocks contains the sphere bundles $U_{F'} \cap f^{-1}(c+\delta)$ and $S_{F'} \cap f^{-1}(c-\delta)$. In fact without loss of generality we may assume that these spheres bound the disks $\{0\} \times U_\alpha \times D^{\lambda^*_{F'}}$ and $D^{\lambda_{F'}} \times U_\alpha \times \{0\}$ in (3.2).

We will construct the family of maps $\tilde{\sigma}_t$ on the blow-up of the piece of W that gets mapped into one of the V'_α under $\sigma = \sigma_{c+\delta}$. Since the construction uses the flow in a natural way, it will be easy to check that the pieces fit together to form a globally defined family $\tilde{\sigma}_t : \widetilde{W} \times [c - \delta, c + \delta] \longrightarrow X$. Abusing notation, we will continue to refer to the piece we are considering as W.

For simplicity of language, we will also assume that $c = 0$ and that the function f is given on V'_α by $f(u, x, v) = |v|^2 - |u|^2$. Inside V'_α consider the region V_α determined by the equations

$$|f(u, x, v)| \leq \delta \text{ and } |u||v| \leq \delta'$$

for some small $\delta' > 0$ (the size of this cut-off is not really essential, since we are only interested in what happens near F'). Note that V_α has 'incoming' and 'outgoing' boundaries with respect to the *forward* time flow, which we denote by A and B respectively. Formally, these are given by

$$A = \{(u, x, v) : f(u, x, v) = -\delta\} \text{ and } B = \{(u, x, v) : f(u, x, v) = \delta\}$$

The special subsets $S_{F'} \cap A$ and $U_{F'} \cap B$ are denoted by A_0 and B_0 respectively and are given in our coordinates by

$$A_0 = \{(u, x, 0) : |u| = \delta\} \text{ and } B_0 = \{(0, x, v) : |v| = \delta\}$$

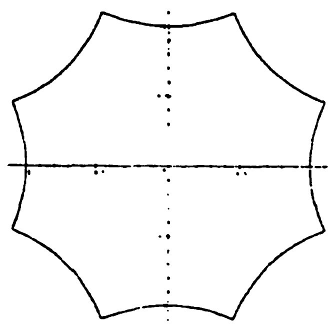

Figure 3.1: A slice of V_α for fixed x, drawn in the $(|u|, |v|)$-plane

Now we will give a proof of the assertions in the case where Z is just a point.

First let us define a map $\Psi : [0, \delta'] \times [-\delta, \delta] \longrightarrow \mathbb{R}^2$ as

$$(q, t) \mapsto (r(q,t), s(q,t)) = \left(\sqrt{\frac{\sqrt{t^2 + 4q^2} + t}{2}}, \sqrt{\frac{\sqrt{t^2 + 4q^2} - t}{2}} \right) \qquad (3.3)$$

Observe that Ψ is continuous everywhere and smooth except at $(0,0)$. It also has the property that $r(q,t)s(q,t) = q$ and $r^2(q,t) - s^2(q,t) = t$. That means one can think of Ψ as a parametrization of the gradient flow lines of the function $r^2 - s^2$ on $\mathbb{R}^2$ (characterized by q being constant) by the value t of the function f.

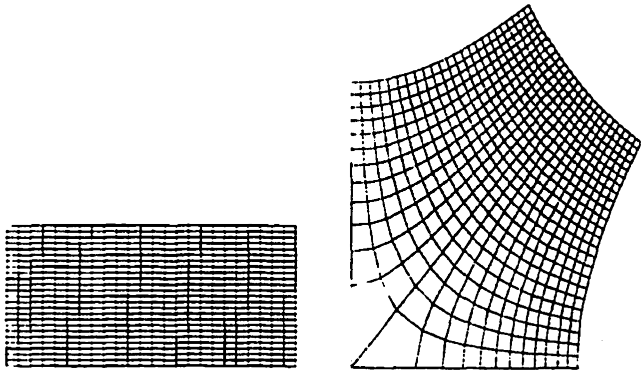

Figure 3.2: Coordinate lines for q and t in the domain and range of Ψ

Now consider the oriented blow-up $\widetilde{V}_\alpha := S^{\lambda_{F'}-1}\times[0,1]\times U_\alpha\times S^{\lambda^*_{F'}-1}\times[0,1]$ of V_α along the set $D \subset V_\alpha$ defined by $|u||v| = 0$. We have an obvious projection from $\widetilde{V}_\alpha$ to V_α given by

$$(\hat{u}, s, x, \hat{v}, r) \mapsto (\hat{u}s, x, \hat{v}r).$$

Notice that the lift of the gradient-like flow to $\widetilde{V}_\alpha$ is given as

$$\widetilde{\varphi}_\tau(\hat{u}, s, x, \hat{v}, r) = (\hat{u}, se^{-\tau}, x, \hat{v}, re^{\tau}).$$

Inside $\widetilde{V}_\alpha$ we have the lifts $\widetilde{A}$ and $\widetilde{B}$ of A and B characterized by $r^2 - s^2 = -\delta$ and $r^2 - s^2 = \delta$ respectively. Within them, we have $\widetilde{A}_0$ and $\widetilde{B}_0$ given by $r = 0$ and $s = 0$, respectively.

Recall that by the assumption of the lemma, we are given $\sigma_{c+\delta} = \sigma_\delta$ mapping (a piece of) W into B. Clearly, we can lift σ_δ to a map $\hat{\sigma}_\delta$ that makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\hat{\sigma}_\delta} & \widetilde{B} \subset \widetilde{V}_\alpha \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\sigma_\delta} & B \subset V_\alpha \end{array}$$

Since σ_δ is completely transverse to B_0, the lift $\hat{\sigma}_\delta$ will be completely transverse to $\widetilde{B_0}$. Now we extend $\hat{\sigma}_\delta$ to $\widetilde{W} \times [-\delta, \delta]$ by using Ψ in the s and r coordinates, i.e.

$$\hat{\sigma}_t(\widetilde{w}) = (\hat{u}(\widetilde{w}), s(q(\widetilde{w}), t), x, \hat{v}(\widetilde{w}), r(q(\widetilde{w}), t)),$$

where $q(\widetilde{w}) = r(\hat{\sigma}_\delta(\widetilde{w}))s(\hat{\sigma}_\delta(\widetilde{w}))$ and we think of $\hat{u}$, x and $\hat{v}$ as the components of $\hat{\sigma}_\delta$. Finally, let $\widetilde{\sigma}_t(\widetilde{w}) := \pi(\hat{\sigma}_t(\widetilde{w}))$. Then (1) is satisfied by definition, and (2) follows from the fact that the lift of f to $\widetilde{V}_\alpha$ looks like $r^2 - s^2$. It remains to explain (3) and (4). Notice first that $\widetilde{\sigma}_{-\delta}(\widetilde{w}) \in A_0$ if and only if $\sigma_\delta(w) \in B_0$. Since on the complement of any neighborhood of B_0 the flow provides a smooth product structure for the backward image under the flow, and transversality to the unstable manifolds is preserved under the flow, we see that for all points in the image of $\widetilde{\sigma}_{-\delta}$ contained in $A \setminus A_0$ the transversality assertion still holds. So us let fix $w_0 \in W_r \subset W$ such that $\sigma_\delta(w_0) = (0, x_0, v_0) \in B_0$. Here recall that W_r is the stratum of 'corners of order r' in W. Then its preimage $\pi^{-1}(w)$ in $\widetilde{W_r}$ consists of a whole sphere $S^{\lambda_{F'}-1}$ mapping diffeomorphically onto the set $\{(\hat{u}, 0, x_0, \hat{v}_0, \delta) \ : \ \hat{u} \in S^{\lambda_F-1}\} \subset \widetilde{B}$ under $\hat{\sigma}_\delta$. Now observe that this sets gets mapped onto $\{(\hat{u}, \delta, x_0, \hat{v}_0, 0) \ : \ \hat{v} \in S^{\lambda_F-1}\} \subset \widetilde{A}$ under $\hat{\sigma}_{-\delta}$. Since the kernel of the differential of the projection from $\widetilde{A}$ to A is transverse to this set, we arrive at the conclusion that for any point in $\widetilde{W_r}$ mapping into A_0 under $\widetilde{\sigma}_{-\delta}$, the image of $T_{\tilde{w}}\widetilde{W}_r$ under the differential of $\widetilde{\sigma}_{-\delta}$ contains the tangent space to the sphere in the stable set of the critical point $x \in F'$ corresponding to it. Since this sphere is transverse to all unstable bundles by the assumption that the flow satisfies the generalized Smale condition, we have proven (3).

The only problem for smoothness of the family $\widetilde{\sigma}_t$ stems from the fact that the map Ψ defined in (3.3) is not smooth at the origin. The trick to fix this is to change the domain of the map to be the graph $\Gamma \subset [0,\delta'] \times [-\delta,\delta] \times [0,1] \times [0,1]$ of Ψ, formally given as

$$\Gamma := \{(q,t,s,r) \ : \ r^2 - s^2 = t, rs = q, 0 \leq q \leq \delta', -\delta \leq t \leq \delta\}.$$

Notice that Γ is a smooth manifold with a corner at $(0,0,0,0)$, because it is also the graph of Ψ^{-1}, which is a well-defined and smooth map on the part of the first quadrant in $\mathbb{R}^2$ which is the image of Ψ. In this smooth structure on the domain, the map Ψ is trivially smooth, since it is just projection on the last two coordinates. Thus we have replaced the singularity of the map by a corner in the domain. We claim that the same idea works in our slightly more general situation. Namely, consider the set $G \subset \widetilde{W} \times [-\delta,\delta] \times \widetilde{V}_\alpha$ defined by the conditions

$$\begin{aligned} G := \{(\widetilde{w}, t, \hat{u}, s, x, \hat{v}, r) \in & \widetilde{W} \times [-\delta, \delta] \times \widetilde{V}_\alpha \ : \\ & \hat{u} = \hat{u}(\hat{\sigma}_\delta(\widetilde{w})) \\ & \hat{v} = \hat{v}(\hat{\sigma}_\delta(\widetilde{w})) \\ & \hat{x} = \hat{x}(\hat{\sigma}_\delta(\widetilde{w})) \\ & t = r^2 - s^2 \\ & rs = r(\hat{\sigma}_\delta(\widetilde{w}))s(\hat{\sigma}_\delta(\widetilde{w}))\} \end{aligned}$$

Here we recall that $(\hat{u}, r, x, \hat{v}, s)$ are coordinates on $\widetilde{V}_\alpha = S^{\lambda_{F'}-1} \times [0,1] \times U_\alpha \times S^{\lambda_F^*-1} \times [0,1]$, and $\hat{\sigma}_\delta : \widetilde{W} \longrightarrow \widetilde{B} \subset \widetilde{V}_\alpha$ is the lift of our original map $\sigma_\delta : W \longrightarrow B$. This set G is homeomorphic to $\widetilde{W} \times [-\delta, \delta]$, because it is the graph of the family $\hat{\sigma}_t$ that extends $\hat{\sigma}_\delta$ to $\widetilde{W} \times [-\delta, \delta]$. If we can prove that it aquires the structure of a smooth manifold with corners from this embedding, then $\widetilde{\sigma}_t$ will be a smooth map in this setting, because it is just the projection onto $\widetilde{V}_\alpha$ followed by projection to V_α. The only points where there is a question about smoothness are points $p = (\widetilde{w}, t, \hat{u}, s, x, \hat{v}, r)$ where the component $\widetilde{w} \in \widetilde{W}$ of p is contained in the blow-up locus.

If $\widetilde{w}_0 \in \widetilde{W}$ is a point in the blow-up locus, then it is mapped under $\hat{\sigma}_\delta$ into $\widetilde{B}_0$. We claim that the function $q = r(\hat{\sigma}_\delta(\widetilde{w}))s(\hat{\sigma}_\delta(\widetilde{w}))$ is smooth with non-zero derivative in some neighborhood of $\widetilde{w}_0$ in $\widetilde{W}$. The smoothness is clear from the smoothness of $\hat{\sigma}_\delta$. The derivative of q at $\widetilde{w}_0$ is $D(r \circ \hat{\sigma}_\delta \cdot s \circ \hat{\sigma}_\delta)(\widetilde{w}_0) = D(r \circ \hat{\sigma}_\delta)(\widetilde{w}_0) \cdot s(\hat{\sigma}_\delta(\widetilde{w}_0)) + r(\hat{\sigma}_\delta(\widetilde{w}_0)) \cdot D(s \circ \hat{\sigma}_\delta)(\widetilde{w}_0)$, where the first term is 0 because $s(\hat{\sigma}_\delta(\widetilde{w}_0)) = 0$ and the second term is non-zero because $\hat{\sigma}_\delta$ is transverse to $\widetilde{B}_0$. Therefore we can use q as one component of a local coordinate system

$(w', q) : D^{k-1} \times [0, \varepsilon) \longrightarrow W'(\widetilde{w}_0)$ for a neighborhood W' of $\widetilde{w}_0$ in $\widetilde{W}$. Over $W' \times [-\delta, \delta]$ we can now consider two sets. The first one is the graph of the three smooth functions $\hat{u}$, x and $\hat{v}$, namely

$$X := \{(w', q, t, \hat{u}, x, \hat{v}) \subset D^{k-1} \times [0, \varepsilon) \times [-\delta, \delta] \times S^{\lambda_{F'}-1} \times U_\alpha \times S^{\lambda^*_{F'}-1} :$$

$$\hat{u} = \hat{u}(\tilde{\sigma}_\delta(w', q))$$

$$\hat{v} = \hat{v}(\tilde{\sigma}_\delta(w', q))$$

$$\hat{x} = \hat{x}(\tilde{\sigma}_\delta(w', q)) \qquad \}$$

The second set is just the smooth manifold with corners

$$\Gamma' := \{(w', q, t, s, r) \in D^{k-1} \times \Gamma \ : \ 0 \leq q < \varepsilon\}.$$

With all this notation, the part G' of G that projects onto the neighborhood $W' \cong D^{k-1} \times [0, \varepsilon) \subset \widetilde{W}$ is the fibre product of X and Γ' over W'. Hence G is a smooth manifold with corners as claimed.

This completes the proof of Lemma 3.3 in the case when Z is a point. The definition of the map $\widetilde{\sigma}_t$ in the general case works exactly as in this special case. The proof of (3) also goes through as before, because both the hypothesis and the conclusion involve only one fibre of p anyway. The proof of (4) requires a little more care, and here we have to make use of the fact that the blow-up $\widetilde{W}$ of W is fibered by the blow-ups of the fibres $\pi^{-1}(z)$, where $z \in Z$. □

Let us make another observation, which will become useful in the next chapter.

Remark 3.6 *It is not strictly necessary for this argument that the image of the initial map* $\sigma : W \longrightarrow X$ *be contained in one level set* $f^{-1}(c + \delta)$.

Instead of being defined on all of $\widetilde{W} \times [c-\delta, c+\delta]$, in this case the map $\widetilde{\sigma}$ is defined on the smaller closed subset of the form

$$\{(\widetilde{w}, t) \ : \ \widetilde{w} \in \widetilde{W}, t \leq f(\sigma(\pi(\widetilde{w})))\} \subset \widetilde{W} \times [c-\delta, c+\delta]$$

Notice that this is just the region below the graph of $f \circ \sigma \circ \pi : \widetilde{W} \longrightarrow \mathbb{R}$, which is essentially a smooth submanifold with corners in $\widetilde{W} \times [c-\delta, c+\delta]$ (even with the new differential structure).

Let us give an example to illustrate the sets $\widetilde{S_F}$ that arise in the construction described in Lemma 3.3.

Example 3.7 *We will consider the function on the sphere $S^3 \subset \mathbb{R}^4$ given by*

$$f(x_1, x_2, x_3, x_4) = x_2^2 + x_3^2 + 3x_4^2.$$

It has five critical manifolds, namely $F_0^\pm = \pm(1,0,0,0)$, $F_3^\pm = \pm(0,0,0,1)$ as well as $F_1 = \{x_2^2 + x_3^2 = 1\}$. Notice that the function is self-indexing in the sense that $\lambda_{F_i} = f(F_i) = i$. Let us study the various stable sets S_i and their 'blow-ups' $\widetilde{S}_i$. Of course at the minimum points, we have $S_0^\pm = F_0^\pm$. The stable set S_p for a point $p \in F_1$ is an open segment connecting F_0^+ and F_0^- (and not containing the end points). S_1 is an open cylinder, and $\overline{S_1}$ is the 2-sphere of points whose last coordinate are 0. Since there are no critical manifolds between F_1 and the minimum, no blow-up is necessary, and $\widetilde{S}_1$ is the union of two closed cylinders $(S_+^1 \sqcup S_-^1) \times [0,1]$.

Next, we look at F_3^+ (the case of F_3^- is completely analogous). Here $S_3^+ \cap f^{-1}(2)$ is a 2-sphere, and its intersection with U_1 is a circle $S^1 \subset S^2$.

The blow-up yields a disjoint union of two closed disks $D^+ \sqcup D^-$, so that after the blow-up process we have a map from $\widetilde{S_3^+} \cong (D^+ \sqcup D^-) \times [0,3]$ onto $\overline{S_3^+}$, which is a closed 3-ball. Notice that part of the boundary of $\widetilde{S_3^+}$ consists of the disjoint union of two cylinders $((S^1)^+ \sqcup (S^1)^-) \times [0,3]$, and in the altered smoothness structure two circles of corners appear at $((S^1)^+ \sqcup (S^1)^-) \times \{1\}$, which is precisely the set mapped onto F_1. Also, the interior of $D^\pm \times [0,3]$ is mapped onto the set of points whose forward time limit is F_3^+ and whose backward time limit is $F_0^\pm$, respectively.

Notice that in this example the stable manifolds form a Whitney stratification of the underlying manifold (for the definition, see the third section of the appendix). This is a completely general phenomenon.

Proposition 3.8 *Let f be a generalized Morse function with tame gradient-like flow φ_τ satisfying the generalized Smale condition. Then the stable sets $\{S_i\}$ form a Whitney stratification of X.*

Proof: This follows more or less immediately from our description of the closure of a stable manifold S_F as the image of a smooth manifold with corners under a smooth map. To verify the Whitney conditions, we lift the sequence of points $p_n \in S_F$ approaching some point $p \in \overline{S_F}$ to a sequence of points q_n in $\widetilde{S}_F$. By compactness of $\widetilde{S}_F$, we can extract a subsequence converging to q. In $\widetilde{S}_F$ the Whitney conditions are trivially satisfied for any such sequence, and so the result follows if we can prove that the differential $\widetilde{\imath}_{S_F *}$ at points q in the boundary that get mapped to $\overline{S_F} \setminus S_F$ is sufficiently non-degenerate. This is done in essentially the same way as the proof of Lemma 3.3, part (3). □

Recall that one of the goals in this chapter was to prove Theorem 3.1. As a next step towards this goal, we prove

Proposition 3.9 *Let φ_T be a tame gradient-like flow for some generalized Morse function f. If φ_T satisfies the generalized Smale condition, then it is a finite volume flow.*

Proof: This argument, adapted from §11 of [HL97b], is included here for completeness. The idea is to find a new Morse function g on $\mathbb{R} \times X \times X$ and a set $\mathcal{T}'$ whose projection onto $X \times X$ is equal to the total graph T of the flow φ_T, and show that $\mathcal{T}'$ is a set of the form described in Remark 3.5. More precisely, we consider the generalized Morse function

$$g(s, x, y) = \frac{1}{2}s^2 - f(y)$$

defined on $\mathbb{R} \times X \times X$ with the standard product metric, so that the gradient flow is given by

$$\psi_T(s, x, y) = (e^T s, x, \varphi_{-T}(y)).$$

The critical sets of g are of the form

$$P_F = \{0\} \times X \times \{F\}$$

where F ranges over the critical manifolds of f. The stable and unstable manifolds are given by

$$\mathbf{S}_F = 0 \times X \times U_F \qquad \text{and} \qquad \mathbf{U}_F = \mathbb{R} \times X \times S_F.$$

The first thing to notice is that g is a generalized Morse-Smale function, because f is. Now define the set

$$\mathcal{T} = \{(e^{-\tau}, x, \varphi_\tau(x)) \in \mathbb{R} \times X \times X \mid -\infty < \tau < \infty \text{ and } x \in X\}$$

and notice that it is just the union of all the flow lines of ψ_τ passing through $\Delta = \{(1, x, x) \mid x \in X\} \subset \mathbb{R} \times X \times X$. Next observe that T is contained in the image of $\mathcal{T}' = \mathcal{T} \cap g^{-1}((-b + \frac{1}{2}, b])$ under projection onto the last two factors, where $b = \max|f| + \frac{1}{2}$. Therefore it is enough to show that $\mathcal{T}'$ has finite (n+1)-volume. In order to use Remark 3.5 we need to know first that $\mathcal{T}$ is transverse to all the $\mathbf{U}_F$, where F ranges over the critical manifolds of f. Clearly $\mathcal{T} \cap \mathbf{U}_F$ is invariant under the flow, so it is enough to check the transversality at all intersection points contained in Δ. But here $T\mathbf{U}_F$ contains the tangent spaces to both $\mathbb{R}$ and the first X-factor, which together with the tangent space to the diagonal $T\Delta \subset T\mathcal{T}$ span $T(\mathbb{R} \times X \times X)$.

Finally we observe that the intersection $\mathcal{T}(b) = \mathcal{T} \cap g^{-1}(b)$ is compact. For otherwise there would exist a sequence $\{(s_i, x_i)\}$ satisfying

$$g(e^{-s_i}, x_i, \varphi_{s_i}(x_i)) = \frac{1}{2}e^{-2s_i} - f(\varphi_{s_i}(x_i)) = b$$

and having no convergent subsequence. Clearly in such a sequence we would have $s_i \longrightarrow \infty$, and so $f(\varphi_{s_i}(x_i)) \longrightarrow b$, which is impossible by the definition of b. Thus Remark 3.5, together with the fact that g has only finitely many critical values in the interval $(-b + \frac{1}{2}, b)$, proves that $\mathcal{T}'$ is the image of a smooth compact manifold with corners under a smooth map. In particular, it has finite volume, and so the same is true for T as well. □

Now we can start to prove Theorem 3.1. Notice that the total graph T of the flow is the mass limit as $c \longrightarrow \infty$ of the pieces

$$T_c := \{(x, \varphi_T(x)) \in X \times X \mid t \in (0, c]\}.$$

Since we now know that the volume of these is bounded (by the volume of T), the equation of currents $\partial T = \Delta - P$ in Theorem 3.1 follows from the fact that $\partial T_c = \Delta - \text{graph}(\varphi_c)$ and the continuity of the boundary operator. It remains to prove the claim about the structure of P. For a critical point p, let $\mathcal{U}_p$ be the set of all points in X that are connected to p by a backward, possibly broken, flow line.

Lemma 3.10 *With the notation as in Theorem 3.1, the current P has support in $\bigcup S_i \times_{F_i} \widetilde{\mathcal{U}}_i$, where*

$$S_i \times_{F_i} \widetilde{\mathcal{U}}_i := \{(x, y) \subset X \times X : \exists\, p \in F_i \text{ such that } x \in S_p \text{ and } y \in \mathcal{U}_p\}.$$

Proof: Lemma 2.10 in [HL97b]. □

Lemma 3.11 *Let φ_T be a tame gradient-like flow for some generalized Morse function f, which satisfies the generalized Smale condition. Then for all $p \in Cr(f)$ we have $\dim(\mathcal{U}_p \setminus U_p) < \lambda_p^*$.*

Proof: First note that

$$\mathcal{U}_p \setminus U_p = \bigcup_{p \prec F} (\mathcal{U}_p \cap U_F),$$

so it is enough to show that $\dim(\mathcal{U}_p \cap U_F) < \lambda_{F_p}^*$ for each $F \succ p$. Recall that the critical manifolds have a natural partial ordering, given by $F \prec F'$ if there is a (possibly piecewise) flow line from F to F'. Let F' be a minimal element in the set $\mathcal{S} := \{F | F_p \prec F, F_p \neq F\}$. In this case there exist only proper flow lines from F_p to F and therefore $\mathcal{U}_p \cap F' = \pi_{F'}(U_p \cap S_{F'})$. By the generalized Morse-Smale condition we have that

$$\dim(U_p \cap S_{F'}) = \lambda_p^* + \lambda_{F'} + n_{F'} - n.$$

U_p meets fibres of the projection $\pi_{F'} : S_{F'} \to F'$ in flow lines, hence

$$\begin{aligned}\dim(\mathcal{U}_p \cap F') &= \dim \pi_{F'}(U_p \cap S_{F'}) \\ &< \lambda_p^* + \lambda_{F'} + n_{F'} - n\end{aligned}$$

On the other hand,

$$\begin{aligned}\dim(\mathcal{U}_p \cap U_{F'}) &\leq \dim(\mathcal{U}_p \cap F') + \lambda_{F'}{}^* \\ &< \lambda_p^* + \lambda_{F'} + n_{F'} - n + \lambda_{F'}{}^* \\ &= \lambda_p^* + 0\end{aligned}$$

as claimed.

Now continue inductively, picking a minimal element F'' of the remaining set $\mathcal{S}' = \mathcal{S} \setminus \{F'\}$ and so on. At stage k the piece of $\mathcal{U}_p$ intersecting $S_{F^{(k)}}$ can consist of part of U_p (to which the above argument applies directly) and pieces of the form $\mathcal{U}_p \cap U_{F^{(l)}}$ for $0 < l < k$. From the previous inductive steps we know that the latter are families of fibres of $U_{F^{(l)}}$ of total dimension less than λ_p^*. Each of these fibres is (by an argument analogous to the above) replaced by a (sub)family of total dimension smaller than its own, thus further decreasing the total dimension of the family. □

It follows from the two lemmas and Federer's flat support theorem (Proposition A.1) that supp $P \subset \bigcup S_i \times_{F_i} U_i$. Since the pieces on the right are locally closed and of the same dimension as P, we conclude that in fact $P = \sum n_i S_i \times_{F_i} U_i$. The local multiplicity is computed as in §2 of [HL97b], and this completes the proof of Theorem 3.1. □

Chapter 4

The Operator $\mathbb{P}$

In this chapter we will make use of currents C on the product $X \times X$ as kernels for operators $\mathbb{C} : \Omega^*(X) \longrightarrow \mathcal{D}_*(X)$. Some basic facts on currents and the kernel calculus are collected in the appendix.

We proceed to state the first version of the main theorem of this chapter.

Theorem 4.1 *Under the assumptions of Theorem 3.1, the operator $\mathbb{T}$ associated to the current T gives a chain-homotopy between the inclusion $\mathbb{I}$ of differential forms into currents and the operator $\mathbb{P}$ associated with P, that is*

$$\mathbb{T} \circ \mathbf{d} + \mathbf{d} \circ \mathbb{T} = \mathbb{I} - \mathbb{P}, \tag{4.1}$$

Moreover, the action of the operator $\mathbb{P}$ on differential forms is given by

$$\mathbb{P}(\alpha) = \sum \pi_{S_i}^*(\pi_{U_i *}(\alpha \wedge U_i))[S_i]. \tag{4.2}$$

Proof: The chain homotopy (4.1) is a direct translation of the current equation (3.1). To verify the formula for $\mathbb{P}$, consider the diagram

$$\begin{array}{ccc} \widetilde{S}_i \times_{F_i} \widetilde{U}_i & \xrightarrow{\pi_2} & \widetilde{U}_i \\ \pi_1 \downarrow & & \downarrow \pi_{U_i} \\ \widetilde{S}_i & \xrightarrow{\pi_{S_i}} & F_i \end{array}$$

where $\widetilde{S}_i$ and $\widetilde{U}_i$ are the smooth families of compact manifolds with corners described in Theorem 3.4, and the map $\widetilde{S}_i \times_{F_i} \widetilde{U}_i \longrightarrow X \times X$ given by the fibre product of the maps $\widetilde{i}_{S_i}$ and $\widetilde{i}_{U_i}$ is smooth onto $\overline{S_i \times_{F_i} U_i}$. Since all the maps in the diagram have compact fibres, both pull-back and push-forward of currents make sense (cf. Appendix A), and the claim now follows from the commutativity of the diagram. □

Now assume that X, all the F_i as well as the bundles U_i and S_i carry R-orientations, where R is either $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p. As usual, a k-dimensional R-chain on X is a finite formal sum of smooth maps $\sigma : \Delta^k \longrightarrow X$ of the k-simplex into X with coefficients in R. Recall that such a map is completely transverse to a locally closed embedded submanifold $Y \subset X$ if its restriction to each face of each simplex is transverse in the usual way.

Denote by $C_*^{tr}(X; R)$ the set of smooth chains σ transverse to all the U_i. We saw in Proposition 3.8 that the U_i form a Whitney stratification of X, so that Proposition A.7 implies that $C_*^{tr}(X; R)$ is an open and dense subset of all smooth R-chains. In fact standard transversality arguments show that any smooth R-chain can be moved into $C_*^{tr}(X; R)$ by composing it with a small diffeomorphism of X. Since $C_*^{tr}(X; R)$ is by definition invariant under ∂, it can be used to compute the homology of X with coefficients in R. These chains are also nice in another respect.

Proposition 4.2 *The domain of the operator* $\mathbb{P}$ *extends to include* $C_\bullet^{tr}(X;R)$.

Before we give a proof of this proposition, let us introduce the notion of the 'stable bundle of a chain'. Recall again from Chapter 3 that the closure of the stable bundle of a critical manifold F is the image of a smooth family of manifolds with corners $\widetilde{S}_F$ over F under a smooth map $\widetilde{i}_{S_F}$ (cf. Theorem 3.4).

Definition 4.3 *Let* $\sigma : C \longrightarrow F$ *be a smooth R-chain in a critical manifold* F. *Here we think of* C *as a finite formal sum of standard simplices with coefficients in R. Then we define the* **stable bundle** S_σ **of the** R**-chain** σ *to be the image of the fibre product* $\widetilde{S_C} = \{(c,s) \in C \times \widetilde{S_F} \ : \ \sigma(c) = \pi(s)\}$ *under the obvious map into* X *(projection onto the second factor, followed by* $\widetilde{i}_{S_F}$*).* S_σ *defines an integral R-current.*

Proof: (of Proposition 4.2) The proof will rely on the description of the backward flow image of a chain as in Lemma 3.3 and Remark 3.6. For simplicity, we will work with only one singular simplex $\sigma : \Delta^k \longrightarrow X$ transverse to all the U_i. The operator $\mathbb{P}$ can be written as a sum $\mathbb{P} = \sum \mathbb{P}_i$, where $\mathbb{P}_i$ is the part corresponding to $P_i = S_i \times_{F_i} U_i$. We will argue that the 'push-pull' description of the operator for forms as in formula (4.2) of Theorem 4.1 can be extended to such a simplex as follows.

Let F be a maximal element in the partially ordered set of critical manifolds F_i such that the image of σ intersects U_i. Since $\sigma(\Delta^k)$ is compact and misses $\overline{U_F} \setminus U_F$, the intersection is contained in some compact subset of U_F. On any such subset the projection π_{U_F} to F given by the backward flow is smooth. Because of the transversality, $\sigma(\Delta^k) \cap U_F$ is itself a smooth chain, so that the pushforward $\pi_{U_F *}(\sigma(\Delta^k) \cap U_F)$ is a well-defined chain σ_F on F. The pull-back map $\pi^*_{S_F}$ applied to the current of integration over σ_F yields the stable bundle of this chain as described above. This is the image of σ under $\mathbb{P}_F$.

Recall that in Lemma 3.3 we described how to construct a new domain $\widetilde{\Delta^k}$ and a new map $\widetilde{\sigma}_{c-\delta}$ which parametrizes the closure of the intersection of some regular level set past F with the backward flow image of $\sigma(\Delta^k)$. The map $\widetilde{\sigma}_{c-\delta}$ is still transverse to all U_i, but its image does not meet U_F, so that we can repeat the procedure finitely many times to get an inductive description of the image of the simplex $\sigma : \Delta^k \longrightarrow X$ under $\mathbb{P}$ as

$$\mathbb{P}(\sigma) = \sum_i S_{\sigma_{F_i}},$$

where σ_{F_i} is a map from some submanifold D_i of a blow-up of Δ^k to F_i. Notice that D_i can be thought of as the proper transform of the closure of $\sigma^{-1}(U_i)$ under the blow-up procedure. □

Now we are ready for the geometric version of the main theorem of this chapter.

Theorem 4.4 *Let f be a generalized Morse function and let φ_T be a tame gradient-like flow for f which satisfies the generalized Smale condition. Then the operator $\mathbb{P}$ associated to the degeneration P of the diagonal in $X \times X$ by φ_T and acting on a map $\sigma : \Delta^k \longrightarrow X$ in $C^{tr}_*(X; R)$ by*

$$\mathbb{P}(\sigma) = \sum_{F_i} S_{\sigma_{F_i}} \tag{4.3}$$

is chain homotopic to the inclusion of $C_*^{tr}(X;R)$ into integral R-currents.

Furthermore, the image of the operator $\mathbb{P}$ when applied to $C_*^{tr}(X;R)$ is precisely given by the set $\mathcal{S}(f;R)$ of stable bundles of smooth R-chains in the various critical manifolds F_i of the function f.

Proof: It remains to prove the last statement. In the proof of the previous proposition we saw that the image of $\mathbb{P}$, when applied to $C_*^{tr}(X;R)$, is contained in the set of stable bundles of smooth R-chains in the various critical manifolds. On the other hand, the stable ε-disk bundle of any chain in some critical manifold F is a chain in $C_*^{tr}(X;R)$, and its image is just the stable bundle of this chain, so that every stable bundle occurs as an image. □

Remark 4.5 *Since $\mathbb{P}$ commutes with* d, *$\mathcal{S}(f;R)$ forms a subcomplex of the complex of integral R-currents.*

In what follows, we will mostly work with the geometric picture and the action of the operator $\mathbb{P}$ on $C_*^{tr}(X;R)$. From the description of the image of $\mathbb{P}$ we immediately get a representation of $H_*(X;R)$.

Corollary 4.6 *The differential ∂_f on the complex $\mathcal{S}(f;R)$ of stable bundles over chains in the critical sets is given by*

$$\partial_f(S_\sigma) = \sum_i \mathbb{P}_i(S_\sigma(\varepsilon)),$$

where $S_\sigma(\varepsilon)$ is the boundary of the stable ε-disk bundle of σ.

Furthermore, $H_*(\mathcal{S}(f;R),\partial_f) = H_*(X;R)$.

Notice that there is a filtration $\emptyset = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{S}_f$ of this complex, where $\mathcal{F}_i$ is defined to be the subset of stable bundles of chains contained in critical manifolds of index $\leq i$. This is a special case of the filtration already considered by Bott [Bot54]. We can form the spectral sequence associated to this filtration, which by general principles converges to the associated graded group of $H_*(X)$. In particular, the E^1-term of the sequence has entries

$$\begin{aligned} E^1_{p,q} &= H_{p+q}(\mathcal{F}_p, \mathcal{F}_{p-1}) \\ &= \bigoplus_{\lambda_i = p} H_{p+q}(S_{F_i}, S^o_{F_i}) \\ &= \bigoplus_{\lambda_i = p} H_q(F_i), \end{aligned}$$

where $S^o_{F_i}$ is the set of non-zero vectors in the bundle S_{F_i} and the last equality follows from the fact that the S_i are R-oriented bundles. From the existence of this spectral sequence one immediately recovers the well-known Morse inequalities, which state that there exists a polynomial $Q(t)$ with non-negative integer coefficients such that

$$M_{f,K}(t) - P_{M,K}(t) = (1+t)Q(t)$$

Here K is a field of characteristic p or 0 depending on R, $P_{M,K}$ is the Poincare polynomial of M for this field and $M_{f,K}$ is the Morse polynomial of f over K defined as

$$M_{f,K}(t) = \sum_{F \in \pi_0(Cr(f))} t^{\lambda_F} P_{F,K}(t)$$

The new element in our approach is the explicit geometric picture for the boundary maps. This can be used to make explicit calculations, as is shown in examples in the next chapter.

We conclude this chapter with a few special cases. One of them arises when all of the pieces $P_i = S_i \times_{F_i} U_i$ are closed as currents. The following theorem shows that this can be taken as the definition of a **perfect (generalized) Morse function**. It has the advantage over the usual definition of being intrinsic in the sense that it does not depend on knowing the homology of X.

Theorem 4.7 *If $\partial P_i = 0$ for all the pieces $P_i = S_i \times_{F_i} U_i$, then $H_*(X; R) = \oplus H_{*-\lambda_i}(F_i; R)$.*

Proof: Since $\partial P_i = \partial S_i \times_{F_i} U_i + S_i \times_{F_i} \partial U_i$ and these two currents have disjoint supports, they both vanish individually. This means that for almost all $p \in F_i$ $\partial S_p = \partial U_p = 0$ as currents. It follows from the continuity that in fact this is true for all $p \in F_i$. This now implies that all higher differentials in the spectral sequence are zero. □

Remark 4.8 *Notice that one way to guarantee that all the P_i are closed as currents is to impose the condition that if $F \prec F'$, then $\lambda_F + n_F < \lambda_{F'} - 1$. In this case Federer's flat support theorem (Theorem A.1) shows that the stable bundle of F is too small in dimension to support a boundary even for a fibre of $S_{F'}$. Thus we recover Cor. 6.5 of [HL97b].*

There are other cases when the complex S_f is particularly nice, and we get an immensely simplified model of the homology of X. Call two critical manifolds F and F' **adjacent**, if there are flow lines between them, but no broken flow lines.

Theorem 4.9 *Let $f : X \longrightarrow \mathbb{R}$ be a generalized Morse function with tame gradient-like flow φ_T satisfying the generalized Smale condition. Suppose that X, the F_i as well as all stable and unstable bundles are orientable for some coefficient ring R as before. Assume also that $\lambda_i + n_i < \lambda_k - 1$ whenever $F_i \prec F_j \prec F_k$. Then*

$$H_*(X; R) = H_*(\mathcal{M}_*(f), \partial_f)$$

where $\mathcal{M}_k(f) := \oplus H_{k-\lambda_i}(F_i)$ and the piece of the boundary map $(\partial_f)_{ji}$: $H_{*-\lambda_j}(F_j) \longrightarrow H_{*-\lambda_i-1}(F_i)$ is given for adjacent critical sets $F_i \prec F_j$ by

$$(\partial_f)_{ji}([c]) := \pi_{U_i *}(U_i \cap S_c \cap f^{-1}(\frac{f(F_i) + f(F_j)}{2}))$$

and is zero otherwise.

Proof: In this case all possible boundary currents are supported in stable manifolds of adjacent critical manifolds. That implies that all higher boundary maps in the spectral sequence can be collected into one boundary map on a complex whose chain groups are the groups in the E^1-term of the spectral sequence. □

Example 4.10 *Let us return to the function on $S^3 \subset \mathbb{R}^4$ from Example 3.7, given by*

$$f(x_1, x_2, x_3, x_4) = x_2^2 + x_3^2 + 3x_4^2.$$

In this case, the assumptions of Theorem 4.9 are satisfied, because $\lambda_0 + n_0 = 0 + 0 < 2 = \lambda_3 - 1$. The complex $\mathcal{M}_k(f)$ has the form

$$H_0(F_3^+) \oplus H_0(F_3^-) \xrightarrow{\partial} H_1(F_1) \xrightarrow{0} H_0(F_1) \xrightarrow{\partial} H_0(F_0^+) \oplus H_0(F_0^-),$$

where the middle boundary map is trivially 0. From our description of the stable sets of the various critical manifolds in chapter 3 we see that the generators of both $H_0(F_3^\pm)$ get mapped onto the generator of $H_1(F_1)$, and similarly the generator of $H_0(F_1)$ gets mapped to the sum (or difference, depending on the choice of orientations) of the generators of $H_0(F_0^\pm)$, so that the homology of the complex yields the homology of S^3 as expected.

Chapter 5

Applications and Examples

In this chapter I will outline a few examples of how the techniques of the previous chapters might be employed to obtain actual information in specific cases. Some general statements about equivariant (co)homology are also given.

5.1 Fibrations

The first interesting example to mention are fibre bundles $F \longrightarrow X \longrightarrow B$. We can choose an arbitrary connection (i.e. a collection of horizontal spaces at each point in the total space) and produce a metric on X such that the projection to B is a Riemannian submersion. We assume that the function $f : X \longrightarrow \mathbb{R}$ is the pull-back of a Morse-Smale function on the base B. Then the critical set for f is just the inverse image under projection of the critical set in B, and the gradient flows of the functions on X and B will be intertwined by the projection. Furthermore we may assume that the metric on B is chosen so that the gradient flow is tame, which ensures that the gradient flow on X will also be tame. Since all the normal bundles to the critical sets are trivial (and assuming that the fibre F carries an orientation), the coefficient ring R can be chosen to be the integers $\mathbb{Z}$, and the spectral sequence of the last chapter is easily seen to a be geometric version of the Leray-Serre spectral sequence of the fibration.

Example 5.1 *We want to apply Theorem 4.9 to the fibration $SO(n) \longrightarrow SO(n+1) \longrightarrow S^n$ and inductively compute the integer homology of $SO(n+1)$ from the homology of $SO(n)$.*

To fix notation, let the embedding of $SO(n)$ into $SO(n+1)$ be the one compatible with the inclusion $\mathbb{R}^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$, and let the action of $SO(n)$ on $SO(n+1)$ be given by right multiplication. Then the projection to S^n can be taken to be the map which assigns to a matrix its last row. Pulling back the standard height function on S^n, $h(x) = x_{n+1}$, to the fibration, we obtain the generalized Morse-Smale function given by $f(A) = a_{n+1,n+1}$. It has two critical manifolds diffeomorphic to $SO(n)$, and building the complex as in Theorem 4.9, we need to compute the boundary maps

$$\partial_r : H_r(SO(n)_{\max}) \longrightarrow H_{r+n-1}(SO(n)_{\min}).$$

We will start with ∂_0. Notice that on the sphere S^n the flow lines for the standard round metric are (reparameterized) geodesics. Hence the same will be true for the lifts to $SO(n+1)$, because the gradient vector field here is horizontal and the bundle projection is a Riemannian submersion. One checks that the lift with end point $Id \in SO(n+1)$ of the flow line on S^n through

$(0, 0, \cdots, 0, 1, 0)$ is given by

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos(t) & \sin(t) \\ 0 & 0 & 0 & -\sin(t) & \cos(t) \end{pmatrix},$$

where $\pi \leq t \leq 2\pi$. Similarly the lift of the flow line through $(0, 0, \cdots, 0, -1, 0)$ is given by $\gamma(t)$, $\pi \geq t \geq 0$. In particular, these two flow lines together form a closed geodesic in $SO(n+1)$, so they have the same starting point in $SO(n)_{\min}$. Now we observe that conjugation by an element in $SO(n)$ acts transitively on the flow lines to $Id \in SO(n+1)$, so that certainly all pairs of antipodal points are identified at the boundary of the stable n-cell at Id. By a short calculation one checks that these are the only identifications, hence the closure of this stable fibre is homeomorphic to $\mathbb{R}P^n$. It turns out that it is precisely the translation of the image of the canonical embedding of $\mathbb{R}P^n$ by a fixed reflection, say $diag(1, 1, \cdots, 1, -1)$. Recall that the canonical embedding of $\mathbb{R}P^n$ into $SO(n+1)$ is given by assigning to a line in $\mathbb{R}^{n+1}$ the reflection through its orthogonal hyperplane. Notice that in particular when $n = 2k - 1$ is odd, we find that the boundary of one and hence all stable fibres is zero as a current, and so

$$H_*(SO(2k)) \cong H_*(SO(2k-1)_{\min}) \oplus H_{*-(2k-1)}(SO(2k-1)_{\max}),$$

at least as additive groups. The case when $n = 2k$ is more interesting. Here the image of ∂_0 is a (twice covered) copy of $\mathbb{R}P^{2k-1} \subset SO(2k)_{\min}$, given by intersecting the translated canonical embedding of $\mathbb{R}P^{2k} \subset SO(2k+1)$ as above with $SO(2k)_{\min}$. To complete the description of the boundary maps, we denote the generators of $H_*(SO(2k-1))$ by $\alpha_1, ..., \alpha_r$. Then by induction $H_*(SO(2k))$ is generated by $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r$, where we think of α_i as the image of the corresponding generator of $H_*(SO(2k-1))$ under the embedding as $SO(2k-1)_{\min}$ and $\beta_i = \alpha'_i \times \mathbb{R}P^{2k-1}$ as the stable bundle inside $SO(2k)$ of $\alpha'_i \subset SO(2k-1)_{\max}$. Notice that if α_1 denotes the generator of $H_0(SO(2k-1)_{\min}) \cong H_0(SO(2k))$, what we have shown so far is that $\partial_0\alpha_1 = 2\beta_1$. More generally it is true that

$$\partial\alpha_i = 2\beta_i \in H_*(SO(2k)_{\min})$$

$$\partial\beta_i = 0$$

For the first relation observe that the boundaries of all the fibres for points in $SO(2k-1)_{\min} \subset SO(2k)_{\max}$ are disjoint. For the second relation it is sufficient to show that $\partial\beta_1 = 0$. The stable bundle of β_1 is the image of $\mathbb{R}P^{2k-1} \times \mathbb{R}P^{2k}$, so it's boundary is (twice) the image of $\mathbb{R}P^{2k-1} \times \mathbb{R}P^{2k-1}$, which we claim is zero. First notice that the image is just the set of all elements in $SO(2k)_{\min}$ which are products of 2 reflections in hyperplanes. These are rotations about a codimension 2 plane, and for any such rotation there is a 1-parameter family of pairs of reflections with the required product. (This is a simple exercise in planar geometry.)

To summarize our results for $n = 2k$, let us denote by $Ker_*(2k-1)$ the kernel of multiplication by 2 in $H_*(SO(2k-1))$, and let $CoKer_*(2k-1)$ be the cokernel. Then

$$\begin{aligned} H_*(SO(2k+1)) = & H_*(SO(2k-1)) \oplus \\ & CoKer_{*-(2k-1)}(SO(2k-1)) \oplus \\ & Ker_{*-(2k)}(SO(2k-1)) \oplus \\ & H_{*-(4k-1)}(SO(2k-1)) \end{aligned}$$

Example 5.2 *The first two non-trivial cases are* $SO(4)$ *and* $SO(5)$*. Here we have*

$$H_k(SO(4)) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 6 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 3 \\ \mathbb{Z}_2 & \text{if } k = 1, 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_k(SO(5)) = \begin{cases} \mathbb{Z} & \text{if } k = 0,7,10 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } k = 3 \\ \mathbb{Z}_2 & \text{if } k = 1,4,5,6,8 \\ 0 & \text{otherwise} \end{cases}$$

5.2 Equivariant Morse theory

In this section, let X be a compact manifold acted on by a compact Lie group G. We will call a generalized Morse function that is G-invariant an **equivariant Morse function.** If the Hessian is non-degenerate in the normal directions at each critical orbit, we will refer to the function as *strong* equivariant Morse function. A strong equivariant Morse function has a discrete set of critical orbits, hence a finite number of them. The following result is due to Wasserman [Was69].

Proposition 5.3 *Equivariant Morse functions always exist and are dense in the set of smooth G-invariant functions in the Whitney topology induced from smooth functions.*

Although the generalized Smale condition can be shown to hold for the gradient flow (associated to some G-invariant metric) of many equivariant Morse functions, simple examples show that it cannot always be satisfied.

Example 5.4 *Consider* $G = SO(n)$ *acting on* $S^n \times S^1$ *by rotation on* S^n*. Let* $g : S^n \longrightarrow \mathbb{R}$ *be the standard height function which is invariant under the action of* $SO(n)$*, shifted by some constant to make it strictly positive. Let* h *be the standard height function on the circle, and define* $f : S^n \times S^1 \to \mathbb{R}$ *as the product of* g *and* h*.*

It is smooth and has non-degenerate critical points of orders 0, 1, n and $n+1$. They are all contained in the fixed point set, which is a union of two circles. The gradient flow of f for any G-invariant metric will have to leave this fixed point set invariant, and in particular will have flow lines from the critical point of index n to the critical point of index 1, clearly contradicting transversality.

As an illustration, consider the case $n = 1$. Here of course we will have to use $G = O(1)$ to get a non-trivial action. Choose coordinates $(\varphi, \psi) \in [0, 2\pi) \times [0, 2\pi)$ on the torus and let the function be defined as $f(\varphi, \psi) := cos\varphi(2 + sin\psi)$. It corresponds to the height function in the embedding of the torus into $\mathbb{R}^3$ pictured below. Clearly the function is invariant under the reflection $(\varphi, \psi) \mapsto (\varphi, -\psi)$, which in the picture corresponds to a reflection in the yz-plane. The two critical points of index 1 are joined by two segments of fixed points. Since these have to be invariant under the gradient flow for any invariant metric, the stable manifold of one of them will intersect the unstable manifold of the other.

Assuming that a given equivariant function is strong generalized Morse-Smale, we can use the standard Borel construction to obtain information about the equivariant homology of X, which proceeds as follows.

Figure 5.1: Another embedding of the torus in $\mathbb{R}^3$

Let EG be a model for the universal contractible simply-connected space with free G-action. It is unique up to homotopy and will in general be infinite dimensional. Let $EG_1 \subset EG_2 \subset EG_3 \subset \ldots$ be an increasing filtration of EG by k-connected, finite-dimensional free G-spaces. Denote the quotients EG/G and EG_k/G by BG and BG_k respectively. These are models for the classifying space of the group G and its approximations. Now starting from X, we can form the free G-space $\widetilde{X} = X \times EG$ with the diagonal action, and one definition of the equivariant homology of X amounts to setting

$$H_*^G(X) := H_*(\widetilde{X}/G)$$

It is a well known fact [AB82] that an equivariant Morse function f on X pulls back to $\widetilde{X}$ and also descends (because it is equivariant) to a generalized Morse function on $\widetilde{X}/G$, and the same is true for all approximation spaces $\widetilde{X_k} = X \times EG_k$ and their quotients. The only additional observation necessary to apply our theory is that, if the original function f on X was generalized Morse-Smale, then the same will hold for the functions $\widetilde{f_k}$ defined on $\widetilde{X_k}$, provided we use a product metric. The critical manifolds of $\widetilde{f_k}$ are of the form $\mathcal{O}_i \times EG_k$, where $\mathcal{O}_i = G/H_i$ is one of the isolated critical orbits of f. If we also insist that the product metric on $\widetilde{X_k}$ be G-invariant, then $\widetilde{f_k}$ descends to a generalized Morse-Smale function (with respect to the pushed down metric) on $X_k = \widetilde{X_k}/G$. Here the critical manifolds of f_k are of the form $(\mathcal{O}_i \times EG_k)/G = (G/H_i \times EG_k)/G = EG_k/H_i$. Since the construction is natural with respect to the inclusions $EG_k \to EG_{k+1}$, we can take the limit of the spectral sequences as $k \to \infty$. In particular, we obtain a version of our spectral sequence with E^1_G-term

$$\begin{aligned}(E^1_G)_{p,q} &= \bigoplus_{\lambda_i = p} H^G_q(\mathcal{O}_i; R) \\ &= \bigoplus_{\lambda_i = p} H_q(BH_i; R)\end{aligned}$$

Here λ_i is the index of the critical orbit $\mathcal{O}_i = G/H_i$, and as before the coefficient ring R is chosen so that all critical orbits as well as the stable and unstable bundles carry R-orientations. Computations can usually be carried out in the finite-dimensional approximations. There are also more immediate consequences of the existence of our spectral sequence.

Corollary 5.5 *Let G be any compact connected Lie group, and let $f : X \to \mathbb{R}$ be a generalized G-equivariant Morse function for some G-invariant metric on X such that its gradient flow is tame and satisfies the generalized Smale condition. If the critical points of f are isolated and of even index, then the G-equivariant homology of X is a free $H_*(BG)$-module with one generator of degree λ for each critical point of f with index λ.*

Example 5.6 *Let S^1 act on $\mathbb{C}P^n$ by*

$$t \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : tz_1 : \ldots : t^n z_n]$$

Then the function $f([z]) := \frac{1}{|z|^2} \sum k|z_k|^2$ is an invariant function with isolated critical points and the above corollary applies to compute the S^1-equivariant homology of $\mathbb{C}P^n$ as a free module over $H_*(\mathbb{C}P^\infty)$ with generators in all even dimensions between 0 and 2n.

More generally, Corollary 5.5 applies to arbitrary torus actions on flag manifolds with isolated critical points (the moment map of a suitably chosen $S^1 \subset T$ will be the required Morse function).

5.3 Cup product

In this section we will discuss how the degeneration of higher diagonals gives rise to operator representations for cup products on X. This is not strictly an application of the results of earlier chapters, but rather a natural extension. For an approach to this question using more than one function compare [BC94].

The kernel of the cup product on a manifold is given by the triple diagonal in $X \times X \times X$. We can use the gradient flow to deform this into a sum of (fibre products) of currents as before. The main point is that the flow on $X \times X \times X$ given by

$$\Phi_\tau(x, y, z) = (x, \varphi_\tau(y), \varphi_{2\tau}(z))$$

has finite volume, which gives us the following

Proposition 5.7 *Let f be a generalized Morse-Smale function on a Riemannian manifold X, and let φ_T be its gradient flow for some canonically flat metric. Then there is an equation of currents*

$$\partial T = \Delta_3 - P$$

where $T = \{(x, \varphi_T(x), \varphi_{2T}(x)) \in X \times X \times X | t \in (0, \infty)\}$ is the total graph of the flow, Δ_3 is the triple diagonal and P is the current given by

$$P = \sum_{F_i, F_j \in \pi_0(Cr(f))} [S_i] \times_{F_i} [U_i \cap S_j] \times_{F_j} [U_j].$$

In particular, when f is actually a Morse-Smale function, this simplifies to

$$P = \sum_{p,q \in Cr(f)} [S_p] \times [U_p \cap S_q] \times [U_q].$$

As an immediate corollary, we get an estimate for the cup-length of a manifold X from the maximal number of break points of a flow line:

Corollary 5.8 *Let f be a Morse-Smale function on a Riemannian manifold X. Consider the number c defined as*

$$c = \max_{\{\text{ broken flow lines } \gamma\}} \#\{ \text{ break points in } \gamma \}$$

Then the cup-length of X is at most $c + 1$.

Proof: The Corollary follows from the proposition once we observe that we can compose P with itself and the kernel of the composition is given by

$$\begin{aligned} \tilde{P} &= pr_{1245*}(pr_{123}^*(P) \cap pr_{345}^*(P)) \\ &= \sum_{p,q,r \in Cr(f)} [S_p] \times [U_p \cap S_q] \times [U_q \cap S_r] \times [U_r] \end{aligned}$$

in the quadruple product of X with itself. This can then again be composed with P and so on. All of the intersections as well as the last unstable manifold will have to be positive dimensional to realize a maximal cup product of cohomology classes of positive degree. The result follows. □

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Appendix A

Background Material

In this appendix we collect some background material on currents, the kernel calculus and Whitney stratifications. It is included mainly for completeness and to set up notation. References for the first two sections are [Fed69, Sch66, HP79]; a reference for the third section is [Whi65], and a more general statement then Theorem A.8 is proven in [Mey67].

A.1 Basics on currents

Let $\mathcal{O}(X)$ denote the orientation bundle of the manifold X, a principal $\mathbb{Z}_2$-bundle that is trivial if and only if X is orientable. A twisted differential form on X is a section of the bundle $\Lambda^\bullet TX \otimes_{\mathbb{Z}_2} \mathcal{O}(X)$. Twisted forms of degree n are usually called densities and can be integrated over X without the need for an orientation. In general, a twisted form of degree k can be integrated over a (non-oriented) locally closed submanifold of dimension k with finite volume. We will denote the space of twisted k-forms by $\widetilde{\Omega}^k(X)$. The space $\mathcal{D}_k(X)$ of currents of degree k is defined to be the topological dual space of $\widetilde{\Omega}^{n-k}(X)$. Notice that currents of degree k are a natural generalization of differential forms of degree k, as for every $\alpha \in \Omega^k(X)$ we can define the current C_α by

$$C_\alpha(\widetilde{\beta}) := \int_X \alpha \wedge \widetilde{\beta} \qquad \text{for all } \widetilde{\beta} \in \widetilde{\Omega}^{n-k}(X).$$

Since $\int_X \mathbf{d}\alpha \wedge \widetilde{\beta} = (-1)^{\deg \alpha + 1} \int_X \alpha \wedge \widetilde{\beta}$, the natural extension of the exterior derivative on differential forms to currents is given by

$$(\mathbf{d}C)(\widetilde{\beta}) = (-1)^{k+1} C(\mathbf{d}\widetilde{\beta}) \qquad \text{for all } C \in \mathcal{D}_k(X).$$

Let S^{n-k} be a locally closed submanifold of X which has finite volume and suppose $\mathcal{O}_S = \mathcal{O}_{X|S}$, so that the pull-back of a twisted form from X yields a twisted form on S. Then we can associate a current $[S] \in \mathcal{D}_k(X)$ to the submanifold defined as

$$[S](\widetilde{\beta}) := \int_S i^*(\widetilde{\beta}),$$

where $i : S \longrightarrow X$ is the inclusion map. More generally, instead of an inclusion, i could just be any smooth map from a *compact* manifold (potentially with boundary and corners) of dimension $n-k$ into X which pulls back the orientation bundle. This suggests to define the dimension of a current of degree k as $n-k$. The boundary map ∂ on currents is given, in a way compatible to Stokes' theorem, by $(\partial C)(\widetilde{\beta}) := C(\mathbf{d}\widetilde{\beta})$.

Recall that the support of a differential form (twisted or not) is the closure of the set where it does not vanish. The support supp C of a current C is by definition the complement of the largest open set $U \subset X$ which has the property that $T(\widetilde{\beta}) = 0$ for all $\widetilde{\beta}$ with support contained in U.

A k-dimensional rectifiable set is a subset $S \subset X$ with finite k-dimensional Hausdorff measure $\mathcal{H}_k(S)$ such that there exists a countable union of Lipschitz maps from $\mathbb{R}^k$ to X whose images contain $\mathcal{H}_k$-almost every point of S. (in fact one may replace Lipschitz by C^1 in this statement). These sets still define currents by integration, and the space of rectifiable currents is the span over the positive integers of currents of integration over rectifiable sets. A current C such that both C and ∂C are rectifiable is called an integral current. Our examples of such creatures arise as closed submanifolds or images of manifolds with boundary under smooth maps. Integral currents are a special class of locally flat currents, for which we have the following two important facts (cf [Fed69]).

Proposition A.1 (Flat support Theorem) *Let C be a locally flat current of dimension k supported in some locally closed submanifold S of X of dimension $l < k$. Then $C = 0$.*

Proposition A.2 (Constancy Theorem) *Let C be a locally flat current of dimension k with $\partial C = 0$ and support in some locally closed submanifold S of X of dimension k. Then $C = a \cdot [S]$ for some real constant a.*

Given a current C and a differential form $\alpha \in \Omega^*(X)$, their wedge product is defined as a current by

$$(C \wedge \alpha)(\widetilde{\beta}) := C(\alpha \wedge \widetilde{\beta}).$$

Let $\psi : X^n \longrightarrow Y^m$ be a map between compact manifolds. Then the push-forward $\psi_*(C) \in \mathcal{D}_k(Y)$ of a current $C \in \mathcal{D}_{k+n-m}(X)$ is defined by duality with the pull-back of a form, namely

$$\psi_*(C)(\widetilde{\beta}) = C(\psi^*(\widetilde{\beta})) \qquad \text{for } \widetilde{\beta} \in \Omega^{m-k}(Y)$$

If the map ψ is a Riemannian submersion, we can also define the push-forward (or integral over the fibre) $\psi_*(\widetilde{\beta}) \in \Omega^{k-n+m}(Y)$ of a form $\widetilde{\beta} \in \Omega^k(X)$ by

$$\psi_*(\widetilde{\beta}) = \int \psi_*(V \lrcorner \widetilde{\beta}) dVol(x),$$

where V is a unit $(n-m)$-vector field tangent to the fibres of ψ, $dVol(x)$ is a volume form in the fibres of ψ and ψ_* on the right stands for the map on forms induced by the identification of the horizontal space at $x \in X$ with the tangent space at $\psi(x)$. The pull-back $\psi^*(C) \in \mathcal{D}_k(X)$ of a current $C \in \mathcal{D}_k(Y)$ is then given as

$$\psi^*(C)(\widetilde{\beta}) = C(\psi_*(\widetilde{\beta})) \qquad \text{for } \widetilde{\beta} \in \Omega^{m-k}(X)$$

A.2 Kernel calculus

There is a general correspondence between currents C on a product $X \times Y$ and operators $\mathbb{C} : \Omega^*(X) \longrightarrow \mathcal{D}_*(Y)$ which was developed in detail in [HP79]. Here we just collect some basic definitions and facts needed in our special case.

Definition A.3 *Given a current $C \in \mathcal{D}_*(X \times X)$, we can associate to it an operator $\mathbb{C} : \widetilde{\Omega}^*(X) \longrightarrow \mathcal{D}_*(X)$ so that, given any smooth (untwisted)* *differential form* $\alpha \in \Omega^*(X)$, *the resulting current* $\mathbb{C}(\alpha)$ *acts on a twisted form* $\tilde{\beta} \in \tilde{\Omega}^*(X)$ *by*

$$\begin{aligned}\mathbb{C}(\alpha)(\tilde{\beta}) &:= C(\pi_1^*(\alpha) \wedge \pi_2^*(\tilde{\beta})) \\ &= \pi_{2*}(C \wedge \pi_1^*(\alpha))(\tilde{\beta})\end{aligned}$$

If the degree of C is $n - d$ (and therefore it's dimension is $n + d$), the associated operator $\mathbb{C}$ will take forms of degree p to currents of degree $p - d$. The cases of most interest to us will be $d = 0$ and $d = 1$.

Example A.4 *The operator associated to the diagonal* $\Delta \subset X \times X$ *is given by the inclusion map* $\mathbb{I} : \Omega^k(X) \longrightarrow \mathcal{D}_k(X)$ *of forms into currents as described above.*

Example A.5 *More generally, the graph of a map* $\varphi : X \longrightarrow X$ *induces the pull-back operator* $\mathbb{C}_\varphi(\alpha) = \varphi^*(\alpha)$.

Proposition A.6 *Given a current equation* $\partial C = A - B$ *in* $\mathcal{D}_*(X \times X)$, *the corresponding operators are related by* $\mathbb{C} \circ \mathbf{d} + (-1)^{\deg C + 1}\mathbf{d} \circ \mathbb{C} = \mathbb{A} - \mathbb{B}$.

Proof: A calculation using the definitions yields

$$\begin{aligned}(\partial C)(\pi_1^*(\alpha) \wedge \pi_2^*(\tilde{\beta})) &= C(\mathbf{d}\pi_1^*(\alpha) \wedge \pi_2^*(\tilde{\beta})) \\ &= C(\pi_1^*(\mathbf{d}\alpha) \wedge \pi_2^*(\tilde{\beta})) + (-1)^{\deg \alpha}(C \wedge \pi_1^*(\alpha))(\pi_2^*(\mathbf{d}\tilde{\beta})) \\ &= \mathbb{C}(\mathbf{d}\alpha)(\tilde{\beta})) + (-1)^{\deg \alpha + \deg(C \wedge \alpha) + 1}\mathbf{d}\mathbb{C}(\alpha)(\tilde{\beta}) \\ &= (\mathbb{C} \circ \mathbf{d})(\alpha)(\tilde{\beta}) + (-1)^{\deg C + 1}\mathbf{d}\mathbb{C}(\alpha)(\tilde{\beta})\end{aligned}$$

□

A.3 Whitney stratifications

A stratification $\mathcal{S}$ of a subset Y contained in some manifold X consists of a collection of smooth manifolds $\{S_k\}$, indexed by dimension, such that X is the disjoint of the S_k and $\bigcup_{l\leq k} S_l$ is a closed subset of Y for each k.

A Whitney stratification satisfies the following additional conditions for any two strata such that $S_{k_1} \cap \overline{S_{k_2}} \neq \emptyset$ (the statements are local and assume that we have identified some open subset of the ambient manifold with $\mathbb{R}^N$).

(W1) Given any sequence $\{p_n\} \subset S_{k_2}$ with $\lim p_n = p \in S_{k_1}$, if $\lim T_{p_n} S_{k_2}$ exists, it contains $T_p S_{k_1}$.

(W2) Given any sequence $\{p_n\} \subset S_{k_2}$ with $\lim p_n = p \in S_{k_1}$, $\lim \overline{p_n p} \subset \lim T_{p_n} S_{k_2}$ whenever both limits exist (here $\overline{p_n p}$ is the line spanned by p_n and p).

Proposition A.7 *The set of smooth maps from a manifold M into X transverse to some Whitney stratified subset $Y \in X$ is open and dense in the Whitney topology.*

Proof: The argument is by induction on the dimension of the strata. The lowest dimensional stratum is a closed submanifold, so the usual argument (see e.g. [Hir76]) assures us that maps transverse to it are open and dense. Assume that maps transverse to $\cup_{k<l} S_k$ are open and dense. By the first Whitney condition, any such map is also transverse to S_l in a neighborhood of its boundary. But the complement S_l' of that neighborhood is a compact submanifold of X, and so maps transverse to it are also open and dense. □

A.4 The Morse Lemma for generalized Morse functions

Theorem A.8 *Let $f : X \longrightarrow \mathbb{R}$ be a generalized Morse function , and let F be a connected component of its critical set. Then there exists a normal bundle with a splitting $N = N^+ \oplus N^-$ and a metric such that the function is given as $f(u,x,v) = f(F) + |v|^2 - |u|^2$. In particular, the gradient flow for this metric is locally of the form*

$$\varphi_\tau(u,x,v) = (e^{-\tau}u, x, e^\tau v).$$

Remark A.9 *A proof of this fact in the setting of Hilbert manifolds can be found in [Mey67].*

Proof: Choose any normal bundle structure $\pi : N \longrightarrow F$. Then we can write

$$f(x,n) = n^t Q(x,v)n,$$

where $Q(x,n)$ is a symmetric bilinear form on $T_{(x,n)}N_x \cong N_x$ gotten by the usual trick of using Taylor's formula in the fibre coordinates. Since f is assumed to be non-degenerate along F, we can arrange for Q along the zero section to be equal to the diagonal matrix $I_{\lambda_F,\lambda_F^*}$ with λ_F entries being -1 and the remaining λ_F^* being 1 by a fibrewise linear transformation.

Now consider the following diagram:

$$\begin{array}{ccc} \pi^*(GL(N)/O_{\lambda_F,\lambda_F^*}(N)) & \xrightarrow{\cong} & \pi^*(S^2(N)) \\ \downarrow & & \downarrow \\ N & \xrightarrow{id} & N \end{array}$$

Here $S^2(N)$ denotes the bundle of symmetric bilinear forms on the fibres of N, and both bundles are pulled back by the projection $\pi : N \longrightarrow F$. We write $\dot{\simeq}$ to express the fact that there is a local diffeomorphism taking some neighborhood U of the section $I \cdot O_{\lambda_F,\lambda_F^*}(N)$ of $\pi^*(GL(N)/O_{\lambda_F,\lambda_F^*}(N))$ onto a neighborhood U' of the section $I_{\lambda_F,\lambda_F^*}$ in $\pi^*(S^2(N))$. This map is given by

$$g \cdot O_{\lambda_F,\lambda_F^*} \mapsto < I_{\lambda_F,\lambda_F^*} g(\cdot), g(\cdot) >,$$

where $< \cdot, \cdot >$ denotes the metric on N. Notice that Q defined above is a section of $\pi^*(S^2(N))$ which for $v \in N$ contained in some small neighborhood $N(\varepsilon)$ of the zero section is contained in U'. We can use the local diffeomorphism to lift $Q|_{N(\varepsilon)}$ to a section $\tilde{Q}$ of $\pi^*(GL(N)/O_{\lambda_F,\lambda_F^*}(N))|_{N(\varepsilon)}$. We can lift $\tilde{Q}$ further to a section $G : N(\varepsilon) \longrightarrow \pi^*(GL(N))|_{N(\varepsilon)}$, because over any sufficiently small set in U' the bundle $\pi^*(GL(N)) \longrightarrow \pi^*(GL(N)/O_{\lambda_F,\lambda_F^*})$ can be trivialized.

Finally, consider the map $\Gamma : N(\varepsilon) \longrightarrow N$ defined as

$$w = \Gamma(n) := G(x, n) \cdot v$$

and observe that for ε small enough it is a diffeomorphism onto its image. Tracing back the construction, we find that

$$f(w) = w^t I_{\lambda_F,\lambda_F^*} w,$$

and now it is easy to construct the required metric to complete the proof of the statement. □