Kähler Metrics of Positive Scalar Curvature on Ruled Surfaces

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Abstract of the Dissertation

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We study a way of constructing Kähler metrics of positive scalar curvature on some blown-up ruled surfaces of any genus. These are shown to have an explicit form on ruled surfaces blown up at most twice successively. Our surfaces are generic in the sense that they make up a dense set in the deformations of a given ruled surface. For completeness we include a previous result on the existence of Kähler metrics with positive scalar curvature on large blow-ups.

To my parents, Susann and SangHoon

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Chapter 1

Introduction

It has constantly been of great interest for geometers to study how a geometric condition governs the topology of the manifold. Here, we consider the existence of Kähler metrics with positive scalar curvature as the geometric condition on a compact complex surface of Kähler type.

It is known that not every Kähler surface admits such a metric.

Proposition 1.1 *If a Kähler manifold has a positive scalar curvature, then the plurigenera* $p_m := \dim H^0(M, \mathcal{O}(K^{\otimes m}))$ *vanish for all* m*, i.e. its Kodaira dimension* $\limsup \frac{\log p_m}{\log m}$ *is* $-\infty$*.*

Thus, by the surface classification theory, we are left with rational surfaces and ruled surfaces. As a matter of fact, even if we loosen the condition by merely requiring the existence of *Riemannian* metrics with positive scalar curvature, the same is true by recent works of LeBrun [16](for minimal surfaces) and Friedman-Morgan [1] using Seiberg-Witten theory.

Theorem 1.2 *If there exists a Riemannian metric of positive scalar curvature on a Kähler surface, then it is rational or ruled, not necessarily minimal.*

Note that this metric doesn't have to be parallel or hermitian with respect to the complex structure as a Kähler metric is.

On the other hand, Hitchin [5] constructed Kähler metrics of positive scalar curvature on rational surfaces obtained by blowing up a minimal model at finitely many distinct points. Even though these don't cover all rational surfaces, they are generic in the sense of deformation of the complex structure.

Gathering all the pieces and evidence [3, 20, 23, 1, 16], the following was conjectured in [16]:

Conjecture 1 *Let X be a compact complex surface of Kähler type. Then the following are equivalent:*

(a) X admits a Riemannian metric of positive scalar curvature;

(b) X admits a Kähler metric of positive scalar curvature;

(c) X is either rational or ruled.

The missing implication is (c) $\Rightarrow$ (b). Since every rational surface, except for $\mathbf{C}P^2$, can be thought of as ruled, we only need to look at ruled surfaces. Moreover it is known that a minimal ruled surface admits a Kähler metric of positive scalar curvature. [24] So it's enough to consider blown-up ruled surfaces.

In this paper, we will construct Kähler metrics of positive scalar curvature on the ruled surfaces which can be obtained by blowing up at most twice successively from a minimal model with arbitrarily many distinct blown-up points. This means that as a *smooth* manifold any blown-up ruled surface admits such a metric. In fact, for a dense subset of complex structures on the surface, the existence holds true. We also know from [6] that if the surface is blown up sufficiently many times a different method leads us to the existence of such a metric.

The rest of the paper is structured as follows:
in Chapter 2 we set up our curvature convention and discuss metrics on minimal ruled surfaces. Blowing up is explained in Chapter 3 and we also study special features of ruled surfaces in order to simplify our problems later. At the end of this chapter we explain precisely in what sense we have genericity of the existence of Kähler metrics of positive scalar curvature. In Chapter 4, we introduce Hitchin's construction of metrics on a space blown up at finitely many points. Then in Chapter 5, it is shown that there exist Kähler metrics of positive scalar curvature on a ruled surface as long as it doesn't contain a region which was essentially blown up more than twice from a minimal model. Lastly, Chapter 6 deals with the somewhat complementary cases of surfaces blown up sufficiently many times. Here the author essentially illustrates the proof of Theorem B in [6].

Chapter 2

Minimal ruled surfaces

2.1 Curvature convention and other notations

Let's assume $\{z^\alpha\}$ is in use for local coordinates around a point p in this section. Then we let

$$\partial_\alpha f = \frac{\partial f}{\partial z^\alpha} \quad \text{and} \quad \bar{\partial}_\alpha f = \frac{\partial f}{\partial \bar{z}^\alpha}.$$

Meanwhile,

$$\partial f = \frac{\partial f}{\partial z^1} dz^1 + \cdots + \frac{\partial f}{\partial z^n} dz^n \quad \text{and} \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}^1} d\bar{z}^1 + \cdots + \frac{\partial f}{\partial \bar{z}^n} d\bar{z}^n$$

where n is the complex dimension of the manifold.

When ω is a Kähler metric, we can write

$$\omega = i \sum g_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta.$$

$g^{\bar{\beta}\alpha}$ is defined to be the inverse of $g_{\alpha\bar{\beta}}$:

$$g_{\alpha\bar{\beta}} g^{\bar{\beta}\gamma} = \delta_\alpha^\gamma.$$

The curvature tensor R of g is

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = g^{\bar{\epsilon}\tau}\partial_\gamma g_{\alpha\bar{\epsilon}}\bar{\partial}_\delta g_{\tau\bar{\beta}} - \partial_\gamma\bar{\partial}_\delta g_{\alpha\bar{\beta}}$$

and in particular, $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\partial_\gamma\bar{\partial}_\delta g_{\alpha\bar{\beta}}$ when all the first derivatives of the metric vanish.

The *Ricci curvature* is defined by

$$ric_{\alpha\bar{\beta}} = g^{\bar{\delta}\gamma}R_{\gamma\bar{\delta}\alpha\bar{\beta}};$$

the *scalar curvature* is defined by

$$s = g^{\bar{\beta}\alpha}ric_{\alpha\bar{\beta}}.$$

It is useful to know the direct way to compute the scalar curvature when you don't need the full curvature tensor.

$$\begin{aligned} s &= -g^{\bar{\beta}\alpha}\partial_\alpha\bar{\partial}_\beta \log\det g \ , \text{ or} \\ s\,d\mathrm{vol}_g &= -\omega\wedge i\partial\bar{\partial}\log\left(\frac{\omega\wedge\omega}{\eta}\right) \end{aligned}$$

where $d\mathrm{vol}_g$ is the volume form of g and η is the coordinate volume form.

The *holomorphic sectional curvature* and the Ricci curvature at p in the direction of z^α are also defined:

$$\begin{aligned} K(p)(z) &= R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)z^\alpha\bar{z}^\beta z^\gamma\bar{z}^\delta/r^4; \\ ric(p)(z) &= ric_{\alpha\bar{\beta}}(p)z^\alpha\bar{z}^\beta/r^2 \end{aligned}$$

where $r^2 = g_{\alpha\bar{\beta}}(p)z^\alpha\bar{z}^\beta$.

2.2 Yau's metrics

First we need a suitable definition of a minimal ruled surface.

Definition 1 *A complex surface is a* minimal ruled surface *if it can be written as the total space of a* $\mathbb{C}P^1$*-bundle over a Riemann surface.*

In [24], Yau proved any minimal ruled surface admits a Kähler metric with positive scalar curvature. It can be shown that every minimal ruled surface X is associated with a rank 2 vector bundle V over the Riemann surface C(see for example, [4]) such that

$$X = \mathbf{P}(V) \xrightarrow{\pi} C.$$

Then Yau's metric is written as

$$\omega = \pi^* \omega_C + \epsilon\, i\partial\bar{\partial} \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle$$

where ω_C is a Kähler metric on C and $\langle \cdot, \cdot \rangle$ is a hermitian metric on $V \backslash \{0\}$. We will see that when $\epsilon\ (> 0)$ is small, ω is Kähler with positive scalar curvature.

Let z^1 be the local coordinate for C and z^2 the inhomogeneous coordinate for the locally trivialized fiber. From the previous section, the scalar curvature of ω has the same sign as

$$-\omega \wedge i\partial\bar{\partial} \log \left(\frac{\omega \wedge \omega}{\eta} \right)$$

where η is the standard coordinate volume form. We use

$$\omega \wedge \omega = 2\epsilon\ i\pi^* \omega_C \wedge \partial_2 \bar{\partial}_2 \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle\, dz^2 \wedge d\bar{z}^2 - \epsilon^2 \partial\bar{\partial} \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle \wedge \partial\bar{\partial} \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle$$

to see that when ϵ is small, the dominating term from above is

$$-\pi^*\omega_C \wedge i\partial_2\bar{\partial}_2 \log\left(\frac{\pi^*\omega_C \wedge \partial_2\bar{\partial}_2 \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle\, dz^2 \wedge d\bar{z}^2}{\eta}\right) dz^2 \wedge d\bar{z}^2.$$

If $z^2 = v^2/v^1$ and if we make v^1 constant, we can write $\langle \mathrm{v}\ ,\ \mathrm{v}\rangle = v^1\bar{v}^1(h_{11} + h_{21}z^2 + h_{12}\bar{z}^2 + h_{22}z^2\bar{z}^2)$ where h_{ij} are functions of z^1 and

$$\begin{aligned}\partial_2\bar{\partial}_2 \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle &= \partial_2\bar{\partial}_2 \log(h_{11} + h_{21}z^2 + h_{12}\bar{z}^2 + h_{22}z^2\bar{z}^2)\\ &= \frac{\det h}{(h_{11} + h_{21}z^2 + h_{12}\bar{z}^2 + h_{22}z^2\bar{z}^2)^2}.\end{aligned}$$

If $\omega_C = g_C\, dz^1 \wedge d\bar{z}^1$, we have

$$\begin{aligned}&-\pi^*\omega_C \wedge i\partial_2\bar{\partial}_2 \log\left(\frac{\pi^*\omega_C \wedge \partial_2\bar{\partial}_2 \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle\, dz^2 \wedge d\bar{z}^2}{\eta}\right) dz^2 \wedge d\bar{z}^2\\ &= -ig_C\partial_2\bar{\partial}_2 \log\left[\frac{g_C \det h}{(h_{11} + h_{21}z^2 + h_{12}\bar{z}^2 + h_{22}z^2\bar{z}^2)^2}\right] dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2\\ &= 2ig_C\partial_2\bar{\partial}_2 \log\langle \mathrm{v}\ ,\ \mathrm{v}\rangle\, dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2\\ &= 2ig_C\frac{\det h}{(h_{11} + h_{21}z^2 + h_{12}\bar{z}^2 + h_{22}z^2\bar{z}^2)^2}\, dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2,\end{aligned}$$

which is a positive multiple of the volume form. This metric ω will be used on the minimal model of a ruled surface in Chapter 5.

Chapter 3

Minimal models and blowing up

3.1 Blowing up

Blowing up is a procedure of obtaining another complex manifold from a given one. In particular, it plays a crucial role in the classification of complex surfaces. Here, we will define the blow-up of a complex manifold at a point.

Let M be a complex manifold of dimension n and p a point on M. Then there is an open neighborhood U of p that is isomorphic to an open set in $\mathbf{C}^n$ containing the origin via some chart ψ with $\psi(p) = 0$. If we identify these two open sets via $\{z^i\}_{i=1}^n$ as local coordinates and consider $\mathbf{C}P^{n-1} \times U \to U$ given by the projection onto the second factor, let

$$\tilde{U} = \left\{ \left([l^i], (z^i)\right) \in \mathbf{C}P^{n-1} \times U \mid l^i z^j = l^j z^i, \ i, j = 1, \cdots, n \ \right\}$$

and $\alpha : \tilde{U} \to U$ be the restriction of the above projection. Note that

$$\alpha^{-1}(p) = \mathbf{C}P^{n-1} \times \{p\} \cong \mathbf{C}P^{n-1}$$

and $\alpha : \tilde{U} \setminus \alpha^{-1}(p) \to U \setminus \{p\}$ is an isomorphism.

Now we can construct a new complex manifold

$$\widetilde{M} := (M \setminus \{p\}) \bigcup_{\alpha} \widetilde{U}$$

using α to identify $U \setminus \{p\}$ with $\widetilde{U} \setminus \alpha^{-1}(p)$. Let $\alpha : \widetilde{U} \to U$ be extended to $\beta : \widetilde{M} \to M$ by identity elsewhere. $\widetilde{M}$ together with β is the *blow-up of M at p* and has an *exceptional divisor* $E \cong \mathbf{C}P^{n-1}$ replacing p. If $[E]$ is the associated line bundle of E, then $[E]|_E \cong \mathcal{O}(-1)$, the tautological line bundle over $\mathbf{C}P^{n-1}$ since $\widetilde{U}$ is isomorphic to the total space of $\mathcal{O}(-1)$. In particular, if M is a surface then E is a curve with *self intersection number*

$$E \cdot E = \int_E c_1([E]) = -1.$$

We can equip $\widetilde{U}$ with n local coordinate charts that are related to those of U, namely, $\widetilde{U} = \cup_{i=1}^{n} \widetilde{U}_i$ where

$$\widetilde{U}_i := \left\{ \left([l^i], \{z^i\}\right) \in \widetilde{U} \mid l^i \neq 0 \right\} \text{ has}$$

$$\left(z^1/z^i, \cdots, z^{i-1}/z^i, z^i, z^{i+1}/z^i, \cdots, z^n/z^i\right) \text{ as its local coordinates.}$$

Hence in each $\widetilde{U}_i$, E is represented by the equation $(z^i = 0)$.

We can also see that a curve C passing through p in the direction of $(z^\alpha/z^i)_{\alpha=1,\cdots,i-1,i+1,\cdots,n}$ on M will become a curve $\widetilde{C}$ on $\widetilde{M}$ which meets E at the point corresponding to that direction. More precisely, the *proper transform* $\widetilde{C} := \overline{\beta^{-1}(C \setminus \{p\})}$ will satisfy

$$\widetilde{C} \cap E = \left\{ (z^1/z^i, \cdots, z^{i-1}/z^i, 0, z^{i+1}/z^i, \cdots, z^n/z^i) \right\}.$$

If M is a surface, the self intersection number of $\widetilde{C}$ is less than that of C by 1. For more account of these, see [2].

3.2 Properties of ruled surfaces

A *minimal surface* is a compact complex surface that has no embedded CP^1 with self intersection -1. A *ruled surface* is obtained by blowing up a minimal ruled surface at finitely many points possibly many times.

We will see now that this ninimal model is not unique for a ruled surface and that there are different ways of blowing down a ruled surface to a minimal one. This is because the proper transform of the fiber passing through the center of blow-up gives us another rational curve with self intersection -1 and when we blow down this curve, we get another minimal surface in general. The following example shows a typical situation.

Example

Let C be a curve with genus greater than zero. Consider the ruled surface $X = C \times \mathbf{C}P^1$ blown up at a point p. Think of C as a section not meeting p and let F be the fiber passing through p. Since $F \cdot F = 0$, the proper transform $\widetilde{F}$ will have $\widetilde{F} \cdot \widetilde{F} = -1$. Now blow down $\widetilde{F}$ to get a new ruled surface $\hat{X}$, where the image of E becomes a fiber class. Indeed, $\hat{X}$ is different from $C \times \mathbf{C}P^1$ because now there is a section $\hat{C}$ whose self intersection number is 1. Furthermore we claim that $\hat{X}$ is minimal. To see this, suppose D is a smooth rational curve in $\hat{X}$ with $D \cdot D = -1$. Since $\hat{C}$ and F span the space of numerical equivalence classes of divisors on $\hat{X}$(see [4], for example), let $D \equiv a\hat{C} + bF$ for some $a, b \in \mathbf{Z}$. Using $\hat{C} \cdot F = 1$, $F \cdot F = 0$, and $\hat{C} \cdot \hat{C} = 1$,

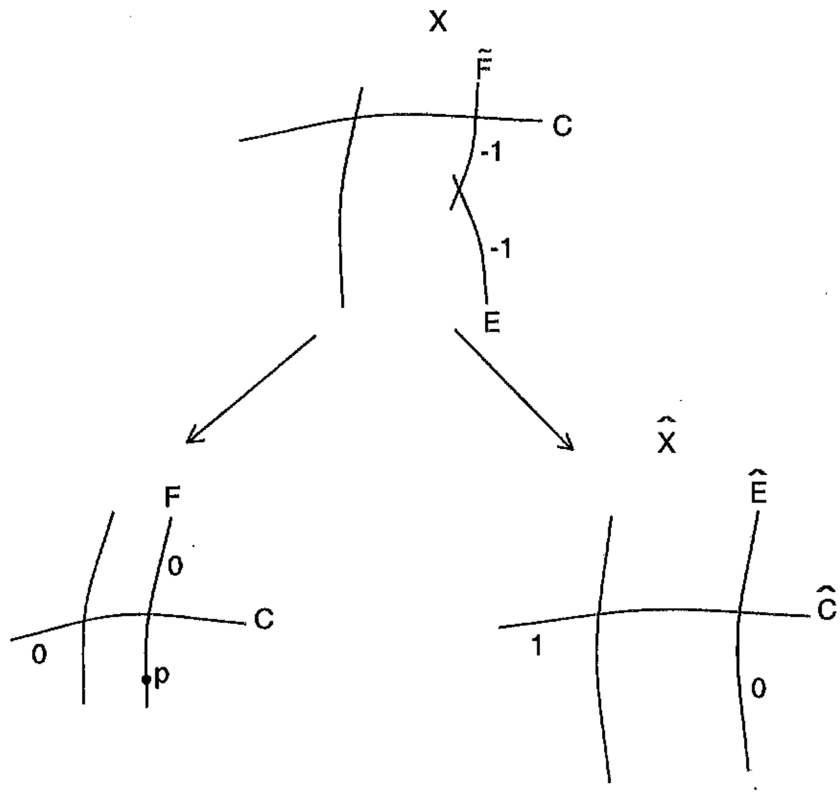

Figure 3.1: Two exceptional curves in X.

we see that a and b must satisfy

$$\begin{aligned} -1 &= (a\hat{C} + bF) \cdot (a\hat{C} + bF) \\ &= a(a + 2b), \end{aligned}$$

and since D cannot be a multiple of fiber, $a = D \cdot F > 0$. Thus we have $a = 1, a + 2b = -1$, so $b = -1$. But then $\hat{C} - F \equiv D$ is a section whose image is a rational curve, which is impossible unless the base $\hat{C}$ is $\mathbf{C}P^1$. Therefore $\hat{X}$ is minimal and, depending on which curve you blow down, both $\hat{X}$ and $C \times \mathbf{C}P^1$ are minimal models for X. $\square$

Another phenomenon we will see in a ruled surface is the fact that a surface which is obtained by blowing up a minimal one successively many times might be blown down to a minimal surface in one step by looking at different exceptional curves. Since blowing up a point is a local process, this is obviously not a problem for a surface successively blown up at the points whose images in the original minimal surtace are distinct. So we only need to consider the case where we blow up a point on the exceptional curve that came from a previous blow-up. The following example should be sufficient to illustrate when we can blow down a successive blow-up in one step.

Example

Let's consider a minimal ruled surface with genus of the base curve greater than zero. Blow up a point p_1 on it and call the exceptional curve E_1. Let $\widetilde{F}$ be the proper transform of the fiber passing through p_1. E_1 will meet $\widetilde{F}$ at the *fiber point.* Now pick any other point p_2 on E_1 and blow it up. This will produce a new exceptional curve E_2 and there is also another rational curve with self intersection -1, namely $\widetilde{F}$.

Since $\widetilde{F}$ and E_2 are disjoint, we can blow down these two in one step and get a surface which has a curve $\hat{E}_1$ with self intersection zero. To show that there is no more rational curve to blow down, set $D \cdot D = -1$ where $D \equiv a\widetilde{C} + bF$ as in the previous example. Then

$$-1 = a\left(a(C \cdot C - 1) + 2b\right)$$

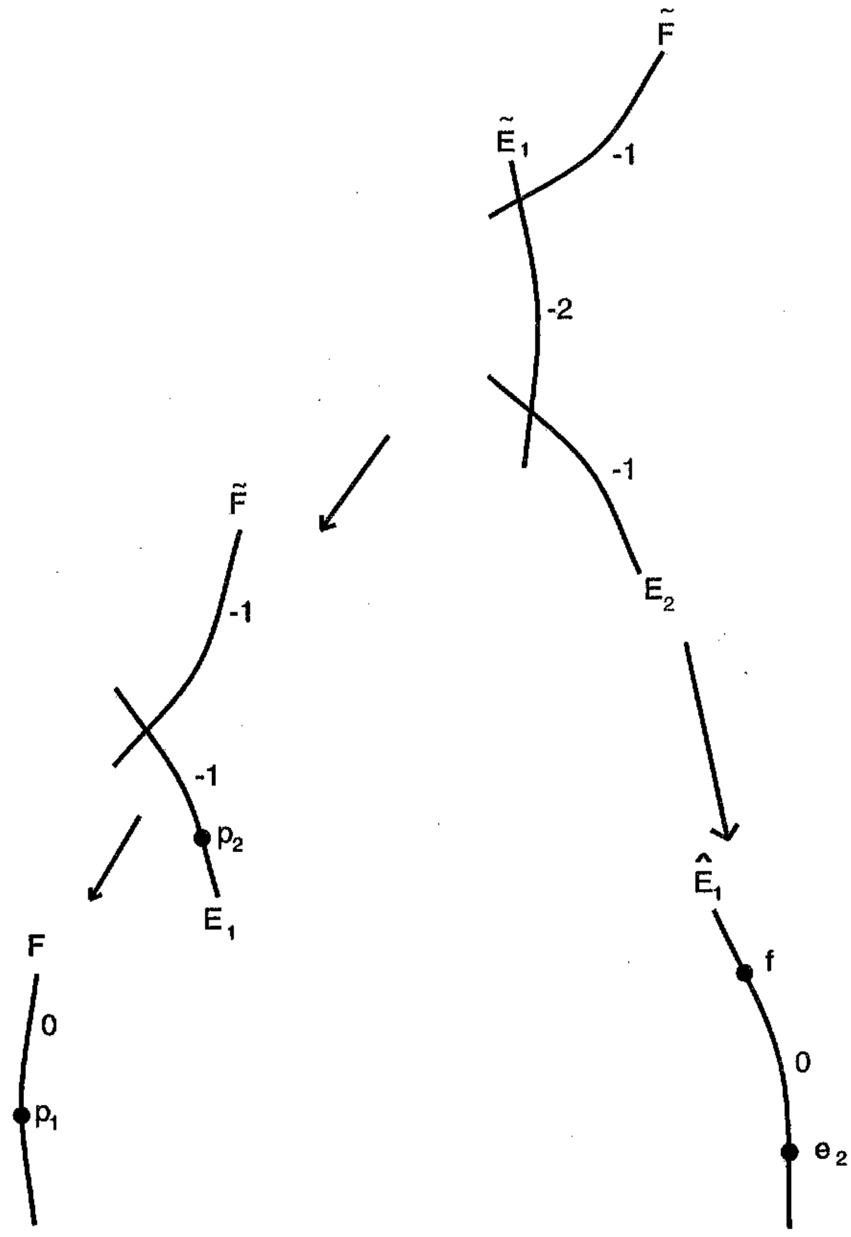

Figure 3.2: This is not an essentially-successive blow-up.

and hence $a = D \cdot F = 1, b = -\frac{C \cdot C}{2}$. Even if $C \cdot C$ is an even integer, $D \equiv \tilde{C} - \frac{C \cdot C}{2} F$ would be a section whose genus is not zero by construction.

□

If we don't consider these successive blow-ups as *essentially-successive blow-ups*, a surface apparently blown up eight times, say, could have been blown up essentially twice. As is obvious in the previous picture, the essetially-twice blowing up happens when the fiber point is chosen to be the center of the second blow-up.

3.3 Deformations of complex structures

In this section, we study how the blow-up structure of a ruled surface behaves under small deformations of the complex structure. First we set up the language of deformations.

Definition 2 *A* holomorphic family *of compact complex manifolds parametrized by a connected complex manifold B is a complex manifold Z together with a holomorphic map $f : Z \to B$ such that Z is a differentiable fiber bundle over B.*

By definition every fiber of the family is a complex manifold of the same dimension. If X is biholomorphic to $f^{-1}(x)$ for some $x \in B$, then Z is a *family of deformations of X.*

Definition 3 *A holomorphic family $Z \to B$ is* semi-universal *if given any family $Z' \to B'$ there exists a holomorphic map $g : B' \to B$ such that Z' is isomorphic to the induced family g^*Z.*

By the celebrated works of Kodaira-Spencer [11, 12] and Kuranishi [13, 14], for every compact complex manifold X there exists a semi-universal (relative to local deformations) family of deformations of X.[1] For a rational surface, Hitchin explicitly constructed a semi-universal family from that of a minimal model of the given surface. We extend this result [5, Proposition 6.1] to ruled surfaces.

Proposition 3.1 *Let X be a ruled surface. Then there exists a semi-universal (relative to local deformations) family $Z \to B$ for X such that there is an open dense set $U \subset B$ so that the fibers over U are ruled surfaces obtained by blowing up distinct points on a minimal model.*

Proof. Let $W \xrightarrow{f} A$ be a family of deformations of a surface. Then we can form a family of deformations $V(W) \to W$ where the fiber over $w \in f^{-1}(u) \subset W$ is $f^{-1}(u)$ with the point w blown up. In other words, $V(W)$ is the fiber product $W \times_A W$ with the diagonal blown up. The projection onto W is simply the projection onto the first factor in $W \times_A W$. We can check that $V(W)$ does have a differentiable fiber bundle structure over W using the fact that the fibers of

[1]When $H^2(X, \Theta) \neq 0$ where Θ is the sheaf of holomorphic vector fields on X, the parameter space could have a singularity at the point that corresponds to X and hence our definition of holomorphic family should be weakened.

f are all diffeomorphic and so are their one point blow-ups. If we iterate this process we get a family $V^m(W) \to V^{m-1}(W)$ and any surface obtained from a fiber of $W \to A$ by successively blowing up m points is contained in this family. We observe that those surfaces obtained by blowing up a point on an exceptional curve produced earlier form a subspace of codimension at least one. Therefore, there is an open dense set in $V^{m-1}(W)$ over whicn the fibers are obtained by blowing up m distinct points on the fibers of W.

We apply this to our situation and take W to be a trivial deformation for the rigid minimal models $\mathbf{C}P^2$ and $\mathbf{C}P^1 \times \mathbf{C}P^1$, and a semi-universal family for other minimal ruled surfaces which is known to exist in general by Kuranishi. If X is obtained by blowing up k times from a minimal model, take $Z = V^k(W)$ and $B = V^{k-1}(W)$. Since the ruled structure is preserved under small deformations, fibers of $Z \to B$ are all ruled surfaces.

Finally, to show that Z is semi-universal relative to small deformations, let $Y \xrightarrow{f} A$ be a family of deformations of $X = f^{-1}(u)$ and suppose X has an exceptional curve of self-intersection -1. Such a curve is stable under small deformations and there exists a neighborhood N of u over which these exceptional curves can be simultaneously blown down. Hence we get Y', a family of deformations of X with its exceptional curve blown down. Note that $N \hookrightarrow Y'$, sending v to the blown-down point in the blow-down of $f^{-1}(v)$, induces the family Y from $V(Y') \to Y'$. We continue ths way and get a commutative diagram. Then we see that Y can be induced over some neighborhood N' from $V^k(Y'') \to Y''$ where Y'' is a family of deformations of a minimal model. Since we have a semi-universal family W this is induced from W and hence $Y|_{N'}$ is induced from $V^k(W)$. ∎

So roughly speaking, the constructed semi-universal family for X is just the semi-universal family for the minimal model of X plus the deformations given by the configuration of blow-ups.

Chapter 4

Metrics on a blown-up space

Here we discuss the kind of metrics we want to put on the blown-up space to make the scalar curvature positive following Hitchin's ideas. These metrics are also the ones Kodaira used for his famous Embedding Theorem. The scalar curvature of the space blown up at a point will be computed in terms of the various curvatures of the original space.

Let $\widetilde{X} \xrightarrow{\beta} X$ be the blow-up of X at a point p. Also, let $\varphi : X \to [0,1]$ be a smooth cut-off function such that $\varphi \equiv 1$ on U'' and $\varphi \equiv 0$ outside U' where $p \in U'' \subset U' \subset U \subset X$ and U has local geodesic coordinates $\{z^\alpha\}$ around p. Then, if ω is the given Kähler metric on X, consider the following metric on $\widetilde{X}$:

$$\widetilde{\omega} = \beta^*\omega + t\, i\partial\bar{\partial}\left[(\beta^*\varphi)\log\|z\|^2\right]. \tag{4.1}$$

If $t\,(> 0)$ is sufficiently small, $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$.

We'd like to show that if ω has positive scalar curvature s, then $\widetilde{\omega}$ also has positive scalar curvature $\widetilde{s}$ for small t under some extra conditions on the curvatures of ω. Since $\widetilde{\omega} = \beta^*\omega$ outside U', $\widetilde{s} = s$ outside U'. On $\widetilde{X} \setminus \beta^{-1}(U'')$

which is a compact set, one can make t small enough so that $\widetilde{s}$ is positive since it approaches s as t goes to zero. If we can show that $\widetilde{s} > \lambda > 0$ on $\beta^{-1}(U'') \setminus E$ for some λ and small t, then $\widetilde{s} > 0$ on $\beta^{-1}(U'')$ by continuity and we are done. Using the isomorphism $\beta^{-1}(U'') \setminus E \overset{\beta}{\cong} U'' \setminus \{p\}$, we need to prove that $\widetilde{s} > \lambda > 0$ on $U'' \setminus \{p\}$ with all computations thought of as being done on the deleted neighborhood of p on X.

On $U'' \setminus \{p\}$ with geodesic coordinates $\{z^\alpha\}$, we can write $\widetilde{\omega}$ as

$$\widetilde{\omega} = \omega + t\, i\partial\bar{\partial} \log \|z\|^2$$

since $\varphi \equiv 1$ on U''. The corresponding metric tensor is

$$\widetilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{t}{r^2}\left(\delta_{\alpha\beta} - \frac{\bar{z}^\alpha z^\beta}{r^2}\right)$$

where we can expand the metric g on X as a Taylor series about p relative to the geodesic coordinates:

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} - R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) z^\gamma \bar{z}^\delta + \cdots. \tag{4.2}$$

The rest of the coefficients are derivatives of the curvature tensor at 0, i.e. at p and $(g_{\alpha\bar{\beta}} - \delta_{\alpha\beta})/r^2 =: C_{\alpha\beta}$ is bounded on U''. If we introduce the matrix $P_{\alpha\beta} := \bar{z}^\alpha z^\beta / r^2$, then P is the projection onto the vector z^α and in particular, $P^2 = P$ and trace $P = 1$.

Using matrix notation, we rewrite the metric $\widetilde{g}$ as:

$$\begin{aligned} \widetilde{g} &= 1 + r^2 C + \frac{t}{r^2}(1 - P) \\ &= \frac{r^2 + t}{r^2}\left(1 - \frac{tP}{r^2 + t} + \frac{r^4 C}{r^2 + t}\right) \\ &= \frac{r^2 + t}{r^2}\left(1 - \frac{tP}{r^2 + t}\right)\left(1 + \frac{tr^2 PC}{r^2 + t} + \frac{r^4 C}{r^2 + t}\right). \end{aligned} \tag{4.3}$$

Since

$$\tilde{s} = -\tilde{g}^{\bar{\beta}\alpha}\partial_\alpha\bar{\partial}_\beta \log\det\tilde{g}, \tag{4.4}$$

we need to compute $\tilde{g}^{-1}$ and $\det\tilde{g}$.

We use $P^2 = P$ and

$$(1+\alpha P)^{-1} = 1 - \frac{\alpha}{1+\alpha}P$$

where $|\alpha| < 1$ to get

$$\tilde{g}^{-1} = \left(1 + \frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right)^{-1}\left(\frac{r^2}{r^2+t} + \frac{tP}{r^2+t}\right). \tag{4.5}$$

Note that

$$\left\|\frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right\| \leq 2r^2\|C\| \tag{4.6}$$

and $\frac{r^2}{r^2+t} < 1$, $\frac{t}{r^2+t} < 1$ and therefore $\tilde{g}^{-1}$ is bounded if $r^2 < \frac{1}{2\|C\|}$ which is independent of t.

Using

$$\det\left(1 - \frac{tP}{r^2+t}\right) = 1 - \frac{t}{r^2+t} = \frac{r^2}{r^2+t},$$

$$\det\tilde{g} = \left(\frac{r^2+t}{r^2}\right)^{n-1}\det\left(1 + \frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right), \tag{4.7}$$

where n is the real dimension of the space. So,

$$\log\det\tilde{g} = (n-1)\left(\log(r^2+t) - \log r^2\right) + \log\det\left(1 + \frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right). \tag{4.8}$$

Let's compute the contribution of the last term, $\log\det\left(1 + \frac{tr^2PC}{r^2+t} + \frac{r^4C}{r^2+t}\right) =:$ H.

Lemma 4.1

$$\partial_\alpha\bar\partial_\beta H = -\partial_\alpha\bar\partial_\beta\left(\frac{1}{r^2(r^2+t)}(tR_{\rho\bar\sigma\gamma\bar\delta}(0)z^\rho\bar z^\sigma z^\gamma\bar z^\delta + r^4 ric_{\gamma\bar\delta}(0)z^\gamma\bar z^\delta)\right) + O(r^2).$$

Here, $f\in O(r^n)$ means $|f|<Ar^n$ as $r\to 0$ where A is independent of t.

Proof. Let $G=\det\left(1+\frac{1}{r^2(r^2+t)}(tr^4PC+r^6C)\right)$. Then $\partial_\alpha\bar\partial_\beta H = \frac{1}{G}\partial_\alpha\bar\partial_\beta G - \frac{1}{G^2}\partial_\alpha G\bar\partial_\beta G$. From 4.6,

$$\begin{aligned} G &= 1 + \text{ trace }\left(\frac{1}{r^2(r^2+t)}(tr^4PC+r^6C)\right)\\ &+ \text{ terms of the form } \frac{t^m r^{4m} r^{6n} a_1\cdots a_m b_1\cdots b_n}{r^{2(m+n)}(r^2+t)^{m+n}} \end{aligned}$$

with $m+n>1$, where a_i are entries of the matrix PC and b_i of C and hence they are bounded. Here, $\frac{t^m r^{4m} r^{6n}}{r^{2(m+n)}(r^2+t)^{m+n}} < \frac{r^{2m+4n}}{(r^2+t)^n} < r^{2m+2n}$ and the extra terms are of order $O(r^4)$.

Let $f=\frac{1}{r^2(r^2+t)}$. Then

$$\begin{aligned} \partial_\alpha f &= -\frac{\bar z^\alpha(2r^2+t)}{r^4(r^2+t)^2}\\ \partial_\alpha\bar\partial_\beta f &= -\frac{\delta_{\alpha\beta}(2r^2+t)}{r^4(r^2+t)^2} + \frac{2\bar z^\alpha z^\beta}{r^6(r^2+t)^3}(3r^4+3r^2t+t^2) \text{ and}\\ \|\partial_\alpha f\| &< \frac{2}{r^3(r^2+t)}\\ \|\partial_\alpha\bar\partial_\beta f\| &< \frac{2}{r^4(r^2+t)} + \frac{2}{r^4(r^2+t)}(3+3+1) = \frac{16}{r^4(r^2+t)}. \end{aligned}$$

So one differentiation reduces the order in r by 1. Hence, $G=1+O(r^2)$, $\partial_\alpha G\in O(r)$ and $\partial_\alpha\bar\partial_\beta G$ is bounded independent of t.

Using these estimates,

$$\partial_\alpha\bar\partial_\beta H = \partial_\alpha\bar\partial_\beta\left(\text{ trace }(tr^4PC+r^6C)/r^2(r^2+t)\right) + O(r^2).$$

From 4.2,

$$C_{\rho\sigma} = -R_{\rho\bar{\sigma}\gamma\bar{\delta}}(0)z^{\gamma}\bar{z}^{\delta}/r^2 + O(r^2), \tag{4.9}$$

so

$$\text{trace } (r^4 PC) = -R_{\rho\bar{\sigma}\gamma\bar{\delta}}(0)z^{\rho}\bar{z}^{\sigma}z^{\gamma}\bar{z}^{\delta} + O(r^6)$$

and

$$\text{trace } (r^2 C) = -ric_{\gamma\bar{\delta}}(0)z^{\gamma}\bar{z}^{\delta} + O(r^4).$$

Keeping track of order in r, we obtain the lemma. ∎

Going back to 4.8, we have

$$\begin{aligned}\partial_\alpha\bar{\partial}_\beta \log\det\tilde{g} = (n-1)&\left(\frac{\delta_{\alpha\beta}}{r^2+t} - \frac{\bar{z}^{\alpha}z^{\beta}}{(r^2+t)^2} - \frac{\delta_{\alpha\beta}}{r^2} + \frac{\bar{z}^{\alpha}z^{\beta}}{r^4}\right)\\ &- \partial_\alpha\bar{\partial}_\beta\left(\frac{1}{r^2(r^2+t)}(tR_{\rho\bar{\sigma}\gamma\bar{\delta}}(0)z^{\rho}\bar{z}^{\sigma}z^{\gamma}\bar{z}^{\delta} + r^4 ric_{\gamma\bar{\delta}}(0)z^{\gamma}\bar{z}^{\delta})\right)\\ &+ O(r^2).\end{aligned} \tag{4.10}$$

From 4.5 we estimate

$$\begin{aligned}\tilde{g}^{-1} \;=\; &\left(\frac{r^2}{r^2+t} + \frac{tP}{r^2+t}\right) - \left(\frac{r^6 C}{(r^2+t)^2} + \frac{tr^4 PC}{(r^2+t)^2} + \frac{tr^4 CP}{(r^2+t)^2} + \frac{t^2 r^2 PCP}{(r^2+t)^2}\right)\\ + \;& O(r^4).\end{aligned} \tag{4.11}$$

Also, notice that we can safely replace $C_{\alpha\beta}$ in 4.11 by $B_{\alpha\beta} = -R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0)z^{\gamma}\bar{z}^{\delta}/r^2$ from 4.9.

If we multiply 4.11 and 4.10 and take the trace according to 4.4, then

$$\tilde{s} = s_0 + s_1 + O(r^2) \tag{4.12}$$

where s_0 is independent of curvatures of g and s_1 is a linear combination of various curvatures of g at p.

Lemma 4.2

$$s_0 = (n-1)(n-2)t/(r^2+t)^2$$

and hence is zero for surfaces.

Proof.

$$\begin{aligned}
s_0 &= -\text{ trace }\left(\frac{r^2}{r^2+t}+\frac{tP}{r^2+t}\right)\left(\frac{1}{r^2+t}-\frac{r^2P}{(r^2+t)^2}-\frac{1}{r^2}+\frac{P}{r^2}\right)(n-1)\\
&= -(n-1)\text{ trace }\left(\frac{r^2}{(r^2+t)^2}-\frac{r^4P}{(r^2+t)^3}-\frac{1}{r^2+t}+\frac{P}{r^2+t}+\frac{tP}{(r^2+t)^2}\right.\\
&\quad\left.-\frac{tr^2P}{(r^2+t)^3}\right)\\
&= -(n-1)\text{ trace }\left(-\frac{t}{(r^2+t)^2}+\frac{2tP}{(r^2+t)^2}\right)\\
&= (n-1)\left(\frac{nt}{(r^2+t)^2}-\frac{2t}{(r^2+t)^2}\right)\\
&= (n-1)(n-2)t/(r^2+t)^2.
\end{aligned}$$

∎

Lemma 4.3

$$\begin{aligned}
s_1 &= \frac{1}{(r^2+t)^3}\left[r^4(r^2+t)s(0)+2tr^2\left((n+2)r^2+4t\right)ric(0)(z)\right.\\
&+ \left. t\left(-4nr^4-(2n+3)r^2t+t^2\right)K(0)(z)\right].
\end{aligned}$$

Proof. In 4.11, terms that produce curvature expressions are

$$-\frac{1}{(r^2+t)^2}\left(r^6B+tr^4PB+tr^4BP+t^2r^2PBP\right).$$

This contributes one term a_1 to s_1 and the two curvature expressions in 4.10 contribute a_2 and a_3:

$$\begin{aligned} a_1 &= \frac{n-1}{(r^2+t)^2} \text{ trace } \left(r^6 B + tr^4 PB + tr^4 BP + t^2 r^2 PBP\right) \left(\frac{1}{r^2+t}\right. \\ &- \left.\frac{r^2 P}{(r^2+t)^2} - \frac{1}{r^2} + \frac{P}{r^2}\right). \end{aligned}$$

Here, trace $B = -ric(0)(z)$ where $ric(0)(z) := ric_{\alpha\bar{\beta}}(0) z^\alpha \bar{z}^\beta / r^2$,

$$\begin{aligned} \text{and trace } BP &= \text{ trace } PB = \text{ trace } PBP = \text{ trace } BP^2 = \text{ trace } PBP^2 \\ &= -R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) z^\alpha \bar{z}^\beta z^\gamma \bar{z}^\delta / r^4 =: -K(0)(z). \end{aligned}$$

So,

$$\begin{aligned} a_1 = \frac{n-1}{(r^2+t)^2} &\left\{ ric(0)(z) \left(-\frac{r^6}{r^2+t} + r^4\right)\right. \\ &\left.+K(0)(z)\left(\frac{r^8}{(r^2+t)^2} - r^4 - \frac{2tr^4}{r^2+t} + \frac{2tr^6}{(r^2+t)^2} - \frac{t^2 r^2}{r^2+t} + \frac{t^2 r^4}{(r^2+t)^2}\right)\right\} \\ = \;& \frac{n-1}{(r^2+t)^3}\left(tr^4 ric(0)(z) - tr^2(2r^2+t)K(0)(z)\right). \end{aligned}$$

Now,

$$a_2 = \left(\frac{r^2 \delta_{\alpha\beta}}{r^2+t} + \frac{tz^\alpha \bar{z}^\beta}{r^2(r^2+t)}\right) \partial_\alpha \bar{\partial}_\beta \left(\frac{tu}{r^2(r^2+t)}\right)$$

where $u = R_{\rho\bar{\sigma}\gamma\bar{\delta}}(0) z^\rho \bar{z}^\sigma z^\gamma \bar{z}^\delta = r^4 K(0)(z)$. We need the following derivatives:

$$\begin{aligned} \partial_\alpha u &= R_{\alpha\bar{\sigma}\gamma\bar{\delta}}(0) \bar{z}^\sigma z^\gamma \bar{z}^\delta + R_{\rho\bar{\sigma}\alpha\bar{\delta}}(0) z^\rho \bar{z}^\sigma \bar{z}^\delta = \left(R_{\alpha\bar{\sigma}\gamma\bar{\delta}}(0) + R_{\gamma\bar{\sigma}\alpha\bar{\delta}}(0)\right) \bar{z}^\sigma z^\gamma \bar{z}^\delta, \\ \bar{\partial}_\beta u &= \left(R_{\rho\bar{\beta}\gamma\bar{\delta}}(0) + R_{\rho\bar{\delta}\gamma\bar{\beta}}(0)\right) z^\rho z^\gamma \bar{z}^\delta, \\ \partial_\alpha \bar{\partial}_\beta u &= \left(R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) + R_{\alpha\bar{\delta}\gamma\bar{\beta}}(0) + R_{\gamma\bar{\beta}\alpha\bar{\delta}}(0) + R_{\gamma\bar{\delta}\alpha\bar{\beta}}(0)\right) z^\gamma \bar{z}^\delta. \end{aligned}$$

Together with the derivatives of f in Lemma 4.1, we have

$$\partial_\alpha \bar{\partial}_\beta (fu) = u \partial_\alpha \bar{\partial}_\beta f + \partial_\alpha f \bar{\partial}_\beta u + \partial_\alpha u \bar{\partial}_\beta f + f \partial_\alpha \bar{\partial}_\beta u$$

$$
\begin{aligned}
&= K(0)(z)\left(-\frac{\delta_{\alpha\beta}(2r^2+t)}{(r^2+t)^2}+\frac{2\bar{z}^\alpha z^\beta}{r^2(r^2+t)^3}(3r^4+3r^2t+t^2)\right)\\
&\quad -\frac{2r^2+t}{r^4(r^2+t)^2}\left(R_{\rho\bar{\beta}\gamma\bar{\delta}}(0)+R_{\rho\bar{\delta}\gamma\bar{\beta}}(0)\right)z^\rho\bar{z}^\alpha z^\gamma\bar{z}^\delta\\
&\quad -\frac{2r^2+t}{r^4(r^2+t)^2}\left(R_{\alpha\bar{\sigma}\gamma\bar{\delta}}(0)+R_{\gamma\bar{\sigma}\alpha\bar{\delta}}(0)\right)z^\beta\bar{z}^\sigma z^\gamma\bar{z}^\delta\\
&\quad +\frac{1}{r^2(r^2+t)}\left(R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0)+R_{\alpha\bar{\delta}\gamma\bar{\beta}}(0)+R_{\gamma\bar{\beta}\alpha\bar{\delta}}(0)+R_{\gamma\bar{\delta}\alpha\bar{\beta}}(0)\right)z^\gamma\bar{z}^\delta.
\end{aligned}
$$

After some computation we get

$$
a_2=\frac{4tr^2}{(r^2+t)^2}ric(0)(z)+\frac{tK(0)(z)}{(r^2+t)^3}\left(-nr^2(2r^2+t)-2r^4-4r^2t+t^2\right).
$$

Next,

$$
a_3=\left(\frac{r^2\delta_{\alpha\beta}}{r^2+t}+\frac{tz^\alpha\bar{z}^\beta}{r^2(r^2+t)}\right)\partial_\alpha\bar{\partial}_\beta(fv)
$$

where $v=r^4ric_{\gamma\bar{\delta}}(0)z^\gamma\bar{z}^\delta=r^6ric(0)(z)$. Proceeding as above,

$$
\begin{aligned}
\partial_\alpha v &= 2r^2ric_{\gamma\bar{\delta}}(0)\bar{z}^\alpha z^\gamma\bar{z}^\delta+r^4ric_{\alpha\bar{\delta}}(0)\bar{z}^\delta,\\
\bar{\partial}_\beta v &= 2r^2ric_{\gamma\bar{\delta}}(0)z^\beta z^\gamma\bar{z}^\delta+r^4ric_{\gamma\bar{\beta}}(0)z^\gamma,\\
\partial_\alpha\bar{\partial}_\beta v &= 2ric_{\gamma\bar{\delta}}(0)\bar{z}^\alpha z^\beta z^\gamma\bar{z}^\delta\\
&\quad +2r^2\left(\delta_{\alpha\beta}ric_{\gamma\bar{\delta}}(0)z^\gamma\bar{z}^\delta+ric_{\gamma\bar{\beta}}(0)\bar{z}^\alpha z^\gamma+ric_{\alpha\bar{\delta}}(0)z^\beta\bar{z}^\delta\right)+r^4ric_{\alpha\bar{\beta}}(0).
\end{aligned}
$$

$$
\begin{aligned}
a_3 = &\left(\frac{r^2\delta_{\alpha\beta}}{r^2+t}+\frac{tz^\alpha\bar{z}^\beta}{r^2(r^2+t)}\right)\times\\
&\left[ric(0)(z)\left(-\frac{r^2\delta_{\alpha\beta}(2r^2+t)}{(r^2+t)^2}+\frac{2\bar{z}^\alpha z^\beta}{(r^2+t)^3}(3r^4+3r^2t+t^2)\right)\right.\\
&-\frac{\bar{z}^\alpha(2r^2+t)}{(r^2+t)^2}\left(2ric(0)(z)z^\beta+ric_{\gamma\bar{\beta}}(0)z^\gamma\right)\\
&-\frac{z^\beta(2r^2+t)}{(r^2+t)^2}\left(2ric(0)(z)\bar{z}^\alpha+ric_{\alpha\bar{\delta}}(0)\bar{z}^\delta\right)\\
&\left.+\frac{2}{r^2+t}\left(ric(0)(z)\left(\bar{z}^\alpha z^\beta+\delta_{\alpha\beta}r^2\right)+ric_{\gamma\bar{\beta}}(0)\bar{z}^\alpha z^\gamma+ric_{\alpha\bar{\delta}}z^\beta\bar{z}^\delta\right)+\frac{r^2ric_{\alpha\bar{\beta}}(0)}{r^2+t}\right]
\end{aligned}
$$

$$= \frac{tr^2 ric(0)(z)}{(r^2+t)^3}\left((n+1)r^2+4t\right)+\frac{r^4 s(0)}{(r^2+t)^2}.$$

Finally,

$$\begin{aligned} s_1 &= a_1+a_2+a_3 \\ &= \frac{1}{(r^2+t)^3}\Big[r^4(r^2+t)s(0)+2tr^2\left((n+2)r^2+4t\right)ric(0)(z) \\ &+ t\left(-4nr^4-(2n+3)r^2t+t^2\right)K(0)(z)\Big] \end{aligned}$$

as stated. ∎

Corollary 4.4 *If X is a complex surface, then*

$$s_1 = \frac{1}{(r^2+t)^2}\left[r^4 s(0)+8tr^2 ric(0)(z)+t(-8r^2+t)K(0)(z)\right].$$

Proof. Set $n=2$ in the lemma. ∎

Now that we have $\tilde{s} = s_1 + O(r^2)$ for surfaces, if $s_1 > \lambda_1 > 0$ for some λ_1, there is a δ which is independent of t such that $r^2 < \delta$ implies $\tilde{s} > \lambda > 0$ for some λ. Taking $U'' = \{z \in U | r^2 < \delta\}$ in the construction of $\tilde{g}$, we have $\tilde{s} > \lambda > 0$ on $U'' \setminus \{p\}$, which is exactly what we aimed for in the beginning. So we need to show $s_1 > \lambda_1 > 0$ to prove that $\tilde{s} > 0$ on the entire $\widetilde{X}$. We will use this method to obtain the main result in the following chapter.

Chapter 5

Curvature of ruled surfaces

5.1 Blowing up at distinct points

If $\widetilde{X}$ is a ruled surface obtained from a minimal model $X = \mathbf{P}(V) \xrightarrow{\pi} C$ by blowing up finitely many distinct points, then we will consider the metric $\tilde{\omega}$ constructed in Chapter 4 with a cut-off function centered around each blown-up point. Here, we use Yau's metric for the minimal model. Since the construction of $\tilde{g}$ is local, we can take cut-off functions such that they don't overlap with each other. Therefore, the local computation around each blown-up point will be the same as that of the case of blowing up at one point and we might as well consider X blown up at one point, p.

Then with the same notation as before,

$$\tilde{\omega} = \beta^*\omega + t\, i\partial\bar{\partial}\left[(\beta^*\varphi)\log r^2\right], \tag{5.1}$$

where $\omega = \pi^*\omega_C + \epsilon\, i\partial\bar{\partial}\log\langle,\rangle$ is Yau's metric from Chapter 2.

We are going to make ω more specific by requiring the following:
Choose ω_C such that regardless of the genus of C the metric around p locally

looks like the standard Fubini-Study metric on $\mathbf{C}P^1$. We can achieve this for example by deforming the uniform metric on C conformally around p until it is the Fubini-Study metric on a neighborhood U of p. We also take a local trivialization of V on a neighborhood of $\pi(p)$ and let $\langle , \rangle$ be the standard inner product on the fiber.

If we call the local inhomogeneous coordinates for C and the fiber z^1 and z^2, respectively, then by making U smaller if necessary, we can express this metric as

$$\omega = i\partial\bar{\partial}\left[\log(1+|z^1|^2) + \epsilon\log(1+|z^2|^2)\right]$$

on U. Notice that this is just the product metric on $\mathbf{C}P^1 \times \mathbf{C}P^1$ with the fiber-shrinking parameter ϵ. We can use this simple local expression for ω in the computation of $\tilde{s}$ from the last chapter if the support of the cut-off function is chosen to lie in U.

Proposition 5.1 *The scalar curvature of $\widetilde{X}$ with the above $\tilde{\omega}$ is positive.*

Proof. From the last paragraph of the previous chapter, all we need to show is $s_1 > \lambda_1 > 0$ for some constant λ_1. Let's compute the full curvature of ω to see the various curvatures that appear in s_1.

First, we have $\omega = i\,(g_{1\bar{1}}\,dz^1 \wedge d\bar{z}^1 + g_{2\bar{2}}\,dz^2 \wedge d\bar{z}^2)$ where

$$\begin{aligned} g_{1\bar{1}} &= \frac{1}{(1+|z^1|^2)^2} \\ g_{2\bar{2}} &= \frac{\epsilon}{(1+|z^2|^2)^2}, \end{aligned}$$

and their derivatives are as follows:

$$\begin{aligned}
\partial_1 g_{1\bar{1}} &= -\frac{2\bar{z}^1}{(1+|z^1|^2)^3}, \\
\bar{\partial}_1 g_{1\bar{1}} &= -\frac{2z^1}{(1+|z^1|^2)^3}, \\
\partial_1\bar{\partial}_1 g_{1\bar{1}} &= -\frac{2(1-2|z^1|^2)}{(1+|z^1|^2)^4},
\end{aligned}$$

and similarly for $g_{2\bar{2}}$ with additional factor of ϵ. All other derivatives are zero.

Since we only need the curvature at p ($z^1 = z^2 = 0$), the first derivatives all vanish and

$$\begin{aligned}
R_{1\bar{1}1\bar{1}}(0) &= -\partial_1\bar{\partial}_1 g_{1\bar{1}}(0) = 2, \\
R_{2\bar{2}2\bar{2}}(0) &= 2\epsilon, \\
g^{\bar{1}1}(0) &= 1, \\
g^{\bar{2}2}(0) &= \frac{1}{\epsilon}.
\end{aligned}$$

Now we get the curvatures at p:

$$\begin{aligned}
K(0)(z) &= \frac{2}{r^4}(|z^1|^4 + \epsilon|z^2|^4) \text{ where } r^2 = |z^1|^2 + \epsilon|z^2|^2, \\
ric_{1\bar{1}}(0) &= g^{\bar{1}1} R_{1\bar{1}1\bar{1}}(0) = 2, \\
ric_{2\bar{2}}(0) &= g^{\bar{2}2} R_{2\bar{2}2\bar{2}}(0) = \frac{1}{\epsilon}\cdot 2\epsilon = 2, \\
ric(0)(z) &= \frac{2}{r^2}(|z^1|^2 + |z^2|^2), \\
s(0) &= g^{\bar{1}1} ric_{1\bar{1}}(0) + g^{\bar{2}2} ric_{2\bar{2}}(0) = 2\left(1 + \frac{1}{\epsilon}\right).
\end{aligned}$$

Using the above,

$$\begin{aligned}
s_1 &= \frac{1}{(r^2+t)^2}\left\{2\left(1+\frac{1}{\epsilon}\right) r^4 + 16t(|z^1|^2 + |z^2|^2) \right. \\
&\quad \left. + \frac{2t}{r^4}(-8r^2 + t)(|z^1|^4 + \epsilon|z^2|^4)\right\}. \qquad (5.2)
\end{aligned}$$

Let $x = |z^1|^2$ and $y = |z^2|^2$, then $r^2 = x + \epsilon y$ in the following:

$$\begin{aligned}
r^4(r^2+t)^2(s_1-\lambda_1) &= 2\left(1+\frac{1}{\epsilon}\right)r^8 + 16tr^4(|z^1|^2+|z^2|^2) \\
&\quad + 2t(-8r^2+t)(|z^1|^4+\epsilon|z^2|^4) - \lambda_1 r^4(r^2+t)^2 \\
&= 2\left(1+\frac{1}{\epsilon}\right)r^4(x+\epsilon y)^2 + 16tr^2(x+\epsilon y)(x+y) + 2t(-8r^2+t)(x^2+\epsilon y^2) \\
&\quad -\lambda_1(r^2+t)^2(x+\epsilon y)^2 \\
&= x^2\left\{\left(2\left(1+\frac{1}{\epsilon}\right)-\lambda_1\right)r^4 - 2\lambda_1 r^2 t + (2-\lambda_1)t^2\right\} \\
&\quad + 2xy\left\{\left(2(1+\epsilon)-\epsilon\lambda_1\right)r^4 + \left(8(1+\epsilon)-2\epsilon\lambda_1\right)r^2 t - \epsilon\lambda_1 t^2\right\} \\
&\quad + y^2\left\{\left(2\epsilon(1+\epsilon)-\lambda_1\epsilon^2\right)r^4 - 2\lambda_1\epsilon^2 r^2 t + (2\epsilon-\lambda_1\epsilon^2)t^2\right\} \\
&=: Ex^2 + 2Fxy + Gy^2.
\end{aligned} \tag{5.3}$$

We will show that for sufficiently small λ_1, we have $E, G > 0$, $F^2 - EG < 0$ and therefore prove $s_1 > \lambda_1$. Notice that E, F, and G are all quadratic expressions in r^2 and t, and that their first coefficients are positive if $\lambda_1 < 2\left(1+\frac{1}{\epsilon}\right)$.

The discriminant of E is

$$\lambda_1^2 - \left(2\left(1+\frac{1}{\epsilon}\right)-\lambda_1\right)(2-\lambda_1) = -4\left(1+\frac{1}{\epsilon}\right) + 2\left(2+\frac{1}{\epsilon}\right)\lambda_1$$

and is negative if $\lambda_1 < \frac{2\left(1+\frac{1}{\epsilon}\right)}{2+\frac{1}{\epsilon}}$. So $E > 0$ for small λ_1.

Similarly, G has negative discriminant $2\epsilon^3(2+\epsilon)\lambda_1 - 4\epsilon^2(1+\epsilon)$ and $G > 0$ if $\lambda_1 < 2\left(1+\frac{1}{\epsilon}\right)/(2+\epsilon)$.

But for F, the discriminant is $\left(4(1+\epsilon)-\epsilon\lambda_1\right)^2 + \epsilon\lambda_1\left(2(1+\epsilon)-\epsilon\lambda_1\right) = 2(1+\epsilon)(8(1+\epsilon)-3\epsilon\lambda_1)$ and it's positive if $\lambda_1 < 8/3\left(1+\frac{1}{\epsilon}\right)$. Since the first two coefficients are positive, F is negative if r^2/t is small enough and $F > -\epsilon\lambda_1 t^2$.

Where r^2/t is large and $F \geq 0$, every term in 5.3 is positive and we're done. Thus, let's assume $0 > F > -\epsilon\lambda_1 t^2$.

Now we should compute the minimum values of E and G.

$$E \geq t^2 \left(-\frac{\lambda_1^2}{2\left(1+\frac{1}{\epsilon}\right) - \lambda_1} + 2 - \lambda_1 \right) > t^2(-1 + 2 - \lambda_1) > 0$$

if $\lambda_1 < \min\{1, k(\epsilon)\}$ for some constant k which depends on ϵ.

Similarly,

$$G \geq t^2 \left(-\frac{\lambda_1^2 \epsilon^2}{2\left(1+\frac{1}{\epsilon}\right) - \lambda_1} + 2\epsilon - \epsilon^2\lambda_1 \right) > t^2(-\epsilon^2 + 2\epsilon - \epsilon^2\lambda_1) > 0$$

if $\lambda_1 < \min\{\frac{2}{\epsilon} - 1, k(\epsilon)\}$ where we can assume $\frac{2}{\epsilon} - 1 > 0$ if ϵ is small enough.

Therefore,

$$F^2 - EG < t^4 \left(\epsilon^2\lambda_1^2 - (\lambda_1 - 1)(\epsilon^2\lambda_1 + \epsilon^2 - 2\epsilon) \right) = t^4(2\epsilon\lambda_1 + \epsilon^2 - 2\epsilon) < 0$$

if $\lambda_1 < 1 - \epsilon/2$, as desired. ∎

Corollary 5.2 *There exist Kähler metrics of positive scalar curvature on a ruled surface which is obtained from a minimal model by blowing up at distinct points.*

5.2 Successive blow-ups

Next, we extend the construction over to a essentially-twice-successively-blown-up ruled surface. As shown in Chapter 3, this case has to do with blowing up the fiber point on an exceptional curve.

We start with $\widetilde{X}$ from the previous section and take local coordinates $w^1 = z^1/z^2, w^2 = z^2$ around the fiber point on the exceptional curve. Then we blow up $\widetilde{X}$ at 0 and consider the metric

$$\tilde{\omega} + \tilde{t}\, i\partial\bar{\partial}(\tilde{\varphi}\log r^2)$$

on the blown-up space as constructed in Chapter 4 . So $\widetilde{X}$ is equipped with the metric expressed as

$$\begin{aligned}\tilde{\omega} &= i\partial\bar{\partial}\left[\log(1+|z^1|^2) + \epsilon\log(1+|z^2|^2) + t\log(|z^1|^2 + \epsilon|z^2|^2)\right] \\ &= i\partial\bar{\partial}\left[\log(1+|w^1w^2|^2) + \epsilon\log(1+|w^2|^2) + t\log(|w^1|^2+\epsilon)\right]\end{aligned}$$

on $\beta^{-1}(U'')$.

We proceed as in the previous section and compute various curvatures of $\tilde{\omega}$ at the point to blow up ($w^1 = w^2 = 0$). First we compute the components of the metric $\tilde{\omega} = i\sum g_{\alpha\bar{\beta}}\, dw^\alpha \wedge d\bar{w}^\beta$:

$$\begin{aligned} g_{1\bar{1}} &= \frac{|w^2|^2}{(1+|w^1w^2|^2)^2} + \frac{t\epsilon}{(|w^1|^2+\epsilon)^2}, \\ g_{1\bar{2}} &= \frac{\bar{w}^1 w^2}{(1+|w^1w^2|^2)^2} = \overline{g_{2\bar{1}}}, \\ g_{2\bar{2}} &= \frac{|w^1|^2}{(1+|w^1w^2|^2)^2} + \frac{\epsilon}{(1+|w^2|^2)^2}. \end{aligned}$$

We also need the following:

$$\begin{aligned} \det g(0) &= t, \\ g^{\bar{1}1}(0) &= \epsilon/t, \\ g^{\bar{1}2}(0) &= 0 = g^{\bar{2}1}(0), \\ g^{\bar{2}2}(0) &= 1/\epsilon, \end{aligned}$$

$$\begin{aligned}
\bar{\partial}_1 g_{1\bar{1}} &= -2w^1 \left(\frac{|w^2|^4}{(1+|w^1w^2|^2)^3} + \frac{t\epsilon}{(|w^1|^2+\epsilon)^3} \right), \\
\bar{\partial}_1 g_{2\bar{1}} &= -2(w^1)^2 \bar{w}^2 \frac{|w^2|^2}{(1+|w^1w^2|^2)^3}, \\
\bar{\partial}_1 g_{1\bar{2}} &= w^2 \frac{1-|w^1w^2|^2}{(1+|w^1w^2|^2)^3}, \\
\bar{\partial}_2 g_{2\bar{2}} &= -2w^2 \left(\frac{|w^1|^4}{(1+|w^1w^2|^2)^3} + \frac{\epsilon}{(1+|w^2|^2)^3} \right),
\end{aligned}$$

and these derivatives all vanish at 0. Thus, we have

$$\begin{aligned}
R_{1\bar{1}1\bar{1}}(0) &= -\partial_1 \bar{\partial}_1 g_{1\bar{1}}(0) = 2t/\epsilon^2, \\
R_{1\bar{2}1\bar{1}}(0) &= -\partial_1 \bar{\partial}_2 g_{1\bar{1}}(0) = 0, \\
R_{1\bar{2}1\bar{2}}(0) &= -\partial_1 \bar{\partial}_2 g_{1\bar{2}}(0) = 0, \\
R_{1\bar{1}2\bar{2}}(0) &= -\partial_1 \bar{\partial}_1 g_{2\bar{2}}(0) = -1, \\
R_{1\bar{2}2\bar{2}}(0) &= -\partial_1 \bar{\partial}_2 g_{2\bar{2}}(0) = 0, \\
R_{2\bar{2}2\bar{2}}(0) &= -\partial_2 \bar{\partial}_2 g_{2\bar{2}}(0) = 2\epsilon,
\end{aligned}$$

and the rest of the components can be obtained by symmetry of the tensor and the metric.

Now we compute the various curvatures:

$$\begin{aligned}
K(w)(0) &= \left(\frac{2t}{\epsilon^2}|w^1|^4 - 4|w^1w^2|^2 + 2\epsilon|w^2|^4 \right) / r^4 \\
&\quad \text{where } r^2 = \frac{t}{\epsilon}|w^1|^2 + \epsilon|w^2|^2 =: \frac{t}{\epsilon}x + \epsilon y, \\
ric_{1\bar{1}}(0) &= g^{\bar{1}1} R_{1\bar{1}1\bar{1}}(0) + g^{2\bar{2}} R_{2\bar{2}1\bar{1}}(0) = 2/\epsilon - 1/\epsilon = 1/\epsilon, \\
ric_{1\bar{2}}(0) &= g^{\bar{1}2} R_{2\bar{1}1\bar{2}}(0) = 0, \\
ric_{2\bar{2}}(0) &= g^{\bar{1}1} R_{1\bar{1}2\bar{2}}(0) + g^{\bar{2}2} R_{2\bar{2}2\bar{2}}(0) = -\epsilon/t + 2, \\
ric(w)(0) &= \left(\frac{1}{\epsilon}|w^1|^2 + \left(2 - \frac{\epsilon}{t}\right)|w^2|^2 \right) / r^2,
\end{aligned}$$

$$s(0) = 2/\epsilon.$$

Here, by the way, we can check that the answers for $s(0)$ from above and from 5.2 are consistent.

Finally, from 4.4,

$$\begin{aligned} s_1 &= \frac{1}{(r^2+\tilde{t})^2}\left\{2r^4/\epsilon + 8\tilde{t}\left(\frac{1}{\epsilon}|w^1|^2 + (2-\epsilon/t)|w^2|^2\right)\right. \\ &\quad \left. + \frac{\tilde{t}}{r^4}(-8r^2+\tilde{t})\left(\frac{2t}{\epsilon^2}|w^1|^4 - 4|w^1w^2|^2 + 2\epsilon|w^2|^4\right)\right\}. \end{aligned}$$

We consider the positivity of the following:

$$\begin{aligned} r^4(r^2+\tilde{t})^2(s_1-\lambda_1) &= \frac{2}{\epsilon}r^4\left(\frac{t}{\epsilon}x+\epsilon y\right)^2 + 8\tilde{t}r^2\left(\frac{t}{\epsilon}x+\epsilon y\right)(x/\epsilon+(2-\epsilon/t)y) \\ &\quad + \tilde{t}(-8r^2+\tilde{t})\left(\frac{2t}{\epsilon^2}x^2 - 4xy + 2\epsilon y^2\right) - \lambda_1(r^2+\tilde{t})^2\left(\frac{t}{\epsilon}x+\epsilon y\right)^2 \\ &= x^2\left\{\frac{2t^2}{\epsilon^3}r^4 - \frac{8t}{\epsilon^2}r^2\tilde{t} + \frac{2t}{\epsilon^2}\tilde{t}^2 - \lambda_1\frac{t^2}{\epsilon^2}(r^2+\tilde{t})^2\right\} \\ &+ 2xy\left\{\frac{2t}{\epsilon}r^4 + 8(2+t/\epsilon)r^2\tilde{t} - 2\tilde{t}^2 - \lambda_1 t(r^2+\tilde{t})^2\right\} \\ &+ y^2\left\{2\epsilon r^4 - \frac{8\epsilon^2}{t}r^2\tilde{t} + 2\epsilon\tilde{t}^2 - \lambda_1\epsilon^2(r^2+\tilde{t})^2\right\} \\ &= x^2\left\{\frac{t^2}{\epsilon^2}(2/\epsilon-\lambda_1)r^4 - \frac{2t}{\epsilon^2}(4+\lambda_1 t)r^2\tilde{t} + \frac{t}{\epsilon^2}(2-\lambda_1 t)\tilde{t}^2\right\} \\ &+ 2xy\left\{t(2/\epsilon-\lambda_1)r^4 + 2(8+4t/\epsilon-\lambda_1 t)r^2\tilde{t} - (2+\lambda_1 t)\tilde{t}^2\right\} \\ &+ y^2\left\{\epsilon^2(2/\epsilon-\lambda_1)r^4 - \frac{2\epsilon^2}{t}(4+t\lambda_1)r^2\tilde{t} + \epsilon^2(2/\epsilon-\lambda_1)\tilde{t}^2\right\} \\ &=: Ex^2 + 2Fxy + Gy^2. \end{aligned}$$

As in the previous section, we'll show $E, G > 0$ and $F^2 - EG < 0$ for suitable values of t/ϵ. First, we make the first coefficients positive by making $\lambda_1 < 2/\epsilon$.

E has discriminant $(4+\lambda_1 t)^2 - t\,(4/\epsilon - 2\lambda_1(t/\epsilon+1) + t\lambda_1^2) = 2\lambda_1(t^2/\epsilon + 5t) + 4(4-t/\epsilon)$ which is negative if $\lambda_1 < \frac{2(t/\epsilon-4)}{t^2/\epsilon+5t}$ where we need to impose the condition $t/\epsilon > 4$. So $E > 0$.

G is also positive since its discriminant is $\epsilon^2(4/t+\lambda_1)^2 - 4 + 4\lambda_1\epsilon - \epsilon^2\lambda_1^2 = 4\lambda_1(\epsilon + 2\epsilon^2/t) + 4(4\epsilon^2/t^2 - 1)$ which is negative if λ_1 is small enough in terms of ϵ and t.

F has discriminant $4(16+17t/\epsilon+4t^2/\epsilon^2) - 6t(t/\epsilon+3)\lambda_1$ which is positive if λ_1 is small in terms of ϵ and t. For the same reason as in the previous section, we assume $F < 0$. We will use this inequality in the following computation.

We compute $F^2 - EG$ directly:

$$
\begin{aligned}
F^2 - EG &= 2(t/\epsilon+3)\tilde{t}\left[8t(2/\epsilon - \lambda_1)r^6 + \left(2(15t/\epsilon+16) - 15t\lambda_1\right)r^4\tilde{t}\right. \\
&\quad \left. - 2(4+3t\lambda_1)r^2\tilde{t}^2\right] + \tilde{t}^4\left(4(1-t/\epsilon) + 2t(t/\epsilon+3)\lambda_1\right) \\
&< 2\tilde{t}^2\left[(t/\epsilon+3)(-96 - 34t/\epsilon + t\lambda_1)r^4 + 2(t/\epsilon+3)(4+t\lambda_1)r^2\tilde{t}\right. \\
&\quad \left. + \left(2(1-t/\epsilon) + t(t/\epsilon+3)\lambda_1\right)\tilde{t}^2\right]
\end{aligned}
$$

This is again a quadratic expression in r^2 and $\tilde{t}$ with negative first coefficient for small λ_1. Also for λ_1 small enough in terms of t and ϵ, the discriminant $2t\lambda_1(17t^2/\epsilon^2 + 104t/\epsilon + 155) - 4(17t^2/\epsilon^2 + 27t/\epsilon - 60)$ is negative under our condition $t/\epsilon > 4$ that was imposed to ensure the positivity of E and G. Therefore, $F^2 - EG < 0$ for sufficiently large t/ϵ and sufficiently small λ_1. Thus, $s_1 > \lambda_1 > 0$ and we have the following results.

Proposition 5.3 *There exists Kähler metrics of positive scalar curvature on a ruled surface that is essentially-successively-blown-up at most twice from a minimal surface.*

Removing the technical terms, we have

Theorem 5.4 *There exists Kähler metrics of positive scalar curvature on a ruled surface that is blown up twice from a minimal surface.*

Chapter 6

Other blown-up surfaces

If one allows blowing up many times, a totally different method produces surfaces with a Kähler metric of positive scalar curvature. We'll show that a ruled surface admits such metrics if it is blown up sufficiently many times in an appropriate way.

Given a ruled surface, first we aim for the existence of a Kähler metric of zero scalar curvature (scalar-flat Kähler, in short). Our strategy is then to deform the metric to achieve positive scalar curvature. To be precise, we use the following result [6, Theorem 3] for the existence of a scalar-flat metric:

Theorem 6.1 *Let (X, J) be a ruled surface. Then (X, J) has blow-ups $(\widetilde{X}, \widetilde{J})$ which admit scalar-flat Kähler metrics.*

Take one of these blow-ups $\widetilde{X}$. We'd like to use the following [17, Corollary 1]:

Theorem 6.2 *Let (M,J,ω) be a compact Kähler manifold which is scalar-flat but not Ricci-flat. Suppose, moreover, that every global holomorphic vector* *field on (M,J) is covariantly constant with respect to ω. Then for every real constant $c \in \mathbf{R}$, there is a Kähler metric on M with scalar curvature $s \equiv c$.*

To satisfy the condition on holomorphic vector fields, we need:

Lemma 6.3 *Let M be a complex manifold with a nontrivial holomorphic vector field ξ. Then $\widetilde{M}$ which is obtained by blowing up M at a nonzero point of ξ doesn't admit a vector field that agrees with ξ on the complement of the exceptional divisor.*

Proof. For simplicity, let's work with surfaces. Say $\xi(p) \neq 0$. In a local coordinate chart $\{z_i\}$ around p on M, we can assume $\xi = \frac{\partial}{\partial z_1}$ without loss of generality. Blow up M at p to get $\widetilde{M}$. As usual, $\{z_i\}$ gives rise to coordinate charts around the exceptional divisor E on $\widetilde{M}$: $\{w_1 = z_1/z_2, w_2 = z_2\}$ and $\{w_1' = z_1, w_2' = z_2/z_1\}$. If $\widetilde{M}$ inherits ξ on the complement of E, it must look like

$$\frac{\partial}{\partial z_1} = \frac{1}{w_2}\frac{\partial}{\partial w_1} = \frac{\partial}{\partial w_1'} - \frac{w_2'}{w_1'}\frac{\partial}{\partial w_2'}.$$

Thus, in each coordinate chart, the vector field cannot be extended onto E. ∎

So we get rid of the vector field each time we blow up at a nonzero point of it. Our scalar-flat $\widetilde{X}$ may have to be blown up more to have all holomorphic vector fields vanish and now this surface certainly satisfies the vector field assumption in Theorem 6.2. But we also have to have a scalar-flat Kähler metric on this new non-minimal surface, so we invoke the result of Kim-Pontecorvo [7, Theorem 5.2] adapted to our situation :

Theorem 6.4 *Let M be a compact non-minimal scalar-flat Kähler surface with $c_1^{\mathbf{R}}(M) \neq 0$. Then the blow-up of M at any collection of points (distinct or not) admits scalar-flat Kähler metrics.*

Now we have a blown-up ruled surface which is scalar-flat and has no nontrivial holomorphic vect. fields. Theorem 6.2 says there is a Kähler metric on it with any given positive constant as its scalar curvature, which is more than we wanted.

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