Stochastic Version of the Selberg Trace Formula

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Abstract of the Dissertation

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We introduce a stochastic analog of the Selberg Trace Formula for compact negatively-curved Riemann surfaces. Motivated from natural questions surrounding Gutzwiller's observations [8] and Takhtajan's conjectures regarding the associated Selberg Zeta Function, we investigate the "local theory" of this formula. This reduces to certain volume computations for noncontractible loops over a cylinder.

We first identify a special class of cylinders and discover a Feynman-Kac formula for the relevant volume $\mu(t)$. Employing

the well-known relationship between the Korteweg-deVries equation and the spectral theory of Hill's equation [13] [14], we construct volume-preserving deformations of cylinders within this class. The collection of all such cylinders having identical volumes is shown to be a torus of (generically) infinite dimension.

Second, we employ the so-called Malliavin Calculus to obtain a general expression for the relevant volume. The equation is modelled on the Feynman-Kac formula of the special case. We extend the techniques developed by S. Watanabe in [22] and obtain an asymptotic expansion for $\mu(t)$ as $t \to 0$; the "one-loop" term of the series is calculated.

To my family, to Andy, and to Han

List of Figures

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Chapter 1

Introduction

1.1 Brownian Motion on a Manifold

Let (M, g) be a complete Riemannian manifold. Under ordinary circumstances, M admits a fundamental solution $p \in C^\infty([0,\infty) \times M \times M)$ to the probabilist's *diffusion equation.* Locally,

$$\frac{\partial p}{\partial t}(t, \cdot, y) = \frac{1}{2\sqrt{g}}\, \partial_i \sqrt{g}\, g^{ij}\, \partial_j\; p(t, \cdot, y) = \frac{1}{2}\Delta_M p(t, \cdot, y) \quad \forall y,$$

$$\lim_{t \to 0} p(t, x, y)\sqrt{g}(y) = \delta_x(y).$$

(Unless noted otherwise, Einstein's summation convention is understood.) If the Ricci curvature is bounded from below, Yau's theorem [9] guarantees the existence and uniqueness of this so-called *heat kernel.* It enjoys the following global properties: $p_t(x, y) = p_t(y, x)$, $\int_M p_t(x, y) dV(x) = 1$, and it satisfies the *Chapman-Kolmogorov* equation:

$$\int_M p_t(x,y)p_{t-s}(y,z)\,dV(y) = p_t(x,z).$$

(Locally, $dV(y) = \sqrt{g}\,dy$.)

The *Brownian motion* for (M, g) is the observation process on

$$(\mathbb{P}(M), \mu) \stackrel{\text{def}}{=} (C([0,\infty), M), \mu),$$

where $d\mu = dV(x) \times d\mu_x$ and μ_x is the unique probability measure on $\mathbb{P}_x(M) = \{\, X \in C([0,\infty), M) \mid X_0 \stackrel{\text{def}}{=} X(0) = x \,\}$ determined by its values on *cylindrical sets*:

$$\mu_x(\{\, X \mid X_{t_j} \stackrel{\text{def}}{=} X(t_j) \in U_j, 0 < t_1 < \cdots < t_n \,\}) =$$
$$\int_{U_1} \cdots \int_{U_n} p_{t_1}(x, x_1)p_{t_2-t_1}(x_1,x_2)\cdots p_{t_n-t_{n-1}}(x_{n-1},x_n)\,dV(x_1)\cdots dV(x_n).$$

It turns out that $\mu(\{\, X \mid \frac{dX}{dt} \text{ exists for some } t \,\}) = 0$, which exhibits the rather complicated nature of Brownian motion. Nevertheless, μ can be approximated by reasonably well-behaved objects. Using g we can equip each tangent plane T_xM with a centered Gaussian distribution, denoted $e^{-\|v_x\|^2/2t}dv_x/(2\pi t)^{\dim M/2}$. We exponentiate this measure to M and define q as its Radon-Nikodym derivative with respect to dV

$$\exp_*(e^{-\|v_x\|^2/2t}dv_x/(2\pi t)^{\dim M/2}) = q_t(x,y)\,dV(y).$$

The following relationship between q and p is well-known [5], [6], [19]

$$\mu_x(\{ X \mid X_{t_j} \in U_j \}) = \lim_{\substack{\|\Delta s\| \to 0 \\ \{s_i\} \text{ refines } \{t_j\}}} \int_{V_1} \cdots \int_{V_m} q_{t_1}(x, x_1) q_{t_2 - t_1}(x_1, x_2) \cdots$$

$$\cdots q_{t_m - t_{m-1}}(x_{m-1}, x_m)\, dV(x_1) \cdots dV(x_m) \qquad V_i \stackrel{\text{def}}{=} \begin{cases} U_j, & \text{if } s_i = t_j \\ M & \text{otherwise.} \end{cases}$$

It is useful to view this relationship from another viewpoint [5]. With a bit of thought, it is not too hard to see that the general term on the right side can be viewed as the definition of a measure on the space of piecewise geodesic curves. Developing these broken geodesics via the Levi-Civita connection to the space of piecewise linear paths in T_xM, Donsker's version of the central limit theorem implies that the measures so-induced converge weakly to the *canonical Wiener measure* on $C([0,\infty), T_xM)$. The optimist would hope that this construction would also define (in the limit) a measure-theoretic isomorphism between the canonical Wiener process and the Brownian motion for (M, g) subject to the initial condition that $X_0 = x$. The limiting map, otherwise called the *stochastic development map*, was discovered in the early 1970's by Eels, Elworthy and Malliavin.

If the tangent bundle is trivial, a more direct method is available. This is the method we will employ throughout the thesis. Supposing that

$\{A_i\}$ form a global orthonormal framing of TM, we define the vector field $\sum a^i A_i \stackrel{\text{def}}{=} 1/2(\Delta_M - \sum A_i^2)$. Then the solution to the *Fisk-Stratonovitch* Stochastic Differential Equation (SDE)

$$dY = A_i(Y) \circ (dw^i + a^i(Y)dt)$$

will provide such an isomorphism. Using the *Girsanov-Cameron-Martin* (GCM) theorem [9][10], we can shuffle the drift term $a^i dt$ into the measure and reduce the requisite SDE to $dY = A_i(Y) \circ dw^i$. Then Brownian motion on M will be the image of ZdP under Y, where dP is canonical Wiener measure and

$$Z = \exp\left(\int a^i(Y)dw^i - 1/2 \int a^i(Y)^2 dt\right).$$

This technique is valid under relatively mild assumptions on the coefficients; for our purposes the a_i's are uniformly bounded and the GCM theorem applies.

A further reduction is possible whenever $\sum a^i A_i = \nabla\phi$. In this case, we can eliminate the stochastic integral appearing in Z via the Itô formula and the associativity of $\circ$:

$$\int_0^t a^i(Y)dw^i = \int_0^t A_i(\phi)dw^i \tag{1.1.1}$$

$$= \int_0^t A_i(\phi) \circ dw^i - \frac{1}{2}\int_0^t d(A_i(\phi)) \cdot dw^i \tag{1.1.2}$$

$$= \phi(Y_t) - \phi(Y_0) - \frac{1}{2}\int_0^t A_i^2(\phi)ds \tag{1.1.3}$$

Essentially this is the theory of the *Smoluchowski* equation [10] in a geometric setting. We exploit this technique in the proof of theorem 4.2 to evaluate appropriate path integrals as iterated integrals without appeal to filtering theory (see theorem 7).

1.2 Loop Space

We can also consider a similar program on the loop space $\Omega^T(M) = C(\mathbb{R}/T\mathbb{Z}, M)$, with some notable differences. For $0 \leq t_1 < \cdots < t_n < T$, the measure $\mu(T)$ satisfies

$$\mu(T)(\{\, X \mid X_{t_j} \in U_j \,\}) = \int_{U_1} \cdots \int_{U_n} p_{T+t_1-t_n}(x_n, x_1) p_{t_2-t_1}(x_1, x_2) \cdots \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n)\, dV(x_1) \cdots dV(x_n). \tag{1.1.4}$$

In particular, note that the map $X_t \mapsto X_{t+s}$ preserves $\mu(T)$, and

$$\mu(T)(\Omega^T) = \int_M p_T(x,x)dV(x).$$

Decomposing Ω^T into connected components (i.e. free homotopy classes $[\gamma]$ of loops in M), we have a *stochastic trace* formula

$$\sum_{[\gamma]} \mu(T)([\gamma]) = \int_M p_T(x,x)dV(x). \tag{1.1.5}$$

Example. Let $M = \mathbb{R}/\mathbb{Z}$; then the method of images shows that

$$\mu(T/2\pi)(\{\text{index } \gamma = n\}) = e^{-\pi n^2/T}/\sqrt{T}.$$

The Fourier expansion $p_t(x,y) = \sum e^{-\pi(n^2 t + 2n(x-y)\sqrt{-1})}$ implies 1.1.5 is simply the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/T} = \sqrt{T}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 T}.$$

In contrast to the Brownian motion for M, isomorphisms between loop spaces will exist only in exceptional cases. The aforementioned isomorphisms arising as solutions to certain SDE's will not map loops to loops, not even in approximation. Furthermore, as a subset of the full path space, $\mu(\Omega^T) = 0$

which also complicates matters. To apply the theory of SDE's to loop spaces generally requires application of the so-called Malliavin calculus, which we require in chapter 4. However, in certain special cases we can stick to more traditional techniques of probability theory, as seen in chapter 3.

A great deal is known about the behavior of $\mu(T)$ on the contractible loops, so we instead focused attention on the noncontractible loops. We will also restrict our analysis to surfaces of negative curvature. There are enough interesting questions even in this setting, and this thesis just 'scratches the surface'. We describe here one well-known example. As pointed out by the physicist Gutzwiller [8], one can observe some striking implications when a formal *stationary phase* calculation is performed on an associated Feynman integral over the loop space. In the mathematical setting we will employ, this amounts to calculating the short-time asymptotics of $\mu(t)$ to first approximation. In order to explain some of the consequences of Gutzwiller's observations, we will use standard terminology from quantum mechanics, but write the equations in their *Wick-rotated* form (i.e. replacing t by $\sqrt{-1}t$).

1.3 Gutzwiller's semiclassical approximation

With M a compact Riemann surface of negative curvature, Gutzwiller [8] calculated the *Wentzel-Kramers-Brillouin* semiclassical approximation to the partition density $\langle x, t_i | x, t_f \rangle = p_{t_f - t_i}(x, x)$ and obtained the zeroth-order approximation for μ as $t \to 0$ (again labeling a free homotopy class via its unique shortest closed geodesic γ)

$$\mu(t)([\gamma]) \sim \frac{e^{-\ell^2(\gamma)/2t}\ell_0(\gamma)}{\sqrt{2\pi t}} \times \frac{1}{\sqrt{\det(I - P(\gamma))}}.$$

Here $\ell =$ *length* , $\ell_0 =$ *primitive length* and $P =$ *Poincarè return map.* Following his lead, by writing the Feynman integral for $\mu(t)$ and using the translation invariance of the 'Lebesgue measure' $\mathcal{D}c$, we can formally compute

$$\begin{aligned}\mu(t)([\gamma]) &= \int_{c \sim \gamma} e^{-S(c)} \mathcal{D}c \\ &\sim_{t \to 0} \int_{\text{fluctuations } \chi} e^{-S(\gamma) - 1/2\ \delta^2 S(\chi)} \mathcal{D}\chi \\ &= e^{-S(\gamma)} \int_{\chi \in T_\gamma M} e^{-1/2\ \delta^2 S(\chi)} \mathcal{D}\chi,\end{aligned}$$

where $S(c(t)) = 1/2 \int_0^t |\dot{c}(t)|^2\, dt$ is the *action* of the curve $c(t)$ and $\delta^2 S$ is its Hessian matrix along γ. While the derivation is purely formal, the final expression makes perfect sense. For simplicity, suppose $|\dot\gamma| = 1$. Now with the help of a parallel orthonormal frame along γ (say the first vector is $\dot\gamma$) and letting $K(t)$ denote the curvature along γ, we obtain

$$\int_{\chi\in T_\gamma M} e^{-1/2\ \delta^2 S(\chi)}\mathcal{D}\chi$$
$$= \int_{x(0)=x(t)} e^{-1/2\int_0^t \dot{x}^2\,dt}\mathcal{D}x \int_{y(0)=y(t)} e^{-1/2\int_0^t \dot{y}^2(t)-K(t)y^2(t)\,dt}\mathcal{D}y.$$

These Feynman integrals admit two interpretations. As $e^{-1/2\int_0^t \dot{x}^2\,dt}\mathcal{D}x$ defines a Brownian motion, by grouping the kinetic terms and the Lebesgue measure together, the path integrals become mathematically meaningful. In particular, the first integral can be interpreted as the volume of the homotopically trivial loops for the Brownian motion on $\mathbb{R}/\ell_0(\gamma)\mathbb{Z}$. By the previous example this equals $\ell_0(\gamma)/\sqrt{2\pi t}$.

The second interpretation is to note that the integrands are Gaussian, which should mean that

$$\int_{\chi\in T_\gamma M} e^{-1/2\ \delta^2 S(\chi)}\mathcal{D}\chi = \frac{1}{\sqrt{\det(-\frac{d^2}{dt^2})}}\frac{1}{\sqrt{\det(-\frac{d^2}{dt^2}-K)}},$$

where some regularization of the determinants appearing on the right is necessary. The choice which keeps the consistency between the various interpretations is the $\zeta-$function regularization (dealing with *zero-modes* appropriately). Using this method, basic facts about Riemann's zeta function show that the first factor is $\ell_0(\gamma)/\sqrt{2\pi t}$, which agrees with the first interpretation. Hence, the following relation

$$\frac{1}{\sqrt{\det(I-P(\gamma))}} = \int_{\Omega^t(\mathbb{R})} e^{-1/2\int_0^t K(s)y^2(s)\,ds}\mu(t)(dy) = \frac{1}{\sqrt{\det_\zeta(-\frac{d^2}{dt^2}-K)}} \tag{1.1.6}$$

should hold; i.e. the WKB, path-integral, and field theoretic expressions agree. There are now many proofs of this equation; we will give a brief sketch of one approach. Denoting the three terms above as WKB, GAUSSIAN, and DETERMINANT respectively, we write 1.1.6 as

$$\text{WKB} = \text{GAUSSIAN} = \text{DETERMINANT}.$$

Sketch. Without loss of generality, we may suppose $t = 1$. Recall that the $\zeta-$ determinant of an unbounded self-adjoint operator A with a purely discrete spectrum $\{\lambda_j\}$ is (assuming analytic continuation is possible)

$$\det{}_\zeta(A) \stackrel{\text{def}}{=} e^{-\zeta'(0)}, \quad \text{where } \zeta(s) = \sum_{j=0}^{\infty} \lambda_j^{-s}.$$

When $K(t)$ is constant, one verifies that $\zeta(s) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}(4\pi^2 n^2 - K)^{-s}$ has $\zeta'(0) = 2\log(2\sinh(\frac{\sqrt{-K}}{2}))$. As $\sqrt{\det(I-P)} = 2\sinh(\frac{\sqrt{-K}}{2})$, we have WKB = DETERMINANT when $K(t) < 0$ is constant. Also, by the Feynman-Kac formula for the harmonic oscillator,

$$\text{GAUSSIAN} = \operatorname{tr} e^{1/2(\frac{d^2}{dy^2}+Ky^2)} = \sum_{n=0}^{\infty} e^{-(n+1/2)\sqrt{-K}}$$
$$= \frac{e^{\sqrt{-K}/2}}{1-e^{\sqrt{-K}}} = \frac{1}{2\sinh(\sqrt{-K}/2)}.$$

Hence WKB=GAUSSIAN=DETERMINANT whenever $K(t) < 0$ is constant. For $K(t) = \text{Const} + zq(t)$, standard techniques of Sturm-Liouville theory [11] and Gaussian processes [16] show that $\text{WKB}^{-2}(z)$, $\text{GAUSSIAN}^{-2}(z)$ and $\text{DETERMINANT}^{-2}(z)$ extend to be entire functions of order $1/2$ with the same zeros (the standard argument uses Fredholm determinants) [4]; hence they are equal for all $z \in \mathbb{C}$ by Hadamard's theorem. $\square$

Upon close inspection, we observe that none of these arguments make use of Brownian motion in a nontrivial fashion. On the other hand, Gelfánd and Yaglom [7] suggest such a method for proving that WKB=GAUSSIAN, which was first introduced by Cameron and Martin in 1945 [2] (albeit for a different purpose). With Itô's formula, we give a simple description of how it works.

Asume again that $t = 1$. Lift K to $\mathbb{R}$ and let ϕ solve $-K = (\phi')^2 + \phi''$. (ϕ exists whenever $K < 0$; we discuss the solvability of the *Ricatti* equation in chapter 3.) A direct computation shows that $\sqrt{\det(I-P)} =$

$2\sinh(\frac{\phi(0)-\phi(1)}{2})$. Letting δ denote Dirac's delta-function and taking our cue from the Cameron-Martin Formula, we write

$$\int_{\Omega^1(\mathbb{R})} e^{-1/2\int_0^t K(s)y^2(s)\,ds}\mu(1)(dy)$$
$$= \int_{\mathbb{P}(\mathbb{R})} e^{-\int_0^1 \phi'(t)y(t)dy(t) - 1/2\int_0^1 \phi''(t)y^2(t)dt}\delta(y(1)-y(0)) \times$$
$$\times\, e^{\int_0^1 \phi'(t)y(t)dy(t) - 1/2\int_0^1 \phi'^2(t)y^2(t)dt} d\mu(y).$$

By Itô's formula,

$$\int_0^1 \phi'(t)y(t)dy(t) = 1/2\int_0^1 \phi'(t)d(y^2(t)-t)$$
$$= 1/2[\phi(0)-\phi(1)] + 1/2[\phi'(1)y^2(1) - \phi'(0)y^2(0)] - 1/2\int_0^1 \phi''(t)y^2(t)dt.$$

Since ϕ' is periodic, we obtain

$$\int_{\Omega^1(\mathbb{R})} e^{-1/2\int_0^t K(s)y^2(s)\,ds}\mu(1)(dy)$$
$$= e^{1/2\,(\phi(1)-\phi(0))}\int_{\mathbb{P}(\mathbb{R})} \delta(y(1)-y(0))e^{\int_0^1 \phi'(t)y(t)dy(t) - 1/2\int_0^1 \phi'^2(t)y^2(t)dt} d\mu(y).$$

Recognizing the integrand as the Jacobian of the map

$$y \mapsto Y(t) = y(t) - \int_0^t \phi'(t)y(t)dt,$$

computing the integral above reduces to inverting this map. It is easy to check that the inverse is given by [7]

$$Y \mapsto y(t) = Y(t) + e^{\phi(t)}\int_0^t e^{-\phi(s)}\phi'(s)Y(s)ds;$$

hence

GAUSSIAN

$$= e^{1/2\,(\phi(1)-\phi(0))} \int_{\mathbb{P}(\mathbb{R})} \delta(y(Y)(1) - Y(0))\, d\mu(Y)$$

$$= e^{1/2\,(\phi(1)-\phi(0))} \int_{\mathbb{P}_0(\mathbb{R})} \int_{-\infty}^{\infty} \delta((1 - e^{\phi(0)-\phi(1)})y + \text{independent of } y)\; dy\, d\mu_0(Y)$$

$$= \frac{e^{1/2\,(\phi(1)-\phi(0))}}{1 - e^{\phi(0)-\phi(1)}} = \frac{1}{\sqrt{\det(1-P)}} = \text{ WKB.}$$

One can view our approach in the thesis as an extension of this method to obtain an exact expression for $\mu(t)$. The results are contained in section 4.2. Takhtajan suggests that a further extension might be used to investigate the analog of 1.1.6 in dimension 2

$$\frac{1}{\sqrt{\Gamma_\infty(s) \prod_{\mathfrak{p}} \prod_{k=0}^{\infty} 1 - e^{-\ell(\mathfrak{p})(s+k)}}} = \frac{1}{\sqrt{\det_\zeta(-\Delta_M + s(1-s))}}$$

$$= \int_{\{\psi\}} e^{-1/2\, s(1-s) \int_M \psi^2 dV} \times e^{1/2 \int_M \psi \Delta_M \psi dV}\, \mathcal{D}\psi. \quad (1.1.7)$$

Here Γ_∞ is a known expression involving Barnes' double-gamma function [12]. For the first equality, see [3] [15]. What is absent from the literature is a *good* definition of the Gaussian integral above and a *proof* of the second equality (typically the Gaussian integral above is *defined* by this equation). In fact, Takhtajan has suggested that a direct method similar to the Cameron-Martin technique described by Gelfánd and Yaglom might be possible in the region where the product in 1.1.7 converges. This remains a conjecture; we are content with developing a bit of the "local theory" in this thesis.

When is the semiclassical approximation exact?

Literally speaking, the answer to this question in this context is "never". However, Gutzwiller noticed that for surfaces of constant negative curvature K, the semiclassical approximation is off by a simple factor $e^{Kt/8}$. (The role and nature of this *quantum correction* has not been adequately explained in the literature. We believe that the methods here paint a reasonable picture of its origin.) Colloquially, the Gutzwiller approximation

$$\mu(t)([\gamma]) \sim_{t\to 0} \frac{e^{-S(\gamma)}}{\sqrt{\det_\zeta \delta^2 S}} e^{K(\gamma)t/8}, \quad \text{where } K(\gamma) = \frac{1}{\ell(\gamma)} \int_0^t K(\gamma(t))dt,$$

is exact for metrics of constant curvature.

It is natural to ask how good this approximation is for general metrics and whether or not a converse statement holds. Although the Gutzwiller approximation is very good for metrics with a certain symmetry property (see corollary 2 and theorem 4 in chapter 3), for general metrics it is no better than the zeroth order (semiclassical) approximation. Nevertheless, it seems plausible that the converse might hold for these metrics as well. Since the correction term is analytic in t, if it were exact $\mu(t)(\Omega^t)$ would depend only upon the germ of the metric along γ (see theorem 4.2) by considering the full asymptotic series for $\mu(t)$ as $t \to 0$. We are not aware of an argument which completes this observation to prove the converse for the general case. The converse for the special case is a consequence of corollary 2.

Chapter 2

Preliminaries

2.1 Trace Formula

Let (M, ds^2) be a compact Riemann surface of strictly negative curvature $K < 0$. Let $\mathcal{F}$ be a fundamental domain for the action of $G = \pi_1(M)$ on the unversal cover $\mathbb{R}^2$. Lifting the metric on M to $\mathbb{R}^2$, it is well-known that the heat kernel p_t on M is obtained from the heat kernel $\tilde{p}_t$ on $(\mathbb{R}^2, ds^2)$ via the method of images

$$p_t(x, y) = \sum_{g \in G} \tilde{p}_t(\tilde{x}, g\tilde{y})$$

where $\tilde{x}$ and $\tilde{y}$ are the lifts of x, y lying in $\mathcal{F}$.

For fixed $g \in G$ define the stabilizer of g by

$$\Gamma_g \stackrel{\text{def}}{=} \{ g' \in G \mid gg' = g'g \} \cong \mathbb{Z}.$$

We take $\mathfrak{p}(g)$ to be the generator of Γ_g that satisfies $\mathfrak{p}^n = g$ for $n > 0$. Since

$$\tilde{p}_t(\tilde{x}, \tilde{y}) = \tilde{p}_t(g\tilde{x}, g\tilde{y}) \qquad \forall g \in G,$$

Selberg computes as follows [20] (here $[g]$ is conjugacy class of g)

$$\int_M p_t(x,x)dV(x) = \sum_{g\in G} \int_{\mathcal{F}} \tilde{p}_t(\tilde{x}, g\tilde{x})dV(\tilde{x}) \tag{2.2.1}$$

$$= \sum_{g'\in[g]} \sum_{G/\Gamma_g} \int_{g'\mathcal{F}} \tilde{p}_t(\tilde{x}, g\tilde{x})dV(\tilde{x}) \tag{2.2.2}$$

$$= \int_{\mathcal{F}} \tilde{p}_t(\tilde{x}, \tilde{x})dV(\tilde{x}) + \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \int_{\bigcup_{g\in G/\langle\mathfrak{p}\rangle} g\mathcal{F}} \tilde{p}_t(\tilde{x}, \mathfrak{p}^n\tilde{x})dV(\tilde{x}). \tag{2.2.3}$$

We observe that $\mathcal{F}_{\mathfrak{p}} \stackrel{\text{def}}{=} \bigcup_{G/\langle\mathfrak{p}\rangle} g\mathcal{F}$ is a fundamental domain for the action of $\langle\mathfrak{p}\rangle$ on $\mathbb{R}^2$. Letting $\{\lambda_j\} = \text{spec}(-\Delta_M)$, we have the trace formula

$$\sum_{j=0}^{\infty} e^{-\lambda_j t/2} = \int_{\mathcal{F}} \tilde{p}_t(\tilde{x}, \tilde{x})dV(\tilde{x}) + \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \int_{\mathcal{F}_{\mathfrak{p}}} \tilde{p}_t(\tilde{x}, \mathfrak{p}^n\tilde{x})dV(\tilde{x}). \tag{2.2.4}$$

Recall that two closed curves on M are freely homotopic if and only if their lifts to $\mathbb{R}^2$ define conjugate elements in G (this is independent of all the necessary choices). This means that precisely the same manipulations employed in 2.2.1-2.2.3 work for computing the measure of any cylindrical set 1.1.4. Comparing the resulting expressions, we have

Theorem 1. *There is a natural isomorphism between the measure theory of the loops on (M, ds^2) in the free homotopy class defined by $[\mathfrak{p}^n]$ and the loops* *that wind n-times around the cylinder* $(\mathbb{R}^2/\langle\mathfrak{p}\rangle, ds^2)$. *Furthermore, letting* $\gamma(\mathfrak{p}^n)$ *denote the unique closed geodesic on* M *whose free-homotopy class* $[\gamma(\mathfrak{p}^n)]$ *corresponds to the conjugacy class* $[\mathfrak{p}^n]$, *we have*

$$\int_{\mathcal{F}_\mathfrak{p}} \tilde{p}_t(\tilde{x}, \mathfrak{p}^n\tilde{x})dV(\tilde{x}) = \mu(t)([\gamma(\mathfrak{p}^n)]). \tag{2.2.5}$$

Notation. Throughout this thesis, γ will denote a *closed geodesic* on M, $\ell(\gamma)$ is its *length*, and $\ell_0(\gamma)$ its *primitive length*. In terms of $\mathfrak{p}$, we have $\ell_0(\gamma(\mathfrak{p}^n)) = \ell(\gamma(\mathfrak{p}))$.

It is convenient to choose a covering map specifically tailored to $\mathbb{R}^2/\langle\mathfrak{p}\rangle$. We use horocyclic coordinates for this purpose. As we do not compare different $\mathfrak{p}$'s (and this letter is already overused in this thesis) we will supress it in the remainder.

2.2 Horocyclic coordinates

Identifying the lift of a closed geodesic γ with the x-axis, the metric in horocyclic coordinates takes the form (cf. [1])

$$ds^2 = dx^2 + e^{2\phi(x,y)}dy^2, \quad \phi(0,y) = 0;$$

the Gaussian curvature is $K = -(\phi_x^2 + \phi_{xx})$. Such coordinates always exist when $K \leq 0$ and are easily constructed. As the 1-form $-dr$ is harmonic in any set of geodesic polar coordinates, fix any geodesic on $\mathbb{R}^2$ and consider the family of 1-forms on $\mathbb{R}^2$ constructed by exponentiating $-dr$ at each point along the geodesic. One readily verifies that these forms converge locally uniformly as the point along the geodesic goes to infinity. The limiting 1-form is harmonic (hence smooth) and corresponds to dx above (of course, $dz = dx + \sqrt{-1}e^{\phi}dy$ is the underlying complex structure). The smoothness of ϕ is immediate.

Setting $\phi(0, y) = 0$ fixes a scale for the horocycles (or *spheres at infinity*) $\{x = \text{ const.}\}$, and we adopt this convention throughout.

Proposition 1. *Every orientation-preserving isometry that preserves the x-axis is of the form*

$$(x, y) \mapsto (x - l, \psi(y))$$

for some $l \in \mathbb{R}$ and ψ a diffeomorphism of $\mathbb{R}$. Furthermore, we can express ψ in terms of ϕ:

$$\psi(y) = \int_0^y e^{\phi(l,\tau)} d\tau.$$

Proof. Any isometry fixing the x-axis fixes the horocycles $\{x = \text{const}\}$ as well as the vector field $A_1 = \partial/\partial x$; hence it is determined by its effect on any single horocycle. This shows that the isometry is of the stated form $(x, y) \mapsto (x - l, \psi(y))$. As for the expression for ψ, note that

$$dx^2 + e^{2\phi(x-l,\psi(y))}\psi'(y)^2 dy^2 = dx^2 + e^{2\phi(x,y)}dy^2 \Rightarrow \psi'(y) = e^{\phi(l,y)}.$$

□

Let $\sigma = e^{-\phi}$ and define the vector fields

$$A_1 \stackrel{\text{def}}{=} \frac{\partial}{\partial x}, \quad A_2 \stackrel{\text{def}}{=} \sigma\frac{\partial}{\partial y}, \quad 2B \stackrel{\text{def}}{=} \phi_x\frac{\partial}{\partial x}.$$

Then $\{A_1, A_2\}$ form an orthonormal frame and $2B = A_1^2 + A_2^2 - \Delta$. The Christoffel symbols for the covariant derivative of the Levi-Civita connection on the tangent bundle are

$$\Gamma_{11}{}^1 = \Gamma_{11}{}^2 = \Gamma_{12}{}^1 = \Gamma_{21}{}^1 = 0$$

$$\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = \phi_x$$

$$\Gamma_{22}{}^1 = -\phi_x e^{2\phi}, \Gamma_{22}{}^2 = \phi_y.$$

Therefore

$$\nabla_{A_1}A_1 = 0, \quad \nabla_{A_1}A_2 = \phi_x A_2, \quad \nabla_{A_2}A_2 = -2B. \tag{2.2.6}$$

2.3 Geometric Bounds

Taking $[0, \ell_0(\gamma)) \times \mathbb{R}$ as fundamental domain, the map

$$(x, y) \mapsto (x - \ell_0(\gamma), \int_0^y e^{\phi(\ell_0(\gamma),\tau)} d\tau) \tag{2.2.7}$$

is an isometry of $(\mathbb{R}^2, dx^2 + e^{2\phi} dy^2)$, and the vector fields $A_1, A_2, 2B$ remain invariant.

Since $g = e^{\phi}$ is a positive decreasing solution to $g_{xx} + Kg = 0, g(0, y) = 1$, the standard comparison technique leads to the following

$$0 < a^2 \leq -K \leq b^2 \text{ everywhere} \quad \Rightarrow \quad 0 < a \leq -\phi_x \leq b \text{ everywhere.} \tag{2.2.8}$$

In particular this implies that for any positive $c < a$, there is a constant A with $g(x, y) \leq Ae^{-cx}$; A is independent of y.

Proposition 2. *Let $0 < c < a \leq \sqrt{-K} \leq b$. Suppose that all covariant derivatives of K are uniformly bounded with respect to ds^2. Then there exist constants $A = A(n, c)$ such that the following inequality holds for $x > 0, n \geq 0$:*

$$|\frac{\partial^n}{\partial y^n} e^{\phi}(x, y)| \leq A(n, c)e^{-cx}. \tag{2.2.9}$$

Proof. As we have already shown this estimate for $n = 0$, we will prove it for general n by induction. Assuming 2.2.9 is true for each $n < N$, the function $G \stackrel{\text{def}}{=} e^{cx}\frac{\partial^N}{\partial y^N}e^{\phi}$ satisfies

$$G_{xx} - 2cG_x + (c^2 + K)G = F, G(0, y) = 0.$$

It follows from the induction hypothesis (and the uniform bounds for K and its covariant derivatives) that F is smooth and bounded for $x > 0$. Suppose for contradiction that G is unbounded; then (suppressing y)

$$E \stackrel{\text{def}}{=} \{\, x > 0 : G_x/G(x) > 0, (a^2 - c^2)|G(x)| > |F| \,\}$$

is open and nonempty. Now $(x, y) \in E$ implies $(x+r, y) \in E$ for every $r > 0$, and $\lim_{x\to\infty} |G(x, c)| = \infty$ if E intersects the line $y = c$. Choose $y_1 < y_2$ such that $y_1 < c < y_2$ implies $E \cap \{(x, c) \mid x > 0\}$ is nonempty. This means that

$$\lim_{x\to\infty} |\int_{y_1}^{y_2} G(x, y)\, dy| = \infty.$$

Since this contradicts the induction hypothesis, G must be bounded for $x > 0$. $\square$

Expressing the derivatives of $\sigma = e^{-\phi}$ in terms of e^{ϕ}, we easily obtain the following corollary.

Corollary 1. *Let* $\sigma = e^{-\phi}$, $0 < c < a \leq \sqrt{-K} \leq b$. *Suppose that all covariant derivatives of* K *are uniformly bounded with respect to* ds^2. *Then there exist constants* $B = B(n,c)$ *such that the following inequality holds for all* $x > 0, n \geq 0$*:*

$$|\frac{\partial^n}{\partial y^n}\sigma(x,y)| \leq B(n,c)e^{bx+n(b-c)x}. \tag{2.2.10}$$

For $f(y)$ satisfying $y = \int_0^{f(y)} e^{\phi(\ell,\tau)}d\tau$, we have from 2.2.7

$$\sigma(x, f(y))e^{\phi}(\ell, f(y)) = \sigma(x-\ell, y). \tag{2.2.11}$$

Since $f'(y) = \sigma(\ell, f(y))$, the following *geometric bounds* follow immediately from the proposition and its corollary.

Theorem 2. *Let* $\sigma = e^{-\phi}$ *and suppose that* $0 \leq a < \sqrt{-K} \leq b$, *that all covariant derivatives of* K *are uniformly bounded with respect to* ds^2, *and that 2.2.7 is an isometry. Then there exist constants* $R(n), r(n)$ *such that the following inequality holds for all* $(x,y) \in \mathbb{R}^2, n \geq 0$*:*

$$|\frac{\partial^n}{\partial y^n}\sigma(x,y)| \leq Re^{r|x|}. \tag{2.2.12}$$

2.4 Basic Setup

For convenience of the reader, we collect notations and minimal assumptions necessary for the remainder of the thesis.

γ a closed geodesic of M which lifts to the x-axis,

$ds^2 = dx^2 + e^{2\phi}dy^2$ has $\phi(0, y) = 0$,

$K = -\phi_{xx} - \phi_x^2$ has uniformly bounded covariant derivatives in ds^2,

$\ell(\gamma) = n\ell_0(\gamma)$ relates length and primitive length of γ,

$\alpha = -\phi(\ell_0, 0)$ satisfies $\sqrt{1 - P_\gamma} = 2\sinh\frac{n\alpha}{2}$,

$\sigma = e^{-\phi}$,

$(x, y) \mapsto (x - \ell, \int_0^y e^{\phi(\ell,\tau)}d\tau)$ is an isometry.

Chapter 3

Special Case

3.1 Feynman-Kac formula

In this chapter, we focus attention on the special case $\frac{\partial \phi}{\partial y} = 0$. We make no claim that nontrivial examples of such metrics arise from compact Riemann surfaces. Rather, the results in this section should be viewed as a detailed investigation of the volume of loop space over cylinders which posess a special symmetry.

Theorem 3 (Feynman-Kac formula). *For $t > 0$, the following equation holds:*

$$\mu(t)([\gamma]) = \frac{e^{\ell(\gamma)^2/2t}}{2\sinh(n\alpha/2)\sqrt{2\pi t}} \int_0^{\ell_0(\gamma)} \mathbb{E}(e^{-t/2 \int_0^1 q(x+s\ell(\gamma)+\sqrt{t}\omega(s))\,ds})\,dx \quad (3.3.1)$$

where $q = T(\phi/2)$, $\ell_0(\gamma)$ is the length of the primitive geodesic generating γ,

$n = \ell(\gamma)/\ell_0(\gamma)$, and $\mathbb{E}$ denotes expectation with respect to pinned Brownian motion $\omega(0) = \omega(1) = 0$.

Proof. Let $p_t(x_1, y_1; x_2, y_2)$ denote the symmetric fundamental solution of the heat equation on $\mathbb{R}^2$ generated by

$$1/2 \ \cdot \ \text{the Laplacian for the metric } ds^{\,2} = dx^{\,2} + e^{2\phi(x)}\, dy^{\,2}.$$

Translations in the y-direction preserve the metric and hence preserve p_t. In fact, the expression

$$\tilde{p}_t(x_1, x_2) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} p_t(x_1, 0; x_2, z) e^{\phi(x_2)}\, dz$$

is the (non-symmetric) fundamental solution for the the one-dimensional Smoluchowski equation [10] $u_t = \frac{1}{2} u_{xx} + \frac{\phi'}{2} u_x$. If $f(x)$ is a smooth bounded function, this means

$$u_t = \frac{1}{2} u_{xx} + \frac{\phi'}{2} u_x, \quad u|_{t=0} = f \quad \Longrightarrow \quad u(t, x_1) = \int_{-\infty}^{\infty} \tilde{p}_t(x_1, x_2) f(x_2)\, dx_2.$$

However, it is a well-known fact [10] [16] that the fundamental solution for the Smoluchowski equation can be expressed in terms of pinned Brownian motion:

$$\tilde{p}_t(x_1, x_2) = \frac{e^{-(x_2 - x_1)^2/2t}}{\sqrt{2\pi t}} e^{\frac{1}{2}(\phi(x_1) - \phi(x_2))}\, \mathbb{E}\big(e^{-t/2 \int_0^1 q(x_1 + s(x_2 - x_1) + \sqrt{t}\omega_s)\, ds}\big) \quad (3.3.2)$$

where $q \stackrel{\text{def}}{=} T(\phi/2)$. With these facts, the rest of the proof follows from straightforward change of variables; with $n = \ell(\gamma)/\ell_0(\gamma)$,

$$\begin{aligned}
\mu(t)([\gamma]) &= \int_0^{\ell_0(\gamma)} \int_{-\infty}^{\infty} p_t(x, y; x + \ell(\gamma), e^{n\alpha} y) e^{\phi(x)}\, dy\, dx \\
&= \int_0^{\ell_0(\gamma)} \int_{-\infty}^{\infty} p_t(x, 0; x + \ell(\gamma), (e^{n\alpha} - 1) y) e^{\phi(x)}\, dy\, dx \\
&= \frac{1}{e^{n\alpha} - 1} \int_0^{\ell_0(\gamma)} \int_{-\infty}^{\infty} p_t(x, 0; x + \ell(\gamma), z) e^{\phi(x)}\, dz\, dx \\
&= \frac{1}{1 - e^{-n\alpha}} \int_0^{\ell_0(\gamma)} \int_{-\infty}^{\infty} p_t(x, 0; x + \ell(\gamma), z) e^{\phi(x + \ell(\gamma))}\, dz\, dx.
\end{aligned}$$

The last line follows from the *fact* that $\phi(x + \ell(\gamma)) + n\alpha = \phi(x)$. After substituting 3.3.2 for the z-integration, we can apply this fact once again to obtain 3.3.1. □

Corollary 2. *We have the following bounds, with equality on either side if and only if $q = T(\phi/2)$ is constant:*

$$\begin{aligned}
\frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}} \frac{\ell_0(\gamma)}{2\sinh(n\alpha/2)} \int_0^1 e^{-tq(x)/2}\, dx \quad &\geq \mu_t([\gamma]) \\
\mu_t([\gamma]) &\geq \frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}} \frac{\ell_0(\gamma)}{2\sinh(n\alpha/2)} e^{tK(\gamma)/8}. \qquad (3.3.3)
\end{aligned}$$

Proof. Both bounds follow from Jensen's inequality. We argue only the lower bound as the proof for the upper bound is similar. We have

$$\frac{1}{\ell_0(\gamma)} \int_0^{\ell_0(\gamma)} \mathbb{E}(e^{-\frac{t}{2}\int_0^1 q(x+s\ell(\gamma)+\sqrt{t}\omega(s))\,ds})\,dx$$
$$\geq e^{-\frac{t}{2\ell_0(\gamma)}\int_0^{\ell_0(\gamma)} \mathbb{E}(\int_0^1 q(x+s\ell(\gamma)+\sqrt{t}\omega(s))\,ds)\,dx}.$$

Since $q \in C^\infty(\mathbb{R}/\ell_0(\gamma)\mathbb{Z})$, switching the order of integration gives the lower bound because

$$\frac{1}{\ell_0(\gamma)} \int_0^{\ell_0(\gamma)} q(x)\,dx = -\frac{K(\gamma)}{4}.$$

If the lower bound is attained, we argue that this means $\int_0^1 q(x+s\ell(\gamma)+\sqrt{t}\omega(s))\,ds$ is independent of (x,ω). To this, we apply $\frac{\partial^2}{\partial x \partial \sqrt{t}}$ in different orders and obtain

$$0 = \int_0^1 q''(x+s\ell(\gamma)+\sqrt{t}\omega(s))\omega(s)\,ds.$$

Now taking $t \searrow 0$, that this equation holds for all ω implies first that $q''(x) = 0$; but since q is periodic, this forces $q'(x) = 0$ as well.

We complete the argument by proving $\int_0^1 q(x+s\ell(\gamma)+\sqrt{t}\omega(s))\,ds$ is independent of (x,ω) whenever the lower bound is achieved. If the lower bound is attained for q, it is also attained for $q - c$ for any constant c. In particular, by taking $c = q_{max}$, we may assume $q \leq 0$ without loss of generality. Set

$$A \stackrel{\text{def}}{=} -\frac{t}{2}\int_0^1 q(x + s\ell(\gamma) + \sqrt{t}\omega(s))\,ds \geq 0 \tag{3.3.4}$$

$$d\nu \stackrel{\text{def}}{=} d(\text{pinned Brownian motion}) \times dx/\ell_0(\gamma). \tag{3.3.5}$$

Since A is continuous, it suffices to prove A is constant ae(ν). By considering the Maclaurin expansion of exp, Hölder's inequality implies

$$\int A^n\,d\nu = (\int A\,d\nu)^n$$

for every $n > 0$. In particular, with $n = 2$ we see that A is constant ae(ν), as desired. □

Another important consequence of the Feynman-Kac formula is that we can study the invariance and asymptotics of $\mu(t)$ through the right side of 3.3.1 and in particular through q (it turns out that the right side depends on q only through the spectrum of $Q = -\frac{d^2}{dx^2} + q$ on $L^2(\mathbb{R}/\ell_0(\gamma)\mathbb{Z})$). To obtain short-time asymptotics, one replaces the exponential function by its Maclaurin series and expands q in a Taylor series at $x + \ell(\gamma)s$. This reduces the problem to computing certain moments of the pinned Brownian motion, the result of which is contained in item (1) of the following theorem.

Theorem 4. *Let $K(\gamma)$=mean value of K along γ.*

1. *$\mu(t)$ posesses an asymptotic expansion of the form*

$$\mu(t)([\gamma]) = \frac{e^{-\ell(\gamma)^2/2t}}{2\sinh(n\alpha/2)\sqrt{2\pi t}}(1 + A_1 t + A_2 t^2 + \ldots)$$

with $A_1 = K(\gamma)/8$, $A_2 = K(\gamma)^2/128$.

2. *There is a (generically infinite-dimensional) torus of distinct metrics on the cylinder $\mathbb{R}^2/\langle \mathfrak{p} \rangle$ with the same $\mu(t)([\gamma^n])$, for all n, t.*

Remark. In particular, item (1) shows that, to order t^2, the short-time asymptotics of $\mu(t)$ are the same as those of the lower bound in 2 (the third-order terms differ). In the remainder of this chapter, we focus on the proof of item (2). We need to introduce the standard notations in the theory of Hill's equation, some of which conflict with our current ones. To simplify matters, throughout the next section we will assume $\ell_0(\gamma) = 1$ and make no explicit references to μ_t nor to its associated Laplace operator.

3.2 Hill's equation and the isospectral flow

Throughout this section, we will use the following conventions:

$\phi = \phi(x) \in C^\infty(\mathbb{R})$	$S^1 = \mathbb{R}/\mathbb{Z}$
$\ell_0(\gamma) = 1$	$q = T(\phi/2) \in C^\infty(S^1)$
$T(\phi) \stackrel{\text{def}}{=} \phi'' + \phi'^2$	$K = -T(\phi) \in C^\infty(S^1)$
$\phi(0) = 0$	$Q \stackrel{\text{def}}{=} -\dfrac{d^2}{dx^2} + q$
$\alpha = \phi(x) - \phi(x+1)$ is constant	$L \stackrel{\text{def}}{=} -\dfrac{1}{2}\dfrac{d^3}{dx^3} + q\dfrac{d}{dx} + \dfrac{d}{dx}q$

Proposition 3. *Suppose $\phi \in C^\infty(\mathbb{R})$ satisfies $\phi(x) = \alpha + \phi(x+1)$. Let $g = e^{-\phi}$. Then the following diagram*

$$\begin{array}{ccc} C^\infty(S^1) & \xrightarrow{-\frac{d^2}{dx^2}+T(\phi)} & C^\infty(S^1) \\ {\scriptstyle \cdot g}\big\downarrow & & \big\downarrow{\scriptstyle \cdot g} \\ C^\infty(S^1)\cdot e^{-\alpha x} & \xrightarrow{-\frac{d^2}{dx^2}-2\phi'\frac{d}{dx}} & C^\infty(S^1)\cdot e^{-\alpha x} \end{array}$$

commutes.

Proof. Direct computation. □

Lemma 1. *The Ricatti Transform* T *is injective when viewed as a map*

$$T : \{\, \phi \in C^\infty(\mathbb{R}) \mid \phi(0) = 0,\ \phi(x) = \alpha + \phi(x+1) \,\} \longrightarrow C^\infty(S^1).$$

Proof. Suppose $T(f) = T(g)$ and consider $h \stackrel{\text{def}}{=} f - g \in C^\infty(S^1)$. Then h' solves the differential equation $(h')' = (f' + g')h'$, and therefore $h' =$

$C \exp(f+g)$. Integrating over a period gives $C = 0$ and proves the lemma.

□

We need a convenient characterization of the image. The main point is that $g = e^{\phi/2}$ satisfies $g(x+1) = e^{-\alpha/2} g(x)$, $g'' = T(\phi/2)g$. The spectral theory of Hill's equation has much to say about this situation.

3.2.1 The Discriminant Δ

For $q \in C^\infty(S^1)$, let $q' = dq/dx$, $Q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$, and $L \stackrel{\text{def}}{=} -\frac{1}{2}\frac{d^3}{dx^3} + q\,\frac{d}{dx} + \frac{d}{dx}\,q$. With $\lambda \in \mathbb{C}$, consider a complex basis (y_1, y_2) in $C^\infty(\mathbb{R}, \mathbb{C})$ for $\ker(Q - \lambda)$ such that the matrix

$$M(x,\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ y_1'(x,\lambda) & y_2'(x,\lambda) \end{pmatrix}$$

satisfies $M(0,\lambda) = I$. Notice that this forces $\det(M(x,\lambda)) = 1$. Define the *discriminant* $\Delta(\lambda) \stackrel{\text{def}}{=} \operatorname{tr} M(1,\lambda) = y_1(1,\lambda) + y_2'(1,\lambda)$.

Lemma 2. *y_1, y_2 and Δ are entire functions of λ of order* $1/2$.

Proof. If $\bar\partial$ denotes the Cauchy-Riemann operator (with repspect to λ), then $Q\bar\partial M = \lambda \bar\partial M$. Since $\bar\partial M(0,\lambda) = 0$, by uniqueness we must have $\bar\partial M(x,\lambda) = 0$. The proof that the order of these functions is $1/2$ comes from the Picard iteration scheme [11] and is omitted.

□

Lemma 3. *Fix $a \in \mathbb{R}$. There exists a solution to $Qg = \lambda g$ with $g(x) = e^a g(x+1)$ if and only if $\Delta(\lambda) = 2\cosh a$.*

Proof. If g is a solution and $g(x) = e^a g(x+1)$, writing g as a linear combination of y_1 and y_2 and evaluating at $x = 0, 1$ gives $\det(M(1,\lambda) - e^{-a}) = 0 \Leftrightarrow \Delta(\lambda) = 2\cosh a$. To get the converse, reading this argument in reverse will produce a $g \neq 0$ that solves $Qg = \lambda g$ with $g(1) = e^a g(0)$ and $g'(1) = e^a g'(0)$. By the periodicity of q, $e^a g(x+1) \in \ker(Q - \lambda)$ and has the same initial conditions as does g. □

Theorem 5. *The solutions of $\Delta(\lambda) = 2$ form the spectrum of Q acting on $L^2(S^1)$. If λ_0 is the least of these, then*

1. $\lambda_0 \geq q_{min} = \min_{0 \leq x \leq 1} q(x)$

2. $\Delta(\lambda) > 2$ *whenever* $\lambda < \lambda_0$

3. *if* $\lambda \leq \lambda_0$, *then* $Qg = \lambda g$ *admits a positive solution.*

Proof. Since $\int_0^1 f(x)(Q - q_{min})f(x)\,dx$ is nonnegative and Q is essentially self-adjoint on $C^\infty(S^1)$, (1) is immediate. Now

$$\lambda \leq q_{min} \Rightarrow y_1(x,\lambda) \geq 1, y_2(1,\lambda) > 1,$$

thus $\Delta(\lambda) > 2$. Since the least root of $\Delta - 2$ is λ_0, this proves (2).

Suppose $\lambda < \lambda_0$, then by (2) and 3 we have a basis for $\ker(Q - \lambda)$ of the form $(e^{ax}f_+(x), e^{-ax}f_-(x))$, where $f_+, f_- \in C^\infty(S^1)$.

Claim. $g = c_+e^{ax}f_+(x) + c_-e^{-ax}f_-(x)$ *is positive for some choice of* c_+, c_- *iff both* f_+, f_- *are root free.*

The converse is trivial; to argue the contrapositive, for definiteness say $a > 0$, f_+ has roots, and f_- is positive. (If both had roots, any g would have roots also by examining the asymptotics.) The roots are simple by uniqueness of solutions for ODE; since f_+ is periodic, it must have at least one root where f_+' is negative and another where f_+' is positive. But the Wronskian W of our basis evaluated at a root r of f_+ is $-f_+'(r)f_-(r)$. The existence of these two roots contradicts the fact that $W' = 0$; this proves the claim.

Let $S = \{\, \lambda < \lambda_0 \mid Qg = \lambda g$ admits a positive solution$\,\}$; we have already seen that $(-\infty, q_{min}] \subset S$, in particular $S \neq \emptyset$. For $\lambda_* \in S$ we may assume the positive solution is of the form $g = e^{a_* x}f_*(x)$ for $f_* \in C^\infty(S^1)$

with $f(1) = 1$. Since $\lambda_* < \lambda_0$, $a_* \neq 0$ by (2). The implicit function theorem allows us to express $a = a(\lambda)$ and hence $f = f(\lambda)$ as continuous (in fact analytic) functions for λ near λ_0. Because f_* has no roots, neither will $f(\lambda)$ for λ sufficiently close to λ_*. This proves that S is open; we will now argue that $S \cup \{\lambda_0\}$ is closed.

It is easy to see from this argument that any limit point λ of S must admit a nonnegative solution g of $Qg = \lambda g$ with $g(0) = 1$ and $g(x) = e^a g(x+1)$, $2\cosh a = \Delta(\lambda)$. If x is a root of g, then $g(x) = g'(x) = 0$, and by uniqueness of solutions $g = 0$. As this contradicts the fact that $g(0) = 1$, we see that $g > 0$ and S is closed. Therefore $S = (-\infty, \lambda_0]$ and the proof of the theorem is complete.

□

Corollary 3. *Fix $\alpha \geq 0$. The map $\phi \mapsto T(\phi/2)$ is a diffeomorphism between* $\{\, \phi \in C^\infty(\mathbb{R}) \mid \phi(0) = 0,\ \phi(x) = \alpha + \phi(x+1) \,\}$ *and*

$$\{\, q \in C^\infty(S^1) \mid \Delta(0) = 2\cosh \alpha/2,\ \lambda_0 > 0 \,\}.$$

Proof. Since $\int_0^1 f \cdot Qf(x)\,dx = \int_0^1 (f' - \frac{1}{2}\phi' f)^2\,dx > 0$ for $0 \neq f \in C^\infty(S^1)$ and $q = T(\phi/2)$, by 1 we need only check that the range of this map is as advertised. (The smoothness of T is obvious from its definition; it will follow from the implicit function theorem -using suitable Sobolev norms- that its inverse is smooth as well. We omit the standard argument.)

Suppose $\lambda_0(q) > 0$. Then by theorem 5 we have a positive solution $g = e^{\phi/2}$ satisfying $Qg = 0$, $g(0) = 1$, $g(x) = e^{\alpha/2} g(x+1)$ with $\Delta(0) = 2\cosh\alpha/2$. In terms of ϕ, this writes as $q = T(\phi/2)$, $\phi(0) = 0$, $\phi(x) = \alpha + \phi(x+1)$. □

Corollary 4. *λ_0 is a simple eigenvalue with a corresponding positive eigenfunction.*

Proof. The cone of solutions to $Qg = \lambda_0 g$ that are positive on $[0, 1]$ contains a unique ray of periodic solutions interior to the cone (take $\alpha = 0$ in the previous corollary.) As any line parallel to the ray must meet this cone, the line through any periodic solution must contain this ray. □

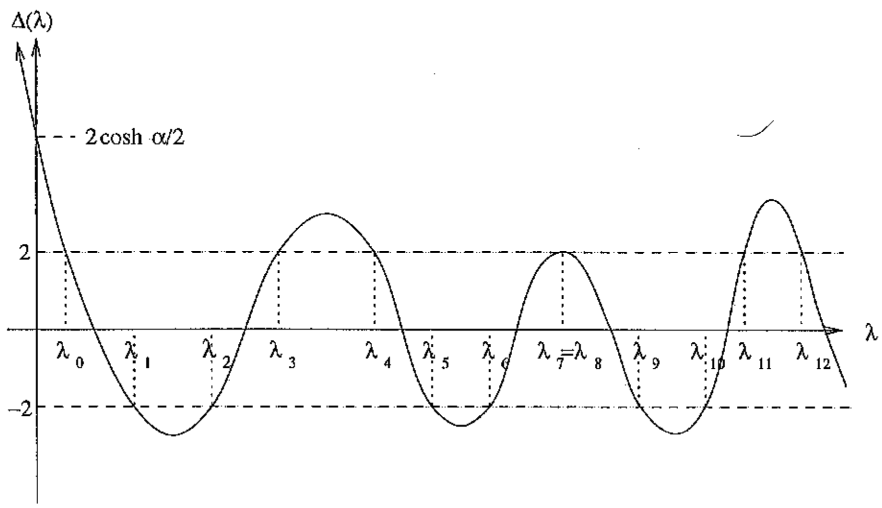

Figure 3.1: sketch of $\lambda \mapsto \Delta(\lambda)$ for a typical $q = T(\phi/2)$

Let $M = M(q) \stackrel{\text{def}}{=} \{\, \tilde{q} \mid \Delta(q) = \Delta(\tilde{q}) \,\}$ denote the *isospectral torus* of q. Let us briefly justify this terminology. $\{\lambda_j\}$ denotes the *periodic spectrum* of Q, i.e.

$$\{\lambda_j\} \stackrel{\text{def}}{=} \{\, \lambda \mid Qf = \lambda f \text{ admits a periodic solution} \,\} = \{\, \lambda \mid \Delta^2(\lambda) - 4 = 0 \,\}$$

The roots of $\Delta(q) - 2$ form the spectrum of Q on $L^2(S^1)$, and

$$\Delta(q) = C \prod_{\substack{j=0,3 \\ \text{mod } 4}} 1 - \frac{\lambda}{\lambda_j}$$

since $\Delta(q)$ is of order $1/2$ (C is determined by $\Delta \sim 2\cos\sqrt{\lambda}$ as $\lambda \to -\infty$). Therefore $\Delta(q) = \Delta(\tilde{q})$ iff Q and $\tilde{Q}$ have the same spectrum on $L^2(S^1)$.

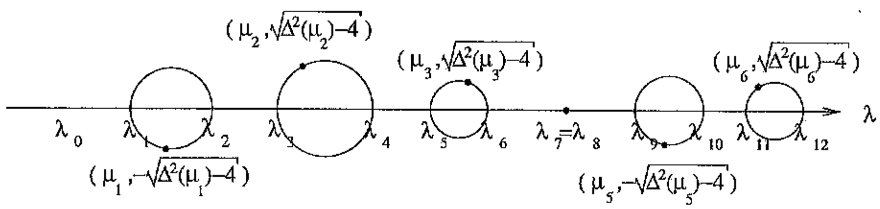

Figure 3.2: parametrization of the isospectral torus for above $q = T(\phi/2)$

The parametrization of M as in [14], [13] is presented here. Let $\{\mu_j\}$ denote the spectrum of the *Dirichlet* problem for Q, i.e.

$$\{\mu_j\} \stackrel{\text{def}}{=} \{\, \mu \mid Qf = \mu f \text{ admits a solution with } f(0) = f(1) = 0 \,\}$$

$$= \{\, \mu \mid y_2(1, \mu) = 0 \,\}$$

With a bit of linear algebra, one can show that $\lambda_{2j-1} \leq \mu_j \leq \lambda_{2j}$. Moreover, $\Delta(\mu_j) = y_2'(1,\mu_j) + 1/y_2'(1,\mu_j)$ can be solved for $y_2' = 1/2(\Delta \pm \sqrt{\Delta^2 - 4})$.

The ambiguity in the sign of the radical turns out to play a decisive role. Borg essentially showed that $q \mapsto (\{\lambda_j\}, \{(\mu_j, \pm_j \sqrt{\Delta^2(\mu_j) - 4})\})$ with $y_2'(1,\mu_j) = 1/2(\Delta(\mu_j) \pm_j \sqrt{\Delta^2(\mu_j) - 4})$ is injective. By fixing the periodic spectrum $\{\lambda_j\}$, we get a parametrization of M. In [14], [13] the authors prove that this map is a homeomorphism between M and the torus of figure 3.2. In particular, M is compact and bounded in every Sobolev norm on $C^\infty(S^1)$.

Geometry of the isospectral torus Q

It is helpful to view $\Delta = \Delta(\lambda, q)$ as a family of functionals (=functions of q) parametrized by λ. For any two functionals F and G, the *Poisson-Gardner* bracket is

$$\{F, G\} \stackrel{\text{def}}{=} \int_0^1 \frac{\delta F}{\delta q(x)} \frac{d}{dx} \frac{\delta G}{\delta q(x)}\, dx,$$

where the variational derivatives are defined by

$$\left.\frac{d}{dt}\right|_{t=0} F(q + t\dot{q}) = \int_0^1 \frac{\delta F}{\delta q(x)} \dot{q}(x)\, dx. \tag{3.3.6}$$

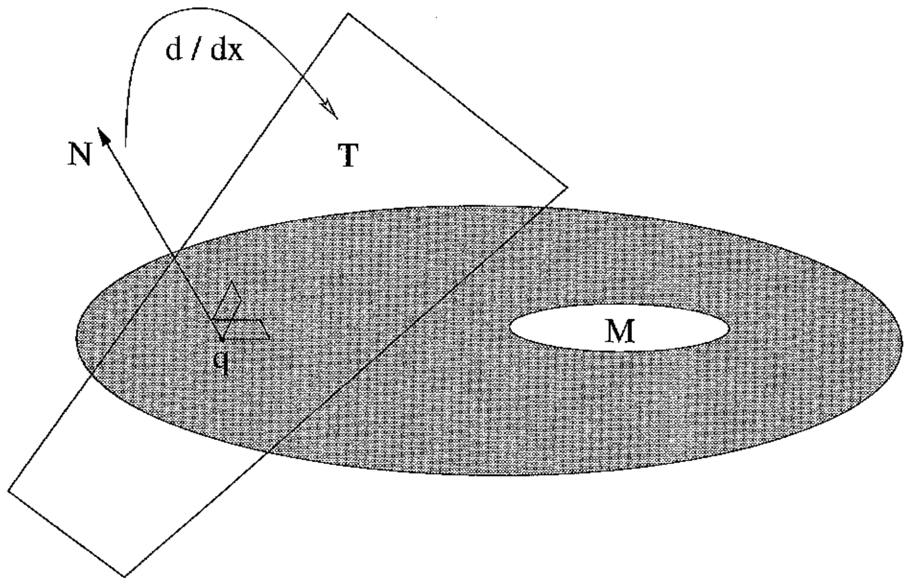

Figure 3.3: d/dx carries N to T

Among other things, we will show Δ is a commuting family of Hamiltonians, whose flow is confined to the isospectral tori M which partition $C^\infty(S^1)$.

Lemma 4. *Let $p_t(x,y)$ denote the fundamental solution for $u_t = -Qu$ on $L^2(\mathbb{R})$. With $L \stackrel{def}{=} -\frac{1}{2}\frac{d^3}{dx^3} + q\,\frac{d}{dx} + \frac{d}{dx}\,q$ and $n \in \mathbb{Z}$,*

$$-2\frac{\partial^2}{\partial x \partial t} p_t(x, x+n) = Lp_t(x, x+n)$$

Proof. Define

$$f_n(t,x) \stackrel{\text{def}}{=} \begin{cases} p_t(x,x) - \frac{1}{\sqrt{4\pi t}}, & \text{if } n = 0, \\ p_t(x, x+n), & \text{otherwise.} \end{cases} \tag{3.3.7}$$

It suffices to check the equation for $f_n(t)$. Note that $f_n(t,x) \to 0$ as $t \to 0$; hence, after applying the Laplace transform to both sides and integrating by parts, we need to verify that following equation is true for all n:

$$2\lambda \frac{d}{dx} g_\lambda(x, x+n) = L g_\lambda(x, x+n).$$

Here g_λ is the Green's function for $Q - \lambda$, the standard construction of which shows that $g_\lambda(x, x+n)$ is a quadratic polynomial in $y_1(\lambda), y_2(\lambda)$. It is straightforward to check the following

Fact.

$$[L - 2\lambda \frac{d}{dx}]\ (c_{11} y_1 y_1 + c_{12} y_1 y_2 + c_{22} y_2 y_2) = 0. \tag{3.3.8}$$

□

3.2.2 Some calculus

Extend the pinned Brownian motion to $C(\mathbb{R})$ by declaring $w(s)$ to be periodic. Now observe that the map $w \mapsto w(\cdot + \sigma) - w(\sigma)$ is a measure-theoretic isomorphism for all values of σ. (This is easily checked by computing the covariance $C(s,t) = (1-t)s, 0 < s < t < 1$.) Hence

$$\mathbb{E}(e^{-\tau \int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\, ds}) = \mathbb{E}(e^{-\tau \int_0^1 q(x+ns+\sqrt{\tau}[\omega(s+\sigma)-\omega(\sigma)])\, ds}) \tag{3.3.9}$$

Proposition 4.

$$\frac{\delta}{\delta q(x)}\int_0^1 \mathbb{E}(e^{-\tau\int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\,ds})\,dx = -\tau\mathbb{E}(e^{-\tau\int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\,ds})$$

Proof.

$$-\frac{1}{\tau}\frac{d}{dt}\bigg|_{t=0} LHS(q+t\dot{q}) = \tag{3.3.10}$$

$$= \mathbb{E}(\int_0^1\int_0^1 \dot{q}(x+n\sigma+\sqrt{\tau}\omega(\sigma))\,d\sigma\; e^{-\tau\int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\,ds}\,dx) \tag{3.3.11}$$

$$= \mathbb{E}(\int_0^1\int_0^1 \dot{q}(x+n\sigma+\sqrt{\tau}\omega(\sigma))e^{-\tau\int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\,ds}\,dx\,d\sigma) \tag{3.3.12}$$

$$= \int_0^1\int_0^1 \dot{q}(x)\mathbb{E}(e^{-\tau\int_0^1 q(x+n(s-\sigma)+\sqrt{\tau}[\omega(s)-\omega(\sigma)])\,ds})\,dx\,d\sigma. \tag{3.3.13}$$

$$\tag{3.3.14}$$

Since the integrand in the exponential is periodic in s, we may change s to $s+\sigma$ without affecting the result. The proposition follows immediately from 3.3.6, 3.3.9. □

The following proposition is from [13].

Proposition 5.

$$\frac{\delta\Delta(\lambda)}{\delta q(x)} = [y_2'(1,\lambda) - y_1(1,\lambda)]y_1(x,\lambda)y_2(x,\lambda) + \\ - y_1'(1,\lambda)y_2^2(x,\lambda) + y_2(1,\lambda)y_1^2(x,\lambda) \tag{3.3.15}$$

This is precisely $y_2(1,\lambda)$ computed for the function $q(\cdot + x)$.

Remark. $\frac{\delta\Delta(\lambda)}{\delta q}$ lies in the kernel of $L - 2\lambda d/dx$ by 3.2.1.

Proposition 6. *Let* $F_n(\tau, q) \stackrel{def}{=} \mathbb{E}(e^{-\tau \int_0^1 q(x+ns+\sqrt{\tau}\omega(s))\,ds})$.

$$\{F_m(\sigma), F_n(\tau)\} = 0 \tag{3.3.16}$$

$$\{\Delta(\lambda), \Delta(\mu)\} = 0 \tag{3.3.17}$$

$$\{F_n(\tau), \Delta(\lambda)\} = 0 \tag{3.3.18}$$

Proof. The relevant variational derivatives are smooth periodic functions, and $L = -\frac{1}{2}\frac{d^3}{dx^3} + q\,\frac{d}{dx} + \frac{d}{dx}\,q$ is skew-symmetric on $C^\infty(S^1)$. We exploit this fact now.

Now $\{F_m(\sigma), F_n(\tau)\}$ is proportional to $\int_0^1 f_m(\sigma, x)\frac{d}{dx}f_n(\tau, x)\,dx$ (see 3.3.7). This expression is 0 if $\tau = 0$ or $\sigma = 0$, and otherwise satisfies

$$\begin{aligned}
\frac{d}{d\epsilon}\int_0^1 f_m(\sigma+\epsilon)\frac{d}{dx}f_n(\tau+\epsilon)\;dx &= \int_0^1 \frac{\partial f_m}{\partial \sigma}\frac{\partial f_n}{\partial x} - f_m\frac{\partial^2 f_n}{\partial \tau \partial x}\;dx \\
&= -\int_0^1 \frac{\partial^2 f_m}{\partial \sigma \partial x} f_n + f_m \frac{\partial^2 f_n}{\partial \tau \partial x}\;dx \\
&= \frac{1}{2}\int_0^1 Lf_m\, f_n + f_m\, Lf_n\;dx \text{ by 3.3.7} \\
&= 0 \text{ by the skew-symmetry of } L.
\end{aligned}$$

Taking $\epsilon \searrow -\min(\sigma, \tau)$ proves 3.3.16.

$$\begin{aligned}
2\mu\{\Delta(\lambda), \Delta(\mu)\} &= 2\mu \int_0^1 \frac{\delta\Delta(\lambda)}{\delta q(x)} \frac{d}{dx} \frac{\delta\Delta(\mu)}{\delta q(x)}\, dx \\
&= \int_0^1 \frac{\delta\Delta(\lambda)}{\delta q(x)}\, L \frac{\delta\Delta(\mu)}{\delta q(x)}\, dx \\
&= -\int_0^1 L \frac{\delta\Delta(\lambda)}{\delta q(x)} \frac{\delta\Delta(\mu)}{\delta q(x)}\, dx \\
&= -2\lambda\{\Delta(\mu), \Delta(\lambda)\} \\
&= 2\lambda\{\Delta(\lambda), \Delta(\mu)\}.
\end{aligned}$$

Hence $\{\Delta(\lambda), \Delta(\mu)\} = 0$ and 3.3.17 is verified. Similar methods show

$$\begin{aligned}
\frac{d}{d\epsilon} \int_0^1 \frac{\delta\Delta(\lambda)}{\delta q(x)} \frac{d}{dx} f_n(\tau + \epsilon)\, dx &= -\frac{1}{2} \int_0^1 \frac{\delta\Delta(\lambda)}{\delta q(x)}\, L f_n(\tau + \epsilon)\, dx \\
&= \lambda \int_0^1 \frac{\delta\Delta(\lambda)}{\delta q(x)} \frac{d}{dx} f_n(\tau + \epsilon)\, dx.
\end{aligned}$$

Taking $\epsilon \searrow -\tau$ proves 3.3.18. □

Theorem 6 (McKean-Trubowitz). *The Hamiltonian vector field*

$$q \mapsto \frac{d}{dx} \frac{\delta\Delta(\lambda)}{\delta q(x)}$$

induces a smooth flow in $C^\infty(S^1)$; this flow preserves M.

Corollary 5 (McKean-Trubowitz). *The (non-Hamiltonian) vector field*

$$q \mapsto \frac{d}{dx} \frac{\delta\Delta(\lambda)}{\delta q(x)}\bigg|_{\lambda = \mu_j}$$

induces a smooth flow in $C^\infty(S^1)$; this flow preserves M. Under this flow, μ_i satisfies

$$\frac{d}{dt}\mu_i(q(t)) = \begin{cases} \frac{1}{2}\sqrt{\Delta^2(\mu_j) - 4} \text{ for } i = j, \\ 0 \text{ otherwise }. \end{cases}$$

Remark. Note that the vector field is not a symplectic gradient since μ_j depends on q. Explicitly,

$$\frac{\delta\Delta(\mu_j)}{\delta q(x)} = \left.\frac{\delta\Delta}{\delta q(x)}\right|_{\mu_j} + \frac{\partial\Delta}{\partial\lambda}\frac{\delta\mu_j}{\delta q(x)}$$

We deduce that $\mathbb{E}(\exp[-\tau\int_0^1 q(x+ns+\sqrt{\tau}\omega(s))]\,ds)$ is constant on M.

3.3 Proof of theorem 4

We can now complete the proof of theorem 4. Fix ϕ, α and compute $q = T(\phi/2)$, $\Delta(q)$. Since we have characterized both the range of T and the isospectral manifold $M(q)$ in terms of Δ, we conclude that M lies in the image of T and every point in the preimage of M equips every loop space over the cylinder with the same total volume.

3.4 Zeta function

We conclude this chapter with some remarks on an associated (Selberg-type) local zeta function. To simplify the formulas, we fix ϕ, α as above and suppose $\ell_0(\gamma) = \ell(\gamma) = 1$.

For $\Re\lambda \leq \lambda_0$ put $\nu(\lambda) \stackrel{\text{def}}{=} 1/2(\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}) \leq 1$ and define

$$\mathcal{Z}(\lambda) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} 1 - \nu(\lambda) e^{-\alpha(k+1/2)}.$$

Then $\mathcal{Z}'/\mathcal{Z}$ is the Laplace transform of $\sum_{n>0} \mu(t)([\gamma^n])$.

Proof.

$$\frac{\mathcal{Z}'}{\mathcal{Z}} = \frac{d\nu}{d\lambda} \sum_{k=0}^{\infty} \frac{e^{-\alpha(k+1/2)}}{1 - \nu e^{-\alpha(k+1/2)}} \tag{3.3.19}$$

$$= \frac{d\nu}{d\lambda} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \nu^n e^{-\alpha(n+1)(k+1/2)} \tag{3.3.20}$$

$$= \frac{d\nu}{d\lambda} \sum_{n=1}^{\infty} \nu^{n-1} \sum_{k=0}^{\infty} e^{-\alpha n(k+1/2)} \tag{3.3.21}$$

$$= \frac{d\nu}{d\lambda} \sum_{n=1}^{\infty} \frac{\nu^{n-1} e^{-\alpha n/2}}{1 - e^{-\alpha n}} \tag{3.3.22}$$

$$= \sum_{n=1}^{\infty} \frac{\nu^{n-1}}{2 \sinh n\alpha/2} \frac{d\nu}{d\lambda}. \tag{3.3.23}$$

The n-th summand is the Laplace transform of $\mu(t)([\gamma^n])$; for details see [18]. □

Chapter 4

General Case

In this chapter we develop the general formula for $\mu(t)([\gamma])$ from the viewpoint acquired in the previous chapter. Namely, we seek an expression of the form

$$\mu(t)([\gamma]) = \frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}} \int_0^{\ell_0(\gamma)} \mathbb{E}(F(\sqrt{t}, x, w^1))\, dx$$

where once again $\mathbb{E}$ denotes expectation with respect to pinned Brownian motion. As in the special case, we expect F to be adapted, smooth in (w^1, x), and it should result from integrating out the w^2 dependence. Our goal in this chapter is to obtain an explicit representation for F. In preparation, we construct a 'pathwise' solution to the SDE $dY_t(w^2) = \sigma(\cdot, Y_t(w^2))dw^2$.

4.1 Pathwise solution

Theorem 7. *Let $\mathbb{E}_2$ denote expectation with respect to the canonical Wiener process w_t^2, with w^2 independent of w^1. For every x, y, w^1, there exists a unique strong solution $Y \stackrel{def}{=} Y(t, x, y, w^1, w^2)$ to the SDE*

$$dY_t = \sigma(x + \ell t + w_t^1, Y_t)dw_t^2, \qquad Y_0 = y \tag{4.4.1}$$

on $(\mathcal{W}_2, P_2, \bar{\mathcal{B}_t})$ which has the following additional properties:

1. *For fixed x, y, $Y : [0,1] \times \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{W}_1) \bar{\times} \mathcal{B}(\mathcal{W}_2)$ measurable and adapted to $\mathcal{B}_t \bar{\times} \mathcal{B}_t$. The overbar denotes the completion with respect to $P_1 \times P_2$.*

2. *If $\|v^1\|, \|w^1\| \leq r$, then*

   $$\mathbb{E}_2(\sup_{s \leq t} |Y_s(\xi, y, v^1) - Y_s(x, y, w^1)|^2) \leq K\|(v^1 + \xi) - (w^1 + x)\|^2 e^{Ar+Bt}$$

   for constants A, B, K depending only on σ.

3. *$t \mapsto Y_t$ is a uniformly bounded continuous curve in $L^2(P_1 \times P_2)$.*

4. *$Y : [0,1] \times \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$ is the strong solution of the SDE 4.4.1 on $\mathcal{B}_t \bar{\times} \mathcal{B}_t, P_1 \times P_2$.*

5. *$P_2(\{y \mapsto Y_t(x, y, w^1, w^2)$ is a diffeomorphism $\forall t \leq 1\}) = 1$.*

6. $(x, y, w^1) \mapsto Y_t(x, y, w^1, \cdot) \in C^\infty(\mathbb{R}^2 \times \mathcal{W}, D^\infty(\mathbb{W}))$ *is uniformly nondegenerate.*

Proof. On the space of progressively measurable functions $F : [0,1] \times \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$, set

$$\|F\|_r(t) \stackrel{\text{def}}{=} \sup_{\|w^1\| \leq r} (\mathbb{E}_2(\sup_{0 \leq s \leq t} F_s^2))^{1/2}. \tag{4.4.2}$$

We will construct Y using the Picard method on $\mathcal{W}_2$. The key estimate, acquired from Doob's martingale theory and Itô's formula, is the following [9]:

$$\mathcal{I}(F)_t \stackrel{\text{def}}{=} \int_0^t \sigma(x + w_s^1, F_s) dw_s^2 \implies$$
$$\|\mathcal{I}(F) - \mathcal{I}(G)\|_r^2(t) \leq 4 \sup_{x,y} |\sigma_y(x,y)|^2 \int_0^t \|\mathcal{F} - \mathcal{G}\|_r^2(s)\, ds \tag{4.4.3}$$

(This is also a trivial application of Burkholder's inequality with $p = 2$.) Upon iteration, with $M = 4 \sup_{(x,y)} |\sigma_y(x,y)|^2$

$$\|\mathcal{I}^n(F) - \mathcal{I}^n(G)\|_r^2(t) \leq \frac{M^n t^n}{n!} \|F - G\|_r^2(t). \tag{4.4.4}$$

This shows that for fixed w^1, the Picard iteration scheme Y^n on $\mathcal{W}_2$ starting with $Y^1 = y$ is a Cauchy sequence in every $\|\cdot\|_r$. Let

$$E = \{\, (w^1, w^2) \mid Y^n(w^1, w^2) \text{ converges uniformly on } [0,1] \,\}$$

and define

$$Y_t(w_1,w_2) \stackrel{\text{def}}{=} \begin{cases} \lim_{n\to\infty} Y_t^n(w_1,w_2), & (w^1,w^2)\in E \\ 0, & \text{otherwise.} \end{cases}$$

From 4.4.4, the Borel-Cantelli theorem shows that $P_2(E(w^1,\cdot)) = 1$ for every $w^1 \in \mathcal{W}$, and hence E has full measure on the product by the Fubini theorem. As the interates constructed above constitute a version of the usual Picard iterates on $\mathcal{W}_1 \times \mathcal{W}_2$, Y constitutes a strong solution [9] to the SDE 4.4.1 as viewed on either $\mathcal{W}_2$ with w^1 fixed, or on the full product $\mathcal{W}_1 \times \mathcal{W}_2$.

From our geometric bounds of chapter 1, it follows that there exist positive constants satisfying $me^{rx} \le \sigma_x(x+\ell, y) \le Me^{Rx}$. Observe that by applying Burkholder's inequality once again, we get the following estimate

$$\mathbb{E}_2\big(\sup_{0\le s\le t} |Y_s(v^1) - Y_s(w^1)|^2\big)^{1/2} \le Kt\, e^{A\max(\|v^1(s)\|,\|w^1\|)}\|v^1 - w^1\| + \\ M^{1/2}\Big(\int_0^t \mathbb{E}_2\big(\sup_{0\le s\le t}|Y_s(v^1) - Y_s(w^1)|^2\big)\,ds\Big)^{1/2}. \quad (4.4.5)$$

Squaring both sides and using $(a+b)^2 \le 2(a^2+b^2)$, Gronwall's inequality implies that for $0 \le t \le 1$

$$\mathbb{E}_2\big(\sup_{0\le s\le t}|Y_s(v^1) - Y_s(w^1)|^2\big) \le 2K^2\|v^1 - w^1\|^2 e^{2A\max(\|v^1\|,\|w^1\|)+2Mt}. \quad (4.4.6)$$

In particular, $Y : [0,1]\times\mathcal{W}_1 \to L^2(\mathcal{W}_2, P_2)$ is continuous with at worst exponential growth in w^1. By Fernique's theorem, we conclude that

$$\mathbb{E}_1 \mathbb{E}_2 (\sup_{0 \le t \le 1} Y_t^2) < \infty. \tag{4.4.7}$$

As σ_y is uniformly bounded, we may appeal to the uniqueness theorem for strong solutions of SDE's with Lipschitz coeffecients to see that any other solution must agree in probability with Y at first on $\{\|w^1\| \le r\}$ and then a.e. by taking $r \to \infty$. We have now established (1)-(4) completely.

As to the remaining items, (5) is a restatement of the diffeomorphism theorem in this setting [9]. That $Y_t(y, w^1) \in D^\infty(\mathcal{W}_2)$ is a well-known (see [22],[21]) consequence of the uniform bounds on the derivatives of σ in any strip $\{|x| < r\}$. That this map is C^∞ is an immediate consequence of the approximation theorem for solutions to SDE's, together with (2). The standard method of variation of parameters shows that the derivatives, if they existed, would constitute solutions to linear SDE's. The coefficients are of lower order and satisfy uniform Lipschitz estimates in a neighborhood of y, w^1.

For induction, we assume that this is so for directional derivatives of order $\le n$. Let F be an n-th order derivative, and formally differentiate the SDE satisfied by F to obtain G. The approximation theorem [9] says that the polygonal approximations to SDE's with Lipschitz coeffecients will converge to the solution in every $\|\cdot\|_{p,s}$ norm. The polygonal approximations

F_n to F are easily seen to be differentiable, and their derivatives constitute polygonal approximations to G. Now $F_n \to F$ and $\delta F_n \to G$ implies F is differentiable with derivative G, whenever the F_n's are uniformly Lipschitz. The induction hypothesis shows that this is precisely the sitation. Together with the standard estimate (again via Gronwall's inequality) [17]

$$\mathbb{E}_2\big(\sup_{0\leq s\leq t} |Y_s(y_1, w^1) - Y_s(y_2, w^1)|^2\big) \leq \text{const} \cdot |y_1 - y_2|^2,$$

(2) shows that the induction hypothesis is satisfied for $n = 0$.

It remains to prove the uniform nondegeneracy of Y_t. Suppressing for the moment y, w^1, recall that the derivative $DY_t : \mathcal{W}_2 \to \mathcal{H}$ satisfies

$$L^2 - \lim_{a\to 0} \frac{Y_t(w + ah) - Y_t(w)}{a} = \langle DY_t, h\rangle_{\mathcal{H}} \quad \forall h \in \mathcal{H}.$$

Letting Ξ_t^h denote this directional derivative of Y_t with respect to h, we can write down as SDE for Ξ

$$d\Xi_t^h = \sigma_y \Xi_t^h dw^2 + \sigma \dot{h} dt. \tag{4.4.8}$$

Since $\Xi_0^h = 0$, by uniqueness of solutions we must have

$$\begin{aligned}\Xi_t^h &= \int_0^t e^{\int_s^t \sigma_y dw^2 - 1/2 \int_s^t \sigma_y^2 d\tau} \sigma \dot{h}\, ds \\ &= \int_0^1 \dot{h}(s)\ \chi_{\{s\leq t\}}\ \sigma e^{\int_s^t \sigma_y dw^2 - 1/2\int_s^t \sigma_y^2 d\tau}\, ds. \end{aligned} \tag{4.4.9}$$

Hence, as an element of $\mathcal{H}$ the gradient of Y_t should be given by the curve

$$DY_t(s) = \int_0^{\min(s,t)} \sigma \exp\,(\int_s^t \sigma_y dw^2 - 1/2 \int_s^t \sigma_y^2 du) d\tau, \tag{4.4.10}$$

and in fact it is so. The proof again employs the polygonal approximations [21]; the key difference between the argument above and the argument needed to complete the induction step here is that *no* Lipschitz estimate is possible. Indeed, Y_t is not even continuous in w_1. One instead needs to prove exactly that the gradient D (or equivalently the so-called number operator) is *closed.* For full details, see e.g.[21].

The Malliavin covariance matrix is easily computed to be

$$C_t \stackrel{\text{def}}{=} \langle DY_t, DY_t \rangle_{\mathcal{H}} = \int_0^t \sigma^2 e^{2\int_s^1 \sigma_y dw^2 - \int_s^1 \sigma_y^2 dt} ds. \tag{4.4.11}$$

To check nondegeneracy, recall from the geometric bounds of chapter 2 that σ_y and σ are bounded uniformly whenever x lies in a bounded set. Therefore

$$\mathbb{E}_2((C_t)^{-p}) \leq R^{2p} e^{rp\|w^1\|}, \tag{4.4.12}$$

for some *geometric* constants r, R. This expression is P_1-integrable by Fernique's theorem. □

By judicious placement of ϵ in the theorem and its proof, we can similarly construct the solution Y^ϵ to the equation

$$Y_t^\epsilon(x, y, w^1, w^2) = \int_0^y e^{\phi(\ell, \epsilon\tau)}\, d\tau + \int_0^t \sigma(x + s\ell + \epsilon w_s^1, \epsilon Y_s^\epsilon)\, dw_s^2. \quad (4.4.13)$$

Corollary 6. *For $\epsilon \in \mathbb{R}$, define Y^ϵ as the solution to 4.4.13 constructed via Picard iteration as in theorem 7. In addition to the conclusions of theorem 7, we also have*

1. $\epsilon \mapsto Y_t^\epsilon(x, y, w^1) \in C^\infty(\mathbb{R}, D^\infty) \quad \forall t, x, y, w^1.$

2. *Y_t^ϵ is nondegenerate in the sense of 4.4.12 uniformly on $|\epsilon| < R$.*

3. $P_2(\{Y_t^{-\epsilon}(x, y, w^1, w^2) = -Y_t^\epsilon(x, -y, -w^1, -w^2) \quad \forall t\}) = 1.$

Proof.

$$C_t^\epsilon = \langle DY_t^\epsilon, DY_t^\epsilon \rangle_{\mathcal{H}} = \int_0^t \sigma^2 e^{2\epsilon \int_s^1 \sigma_y dw^2 - \epsilon^2 \int_s^1 \sigma_y^2 dt} ds.$$

□

4.2 Formula for $F(\epsilon, x, w^1)$

Let δ be Dirac's delta function on $\mathbb{R}$, define Y_t^ϵ as in corollary 6 and put

$$X_t^\epsilon(x, w^1) = x + \ell t + \epsilon w_t^1, \qquad V = \frac{1}{2}(\phi_{xx} + \sigma\sigma_{yy}) + \frac{1}{4}(\phi_x^2 + \sigma_y^2).$$

Main Theorem. *Define* $F \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathcal{W}_1)$ *by*

$$F(\epsilon, x, w^1) \stackrel{def}{=} \int_{-\infty}^{\infty} e^{\phi(\ell,\epsilon y)/2} \mathbb{E}_2 \left(e^{\epsilon \int_0^1 \sigma_y(X_s^\epsilon, \epsilon Y_s^\epsilon) dw_s^2 - \frac{\epsilon^2}{2} \int_0^1 V(X_s^\epsilon, \epsilon Y_s^\epsilon) ds} \delta(y - Y_1^\epsilon)\right) dy. \tag{4.4.14}$$

Then we have

$$\mu(t)([\gamma]) = \frac{e^{-\ell(\gamma)^2/2t}}{\sqrt{2\pi t}} \int_0^{\ell_0(\gamma)} \mathbb{E}(F(\sqrt{t}, x, w^1))\, dx.$$

Moreover, $\mathbb{E}(F(\epsilon, x)) = \mathbb{E}(F(-\epsilon, x))$ *and* $\mathbb{E}(F(\epsilon, x))$ *posesses an asymptotic expansion in* ϵ^2 *as* $\epsilon \to 0$*. Explicitly*

$$\mathbb{E}(F(\epsilon, x)) \sim \frac{1}{2\sinh(n\alpha/2)}\left(1 + a_1(x)\frac{\epsilon^2}{2} + a_2(x)\frac{\epsilon^4}{4!} + \cdots\right),$$

where the a_i*'s are expressible as certain integrals of the germ of* σ *along the* x*-axis. In particular,*

$$a_1(x) = c_0 + c_1 \int_0^1 \sigma^2(x + \ell s, 0) ds + c_2 \int_0^1 \sigma\sigma_y(x + \ell s, 0) ds + \\ + c_3 \int_0^1 \sigma_y^2(x + \ell s, 0) ds + c_4 \int_0^1 \sigma\sigma_{yy}(x + \ell s, 0) ds, \tag{4.4.15}$$

and constants c_i *are given by*

1. $c_0 = \frac{\phi_{yy}}{2} + \frac{\phi_y^2}{4} + \frac{K(\gamma)}{4}$

2. $c_1 = \frac{e^{2\phi}}{(1-e^{\phi})^4}(3\phi_y^2 + \phi_{yy})$

3. $c_2 = \frac{2e^{\phi}}{(1-e^{\phi})^2}(3\phi_y e^{3\phi} - 6\phi_y e^{2\phi} + (3\phi_y - 1)e^{\phi} + 2\phi_y + 1)$

4. $c_3 = \frac{2e^{2\phi}}{(1-e^{\phi})^2} + \frac{3}{4}$

5. $c_4 = \frac{-1}{2(1-e^{\phi})^2}(e^{2\phi} - 6e^{\phi} + 1)$

Here ϕ and its derivatives are evaluated at $(\ell, 0)$.

Proof. First we will assume $\frac{\partial^n}{\partial y^n}\sigma$ is bounded for each $n > 0$. Now the Brownian motion on $(\mathbb{R}^2, dx^2 + e^{2\phi}dy^2)$ is the image of $dP_1 \times dP_2$ under the solution $(\mathfrak{x}(w^1, w^2), \mathfrak{y}(w^1, w^2))$ of the SDE

$$\begin{aligned} d\mathfrak{x}_t &= dw_t^1 + \frac{1}{2}\phi_x(\mathfrak{x}_t, \mathfrak{y}_t)dt \\ d\mathfrak{y}_t &= \sigma(\mathfrak{x}_t, \mathfrak{y}_t)[dw_t^2 + \frac{1}{2}\sigma_y(\mathfrak{x}_t, \mathfrak{y}_t)dt]. \end{aligned} \tag{4.4.16}$$

Equivalently, by the Girsanov-Cameron-Martin theorem, it is the image of $ZdP_1 \times dP_2$ under the solution of the SDE

$$\begin{aligned} dX_t &= dw_t^1 \\ dY_t &= \sigma(X_t, Y_t)dw_t^2. \end{aligned} \tag{4.4.17}$$

where

$$Z_t = \exp[1/2 \int_0^t \phi_x(X_s, Y_s)dw_s^1 + 1/2 \int_0^t \sigma_y(X_s, Y_s)dw_s^2 - 1/8 \int_0^t \phi_x^2(X_s, Y_s) + \sigma_y^2(X_s, Y_s)\, ds]. \tag{4.4.18}$$

Once again, we can rewrite this using Itô 's formula to obtain

$$Z_t = \exp(\frac{1}{2}[\phi(X_t, Y_t) - \phi(X_0, Y_0)] + \int_0^t \sigma_y(X_s, Y_s) dw_s^2 - \frac{1}{2}\int_0^t V(X_s, Y_s) ds). \quad (4.4.19)$$

To solve 4.4.17, we may take $X_t = x + w_t^1$ and Y_t as constructed in theorem 7; thus Z_t admits a "pathwise" version (with respect to w^1) just as Y_t does. Therefore, the heat kernel $\tilde{p}$ for $(\mathbb{R}^2, dx^2 + e^{2\phi}dy^2)$ admits the following representation:

$$\tilde{p}_t(x_1, y_1; x_2, y_2)e^{\phi(x_2,y_2)} = \mathbb{E}_1(\delta(x_1 - x_2 + w_t^1)\mathbb{E}_2[Z_t\delta(Y_t(x_1, y_1, w^1, w^2) - y_2)]). \quad (4.4.20)$$

We wish to compute $\int_{-\infty}^{\infty} \tilde{p}_t(x, \int_0^y e^{\phi(\ell,\tau)}d\tau; x+\ell, y)e^{\phi(x+\ell,y)}dy$. Rescaling by $y \mapsto \epsilon y$ gives

$$\int_{-\infty}^{\infty} \tilde{p}_t(x, \int_0^y e^{\phi(\ell,\tau)}d\tau; x+\ell, y)e^{\phi(x+\ell,y)}dy = \int_{-\infty}^{\infty} \mathbb{E}_1(\delta(\ell - w_t^1)\mathbb{E}_2[Z_t(X,Y)\delta(\frac{1}{\epsilon}Y_t(x, \epsilon\int_0^y e^{\phi(\ell,\epsilon\tau)}d\tau, w^1, w^2) - y)])\, dy. \quad (4.4.21)$$

Now the standard scaling techniques [22] with $t = \epsilon^2$ allow us to rewrite this as

$$\int_{-\infty}^{\infty} \tilde{p}_{\epsilon^2}(x, \int_0^y e^{\phi(\ell,\tau)} d\tau; x+\ell, y) e^{\phi(x+\ell,y)} dy =$$

$$\int_{-\infty}^{\infty} \mathbb{E}_1(\delta(\ell - \epsilon w_1^1) \mathbb{E}_2[Z_1(X^\epsilon, Y^\epsilon)\delta(Y_1^\epsilon(x, \int_0^y e^{\phi(\ell,\epsilon\tau)} d\tau, w^1, w^2) - y)]) \, dy \tag{4.4.22}$$

where X^ϵ, Y^ϵ are as defined above. This leads directly to 4.4.14; the usual large deviation estimates justify the fact that the Taylor series for F may be integrated over w^1 to obtain the short-time asymptotic expansion for μ. The remaining statements of the theorem follow naturally, save for the computation of a_1.

We can dispense with the unnecessary boundedness assumption on $\frac{\partial^n}{\partial y^n}\sigma$ by using the geometric bounds of chapter 2. The suspect point in the argument above is the application of the Girsanov-Cameron-Martin theorem; however by restricting attention to sample paths of the Brownian motion for $(\mathbb{R}^2, ds^2)$ which remain in a strip $\{|x| < r\}$, the theorem applies as above. Since every Borel set in $C([0,\infty), \mathbb{R}^2)$ can be written as the union of sets of this form (we have implicitly used Yau's result here), the boundedness assumption is superfluous.

The calculation of a_1 is based on the integration-by-parts formulas standard in the Malliavin calculus. As the calculation is rather involved, we provide only a recipe for producing a_1. First we obtain the following formulas for the derivatives of Y^ϵ at $\epsilon = 0$:

$$Y_t = e^{\phi(\ell,0)}y + \int_0^t \sigma dw_s^2 \tag{4.4.23}$$

$$\frac{\partial Y}{\partial \epsilon} = \phi_y(\ell,0)e^{\phi(\ell,0)}\frac{y^2}{2} + \int_0^t [\sigma_x w_s^1 + \sigma_y Y_s]dw_s^2, \tag{4.4.24}$$

and

$$\frac{\partial^2 Y}{\partial \epsilon^2} = [\phi_{yy}(\ell,0) + \phi_y(\ell,0)^2]e^{\phi(\ell,0)}\frac{y^3}{3} + \\ + \int_0^t [\sigma_{xx}(w_s^1)^2 + 2\sigma_x\sigma_y w_s^1 Y_s + \sigma_{yy}(Y_s)^2 + 2\sigma_y\frac{\partial Y_s}{\partial \epsilon}]\, dw_s^2. \tag{4.4.25}$$

Next, we compute derivatives of Z at $\epsilon = 0$:

$$Z_t = e^{\phi(\ell,0)/2} \tag{4.4.26}$$

$$\frac{\partial Z_t}{\partial \epsilon} = [\frac{\phi_y(\ell,0)}{2}y + \int_0^t \sigma_y dw_s^2]e^{\phi(\ell,0)/2}, \tag{4.4.27}$$

and

$$\frac{\partial^2 Z_t}{\partial \epsilon^2} = [(\frac{\phi_y(\ell,0)}{2}y + \int_0^t \sigma_y dw_s^2)^2 + \frac{\phi_{yy}}{2}y^2 + \\ + 2\int_0^t [\sigma_{xy}w^1 + \sigma_{yy}Y_s]dw_s^2 - \int_0^t V ds]e^{\phi(\ell,0)/2}. \tag{4.4.28}$$

Finally, let $D_\sigma F$ denote the derivative of F in the direction of $\int_0^t \sigma ds \in \mathcal{H}$ ($\mathcal{H}$ is the Cameron-Martin subspace of $\mathcal{W}_2$). Let D_σ^* denote it's adjoint with respect to the Wiener measure P_2. Integrating by parts, we have for $\epsilon = 0$

$$\mathbb{E}_2[F\frac{d}{d\epsilon}\delta(Y_1^\epsilon - y)] = \frac{1}{\int_0^1 \sigma^2 ds}\mathbb{E}_2[\delta(Y_1 - y)D_\sigma^*(F\frac{\partial Y}{\partial \epsilon})]. \qquad (4.4.29)$$

The usefulness of this equation comes from writing $F\frac{\partial Y}{\partial \epsilon}$ as a polynomial in y. If we integrate over y first, this effectively kills the constant term (this eliminates roughly 20 terms!) and puts $y = (1 - e^{\phi(\ell,0)})^{-1}\int_0^1 \sigma dw_s^2$. A long calculation using these facts will produce the desired expression for a_1. $\square$

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Common Notation

Brownian Motion

$\mathcal{W}_1 = (\{w_t \in C([0,\infty),\mathbb{R}) \mid w_0 = 0\}, \mathcal{B}_t, \mathcal{P}_1(dw^1), \mathbb{E}_1)$ Wiener process

$\mathcal{W}_2 = (\{w_t \in C([0,\infty),\mathbb{R}) \mid w_0 = 0\}, \mathcal{B}_t, \mathcal{P}_2(dw^2), \mathbb{E}_2)$ is independent of $\mathcal{W}_1$

$\mathbb{E}(F(w)) = \mathbb{E}_1(F(w) \mid w_1 = 0)$ expectation w.r.t. pinned Brownian motion

$\int_0^t a_s(w)\, dw_s$ Itô integral of nonanticipating a with $P_1(\{\int_0^t a_s^2(w^1)ds < \infty\}) = 1$

$\int_0^t a_s(w) \circ dw_s$ Fisk-Stratonovitch integral for semimartingale a

$\mathbb{P}(M) \stackrel{\text{def}}{=} C([0,\infty), M)$ path space

$\mathbb{P}_x(M) = \{\, X \in C([0,\infty), M) \mid X_0 \stackrel{\text{def}}{=} X(0) = x \,\}$

$\Omega^t(M) = C(\mathbb{R}/t\mathbb{Z}, M)$ loop space

μ a measure on path space

$\mu(t)$ a measure on loop space

Surface Geometry

(M, ds^2) compact Riemann surface with metric ds^2

γ closed geodesic of M

G fundamental group of M

$\mathfrak{p}$ primitive element of G

K Gaussian curvature of ds^2

∇ Levi Civita connection

$ds^2 = dx^2 + e^{2\phi} dy^2$

ℓ, ℓ_0 length, primitive length

$\alpha = -\phi(\ell_0, 0)$

$\sigma = e^{-\phi}$

HILL'S EQUATION

$T(\phi) = \phi'' + \phi'^2$

$K = -T(\phi)$

$q = T(\phi/2)$

$Q = -\frac{d^2}{dx^2} + q$

$L = -\frac{1}{2}\frac{d^3}{dx^3} + q\frac{d}{dx} + \frac{d}{dx}q$

$\alpha = -\phi(1)$

y_1, y_2 basis for $\ker(Q - \lambda)$ in $C^\infty(\mathbb{R})$

$M(\lambda)$ Monodromy matrix for $Q - \lambda$

$\Delta(\lambda) = \operatorname{tr} M(\lambda) = y_1(1) + y_2'(1)$ the discriminant

ν eigenvalue of $M(\lambda)$

Malliavin Calculus (following Watanabe [22])

$\mathcal{H} = (\{h \in \mathcal{W}_2 \mid \int_0^1 (\frac{dh}{dt})^2 dt < \infty\}, \langle h, k\rangle_{\mathcal{H}} = \int_0^1 \dot{h}\dot{k} dt)$ Cameron-Martin space

$-\#$ the Ornstein-Uhlenbeck operator on $L^2(\mathcal{W}_2, P_2)$

$\|F\|_{p,s} = \mathbb{E}_1(\|(1+\#)^{s/2} F\|_{\mathcal{E}}^p)^{1/p}$ where $F : \mathcal{W} \to \mathcal{E} =$ auxiliary Hilbert space

$D_{p,s}(\mathcal{E}) = \{F \mid \|F\|_{p,s} < \infty\}$

$D^\infty(\mathcal{E}) = \bigcap_{p>1} \bigcap_s D_{p,s}(\mathcal{E})$

DF for $F \in D^\infty$ satisfies $\|F(w+h) - F(w) - \langle DF, h\rangle_{\mathcal{H}}\|_{p,s} = o(\|h\|_{\mathcal{H}})$

D^* the adjoint of $D : D^\infty(\mathcal{E}) \to D^\infty(\mathcal{E} \otimes \mathcal{H})$

$\# = D^* D$

$D_h F = \langle DF, h\rangle_{\mathcal{H}}$

$D_h^* F = \int_0^1 h dw_s - D_h F \quad (\mathcal{E} = \mathbb{R})$

$[D_h, D_k^*] = \langle h, k\rangle_{\mathcal{H}}$