Quantum Groups, Screening Operators, and q-de Rham Cocycles

A Dissertation Presented

by

Abdellah Sebbar

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1997

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation

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The representation theory of Virasoro algebras, established by Feigin and Fuchs, gives a way to obtain intertwining operators between the Fock space representations out of the top homology classes of certain one dimensional local systems over configuration spaces. A similar construction exists for affine algebras, the main tool in this construction being the so-called screening operators. Ginzburg and Schechtman made the remark that in fact these screening operators contain more information. Namely they provide certain canonical cocycles of the Virasoro (resp. affine)

operator-valued local systems on the above configuration spaces. This gives the canonical morphisms from all homology groups of the local systems into Ext-spaces between the above Fock space representations. The purpose of this thesis is to investigate this connection between the geometry of configuration spaces and representation theory in the case of quantum groups. After giving some constructions and results about Hopf algebras, we treat the cases of the quantized enveloping algebras of a semisimple Lie algebra, and of the affine algebra $\widehat{\mathfrak{sl}}_2$. More detailed study is done for the case of $\mathfrak{sl}_2$; which includes also the case when the deformation parameter is a root of unity.

To my mother

To l. Fatima and Allal

To s. Ahmed and his family

To all my family, for everything. And to Leila.

In memory of Sh. Alpha-him Jobe,

who answered a call so soon,

but for a reason.

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Acknowledgements

I would like to thank Professor Anthony Knapp for being my advisor during the past four years. He was willing to help me and to give me advices when I needed them. Many thanks to Professor Vadim Schechtman who introduced me to the subject of Quantum Groups and directed my research. He was a source of help and motivation during my first steps.

Many people in Stony Brook were very kind to me. Among them Professor Chih-Han Sah, who was very helpful and with whom I enjoyed lengthy and fruitful mathematical discussions. Professor Leon Takhtajan was always encouraging and willing to listen to my questions. My friend Hamid Bougourzi, was always there to clarify some of my questions, especially those technicalities which deal with the physics side of the story. Thanks to Professor Susan Montgomery from the University of Southern California, who clarified one of my questions. I am greatful to my friend Ramón del Castillo who had a significant impact on the preparation of the text. He shared with me his time and energy and his outstanding expertise with the computers. And thanks to all my friends in the mathematics department, and to the secretaries who were kind and helpful.

Outside Stony Brook, I am greatful to the Fulbright program, from which I was supported during all my graduate studies. And it is my privilege to thank the staff of the Moroccan-American Commission for Educational and Cultural Exchange, the staff of Amideast in Washington D.C. and In Rabat.

I would not forget the warm friendship I had with Professor Hamid Sultan and Professor Ismael Zahed, and the same goes for all the members of the IALI. Finally, it is my pleasure to acknowledge the help of my dear brother Ahmed Sebbar, for his paternal support during all the times, for being there in times of stress, and for setting the example for me in everything.

Introduction

1. The term quantum group is given to Hopf algebras that are associated with objects connected with algebraic groups. This includes the algebra of regular functions over an algebraic group, and the universal enveloping algebra of a Lie algebra. Important examples are the deformations of the enveloping algebras of Kac-Moody Lie algebras using a parameter q, introduced by Drinfeld and Jimbo, in terms of generators and relations. They provide a wide variety of Hopf algebras which are neither commutative nor cocommutative. When $q \to 1$, it is expected that one recovers the universal enveloping algebra. For a finite dimensional algebra, Fadeev, Reshetikhin and Takhtajan gave a second realization of the quantum group associated with a finite dimensional Lie algebra by means of solutions of the Yang-Baxter equation. This realization can be seen as an analogue to the matrix realization of classical Lie algebras. Later they showed how to extend their construction to the case of a loop algebra using a solution of the Yang-Baxter equation depending on a parameter. A central extension of this quantized loop algebra was established later by Reshetikhin and Semenov-Tian-Shansky. Meanwhile, Drinfeld gave another realization of the quantum affine algebra, which is an analogue to the loop realization of an affine algebra, that he showed to be equivalent to the Drinfeld-Jimbo realization. The equivalence between the Drinfeld realization and the second realization above was established by Frenkel and Ding using the Gauss decomposition. These equivalences are simply algebra isomorphisms, and in general not Hopf algebra isomorphisms.

2. Although these quantum groups have a strong connection with statistical mechanics, conformal field theory and knot theory, they are, according to Lusztig, simply a new development in Lie theory. He showed that the algebras of Drinfeld and Jimbo have a natural form over $\mathbb{Z}[q, q^{-1}]$, which specializes for $q = 1$ to the Kostant form of the classical enveloping algebras. When q is a root of unity, quantum versions of the semisimple groups over fields of positive characteristic are obtained. In terms of representation theory, everything depend on whether q is a root of unity or not. More precisely, if q is not a root of unity, then the finite dimensional representations of the quantized algebras present no real difference with the representations of the Lie algebra except for the fact that for each representation of the latter, there are 2^r representations for the quantized algebra, where r is the number of simple roots, which is due to the choice of r signs. If q is a root of unity, then the quantized enveloping algebra behaves like the enveloping algebra over a field of prime characteristic p (at least for $p \geq 3$). Lusztig also developed the theory of highest weight modules and Verma modules.

3. A natural question arises then: How much of the classical theory of Lie algebras, their representations, and their subsequent applications to other fields can be considered for quantization? And what are the tools that will substitute the classical ones developed over the years for their study? Hinting for instance to Lusztig's use of the theory of quivers and perverse sheaves to construct canonical bases and the use of Kashiwara theory.

The aim of this work is to study some aspects of these questions. The classical theory which is considered can be described as follows:

In their work on representation theory of Virasoro algebras, Feigin and Fucks gave a way to obtain intertwining operators between the "Fock space" representations of these algebras out of the top homology classes of certain one dimensional local systems over configuration spaces. A similar construction exists for affine Kac-Moody Lie algebras. These intertwining operators were built up from the so called "screening operators". In a recent work, Ginzburg and Schechtman made the remark that in fact these screening operators contain more information. They provide canonical cocycles of the affine Kac-Moody algebra with coefficients in the de Rham complex of an operator-valued local system on the configuration space. This makes it possible to obtain canonical morphisms from higher homology groups of the above local system to appropriate Ext-groups between the Fock modules.

4. We investigate a quantum analogue of these constructions in the case of the deformation of the enveloping algebra of a semisimple Lie algebra, and in the case of the affine algebra $\widehat{\mathfrak{sl}}_2$. In the first case, the representations considered are the Verma modules; in the second case, we consider certain highest weight representations that are q-analogues of the so called Wakimoto modules. We construct a family of operators between these modules which satisfy certain difference equations and certain cocycle conditions. These equations are built using a family of q-difference operators which generate a flat connection in a 1-dimensional vector bundle over the n-dimensional torus. This connection depends on the representations considered. We consider a "q-de Rham" complex of the spaces of formal algebraic differential forms over the n-torus. The homology groups of this complex can be regarded as the homology groups of the n-torus with coefficients in a local system with stalk $\mathbb{C}$. From these data, we construct the canonical "q-de Rham" cocycles, and consequently, we obtain the canonical maps between the homology of the local systems and the Ext-spaces between the representations in question.

Of course, one has to make sense of all these objects in the quantum case. The Hopf algebra structure of the quantum groups and the q-calculus are important ingredients in these constructions. In Chapter I, we establish some results concerning the actions of Hopf algebras on modules, and we introduce two important objects for all this work, namely a certain bracket which play a fundamental role, and a cochain complex that will lead to the Ext-spaces. In Chapter II, we treat the case of the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, and we construct nontrivial 1-cocycles which lead to all canonical maps between homology spaces and Ext-spaces. These maps turn out to be isomorphisms. As a consequence we obtain nontrivial intertwining operators between the Verma modules. The case of q being a root of unity is also treated. In Chapter III, we generlize the above construction to the case of a semisimple Lie algebra. A type of Kashiwara operators is used in the constructions of the Vertex operators, and composition series of these operators are considered.

In Chapter IV we consider the case of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$

in the Drinfeld realization. We present a "free field" realization for this algebra in terms of certain completions of Heizenberg algebras, and we present the quantum screening operators. Afterward, we solve the main difference equation, and we construct the canonical cocycles in the simple case and in the composition case.

Chapter I. Hopf algebras and their actions

In this chapter we recall the definition of Hopf algebras, and we present some constructions and results related to them. All the algebraic structures will be considered over the field of complex numbers.

1. Preliminaries

An algebra is given by a triple (B, μ, η) where B is a vector space, and $\mu : B \otimes B \longrightarrow B$ and $\eta : \mathbb{C} \longrightarrow B$ are linear maps which make the following diagrams commutative:

Associativity axiom:

$$\begin{array}{ccc} B \otimes B \otimes B & \xrightarrow{\mu \otimes \mathrm{id}} & B \otimes B \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes \mu} & & \Big\downarrow{\scriptstyle \mu} \\ B \otimes B & \xrightarrow{\mu} & B \end{array}$$

Unit axiom:

$$\begin{array}{ccccc} \mathbb{C} \otimes B & \xrightarrow{\eta \otimes \mathrm{id}} & B \otimes B & \xleftarrow{\mathrm{id} \otimes \eta} & B \otimes \mathbb{C} \\ & \cong \searrow & \downarrow{\scriptstyle \mu} & \swarrow \cong & \\ & & B & & \end{array}$$

A coalgebra is obtained by reversing the arrows of the above diagrams; it is a triple (C, Δ, ε), where C is a vector space and $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathbb{C}$ are linear maps which make the following diagrams commutative:

Coassociativity axiom:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C \otimes C \otimes C \end{array}$$

Counit axiom:

$$\begin{array}{ccccc} \mathbb{C} \otimes C & \xleftarrow{\varepsilon\otimes\mathrm{id}} & C \otimes C & \xrightarrow{\mathrm{id}\otimes\varepsilon} & C \otimes \mathbb{C} \\ & \cong\nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow\cong & \\ & & C & & \end{array}$$

The map Δ is called the comultiplication and ε is called the counit map.

A *bialgebra* is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ where (H, μ, η) is an algebra and (H, Δ, ε) is a coalgebra satisfying the following equivalent conditions:

(1) The maps $\mu : H \otimes H \longrightarrow H$ and $\eta : \mathbb{C} \longrightarrow H$ are morphisms of coalgebras.

(2) The maps $\Delta : H \longrightarrow H \otimes H$ and $\varepsilon : H \longrightarrow \mathbb{C}$ are morphisms of algebras.

Here, a morphism of algebras $f : (H, \mu, \eta) \longrightarrow (H', \mu', \eta')$ is a linear map $f : H \longrightarrow H'$ satisfying:

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \eta = \eta'.$$

And a morphism of coalgebras $g : (H, \Delta, \varepsilon) \longrightarrow (H', \Delta', \varepsilon')$ is a linear map $g : H \longrightarrow H'$ satisfying:

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \varepsilon = \varepsilon' \circ g.$$

We will a adopt the Sweedler notation for the comultiplication, that is, if $x \in H$, then

$$\Delta(x) = \sum_{(x)} x' \otimes x''.$$

Given an algebra (B, μ, η) and a coalgebra (C, Δ, ε), we define a bilinear map on the vector space $\mathrm{Hom}(C, B)$ called the *convolution* as follows: if f and g are elements of $\mathrm{Hom}(C, B)$, then $f * g$ is the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\mu} B.$$

Using the Sweedler notation, we have:

$$(f * g)(x) = \sum_{(x)} f(x')g(x'').$$

Now, if $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, an endomorphism A of H is called an *antipode* for H if

$$A * \mathrm{id}_H = \mathrm{id}_H * A = \eta \circ \varepsilon.$$

Finally, a *Hopf algebra* is a bialgebra with antipode.

The properties of the antipode are summarized in the following theorem, for a proof, see [24] or [15].

THEOREM 1.1. *Let $(H, \mu, \eta, \Delta, \varepsilon, A)$ be a Hopf algebra, then:*

(1) *A is a bialgebra morphism from H to H^{op}, i.e. we have*

$$A(xy) = A(y)A(x)\ , \quad \varepsilon \circ A = \varepsilon,$$

and

$$(A \otimes A)\Delta = \Delta^{op}, \quad A(1) = 1,$$

where the comultiplication for H^{op} *is given by:* $\Delta^{op} = \tau \circ \Delta$, *with* $\tau(a \otimes b) = b \otimes a$.

(2) *The following statements are equivalent :*

(a) $A^2 = \mathrm{id}_H$

(b) *for all* $x \in H$ *we have*

$$\sum_{(x)} A(x'')x' = \varepsilon(x).$$

(c) *for all* $x \in H$ *we have*

$$\sum_{(x)} x''A(x') = \varepsilon(x).$$

In addition, if H is commutative (i.e. $\mu(a \otimes b) = \mu(b \otimes a)$) or cocommutative (i.e. $\Delta(a) = \tau \circ \Delta(a)$), then $A^2 = \mathrm{id}_H$.

For practical reasons, we summarize the axiomatic relations of the maps Δ, A and ε in the following equations:

$$\sum_{(x)} x' \otimes (x'')' \otimes (x'')'' = \sum_{(x)} (x')' \otimes (x')'' \otimes x'' \quad \text{(Coassociativity axiom).}$$

$$(1.1) \qquad \sum_{(x)} x'\varepsilon(x'') = \sum_{(x)} \varepsilon(x')x'' = x \quad \text{(Counit axiom).}$$

$$(1.2) \qquad \sum_{(x)} x'A(x'') = \sum_{(x)} A(x')x'' = \varepsilon(x).1 \quad \text{(Antipode axiom).}$$

2. Actions on Hom-spaces

The modules considered in this section are algebra left-modules, we will not deal with the notion of comodules.

Let H be a Hopf algebra over $\mathbb{C}$ and let Δ, A and ε be the comultiplication, the antipode and the counit respectively. If $\mathcal{M}$ and $\mathcal{N}$ are two H-modules, then one can define a structure of left module on both $\mathcal{M}\otimes\mathcal{N}$ and $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ by:

$$x \cdot (m_1 \otimes m_2) = \sum_{(x)} (x'm_1) \otimes (x''m_2) \quad (m_1, m_2 \in \mathcal{M},\ x \in H),$$
$$(x \cdot f)(m) = \sum_{(x)} x' f(A(x'')m) \quad (m \in \mathcal{M},\ f \in \mathrm{Hom}(\mathcal{M},\mathcal{N}),\ x \in H).$$

Each vector space carries a structure of H-module through the map ε. Therefore, the dual space $\mathcal{M}^* = \mathrm{Hom}(\mathcal{M},\mathbb{C})$ carries a structure of H-module. It is given by

$$\begin{aligned}(x \cdot \phi)(m) &= \sum_{(x)} \varepsilon(x')\phi(A(x'')m) \\ &= \phi(m)\sum_{(x)} \phi(A(\varepsilon(x')x'')m) \\ &= \phi(A(x)m) \quad \text{using (1.1)}.\end{aligned}$$

If $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are three modules, we would like to factorize the action of elements of H on maps in $\mathrm{Hom}(\mathcal{M},\mathcal{P})$ which are compositions of maps from $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ and $\mathrm{Hom}(\mathcal{N},\mathcal{P})$. We need the following lemma, whose proof was outilined to me by S. Montgomery. To simplify the notations, we change the superscripts $'$ and $''$ to subscripts with arabic numbers whenever more than one is involved.

LEMMA 2.1. *If H is a Hopf algebra, then for every x in H, we have:*

$$\sum_{(x)} x_1 \otimes 1 \otimes x_2 = \sum_{(x)} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2}. \tag{2.1}$$

PROOF. We prove the identity by applying the coassociativity of the map Δ several times. If $x \in H$, then by the coassociativity, we have:

$$x_1 \otimes x_{2,1} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_2.$$

Now, applying $1 \otimes \Delta \otimes 1$ to left handside, and the $1 \otimes 1 \otimes \Delta$ to the right handside, we obtain, by coassociativity:

$$x_1 \otimes x_{2,1,1} \otimes x_{2,1,2} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_{2,1} \otimes x_{2,2}.$$

Therefore,

$$\begin{aligned} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2} &= x_1 \otimes A(x_{2,1,1})x_{2,1,2} \otimes x_{2,2} \\ &= x_1 \otimes \varepsilon(x_{2,1}) \otimes x_{2,2} \\ &= x_1 \otimes 1 \otimes x_2 \end{aligned}$$

using (1.1) twice. This proves the lemma. □

PROPOSITION 2.2. (Composition lemma). *If $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are three H-modules, then for every $f \in \operatorname{Hom}(\mathcal{N}, \mathcal{P})$ and for every $g \in \operatorname{Hom}(\mathcal{M}, \mathcal{N})$ and $x \in H$, we have:*

$$x \cdot (f \circ g) = \sum_{(x)} (x' \cdot f) \circ (x'' \cdot g).$$

PROOF. The relation can be written as:

$$x_1 f(g(A(x_2)m)) = x_{1,1} f(A(x_{1,2})x_{2,1} g(A(x_{2,2}m))) \quad (m \in \mathcal{M}).$$

Which follows from (2.1) in the above lemma after applying $1 \otimes 1 \otimes A$ to its two sides. □

The composition lemma seems to be just a consequence of the axiomatic definition of the Hopf algebra, especially from the coassociativity. For the sake of completeness, we give another interpretation and a proof not involving the coassociativity, and which deals more with the modules themselves:

The composition map $(f,g)\longrightarrow f\circ g$ is bilinear, it induces a linear map

$$\mathrm{Hom}(\mathcal{N},\mathcal{P})\otimes\mathrm{Hom}(\mathcal{M},\mathcal{N})\overset{\circ}{\longrightarrow}\mathrm{Hom}(\mathcal{M},\mathcal{P}).$$

The statement of the proposition is equivalent to the fact that the composition map is H-linear. Let us define the following maps:

$$\Phi_{\mathcal{M},\mathcal{N}}:\mathcal{N}\otimes\mathcal{M}^*\longrightarrow\mathrm{Hom}(\mathcal{M},\mathcal{N})$$

given by

$$\Phi_{\mathcal{M},\mathcal{N}}(n\otimes\phi)(m)=\phi(m)n\quad(n\in\mathcal{N},\ m\in\mathcal{M},\ \phi\in\mathcal{N}^*),$$

and the evaluation map $\mathrm{ev}_{\mathcal{M}}:\mathcal{M}\otimes\mathcal{M}^*\longrightarrow\mathbb{C}$ given by $\mathrm{ev}_{\mathcal{M}}(m\otimes\phi)=\phi(m)$. Then the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{P}\otimes\mathcal{N}^*\otimes\mathcal{N}\otimes\mathcal{M}^* & \xrightarrow{\mathrm{id}\otimes\mathrm{ev}_{\mathcal{N}}\otimes\mathrm{id}} & \mathcal{P}\otimes\mathcal{M}^* \\
\Big\downarrow{\scriptstyle \Phi_{\mathcal{N},\mathcal{P}}\otimes\Phi_{\mathcal{M},\mathcal{N}}} & & \Big\downarrow{\scriptstyle \Phi_{\mathcal{M},\mathcal{P}}} \\
\mathrm{Hom}(\mathcal{N},\mathcal{P})\otimes\mathrm{Hom}(\mathcal{M},\mathcal{N}) & \xrightarrow{\ \circ\ } & \mathrm{Hom}(\mathcal{M},\mathcal{P}).
\end{array}$$

Indeed, let $m\in\mathcal{M}$, $n\in\mathcal{N}$, $p\in\mathcal{P}$, $\phi\in\mathcal{N}^*$ and $\psi\in\mathcal{M}^*$, then we have:

$$\begin{aligned}
\left[\circ(\Phi_{\mathcal{N},\mathcal{P}}\otimes\Phi_{\mathcal{M},\mathcal{N}})(p\otimes\phi\otimes n\otimes\psi)\right](m) &= \left[\circ(\Phi_{\mathcal{N},\mathcal{P}}(p\otimes\phi)\otimes\Phi_{\mathcal{M},\mathcal{N}}(n\otimes\psi))\right] \\
&= \Phi_{\mathcal{N},\mathcal{P}}(p\otimes\phi)\circ\Phi_{\mathcal{M},\mathcal{N}}(n\otimes\psi)(m) \\
&= \psi(m)\phi(n)p.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\Phi_{\mathcal{N},\mathcal{P}}(\mathrm{id}\otimes\mathrm{ev}_{\mathcal{N}}\otimes\mathrm{id})(p\otimes\phi\otimes n\otimes\psi)(m) &= \phi(n)\Phi_{\mathcal{M},\mathcal{P}}(p\otimes\psi)(m) \\
&= \phi(n)\psi(m)p,
\end{aligned}$$

which shows that the diagram is commutative. If $\mathcal{N}$ is finite dimensional, then $\Phi_{\mathcal{N},\mathcal{P}}$ and $\Phi_{\mathcal{M},\mathcal{N}}$ are isomorphisms, therefore, we can express the composition map in terms of $\Phi_{\mathcal{N},\mathcal{P}}$, $\Phi_{\mathcal{M},\mathcal{N}}$, $\Phi_{\mathcal{M},\mathcal{P}}$ and the evaluation map. To conclude that the composition map is H-linear, it is enough to show that the maps $\Phi_{\mathcal{M},\mathcal{N}}$ and $\mathrm{ev}_{\mathcal{M}}$ are H-linear. Let $x \in H$, $m \in \mathcal{M}$, $n \in \mathcal{N}$, $\phi \in \mathcal{M}^*$ and $\psi \in \mathcal{N}^*$, then:

$$\begin{aligned}
\Phi_{\mathcal{M},\mathcal{N}}\left(x \cdot (n \otimes \phi)\right)(m) &= \sum_{(x)} \Phi_{\mathcal{M},\mathcal{N}}(x'n \otimes x'' \cdot \phi)(m) \\
&= \sum_{(x)} (x'' \cdot \phi)(m) x'' n \\
&= \sum_{(x)} \phi(A(x'')m) x'n, \\
&= \sum_{(x)} x' \Phi_{\mathcal{M},\mathcal{N}}(n \otimes \phi)(A(x'')m) \\
&= x \cdot \Phi_{\mathcal{M},\mathcal{N}}(n \otimes \phi)(m),
\end{aligned}$$

which shows that $\Phi_{\mathcal{M},\mathcal{N}}$ is H-linear, and so is $\mathrm{ev}_{\mathcal{M}}$ because:

$$\begin{aligned}
\mathrm{ev}_{\mathcal{M}}\left(x \cdot (\psi \otimes n)\right) &= \sum_{(x)} \mathrm{ev}_{\mathcal{M}}(x' \cdot \psi \otimes x'' n) \\
&= \sum_{(x)} x' \cdot \psi(x'' n) \\
&= \sum_{(x)} \psi(A(x') x'' n) \\
&= \varepsilon(x) \psi(n). \quad \square
\end{aligned}$$

Note that the result is still true if one assumes that $\mathcal{M}$ and $\mathcal{P}$ are finite dimensional and not $\mathcal{N}$.

3. A bracket and a cochain complex

Let H be a Hopf algebra over $\mathbb{C}$ and let Δ, ε, A be respectively the comultiplication, the counit and the antipode.

We define the bilinear map $< \cdot, \cdot >$ on $H \otimes H$ by

$$< x, y >= \sum_{(x)} x' y A(x'') - \varepsilon(x) y \quad (x, y \in H). \tag{3.1}$$

This bracket satisfies the following properties:

PROPOSITION 3.1. *For every x, y, z in H, we have :*

(1) $< xy, z >=< x, < y, z >> + \varepsilon(x) < y, z > + \varepsilon(y) < x, z >$,

(2) $\varepsilon(< x, y >) = 0$,

(3) $A^2(< x, y >) =< A^2(x), A^2(y) >$.

PROOF. We have

$$\begin{aligned}
< xy, z > &= \sum_{(x),(y)} x' y' z A(y'') A(x'') - \varepsilon(xy) z \\
&=< x, \sum_{(y)} y' z A(y'') > + \varepsilon(x) \sum_{(y)} y' z A(y'') - \varepsilon(x)\varepsilon(y) z \\
&=< x, < y, z >> + < x, \varepsilon(y) z > + \varepsilon(x) < y, z > \\
&=< x, < y, z >> + \varepsilon(y) < x, z > + \varepsilon(x) < y, z >,
\end{aligned}$$

which proves the first relation. We also have:

$$\begin{aligned}
\varepsilon(< x,y >) &= \sum_{(x)} \varepsilon(x')\varepsilon(y)\varepsilon(A(x'')) - \varepsilon(xy) \\
&= \varepsilon(y)\varepsilon\left(\sum_{(x)} x'A(x'')\right) - \varepsilon(xy) \\
&= \varepsilon(y)\varepsilon(\varepsilon(x).1) - \varepsilon(xy) \quad \text{by definition of } \Delta \text{ and } \varepsilon \\
&= 0,
\end{aligned}$$

which proves (2). To prove (3), we use the fact that A is an anti homomorphism, and the fact that $\Delta(A(x)) = \sum_{(x)} A(x'') \otimes A(x')$, and that $\varepsilon(A(x)) = \varepsilon(x)$. We have

$$\begin{aligned}
\Delta(A^2(x)) &= \sum A^2(x)' \otimes A^2(x)'' \\
&= \sum A(A(x)'') \otimes A(A(x)') \\
&= (A \otimes A)\left(\sum A(x)'' \otimes A(x)'\right) \\
&= (A^2 \otimes A^2)\left(\sum x' \otimes x''\right) \\
&= \sum A^2(x') \otimes A^2(x'').
\end{aligned}$$

Therefore,

$$\begin{aligned}
< A^2(x), A^2(y) > &= \sum A^2(x')A^2(y)A(A^2(x'')) \\
&= A^2\left(\sum x'yA(x'')\right) \\
&= A^2(< x,y >). \quad \square
\end{aligned}$$

Note that $< 1,x >=< x,1 >= 0$ for every $x \in H$. And if H is commutative then $< x,y >= 0$ for every $x,y \in H$, while if H is cocommutative, then

$$< x, A(y) >= A < x,y > .$$

If $\mathcal{M}$ is a left H-module, then $m \in \mathcal{M}$ is called an *H-invariant* if $xm = \varepsilon(x)m$ for every $x \in H$. If $\mathcal{N}$ is another H-module, we set

$$(3.2) \qquad < x, \phi >= x \cdot \phi - \varepsilon(x)\phi \quad x \in H,\ \phi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N}).$$

Notice the analogy with the definition of $< x, y >$. The relation (1) in the above proposition is still valid when we substitute z by ϕ. We rewrite it as

(3.3)

$$< xy, \phi >= x\cdot < y, \phi > +\varepsilon(y) < x, \phi >, \quad x, y \in H,\ \phi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N}).$$

We also have $< 1, \phi >= 0$ for every ϕ. We say that ϕ is an *invariant* if $< x, \phi >= 0$ for every $x \in H$.

PROPOSITION 3.2. *We have:*

$$< x, \phi\psi >= \sum_{(x)} < x', \phi >< x'', \psi > + < x, \phi > \psi + \phi < x, \psi > .$$

PROOF. We have

$$< x, \phi\psi >= \sum_{(x)} (x' \cdot \phi)(x'' \cdot \psi) - \varepsilon(x)\phi\psi.$$

The proposition follows from the composition lemma, from (1.1) and the fact that $\varepsilon(x) = \sum_{(x)} \varepsilon(x')\varepsilon(x'')$, which follows also from (1.1). □

Let $\mathcal{M}$ be a H-module, and let us consider the following complex:

$$C^\bullet = C(H^{\otimes\bullet}, \mathcal{M}) : 0 \longrightarrow \mathcal{M} \longrightarrow \mathrm{Hom}(H, \mathcal{M}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(H^{\otimes n}, \mathcal{M}) \longrightarrow \cdots,$$

and the linear map

$$d : \mathrm{Hom}(H^{\otimes n-1}, \mathcal{M}) \longrightarrow \mathrm{Hom}(H^{\otimes n}, \mathcal{M})$$

defined as follows:

If $\phi \in \mathrm{Hom}(H^{\otimes n-1}, \mathcal{M})$ and $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in H^{\otimes n}$, then

$$\begin{aligned} d\phi(x_1, x_2, \ldots, x_n) &= x_1 \cdot \phi(x_2, \ldots, x_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n) \\ &+ (-1)^n \phi(x_1, x_2, \ldots, x_{n-1}) \varepsilon(x_n). \end{aligned}$$

PROPOSITION 3.3. *We have $d^2 = 0$, i.e. $(C^\bullet, d)$ is a cochain complex.*

Let us introduce the following notations:

$$d_n^0 \phi(x_1, \ldots, x_n) = x_1 \cdot \phi(x_2, \ldots, x_n),$$

$$d_n^n \phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_{n-1}) \varepsilon(x_n).$$

And for $1 \leq i \leq n-1$:

$$d_n^i \phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_n),$$

so that we have:

$$d = \sum_{i=0}^{n} (-1)^i d_n^i.$$

The next two lemmas conclude the proof of the proposition .

LEMMA 3.4. *We have*

$$d \circ d = \sum_{i<j} (-1)^{i+j} (d_{n+1}^j d_n^i - d_{n+1}^i d_n^{j-1}).$$

PROOF. We have:

$$\begin{aligned} d \circ d &= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} d_{n+1}^j d_n^i \\ &= \sum_{i<j} (-1)^{i+j} d_{n+1}^j d_n^i + \sum_{i \geq j} (-1)^{i+j} d_{n+1}^j d_n^i. \end{aligned}$$

By changing the indices, we get:

$$\sum_{i\geq j}(-1)^{i+j}d_{n+1}^{j}d_{n}^{i}=-\sum_{i<j}(-1)^{i+j}d_{n+1}^{i}d_{n}^{j-1}.$$

This proves the lemma. □

LEMMA 3.5. *For $0\leq i<j\leq n-1$, we have:*

$$d_{n+1}^{j}d_{n}^{i}=d_{n+1}^{i}d_{n}^{j-1}$$

for $0\leq i<j\leq n-1$.

PROOF. We will treat different cases:

(*i*) $i=0$:

If $j=1$, then

$$\begin{aligned} d_{n+1}^{1}d_{n}^{0}\phi(x_1,\ldots,x_{n+1}) &= d_{n}^{0}\phi(x_1x_2,x_3,\ldots,x_{n+1}) \\ &= (x_1x_2)\cdot\phi(x_3,\ldots,x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} d_{n+1}^{0}d_{n}^{0}\phi(x_1,\ldots,x_{n+1}) &= x_1\cdot d_{n}^{0}\phi(x_2,\ldots,x_{n+1}) \\ &= x_1\cdot(x_2\cdot\phi(x_3,\ldots,x_{n+1})) \\ &= (x_1x_2)\cdot\phi(x_3,\ldots,x_{n+1}). \end{aligned}$$

If $2\leq j\leq n$, then

$$\begin{aligned} d_{n+1}^{0}d_{n}^{j-1}\phi(x_1,\ldots,x_{n+1}) &= x_1\cdot d_{n}^{j-1}\phi(x_2,\ldots,x_{n+1}) \\ &= x_1\cdot\phi(x_2,\ldots,\ldots,x_jx_{j+1},\ldots,x_{n+1}), \end{aligned}$$

and

$$d_{n+1}^{j} d_n^0 \phi(x_1, \ldots, x_{n+1}) = d_n^0 \phi(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$
$$= x_1 \cdot \phi(x_2, \ldots, x_j x_{j+1}, \ldots, x_{n+1}).$$

If $j = n+1$, then

$$d_{n+1}^{n+1} d_n^0 \phi(x_1, \ldots, x_{n+1}) = x_1 \cdot \phi(x_2, \ldots, x_n) \varepsilon(x_{n+1})$$
$$= d_{n+1}^0 d_n^n \phi(x_1, \ldots, x_{n+1}).$$

(ii) $i \geq 1$ and $j \geq i+2$:

This case is straightforward since the indices $(i, i+2)$ and $(j, j+1)$ are disjoint for $j \leq n$, and the identity is clear for $j = n+1$.

(iii) $1 \leq i = j-1 \leq n-1$:

$$\begin{aligned} d_{n+1}^{i+1} d_n^i \phi(x_1, \ldots, x_{n+1}) &= d_n^i \phi(x_1, \ldots, x_{i+1} x_{i+2}, \ldots, x_{n+1}) \\ &= \phi(x_1, \ldots, x_i x_{i+1} x_{i+2}, \ldots, x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} d_{n+1}^{i} d_n^i \phi(x_1, \ldots, x_{n+1}) &= d_n^i \phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\ &= \phi(x_1, \ldots, x_i x_{i+1} x_{i+2}, \ldots, x_{n+1}). \end{aligned}$$

(iv) $i = 1,\ j = n+1$:

This case follows from the fact that ε is an algebra homomorphism. □

One can choose the coefficients of the cochains in $\mathrm{Hom}(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are two H-modules. Thus we obtain a complex

$$C^\bullet(H,\mathcal{M},\mathcal{N}) = \mathrm{Hom}(H^{\otimes\bullet},\mathrm{Hom}(\mathcal{M},\mathcal{N})).$$

In this case, the zeroth cohomology space is the space of H-invariants. Indeed, if $\phi \in \mathrm{Hom}(\mathcal{M},\mathcal{N})$, then $d\phi(x) =< x,\phi >$. Therefore, if $d\phi = 0$, then ϕ is H-invariant. We will see that in the case of quantum groups, the space of invariants coincides with the space of intertwiners. More generally, the cohomology spaces are the Ext-spaces $\mathrm{Ext}^\bullet_H(\mathcal{M},\mathcal{N})$.

Chapter II. The algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, Intertwiners and Cocycles

4. The main constructions

The quantum group $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ is the associative algebra generated by four variables E, F, $K^{\pm 1}$ and the relations :

$$KK^{-1} = K^{-1}K = 1, \tag{4.1}$$

$$KE = q^2 EK, \tag{4.2}$$

$$KF = q^{-2} FK, \tag{4.3}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{4.4}$$

The associative algebra $\mathcal{U}_q$ has a structure of Hopf algebra. The comultiplication, the counit and the antipode are given by:

$$\Delta(E) = 1 \otimes E + E \otimes K,$$

$$\Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$

$$\varepsilon(E) = \varepsilon(F) = 0 \quad \text{and} \quad \varepsilon(K^{\pm 1}) = 1$$

and

$$A(E) = -EK^{-1}\ , \quad A(F) = -KF\ , \quad A(K^{\pm 1}) = K^{\mp 1}.$$

If λ is a nonzero complex number, $\mathcal{M}(\lambda)$ will denote the Verma module over $\mathcal{U}_q$ with highest weight λ. This module is generated by a nonzero vector v_λ satisfying :

$$Ev_\lambda = 0\ , \quad Kv_\lambda = q^\lambda v_\lambda.$$

The action of the generators of $\mathcal{U}_q$ on the Verma module $\mathcal{M}(\lambda)$ is summarized in the following proposition, which one can easily prove by induction.

PROPOSITION 4.1. *Let v_λ be the highest weight vector of $\mathcal{M}(\lambda)$, then :*

$$K^n F^a v_\lambda = q^{n(\lambda-1)} F^a v_\lambda \quad (a \in \mathbb{N},\ n \in \mathbb{Z}),$$
$$E^n F^a v_\lambda = \prod_{k=0}^{n-1} [a-k][\lambda - a + k + 1] F^{a-n} v_\lambda \quad (a \in \mathbb{N}, n \in \mathbb{N}),$$

with the convention that $F^a v_\lambda = 0$ if $a < 0$. □

Let us consider two complex numbers λ and λ', and let us consider the $\mathbb{C}$-linear map

$$V_n : \mathcal{M}(\lambda' - 1) \longrightarrow \mathcal{M}(\lambda - 1) \quad (n \in \mathbb{N})$$

given by

$$V_n(F^a v_{\lambda'-1}) = F^{a+n} v_{\lambda-1} \quad (a \in \mathbb{N},\ n \in \mathbb{N}).$$

By direct computations, we obtain :

PROPOSITION 4.2. *The pairing of V_n with the generators of $\mathcal{U}_q$ is given by:*

$$< E, V_n > (F^a v_{\lambda'-1}) = q^{-\lambda'+2a+1}([a+n][\lambda - a - n] - [a][\lambda' - a])F^{a+n-1}v_{\lambda-1},$$

$$< F, V_n > (F^a v_{\lambda'-1}) = (1 - q^{\lambda'-\lambda+2n})F^{a+n+1}v_{\lambda-1},$$

$$< K^m, V_n > (F^a v_{\lambda'-1}) = (-1 + q^{m(\lambda'-\lambda-2n)})F^{a+n}v_{\lambda-1},$$

for every $n, a \in \mathbb{N}$ and $m \in \mathbb{Z}$. □

If $\alpha \in \mathbb{C}$, we define a twisted differential $d_\alpha : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]]\frac{dz}{z}$ linearly by:

$$d_\alpha(z^n) = [n + \alpha]z^n \frac{dz}{z},$$

where z is a formal variable. Let $\mathcal{M}(\lambda - 1)[[z^{-1}]]$ be the module of Laurent series in z^{-1} with coefficients in $\mathcal{M}(\lambda - 1)$, and consider the operator :

$$V(z) = \sum_{n \geq 0} V_n z^{-n-1} dz : \mathcal{M}(\lambda' - 1) \longrightarrow \mathcal{M}(\lambda - 1)[[z^{-1}]]\frac{dz}{z}.$$

We would like to find a number $\alpha \in \mathbb{C}$ and an operator

$$V(E, z) = \sum_{n \geq 0} V_n(E) z^{-n} : \mathcal{M}(\lambda' - 1) \longrightarrow \mathcal{M}(\lambda - 1)[[z^{-1}]]$$

such that :

$$< E, V(z) >= d_\alpha V(E, z). \tag{4.5}$$

This equation is equivalent to :

$$< E, V_n >= [-n + \alpha]V_n(E) \quad (n \in \mathbb{N}).$$

Applying this to $F^a v_{\lambda'-1}$ for a nonnegative integer a, we obtain :

$$[-n + \alpha]V_n(F^a v_{\lambda'-1}) = q^{-\lambda'+2a+1}\left([a+n][\lambda - a - n] - [a][\lambda' - a]\right)F^{a+n-1}v_{\lambda-1}.$$

We look for a number a' depending on a such that, for every n, we have:

$$[a+n][\lambda - a - n] - [a][\lambda' - a] = [-n + \alpha][n + a'].$$

After multiplying by $(q-q^{-1})^2$, the right hand side gives:

$$-q^{2n-\alpha+a'} - q^{-2n+\alpha-a'} + q^{\alpha+a'} + q^{-\alpha-a'},$$

and the left hand side gives:

$$q^{2n+2a-\lambda} - q^{-2n-2a+\lambda} + q^{2a-\lambda'} + q^{-2a+\lambda'} + q^{\lambda} + q^{-\lambda} - q^{\lambda'} - q^{-\lambda'}.$$

Identifying the powers containing $2n$ (q is not a root of unity), we obtain $-\alpha + a' = 2a - \lambda$, hence $a' = 2a + \alpha - \lambda$. Now identifying the powers containing $2a$, we get $\alpha + a' = 2a - \lambda'$. This gives $2\alpha = \lambda - \lambda'$. We must also have $q^{\lambda} + q^{-\lambda} = q^{\lambda'} + q^{-\lambda'}$, which gives $q^{\lambda} = q^{\pm\lambda'}$. If $q^{\lambda} = q^{\lambda'}$, then we are dealing with the same Verma module since it is q^{λ} which is involved in the highest weight condition. Therefore without loss of generality, we can assume that $\lambda = \pm\lambda'$.

From now on, we suppose $\lambda = -\lambda'$, hence $a' = 2a$ and $\alpha = \lambda$. Thus

$$< E, V_n >= [-n+\lambda]V_n(E),$$

where

$$V_n(E)(F^a v_{-\lambda-1}) = q^{\lambda+2a+1}[n+2a]F^{a+n-1}v_{\lambda-1}. \tag{4.6}$$

By doing the same construction for the other generators F, K, K^{-1}, and using Proposition 4.2, we have:

PROPOSITION 4.3. *For $X = E$, F, $K^{\pm 1}$, one can define operators*

$$V_n(X) : \mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)$$

for every $n \in \mathbb{N}$, given by:

$$V_n(E)(F^a v_{-\lambda-1}) = q^{\lambda+2a+1}[n+2a]F^{a+n-1}v_{\lambda-1},$$

$$V_n(F)(F^a v_{-\lambda-1}) = q^{n-\lambda}(q-q^{-1})F^{a+n+1}v_{\lambda-1},$$

$$V_n(K^{\pm 1})(F^a v_{-\lambda-1}) = -q^{\pm(-n+\lambda)}(q-q^{-1})F^{a+n}v_{\lambda-1},$$

and which satisfy:

$$< X, V_n >= [-n+\lambda]V_n(X). \quad \square$$

REMARK 4.1. If we omit the term with $\varepsilon(x)$ in the definition of the pairing $< \cdot, \cdot >$, $V_n(K^{\pm 1})$ cannot be defined, at the same time, $K^{\pm 1}$ are the only generators for which ε is not zero.

Next, we need to define the operator $V_n(x)$ for every $x \in \mathcal{U}_q$.

PROPOSITION 4.4. *Let $\mathfrak{f}$ be the free associative algebra generated by E, F and $K^{\pm 1}$. Then for every $x \in \mathfrak{f}$ and $n \in \mathbb{N}$, there exists an operator $V_n(x)$: $\mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)$ satisfying:*

$$< x, V_n >= [-n+\lambda]V_n(x). \tag{4.7}$$

PROOF. Assume that for x and y in $\mathfrak{f}$, and for every $n \in \mathbb{N}$, we can define $V_n(x)$ and $V_n(y)$ satisfying (4.7), then for very $n \in \mathbb{N}$, we have:

$$\begin{aligned} < xy, V_n > &= x \cdot < y, V_n > + \varepsilon(y) < x, Vn > \quad \text{using (3.3)} \\ &= [-n+\lambda](x \cdot V_n(y) + \varepsilon(y)V_n(x)). \end{aligned}$$

We set $V_n(xy) = x \cdot V_n(y) + \varepsilon(y)V_n(x)$, then we have

$$< xy, V_n >= [-n+\lambda]V_n(xy).$$

Since, for x being one of the generators of $\mathfrak{f}$, $V_n(x)$ exists and satisfies (4.7), the proposition follows. □

Now we extend the definition of $V_n(x)$ to $\mathcal{U}_q$.

PROPOSITION 4.5. *For every x in $\mathcal{U}_q$ and for every $n \in \mathbb{N}$, there exists an operator $V_n(x) : \mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)$ satisfying the relation* (4.7).

PROOF. Recall that on $\mathfrak{f}$ we have:

$$V_n(xy) = x \cdot V_n(y) + \varepsilon(y) V_n(x). \tag{4.8}$$

In view of the above proposition, we need to prove that this relation is compatible with the defining relations of the algebra $\mathcal{U}_q$. For the relation (4.1), we have:

$$\begin{aligned} V_n(KK^{-1}) &= K \cdot V_n(K^{-1}) + \varepsilon(K^{-1}) V_n(K) \\ &= K V_n(K^{-1}) K^{-1} + V_n(K). \end{aligned}$$

Hence

$$\begin{aligned} V_n(KK^{-1})(F^a v_{-\lambda-1}) &= K V_n(K^{-1})(q^{2a+\lambda+1} F^a v_{-\lambda-1}) + q^{-n+\lambda}(q-q^{-1}) F^{a+n} v_{\lambda-1} \\ &= \left(-q^{2a+\lambda+1} q^{n-\lambda}(q-q^{-1}) K + q^{-n+\lambda}(q-q^{-1})\right) F^{a+n} v_{\lambda-1} \\ &= 0. \end{aligned}$$

On the other hand, it is clear that $V_n(1) = 0$. Similar calculations hold for $K^{-1}K = 1$.

For the relation (4.2), we have:

$$\begin{aligned} V_n(KE)(F^a v_{-\lambda-1}) &= K \cdot V_n(E)(F^a v_{-\lambda-1}) \\ &= K V_n K^{-1}(F^a v_{-\lambda-1}) \\ &= q^{2a+3\lambda-2n+3}[n+2a]F^{a+n-1}v_{\lambda-1}. \end{aligned}$$

And

$$V_n(EK)(F^a v_{-\lambda-1}) = (E \cdot V_n(K) + V_n(E))(F^a v_{-\lambda-1})$$
$$= -V_n(K)EK^{-1}F^a v_{-\lambda-1} + V_n(K)K^{-1}F^a v_{-\lambda-1} + V_n(E)F^a v_{-\lambda-1}.$$

The coefficient of $F^{a+n-1}v_{-\lambda-1}$ is:

$$\begin{aligned} &q^{2a+\lambda+1}(q-q^{-1})\left(q^{-n+\lambda}[a][\lambda+a] + q^{-n+\lambda}[a+n][\lambda-a-n] + \frac{[n+2a]}{q-q^{-1}}\right) \\ &= q^{2a+\lambda+1}\frac{1}{q-q^{-1}}(q^{-n+2\lambda+2a} - q^{-3n+2\lambda-2a}) \\ &= q^{2a+3\lambda-2n+1}[n+2a]. \end{aligned}$$

Therefore

$$V_n(q^2EK) = q^{2a+3\lambda-2n+1}[n+2a]F^{a+n-1}v_{\lambda-1} = V_n(KE).$$

The compatibility with (4.3) is checked in the same way. Finally,

$$\begin{aligned} &V_n\left(\frac{K-K^{-1}}{q-q^{-1}}\right)(F^a v_{-\lambda-1}) \\ &= \frac{1}{q-q^{-1}}\left(q^{-n+\lambda}(q-q^{-1}) + q^{n-\lambda}(q-q^{-1})\right)F^{a+n}v_{\lambda-1} \\ &= (q^{n-\lambda} + q^{-n+\lambda})F^{a+n}v_{\lambda-1}. \end{aligned}$$

On the other hand:

$$V_n([E,F]) = -V_n(F)EK^{-1} + EV_n(F)K^{-1} + K^{-1}V_n(E)KF - FV_n(E).$$

Applying this to $(q-q^{-1})F^a v_{-\lambda-1}$, we obtain successively the following factors as coefficients of $F^{a+n}v_{\lambda-1}$:

$$(q^{3a+n+1}-q^{a+n+1})(q^{\lambda+a}-q^{-\lambda-a})\ , \quad (q^{3a+2n+2}-q^{a})(q^{\lambda-(a+n+1)}-q^{-\lambda+(a+n+1)})\ ,$$

$$q^{-\lambda+2a+2n+1}(q^{n+2a+2}-q^{-n-2a-2}) \quad \text{and} \quad q^{\lambda+2a+1}(q^{n+2a}-q^{-n-2a}).$$

Adding up these expressions, we get:

$$(q-q^{-1})V_n([E,F])F^a v_{-\lambda-1} = \left(q^{-\lambda+n}(q-q^{-1})+q^{\lambda-n}(q-q^{-1})\right) F^{a+n}v_{\lambda-1},$$

which gives the same expression as $V_n(K-K^{-1})$. □

Recall that we have set:

$$V(z)=\sum_{n\geq 0} V_n z^{-n-1}dz : \mathcal{M}(-\lambda-1)\longrightarrow \mathcal{M}(\lambda-1)[[z^{-1}]]\frac{dz}{z}$$

$$V(x,z)=\sum_{n\geq 0} V_n(x)z^{-n} : \mathcal{M}(-\lambda-1)\longrightarrow \mathcal{M}(\lambda-1)[[z^{-1}]] \quad (x\in \mathcal{U}_q)$$

Using the definition of the twisted differential d_λ and the previous propositions, we have:

THEOREM 4.6. *For every $x\in\mathcal{U}_q$, there exists an operator*

$$V(x,z)=\sum_{n\geq 0} V_n(x)z^{-n} : \mathcal{M}(-\lambda-1)\longrightarrow \mathcal{M}(\lambda-1)[[z^{-1}]]$$

linearly dependent on x such that :

$$< x, V(z) >= d_\lambda V(x,z). \tag{4.9}$$

Moreover, for x, y in $\mathcal{U}_q$, we have :

$$V(xy,z)=x\cdot V(y,z)+\varepsilon(y)V(x,z). \tag{4.10}$$ □

5. Intertwiners and Cocycles

We consider the complex of length one :

$$\Omega^\bullet : 0 \longrightarrow \Omega^0 \xrightarrow{d_\lambda} \Omega^1 \longrightarrow 0$$

where

$$\Omega^0 = \mathbb{C}[[z^{-1}]] \quad , \quad \Omega^1 = \mathbb{C}[[z^{-1}]]\frac{dz}{z}.$$

Recall that d_λ is defined linearly by :

$$d_\lambda(z^{-n}) = [\lambda - n]z^{-n}\ \frac{dz}{z}.$$

The length one is due to the fact that $\mathfrak{sl}_2$ has one simple root .

From this complex and the cochain complex $C^\bullet$ introduced in the first chapter, we construct the following bigraded space:

If i and j are two nonnegative integers, we set :

$$C^{ij}(\mathcal{U}_q, \mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1)) = \mathrm{Hom}(\mathcal{U}_q^{\otimes i}, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1)\otimes\Omega^j))$$

with $\Omega^j = 0$ for $j \geq 2$. We will denote it simply by C^{ij}. Note that C^{ij} is isomorphic to $\mathrm{Hom}(\mathcal{U}_q^{\otimes i} \otimes \mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1) \otimes \Omega^j)$. This bigraded space has a natural structure of a bicomplex. The first differential $d' : C^{ij} \longrightarrow C^{i+1,j}$ is induced by the differential d introduced in the first chapter. The second differential $d'' : C^{ij} \longrightarrow C^{i,j+1}$ is induced by the differential d_λ of $\Omega^\bullet$.

The operator $V(z)$ is an element of C^{01}, we will denote it by $V^{01}(z)$, and the operator $V(x,z)$ ($x \in \mathcal{U}_q$) defines an element $V^{10}(z)$ of the space C^{10} by:

$$V^{10}(z)(x) = V(x,z).$$

The results of the previous section can be interpreted in the following:

PROPOSITION 5.1. *The elements $V^{01}(z)$ and $V^{10}(z)$ satisfy:*

$$d'V^{01}(z) = d''V^{10}(z), \tag{5.1}$$

$$d'V^{10}(z) = 0. \tag{5.2}$$

PROOF. For $x \in \mathcal{U}_q$, we have by definition of d':

$$\begin{aligned} d'(V^{01}(z))(x) &= x \cdot V^{01}(z) - \varepsilon(x)V^{01}(z) \\ &= < x, V^{01}(z) >, \end{aligned}$$

and $d''V^{10}(z)(x) = d_\lambda V(x,z)$. Thus, the first relation is just a consequence of the relation (4.9) of Theorem 4.6. And for $x, y \in \mathcal{U}_q$, we have:

$$\begin{aligned} d'(V^{10}(z))(x \otimes x) &= x \cdot V^{10}(z)(y) - V^{10}(z)(xy) + \varepsilon(y)V^{10}(z)(x) \\ &= x \cdot V(y,z) - V(xy,z) + \varepsilon(y)V(x,z) \\ &= 0, \end{aligned}$$

using the relation (4.10) of Theorem 4.6. □

Let $\mathcal{C}^\bullet = \mathcal{C}^\bullet(\mathcal{U}_q, \mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$ denotes the simple complex associated with the double complex $C^{\bullet\bullet}$, that is:

$$C^n = \bigoplus_{a+b=n} C^{ab} \quad (n \in \mathbb{Z}).$$

Its differential $\mathfrak{d}$ is defined by [3]:

$$\mathfrak{d}|_{C^{ab}} = d' + (-1)^a d''.$$

THEOREM 5.2. *The element $(V^{01}(z), V^{10}(z))$ is a 1-cocycle of the complex $\mathcal{C}^\bullet$.*

PROOF. The element $(V^{01}(z), V^{10}(z))$ is in the space

$$\mathcal{C}^1(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))).$$

Applying $\mathfrak{d}$, we get:

$$\mathfrak{d}(V^{01}(z) + V^{10}(z)) = d'V^{01}(z) + d''V^{01}(z) + d'V^{10}(z) - d''V^{10}(z).$$

And we have $d''V^{01}(z) = 0$ since $\Omega^\bullet$ is of length one. Using Proposition 5.1, we have $d'V^{10}(z) = 0$ and $d'V^{01}(z) = d''V^{10}(z)$. Hence

$$\mathfrak{d}(V^{01}(z) + V^{10}(z)) = 0. \quad \square$$

Let us consider again the complex $\Omega^\bullet$, a monomial z^{-n} is in the Kernel of d_λ if and only if $[-n+\lambda] = 0$. Since q is not a root of unity, this is equivalent to $\lambda = n$. Thus the complex $\Omega^\bullet$ is acyclic if λ is not an integer.

We assume that λ is a nonnegative integer for the rest of this section. Thus the space $\mathcal{H}^0(\Omega^\bullet)$ is a 1-dimensional space generated by the function $z^{-\lambda}$. The space $\mathcal{H}^1(\Omega^\bullet)$ is generated by the class of the form $z^{-\lambda}\,\frac{dz}{z}$.

If we consider the homology spaces $\mathcal{H}_i = \mathcal{H}^{i*}$, then $\mathcal{H}_1$ is a 1-dimensional space generated (for instance) by the linear form:

$$\begin{array}{ccc} \Omega^1 & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & \mathrm{Res}_{z=0}(\omega z^\lambda) \end{array}$$

The space $\mathcal{H}^0$ is generated by the linear form:

$$\begin{array}{ccc} \Omega^0 & \longrightarrow & \mathbb{C} \\ f(z) & \longmapsto & \mathrm{Res}_{z=0}(f(z) z^\lambda\,\frac{dz}{z}). \end{array}$$

PROPOSITION 5.3. *The operator* $\mathrm{Res}_{z=0}(V^{01}(z^\lambda))$ *is an intertwiner. It is the unique* $\mathcal{U}_q$*-homomorphism* : $\mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)$ *sending* $v_{-\lambda-1}$ *to* $F^\lambda v_{\lambda-1}$.

PROOF. From the first chapter, we know that C^0 is a space of $\mathcal{U}_q$-invariants, we need to show that in fact it coincides with the space of intertwiners in our case. Let $\phi \in C^0$, i.e. $d\phi = 0$, we need to show that ϕ intertwines with E, F and $K^{\pm 1}$. Let $m \in \mathcal{M}(-\lambda-1)$, since $< E, \phi > (m) = 0$, we have $-\phi(EK^{-1}m) + E\phi(K^{-1}m) = 0$, therefore $\phi(EK^{-1}m) = E\phi(K^{-1}m)$. This shows that E intertwines with ϕ. Now, if $< K^{-1}, \phi > (m) = 0$, then $K^{-1}\phi(Km) - \phi(m) = 0$, which shows that K intertwines with ϕ. The intertwining property for K^{-1} follows from the invariance of ϕ with K. And if $< F, \phi > (m) = 0$, then $-K^{-1}\phi(KFm) + F\phi(m) = 0$, since K intertwines with ϕ, we see that F does too.

Finally, it is clear that $\mathrm{Res}_{z=0}(V^{01}(z^\lambda))$ sends $v_{-\lambda-1}$ to $F^\lambda v_{-\lambda-1}$ and therefore is a nontrivial operator. Notice that $F^\lambda v_{-\lambda-1}$ is also a singular vector. □

From the discussion preceding the proposition, we have the following :

COROLLARY 5.4. *We have*

$$\mathcal{H}_1(\Omega^\bullet) \cong \mathcal{H}^0\left(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda))\right).$$

PROPOSITION 5.5. *The operator* $\mathrm{Res}_{z=0}(V^{10}(z)\,\frac{dz}{z})$ *is a nontrivial element of the space* $\mathrm{Ext}^1_{\mathcal{U}_q}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$.

PROOF. $\mathrm{Res}_{z=0}(V^{01}z^\lambda \frac{dz}{z})$ is in the space $\mathrm{Hom}\left(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda))\right)$. By Theorem 5.2 it is a 1-cocycle of the algebra $\mathcal{U}_q$ with coefficients in the $\mathcal{U}_q$-module $\mathrm{Hom}_{\mathbb{C}}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$, hence it defines an element of $\mathrm{Ext}^1_{\mathcal{U}_q}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$. It is a nontrivial element, indeed:

$$V^{10}(z)z^\lambda\,\frac{dz}{z}(E) = \sum_{n\geq 0} q^{\lambda+1}[n]z^{-n+\lambda}\frac{dz}{z}.F^{n-1}v_{\lambda-1}.$$

(Recall that $V_n(E)(v_{-\lambda-1}) = q^{\lambda+1}[n]F^{n-1}v_{\lambda-1})$. Therefore

$$\mathrm{Res}_{z=0}(V^{10}(z)z^{\lambda}\,\frac{dz}{z})(E)(v_{-\lambda-1}) = q^{\lambda+1}[\lambda]F^{\lambda-1}.$$

The right handside is not zero since λ is a nonzero integer and q is not a root of unity. □

6. Case when q is a root of unity

The following relations hold in $\mathcal{U}_q$:

$$E^mK^n = q^{-2mn}K^nE^m, \quad F^mK^n = q^{2mn}K^nF^m \quad (m \in \mathbb{Z},\ n \in \mathbb{Z}).$$

$$[E, F^m] = [m]F^{m-1}\frac{q^{-m+1}K - q^{m-1}K^{-1}}{q-q^{-1}} = [m]\frac{q^{m-1}K - q^{-m+1}K^{-1}}{q-q^{-1}}F^{m-1}.$$

$$[E^m, F] = [m]\frac{q^{-m+1}K - q^{m-1}K^{-1}}{q-q^{-1}}E^{m-1} = [m]E^{m-1}\frac{q^{m-1}K - q^{-m+1}K^{-1}}{q-q^{-1}}.$$

If we assume that q is a root of unity of order d ($q \neq \pm 1$), then it follows from the above identities that E^e, F^e and $K^{\pm 1}$ are in the center of $\mathcal{U}_q$, where

$$e = \begin{cases} d & \text{if } d \text{ is odd,} \\ \frac{d}{2} & \text{if } d \text{ is even.} \end{cases}$$

For $\lambda \in \mathbb{C}$, if v_λ is the highest weight vector of the Verma module $\mathcal{M}(\lambda)$ of highest weight λ, then for $k \in \mathbb{Z}$ we have

$$Kv_{\lambda-kd} = q^{\lambda-kd}v_{\lambda-kd} = q^{\lambda}v_{\lambda-kd}.$$

Hence, $v_{\lambda-kd}$ is also of highest weight λ.

By uniqueness of the Verma module of highest weight weight λ, we see that all $\mathcal{M}(\lambda-kd)$, $d \in \mathbb{Z}$), are $\mathcal{U}_q$-isomorphic to $\mathcal{M}(\lambda)$. The isomorphism is given for instance by $F^a v_{\lambda-kd} \longmapsto F^{a+kd}v_\lambda$.

Notice that $F^{kd}v_\lambda$ is a singular vector of $\mathcal{M}(\lambda)$ since F^{kd} commutes with E.

Let us consider the family of operators $V_{n,k}$ ($n \in \mathbb{N}, k \in \mathbb{N}$), defined by:

$$\begin{array}{rccc} V_{n,k}: & \mathcal{M}(-\lambda-1) & \longrightarrow & \mathcal{M}(\lambda-1) \\ & F^a v_{-\lambda-1} & \longmapsto & F^{a+n+kd}v_{\lambda-1}. \end{array}$$

And let us consider the operators:

$$V_k: \mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)[[z^{-1}]]\,\frac{dz}{z} \quad (k \in \mathbb{N})$$

defined by:

$$V_k(z) = \sum_{n \geq 0} V_{n,k} z^{-n-1} dz.$$

It is clear that $V_{n,k}$ acts on $\mathcal{M}(-\lambda-1)$ in the same way the operator V_{n+kd} introduced in the generic case does. For $x \in \mathcal{U}_q$ we have:

$$< x, V_{n,k} > (F^a v_{-\lambda-1}) = < x, V_{n+kd} > (F^a v_{-\lambda-1}).$$

This allows us to use the generic case to obtain the same constructions here. Namely, if we define $V_{n,k}(x)$ to be equal to $V_{n+kd}(x)$ and if we consider the operators:

$$V_k(x,z) = \sum_{n \geq 0} V_{n,k}(x) z^{-n} : \mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)[[z^{-1}]] \quad (k \in \mathbb{Z}),$$

then we have:

PROPOSITION 6.1. *For every k in $\mathbb{N}$, and every x and y in $\mathcal{U}_q$, we have:*

(a) $< x, V_k(z) >= d_\lambda V_k(x,z)$.

(b) $V_k(xy,z) = x \cdot V_k(y,z) + \varepsilon(y) V_k(x,z)$.

We have obtained then an infinite family of 1-cocycles $(V_k^{01}(z), V_k^{10}(z))$ in the simple complex $C^\bullet(\mathcal{U}_q, \mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$. If $\lambda \in \mathbb{N}$, the space $\mathcal{H}^0(\Omega^\bullet)$ is an infinite dimensional, generated by $z^{-\lambda-kd}\,\frac{dz}{z}$, $(k \in \mathbb{N})$.

In the same way, the homology space $\mathcal{H}_1(\Omega^\bullet)$ is generated by the family

$$\left\{ \begin{array}{ccc} \Omega^1 & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & \mathrm{Res}_{z=0}(\omega z^{\lambda+kd}) \end{array}, \quad k \in \mathbb{Z} \right\}.$$

The space $\mathcal{H}^0$ is generated by (the restrictions of) the linear forms

$$\left\{ \begin{array}{ccc} \Omega^0 & \longrightarrow & \mathbb{C} \\ f(z) & \longmapsto & \mathrm{Res}_{z=0}\left(f(z) z^{\lambda+kd}\,\frac{dz}{z}\right) \end{array}, \quad k \in \mathbb{Z} \right\}.$$

The operator $\mathrm{Res}_{z=0}(V_k^{01}(z) z^\lambda)$ in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$, $k \in \mathbb{Z}$, is the unique $\mathcal{U}_q$-homomorphism: $\mathcal{M}(-\lambda-1) \longrightarrow \mathcal{M}(\lambda-1)$ sending $v_{-\lambda-1}$ to $F^{\lambda+kd} v_{\lambda-1}$. This provides us with an infinite family of linearly independent intertwiners.

The map

$$\begin{array}{ccc} \mathcal{H}^1(\Omega^\bullet) & \longrightarrow & \mathcal{H}^0(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))) \\ \left(\begin{array}{ccc} \Omega^1 & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & \mathrm{Res}_{z=0}(\omega z^{\lambda+kd}) \end{array} \right) & \longmapsto & \mathrm{Res}_{z=0}(V_k^{01}(z) z^\lambda) \end{array}$$

realizes a one-one linear map.

The operators

$$\mathrm{Res}_{z=0}(V^{10}(z) z^\lambda\,\frac{dz}{z}) \in \mathrm{Hom}(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))) \quad (k \in \mathbb{N})$$

are 1-cocycles of the algebra $\mathcal{U}_q$ with coefficients in $\mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$, therefore they define an infinite family of linearly independent elements in the space $\mathrm{Ext}^1_{\mathcal{U}_q}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))$.

To clarify this, we apply these operators to F (if we use E or $K^{\pm 1}$ we will obtain 0):

$$V_k^{10} z^\lambda \frac{dz}{z}(F)(v_{-\lambda-1}) = \sum_{n\geq 0} V_{n,k}(F)(v_{-\lambda-1}) z^{-n+\lambda} \frac{dz}{z},$$

and

$$V_{n,k}(F)(v_{-\lambda-1}) = V_{n+kd}(F)(v_{-\lambda-1}) = q^{n-\lambda}(q - q^{-1})F^{a+n+kd-1} v_{\lambda-1}.$$

Therefore

$$\mathrm{Res}_{z=0} V_k^{10}(z) z^\lambda \frac{dz}{z}(F)(v_{-\lambda-1}) = (q - q^{-1})F^{a+\lambda+kd-1} v_{\lambda-1},$$

which is nonzero since $q \neq \pm 1$. The linear independence follows from the appearance of k in the power of F.

Finally, this provides us with a one-one linear map:

$$\mathcal{H}^0(\Omega^\bullet) \longrightarrow \mathrm{Hom}(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(-\lambda-1), \mathcal{M}(\lambda-1))),$$

which sends the linear form

$$\begin{array}{rcl} \Omega^0 & \longrightarrow & \mathbb{C} \\ f(z) & \longmapsto & \mathrm{Res}_{z=0} f(z) z^{\lambda+kd} \frac{dz}{z} \end{array}$$

to $\mathrm{Res}_{z=0} V_k^{10}(z) z^\lambda \frac{dz}{z}$.

The meaning of these maps between homology and cohomology spaces will be clarified in the next chapter.

Chapter III. Generalization to a semisimple Lie algebra

7. The quantum group $\mathcal{U}_q(\mathfrak{g})$

Root systems. Let $(a_{ij})_{1\leq i,j\leq N}$ be an nxn indecomposable matrix with integer entries such that $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$, and let $(d_1, \cdots, d_N)$ be a vector with relatively prime entries such that the matrix $(d_i a_{ij})$ is symmetric and positive definite. Notice that (a_{ij}) is a Cartan matrix of a simple finite dimensional Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra and $\Pi = \{\alpha_i,\ 1 \leq i \leq N\}$ the corresponding root system and $\Pi^\vee = \{\alpha_i^\vee,\ 1 \leq i \leq N\}$ the corresponding coroot system, $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a realization of $\mathfrak{g}$ [13]. There exists a nondegenerate symmetric bilinear $\mathbb{C}$−valued form $(.,.)$ on $\mathfrak{h}$ satisfying:

$$(\alpha_i^\vee, h) = \langle h, \alpha_i \rangle d_i^{-1} \quad \text{for } h \in \mathfrak{h},\ i = 1, \cdots, N.$$

Here, $\langle .,. \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$.

Since $(.,.)$ is nondegenerate, there is an isomorphism $\mu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ defined by

$$\langle h_1, \mu(h) \rangle = (h_1, h) \quad (h_1, h \in \mathfrak{h}).$$

This isomorphism induces a symmetric (nondegenerate) bilinear form $(.,.)$ on $\mathfrak{h}^*$. Thus we have

$$\mu(\alpha_i^\vee) = d_i^{-1}\alpha_i\ ,\quad i = 1, \cdots, N,$$

and

$$(\alpha_i, \alpha_j) = d_i a_{ij} = d_j a_{ji}.$$

Let ρ be the element of $\mathfrak{h}^*$ defined by

$$\langle \alpha_i^\vee, \rho\rangle = 1\ ,\quad i = 1, \cdots, N.$$

Then ρ satisfies $(\rho, \alpha_i) = d_i$.

We define also the fundamental reflexions r_i, $1 \leq i \leq n$, of $\mathfrak{h}^*$ by:

$$r_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda\rangle \quad (\lambda \in \mathfrak{h}^*).$$

In particular: $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Gaussian binomial coefficients. Let q be an indeterminate. For $n \in \mathbb{Z}$, $d \in \mathbb{N}$, we define the $q-$integer $[n]_d$ by :

$$[n]_d = \frac{q^{dn} - q^{-dn}}{q^d - q^{-d}}.$$

If $d = 1$, we denote it simply by $[n]$ and we have $[n]_d = \frac{[nd]}{[d]}$.

We also set

$$[n]_d! = \prod_{s=1}^{n} [s]_d.$$

And we define the $q-$binomial coefficients

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[d]_d \cdots [n-j+1]_d}{[j]_d!} \quad \text{for } j \in \mathbb{Z},\ j \leq n,$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = 0 \quad \text{if } j > n.$$

We have

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = (-1)^j \begin{bmatrix} -n+j-1 \\ j \end{bmatrix}_d$$

and

$$\prod_{s=0}^{n-1}(1+q^{2sd}z) = \sum_{j=0}^{n} q^{dj(n-1)} \begin{bmatrix} n \\ j \end{bmatrix}_d z^j \quad (n \geq 0), \tag{7.1}$$

here z is another indeterminate. It follows that $\begin{bmatrix} n \\ j \end{bmatrix}_d \in \mathbb{Z}[q, q^{-1}]$.

If m and n are in $\mathbb{Z}$ and $j \in \mathbb{N}$, then

$$\begin{bmatrix} m+n \\ j \end{bmatrix}_d = \sum_{k+l=j} q^{d(ml-nk)} \begin{bmatrix} m \\ k \end{bmatrix}_d \begin{bmatrix} n \\ l \end{bmatrix}_d.$$

By putting $z = -1$ in (7.1) and using $[n]_d = [n]_{-d}$ for an integer n, we obtain

$$\sum_{j=0}^{n} q^{dj(1-n)} \begin{bmatrix} n \\ j \end{bmatrix}_d = 0. \tag{7.2}$$

Finally, if x,y are two elements in a $\mathbb{Q}(q)-$algebra such that $xy = q^2yx$, then for any $n \geq 1$, we have the quantum binomial formula:

$$(x+y)^n = \sum_{j=0}^{n} q^{j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} y^j x^{n-j}.$$

The Drinfeld-Jimbo algebra $\mathcal{U}_q$. We assume that q is a generic complex number and we set $q_i = q^{d_i}$. We consider the algebra $\mathcal{U}_q$ defined by the generators E_i, F_i, $K_i^{\pm 1}$ $(1 \leq i \leq n)$ and the relations:

$$K_iK_j = K_jK_i\ ,\quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \tag{7.3}$$

$$K_iE_j = q_i^{a_{ij}}E_jK_i\ ,\quad K_iF_j = q_i^{-a_{ij}}F_jK_i, \tag{7.4}$$

$$[E_i, F_j] = \delta_{ij}\ \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{7.5}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s}E_jE_i^s = 0 \quad if\ i \neq j, \tag{7.6}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF_i^s = 0 \quad if\ i \neq j. \tag{7.7}$$

The last two relations are referred to as the Serre relations. There is a unique algebra involution $\omega : \mathcal{U}_q \longrightarrow \mathcal{U}_q$ such that $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, $\omega(K_i) = K_i^{-1}$.

The following propositions determine the Hopf algebra structure of $\mathcal{U}_q$.

PROPOSITION 7.1. *There is a unique algebra homomorphism $\Delta : \mathcal{U}_q \longrightarrow \mathcal{U}_q \otimes \mathcal{U}_q$ which takes the generators E_i, F_i, $K_i^{\pm 1}$, respectively to the elements $\Delta(E_i)$, $\Delta(F_i)$, $\Delta(K_i^{\pm 1})$ given by:*

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta(K_i) = K_i \otimes K_i\ ,\quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

for $1 \le i \le N$. *Moreover,* Δ *satisfies the coassociativity axiom of a comultiplication.*

PROOF. [19] lemma 3.1.4 and lemma 3.1.10. □

PROPOSITION 7.2. *There is a unique algebra homomorphism* $\varepsilon : \mathcal{U}_q \longrightarrow \mathbb{C}$ *such that* $\varepsilon(E_i) = \varepsilon(F_i) = 0$ *and* $\varepsilon(K_i) = \varepsilon(K_i^{-1}) = 1,\ \ 1 \le i \le N$, *which satisfies the axioms of a counit.*

PROOF. [19] lemma 3.1.11. □

PROPOSITION 7.3. *There is a unique homomorphism of algebras* $A : \mathcal{U}_q \longrightarrow \mathcal{U}_q^{opp}$ *such that*

$$A(E_i) = -K_i^{-1}E_i,\ \ A(F_i) = -F_iK_i,\ \ A(K_i^{\pm 1}) = K_i^{\mp 1} \quad (1 \le i \le N),$$

which satisfies the axioms of an antipode.

PROOF. [19] lemma 3.3.1. □

COROLLARY 7.4. $\mathcal{U}_q$ *is a Hopf algebra with comultiplication* Δ, *counit* ε *and antipode* A. □

8. Differentials and Operators

The maps ∂_i and $_i\partial$. Let $\mathfrak{f}'$ be the free algebra with **1** generated by the F_i's. Let $\mathbb{Z}[\Pi]$ be the root lattice and $\mathbb{N}[\Pi]$ be the submonoid of $\mathbb{Z}[\Pi]$ of all linear combinations of elements of Π with coefficients in $\mathbb{N}$. For any $\alpha = \sum a_i\alpha_i$ in $\mathbb{N}[\Pi]$, we denote by $\mathfrak{f}'_\alpha$ the subalgebra of $\mathfrak{f}'$ spanned by monomials $F_{i_1}F_{i_2}\cdots F_{i_r}$ such that for any i, the number of occurrences of i in the sequence

$i_1, i_2, \ldots, i_r$ is equal to a_i. Each $\mathfrak{f}'_\alpha$ is a finite dimensional vector space and we have a direct sum decomposition $\mathfrak{f}' = \oplus_\alpha \mathfrak{f}'_\alpha$ where α runs over $\mathbb{N}[\mathrm{II}]$. We also have $\mathfrak{f}'_\alpha \mathfrak{f}'_{\alpha'} \subset \mathfrak{f}'_{\alpha+\alpha'}$, $\mathbf{1} \in \mathfrak{f}'_0$ and $F_i \in \mathfrak{f}'_{\alpha_i}$. An element $x \in \mathfrak{f}'$ is said to be homogeneous if it belongs to $\mathfrak{f}'_a$ for some a, we then set $|x| = a$.

We denote by ${}_i\partial$ the linear map ${}_i\partial : \mathfrak{f}' \longrightarrow \mathfrak{f}'$ such that

$$ {}_i\partial(1) = 0 \ , \quad {}_i\partial(F_j) = \delta_{i,j} \quad \text{for all } j $$

and

$$ {}_i\partial(xy) = {}_i\partial(x)y + q^{(|x|,\alpha_i)} x\, {}_i\partial(y) $$

for all homogeneous x, y. Similarly, we denote by ∂_i the linear map $\partial_i : \mathfrak{f}' \longrightarrow \mathfrak{f}'$ such that

$$ \partial_i(1) = 0 \ , \quad \partial_i(F_j) = \delta_{i,j} \quad \text{for all } j $$

and

$$ \partial_i(xy) = q^{(|y|,\alpha_i)} \partial_i(x)y + x\partial_i(y) $$

for all homogeneous x, y. These maps are examples of the so called Kashiwara operators [19].

If $x \in \mathfrak{f}'_\alpha$, then ${}_i\partial(x)$ and $\partial_i(x)$ are in $\mathfrak{f}'_{\alpha-\alpha_i}$ if $a_i \geq 1$ and ${}_i\partial(x) = \partial_i(x) = 0$ if $a_i = 0$.

In [19], it is shown that the maps ${}_i\partial$ and ∂_i leave the radical $\mathcal{I}$ of a certain bilinear inner product on $\mathfrak{f}'$ stable, this radical turns out to contain (even generated by) the Serre relations. Therefore they are also defined on the quotient $\mathfrak{f}'/\mathcal{I}$. Here, we will check directly that ${}_i\partial$ and ∂_i conserve the Serre relations because the inner product won't be of any use in this chapter.

PROPOSITION 8.1. *For k, i, and j in $\{1,\dots,N\}$, $i \neq j$, we have :*

$$(8.1)\qquad {}_k\partial\left(\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}F_i^{1-a_{ij}-s}F_jF_i^s\right)=0,$$

$$(8.2)\qquad \partial_k\left(\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}F_i^{1-a_{ij}-s}F_jF_i^s\right)=0.$$

PROOF. A simple induction shows that

$${}_k\partial(F_j^n)=\partial_k(F_j^n)=\delta_{kj}q_j^{n-1}[n]_jF_j^{n-1}\quad(n\in\mathbb{N}).$$

It follows that for $k\neq i$ and $k\neq j$, the proposition is clear.

If $k=i$:

$${}_k\partial(F_i^{1-a_{ij}-s}F_jF_i^s)=q^{(|F_i^s|,\alpha_i)}{}_k\partial(F_i^{1-a_{ij}-s})F_i^s+F_i^{1-a_{ij}-s}F_j\,{}_k\partial(F_i^s)$$

$$=q^{(|F_i^s|,\alpha_i)+(|F_j|,\alpha_i)}{}_k\partial(F_i^{1-a_{ij}-s})F_jF_i^s+q_i^{s-1}[s]_iF_i^{1-a_{ij}-s}F_jF_i^{s-1}.$$

Since $|F_i^s|=s\alpha_i$, $(\alpha_i,\alpha_i)=2d_i$, $|F_j|=\alpha_j$, and $(\alpha_j,\alpha_i)=d_ia_{ij}$, we have :

$${}_k\partial(F_i^{1-a_{ij}-s}F_jF_i^s)=q_i^s[1-a_{ij}-s]_iF_i^{-a_{ij}-s}F_jF_i^s+q_i^{s-1}[s]_iF_i^{1-a_{ij}-s}F_jF_i^{s-1}.$$

At this point we set $a=1-a_{ij}$, and we have to show that:

$$\sum_{s=0}^{a}(-1)^s\begin{bmatrix}a\\ s\end{bmatrix}_{d_i}\left(q_i^s[a-s]_iF_i^{a-s-1}F_jF_i^s+q_i^{s-1}[s]_iF_i^{a-s}F_jF_i^{s-1}\right)=0.$$

The left handside is equal to

$$\sum_{s=0}^{a-1}(-1)^s\begin{bmatrix}a\\ s\end{bmatrix}_{d_i}[a-s]_iq_i^sF_i^{a-s-1}F_jF_i^s+\sum_{s=1}^{a}(-1)^s\begin{bmatrix}a\\ s\end{bmatrix}_{d_i}[s]_iq_i^{s-1}F_i^{a-s}F_jF_i^{s-1}=$$

$$\sum_{s=0}^{a-1}(-1)^s\begin{bmatrix} a \\ s \end{bmatrix}_{d_i}[a-s]_i q_i^s F_i^{a-s-1}F_jF_i^s - \sum_{s=0}^{a-1}(-1)^s\begin{bmatrix} a \\ s+1 \end{bmatrix}_{d_i}[s+1]_i q^s F_i^{a-s-1}F_jF_i^s$$

which is equal to 0 because $\begin{bmatrix} a \\ s \end{bmatrix}_{d_i}[a-s]_i = \begin{bmatrix} a \\ s+1 \end{bmatrix}_{d_i}[s+1]_i$.

If $k = j$:

$$\begin{aligned} {}_j\partial(F_i^{1-a_{ij}-s}F_jF_i^s) &= q^{(|F_i^s|,\alpha_j)}\partial_i(F_i^{1-a_{ij}-s}F_j)F_i^s \\ &= q_j^{sa_{ij}}F_i^{1-a_{ij}}. \end{aligned}$$

It follows that

$${}_j\partial\left(\sum_{s=0} 1-a_{ij}(-1)^s\begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF_i^s\right)$$

$$= \left(\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} q^{sa_{ij}}\right)F_i^{1-a_{ij}},$$

which is equal to 0 by (7.2). This proves (8.1). The relation (8.2) is obtained in the same way. □

As a consequence of the proposition, we have :

COROLLARY 8.2. *The maps ∂_i and ${}_i\partial$ extend to well defined linear maps on the algebra $\mathfrak{f}$ generated by F_i $(1 \le i \le N)$, and satisfying the Serre relations.* □

The following proposition will be useful in all what follows.

PROPOSITION 8.3. *For $x \in \mathfrak{f}$ homogeneous, we have:*

$$K_i x = q^{-(|x|,\alpha_i)} x K_i \quad (1 \le i \le N) \tag{8.3}$$

$$E_i x = x E_i + \frac{K_{ii}\partial(x) - \partial_i(x) K_i^{-1}}{q_i - q_i^{-1}} \quad (1 \le i \le N). \tag{8.4}$$

PROOF. The relation (8.3) is clear for $x = 1$, and for $x = F_j$, it follows from (7.4) since $(|x|, \alpha_i) = d_i a_{ij}$ and $q^{(|x|,\alpha_i)} = q_i^{a_{ij}}$. Assume that (8.3) is true for homogeneous x' and x'' in $\mathfrak{f}$, then

$$\begin{aligned} K_i(x'x'') &= q^{-(|x'|,\alpha_i)} x' K_i x'' \\ &= q^{-(|x'|+|x''|,\alpha_i)} x'x'' K_i \\ &= q^{-(|x'x''|,\alpha_i)} x'x'' K_i \quad \text{since } |x'x''| = |x'| + |x''|. \end{aligned}$$

Therefore (8.3) is true for any homogeneous $x \in \mathfrak{f}$.

The relation (8.4) is a version of Proposition 3.1.6 in [19]. We will prove it using (8.3). For $x = 1$, it is clear, and for $x = F_j$, it follows from (7.5) since $\partial_i(x) = {}_i\partial(x) = \delta_{ij}$, and

$$\frac{1}{q_i - q_i^{-1}}(K_{ii}\partial(x) - \partial_i(x) K_i^{-1}) = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

Assume that (8.4) is true for homogeneous x' and x'' in $\mathfrak{f}$, then

$$\begin{aligned} &E_i x'x'' - x'x''E_i \\ &= x'E_i x'' + \frac{1}{q_i - q_i^{-1}}(K_{ii}\partial(x') - \partial_i(x')K_i^{-1})x'' - x'x''E_i \\ &= \frac{1}{q_i - q_i^{-1}} x'(K_{ii}\partial(x'') - \partial_i(x'')K_i^{-1}) + \frac{1}{q_i - q_i^{-1}}(K_{ii}\partial(x') - \partial_i(x')K_i^{-1})x'' \\ &= \frac{1}{q_i - q_i^{-1}}\left(q^{(|x'|,\alpha_i)} K_i x' {}_i\partial(x'') - x'\partial_i(x'')K_i^{-1} + K_{ii}\partial(x')x'' - q^{(|x''|,\alpha_i)}\partial_i(x')x''\right) \\ &= \frac{K_{ii}\partial(x'x'') - \partial_i(x'x'')K_i^{-1}}{q_i - q_i^{-1}}. \end{aligned}$$

Therefore (8.4) is true for $x'x''$. This completes the proof. $\square$

REMARK 8.1. since ∂_i and ${}_i\partial$ are linear, the relation (8.4) is in fact valid for every $x \in \mathfrak{f}$.

The operators $V_i(z)$ and $V_i(x,z)$. The notations used here are those of the first subsection. Let us fix $\lambda \in \mathfrak{h}^*$ and let $\mathcal{M}(\lambda)$ denote the Verma module over $\mathcal{U}_q$ of highest weight $\lambda - \rho$ and highest weight vector v_λ satisfying:

$$E_i v_\lambda = 0$$
$$K_i v_\lambda = q^{\langle \lambda-\rho, \alpha_i \rangle} v_\lambda = q^{\langle \lambda-\rho, \alpha_i^\vee \rangle} v_\lambda,$$

for $i = 1, \dots, N$. Recall that the fundamental reflexions r_i of $\mathfrak{h}^*$ were defined by $r_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$. We fix an index $i \in \{1, \dots, N\}$ for the remaining of this section, and we consider $\lambda' = r_i\lambda$. For $j = 1, \dots, N$, we introduce the following $q-$numbers associated with an integer n:

$$\mu_{ij}^{\pm}(n) = \frac{q_i^{\pm a_{ij}(\langle \lambda, \alpha_i^\vee \rangle - n)} - 1}{[\langle \lambda, \alpha_i^\vee \rangle - n]_i} \quad \text{if } n \neq \langle \lambda, \alpha_i^\vee \rangle,$$
$$\mu_{ij}^{\pm}(n) = (q_i - q_i^{-1}) \frac{q_i^{\pm a_{ij}} - 1}{2} \quad \text{if } n = \langle \lambda, \alpha_i^\vee \rangle,$$

and

$$\eta_{ij}(n) = \frac{[(-n + \langle \lambda, \alpha_i^\vee \rangle) a_{ji}]_j}{[-n + \langle \lambda, \alpha_i^\vee \rangle]_i} \, q_j^{1 - \langle \lambda, \alpha_j^\vee \rangle + \langle \lambda, \alpha_i^\vee \rangle a_{ji}}.$$

We have:

$$\eta_{ij}(n) = q_j^{1 - \langle \lambda, \alpha_j^\vee \rangle + \langle \lambda, \alpha_i^\vee \rangle a_{ji}} \, \frac{\mu_{ij}^+ - \mu_{ij}^-}{q_j - q_j^{-1}}$$

and

$$\lim_{q \longrightarrow 1} \eta_{ij}(n) = \lim_{q \longrightarrow 1} \frac{\mu_{ij}^+ - \mu_{ij}^-}{q_j - q_j^{-1}} = a_{ji}.$$

We will consider these expressions as deformations of the entries of the matrix $(a_{ij})_{i,j}$. For each $n \geq 0$, we define an operator $V_{i,n} : \mathcal{M}(\lambda') \longrightarrow \mathcal{M}(\lambda)$ by :

$$V_{i,n}(xv_{\lambda'}) = xF_i^n v_\lambda \quad (x \in \mathfrak{f}).$$

And for x being one of the generators of $\mathcal{U}_q$, we define an operator

$$V_{i,n}(x) : \mathcal{M}(\lambda') \longrightarrow \mathcal{M}(\lambda)$$

given by:

$$V_{i,n}(E_j)(xv_{\lambda'}) = \eta_{ij}(n)\partial_j(x)F_i^n v_\lambda + \delta_{ij}[n]_j x F_i^{n-1} v_\lambda, \tag{8.5}$$

$$V_{i,n}(K_j^{\pm})(xv_{\lambda'}) = \mu_{ij}^{\pm}(n) x F_i^n v_\lambda, \tag{8.6}$$

$$V_{i,n}(F_j) = 0, \tag{8.7}$$

for $1 \leq j \leq N$. Then we have:

PROPOSITION 8.4. *For $x = E_j,\ F_j,\ K_j^{\pm}$ $(1 \leq i \leq N)$, we have :*

$$< x, V_{i,n} >= [-n + \langle \lambda, \alpha_i^\vee \rangle]_i \ V_{i,n}(x). \tag{8.8}$$

PROOF. For x homogeneous in $\mathfrak{f}$ we have:

$$< E_j, V_{i,n} > (xv_{\lambda'}) = E_j V_{i,n}(xv_{\lambda'}) - K_j V_{i,n}(K_j^{-1} E_j x v_{\lambda'}). \tag{8.9}$$

And using (8.4), we have

$$E_j V_{i,n}(xv_{\lambda'}) = E_j x F_i^n v_\lambda = \frac{1}{q_j - q_j^{-1}} (K_j\, {}_j\partial(xF_i^n) - \partial_j(xF_i^n)K_j^{-1}) v_\lambda$$

By definition of ∂_j and ${}_j\partial$, this is equal to (ignoring the factor $\frac{1}{q_j - q_j^{-1}}$):

$$\left(K_j\, {}_j\partial(x)F_i^n + q^{(|x|,\alpha_j)} K_j x\, {}_j\partial(F_i^n) - q^{(|F_i^n|,\alpha_j)} \partial_j(x) F_i^n K_j^{-1} - x\partial_j(F_i^n)K_j^{-1}\right) v_\lambda.$$

The middle two terms give:

$$\frac{1}{q_j - q_j^{-1}}\delta_{ij}[n]_j q_j^{n-1} x K_j F_i^{n-1} v_\lambda - q^{-\langle \lambda-\rho, \alpha_j^\vee \rangle}\delta_{ij}[n]_j q_j^{n-1} x F_j^{n-1}$$
$$= \frac{1}{q_j - q_j^{-1}}\left(q_j^{2(n-1)+\langle \lambda-\rho, \alpha_j^\vee \rangle}\delta_{ij}[n]_j q_j^{n-1} x F_i^{n-1} v_\lambda - q^{-\langle \lambda-\rho, \alpha_j^\vee \rangle}\delta_{ij}[n]_j q_j^{n-1} x F_j^{n-1} v_\lambda\right)$$
$$= \frac{\delta_{ij}}{q_j - q_j^{-1}}\ [n]_j \left(q_j^{-(n-1)+\langle \lambda-\rho, \alpha_i^\vee \rangle} - q_j^{n-1-\langle \lambda-\rho, \alpha_i^\vee}\right) x F_i^{n-1} v_\lambda$$
$$= [-n + \langle \lambda, \alpha_i^\vee \rangle]_i \delta_{ij}[n]_j x F_i^{n-1} v_\lambda.$$

Using (8.3), the second term in (8.9) gives:

$$K_j V_{i,n}(K_j^{-1}E_j) x v_{\lambda'} = \frac{1}{q_j - q_j^{-1}} K_j V_{i,n}({}_j\partial(x) - K_j^{-1}\partial_j(x)K_j^{-1}) v_{\lambda'}$$
$$= \frac{1}{q_j - q_j^{-1}}\left(K_j\, {}_j\partial(x) F_i^n v_\lambda - q_j^{(|\partial_j(x)|, \alpha_j) - 2\langle \lambda'-\rho, \alpha_j^\vee \rangle} K_j V_{i,n} \partial_j(x) v_{\lambda'}\right)$$
$$= \frac{1}{q_j - q_j^{-1}}\left(K_j\, {}_j\partial(x) F_i^n v_\lambda - q_j^{-2\langle \lambda'-\rho, \alpha_j^\vee \rangle + \langle \lambda-\rho, \alpha_j^\vee \rangle} q^{-(|F_i^n|, \alpha_j)} \partial_j(x) F_i^n v_\lambda\right)$$
$$= \frac{1}{q_j - q_j^{-1}}\left(K_j\, {}_j\partial(x) F_i^n v_\lambda - q_j^{1-\langle \lambda, \alpha_j^\vee \rangle + 2\langle \lambda, \alpha_i^\vee \rangle a_{ji} - n a_{ji}} \partial_j(x) F_i^n v_\lambda\right),$$

where we have used the fact that if x is homogeneous then ${}_j\partial(x)$ and $\partial_j(x)$ are also homogeneous, the fact that $(|F_i^n|, \alpha_j) = n(\alpha_i, \alpha_j) = n d_i a_{ij} = n d_j a_{ji}$, and that

$$\langle \lambda', \alpha_j^\vee \rangle = \langle \lambda, \alpha_j^\vee \rangle - \langle \lambda, \alpha_i^\vee \rangle \langle \alpha_i, \alpha_j^\vee \rangle = \langle \lambda, \alpha_j^\vee \rangle - a_{ji}\langle \lambda, \alpha_i^\vee \rangle.$$

Adding up both terms of (8.9), we get :

$$< E_j, V_{i,n} > (x v_{\lambda'}) = [-n + \langle \lambda, \alpha_i^\vee \rangle]_i (\delta_{ij}[n]_j x F_i^{n-1} + \eta_{ij}(n)\partial_j(x) F_i^n) v_\lambda.$$

This proves the proposition for $x = E_j$. For $x = F_j$, the proposition is clear since $< F_j, V_{i,n} >= 0$. And for $x = K_j$ we have:

$$\begin{aligned} < K_j, V_{i,n} > (xv_{\lambda'}) &= K_j V_{i,n} K_j^{-1} x v_{\lambda'} - \varepsilon(K_j) V_{i,n}(xv_{\lambda'}) \\ &= q^{(|x|,\alpha_j)} q^{-(\lambda'-\rho,\alpha_j^\vee)} K_j x F_i^n v_\lambda - x F_i^n v_\lambda \\ &= (q_j^{a_{ji}(\langle\lambda,\alpha_i^\vee\rangle - n)} - 1) x F_i^n v_\lambda \\ &= [-n + \langle\lambda, \alpha_i^\vee\rangle]_i V_{i,n}(K_j^{\pm})(xv_{\lambda'}). \end{aligned}$$

The case $x = K_j^{-1}$ is done similarly. □

A cocycle condition. Having defined the operators $V_{i,n}(.)$ for the generators E_j, F_j, $K_j^{\pm}$, we can also define them for any element of the free associative algebra generated by E_j, F_j and $K_j^{\pm}$, $(1 \le i \le N)$. Indeed, assume this is done for two elements x and y, then using (3.3), we have:

$$< xy, V_{i,n} >= x \cdot < y, V_{i,n} > + \varepsilon(y) < x, V_{i,n} >, \tag{8.10}$$

therefore, if we set

$$V_{i,n}(xy) = x \cdot V_{i,n}(y) + \varepsilon(y) V_{i,n}(x),$$

then

$$< xy, V_{i,n} >= [-n + \langle\lambda, \alpha_i^\vee\rangle]_i V_{i,n}(xy).$$

It remains to extend this construction to the algebra $\mathcal{U}_q$.

PROPOSITION 8.5. *For any x in $\mathcal{U}_q$, there is an operator $V_{i,n}(x)$ from $\mathcal{M}(\lambda')$ to $\mathcal{M}(\lambda)$ for any nonnegative integer n satisfying:*

$$< x, V_{i,n} > = [\, -n + \langle\lambda, \alpha_i^\vee\rangle \,]_i \; V_{i,n}(x).$$

Moreover, for y in $\mathcal{U}_q$, we have

$$V_{i,n}(xy) \;=\; x \cdot V_{i,n}(y) \;+\; \varepsilon(y)\; V_{i,n}(x).$$

PROOF. As in the previous chapter, we need to show that the cocycle condition (8.10) leaves invariant the defining relations of $\mathcal{U}_q$.

We have

$$V_{i,n}(K_k K_l) = K_k V_{i,n}(K_l) K_k^{-1}.$$

Hence

$$\begin{aligned} V_{i,n}(K_k K_l)(xv_\lambda &= \left(\mu_{il}^+(n) q_k^{-\langle\lambda',\alpha_k^\vee\rangle+\langle\lambda,\alpha_k^\vee\rangle}\, q^{-n(\alpha_i,\alpha_k)} + \mu_{ik}^+(n)\right)\; xF_i^n v_\lambda \\ &= \frac{q_i^{(-n+\langle\lambda,\alpha_i^\vee\rangle)(a_{ik}+a_{il})} - 1}{[-n+\langle\lambda,\alpha_i^\vee\rangle]_i}\; xF_i^n v_\lambda, \end{aligned}$$

which is symmetric in k and l. Therefore, $V_{i,n}(K_k K_l) = V_{i,n}(K_l K_k)$. For the relation $K_l K_l^{-1} = K_l^{-1} K_l$, it is straightforward (see Chapter II). This shows the compatibility with (7.3). For the relation (7.6), we have:

$$V_{i,n}(K_k F_l) = K_k \cdot V_{i,n}(F_l) + \varepsilon(F_l) V_{i,n}(K_k) = 0,$$

and for x homogeneous

$$\begin{aligned} V_{i,n}(F_l K_k)(xv_{\lambda'}) &= (F_l \cdot V_{i,n}(K_k))(xv_{\lambda'}) \\ &= (F_l V_{i,n}(K_k) K_k - V_{i,n}(K_k) F_l K_k)\,(xv_{\lambda'}) \\ &= q^{-(|x|,\alpha_k)+(\alpha_k,\lambda'-\rho)}\, \left(\mu_{ik}^+(n) F_l x F_i^n v_\lambda - \mu_{ik}^+(n) F_l x F_i^n v_\lambda\right) \\ &= 0. \end{aligned}$$

Also for x homogeneous, we have:

$$\begin{aligned}V_{i,n}(E_kF_l - F_lE_k)(xv_{\lambda'}) &= -(F_l \cdot V_{i,n}(E_k))(xv_{\lambda'}) \\ &= -\left(F_lV_{i,n}(E_k)K_l - V_{i,n}(E_k)F_lK_l\right)xv'_\lambda \\ &= -q^{-(|x|,\alpha_l)+(\lambda'-\rho,\alpha_l)}\left(F_lV_{i,n}(E_k)xv_{\lambda'} - V_{i,n}(E_k)F_lxv_{\lambda'}\right).\end{aligned}$$

The expression between parentheses is equal to

$$\eta_{ik}(n)F_l\partial_k(x)F_i^nv_\lambda + \delta_{ik}[n]_kF_lxF_i^{n-1}v_\lambda - \eta_{ik}(n)\partial_k(F_lx)F_i^nv_\lambda - \delta_{ik}[n]_kF_lxF_i^{n-1}v_\lambda.$$

Since $\partial_k(F_lx) = q^{(|x|,\alpha_k)}\delta_{kl}x + F_l\partial_k(x)$, we obtain:

$$V_{i,n}(E_kF_l - F_lF_k)(xv_{\lambda'}) = \delta_{kl}q^{(\lambda'-\rho,\alpha_l)}\eta_{ik}(n)xF_i^nv_\lambda.$$

On the other hand,

$$\begin{aligned}\delta_{kl}V_{i,n}\left(\frac{K_k - K_k^{-1}}{q_k - q_k^{-1}}\right)(xv_{\lambda'}) &= \frac{\delta_{kl}}{q_k - q_k^{-1}}\left(V_{i,n}(K_k)xv_{\lambda'} - V_{i,n}(K_k^{-1})xv_{\lambda'}\right) \\ &= \frac{\delta_{kl}}{q_k - q_k^{-1}}\left(\mu_{ik}^+(n) - \mu_{ik}^-(n)\right)xF_i^nv_\lambda \\ &= \delta_{kl}\ q^{(\lambda'-\rho,\alpha_k)}\eta_{ik}(n)xF_i^nv_\lambda,\end{aligned}$$

which shows the compatibility with the relation (7.5). The verification for the remaining relations is done using the same calculations. □

For any $\mathcal{U}_q$−module $\mathcal{M}$, we define a shifted differential

$$d_{\lambda,i} : \mathcal{M}[[z^{-1}]] \longrightarrow \mathcal{M}[[z^{-1}]]\ \frac{dz}{z},$$

given linearly by:

$$d_{\lambda,i}(z^{-n}) = [-n + \langle\lambda, \alpha_i^\vee\rangle]_i\ z^{-n}\ \frac{dz}{z}$$

Let us consider the operators

$$V_i(z) = \sum_{n=0}^{\infty} V_{i,n} z^{-n-1} dz \in \operatorname{Hom}(\mathcal{M}(r_i\lambda), \mathcal{M}(\lambda))\,[[z^{-1}]]\,\frac{dz}{z}$$

and

$$V_i(x,z) = \sum_{n=0}^{\infty} V_{i,n}(x) z^{-n} \in \operatorname{Hom}(\mathcal{M}(r_i\lambda), \mathcal{M}(\lambda))\,[[z^{-1}]] \quad (x \in \mathcal{U}_q).$$

The results of this section can be reformulated in the following theorem.

THEOREM 8.6. *For any x, y in $\mathcal{U}_q$ and $i \in \{1, \dots, N\}$, we have*

$$< x, V_i(z) > = d_{\lambda,i}\; V_i(x,z), \tag{8.11}$$

$$V_i(xy,z) = x \cdot V_i(y,z) + \varepsilon(y) V_i(x,z). \tag{8.12}$$

9. q-de Rham cocycles

In this section we fix an element w of the Weyl group of the Lie algebra $\mathfrak{g}$. We assume that $w = r_{i_a} \dots r_{i_1}$ is a reduced decomposition of w into fundamental reflexions. We also fix an element $\lambda \in \mathfrak{h}^*$ and we consider the following elements of $\mathfrak{h}^*$:

$$\lambda_p = r_{i_{p-1}} \dots r_{i_1} \lambda \quad (1 \le p \le a).$$

Let $\mathcal{A} = \mathbb{C}[[z_1^{-1}, \dots, z_a^{-1}]]$ and let Ω^p, $1 \le p \le a$, be the free $\mathcal{A}$-module generated by $\left\{ \frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_p}}{z_{j_p}}\,,\ 1 \le j_1 < \dots < j_p \le a \right\}$. We consider the sequence of $\mathcal{A}$-modules

$$\Omega^\bullet :\ 0 \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \dots \longrightarrow \Omega^a \longrightarrow 0$$

and we define a linear map $d'' : \Omega^k \longrightarrow \Omega^{k+1}$ as follows:

If

$$(9.1) \qquad \eta = f(z_1^{-1}, \dots, z_a^{-1})\frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}}$$

where f is a monomial in $\mathcal{A}$, and if n_p is the exponent of z_p^{-1} in f, then

$$d''\eta = \sum_{p=1}^{a}[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee}\rangle]_{i_p}\ f(z_1^{-1}, \dots, z_a^{-1})\frac{dz_p}{z_p} \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}}.$$

Using the notations of the previous section, this can be expressed as:

$$d''\eta = \sum_{p=1}^{a} d_{\lambda_p, i_p}\ f(z_1^{-1}, \dots, z_a^{-1}) \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}}.$$

We extend d'' by linearity to any η in Ω^k.

PROPOSITION 9.1. *We have* $d''^2 = 0$, *i.e.* $(\Omega^\bullet, d)$ *is a complex.*

PROOF. Let $f(z) = f(z_1^{-1}, z_2^{-1}, \dots, z_a^{-1})$ be a monomial, and let η be a form given by (9.1). Then

$$d''\eta = \sum_{p=1}^{a}[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee}\rangle]_{i_p}\ f(z)\frac{dz_p}{z_p} \wedge \frac{dz_{j_1}}{z_{j_1}}\wedge, \dots, \wedge\frac{z_{j_k}}{z_{j_k}}$$

And

$$\begin{aligned} d''^2\eta &= \sum_{p=1}^{a}[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee}\rangle]_{i_p} \sum_{s=1}^{a}[-n_s + \langle \lambda_s, \alpha_{i_s}^{\vee}\rangle]_{i_s}\ f(z)\frac{dz_s}{z_s} \wedge \frac{dz_p}{z_p} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}} \\ &= \left(\sum_{p=1}^{a}\sum_{s=1}^{a}[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee}\rangle]_{i_p}[-n_s + \langle \lambda_s, \alpha_{i_s}^{\vee}\rangle]_{i_s} f(z)\frac{dz_s}{z_s} \wedge \frac{dz_p}{z_p}\right) \\ &\qquad \wedge\frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_k}}{z_{j_k}}, \end{aligned}$$

which is easily seen to be equal to zero. □

For each $p = 1, \dots, a$, we consider the operators defined in the previous section:

$$V_p(z_p) : \mathcal{M}(\lambda_{p+1}) \longrightarrow \mathcal{M}(\lambda_p) z_p^{-1}[[z_p^{-1}]]$$

and

$$V_p(x, z_p) : \mathcal{M}(\lambda_{p+1}) \longrightarrow \mathcal{M}(\lambda_p)[[z_p^{-1}]] \quad (x \in \mathcal{U}_q)$$

We recall from Chapter I that we have the following complex :

$$\mathrm{Hom}\left(\mathcal{U}_q^{\otimes \cdot}, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda))\right)$$

and the differential d' of this complex is defined on the cochains as follows:

$$\begin{aligned} d'\phi(x_1, \dots, x_n) = x_1 \cdot \phi(x_2, \dots, x_n) \;&+\; \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \dots, x_i x_{i+1}, \dots, x_n) \\ &+\; (-1)^n \phi(x_1, \dots, x_{n-1}) \varepsilon(x_n) \end{aligned}$$

We let $C^{\bullet\bullet}$ denote the follwing double complex:

$$\mathrm{Hom}\left(\mathcal{U}_q^{\otimes \bullet}, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda) \otimes \Omega^\bullet)\right) \cong \mathrm{Hom}\left(\mathcal{U}_q^{\otimes \bullet} \otimes \mathcal{M}(w\lambda), \mathcal{M}(\lambda) \otimes \Omega^\bullet\right).$$

The first differential for this double complex is d', the second differential is the shifted de Rham differential d'' defined above. In the following, we will construct an a-cocycle in the simple complex associated with $C^{\bullet\bullet}$.

For $x_1, x_2, \dots, x_m$ in $\mathcal{U}_q$ and $p_1, p_2, \dots, p_m$ where $0 \le m \le a$ and $1 \le p_1 < p_2 < \dots < p_m \le a$ we define $\mathcal{G}(x_1, \dots, x_n; p_1, \dots, p_m)$ to be equal to the following expression:

$$\begin{aligned} \sum V_1(z_1) \cdots V_{p_1 - 1}(z_{p_1 - 1}) V_{p_1}(x_1', z_{p_1}) x_1'' \cdot (V_{p_1+1}(z_{p_1+1}) \cdots V_{p_2 - 1}(z_{p_2 - 1}) V_{p_2}(x_2', z_{p_2}) \\ x_2'' \cdot (\dots V_{p_m}(x_m', z_{p_m}) x_m'' \cdot (V_{p_m+1}(z_{p_m+1}) \cdots V_{i_a}(z_a)) \dots)) \\ dz_1 \wedge \hat{d}z_{p_1} \wedge \dots \wedge \hat{d}z_{p_m} \wedge \dots \wedge dz_a \end{aligned}$$

where $\hat{}$ means omission, the summation is over all the terms involved in the comultiplication of $x_1, \cdots, x_m$ in the Sweedler notation. In each summand, we consider initially the composition $V_1(z_1)V_2(z_2) \cdots V_a(z_a)$ and for each p_k we substitute $V_{p_k}(z_{p_k})$ by $V_{p_k}(x'_k, z_{p_k})x''_k \cdot (\ldots$, where x''_k is going to act on all the remaining factors to the right if there are any.

For each m, $0 \leq m \leq a$, we define the operators

$$V^{m,a-m} \in \operatorname{Hom}\left(\mathcal{U}_q^{\otimes m}, \operatorname{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda) \otimes \Omega^{a-m})\right)$$

as follows:

$$\begin{aligned} &V^{m,a-m}(x_1, \ldots, x_m) = \\ &\quad (-1)^{\frac{m(m+1)}{2}} \sum_{1 \leq p_1 < \ldots < p_m \leq a} (-1)^{p_1 + \cdots + p_m} \, \mathcal{G}(x_1, \ldots, x_m; p_1, \ldots, p_m). \end{aligned}$$

In order to clarify these rather complicated expressions, we will give some examples:

$$\begin{aligned} V^{0,a} =& V_1(z_1) \cdots V_a(z_a) \; dz_1 \wedge \ldots dz_a. \\ V^{a,0}(x_1, \ldots, x_n) =& \sum_{(x_1), \ldots, (x_{a-1})} V_1(x'_1, z_1)x''_1 \cdot (V_2(x'_2, z_2)x''_2 \cdot \\ & \left(\ldots \left(V_{a-1}(x'_{a-1}, z_{a-1})x''_{a-1} \cdot V_a(x_a, z_a)\right) \ldots\right)\right). \end{aligned}$$

For $a = 2$:

$$\begin{aligned} V^{0,2} =& V_1(z_1)V_2(z_2)dz_1 \wedge dz_2 \\ V^{1,1}(x) =& \sum_{(x)} (V_1(x', z_1)x'' \cdot V_2(z_2)dz_2 - V_1(z_1)V_2(x, z_2)dz_1) \\ V^{2,0}(x_1, x_2) =& \sum_{(x_1)} V_1(x'_1, z_1)x''_1 \cdot V_2(x_2, z_2). \end{aligned}$$

For $a = 3$:

$$V^{0,3} = V_1(z_1)V_2(z_2)V_3(z_3)dz_1 \wedge dz_2 \wedge dz_3.$$

$$\begin{aligned} V^{1,2}(x) =& \sum_{(x)} V_1(x', z_1)x'' \cdot (V_2(z_2)V_3(z_3))dz_2 \wedge dz_3 \\ &- \sum_{(x)} V_1(z_1)V_2(x', z_2)x'' \cdot V_3(z_3)dz_1 \wedge dz_3 \\ &+ V_1(z_1)V_2(z_2)V_3(x, z_3)dz_1 \wedge dz_2. \end{aligned}$$

$$\begin{aligned} V^{2,1}(x_1, x_2) =& \sum_{(x_1),(x_2)} V_1(x_1', z_1)x_1'' \cdot (V_2(x_2', z_2)x_2'' \cdot V_3(z_3))dz_3 \\ &- \sum_{(x_1)} V_1(x_1', z_1)x_1'' \cdot (V_2(z_2)V_3(x_2, z_3))dz_2 \\ &+ \sum_{(x_1)} V_1(z_1)V_2(x_1', z_2)x_1'' \cdot V_2(x_2, z_3)dz_1. \end{aligned}$$

$$V^{3,0}(x_1, x_2, x_3) = \sum_{(x_1),(x_2)} V_1(x_1', z_1)x_1'' \cdot (V_2(x_2', z_2))x_2'' \cdot V_3(x_3, z_3).$$

We recall the counit axiom from Chapter I:

$$\sum_{(x)} x'\varepsilon(x'') = \sum_{(x)} \varepsilon(x')x'' = x,$$

and the composition lemma:

$$x \cdot (\phi_1 \circ \phi_2) = \sum_{(x)} (x' \cdot \phi_1) \circ (x''\phi_2) \quad (x \in \mathcal{U}_q).$$

PROPOSITION 9.2. *We have*

(1) $d''V^{0,a} = d'V^{a,0} = 0$

(2) $d'V^{k,a-k} = (-1)^k d''V^{k+1,a-k-1}$ *for* $k = 0, \ldots, a-1$.

PROOF. We will prove (1), (2) is proved in the same way after substituting the right hand side using the relation (8.11) of Theorem 8.6.

The fact that $d''V^{0,a} = 0$ is clear since $\Omega^\bullet$ is of length a.

By definition of d' we have

$$d'V^{a,0}(x_1, \dots, x_a, x_{a+1}) = x_1 \cdot V^{a,0}(x_2, \dots, x_{a+1}) \qquad (a)$$

$$+ \sum_{k=1}^{a} (-1)^k V^{a,0}(x_1, \dots, x_k x_{k+1}, \dots, x_{a+1}) \qquad (b)$$

$$+ (-1)^{a+1} \varepsilon(x_{a+1}) V^{a,0}(x_1, \dots, x_a). \qquad (c)$$

Using the composition lemma, we have

$$(a) = \sum_{(x_1), \dots, (x_a)} x_1' \cdot V_1(x_2', z_1) x_1'' \cdot (x_2'' \cdot [\dots])$$

where the terms in $[\dots]$ are the same as in $V^{a,0}(x_2, \dots, x_{a+1})$ except for the first factor. Hence

$$(a) = \sum_{(x_1), \dots, (x_a)} (x_1' \cdot V_1(x_2', z_1)) \left((x_1'' x_2'') \cdot [\dots] \right).$$

Using Theorem 8.6, the summand corresponding to k in (b), for $k < a$, is equal to

$$(-1)^k V_1(x_1', z_1) x_1'' \cdot \Big(V_2(x_2', z_2) x_2'' \cdot \dots x_{k-1}'' \cdot$$

$$\Big(\big(x_k' \cdot V_k(x_{k+1}', z_k) + \varepsilon(x_{k+1}') V_k(x_k', z_k) \big) x_k'' x_{k+1}'' \cdots \Big) \Big).$$

Using the counit axiom, this is equal to

$$(-1)^k V_1(x_1', z_1) x_1'' \cdot \Big(V_2(x_2', z_2) x_2'' \cdot \dots x_{k-1}'' \cdot \big(x_k' \cdot V_k(x_{k+1}', z_k)(x_k'' x_{k+1}'') \cdot (\dots) \big) \Big)$$

$$+ (-1)^k V_k(x_1', z_1) x_1'' \cdot (\dots x_{k-1} \cdot (V_k(x_k', z_k)(x_k'' x_{k+1}) \dots)).$$

Using the composition lemma one more time in the second term, we obtain:

$$(-1)^k V_1(x_1', z_1) x_1'' \cdot \Big(V_2(x_2', z_2) x_2'' \cdot \dots x_{k-1}'' \cdot \big(x_k' \cdot V_k(x_{k+1}', z_k)(x_k'' x_{k+1}'') \cdot (\dots) \big) \Big)$$

$$+ (-1)^k V_k(x_1', z_1) x_1'' \cdot \Big(\dots x_{k-1} \cdot \big(V_k(x_k', z_k) x_k'' \cdot (x_{k+1}' \cdot (\dots) x_{k+1}'' \cdot) \dots \big) \Big).$$

The term corresponding to $k = a$ in (b) is:

$$(-1)^a V^{a,0}(x_1, \dots, x_a x_{a+1}),$$

which is equal to:

$$(-1)^a \sum V_1(x_1', z_1) x_1'' \cdot \Big(\dots (x_{a-1}'' x_a) \cdot V_{a+1}(x_{a+1}, z_a) \dots \Big) +$$

$$(-1)^a \varepsilon(x_{a+1}) V^{a,0}(x_1, \dots, x_a).$$

It follows that all the terms in (b) cancel out except the first and the last, and it is clear that the first term cancels out with (a) and the last one cancels out with (c). $\square$

As consequence of the preceding results, we have the following:

THEOREM 9.3. *The element* $\mathcal{V} = (V^{0,1}, \dots, V^{a,0})$ *is an* $a-$*cocycle in the simple complex associated with* $C^\bullet(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda) \otimes \Omega^\bullet))$.

As a corollary, we have:

THEOREM 9.4. *The cocycle* $\mathcal{V}$ *induces linear maps*

$$f_m : \mathcal{H}^m(\Omega^\bullet)^* \longrightarrow \mathrm{Ext}^{a-m}_{\mathcal{U}_q}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda)) \quad 0 \le m \le a. \quad \square$$

PROOF. Let $\phi_m : \mathcal{H}^m(\Omega^\bullet) \longrightarrow \mathbb{C}$ be a linear form, we extend ϕ to the space $\mathcal{Z}^m(\Omega^\bullet)$ of cocycles up to homotopy. We obtain a linear form $\phi_m : \mathcal{Z}^m(\Omega^\bullet) \longrightarrow \mathbb{C}$. Then we extend ϕ_m to Ω^m by taking it to be zero on a complement of $\mathcal{Z}^m$ in Ω^m. Let $x_1, \dots, x_{a-m}$ be elements of $\mathcal{U}_q$, and set

$$\Phi_m = V^{a-m,m}(x_1 \otimes \dots \otimes x_{a-m}) \in \mathrm{Hom}\left(\mathcal{M}(w\lambda), \mathcal{M}(\lambda) \otimes \Omega^m\right).$$

We consider the composition

$$\begin{array}{ccc} \mathcal{M}(w\lambda) & \xrightarrow{\Phi_m} & \mathcal{M}(\lambda)\otimes\Omega^m \\ & & \big\downarrow {\scriptstyle Id\otimes\phi_m} \\ & & \mathcal{M}(\lambda)\otimes\mathbb{C}\cong\mathcal{M}(\lambda) \end{array}$$

The map obtained is then an element of $\mathrm{Hom}\left(\mathcal{U}_q^{\otimes a-m}, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda))\right)$. Since V is an a−cocycle by the preceding theorem , we have $d'V^{a-m,m} = (-1)^{a-m}d''V^{a-m+1,m-1}$, therefore $d'\left((Id\otimes\phi_m)\circ\Phi_m\right)=0$.

Meanwhile, ϕ_m was chosen up to homotopy, satisfying $d''\circ\phi_m=0$, hence, the resulting map is a cochain in the complex $C^\bullet(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda)))$. And since $d''V^{a-m,m}=(-1)^{a-m-1}d'V^{a-m-1,m+1}$ (by Proposition 9.2), the class of this cochain modulo coboundaries does not depend on the choice of ϕ_m up to homotopy because the above relation implies that the image of $\phi_m\circ d''$is $d'\left((Id\otimes(\phi_m\circ d''))\circ\Phi_m\right)$, hence a coboundary. Since the cohomology spaces of the cochain complex $C^\bullet(\mathcal{U}_q, \mathrm{Hom}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda)))$ are the Ext-spaces $\mathrm{Ext}^\bullet_{\mathcal{U}_q}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda))$, the theorem is proved. □

In order to illustrate this theorem, we assume that the numbers $\langle\lambda_p, \alpha_{i_p}^\vee\rangle$, $p=1,\ldots,a$, are nonnegative integers. The homology space $\mathcal{H}^a(\Omega^\bullet)$ is one-dimensional (q is not a root of unity), generated by (the image of) the linear form $r\in\Omega^{a*}$ defined by

$$r(\eta)=\mathrm{Res}_{z_a=0}\ldots\mathrm{Res}_{z_1=0}\left(z_1^{\langle\lambda_1,\alpha_{i_1}^\vee\rangle}\cdots z_a^{\langle\lambda_a,\alpha_{i_a}^\vee\rangle}\,\eta\right).$$

The element$f_a(r)$, in $\mathrm{Hom}_{\mathcal{U}_q}(\mathcal{M}(w\lambda), \mathcal{M}(\lambda))$, is the unique intertwiner between $\mathcal{M}(w\lambda)$ and $\mathcal{M}(\lambda)$ sending $v_{w\lambda}$ to $F_{i_a}^{\langle\lambda_a,\alpha_{i_a}^\vee\rangle+1}\cdots F_{i_1}^{\langle\lambda_1,\alpha_{i_1}^\vee\rangle+1}\,v_\lambda$.

Chapter IV. The case of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ and the Screening operators

10. The quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$

The affine algebra $\widehat{\mathfrak{sl}_2}$. We recall the definition of the affine algebra $\widehat{\mathfrak{sl}_2}$ and we fix some notations.

Let E, F, H be the standard generators of the Lie algebra $\mathfrak{sl}_2$. For X, Y in $\mathfrak{sl}_2$, we set $(X,Y) = \mathrm{tr}(XY)$, this defines an invariant bilinear form on $\mathfrak{sl}_2$. Thus $(E,F) = (F,E) = 1$, $(H,H) = 2$. We fix a complex number k and we set $B(X,Y) = k(X,Y)$.

The corresponding affine algebra $\widehat{\mathfrak{sl}_2}$ is defined by the generators X_n ($X \in \mathfrak{sl}_2$, $n \in \mathbb{Z}$) and $\mathbf{1}$, and the relations

$$\text{(a)} \quad [X_n, Y_m] = [X,Y]_{m+n} + nB(X,Y)\delta_{m+n,0}.\mathbf{1} \quad (X, Y \in \mathfrak{sl}_2,\ m,n \in \mathbb{Z}).$$

This algebra is realized as a central extension of the loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}]$ by $\mathbb{C}.\mathbf{1}$, and we identify X_n with $X \otimes z^n$.

If we introduce the generating functions (currents) $X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$,

($X \in \mathfrak{sl}_2$) then the formula (a) is equivalent to

$$\text{(b)} \quad X(z)Y(w) = \frac{B(X,Y)}{(z-w)^2} + \frac{[X,Y](w)}{z-w} + \text{ regular part at } \; z = w.$$

The affine algebra $\widehat{\mathfrak{sl}_2}$ is isomorphic to the Kac-Moody algebra ${\mathfrak{sl}_2}^{(1)}$ corresponding to the Cartan matrix of affine type $A_1^{(1)}$. To be more precise, we have to add a derivation to $\widehat{\mathfrak{sl}_2}$, that is an element D satisfying: $[D, X_n] = nX_n$ and $[D, \mathbf{1}] = 0$. And we have the identification

$${\mathfrak{sl}_2}^{(1)} = \mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}.\mathbf{1} \oplus \mathbb{C}.D.$$

Let $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda \oplus \mathbb{Z}\delta$ be the weight lattice and let $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the root lattice endowed with a symmetric bilinear form $(.,.)$ defined by $(\Lambda_0, \lambda_0) = 0$, $(\Lambda_0, \alpha_1) = 0$, $(\Lambda_0, \delta) = 1$, $(\alpha_0, \alpha_1) = 2$, $(\alpha_1, \delta) = 0$, $(\delta, \delta) = 0$, where $\Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}$, $\delta = \alpha_0 + \alpha_1$.

We define $P^* = \mathbb{Z}H_0 \oplus \mathbb{Z}H_1 \oplus \mathbb{Z}D$ as the dual space of P. The dual pairing is defined by

$$\langle H_i, \lambda \rangle = (\alpha_i, \lambda) \qquad (i = 0, 1) \quad \text{for } \lambda \in P.$$

If k is a nonnegative integer we denote by $P_k = \{(k-i)\Lambda_0 + i\Lambda_1,\ i = 0, 1, \ldots, k\}$ the set of dominant integral weights of level k, and we set $\lambda_i = (k-i)\Lambda_0 + i\Lambda_1$.

Deformation of the affine algebra. The quantum affine algebra $U_q({\mathfrak{sl}_2}^{(1)})$ is an associative algebra over $\mathbb{Q}(q)$ with $\mathbf{1}$, where q is a transcendental complex number, generated by e_i, f_i, $i = 0, 1$, and q^h ($h \in P^*$). The defining relations are as follows

$$q^h q^{h'} = q^{h+h'} \qquad q^0 = 1\ ,$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i\ , \qquad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i\ ,$$

$$[e_i, f_j] = \delta_{i,j}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$e_i^3 e_j - [3]e_i^2 e_j e_i + [3]e_i e_j e_i^2 - e_j e_i^3 = 0 \qquad (i \neq j),$$

$$f_i^3 f_j - [3]f_i^2 f_j f_i + [3]f_i f_j f_i^2 - f_j f_i^3 = 0 \qquad (i \neq j).$$

Here $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

We consider the algebra $U_q(\widehat{\mathfrak{sl}_2})$ to be the subalgebra of $U_q({\mathfrak{sl}_2}^{(1)})$ generated by e_i, f_i, $t_i = q^{h_i}$ $(i = 0, 1)$. The algebra $U_q({\mathfrak{sl}_2}^{(1)})$ has a Hopf algebra structure. The comultiplication is given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \ ,$$

$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \quad (i = 0, 1) \ ,$$

$$\Delta(q^h) = q^h \otimes q^h \qquad (h \in P^*).$$

The antipode A is given by

$$A(q^h) = q^{-h} \ , \quad A(e_i) = -t_i^{-1} e_i \ , \quad A(f_i) = -f_i t_i \quad (i = 0, 1) \ .$$

The counit ε is given by

$$\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad (i = 0, 1) \ , \quad \varepsilon(q^h) = 1.$$

This definition of the quantum affine algebra by Chevalley generators is due to Drinfeld and Jimbo [5, 12]. Later, Drinfeld gave another realization for $U_q(\widehat{\mathfrak{sl}_2})$ [6], which is the loop algebra version of the above algebra. In this realization $U_q(\widehat{\mathfrak{sl}_2})$ is an associative algebra generated by $\{E_n, \ n \in \mathbb{Z}\}$, $\{F_n, \ n \in \mathbb{Z}\}$, $\{H_n, \ n \in \mathbb{Z} - \{0\}\}$ and invertible K and $q^{\frac{k}{2}}$ satisfying the following relations

$$[H_n, K] = 0 \ , \quad [H_n, H_m] = \delta_{n+m,0}\frac{1}{n}[2n][k],$$

$$K E_n K^{-1} = q^2 E_n \ , \quad K F_n K^{-1} = q^{-2} F_n,$$

$$[H_n, E_m] = \frac{[2n]}{n} q^{\frac{-|n|k}{2}} E_{n+m} \ , \quad [H_n, F_m] = -\frac{[2n]}{n} q^{\frac{|n|k}{2}} F_{n+m} \ ,$$

$$E_{n+1}E_m - q^2 E_m E_{n+1} = q^2 E_n E_{m+1} - E_{m+1} E_n \ ,$$

$$F_{n+1}F_m - q^{-2} F_m F_{n+1} = q^{-2} F_n F_{m+1} - F_{m+1} F_n \ ,$$

$$[E_n, F_m] = \frac{1}{q - q^{-1}} \left(q^{\frac{k(n-m)}{2}} \psi_{n+m} - q^{\frac{k(m-n)}{2}} \phi_{n+m} \right).$$

Where ψ_n and ϕ_n are related to H_l by

(10.1) $$\sum_{n \in \mathbb{Z}} \psi_n z^{-n} = K \exp\left((q - q^{-1}) \sum_{l=1}^{\infty} H_l z^{-l} \right)$$

(10.2) $$\sum_{n \in \mathbb{Z}} \phi_n z^{-n} = K^{-1} \exp\left(-(q - q^{-1}) \sum_{l=1}^{\infty} H_{-l} z^{l} \right) \ .$$

Here, $\phi_n = \psi_{-n} = 0$ for $n \geq 0$. We define H_0 by the formula

$$K = \exp\left((q - q^{-1}) \frac{H_0}{2} \right).$$

The standard Chevalley generators $\{e_i, f_i, t_i\}$ are given by the identification

(10.3)

$$t_0 = q^k K^{-1} \ , \ t_1 = K \ , \ e_1 = E_0 \ , \ f_1 = F_0 \ , \ e_0 t_1 = F_1 \ , \ t_1^{-1} f_0 = E_{-1}.$$

This identification leads to an algebra isomorphism between the above realizations. Equivalently, the Drinfeld realization can be obtained using the generators E_n, F_n $(n \in \mathbb{Z})$, ϕ_{-n}, ψ_n $(n \in \mathbb{N})$ and $q^{\pm \frac{k}{2}}$. And if we consider the currents

$$E(z) = \sum_{n \in \mathbb{Z}} E_n z^{-n}, \quad F(z) = \sum_{n \in \mathbb{Z}} F_n z^{-n},$$

$$\phi(z) = \sum_{n=0}^{\infty} \phi_{-n} z^{n}, \quad \psi(z) = \sum_{n=0}^{\infty} \psi_n z^{-n} \ ,$$

then the defining relations can be written as

(10.4) $$\phi_0 \psi_0 = \psi_0 \phi_0 = 1,$$

$$(10.5) \qquad \phi(z)\phi(w) = \phi(w)\phi(z)\ , \quad \psi(z)\psi(w) = \psi(w)\psi(z),$$

$$(10.6) \qquad \phi(z)\psi(w) = \frac{(zq^{k-2} - w)(zq^{-k+2} - w)}{(zq^{k+2} - w)(zq^{-k-2} - w)}\psi(w)\phi(z),$$

$$(10.7) \qquad \phi(z)E(w) = \frac{zq^{-\frac{k}{2}+2} - w}{zq^{-\frac{k}{2}} - wq^2}E(w)\phi(z),$$

$$(10.8) \qquad \phi(z)F(w) = \frac{zq^{\frac{k}{2}-2} - w}{zq^{\frac{k}{2}} - wq^{-2}}F(w)\phi(z),$$

$$(10.9) \qquad \psi(z)E(w) = \frac{wq^{-\frac{k}{2}} - zq^2}{wq^{-\frac{k}{2}+2} - z}E(w)\psi(z),$$

$$(10.10) \qquad \psi(z)F(w) = \frac{wq^{\frac{k}{2}} - q^{-2}z}{wq^{\frac{k}{2}-2} - z}F(w)\psi(z),$$

$$(10.11) \quad [E(z), F(w)] = \frac{1}{q - q^{-1}}\left(\delta(\frac{w}{z}q^k)\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z}q^{-k})\phi(wq^{\frac{-k}{2}})\right)$$

$$(10.12) \qquad (z - q^2w)E(z)E(w) = (q^2z - w)E(w)E(z),$$

$$(10.13) \qquad (z - q^{-2}w)F(z)F(w) = (q^{-2}z - w)F(w)F(z),$$

where

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n.$$

These relations are understood to be between formal power series. Since the Drinfeld currents are suitable for bosonization, we will use this realization for the rest of this chapter. Drinfeld also gave the Hopf algebra structure for this current realization [4].

Proposition 10.1. *The algebra $U_q(\widehat{\mathfrak{sl}_2})$ has a Hopf algebra structure given by*

- *Comultiplication* Δ

$$\Delta(q^k) = q^k \otimes q^k,$$

$$\Delta(E(z)) = E(z) \otimes 1 + \phi(zq^{\frac{k}{2}\otimes 1}) \otimes E(zq^{k\otimes 1}),$$

$$\Delta(F(z)) = 1 \otimes F(z) + F(zq^{1\otimes k}) \otimes \psi(q^{1\otimes \frac{k}{2}}),$$

$$\Delta(\phi(z)) = \phi(zq^{-1\otimes \frac{k}{2}}) \otimes \phi(zq^{\frac{k}{2}\otimes 1}),$$

$$\Delta(\psi(z)) = \psi(zq^{1\otimes \frac{k}{2}}) \otimes \psi(zq^{-\frac{k}{2}\otimes 1}),$$

- *Counit* ε

$$\varepsilon(q^k) = \varepsilon(\psi(z)) = \varepsilon(\phi(z)) = 1,$$

$$\varepsilon(E(z)) = \varepsilon(F(z)) = 0.$$

- *Antipode* A

$$A(q^k) = q^{-k},$$

$$A(E(z)) = -\phi(zq^{-\frac{k}{2}})^{-1}E(zq^{-k}),$$

$$A(F(z)) = -F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1},$$

$$A(\phi(z)) = \phi(z)^{-1},$$

$$A(\psi(z)) = \psi(z)^{-1}.$$

If $\phi(z) = \sum_{n=0}^{\infty} \phi_{-n} z^n$ then the formula for $\Delta(\phi(z))$ means that

$$\Delta(\phi(z)) = \sum_{m,n} (q^{\frac{mk}{2}} \phi_{-n} z^n) \otimes (q^{-\frac{nk}{2}} \phi_{-m} z^m).$$

We can also formulate the comultiplication for the coefficients H_n, E_n and F_n.

PROPOSITION 10.2. *The formulas for the comultiplication in the previous proposition are equivalent to the following formulas:*

$$\Delta(H_n) = q^{\frac{|n|k}{2}} \otimes H_n + H_n \otimes q^{-\frac{|n|k}{2}},$$

$$\Delta(E_n) = E_n \otimes 1 + \sum_{m \geq 0} q^{-(2n+m)\frac{k}{2}} \phi_{-m} \otimes E_{n+m},$$

$$\Delta(F_n) = 1 \otimes F_n + \sum_{m \geq 0} F_{n-m} \otimes q^{(m-2n)\frac{k}{2}} \psi_m.$$

PROOF. It is easy to check the equivalence for E and F. It is also easy to see that the formula for $\Delta(H_n)$ gives the formulas for $\Delta(\phi(z))$ and $\Delta(\psi(z))$. Let us prove the other direction for $\phi(z)$. Recall that

$$K\phi(z) = \exp\left(-(q - q^{-1}) \sum_{n=1}^{\infty} H_{-n} z^n\right) = \sum_{n=0}^{\infty} K\phi_{-n} z^n,$$

Taking the derivative of both sides we obtain

$$-(q - q^{-1}) \sum_{n=1}^{\infty} \sum_{m=1}^{n} m H_m K \phi_{m-n} z^{n-1} = \sum_{n=1}^{\infty} n K \phi_{-n} z^{n-1},$$

therefore

$$-n(q - q^{-1}) H_{-n} = n K \phi_{-n} + \sum_{m=0}^{n-1} (q - q^{-1}) H_{-m} K \phi_{m-n} \quad (n \geq 0).$$

Using this relation, we can show by induction on n that the formula for $\Delta(\phi(z))$ implies the formula for $\Delta(H_{-n})$ $(n \geq 0)$. Similarly, the formula for $\Delta(\psi(z))$ implies the formula for H_n $(n \geq 0)$. □

11. Bosonization and q-analog of Wakimoto modules

Free Boson Realization of $U_q(\widehat{\mathfrak{sl}_2})$. Here we introduce the Heisenberg algebra generated by three free boson fields a, b and c. We construct a homomorphism from the quantum affine algebra to the Heisenberg algebra which will enable us to express the Drinfeld generators in terms of the Heisenberg generators.

The generators of the quantum Heisenberg algebra $\mathcal{H}_q(\mathfrak{sl}_2)$ are a_n, b_n, c_n, $n \in \mathbb{Z}$, p_a, p_b, and p_c. The relations are

$$(11.1) \quad [a_n, a_m] = \frac{1}{n}[(k+2)n][2n]\delta_{n+m,0} \ , \quad [a_0, p_a] = \frac{4h}{q-q^{-1}}(k+2),$$

$$(11.2) \quad [b_n, b_m] = -\frac{1}{n}[n]^2\delta_{m+n,0} \ , \quad [b_0, p_b] = \frac{-2h}{q-q^{-1}},$$

$$(11.3) \quad [c_n, c_m] = \frac{1}{n}[n]^2\delta_{m+n,0} \ , \quad [c_0, p_c] = \frac{2h}{q-q^{-1}},$$

where $q = e^h$. The remaining commutators vanish.

We define the completion $\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2)$ of $\mathcal{H}_q(\mathfrak{sl}_2)$ as follows:

$$\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2) = \varprojlim \mathcal{H}_q/I_n \quad (n > 0),$$

where I_n is the left ideal of $\mathcal{H}_q(\mathfrak{sl}_2)$ generated by all the polynomials in a_m, b_m, c_m, $m > 0$, of degrees greater than or equal to n (we set $\deg a_m = \deg b_m = \deg c_m = m$).

We form the generating functions:

$$a_\pm(z) = \pm(q-q^{-1})\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_{\pm n} z^{\mp n}\right),$$

$$b_\pm(z) = \pm(q-q^{-1})\left(\frac{b_0}{2} + \sum_{n=1}^{\infty} b_{\pm n} z^{\mp n}\right),$$

$$b(z) = -\sum_{n\neq 0} \frac{b_n}{[n]} z^{-n} + \frac{q-q^{-1}}{2h} b_0 \log z + p_b,$$

$$c_\pm(z) = \pm(q-q^{-1})\left(\frac{c_0}{2} + \sum_{n=1}^{\infty} c_{\pm n} z^{\mp n}\right),$$

$$c(z) = -\sum_{n\neq 0} \frac{c_n}{[n]} z^{-n} + \frac{q-q^{-1}}{2h} c_0 \log z + p_c.$$

For a real number α we define:

$$a(z;\alpha) = -\sum_{n\neq 0} \frac{a_n}{[(k+2)n]} q^{-\alpha|n|} z^{-n} + \frac{1}{k+2}\left(\frac{q-q^{-1}}{2h} a_0 \log z + p_a\right).$$

Let : : denote the normal ordering of a product of operators defined by moving the creation operators to the left and moving the annihilation operators to the right. In our case the annihilation operators are $\{a_n,\ b_n,\ c_n\ n \geq 0\}$ and the creation operators are $\{a_n,\ b_n,\ c_n,\ p_a,\ p_b,\ p_c,\ n<0\}$. For example

$$: \exp(b(z)) := \exp\left(-\sum_{n<0} \frac{b_n}{[n]} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{b_n}{[n]} z^{-n}\right) e^{p_b} z^{\frac{q-q^{-1}}{2h} b_0}.$$

The following proposition is inspired from [23, 8], with modifications either in the defining relations of the Heisenberg generators, or in the bosonization formulas.

Proposition 11.1. *There is a homomorphism* ω *from* $U_q(\widehat{\mathfrak{sl}_2})$ *to* $\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2)$ *which is defined on generators as follows:*

$$\begin{aligned}
\omega[E'(z)] &= -:e^{b_+(z)-(b+c)(zq)}: + :e^{b_-(z)-(b+c)(zq^{-1})}:,\\
\omega[F'(z)] &= e^{a_+(zq^{\frac{k}{2}+1})} : e^{b_+(zq^{k+2})+(b+c)(zq^{k+1})} :\\
&\quad - e^{a_-(zq^{-\frac{k}{2}-1})} : e^{b_-(zq^{-k-2})+(b+c)(zq^{-k-1})} :,\\
\omega[\psi(z)] &= e^{a_+(zq)} e^{b_+(zq^{\frac{k}{2}})+b_+(zq^{\frac{k}{2}+2})},\\
\omega[\phi(z)] &= e^{a_-(zq^{-1})} e^{b_-(zq^{-\frac{k}{2}})+b_-(zq^{-\frac{k}{2}-2})},
\end{aligned}$$

where $E'(z) = (q-q^{-1})E(z)$ *and* $F'(z) = (q-q^{-1})F(z)$.

LEMMA 11.2. *Let X and Y be two operators such that $[X,Y]$ commutes with X and with Y, then*

$$[X, e^Y] = [X,Y]e^Y \quad \text{and} \quad e^X e^Y = e^Y e^X e^{[X,Y]}.$$

PROOF. Since $[X,Y]$ commutes with X and Y, one shows easily by induction that

$$Y^n X = XY^n - n[X,Y]Y^{n-1} \quad (n \in \mathbb{N}),$$

which implies that

$$[X, e^Y] = [X,Y]e^Y.$$

And

$$\begin{aligned} Xe^Y &= e^Y X + [X, e^Y] \\ &= e^Y X + e^Y [X,Y] \\ &= e^Y (X + [X,Y]). \end{aligned}$$

Iterating, we obtain $X^n e^Y = e^Y (X + [X,Y])^n$ for $n \geq 0$. Therefore

$$e^X e^Y = e^Y e^{X+[X,Y]} = e^Y e^X e^{[X,Y]}. \quad \square$$

LEMMA 11.3. *We have the following commutation relations:*

$$e^{a_+(z)} e^{a_-(w)} = \frac{(w - zq^{k+4})(w - zq^{-k-4})}{(w - zq^k)(w - zq^{-k})} e^{a_-(w)} e^{a_+(z)},$$

$$e^{b_+(z)} e^{b_-(w)} = \frac{(z-w)^2}{(z - wq^2)(z - wq^{-2})} e^{b_-(w)} e^{b_+(z)},$$

$$e^{c_+(z)} e^{c_-(w)} = \frac{(z - wq^2)(z - wq^{-2})}{(z-w)^2} e^{c_-(w)} e^{c_+(z)},$$

$$e^{b_+(z)} : e^{b(w)} := \frac{z - wq}{zq - w} : e^{b(w)} : e^{b_+(z)}, \quad e^{c_+(z)} : e^{c(w)} := \frac{zq - w}{z - wq} : e^{c(w)} : e^{c_+(z)},$$

$$e^{b_-(z)} : e^{b(w)} := \frac{wq - z}{w - zq} : e^{b(w)} : e^{b_-(z)}, \quad e^{c_-(z)} : e^{c(w)} := \frac{w - zq}{wq - z} : e^{c(w)} : e^{c_-(z)},$$

$$: e^{(b+c)(z)} :: e^{(b+c)(w)} :=: e^{(b+c)(w)} :: e^{(b+c)(z)} : .$$

PROOF. We need to compute different brackets.

$$\begin{aligned}[a_+(z), a_-(w)] &= \sum_{n=1}^{\infty} -(q-q^{-1})^2 [a_n, a_{-n}] \left(\frac{w}{z}\right)^n \\ &= \sum_{n=1}^{\infty} \frac{-1}{n} \left(q^{-(k+4)n} + q^{-(k+4)n} - q^{kn} - q^{-kn}\right) \left(\frac{w}{z}\right)^n,\end{aligned}$$

therefore

$$e^{[a_+(z), a_-(w)]} = \frac{(w - zq^{k+4})(w - zq^{-k-4})}{(w - zq^k)(w - zq^{-k})}.$$

And

$$\begin{aligned}[b_+(z), b_-(w)] &= \sum_{n=1}^{\infty} -(q-q^{-1})^2 [b_n, b_{-n}] \left(\frac{w}{z}\right)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n} (q^n - q^{-n})^2 \left(\frac{w}{z}\right)^n,\end{aligned}$$

we deduce

$$e^{[b_+(z), b_-(w)]} = \frac{(z-w)^2}{(z - wq^2)(z - wq^{-2})}.$$

And

$$\begin{aligned}[b_+(z), b(w)] &= [(q-q^{-1})\frac{b_0}{2}, p_b] + \sum_{n=1}^{\infty} \frac{q-q^{-1}}{[n]} [b_n, b_{-n}] \left(\frac{w}{z}\right)^n \\ &= -h + \sum_{n=1}^{\infty} \frac{-1}{n} (q^n - q^{-n}) \left(\frac{w}{z}\right)^n,\end{aligned}$$

therefore

$$e^{[b_+(z), b(w)]} = q^{-1} \frac{z - wq}{z - wq^{-1}} = \frac{z - wq}{zq - w}.$$

Similar calculations give

$$e^{[b_-(z),b(w)]} = \frac{z - wq}{zq - w}.$$

Since $[b_n, b_{-n}] = -[c_n, c_{-n}]$ and $[b_0, p_b] = -[c_0, p_c]$, the brackets involving c are deduced from the above brackets. This proves the lemma. $\square$

For simplicity we will use the same notation for the elements of $U_q(\widehat{\mathfrak{sl}_2})$ and their images in $\widetilde{\mathcal{H}}(\mathfrak{sl}_2)$.

Using the above lemma and the fact that

$$\phi_0 = e^{-\frac{q-q^{-1}}{2}(a_0+b_0+c_0)} \quad \text{and} \quad \psi_0 = e^{\frac{q-q^{-1}}{2}(a_0+b_0+c_0)},$$

one can easily prove that $E(z)$, $F(z)$, $\phi(z)$ and $\psi(z)$ satisfy the defining relations of the algebra $U_q(\widehat{\mathfrak{sl}_2})$, except for the relation involving $[E(z), F(w)]$ which needs an explanation. We look at z and w as complex variables and we set $E'(z) = -E'_+(z) + E'_-(z)$ and $F'(z) = F'_+(z) - F'_-(z)$. Then we have

$$E'_+(z)F'_+(w) = \frac{zq^{-1} - wq^{k+1}}{z - wq^k} \; : E'_+(z)F'_+(w) : \qquad (|z| > |wq^k|),$$

$$E'_-(z)F'_-(w) = \frac{zq - wk^{-k-1}}{z - wq^{-k}} \; : E'_-(z)F'_-(w) : \qquad (|z| > |wq^{-k}|),$$

$$F'_+(w)E'_+(z) = \frac{wq^{k+1} - zq^{-1}}{wq^k - z} \; : F'_+(w)E'_+(z) : \qquad (|z| < |wq^k|),$$

$$F'_-(w)E'_-(z) = \frac{wq^{-k-1} - zq}{wq^{-k} - z} \; : F'_-(w)E'_-(z) : \qquad (|z| < |wq^{-k}|).$$

The other products have no poles, more precisely:

$$E'_+(z)F'_-(w) = F'_-(w)E'_+(z) =: E'_+(z)F'_-(w) : ,$$

$$E'_-(z)F'_+(w) = F'_+(w)E'_-(z) =: E'_-(z)F'_+(w) : .$$

Therefore, for $|z| >> |w|$, we have:

$$
\begin{aligned}
&E'(z)F'(w) = \\
&\quad -\frac{zq^{-1} - wq^{k+1}}{z - wq^k} : E'_+(z)F'_+(w) : \; - \frac{zq - wq^{-k-1}}{z - wq^{-k}} : E'_-(z)F'_-(w) : \\
&\quad + : E'_+(z)F'_-(w) : + : E'_-(z)F'_+(w) :,
\end{aligned}
$$

and for $|z| << |w|$ we have:

$$
\begin{aligned}
&F''(w)F'(z) = \\
&\quad -\frac{zq^{-1} - wq^{k+1}}{z - wq^k} : F'_+(w)E'_+(z) : \; - \frac{zq - wq^{-k-1}}{z - wq^{-k}} : F''_-(w)E'_-(z) : \\
&\quad + : F'_+(w)E'_-(z) : + : F'_-(w)E'_+(z) : .
\end{aligned}
$$

Since the normally ordered product does not depend on the order of the factors, we conclude that $E'(z)F''(w)$ and $F'(w)E'(z)$ have the same analytic continuation. The coefficient of z^{-n-1} in the Laurent expansion of $E'(z)F'(w) - F'(w)E'(z)$ is

$$\frac{1}{2\pi i}\int_{C_R} E'(z)F'(w)z^n dz - \frac{1}{2\pi i}\int_{C_r} F'(w)E'(z)z^n dz,$$

where C_R and C_r are circles on the z-plane of radii $R >> |w|$ and $r << |w|$ respectively, which is equal to the sum of the residues of the common analytic continuation. The latter is equal to

$$(wq^k)^{n+1}(q-q^{-1}) : E'_+(wq^k)F'_+(w) : -(wq^{-k})^{n+1}(q-q^{-1}) : E'_-(wq^{-k})F'_-(w) : .$$

Moreover,

$$: E'_+(wq^k)F'_+(w) := \psi(wq^{\frac{k}{2}})$$

and

$$: E'_-(wq^{-k})F'_-(w) := \phi(wq^{-\frac{k}{2}}).$$

Hence

$$[E'(z),F'(w)] = (q-q^{-1})\left(\delta(\frac{w}{z}q^{k})\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z}q^{-k})\phi(wq^{-\frac{k}{2}})\right),$$

which provides the right formula for $[E(z),F(w)]$. □

REMARK 11.1. The power series $b(z)$ and $c(z)$ are q–analogues of the scalar bosonic fields representing the $\beta\gamma$ system when $q=1$. When $k\neq -2$, the homomorphism ω provides representations of $U_q(\widehat{\mathfrak{sl}_2})$ in the Fock space representation of the Heisenberg algebra (see next section). When $k=-2$ (the critical level), the generators a_n commute among themselves and generate a commutative algebra. Therefore the representations of the quantum affine algebra at the critical level can be realized via ω in a smaller space, which is the tensor product of the Fock representation of the subalgebra of the Heisenberg algebra generated by b_n, c_n, $n\in\mathbb{Z}$, and a one-dimensional representation of the commutative algebra generated by a_n, $n\in\mathbb{Z}$, [8].

Representations. The $U_q(\widehat{\mathfrak{sl}_2})$-modules we will be dealing with are infinite dimensional. As finite dimensional representations, we have the so called *Spin* $\frac{1}{2}$ representations $V^{(l)}$ ($l\in\mathbb{Z}$), of dimension $l+1$, with basis $\{v^{(l)}_m,\ 0\leq m\leq l\}$ given by

$$e_1 v^{(l)}_m = [m]v^{(l)}_{m-1}\,,\quad f_1 v^{(l)}_m = [l-m]v^{(l)}_{m+1}\,,\quad t_1 v^{(l)}_m = q^{l-2m}v^{(l)}_m,$$

$$e_0 = f_1\,,\quad f_0 = e_1\,,\quad t_0 = t_1^{-1}\quad \text{on } V^{(l)},$$

here $v^{(l)}_m = 0$ for $m<0$ or $m>l$.

The infinite dimensional representations of $U_q(\widehat{\mathfrak{sl}_2})$ are created from the Fock module of the Heisenberg algebra via the homomorphism ω. We start by considering the vacuum state Ω of the boson Fock space which satisfies

$$a_n.\Omega = b_n.\Omega = c_n.\Omega = 0 \quad (n \geq 0).$$

We define the vector $\Omega_{r,s}$ by:

$$\Omega_{r,s} = \exp\left(r\frac{p_a}{2(k+2)} + s(p_b + p_c) \right) \ \Omega \quad (r, s \in \mathbb{Z}).$$

Let $\mathcal{F}$ be the free $\mathbb{Q}(q)$-algebra generated by $\{a_n, b_n, c_n, \ n < 0\}$ and let $\mathcal{F}_{r,s}$ be the Fock module defined by

$$\mathcal{F}_{r,s} := \mathcal{F}.\Omega_{r,s}.$$

It is clear that $\phi(z)$ and $\psi(z)$ map $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s} \otimes \mathbb{C}((z))$ and from the simple observation that

$$e^{\pm(p_b+p_c)}\Omega_{r,s} = e^{r\frac{p_a}{2(k+2)}+(s\pm1)(p_a+p_b)}\Omega,$$

we deduce that $E(z)$ maps $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s-1} \otimes \mathbb{C}((z^{-1}))$ and $F(z)$ maps $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s+1} \otimes \mathbb{C}((z^{-1}))$.

We denote by $V(\lambda)$ the Verma module over $U_q(\widehat{\mathfrak{sl}_2})$ of highest weight λ and generated by the highest weight vector v_λ. Thus

$$(11.4) \qquad e_0 v_\lambda = e_1 v_\lambda = 0 \ , \quad t_0 v_\lambda = q^\alpha v_\lambda \ , \quad t_1 v_\lambda = q^\beta v_\lambda,$$

where $\alpha\Lambda_0 + \beta\Lambda_1$ is the classical part of λ; Λ_0 and Λ_1 are the fundamental weights of $\mathfrak{sl}_2$. We refer to (11.4) as the highest weight condition.

PROPOSITION 11.4. *The vector* $\Omega_{r,0}$ *satisfies:*

$$\begin{aligned} E_n\Omega_{r,0} &= F_n\Omega_{r,0} = H_n\Omega_{r,0} = 0 \quad \text{if } n > 0, \\ E_0\Omega_{r,0} &= 0, \\ K\Omega_{r,0} &= q^r\Omega_{r,0}. \end{aligned}$$

PROOF. We have

$$E(z)\Omega_{r,0} = \exp\left(\sum_{n<0}\frac{b_n+c_n}{[n]}(zq)^{-n}\right)\Omega_{r,-1} + \exp\left((q-q^{-1})\sum_{n<0}b_n z^{-n} + \sum_{n<0}\frac{b_n+c_n}{[n]}(zq^{-1})^{-n}\right)\ \Omega_{r,-1},$$

it follows that, for $n \geq 0$, $E_n\Omega_{r,0} = 0$.

We have a similar formula for $F(z)\Omega_{r,0}$ except that, due to the presence of $a_{\pm}(z)$, the first term comes with the factor q^r and the second term comes with the factor q^{-r}, which implies that $F_n\Omega_{r,0} = 0$ for $n > 0$ only.

Also, we have $\psi(z)\Omega_{r,0} = q^r\Omega_{r,0}$, using (10.1) we deduce that $H_n\Omega_{r,0} = 0$ for $n \geq 0$ and $K\Omega_{r,0} = q^r\Omega_{r,0}$. □

COROLLARY 11.5. *The vector $\Omega_{r,0}$ satisfies the highest weight condition* (11.4) *and can be identified with the highest vector v_{λ_r}, where*

$$\lambda_r = (k-r)\Lambda_0 + r\Lambda_1.$$

PROOF. This follows from the proposition and the identification (10.3). □

The following proposition follows from the identification of v_{λ_r} with $\Omega_{r,0}$ and the action of the generators of $U_q(\widehat{\mathfrak{sl}_2})$ on $\Omega_{r,0}$:

PROPOSITION 11.6. *There is an embedding of the highest weight module in a direct sum of Fock modules:*

$$V(\lambda_r) \hookrightarrow \bigoplus_{s\in\mathbb{Z}} \mathcal{F}_{r,s}. \tag{11.5}$$

□

Remark 11.2. The module $V(\lambda_r)$ is reducible as it is known in Conformal Field Theory. If k and r are nonnegative integers, then one can see the reducibility from the facts that $f_0^{k-2r+1}\Omega_{r,0} \neq 0$ and $f_1^{2r+1}\Omega_{r,0} = 0$. To obtain irreducible modules, one has to use the q-analogue of the Felder resolution, see [18].

We set

$$\mathcal{W}_r := \bigoplus_{s\in\mathbb{Z}} \mathcal{F}_{r,s}. \tag{11.6}$$

This Fock space carries a $U_q(\widehat{\mathfrak{sl}_2})$-module structure defined by the bosonization formulae of Proposition 11.1. These are the q-analogues of the *Wakimoto* modules [25].

12. Screening Operators

The operator $S(z)$. The so called screening operator $S(z)$ is used to investigate the irreducible representations of $U_q(\widehat{\mathfrak{sl}_2})$, and to compute correlation functions in Conformal Field Theory. It is an element of $\widetilde{\mathcal{H}}(\mathfrak{sl}_2)$ which acts then on Fock space representations.

Definition 12.1. [1] The screening operator is given by

$$S(z) = \frac{1}{(q-q^{-1})z} : e^{-a(z,\frac{k+2}{2})} : \left(: e^{-b_-(z)-(b+c)(qz)} : - : e^{-b_+(z)-(b+c)(q^{-1}z)} : \right).$$

For simplicity we write $z(q-q^{-1})S(z) = S_1(z) - S_2(z)$.

Proposition 12.1. *The operator $S(z)$ sends $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r-1,s-1}$, and therefore sends the module $\mathcal{W}_r$ to the module $\mathcal{W}_{r-2} \otimes z^{-\frac{r}{k+2}}\mathbb{C}((z^{-1}))$.*

PROOF. $\exp\left(\frac{-p_a}{k+2}\right)$ sends $\Omega_{r,s}$ to $\Omega_{r-1,s}$, $\exp\left(-\frac{p_b+p_c}{2}\right)$ sends $\Omega_{r,s}$ to $\Omega_{r,s-1}$, and

$$\begin{aligned} e^{\frac{-p_a}{k+2}} e^{-\frac{q-q^{-1}}{2h(k+2)} a_0 \log z}\ \Omega_{r,s} &= e^{\frac{-p_a}{k+2}} e^{[-\frac{q-q^{-1}}{2h(k+2)} a_0 \log z, \frac{rp_a}{2(k+2)}]}\ \Omega_{r,s} \\ &= z^{-\frac{r}{k+2}} \Omega_{r-2,s} \quad \Box \end{aligned}$$

Now we compute the pairing between $E(z)$, $F(z)$, $\phi(z)$, $\psi(z)$ and $S(w)$.

PROPOSITION 12.2. *The operator $S(w)$ satisfies:*

$$< E(z), S(w) >= E(z)S(w) - \phi(zq^{\frac{k}{2}})S(w)\phi(zq^{\frac{k}{2}})^{-1}E(z), \tag{12.1}$$

$$< F(z), S(w) >= F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1} - S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}, \tag{12.2}$$

$$< \phi(z), S(w) >= \phi(zq^{\frac{k}{2}})S(w)\phi(zq^{\frac{k}{2}})^{-1} - S(w), \tag{12.3}$$

$$< \psi(z), S(w) >= \psi(zq^{-\frac{k}{2}})S(w)\psi(zq^{-\frac{k}{2}})^{-1} - S(w), \tag{12.4}$$

where these relations are understood to be between formal power series.

PROOF. We have

$$\begin{aligned} \Delta(E(z)) &= E(z) \otimes 1 + \phi(zq^{\frac{k}{2}\otimes 1}) \otimes E(zq^{k\otimes 1}) \\ &= E(z) \otimes 1 + \sum_{m\geq 0,n} \phi_{-m} q^{\frac{k}{2}m-km} z^m \otimes E_n z^{-n}, \end{aligned}$$

hence

$$(\mathrm{id} \otimes A)\Delta(E(z)) = E(z) \otimes 1 + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}m-kn} z^m \otimes A(E_n) z^{-n},$$

therefore

$$
\begin{aligned}
< E(z), S(w) > &= E(z)S(w) + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}m-kn} z^m S(w) A(E_n) z^{-n} \\
&= E(z)S(w) + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}} z^m S(w) A(E_n)(q^k z)^{-n} \\
&= E(z)S(w) - \phi(zq^{\frac{k}{2}}) S(w) \phi(zq^{\frac{k}{2}})^{-1}.
\end{aligned}
$$

This proves (12.1). The proof of the other formulas is similar, i.e. by expanding the currents and applying A to the second component of the tensor product, and using the fact that $q^{\frac{k}{2}}$ is central. □

PROPOSITION 12.3. *We have:*

$$
\begin{aligned}
&(12.5) \quad \phi(z)S(w) &&= S(w)\phi(z) =: \phi(z)S(w) :, \\
&(12.6) \quad \psi(z)S(w) &&= S(w)\psi(z) =: \psi(z)S(w) :, \\
&(12.7) \quad E(z)S(w) &&= S(w)E(z) \ , \quad \textit{and the products have no poles.}
\end{aligned}
$$

PROOF. The relations (12.5) and (12.6) follow from direct computations. (12.7) needs some explanation. Set

$$E_1(z) =: \exp(b_+(z) - (b+c)(zq)) : \quad \text{and} \quad E_2(z) =: \exp(b_-(z) - (b+c)(zq^{-1})) : .$$

Then we have

$$E_1(z)S_1(w) = S_1(w)E_1(z) = q : E_1(z)S_1(w) :,$$

and

$$E_2(z)S_2(w) = S_2(w)E_2(z) = q^{-1} : E_2(z)S_2(w) : .$$

The other products have a pole at $z = w$:

$$E_1(z)S_2(w) = S_2(w)E_1(z) = q\frac{z - wq^{-2}}{z - w} \ : E_1(z)S_2(w) :$$
$$= qw\frac{1 - q^{-2}}{z - w} \ : E_1(w)S_2(w) : + \text{ regular part at } z = w$$

and

$$E_2(z)S_1(w) = S_1(w)E_2(z) = q^{-1}\frac{z - wq^2}{z - w} \ : E_2(z)S_1(w) :$$
$$= q^{-1}w\frac{1 - q^2}{z - w} \ : E_2(w)S_1(w) : + \text{ regular part at } z = w.$$

It follows that

$$E_1(z)S_2(w) + E_2(z)S_1(w) = S_2(w)E_1(z) + S_1(w)E_2(z)$$
$$= w\frac{q - q^{-1}}{z - w}(: E_1(w)S_2(w) : - : E_2(w)S_1(w) :) + \cdots .$$

It is easy to see that $: E_1(w)S_2(w) :=: E_2(w)S_1(w) :$, and (12.7) follows. □

Before investigating how the Fourier coefficients behave with $S(z)$, we introduce the notion of q-difference operator:

For a function $f(z)$ and for $\alpha \in \mathbb{C}$, we define

$$\frac{\mathcal{D}_\alpha}{d_q z} f(z) := \frac{f(zq^\alpha) - f(zq^{-\alpha})}{z(q - q^{-1})}.$$

We can also write

$$\mathcal{D}_\alpha f(z) = \frac{f(zq^\alpha) - f(zq^{-\alpha})}{z(q - q^{-1})} d_q z. \tag{12.8}$$

In particular

$$\mathcal{D}_\alpha z^n = [\alpha n] z^{n-1} d_q z.$$

If f and g are two functions, the q-difference of the products is given by:

$$\begin{aligned}\mathcal{D}_\alpha(f(z)g(z)) &= f(zq^{\alpha})\mathcal{D}_\alpha(g(z)) + \mathcal{D}_\alpha(g(z))f(zq^{-\alpha}) \\ &= f(zq^{-\alpha})\mathcal{D}_\alpha(g(z)) + \mathcal{D}_\alpha(f(z))g(zq^{\alpha}).\end{aligned}$$

For $p, s \in \mathbb{C} - \{0\}$ with $|p| < 1$ and for a function $f(u)$, we define

$$\int_0^{s\infty} f(u)d_pu = s(1-p)\sum_{n=-\infty}^{\infty} f(sp^n)p^n$$

whenever it is convergent. This is a q-difference analogue of the ordinary integration, and it is called the Jackson integral along a q-interval $[0, s\infty]$. This integral has the following property:

$$\int_0^{s\infty} \frac{\mathcal{D}_\alpha}{d_pu} f(u)d_pu = 0.$$

All these notions are extended to multi-variable functions.

THEOREM 12.4. *For every integer n, we have:*

$$\begin{aligned}
&(12.9) & < E_n, S(w) > &= 0, \\
&(12.10) & < \phi_{-n}, S(w) > &= < \psi_n, S(w) >= 0 \quad (n \geq 0), \\
&(12.11) & < F_n, S(w) > &= \frac{\mathcal{D}_{k+2}}{d_qw}\left((wq)^n : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1}\right).
\end{aligned}$$

PROOF. The relations (12.9) and (12.10) follow immediately from Proposition 12.2 and Proposition 12.3. Set $(q-q^{-1})F(zq^{-k}) = F_1(zq^{-k}) - F_2(z^{-k})$, where

$$F_1(zq^{-k}) = e^{a_+(zq^{-\frac{k}{2}+1})} : \exp(b_+(zq^2) + (b+c)(zq)) :$$

and

$$F_2(zq^{-k}) = e^{a_-(zq^{-\frac{3k}{2}-1})} : \exp(b_-(zq^{-2k-2}) + (b+c)(zq^{-2k-1})) : .$$

We look at z and w as complex variables. To normally order $F_1(zq^{-k})S_1(w)$, we need the following brackets:

$$\begin{aligned}&\left[a_+(zq^{-\frac{k}{2}+1}), -a\left(w, \frac{k+2}{2}\right)\right]\\&\quad = \left[\frac{q-q^{-1}}{2}a_0, \frac{-1}{k+2}p_a\right] - \sum_{n=1}^{\infty}(q-q^{-1})\frac{q^{-2n}}{[(k+2)n]}[a_n, a_{-n}]\left(\frac{w}{z}\right)^n\\&\quad = -2h + \sum_{n=1}^{\infty}\frac{1}{n}(q^{-4n}-1)\left(\frac{w}{z}\right)^n.\end{aligned}$$

And $[b_+(zq^2), -b_-(w) - b(qw)] + [b(zq) - p_b, -b_-(w)]$ gives

$$h + \sum_{n=1}^{\infty}\frac{1}{n}(1-q^{-4n})\left(\frac{w}{z}\right)^n.$$

Therefore

$$F_1(zq^{-k})S_1(w) = q^{-1} : F_1(zq^{-k})S_1(w) : .$$

Meanwhile

$$S_1(w)F_1(zq^{-k}) = e^{[\frac{q-q^{-1}}{2}b_0, p_b]} : S_1(w)F_1(zq^{-k}) := q^{-1} : F_1(zq^{-k})S_1(w) : .$$

Similarly, we have:

$$F_2(zq^{-k})S_2(w) = S_2(w)F_2(zq^{-k}) = q : F_2(zq^{-k})S_2(w) : .$$

The other products have poles, indeed for $|w| << |z|$ we have:

$$F_1(zq^{-k})S_2(w) = q^{-1}\frac{z-w}{z-wq^{-2}} : F_1(zq^{-k})S_2(w) :$$

$$F_2(zq^{-k})S_1(w) = q\frac{z-wq^{2k}}{z-wq^{2k+2}} : F_2(zq^{-k})S_1(w) :$$

And for $|z| << |w|$ we have:

$$S_1(w)F_2(zq^{-k}) = q^{-1}\frac{w-zq^{-2k}}{w-zq^{-2k-2}} : S_1(w)F_2(zq^{-k}) :$$

$$S_2(w)F_1(zq^{-k}) = q\frac{w-z}{w-zq^2} : S_2(w)F_1(zq^{-k}) :$$

Since the normally ordered product does not depend on the order of the factors, we deduce from the above relations that $F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1}$ and $S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}$ have the same analytic continuation ($\psi(zq^{-\frac{k}{2}})^{-1}$ contains only positive modes, so it stays to the right when the products are normally ordered).

Therefore, in view of Proposition 12.2, we obtain:

$$\begin{aligned} < F_n, S(w) > &= \frac{1}{2\pi i}\int_{C_R} F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1}z^{n-1}dz \\ &- \frac{1}{2\pi i}\int_{C_r} S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}z^{n-1}dz, \end{aligned}$$

where C_R and C_r are circles of radii $R >> |w|$ and $r << |w|$ respectively. It follows that $< F_n, S(w) >$ is equal the sum of the residues of

$$\begin{aligned} &\frac{z^{n-1}}{(q-q^{-1})^2 w}\frac{q^{-1}(z-w)}{z-wq^{-2}} : F_1(zq^{-k})S_2(w) : \psi(zq^{-\frac{k}{2}})^{-1} + \\ &\frac{z^{n-1}}{(q-q^{-1})^2 w}\frac{q(z-wq^{2k})}{z-wq^{2k+2}} : F_2(zq^{-k})S_1(w)\psi(zq^{-\frac{k}{2}})^{-1}, \end{aligned}$$

therefore

$$\begin{aligned} < F_n, S(w) > = &\frac{1}{(q-q^{-1})w}\Big(-(wq^{-2})^n : F_1(wq^{-k-2})S_2(w)\psi(wq^{-\frac{k}{2}-2})^{-1} : \\ &+ (wq^{2k+2})^n : F_2(wq^{k+2})S_1(w)\psi(wq^{\frac{3k}{2}+2})^{-1} :\Big). \end{aligned}$$

Meanwhile

$$\begin{aligned} : F_1(wq^{-k-2})S_2(w) : &= e^{a_+(wq^{-\frac{k}{2}-1}) - a(w, \frac{k+2}{2})} : \\ &= e^{-a(wq^{-k-2}, -\frac{k+2}{2})}, \end{aligned}$$

Notice the change of sign in the argument of a. Similarly

$$: F_2(wq^{k+2})S_1(w) :=: e^{-a(wq^{k+2}, -\frac{k+2}{2})}.$$

It follows that

$$< F_n, S(w) >= \frac{\mathcal{D}_{k+2}}{d_q w}\left((wq^k)^n : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} :\right),$$

which proves (12.11). □

As a consequence, the bracket of the Jackson integral of $S(w)$ for $p = q^{k+2}$ with the generators of $U_q(\widehat{\mathfrak{sl}_2})$ vanishes. Using(12.11) we have

COROLLARY 12.5. *The Jackson integral,* $\int_0^{s\infty} S(w)d_p w \quad (p = q^{k+2})$ *is an invariant for* $U_q(\widehat{\mathfrak{sl}_2})$. □

Difference equations. We will state and prove two results concerning the screening operators, which deal with the following difference equation:

$$< x, S(w) >= \frac{\mathcal{D}_{k+2}}{d_q w} S(x, w), \tag{12.12}$$

for every $x \in U_q(\widehat{\mathfrak{sl}_2})$.

THEOREM 12.6. *For every* $x \in U_q(\widehat{\mathfrak{sl}_2})$, *there exists a well defined operator* $S(x, w)$ *such that:*

(1) $S(x,w)$ *vanishes on the Fourier coefficients of* $E(z)$, $\phi(z)$, $\psi(z)$, *and on* $q^{\pm\frac{k}{2}}$ *and*

$$S(F_m, w) = (wq)^m : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \quad (m \in \mathbb{Z}).$$

(2) *For every* $x, y \in U_q(\widehat{\mathfrak{sl}_2})$, *we have:*

$$S(xy, w) = x \cdot S(y, w) + \varepsilon(y)S(x, w). \tag{12.13}$$

PROOF. We consider $S(x,w)$ to be linearly dependent on x and defined on the generators as in (1), and we extend it to the free algebra generated by E_m, F_m, ϕ_{-n}, ψ_n ($m \in \mathbb{Z}$, $n \in \mathbb{N}$) and $q^{\pm\frac{k}{2}}$. In order to have $S(x,w)$ well defined for every $x \in U_q(\widehat{\mathfrak{sl}_2})$, we need to show that the relation (12.13) is compatible with the defining relations of $U_q(\widehat{\mathfrak{sl}_2})$. By linearity of $S(x,w)$, we have:

$$S(E(z),w) = S(\phi(z),w) = S(\psi(z),w) = 0, \tag{12.14}$$

and

$$S(F(z),w) = \delta\left(\frac{wq^k}{z}\right) : e^{-a(w,-\frac{k+2}{2})}\psi(zq^{\frac{k}{2}})^{-1} : . \tag{12.15}$$

It is clear that the relation (12.13) is compatible with the relations (10.4)–(10.7), (10.9) and (10.12) since (12.13) vanishes on both sides of these relations (they do not involve $F(z)$). We will prove the compatibility with (10.8) and (10.11).

From (10.8) we have

$$\phi(z_1)F(z_2) = g_k(z)F(z_2)\phi(z_1), \quad \text{with } g_k(z) = \frac{z_1q^{\frac{k}{2}-2} - z_2}{z_1q^{\frac{k}{2}} - z_2q^{-2}}.$$

And

$$\begin{aligned} S(\phi(z_1)F(z_2),w) &= \phi(z_1)\cdot S(F(z_2),w) + \varepsilon(F_2(z_2))S(\phi(z_1),w) \\ &= \phi(z_1q^{\frac{k}{2}})\delta\left(\frac{wq^k}{z_2}\right) : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \phi(z_1q^{\frac{k}{2}})^{-1}. \end{aligned}$$

Meanwhile,

(12.16)

$$\psi(wq^{\frac{k}{2}})^{-1}\phi(z_1q^{\frac{k}{2}})^{-1} = \frac{(z_1^{k+2} - w)(z_1q^{-k-2} - w)}{(z_1q^{k-2} - w)(z_1q^{-k+2} - w)}\phi(z_1q^{\frac{k}{2}})^{-1}\psi(wq^{\frac{k}{2}})^{-1}.$$

And since

$$\left[-a\left(w,-\frac{k+2}{2}\right), -a(z_1q^{\frac{k}{2}-1})\right] = 2h + \sum_{n=1}^{\infty}\left(q^{(k+2)n} - q^{(k-2)n}\right)\left(\frac{z_1}{w}\right)^n,$$

we obtain

$$e^{-a(w,-\frac{k+2}{2})}\phi(z_1q^{\frac{k}{2}})^{-1} = \frac{wq^2 - z_1q^k}{w - z_1q^{k+2}}\phi(zq^{\frac{k}{2}})^{-1}e^{-a(w,-\frac{k+2}{2})}. \tag{12.17}$$

From (12.16) and (12.17) we obtain:

(12.18)

$$S(\phi(z_1)F(z_2), w) = \frac{wq^2 - z_1q^{-k}}{w - z_1q^{-k+2}}\delta\left(\frac{wq^k}{z_2}\right) : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : .$$

We need to compare this expression with $S(g_k(z)F(z_2)\phi(z_1), w)$. Since we know that $S(g_k(z)\phi(z_1), w) = 0$, we have:

$$\begin{aligned} S(g_k(z)F(z_2)\phi(z_1), w) &= \varepsilon(g_k(z)\phi(z_1))S(F(z_2), w) \\ &= g_0(z)S(F(z_2), w). \end{aligned}$$

Now we use techniques from Vertex Operator Calculus [9], and we have:

$$g_0(z)\delta\left(\frac{wq^k}{z_2}\right) = \frac{z_2 - q^{-2}}{q^{-2}z_2 - z_1}\delta\left(\frac{wq^k}{z_2}\right) = \frac{wq^k - z_1q^{-2}}{q^{-2}z_2 - z_1}\delta\left(\frac{wq^k}{z_2}\right).$$

This follows from the fact that:

$$z^{-n}\delta\left(\frac{w}{z}\right) = w^{-n}\delta\left(\frac{w}{z}\right).$$

Therefore

$$S(g_k(z)F(z_2)\phi(z_1), w) = \frac{q^2w - q^{-k}z_1}{w - q^{-k+2}z_1}S(F(z_2), w),$$

which is equal to (12.18). The case of (10.10) is done similarly.

We will use a different approach to prove the compatibility with

$$[E(z), F(w)] = \frac{1}{q - q^{-1}}\left(\delta(\frac{w}{z}q^k)\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z}q^{-k})\phi(wq^{\frac{-k}{2}})\right)$$

Since $S(x, w)$ vanishes if x is the right hand side, we need to show that $S(E(z)F_m, w)$ and $S(F_mE(z))$ $(m \in \mathbb{Z})$, given by (12.13), are equal.

On one hand

$$S(F_mE(z),w) = F_m \cdot S(E(z),w) + \varepsilon(E(z))S(F_m,w)$$
$$= 0.$$

On the other hand

$$\begin{aligned} S(E(z)F_m,w) &= E(z)\cdot S(F_m,w) + 0 \\ &= E(z)S(F_m,w) \\ &- (wq^k)^n \phi(zq^{\frac{k}{2}}) : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \phi(zq^{\frac{k}{2}})^{-1}E(z). \end{aligned}$$

Using (10.9), we have

$$\psi(wq^{\frac{k}{2}})^{-1}E(z) = \frac{w - zq^{-k+2}}{wq^2 - zq^{-k}}E(z)\psi(wq^{\frac{k}{2}})^{-1}.$$

Keeping in mind the relations (12.16) and(12.17), we see that the second term in the expression of $S(E(z)F_m,w)$ is $E(z)S(F_m,w)$ times

$$\frac{(w-q^{k+2}z)(w-q^{-k-2})}{(w-q^{-k+2})(w-q^{k-2}z)}\cdot\frac{wq^2 - zq^k}{w-q^{k+2}}\cdot\frac{wq^{-2}-q^{-k}z}{w-q^{-k-2}z}.$$

This product is easily seen to be equal to 1. Therefore $S(E(z)F_m,w) = 0$. The case of (10.13) is similar. □

THEOREM 12.7. *The screening operator $S(z)$ satisfies:*

$$< x, S(z) >= \frac{\mathcal{D}_{k+2}}{d_q z}S(x,z), \tag{12.19}$$

for every $x \in U_q(\widehat{\mathfrak{sl}_2})$.

PROOF. This is already satisfied by the generators of $U_q(\widehat{\mathfrak{sl}_2})$ by Theorem 12.4. Since both sides are well defined for every $x \in U_q(\widehat{\mathfrak{sl}_2})$, assume that the equality holds for x and y, then, by Theorem 12.6, we have

$$\begin{aligned} < xy, S(z) > &= x\cdot < y, S(z) > +\varepsilon(y) < x, S(z) > \\ &= \frac{\mathcal{D}_{k+2}}{d_q z}(x \cdot S(y,z) + \varepsilon(y) S(x,z)) \\ &= \frac{\mathcal{D}_{k+2}}{d_q z} S(xy, z). \end{aligned}$$

Which proves the theorem. □

13. Universal q-de Rham cocycles

The simple case. From now on, we set $\alpha = k+2$ and we suppose $\alpha \neq 0$ (the level is noncritical), and let r be a complex number. Recall that the operators $z^{\frac{r}{\alpha}} S(z)$ and $z^{\frac{r}{\alpha}} S(x,z)$ ($x \in U_q(\widehat{\mathfrak{sl}_2})$) send W_r to $W_{r-2} \otimes \mathbb{C}((z^{-1}))$. We consider the following complex

$$\text{(13.1)} \qquad \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes \bullet}, \mathrm{Hom}(W_r, W_{r-2})\right).$$

The differential d' of this complex is given by

$$\begin{aligned} d'\phi(x_1, \dots, x_n) =& x_1 \cdot \phi(x_2, \dots, x_n) + \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \dots, x_i x_{i+1}, \dots, x_n) \\ &+ (-1)^n \varepsilon(x_n) \phi(x_1, \dots, x_{n-1}). \end{aligned}$$

The cohomology groups are $\mathrm{Ext}^{\bullet}_{U_q(\widehat{\mathfrak{sl}_2})}(W_r, W_{r-2})$. And the 0-cohomolgy group is exactly the space of $U_q(\widehat{\mathfrak{sl}_2})$-invariants.

Let $\Omega^0 = \mathbb{C}((z^{-1}))$ and $\Omega^1 = \mathbb{C}((z^{-1})) d_q z$ (the space of formal algebraic q-differentials 1-forms). We consider the following complex:

$$\text{(13.2)} \qquad \Omega^{\bullet} \; : \; 0 \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow 0.$$

The differential of this complex is given by

$$d''(f(z)) = \mathcal{D}'_\alpha(f(z)) - [r]\frac{q^{r-\alpha}d_q z}{z},$$

where $\mathcal{D}'_\alpha(f(z)) = \mathcal{D}_\alpha f(zq^\alpha)$ (the last derivative should be read as $f'(zq^\alpha)$ and not as $(f(zq^\alpha))'$).

Let us consider the double complex

$$\mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}, \mathrm{Hom}(W_r, W_{r-2}\otimes\Omega^\bullet)\right) = \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}\otimes W_r, W_{r-2}\otimes\Omega^\bullet\right).$$

which we denote by $C^{\bullet\bullet}$. Let $C^\bullet$ be the simple complex associated with this double complex, that is

$$C^n = \bigoplus_{a+b=n} C^{a,b} \quad (n\in\mathbb{Z}),$$

and its differential is given by $d = d' + (-1)^a d''$ when acting on $C^{a,b}$.

Set

$$S^{01} = q^{2r} z^{\frac{r}{\alpha}} S(zq^\alpha) d_q z,$$

$$S^{10}(x) = z^{\frac{r}{\alpha}} S(x,z) \quad (x\in U_q(\widehat{\mathfrak{sl}_2})).$$

Then (S^{01}, S^{10}) is a one-cochain in C^1, and we have:

THEOREM 13.1. *The cochain (S^{01}, S^{10}) is a one-cocycle in C^1.*

PROOF. We need to prove that

$$d''S^{01} = 0, \tag{13.3}$$

$$d'S^{01}(z) = 0, \tag{13.4}$$

$$d'S^{01} = d''S^{10}. \tag{13.5}$$

(13.3) follows from the fact that the complex $\Omega^\bullet$ has length 1, and (13.4) follows from Theorem 12.6. The relation (13.5) follows from Theorem 13.5 in the following way:

$$\begin{aligned}\mathcal{D}_\alpha(z^{\frac{r}{\alpha}}S(x,z)) &= \mathcal{D}_\alpha(z^{\frac{r}{\alpha}}S(x,zq^{-\alpha}) + (zq^\alpha)^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x,z)\\ &= [r]z^{\frac{r}{\alpha}-1}S(x,zq^{-\alpha}) + q^r z^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x,z),\end{aligned}$$

therefore

$$\begin{aligned}\mathcal{D}'_\alpha(z^{\frac{r}{\alpha}}S(x,z)) &= [r]q^{r-\alpha}S(x,z) + q^{2r}z^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x.zq^\alpha)\\ &= [r]q^{r-\alpha}S(x,z) + q^{2r}z^{\frac{r}{\alpha}} < x, S(zq^\alpha) >,\end{aligned}$$

by Theorem 12.7. We deduce that

$$\begin{aligned}d''(z^{\frac{r}{\alpha}}S(x,z)) &=< x, q^{2r}z^{\frac{r}{\alpha}}S(zq^\alpha) >\\ &= d'S^{01}.\end{aligned}$$

□

Compositions. Let p be a positive integer, and let us consider the ring

$$A_p = \mathbb{C}[[z_1,\dots,z_p]][\prod_{i=1}^p z_i^{-1}];$$

we look at A_p as the ring of functions on the p-th power of the formal punctured disk X_p.

Let Ω^a $(1 \le a \le p)$ denote the space of algebraic q-differential a-forms on the formal variety X_p. Thus, Ω^0 is just A_p, and elements of Ω^a have the form

$$\sum f(z_1,\dots,z_p)d_q z_{i_1}\wedge\dots\wedge d_q z_{i_a} \quad (f(z_1,\dots,z_p)\in A_p).$$

We consider the following complex

$$\Omega^\bullet \ : \ 0\longrightarrow\Omega^0\longrightarrow\ldots\longrightarrow\Omega^p\longrightarrow 0, \tag{13.6}$$

its differential is given by

$$d'' = D'_\alpha - \sum_{i=0}^{p-1} q^{r-\alpha-2i}[r-2i]\frac{d_q z_i}{z_i}.$$

(D'_α is defined above). For $i = 1,\ldots,p$, we set:

$$S'(z_i) = q^{2(r+i-p)} z_i^{r-2(p-i)} S(z_i q^\alpha),$$

$$S'(x, z_i) = z_i^{r-2(p-i)} S(x, z_i).$$

The operators $S'(z_i)$ and $S'(x,z_i)$ are elements of

$$\mathrm{Hom}\left(W_{r-2(i-1)}, W_{r-2i}\otimes A_p\right).$$

We consider the following double complex:

$$\begin{aligned} C^{\bullet\bullet} &= \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}, \mathrm{Hom}(W_r, W_{r-2p}\otimes\Omega^\bullet)\right) \\ &= \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}\otimes W_r, W_{r-2p}\otimes\Omega^\bullet\right). \end{aligned}$$

And for each $m = 0,\ldots,p$, we define the operators

$$S^{m,p-m} \in \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes m}, \mathrm{Hom}(W_r, W_{r-2p}\otimes\Omega^{p-m})\right)$$

as follows:

$$S^{m,p-m}(x_1,\ldots,x_m) =$$
$$(-1)^{\frac{m(m+1)}{2}} \sum_{1\le i_1\le\ldots\le i_m\le p} (-1)^{i_1+\cdots+i_m}\mathcal{S}(x_1,\ldots,x_m; i_1,\ldots,i_m),$$

and $\mathcal{S}$ is given by

$$\sum_{(x_1,\ldots,x_n)} S'(z_1)\cdot\ldots\cdot S'(x'_1, z_{i_1})x''_1\cdot(S'(z_{i_1+1})\cdot\ldots\cdot S'(x'_m, z_{i_m})x''_m$$
$$\cdot(S'(z_{i_m+1})\cdot\ldots\cdot S'(z_p)))d_q z_1\wedge\ldots\wedge\widehat{d_q}z_{i_1}\wedge\ldots\wedge\widehat{d_q}z_{i_m}\wedge\ldots\wedge d_q z_p.$$

In this expression, we consider initially the composition $S'(z_1) \cdot \ldots \cdot S'(z_p)$, and we substitute each $S'(z_{i_k})$ by $S'(x'_k, z_{i_k})x''_k \cdot \ldots$ and x''_k is acting on all the remaining factors on the right if there are any, if not, it is just $S'(x_k, z_{i_k})$ (which happens only when $i_k = p$. And the sum is taken over all the terms involved in the Sweedler notation for the comultiplication of $x_1, \ldots, x_m$.

The element $\mathcal{S} = (\mathcal{S}^{0,p}, \ldots, \mathcal{S}^{p,0})$ is p-cochain in the simple complex associated with the double complex $C^{\bullet\bullet}$.

THEOREM 13.2. *The cochain $\mathcal{S}$ is a p-cocycle.*

We need to prove that

$$d''\mathcal{S}^{0,p} = d'\mathcal{S}^{p,0} = 0,$$

and for $k = 0, \ldots, p-1$

$$d'\mathcal{S}^{k,p-k} = (-1)^k d''\mathcal{S}^{k+1,p-k-1}.$$

These relations follow from Theorem 12.6 and Theorem 12.7 and some lemmas on Hopf algebras. We give the proof in the case $p = 2$ to illustrate the techniques used, the general case is proved exactly in the same way but with lengthy formulas.

Let us assume $p = 2$, we have:

$$\begin{aligned}
\mathcal{S}^{0,2} &= S'(z_1)S'(z_2)d_q z_1 \wedge d_q z_2, \\
\mathcal{S}^{1,1}(x) &= \sum_{(x)} \left(\mathcal{S}'(x', z_1)x'' \cdot S'(z_2)d_q z_2 - S'(z_1)S'(\bar{x}, z_2)d_q z_1\right), \\
\mathcal{S}^{2,0}(x,y) &= \sum_{(x)} S'(x', z_1)x'' \cdot S(y, z_2).
\end{aligned}$$

We will use the counit axiom and the composition lemma several times, without mentioning them. We will drop the sigma in the Sweedler notation in order to simplify the expressions.

Since $\Omega^\bullet$ is of length 1, it is clear that $d''S^{0,2} = 0$. Let us prove that $d'S^{2,0} = 0$. We have

$$\begin{aligned} d'S^{2,0}(x,y,z) = x\cdot(S'(y',z_1)y''\cdot S(z,z_2)) - S(x'y',z_1)(x''y'')\cdot S(z,z_2) \\ + S(x',z_1)x''\cdot S(yz,z_2) - \varepsilon(z)S(x',z_1)x''\cdot S(y,z_2). \end{aligned}$$

Using Theorem 12.6, we get:

$$\begin{aligned} x\cdot(S'(y',z_1)y''\cdot S'(z,z_2)) =& x'\cdot S'(y',z_1)(x''y'')\cdot S'(z,z_2), \\ S'(x'y',z_1)(x''y'')\cdot S'(z,z_2) =& x'\cdot S'(y',z_1)(x''y'')\cdot S'(z,z_2) \\ &+ S'(x',z_1)(x''y)\cdot S'(z,z_2), \\ S'(x',z_1)x''\cdot S'(yz,z_2) =& S'(x',z_1)(x''y)\cdot S'(z,z_2) \\ &+ \varepsilon(z)S'(x',z_1)x''\cdot S'(y,z_2). \end{aligned}$$

Adding up, we get $d'S^{2,0} = 0$.

Now we prove that $d'S^{1,1} = -d''S^{2,0}$:

$$\begin{aligned} d'S^{1,1}(x,y) =& x\cdot(S'(y',z_1)y''\cdot S'(z_2))d_qz_2 - x\cdot(S'(z_1)S'(y,z_2))d_qz_1 \\ &- S'(x'y',z_1)(x''y'')\cdot S'(z_2)d_qz_2 + S'(z_1)S'(xy,z_2)d_qz_1 \\ &+ \varepsilon(y)S'(x',z_1)x''\cdot S'(z_2)d_qz_2 - \varepsilon(y)S(z_1)S'(x'z_2)d_qz_1. \end{aligned}$$

Using Theorem 12.6, we get:

$$d'S^{1,1}(x,y) = - < x', S(z_1) > x'' \cdot S'(y, z_2) d_q z_1$$
$$- S'(x', z_1) x'' \cdot < y, S'(z_2) > d_q z_2.$$

On the other hand:

$$\begin{aligned}\mathcal{D}'_\alpha S^{2,0}(x,y) =& [r] q^{r-\alpha} z_1^{\frac{r}{\alpha}-1} z_2^{\frac{r-2}{\alpha}} S(x', z_1) x'' \cdot S(y, z_2) d_q z_1 \\ &+ q^{2r} z_1^{\frac{r}{\alpha}} z_2^{\frac{r-2}{\alpha}} < x', S(zq^\alpha) > x'' \cdot S(y, z_2) d_q z_1 \\ &+ [r-2] q^{r-2-\alpha} z_1^{\frac{r}{\alpha}} z_2^{\frac{r-2}{\alpha}-1} S(x', z_1) x'' \cdot S(y, z_2) d_q z_2 \\ &+ q^{2(r-2)} z_1^{\frac{r}{\alpha}} z_2^{\frac{r-2}{\alpha}} S(x', z_1) x'' \cdot < x, S(z_2 q^\alpha) > d_q z_2,\end{aligned}$$

where we have used Theorem 12.7. Therefore

$$\begin{aligned}d''S^{2,0}(x,y) =& q^{2r} z_1^{\frac{r}{\alpha}} z_2^{\frac{r-2}{\alpha}} < x', S(zq^\alpha) > x'' \cdot S(y, z_2) d_q z_1 \\ &+ q^{2(r-2)} z_1^{\frac{r}{\alpha}} z_2^{\frac{r-2}{\alpha}} S(x', z_1) x'' \cdot < x, S(z_2 q^\alpha) > d_q z_2 \\ =& < x', S(z_1) > x'' \cdot S'(y, z_2) d_q z_1 \\ &+ S'(x', z_1) x'' \cdot < y, S'(z_2) > d_q z_2,\end{aligned}$$

which show that $d'S^{1,1} = -d''S^{2,0}$.

Finally, we need to prove that $d'S^{0,2} = d''S^{1,1}$:

$$\begin{aligned}
d'S^{0,2}(x) =& x \cdot (S'(z_1)S'(z_2))d_qz_1 \wedge d_qz_2 - \varepsilon(x)S'(z_1)S'(z_2)d_qz_1 \wedge d_qz_2 \\
=& x' \cdot S'(z_1)x'' \cdot S'(z_2)d_qz_1 \wedge d_qz_2 - \varepsilon(x)S'(z_1)S'(z_2)d_qz_1 \wedge d_qz_2 \\
=& < x', S'(z_1) > x'' \cdot S'(z_2)d_qz_1 \wedge d_qz_2 \\
& + \varepsilon(x')S'(z_1)x'' \cdot S'(z_2)d_qz_1 \wedge d_qz_2 - \varepsilon(x)S'(z_1)S'(z_2)d_qz_1 \wedge d_qz_2 \\
=& < x', S'(z_1) > x'' \cdot S'(z_2)d_qz_1 \wedge d_qz_2 \\
& + S'(z_1) < x, S'(z_2) > d_qz_1 \wedge d_qz_2,
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\mathcal{D}'_\alpha(S^{1,1}(x)) =& q^{r-\alpha}[r]q^{2(r-2)}z_1^{\frac{r}{\alpha}-1}z_2^{\frac{r-2}{\alpha}}S(x', z_1)x'' \cdot S(z_2)d_qz_1 \wedge d_qz_2 \\
& + q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}} < x', S(z_1q^\alpha) > x'' \cdot S(z_2)d_qz_1 \wedge d_qz_2 \\
& + q^{2r}[r-2]q^{r-2-\alpha}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}-1}S(z_1)S(x, z_2)d_qz_1 \wedge d_qz_2 \\
& + q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(z_1) < x, S(z_2q^\alpha) > d_qz_1 \wedge d_qz_2,
\end{aligned}$$

therefore

$$\begin{aligned}
d''S^{1,1}(x) =& q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}} < x', S(z_1q^\alpha) > x'' \cdot S(z_2)d_qz_1 \wedge d_qz_2 \\
& + q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(z_1) < x, S(z_2q^\alpha) > d_qz_1 \wedge d_qz_2 \\
=& < x', S'(z_1) > x'' \cdot S'(z_2)d_qz_1 \wedge d_qz_2 \\
& + S'(z_1) < x, S'(z_2) > d_qz_1 \wedge d_qz_2.
\end{aligned}$$

Which proves that $d'S^{0,2} = d''S^{1,1}$. □

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