Intertwining Operators into Cohomology Representations for Semisimple Lie Groups

A Dissertation Presented

by

Robert William Donley, Jr.

to

The Graduate School in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at Stony Brook

August 1996

State University of New York
at Stony Brook

The Graduate School

Robert William Donley, Jr.

We, the dissertation committee for the above candidate for the Doctorate of Philosophy degree, hereby recommend acceptance of this dissertation.

Anthony Knapp
Professor of Mathematics
Dissertation Director

Chih-Han Sah
Professor of Mathematics
Chairman of Defense

Leon Takhtajan
Professor of Mathematics

Folkert Tangerman
Visiting Assistant Professor of Applied Mathematics
& Statistics
Outside Member

This dissertation is accepted by the Graduate School.

Abstract of the Dissertation

Intertwining Operators into Cohomology Representations for Semisimple Lie Groups

by

Robert William Donley, Jr.
Doctor of Philosophy
in
Mathematics

State University of New York
at Stony Brook
1996

The Bott-Borel-Weil theorem allows one to realize all irreducible representations of a compact connected Lie group as cohomology spaces over generalized flag manifolds. Recasting these results via Hodge theory, one can realize all such representations as spaces of strongly harmonic forms over such manifolds. For noncompact semisimple Lie groups with finite center, similar constructions of representations in cohomology spaces are possible, and in such cases the representations are infinite-dimensional. This thesis relates these representations to the Langlands classification.

The irreducible admissible representations of a semisimple Lie group with finite center are classified by Langlands quotients of standard representations induced from cuspidal parabolic subgroups. In this thesis, intertwining operators are constructed from standard representations induced from minimal (cuspidal) parabolic subgroups into spaces of strongly harmonic forms. These strongly harmonic forms are realized on certain infinite dimensional vector bundles over open submanifolds of generalized flag manifolds. A Penrose transform is used to show that the interwining operator is nonzero when considered as a map into the associated Dolbeault cohomology.

To my mother,

Catherine

Table of Contents

Acknowledgements

Since this work represents my apprenticeship in mathematics, I would first like to thank my advisor, Anthony Knapp. There are too many reasons for giving thanks to Tony. Perhaps the best is for being a good role model, in mathematics and in life.

Many conversations have helped influence this work. While Tony did the lion's share of advising, I would also like to thank Michael Eastwood, Toshiyuke Kobayashi, Hisayosi Matumoto, Bent Ørsted, and David Vogan for their time and patience.

Thanks to all of my friends, especially Jim Anderson, Bianca DiBlasio, Renee Burton, Carlos Duran, Paul Friedman, Sandy Gilbert, Ad Harris, Marie-Loiuse Mishelsohn, Joe Schaefer, Maureen Schweers, Rob Stingley, and Leroy Wenstrom.

I would like to thank the Sloan Foundation; my last year at Stony Brook was supported by an Alfred P. Sloan Doctoral Dissertation Fellowship.

I would like to thank the Institut Mittag-Leffler for the wonderful time I spent there (Nov. 95 to May 96). I would also like to thank the organizers and many friends made during the year.

Thanks to the staff and faculty at Stony Brook math department.

Thanks to Anthony DeSalis and David Shale for starting me early.

Finally, thanks to my mother and grandmother. School is finally letting out.

Chapter 1. Introduction

Section 1.1. Motivation

The tradition of constructing representations for connected semisimple Lie groups by cohomology methods began with the Bott-Borel-Weil theorem in [Bo]. Though the Theorem of the Highest Weight was well-known at the time, irreducible representations of compact connected Lie groups were given concretely as sheaf cohomology spaces on vector bundles over generalized flag manifolds.

This result can be interpreted in several ways. Lie algebra cohomology ([Ko1]), Dolbeault cohomology ([S1]), and harmonic forms ([GS]) have all evolved as methods for studying the noncompact case.

Let G be a connected semisimple Lie group with finite center, $\mathfrak{g}_0$ its Lie algebra, and K a maximal compact subgroup. Denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$.

The first major generalization was Schmid's proof of the Kostant-Langlands conjecture ([Ko2], [L1]). In a series of papers beginning in [S1] and ending in [S2], Schmid gave an L^2-cohomology construction of the discrete series; these representations occur in spaces of strongly harmonic L^2-sections on certain line bundles. In many cases, Schmid established a definite relationship between the L^2-cohomology and the associated Dolbeault cohomology.

Meanwhile Langlands [L2] solved the classification problem for irreducible admissible representations using methods of real parabolic induction.

Knapp and Zuckerman revised this classification in [KZ] in terms of discrete series and limits of discrete series. The final result gives the classification in terms of "Langlands quotients" of the standard induced representations induced from cuspidal parabolic subgroups.

Zuckerman tried generalizing Schmid's construction to obtain a wider class of representations in Dolbeault cohomology, but he found the analytic difficulties to be insurmountable using the techniques at hand. Instead, he distilled the algebraic properties of the problem into an analogous theory now known as cohomological induction. Modules constructed via cohomological induction are known as Vogan-Zuckerman modules. Details of the theory can be found in [V2], [K3] and [KV]. Many powerful results have descended from cohomological induction, notably an algebraic classification of irreducible admissible $(\mathfrak{g}, K)$-modules, placement of Vogan-Zuckerman modules in the Langlands classification, and methods for detecting unitarity algebraically.

Recently Wong has overcome the analytic obstacles in Zuckerman's geometric formulation have been overcome. The problem, known as the Maximal Globalization Conjecture, is formulated precisely in [V2] and proven in [Wo]. This conjecture states that the analytic and algebraic cohomologies coincide; the non-vanishing Dolbeault cohomology group for a certain infinite-dimensional vector bundle has a well-behaved Frechet topology and has as its underlying $(\mathfrak{g}, K)$-module the Vogan-Zuckerman module with the corresponding parameters.

Results in [KV] relating the Langlands classification and cohomological induction have been derived using the methods of homological algebra. The purpose of this thesis is to construct integral intertwining operators that demonstrate this relationship directly at the analytic level. Here is a strategy to be used in constructing such operators:

(1.2.1) Pick parameters for a given irreducible representation relevant to the cohomological induction theory.

(1.2.2) Identify the Langlands' parameters of the Vogan-Zuckerman module associated to parameters in (1.2.1) by known theorems. Construct the associated standard representation induced from a cuspidal parabolic subgroup.

(1.2.3) Construct a mapping from the representation in (1.2.2) into the space of strongly harmonic forms associated to the representation in (1.2.1).

(1.2.4) Show that the mapping in (1.2.3) remains nonzero when followed with passage to Dolbeault cohomology.

Section 1.2. Results

The main results of this thesis, which occur as Theorems 5.2.1, 5.2.2, and 5.2.3, address steps (1.2.3) and (1.2.4) when the domain is a standard representation induced from a minimal (cuspidal) parabolic subgroup. This work generalizes techniques found in [BKZ]; work on an aspect of the non-minimal parabolic case can be found in [Ba].

The representations are constructed in Chapter 2, addressing steps (1.2.1) and (1.2.2). Sections 2.1–2.3 collect their defining data and Sections

2.4–2.5 give their constructions.

Chapters 3 and 4 concern cohomology results where the fiber for cohomology is a non-unitary principal series representation. An intertwining operator $\mathcal{S}$ (defined in Theorem 3.1.1) maps an induced representation into a cochain space for cohomology. The image of $\mathcal{S}$ will lie in the kernel of $\bar{\partial}$ by Theorem 3.2.1 and in the kernel of the formal adjoint to $\bar{\partial}$ by Theorem 3.3.5.

In Chapter 4 the operator $\mathcal{S}$ is shown to have nonzero image in Dolbeault cohomology. The technique is to compose $\mathcal{S}$ with a Penrose transform. An explicit formula for this composition is given in Theorem 4.2.4, from which non-vanishing follows in Corollary 4.2.5.

In Chapter 5 the results of Chapters 3 and 4 are shown to hold if the fiber for cohomology is replaced with its irreducible quotient. The quotient is given in terms of the Langlands quotient operator. The analogs of Theorem 3.2.1, Theorem 3.3.5, and Corollary 4.2.5 are respectively Theorem 5.2.1, Theorem 5.2.2, and Theorem 5.2.3.

Showing that the domain of $\mathcal{S}$ descends to the Langlands quotient remains. Results in [Wo] suggest methods of doing so, but we do not pursue them here.

Chapter 2. Constructing Representations

Section 2.1. Preliminaries

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition corresponding to K, let θ be the corresponding Cartan involution of $\mathfrak{g}_0$, and let Θ be the corresponding Cartan involution of G.

Let $G^{\mathbb{C}}$ be the complexification of the adjoint group of G. As a convention, real Lie algebras have zero subscripts; their complexifications do not. We denote the Killing form on $\mathfrak{g}$ by $(\cdot, \cdot)$. For $X, Y \in \mathfrak{g}$, set $\langle X, Y \rangle = (X, \bar{Y})$.

Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a θ-stable Cartan subalgebra in $\mathfrak{g}_0$ with compact part $\mathfrak{t}_0$ and associated Cartan subgroup $H = TA$. We assume that $\mathfrak{a}_0$ is maximally abelian in $\mathfrak{p}_0$. Then the subgroup $M = Z_K(\mathfrak{a}_0)$ is compact, although not necessarily connected. Let M_0 be the identity component of M and let $\mathfrak{m}_0$ be its Lie algebra. $\mathfrak{t}_0$ is maximal abelian in $\mathfrak{m}_0$.

Choose $X \in \mathfrak{t}_0$ and let $L = Z_G(X)$. The subgroup L is connected and Θ-stable. Let $\mathfrak{l}_0$ be the Lie algebra of L. Choose a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ in $\mathfrak{g}$. Then G/L is an open complex submanifold of the compact complex manifold $G^{\mathbb{C}}/Q$, where Q is the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. We identify (via this inclusion) the antiholomorphic tangent space at the identity of G/L with $\mathfrak{u}$.

Section 2.2. Parabolic Subalgebras and Roots

We will need to consider several root systems and here we establish their positive roots by way of parabolic subalgebras. These subalgebras will depend somewhat on the parameters for the Dolbeault cohomology, which we fix in this section in accordance with [KV], Theorem 11.216. Let λ_M be an element of $\mathfrak{t}^*$ which is integral for M and let $\nu \in \mathfrak{a}^*$. Several restrictions will be imposed on λ_M and ν presently. All bundles are holomorphic and homogeneous.

Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{l} \cap \mathfrak{m}$ containing $\mathfrak{t}$. We denote the κ-ortho-complement of $\mathfrak{t}$ in $\mathfrak{b}$ by $\mathfrak{n}_{L\cap M/T}$. The subalgebra $\mathfrak{b}' = \mathfrak{b} \oplus (\mathfrak{u} \cap \mathfrak{m})$ is a Borel subalgebra for $\mathfrak{m}$; we denote its nilpotent radical by $\mathfrak{n}_{M/T}$.

Form any real parabolic subgroup of L with Levi factor $(L \cap M)A$ and denote its nilpotent factor by N_L. Let $(\mathfrak{n}_L)_0$ be the Lie algebra of N_L. Also let $\overline{N}_L$ be the subgroup of G with Lie algebra $\theta(\mathfrak{n}_L)_0 = (\bar{\mathfrak{n}}_L)_0$.

Let $\Delta(\mathfrak{g}, \mathfrak{h})$ denote the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let the root space decomposition be given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space associated to α. We choose positive root systems for $\mathfrak{l}$ and $\mathfrak{g}$. Let

$$\Delta^+(\mathfrak{l}, \mathfrak{h}) = \{\text{roots contributing to } \mathfrak{n}_L \oplus \mathfrak{n}_{L\cap M/T}\}$$

and let

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{l}, \mathfrak{h}) \cup \Delta(\mathfrak{u}, \mathfrak{h}).$$

If $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ that $\mathfrak{h}$ acts on semisimply and $\Delta(\mathfrak{g}', \mathfrak{h})$ is the set of $\mathfrak{h}$-weights, define

$$\delta(\mathfrak{g}') = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{g}')} \alpha.$$

Here are the restrictions on our parameters:

(2.2.1) $\langle \text{Re } \nu, \beta \rangle > 0$ for every positive $\mathfrak{a}$-root β of $\mathfrak{l}$,

(2.2.2) $\lambda_M + \delta(\mathfrak{n}_{L \cap M/T})$ is strictly dominant for $\mathfrak{b}$, and

(2.2.3) $\langle \lambda_M + \delta(\mathfrak{n}_{L \cap M/T}) + \delta(\mathfrak{u} \cap \mathfrak{m}) + \text{Re } \nu, \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$.

We choose a real parabolic subgroup for G with Levi factor MA. The Lie algebra $\mathfrak{n}_0$ of the nilpotent factor N is chosen to satisfy three conditions:

(2.2.4) $\Delta(\mathfrak{n})$ is closed under conjugation,

(2.2.5) $\Delta(\mathfrak{n})$ contains $\Delta^+_{real}(\mathfrak{l}, \mathfrak{h})$, and

(2.2.6) $\langle \text{Re } \nu, \alpha \rangle > 0$ for $\alpha \in \Delta(\mathfrak{n})$.

Section 2.3. Disconnectedness of M

As noted before, the group M is compact but not necessarily connected. The discussion follows Chapter IV of [KV].

The disconnectedness of M is captured by a large Cartan subgroup. $M = M_0 T_M$, where $T_M = N_M(\mathfrak{b})$, the normalizer of $\mathfrak{b}$ in M. The irreducible representations of M are in one-one correspondence with the irreducible dominant representations of T_M. Given the dominant integral weight $\lambda_{\mathfrak{m}}$, the

T_M-orbit yields an irreducible dominant representation of T_M and we call the associated M-representation (σ, V^σ). When M is connected, this is just the theorem of the highest weight. In general, V^σ is a direct sum of irreducible $\mathfrak{m}$-representations; the set of highest weights is a T_M-orbit of λ_M. Let $\langle\cdot,\cdot\rangle_\sigma$ be an M-invariant inner product on V^σ.

Choose a nonzero weight vector $\phi \in V^\sigma$ associated to λ_M; this is a highest weight vector with respect to the given positive ordering for $\mathfrak{m}$. Define the irreducible $(L \cap M)$-representation (τ, V^τ) as the cyclic $L \cap M$-span of ϕ. Denote the projection from V^σ to V^τ by P_τ. When M is linear, $L \cap M$ meets every component of M but this is not true in general. When M is connected, $L \cap M$ is connected and V^τ is the $\mathfrak{l} \cap \mathfrak{m}$-span of ϕ. In general, V^τ is a direct sum of irreducible $\mathfrak{l} \cap \mathfrak{m}$-representations and the set of highest weights are a subset of the T_M-orbit of λ_M. Let $\langle\cdot,\cdot\rangle_\tau$ be the $L \cap M$-invariant inner product on V^τ induced from $\langle\cdot,\cdot\rangle_\sigma$.

One final observation is that

$$P_\tau((\bar{\mathfrak{u}} \cap \mathfrak{m})V^\sigma) = 0. \tag{2.3.1}$$

Each irreducible summand of V^τ as an $\mathfrak{l} \cap \mathfrak{m}$-representation is contained in the irreducible summand of the $\mathfrak{m}$-representation V^σ with the same highest weight. Thus $\bar{\mathfrak{u}} \cap \mathfrak{m}$ cannot map elements of one irreducible summand in V^τ into another; thus it is enought to prove the observation in the case of one summand.

Let V_i^τ be the $\mathfrak{l}\cap\mathfrak{m}$-summand with highest weight λ_i and highest weight vector v_i. Consideration of roots shows that every weight vector of V_i^τ is of the form

$$\gamma = \lambda_i - \sum n_\alpha \alpha$$

where α ranges over the simple roots in $\Delta^+(\mathfrak{l} \cap \mathfrak{m}, \mathfrak{t})$. Application of the $\mathfrak{t}$-simple root vector $E_{-\beta}$ in $\bar{\mathfrak{u}} \cap \mathfrak{m}$ to any vector in V_i^τ gives a weight vector in the corresponding $\mathfrak{m}$-summand V_i^σ; this vector does not have a weight as above. Thus the observation holds.

Section 2.4. The Principal Series Representation

The domain of the operator S will be a non-unitary principal series representation of G. There are several possible ways of defining this representation and in this paper we will use two of them.

Recall the representation (σ, V^σ) of M from the last section. Let ρ_G be half the sums of the $\mathfrak{a}_0$-roots of $\mathfrak{n}_0$. The non-unitary principal series representation

$$(\pi_G, \mathrm{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1))$$

has as a dense subspace (denoted by $C^\infty(G/MAN, \sigma \otimes e^\nu)$)

$$\{f \in C^\infty(K, V^\sigma) : f(gman) = e^{-(\nu+\rho_G) log a}\sigma(m)^{-1} f(g) \quad \text{for } \ m \in M, a \in A \text{and } \ n \in N\}$$

and G acts by

$$(\pi_G(g)f)(x) = f(g^{-1}x).$$

The space is completed with respect to the norm

$$||f||_G^2 = \int_K |f(k)|_\sigma^2 \, dk.$$

This construction is known as the induced picture; this picture is easier to use when the focus is group actions.

For $g \in G$, let $g = \kappa(g)e^{H(g)}n(g)$ be the Iwasawa decomposition for $G = KAN$. We can get an equivalent representation to the previous representation by restricting functions in the dense subspace to K and completing with respect to the same norm. The group action becomes

$$(\pi_G(g)f)(k) = e^{-(\nu+\rho_G)H(g^{-1}k)}f(\kappa(g^{-1}k)).$$

This construction is known as the compact picture and has the benefit of making the space independent of the parameter ν. This space will be used implicitly when computing with minimal K-types.

Section 2.5. The Dolbeault Cohomology Representation

Recall that if (π, W) is a representation of L, one can construct a homogeneous holomorphic line bundle over G/L with coefficients in W. Since our primary interest is the space of sections of this bundle, we omit construction of this bundle; this can be found in [K5]. The sections will be realized instead as functions on G with certain invariance properties.

In the present case W will be infinite dimensional. The representation of L will be a principal series representation induced from the minimal parabolic subgroup $(L\cap M)AN_L$. Let ρ_L be half the sum of the $\mathfrak{a}_0$-roots of $(\mathfrak{n}_L)_0$. In the next section we will construct a one-dimensional representation χ^{-1} of $L\cap M$ which has differential $\delta(\mathfrak{u}\cap\mathfrak{m}) - \delta(\mathfrak{u})$; let $\tau' = \chi\otimes\tau$.

Construct the representation (π_L, W) with

$$W = \mathrm{ind}_{(L\cap M)AN_L}^{L}(\tau' \otimes e^{\nu} \otimes 1)$$

as in Section 5. Here the dense space of smooth vectors is given as

$$\{f \in C^{\infty}(L, V^{\tau'}) : f(lman) = e^{-(\nu+\rho_L)loga}\ \tau'(m)^{-1}f(l)\};$$

the other features are given by replacing G and K with L and $L \cap K$.

Let $s = \dim(\mathfrak{u} \cap \mathfrak{k})$. We are interested in the $(0, s)$-th Dolbeault cohomology group $H_{Dol}^{0,s}(G/L, W)$ of the holomorphic vector bundle $(G/L, W)$. We defer questions of topology to [Wo]. The image of the intertwining operator $\mathcal{S}$ will lie in the space of strongly harmonic forms on this bundle.

Let $C^{0,k}(G/L, W)$ denote the space of smooth $(0, k)$-forms with coefficients in W. These forms can be represented as the space

$$(C^{\infty}(G, W \otimes \wedge^k \mathfrak{u}^*))^L,$$

which consists of smooth functions F from G into $W \otimes \wedge^k \mathfrak{u}^*$ with the property that

$$F(gl) = (\pi_L(l) \otimes Ad(l))^{-1} F(g)$$

where $g \in G$ and $l \in L$. In Section 3.2, we give an explicit formula for the $\bar{\partial}$ operator; since $\bar{\partial}^2 = 0$, we can define the $(0, k)$-th Dolbeault cohomology group

$$H_{Dol}^{0,k}(G/L, W) = (ker\ \bar{\partial}_k)/(im\ \bar{\partial}_{k-1})$$

where $\bar{\partial}_i$ is the restriction of $\bar{\partial}$ to $C^{0,i}(G/L, W)$.

We define $\bar{\partial}^*$ in Section 3.3 formally on all of $C^{0,k}(G/L, W)$ for each k. Define the space of strongly harmonic $(0, k)$-forms

$$\mathcal{H}^{0,k}(G/L, W) = (ker\ \bar{\partial}_k) \cap (ker\ \bar{\partial}_k^*).$$

Since each element of $\mathcal{H}^{0,k}(G/L, W)$ is a cocycle, there is a natural map

$$c : \mathcal{H}^{0,k}(G/L, W) \to H^{0,k}(G/L, W).$$

Chapter 3. The Intertwining Operator and Cohomology

3.1. The Operator $\mathcal{S}$

Before defining the operator $\mathcal{S}$, a form in $\wedge^s\mathfrak{u}^*$ needs to be distinguished. Choose a basis $\{X_\alpha\}$ of $\mathfrak{h}$-root vectors for $\mathfrak{g}$; a subset of these root vectors form a basis for $\mathfrak{u}$ and let $\{\omega_\alpha\}$ denote the corresponding dual basis. Let $\Delta_s = \Delta(\mathfrak{u}\cap\mathfrak{m})\cup\Delta(\mathfrak{u}\cap\bar{\mathfrak{n}})$. As in [BKZ], define

$$\omega_S = \bigwedge_{\alpha\in\Delta_s} \omega_\alpha.$$

We note that ω_S is an $\mathfrak{a}$-weight vector with weight $\rho_G - \rho_L$ and $L\cap M$ acts on ω_S by the character χ^{-1}. χ^{-1} is one-dimensional since L normalizes $\mathfrak{u}$ and M normalizes $\mathfrak{m}$ and $\bar{\mathfrak{n}}$. The $\mathfrak{a}$-weight value follows since

$$\Delta(\mathfrak{n},\mathfrak{a}) = \Delta(\mathfrak{u}\cap\mathfrak{n},\mathfrak{a})\cup\Delta(\mathfrak{n}_L,\mathfrak{a}).$$

Restricted to $\mathfrak{t}$,

$$\begin{aligned}\delta(\mathfrak{u}) &= \delta(\mathfrak{u}\cap\mathfrak{m}) + \delta(\mathfrak{u}\cap\bar{\mathfrak{n}}) + \delta(\mathfrak{u}\cap\mathfrak{n})\\ &= \delta(\mathfrak{u}\cap\mathfrak{m}) + 2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}}),\end{aligned}$$

and thus the $\mathfrak{t}$-weight of χ is given by

$$\begin{aligned}-2(\delta(\mathfrak{u}\cap\bar{\mathfrak{n}}) + \delta(\mathfrak{u}\cap\mathfrak{m})) &= -2(\frac{1}{2}(\delta(\mathfrak{u}) - \delta(\mathfrak{u}\cap\mathfrak{m})) + \delta(\mathfrak{u}\cap\mathfrak{m}))\\ &= \delta(\mathfrak{u}\cap\mathfrak{m}) - \delta(\mathfrak{u}).\end{aligned}$$

Under the usual action by ad, ω_S vanishes when acted upon by elements in $\mathfrak{u}\cap\mathfrak{n}$ or $\mathfrak{u}\cap\mathfrak{m}$. This follows because $ad(X_\alpha)\omega_\beta$ is a (possibly zero) multiple of $\omega_{\beta-\alpha}$ when $X_\alpha \in \mathfrak{g}_\alpha$ and ω_β is the vector dual to X_β in $\mathfrak{u}^*$; since the root spaces have multiplicity one and $ad|_{\mathfrak{u}\cap\mathfrak{m}\oplus\mathfrak{u}\cap\bar{\mathfrak{n}}}$ has no non-trivial fixed vectors in $\mathfrak{u}^*$, each factor of ω_S is mapped to another factor (up to scale) of different weight. Hence the exterior product vanishes.

Note that the above implies $\mathbb{C}\omega_S$ is an irreducible $(L\cap M)AN_L$ representation.

Theorem 3.1.1. The intertwining operator

$$\mathcal{S} : ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1) \to C^{0,s}(G/L, W)$$

is defined by

$$[\mathcal{S}f(x)](l) = P_\tau(f(xl))\ Ad(l)\omega_S$$

where $x \in G$ and $l \in L$.

Proof. First we show that $[\mathcal{S}f(x)]$ has the correct $(L\cap M)A(N_L)$-invariance as a function of L. If $m \in L\cap M, a \in A$ and $n \in N_L$ then

$$[\mathcal{S}f(x)](lman)$$

$$= P_\tau(f(xlman))\ Ad(lman)\,\omega_S$$

$$= P_\tau(f(xlman))\ Ad(lma)\,\omega_S$$

by the highest weight property of ω_S

$$= P_\tau(e^{-(\nu+\rho_G)log\,a}\,\sigma(m)^{-1}\,f(xl))\ e^{(\rho_G-\rho_L)\,log\,a}\,\chi(m)^{-1}Ad(l)\,\omega_S$$

by properties of f and ω_S

$$= e^{-(\nu+\rho_L)log\,a}\chi(m)^{-1}\tau(m)^{-1}\,P_\tau(f(xl))\ Ad(l)\,\omega_S$$

$$= e^{-(\nu+\rho_L)log\,a}\,\tau'(m)^{-1}[(\mathcal{S}f(x))(l)]$$

Hence $[\mathcal{S}f(x)]$ is in $ind^L_{(L\cap M)AN_L}(\tau'\otimes e^\nu\otimes 1)$.

To show that $\mathcal{S}f(x)$ has the correct L-invariance, observe that

$$\begin{aligned}[\mathcal{S}f(xl')](l) &= P_\tau(f(xl'l))Ad(l)\omega_S\\ &= P_\tau(f(xl'l))Ad(l')^{-1}Ad(l'l)\omega_S\\ &= Ad(l')^{-1}[(\mathcal{S}f(x))(l'l)]\\ &= [(\pi_L(l')\otimes Ad(l'))^{-1}\mathcal{S}f(x)](l)\end{aligned}$$

It follows that the image of $\mathcal{S}$ lies in the space of cochains. □

Remark: The formula for $\mathcal{S}$ follows from a composition of maps which we describe below. Note that we include the parameter shifts for clarity.

$$\begin{aligned}ind^G_{MAN}(\sigma\otimes e^{\nu+\rho_G}\otimes 1) &\to ind^G_{(L\cap M)AN_L}(\tau\otimes e^{\nu+\rho_G}\otimes 1)\\ &\to ind^G_L(ind^L_{(L\cap M)AN_L}(\tau\otimes e^{\nu+\rho_G}\otimes 1))\\ &\to ind^G_L(ind^L_{(L\cap M)AN_L}(\tau'\otimes e^{\nu+\rho_L}\otimes 1)\otimes\Lambda^s\mathfrak{u}^*)\end{aligned}$$

The first map is given by projecting the values into V^τ. The second is a double induction isomorphism; this is given by

$$[(\Phi f)(x)](l) = f(xl)$$

for $f \in ind^G_{(L\cap M)AN_L}(\sigma \otimes e^\nu \otimes 1)$. For the third map, we note that since $\mathbb{C}\omega_S$ is an irreducible $(L\cap M)AN_L$-representation, it can be used as the first term in a composition series for $\wedge^s \mathfrak{u}^*$ with respect to $(L\cap M)AN_L$. Note that

$$\begin{aligned} ind^L_{(L\cap M)AN_L}(\tau \otimes e^{\nu+\rho_G} \otimes 1) &\cong ind^L_{(L\cap M)AN_L}((\tau' \otimes e^{\nu+\rho_L} \otimes 1) \otimes \mathbb{C}\omega_S) \\ &\subset ind^L_{(L\cap M)AN_L}(\tau' \otimes e^{\nu+\rho_L} \otimes 1) \otimes \mathbb{C}Ad(L)\omega_S \\ &\subset ind^L_{(L\cap M)AN_L}(\tau' \otimes e^{\nu+\rho_L} \otimes 1) \otimes \wedge^s \mathfrak{u}^*. \end{aligned}$$

The first line follows by checking the parameters for ω_S. The second line follows by a "Mackey isomorphism"; right to left, the map is given by

$$M(f \otimes v)(l) = f(l) \otimes Ad(l)^{-1} v,$$

and in the other direction

$$M'(f)(l) = (1 \otimes Ad(l))f(l).$$

3.2. The Cocycle Property

In this section a formula for the Cauchy-Riemann operator

$$\bar\partial : C^{0,k}(G/L, W) \to C^{0,k+1}(G/L, W)$$

is given for all k and shown to annihilate the image of $\mathcal{S}$ when $k = s$.

For $f \in C^\infty(G/L, W)$ and $\omega \in \wedge^k \mathfrak{u}^*$, define

$$\bar{\partial} f\omega = \sum_{\alpha \in \Delta(\mathfrak{u})} X_\alpha f \ (\omega_\alpha \wedge \omega) + \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u})} f \ ad(X_\alpha)(\omega_\alpha \wedge \omega).$$

We note that although this formula uses the basis of root vectors given in Section 3.1, $\bar{\partial}$ is independent of the complex basis chosen for $\mathfrak{u}$.

Theorem 3.2.1. $\bar{\partial} \circ \mathcal{S} = 0$

This follows immediately from the next two lemmas. The first lemma shows how to remove the dependence on the L-variable and the second shows the vanishing. That vanishing occurs term-by-term with respect to a basis of root vector is suggestive. Fix f in $C^\infty(G/MAN, \sigma \otimes e^\nu)$.

Lemma 3.2.2. For $x \in G$ and $l \in L$,

$$\bar{\partial}(P_\tau(f(\tilde{x}l)) \ Ad(l)\omega_S)|_{\tilde{x}=x} = Ad(l)(\bar{\partial}(P_\tau(f(\tilde{x}))\omega_S)|_{\tilde{x}=xl}).$$

Proof. Before computing, we make an observation. If $X \in \mathfrak{g}_0$ then

$$(3.2.1) \qquad \begin{aligned} Xf(\tilde{x}l)|_{\tilde{x}=x} &= \frac{d}{dt} f(x\, exp(tX)l)|_{t=0} \\ &= \frac{d}{dt} f(xl\, exp(tAd(l)^{-1}X))|_{t=0} \\ &= (Ad(l)^{-1}X) \ f(\tilde{x})|_{\tilde{x}=xl}. \end{aligned}$$

Let $\bar{\partial} = \bar{\partial}_1 + \bar{\partial}_2$ in the obvious manner. Then

$\bar{\partial}_1(P_\tau(f(\tilde{x}l))Ad(l)\omega_S)|_{\tilde{x}=x}$

$$= \sum_{\alpha\in\Delta(\mathfrak{u})} (X_\alpha P_\tau(f(\tilde{x}l))|_{\tilde{x}=x})\ \omega_\alpha \wedge Ad(l)\omega_S$$

$$= \sum_{\alpha\in\Delta(\mathfrak{u})} (Ad(l)^{-1}X)P_\tau(f(\tilde{x}))|_{\tilde{x}=xl}\ Ad(l)(Ad(l)^{-1}\omega_\alpha\ \wedge\ \omega_S)$$

by the above observation

$$= Ad(l)[\sum_{\alpha\in\Delta(\mathfrak{u})} (Ad(l)^{-1}X)P_\tau(f(\tilde{x}))|_{\tilde{x}=xl}\ Ad(l)^{-1}\omega_\alpha\ \wedge\ \omega_S]$$

$$= Ad(l)\ I$$

and

$\bar{\partial}_2(P_\tau(f(\tilde{x}l))\ Ad(l)\omega_S)|_{\tilde{x}=x}$

$$= \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl))\ ad(X_\alpha)(\omega_\alpha \wedge Ad(l)\omega_S)$$

$$= \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl))\ Ad(l)Ad(l)^{-1}\ ad(X_\alpha)Ad(l)\ (Ad(l)^{-1}\omega_\alpha \wedge \omega_S)$$

$$= Ad(l)\ [\frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl))\ ad(Ad(l)^{-1}X_\alpha)(Ad(l)^{-1}\omega_\alpha \wedge \omega_S)]$$

$$= Ad(l)\ II.$$

Thus the left hand side in the Lemma is equal to $Ad(l)\ (I + II)$. But the definition of $\bar{\partial}$ is independent of the complex basis chosen for $\mathfrak{u}$; since $Ad(l)$ is a complex linear isomorphism on $\mathfrak{u}$, the lemma follows. □

Lemma 3.2.3. For $x \in G$, $\bar{\partial}(P_\tau(f(\tilde{x}))\omega_S)|_{\tilde{x}=x} = 0$.

Proof. Using the notation for $\bar{\partial}_1$ and $\bar{\partial}_2$ as in Lemma 1, we have

$$\bar{\partial}_1(P_\tau(f(\tilde{x}))\omega_S)|_{\tilde{x}=x} = \sum_{\alpha\in\Delta(\mathfrak{u})} (X_\alpha P_\tau(f(\tilde{x})))|_{\tilde{x}=x}\, \omega_\alpha \wedge \omega_S$$

If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{m}) \cup \Delta(\mathfrak{u}\cap\bar{\mathfrak{n}})$ then $\omega_\alpha \wedge \omega_S = 0$ by definition of ω_S. If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{n})$ then $Re(X_\alpha)$ and $Im(X_\alpha)$ are in $\mathfrak{n}_0$. Hence $X_\alpha P_\tau(f(\tilde{x}))|_{\tilde{x}=x} = 0$ by complex linearity of the derivative and right N-invariance of f.

Next we have

$$\bar{\partial}_2(P_\tau(f(\tilde{x}))\, \omega_S)|_{\tilde{x}=x} = \sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(x))\, ad(X_\alpha)(\omega_\alpha \wedge \omega_S)$$

If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{m}) \cup \Delta(\mathfrak{u}\cap\bar{\mathfrak{n}})$ then $\omega_\alpha \wedge \omega_S = 0$ by definition of ω_S. If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{n})$ then

$$ad(X_\alpha)(\omega_\alpha \wedge \omega_S) = \omega_\alpha \wedge ad(X_\alpha)\omega_S = 0$$

by the highest weight property of ω_S.

Thus the lemma follows. □

3.3. The Strongly Harmonic Property

The proof that $\bar{\partial}^* \circ S = 0$ closely resembles the proof in the previous section except that some care must be used when computing with the adjoints. Recall the G-invariant Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. This induces a non-degenerate Hermitian form on $\bigwedge \mathfrak{g}$ and in particular a non-degenerate Hermitian form on $\bigwedge \mathfrak{u}$. This form extends to a non-degenerate Hermitian form on $\wedge^k \mathfrak{u}^*$ in a natural way for each k.

We will also need to define an adjoint of the space W as in [KS]. First we change our point of view on W to the compact picture. Recall the Iwasawa decomposition of Section 5. The Iwasawa decomposition of $L = (L \cap K)AN_L$ coincides with the one for $G = KAN$. We will use the same notation.

Recall the norm for W is given by

$$||f||_L^2 = \int_{L \cap K} |f(l)|_\tau^2 \, dl$$

where $f \in W$ and dl is the Haar measure for $L \cap K$. This norm does not in general give a unitary structure for W, but one can use it to get an L-invariant non-degenerate pairing between (π_L, W) and (π'_L, W') where

$$W' = ind_{(L \cap M)AN_L}^L (\tau' \otimes e^{-\bar{\nu}} \otimes 1).$$

The group action for π'_L is given by

$$(\pi'_L(l)f)(k) = e^{-(-\bar{\nu}+\rho_L) \ loga} f(\kappa(l^{-1}k)).$$

Note that W and W' are the same space in the compact picture. The proof of this invariance can be found as Lemmas 20 and 24 in [KuS].

With these pairings in place, one can define the adjoint $\bar{\partial}^*$ to $\bar{\partial}$ as usual by

$$\int_{G/L} \langle f(x), \bar{\partial}^* g(x) \rangle \; dx = \int_{G/L} \langle \bar{\partial} f(x), g(x) \rangle \; dx$$

where f runs through the forms with compact support modulo L.

First we prove the analog of Lemma 3.2.2, writing our formula for $\bar{\partial}^*$ simply as

$$\bar{\partial}^* f\omega = \sum_{\alpha \in \Delta(\mathfrak{u})} (X_\alpha)^* f \, (\omega_\alpha \wedge)^* \omega + \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u})} f (Ad(X_\alpha))^* (\omega_\alpha \wedge)^* \omega.$$

First we compute some adjoint formulas.

Lemma 3.3.1. For $l \in L$,

(a) $Ad(l)^* X = Ad(l)^{-1} X$ for $X \in \mathfrak{u}$.

(b) $(\omega_\alpha \wedge)^* Ad(l) \omega = Ad(l)((Ad(l)^{-1} \omega_\alpha \wedge)^* \omega)$ for $\omega \in \wedge^k \mathfrak{u}^*$.

(c) $ad(X_\beta)^* Ad(l) \omega_\alpha = Ad(l)(ad(Ad(l)^{-1} X_\beta)^* \omega_\alpha)$.

Proof. (a) Let $X, Y \in \mathfrak{u}$. Then

$$\begin{aligned} \langle Ad(l)^* X, Y \rangle &= \langle X, Ad(l) Y \rangle \\ &= \langle Ad(l)^{-1} X, Y \rangle \end{aligned}$$

since l is real.

(b) If $\omega' \in \wedge^{k-1}\mathfrak{u}^*$ then

$$\begin{aligned}\langle(\omega_\alpha\wedge)^* Ad(l)\omega, \omega'\rangle &= \langle Ad(l)\omega, \omega_\alpha \wedge \omega'\rangle \\ &= \langle \omega, Ad(l)^*(\omega_\alpha \wedge \omega')\rangle \\ &= \langle (Ad(l)^{-1}\omega_\alpha\wedge)^*\omega, Ad(l)^*\omega'\rangle \text{ by (a)} \\ &= \langle Ad(l)(Ad(l)^{-1}\omega_\alpha\wedge)^*\omega, \omega'\rangle.\end{aligned}$$

(c) Let $\omega \in \mathfrak{u}^*$ and $X \in \mathfrak{u}$. Then

$$\begin{aligned}\langle ad(X)^* Ad(l)\omega_\alpha, \omega\rangle &= \langle Ad(l)\omega_\alpha, ad(X)\omega\rangle \\ &= \langle \omega_\alpha, Ad(l)^* ad(X)\omega\rangle \\ &= \langle \omega_\alpha, Ad(l)^{-1} ad(X) Ad(l) Ad(l)^{-1}\omega\rangle \\ &= \langle \omega_\alpha, ad(Ad(l)^{-1}X) Ad(l)^*\omega\rangle \\ &= \langle Ad(l)(ad(Ad(l)^{-1}X)^*\omega_\alpha), \omega\rangle. \qquad \square\end{aligned}$$

Again fix $f \in C^\infty(G/MAN, \sigma \otimes e^\nu)$.

Lemma 3.3.2. For $x \in G$ and $l \in L$,

$$\bar\partial^*(P_\tau(f(\tilde{x}l))Ad(l)\omega_S)|_{\tilde{x}=x} = Ad(l)(\bar\partial^*(P_\tau f(\tilde{x})\omega_S)|_{\tilde{x}=xl}).$$

Proof. As in Section 3.2, let $\bar\partial^* = \bar\partial_1^* + \bar\partial_2^*$ in the obvious manner. As before, we consider each sum separately. Again we note that the formula for $\bar\partial^*$ is independent of the $\mathfrak{u}$-basis chosen.

For the first sum, we note from (3.2.1) and from Lemma 3.3.1 (a) that if $X \in \mathfrak{g}_0$ and $l \in L$ then

$$X^* f(\tilde{x}l)|_{\tilde{x}=x} = (Ad(l)^{-1}X)^* f(\tilde{x})|_{\tilde{x}=xl}.$$

We compute

$$\bar\partial_1^*(P_\tau(f(\tilde{x}l))Ad(l)\omega_S)|_{\tilde{x}=x}$$

$$= \sum_{\alpha\in\Delta(\mathfrak{u})} ((X_\alpha)^* P_\tau(f(\tilde{x}l))|_{\tilde{x}=x})\ (\omega_\alpha\wedge)^* Ad(l)\omega_S$$

$$= \sum_{\alpha\in\Delta(\mathfrak{u})} (Ad(l)^{-1}X_\alpha)^* P_\tau(f(\tilde{x}))|_{\tilde{x}=xl}\ Ad(l)(Ad(l)^{-1}\omega_\alpha\ \wedge)^*\ \omega_S)$$

by Lemma 3.3.1 (b) and the above observation

$$= Ad(l)[\sum_{\alpha\in\Delta(\mathfrak{u})} (Ad(l)^{-1}X_\alpha)^*(P_\tau f(\tilde{x}))|_{\tilde{x}=xl}\ (Ad(l)^{-1}\omega_\alpha\ \wedge)^*\ \omega_S]$$

$$= Ad(l)\ I'.$$

For the second sum we have that

$$\bar\partial_2^*(P_\tau(f(\tilde{x}l))Ad(l)\omega_S)|_{\tilde{x}=x}$$

$$= \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl)ad(X_\alpha)^*((\omega_\alpha\wedge)^* Ad(l)\omega_S)$$

$$= \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl))Ad(l)Ad(l)^{-1}ad(X_\alpha)^* Ad(l)((Ad(l)^{-1}\omega_\alpha\wedge)^*\omega_S)$$

by Lemma 3.3.1 (b)

$$= Ad(l)[\frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(xl))ad(Ad(l)^{-1}X_\alpha)^*((Ad(l)^{-1}\omega_\alpha\wedge)^*\omega_S)]$$

by Lemma 3.3.1 (c)

$$= Ad(l)\ II'.$$

As in Lemma 3.2.2, $I' + II'$ is just $\bar{\partial}^*(P_\tau(f(\tilde{x}))\omega_S)|_{\tilde{x}=xl}$ written in terms of the $\mathfrak{u}$-basis $\{Ad(l)^{-1}X_\alpha\}$. Thus the lemma follows. □

To show that $P_\tau(f(x))\omega_S$ lies in the kernel of $\bar{\partial}^*$, it will be enough to compute in terms of root vectors, independent of scale. We list some important observations, the last two of which depend on the fact that $\langle X_\alpha, X_\beta\rangle$ is nonzero if $\beta = -\bar{\alpha}$ and is zero otherwise. We will also refer to $(\omega_\alpha\wedge)^*$ by the more familiar $i(\omega_\alpha)$, the interior product.

Lemma 3.3.3. (a) $\overline{X}_\alpha \in \mathfrak{g}_{\bar{\alpha}}$.

(b) Let $\omega = \omega_{\alpha_1} \wedge ... \wedge \omega_{\alpha_k} \in \wedge^k \mathfrak{u}^*$. With c_α a constant depending only on X_α and $X_{-\overline{\alpha}}$,

$$i(\omega_\alpha)\omega = \begin{cases} (-1)^{m-1} c_{\alpha_j}\omega_{\alpha_1} \wedge ... \wedge \hat{\omega}_{\alpha_m} \wedge ... \wedge \omega_{\alpha_k}, & \text{if } \alpha = -\overline{\alpha}_m \\ 0, & \text{otherwise.} \end{cases}$$

(c) $ad(X_\alpha)^*\omega_\beta$ is a multiple of $\omega_{\beta+\bar{\alpha}}$.

Proof of (c). Assume $\beta + \bar{\alpha}$ is a root. Then

$$\begin{aligned} \langle ad(X_\alpha)^*\omega_\beta, \omega_\gamma\rangle &= \langle \omega_\beta, ad(X_\alpha)\omega_\gamma\rangle \\ &= \langle \omega_\beta, c_{\alpha\gamma}\omega_{\gamma-\alpha}\rangle \\ &\neq 0 \text{ only if } \beta = -\overline{(\gamma-\alpha)}, \end{aligned}$$

where $c_{\alpha\gamma}$ is a constant depending on X_α and X_γ. Thus $ad(X_\alpha)^*\omega_\beta$ is a multiple of $\omega_{\beta+\bar{\alpha}}$. □

Lemma 3.3.4. For $x \in G$, $\bar{\partial}^*(P_\tau f(\tilde{x})\omega_S)|_{\tilde{x}=x} = 0$.

Proof. First consider

$$\bar{\partial}_1^*(P_\tau(f(\tilde{x}))\ \omega_S)|_{\tilde{x}=x} = \sum_{\alpha\in\Delta(\mathfrak{u})} (X_\alpha)^* P_\tau(f(\tilde{x}))|_{\tilde{x}=x} i(\omega_\alpha)\omega_S$$

If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{n})$ then $\overline{X}_\alpha \in \overline{\mathfrak{u}}\cap\mathfrak{n}$ and

$$(X_\alpha)^* P_\tau(f(x)) = (-\overline{X}_\alpha)P_\tau(f(x)) = 0$$

by complex linearity of the derivative and right N-invariance of f. If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{m})$ then $\overline{X}_\alpha \in \overline{\mathfrak{u}}\cap\mathfrak{m}$ and

$$\overline{X}_\alpha P_\tau(f(x)) = -P_\tau(\sigma(\overline{X}_\alpha)f(x)) = 0.$$

The first equality follows from complex linearity of the derivative and the right M-translation property of f; the second follows since $P_\tau((\overline{\mathfrak{u}}\cap\mathfrak{m})V^\sigma) = 0$ by (2.3.1). If $\alpha \in \Delta(\mathfrak{u}\cap\bar{\mathfrak{n}})$ then $i(\omega_\alpha)\omega_S = 0$; since $-\overline{\alpha} \in \Delta(\mathfrak{u}\cap\mathfrak{n})$, $\omega_{-\overline{\alpha}}$ is not a factor of ω_S.

Next

$$\bar{\partial}_2^*(P_\tau(f(\tilde{x}))\ \omega_S)|_{\tilde{x}=x} = \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} P_\tau(f(x))ad(X_\alpha)^* i(\omega_\alpha)\omega_S$$

Again if $\alpha \in \Delta(\mathfrak{u}\cap\bar{\mathfrak{n}})$ then $i(\omega_\alpha)\omega_S = 0$ since $\omega_{-\overline{\alpha}}$ is not a factor of ω_S. If $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{n})\cup\Delta(\mathfrak{u}\cap\mathfrak{m})$ then part (c) in Lemma 3.3.3 shows that $ad(X_\alpha)^*\omega_S = 0$ using the same argument as for the highest weight property of ω_S. □

Combining Lemmas 3.3.2 and 3.3.4 as in Section 3.2, we get

Theorem 3.3.5. $\bar{\partial}^* \circ \mathcal{S} = 0$.

Chapter 4. The Penrose Transform

Section 4.1. Minimal K-Types

For $\mathcal{S}$ to be particularly useful, it will be necessary to show that passing to Dolbeault cohomology does not result in the zero operator. Rather than try to show this directly, a Penrose transform will be used as stated in the Introduction. The definition of this operator depends directly on the relationship between the minimal $L \cap K$-types of W and the minimal K-types of $ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$. This relationship follows from Theorem 11.230 in [KV] and is the reason for imposing condition (2.2.3). Explicit formulas for minimal K-types can be found in [K1].

Minimal K-types were defined in full generality for irreducible admissible representations in [V1]. An important property is that the minimal K-types for standard induced representations always have multiplicity one.

Suppose $ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ has a minimal K-type of type μ. Let (μ, V^{μ}) be an abstract copy with K-invariant inner product $\langle \cdot, \cdot \rangle_{\mu}$. Let (τ_L, V^{τ_L}) be the space of $\mathfrak{u} \cap \mathfrak{k}$-invariants in V^{μ}; this space is an irreducible $L \cap K$-representation. We also note that by an argument similar to the one for (2.3.1),

$$P_{\tau_L}((\bar{\mathfrak{u}} \cap \mathfrak{k})V^{\mu}) = 0 \qquad (4.1.1)$$

where P_{τ_L} is the projection from V^{μ} to V^{τ_L}.

Let ω_C be a nonzero element of $\wedge^{s}(\mathfrak{u} \cap \mathfrak{k})^{*}$; since this is a top form, the space $\mathbb{C}\omega_C$ is a one-dimensional irreducible $L \cap K$-representation χ_L^{-1}.

Define the $L \cap K$-representation $(\tau'_L, V^{\tau'_L})$ by $\tau'_L = \tau_L \otimes \chi_L$. We consider it as a different representation with the same space and inner product as for τ_L. We can arrange spaces such that

$$V^{\tau} = V^{\tau'} \subset V^{\sigma} \subset V^{\mu}$$

and

$$V^{\tau'} \subset V^{\tau'_L} \subset V^{\mu};$$

this can be done since character shifts are accounted for by redefining the group action and not the space. Thus all inner products are restrictions from $\langle \cdot, \cdot \rangle_{\mu}$.

Using the Bott-Borel-Weil theorem, consider the map

$$V^{\mu} \to H^{0,s}(K/(L \cap K), V^{\tau'_L})$$

which sends

$$v \to f_v$$

by

$$f_v(k) = P_{\tau_L}(\mu(k)^{-1}v) \otimes \omega_C.$$

This map is K-equivariant and descends to cohomology since the image lies in the space of top forms. The map to cohomology is an isomorphism. By appealing to the adjoint formula for the $\bar{\partial}$ operator in [GS], one can see that the image vanishes under the adjoint of $\bar{\partial}$ using (4.1.1). Thus these forms are harmonic and give nonzero representatives for cohomology classes by Hodge theory.

What is important here is the connection between μ and τ'_L; we will consider function analogs of these spaces in $ind_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$ and W. τ'_L will occur as a minimal $L \cap K$ type in W and be related to the μ type in the above fashion.

The μ minimal K-type space for $ind_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$ has elements of the form

$$f_v(kan) = e^{-(\nu+\rho_G)\ loga} P_\sigma(\mu(k)^{-1}v) \qquad (kan \in G)$$

where $v \in V^\mu$, and the τ'_L minimal $L \cap K$-type for W has elements of the form

$$g_w(kan) = e^{-(\nu+\rho_L)\ loga} P_{\tau'}(\tau'_L(k)^{-1}w) \qquad (kan \in L)$$

where $w \in V^{\tau'_L}$. Denote these spaces by V_G^μ and $V_L^{\tau'_L}$, respectively.

Parameter shifts are used in order to introduce cohomology; to get nice formulas, they will need to be removed. Define $V_L^{\tau_L}$ as the space of functions

$$\{f_{\chi_L} = \chi_L f \ : \ f \in V_L^{\tau'_L}\}.$$

This is just a function version of V^{τ_L}; left translations by elements of $L \cap K$ act as $\pi_L \otimes \chi_L^{-1}$.

Define

$$I : V_L^{\tau_L} \to V^{\tau_L}$$

by

$$If = \int_{L \cap K} \tau_L(l) f(l)\, dl$$

where dl is the Haar measure on $L \cap K$.

First, this operator intertwines the $L \cap K$-actions; for $f \in V_L^{\tau_L'}$ and $l' \in L \cap K$,

$$\begin{aligned}
\tau_L(l')If &= \tau_L(l') \int_{L\cap K} \tau_L(l)[f(l)]\, dl \\
&= \int_{L\cap K} \tau_L(l'l)[f(l)]\, dl \\
&= \int_{L\cap K} \tau_L(l)[f(l'^{-1}l)]\, dl \\
&\qquad \text{by a change of variables and Haar measure} \\
&= \int_{L\cap K} \tau_L(l)[((\pi_L \otimes \chi_L^{-1})(l')f)(l)]\, dl \\
&= I((\pi_L \otimes \chi_L^{-1})(l')f).
\end{aligned}$$

To see that I is nonzero, note that for $v, w \in V^{\tau_L}$ and

$$f_v(l) = P_\tau(\tau_L(l)^{-1}v) \in V_L^{\tau_L},$$

$$\begin{aligned}
\langle If_v, w\rangle_\mu &= \langle \int_{L\cap K} \tau_L(l) f_v(l)\, dl, w\rangle_\mu \\
&= \langle \int_{L\cap K} \tau_L(l) P_\tau(\tau_L(l)^{-1}v)\, dl, w\rangle_\mu \\
&= \int_{L\cap K} \langle P_\tau(\tau_L(l)^{-1}v), \tau_L(l)^{-1}w\rangle_\mu\, dl \\
&= \int_{L\cap K} \sum_i \langle \tau_L(l)^{-1}v, \phi_i\rangle_\mu \langle \phi_i, \tau_L(l)^{-1}w\rangle_\mu\, dl \\
&\qquad \text{where } \{\phi_i\} \text{ is an orthonormal basis for } V^\tau \\
&= \int_{L\cap K} \sum_i \overline{\langle \tau_L(l)\phi_i, v\rangle_\mu} \langle \tau_L(l)\phi_i, w\rangle_\mu\, dl \\
&= \sum_i \frac{1}{\dim \tau_L} \langle \phi_i, \phi_i\rangle_\mu \overline{\langle w, v\rangle_\mu} \\
&= \frac{\dim \tau}{\dim \tau_L} \langle v, w\rangle_\mu.
\end{aligned}$$

Section 4.2. Non-vanishing in Dolbeault Cohomology

In this section, we will show that $\mathcal{S}$ is non-vanishing as a map into Dolbeault cohomology. To prove this, $\mathcal{S}$ will be composed with a Penrose transform

$$\mathcal{P} : C^{0,s}(G/L, W) \to C^{\infty}(G/K, V^{\mu}).$$

It will be shown that $\mathcal{P}$ vanishes on $\bar{\partial}$-cocycles and hence descends to a map on cohomology (which is also denoted $\mathcal{P}$). This composition will be non-vanishing on the minimal K-type V^{μ}. Hence the image of $\mathcal{S}$ cannot be trivial when passing to cohomology. Further details on Penrose transforms can be found in [BE].

The composition $\mathcal{P} \circ \mathcal{S}$ is of interest in its own right, generalizing the Poisson integrals formula found in [KW], [Ba] and [BKZ].

The argument here adapts that of [BKZ] to the present case. The following implements the Bott-Borel-Weil theorem in the direction

$$H^{0,s}(K/(L \cap K), V^{\tau_L'}) \to V^{\mu}.$$

Let $\{\phi_i'\}$ be an orthonormal basis of V^{μ}. Let $\bar{\omega}_C$ be the complex conjugate of ω_C. Then $\omega_C \wedge \bar{\omega}_C$ is a volume form for $K/(L \cap K)$ and $L \cap K$ acts trivially on it.

Define

$$\psi_i : K \to (V_L^{\tau_L'})^* \otimes \wedge^s(\bar{\mathfrak{u}} \cap \mathfrak{k})^*$$

by

$$[\psi_i(k)](v) = \langle \mu(k)Iv, \phi_i' \rangle_\mu \, \bar{\omega}_C.$$

Note that one can write an element F of $C^{0,s}(K/(L \cap K), V_L^{\tau_L'}))$ in the form $F(k) = f(k) \otimes \omega_C$ where $f(k) \in V_L^{\tau_L}$ for each $k \in K$. With F represented in such a manner, define

$$F \cdot \psi_i : K \to \wedge^{2s}(\mathfrak{k}/(\mathfrak{l} \cap \mathfrak{k}))^*$$

by

$$(F \cdot \psi_i)(k) = \langle \mu(k)I[f(k)], \phi_i' \rangle_\mu \, \omega_C \wedge \bar{\omega}_C.$$

It is easy to see $F \cdot \psi_i$ is right invariant under $L \cap K$ and thus a volume form on $K/L \cap K$.

As in [BKZ], define

$$P : C^{0,s}(K/(L \cap K), V_L^{\tau_L'}) \to V^\mu$$

by

$$P(F) = \sum_i \left(\int_{K/(L \cap K)} F \cdot \psi_i \, dk \right) \phi_i'.$$

This definition allows for some simplification:

Proposition 4.2.1. For $F \in C^{0,s}(K/(L \cap K), V_L^{\tau_L'})$ and f as above,

$$P(F) = \int_K \mu(k)[f(k)](1) \, dk.$$

Proof. We compute that

$$\begin{aligned}
P(F) &= \sum_i \left(\int_{K/L\cap K} \langle \mu(k) I[f(k)], \phi_i' \rangle_\mu \, dk \right) \phi_i' \\
&= \sum_i \int_{K/L\cap K} \langle \mu(k) \int_{L\cap K} \tau_L(l)[f(k)](l) dl, \phi_i' \rangle_\mu \, dk \; \phi_i' \\
&= \sum \int_{K/L\cap K} \int_{L\cap K} \langle \mu(k) \tau_L(l)[f(k)](l) \; dl, \phi_i' \rangle_\mu \; dk \; \phi_i' \\
&= \sum_i \int_{(K/(L\cap K))\times L\cap K} \langle \mu(kl)[f(kl)](1), \phi_i' \rangle_\mu \; dk \; dl \; \phi_i' \\
&= \int_K \mu(k)[f(k)](1) \; dk. \quad \square
\end{aligned}$$

Let $\bar{\partial}_K$ be the Cauchy-Riemann operator on $K/L \cap K$ associated to $\mathfrak{u} \cap \mathfrak{k}$. Compare the next proposition with Proposition 10.1 of [BKZ].

Proposition 4.2.2. P is K-equivariant, is independent of the orthonormal basis $\{\phi_i'\}$ and annihilates the image of $\bar{\partial}_K$.

Proof. The proof will follow entirely from Proposition 4.2.1. Independence of the orthonormal basis used holds immediately.

K-equivariance follows since for $k' \in K$ and $F \in C^{0,s}(K/(L\cap K), V_L^{\tau_L})$ with associated f,

$$\begin{aligned}
\mu(k')P(F) &= \mu(k') \int_K \mu(k)[f(k)](1) \; dk \\
&= \int_K \mu(k'k)[f(k)](1) \; dk \\
&= \int_K \mu(k)[f(k'^{-1}k)](1) \; dk \\
&= P(\mathcal{L}(k')F),
\end{aligned}$$

where $\mathcal{L}$ is the left translation action on $C^{0,s}(K/(L\cap K), V_L^{\tau'_L})$.

We show the $\bar{\partial}_K$-cocycle property. Let $\{E_i\}$ be a basis for $\mathfrak{u}\cap\mathfrak{k}$ with corresponding dual basis $\{\tilde{\omega}_i\}$.

Suppose $f \in C^{0,s-1}(K/(L\cap K), V_L^{\tau'_L})$ is given by

$$f(k) = \sum_{i=1}^{s} f_i(k) \otimes \tilde{\omega}_i^*$$

where $f_i \in C^{0,s}(K/(L\cap K), V_L^{\tau'_L})$ and

$$\tilde{\omega}_i^* = \tilde{\omega}_1 \wedge \cdots \wedge \widehat{\tilde{\omega}_i} \wedge \cdots \tilde{\omega}_s;$$

here $\widehat{}$ denotes omission. Note that

$$\begin{aligned}\bar{\partial}_K f_i(k) \otimes \tilde{\omega}_i^* &= \sum_{j=1}^{s} E_j f_i(k) \otimes \tilde{\omega}_j \wedge \tilde{\omega}_i^* \\ &\quad + \frac{1}{2}\sum_{j=1}^{s} f_i(k) \otimes ad(E_j)(\tilde{\omega}_j \wedge \tilde{\omega}_j^*) \\ &= (-1)^{i-1} E_i f_i(k) \otimes \omega_C.\end{aligned}$$

Vanishing of the second sum follows since $\mathfrak{u}\cap\mathfrak{k}$ annihilates ω_C; this follows since ω_C is a top form and $\mathfrak{u}\cap\mathfrak{k}$ is nilpotent.

Now for $E \in \mathfrak{k}_0$,

$$\begin{aligned}\int_K \mu(k)[(Ef_i)(k)](1)\, dk &= \int_K \mu(k)[\frac{d}{dt} f_i(ke^{tE})](1)\, dk \\ &= \int_K \frac{d}{dt}\mu(ke^{-tE})[f_i(k)](1)\, dk \\ &= \int_K \mu(k)\mu(-E)([f_i(k)](1))\, dk.\end{aligned}$$

But, using complex linearity of the derivative, we have

$$\int_K \mu(k)[(E_i f_i)(k)](1)\, dk = \int_K \mu(k)\mu(-E_i)([f_i(k)](1))\, dk$$
$$= 0$$

since $\mu(\mathfrak{u} \cap \mathfrak{k})V^{\tau_L} = 0$.

After collecting all terms, we see that $P(\bar{\partial}_K f) = 0$. □

Next define

$$R^* : C^{0,s}(G/L, W) \to C^{0,s}(K/(L \cap K), V_L^{\tau_L'})$$

as the pullback of the inclusion

$$R : (K/L \cap K, V_L^{\tau_L'} \otimes \wedge^s \mathfrak{u} \cap \mathfrak{k}) \to (G/L, W \otimes \wedge^s \mathfrak{u}).$$

Since this inclusion is holomorphic, $R^*\bar{\partial} = \bar{\partial}_K R^*$. Note that I factors through R^* in the W component and projects onto ω_C in the $\wedge^s(\mathfrak{u} \cap \mathfrak{k})^*$ component.

Define the Penrose transform $\mathcal{P}$ as

$$(\mathcal{P}F)(x) = P(R^*(\mathcal{L}(x)^{-1}F)).$$

Proposition 4.2.3. The image of $\bar{\partial}$ lies in ker $\mathcal{P}$.

Proof. As in [BKZ], we have

$$\mathcal{P}(\bar{\partial}F)(x) = P(R^*(\mathcal{L}(x)^{-1}\bar{\partial}F)) = P(R^*(\bar{\partial}\mathcal{L}(x)^{-1}F))$$
$$= P(\bar{\partial}_K(R^*\mathcal{L}(x)^{-1}F))$$
$$= 0. \qquad \square$$

Taking the composition with $\mathcal{S}$, we have

$$\mathcal{P} \circ \mathcal{S} : ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1) \rightarrow C^{0,s}(G/K, V^{\mu}).$$

The formula for $\mathcal{P} \circ \mathcal{S}$ allows for simplification.

Theorem 4.2.4. For some nonzero constant c,

$$[\mathcal{P} \circ \mathcal{S} f](x) = c \int_{K} \mu(k) P_{\tau}(f(xk))dk.$$

Proof. Since $\mathcal{P}$ and $\mathcal{S}$ commute with left translations by G, it will be enough to show the theorem when $x = 1$.

Consider

$$(R^{*}\mathcal{S}f)(k)(l) = R^{*}(P_{\tau}[f(kl)]\ Ad(l)\omega_{S})$$

with $k \in K$ and $l \in L \cap K$. The $\wedge^{s}(\mathfrak{u} \cap \mathfrak{k})^{*}$-component is just a nonzero multiple of $Ad(l)\omega_{C}$ since $\langle \omega_{S}, \omega_{C} \rangle \neq 0$; see [BKZ] for more details. Thus the nonvanishing piece of $Ad(l)\omega_{S}$ is a nonzero multiple of $\chi_{L}(l)^{-1}\omega_{C}$.

The $\chi_{L}(l)^{-1}$ can be moved to the W-component (as a $L \cap K$ representation). The domain of the operator I can be extended to this shifted W. Since I is $L \cap K$ equivariant and τ_{L} occurs in the shifted W with multiplicity one, I must annihilate all other $L \cap K$-types. Since the span of these $L \cap K$-types is dense in the shifted W, I already accounts for the projection onto $V_{L}^{\tau_{L}}$.

Collecting the pieces, the previous paragraph gives

$$(\mathcal{P} \circ \mathcal{S} f)(1) = c \int_K \mu(k)(P_\tau[f(k)])\, dk. \quad \square$$

Corollary 4.2.5. $\mathcal{P} \circ \mathcal{S} \neq 0$.

Proof. Let

$$f_\phi(kan) = e^{-((v+\rho_G)log a)} P_\sigma(\mu(k)^{-1}\phi)$$

where $\phi \in V^\tau$ is given as in Section 2.3. Then

$$\begin{aligned}
\langle \mathcal{P} \circ \mathcal{S} f_\phi(1), \phi\rangle_\mu &= \langle c \int_K \mu(k) P_\tau(f_\phi(k))\, dk, \phi\rangle_\mu \\
&= \langle c \int_K \mu(k) P_\tau(P_\sigma(\mu(k)^{-1}\phi)\, dk, \phi\rangle_\mu \\
&= c \int_K \langle P_\tau(\mu(k)^{-1}\phi), \mu(k)^{-1}\phi\rangle_\mu\, dk \\
&= c \int_K \langle P_\tau(\mu(k)^{-1}\phi), P_\tau(\mu(k)^{-1}\phi)\rangle_\mu\, dk \\
&> 0 \qquad \text{since } \langle P_\tau(\phi), P_\tau(\phi)\rangle_\mu > 0. \quad \square
\end{aligned}$$

Chapter 5. Formulation with Quotients

Section 5.1. The Langlands Quotient Operator

Many results concerning cohomological induction or maximal globalizations are stated in terms of irreducible representations. Until now we have not been concerned with irreducibility of W; in this section we introduce the Langlands quotient operator and recast previous results with this operator in place.

The key features of the Langlands' classification for irreducible admissible $(\mathfrak{g}, K)$-modules is that every such module is infinitesimally equivalent to a quotient of some standard induced representation and that the quotient operation can be written as an integral intertwining operator. Reformulated via [KZ], these standard induced representations are of the form

$$ind_{M'A'N'}^{G}(D \otimes e^{\nu} \otimes 1)$$

where $M'A'N'$ is a real cuspidal parabolic, D is a discrete series or limit of discrete series representation on M' and ν occurs in the closed positive Weyl chamber associated to $M'A'N'$. Since we are interested only in the case where MAN is a minimal parabolic, we will not define these notions; see [K2] for further details.

We note that we have not made use of the parameter ν in any proofs; here this parameter becomes important. Reducibility for non-unitary principal series representations is controlled by ν. The choices in (2.2.1) and (2.2.6) guarantee uniqueness of the respective Langlands quotients.

For the choice of ν, the quotient operator for W is given by

$$Qf(l) = \int_{\overline{N}_L} f(l\bar{n})\, d\bar{n}$$

where f is an $L \cap K$-finite vector in W and $d\bar{n}_L$ is the left Haar measure on N_L. Calculating shows that the image of Q lies in

$$W^- = ind^L_{(L\cap M)A\overline{N}_L}(\tau' \otimes e^\nu \otimes 1).$$

We denote the closure of the image of the $L \cap K$-finite vectors in W by $\bar{W}$.

We recall some facts from [V1]. The minimal $L \cap K$-types of W occur with multiplicity one and are independent of the ν parameter. In the compact picture W and W^- are the same space and thus Q can be thought of as an operator on W. For our choice of ν, Q is non-vanishing on these $L \cap K$-types and by Schur's Lemma must act as a nonzero scalar on each type.

Section 5.2. Results with Quotients

We wish to define an associated operator $\mathcal{S}'$ with the quotient operation on W in place. Definition of bundles and operators are as before with W replaced by $\bar{W}$. Results in [BKZ] and [Ba] are formulated with the quotient operator already in place; the 'heuristic principle' in [K4] leads one directly to quotient operators.

Define $\mathcal{S}' = Q \circ \mathcal{S}$; that is, we have a map

$$\begin{aligned} \mathcal{S}' : ind^G_{MAN}(\sigma \otimes e^\nu \otimes 1) &\to ind^G_L(\bar{W} \otimes \wedge^s \mathfrak{u}^*) \\ &\subset ind^G_L(W^- \otimes \wedge^s \mathfrak{u}^*) \end{aligned}$$

given by

$$[S'f(x)](l) = \int_{\overline{N}_L} P_\tau(f(xl\bar{n}))\ Ad(l\bar{n})\omega_S\ d\bar{n}$$

where $f \in ind_{MAN}^G(\sigma \otimes e^\nu \otimes 1)$.

That the image space is correct follows from Section 3.1 and that Q maps W to $\bar{W}$.

Theorem 5.2.1. $\bar{\partial} \circ S' = 0$.

Proof. The integrand vanishes under $\bar{\partial}$ for all $\bar{n} \in \overline{N}_L$ from Lemmas 3.2.2 and 3.2.3. □

Theorem 5.2.2. $\bar{\partial}^* \circ S' = 0$.

Proof. Similar to Theorem 5.2.1, this follows from Lemmas 3.3.2 and 3.3.4. Note that for purposes of adjoints, $\bar{W} \subset W^-$. □

Theorem 5.2.3. S is nonzero in cohomology.

Proof. As noted in the previous section, the quotient operator is a nonzero scalar on the set of minimal $L \cap K$-types. Thus all statements in Chapter 4 will hold with suitable alterations with Q in place. □

References

[Ba] L. Barchini, *Szegö Mappings, Harmonic Forms, and Dolbeault Cohomology*, J. Funct. Anal. **118** (1993), 351–406.

[BKZ] L. Barchini, A. W. Knapp and R. Zierau, *Intertwining operators into Dolbeault cohomology representations*, J. Funct. Anal. **107** (1992), 302–341.

[BE] R. J. Baston and M. G. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory*, Oxford University Press, Oxford, 1989.

[Bo] R. Bott, *Homogeneous Vector Bundles*, Ann. of Math. **66** (1957), 203–248.

[GS] P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253–302.

[K1] A. W. Knapp, *Minimal K-type formula*, Non-Commutative Harmonic Analysis and Lie Groups, Lecture Notes in Mathematics, vol. 1020, Springer-Verlag, New York, 1983, pp. 107–118.

[K2] A. W. Knapp, *Representation Theory of Lie Groups: An Overview Based on Examples*, Princeton Univ. Press, Princeton, 1986.

[K3] A. W. Knapp, *Lie Groups, Lie Algebras and Cohomology*, Princeton Univ. Press, Princeton, 1988.

[K4] A. W. Knapp, *Imbedding Discrete Series in $L^2(G/H)$*, Harmonic Analysis on Lie Groups (Sandjerg Estate, August 26-30, 1991), Copenhagen University Mathematics Institute, Report Series 1991, No. 3, Copenhagen, 1991, pp. 27–29.

[K5] A. W. Knapp, *Introduction to Representations in Analytic Cohomology*, Contemp. Math. **154** (1993), 1–19.

[KV] A. W. Knapp and D. A. Vogan, *Cohomological Induction and Unitary Representations*, Princeton Univ. Press, Princeton, 1995.

[KW] A. W. Knapp and N. R. Wallach, *Szegö kernels associated with discrete series*, Inventiones Math. **34** (1976), 163–200; **62** (1980) 341–346.

[KZ] A. W. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Ann. of Math. **116** (1982), 389-501 (See also **119** (1984), 639).

[Ko1] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329–387.

[Ko2] B. Kostant, *Orbits, symplectic structures and representation theory*, Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoronsha, Co., Tokyo, 1966, pp. 71.

[KuS] R. A. Kunze and E. M. Stein, *Uniformly bounded representations III. Intertwining operators for the principal series on semisimple groups*, Amer. J. Math **89** (1967), 385–442.

[L1] R. P. Langlands, *Dimensions of Spaces of Automorphic Forms*, Algebraic Groups and Discontinuous Subgroups, Proc. Symo. Pure Math., vol. 9, American Mathematical Society, Providence, 1966, pp. 253–257.

[L2] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, preprint, Institute for Advanced Study, Princeton (1973), reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Math Surveys and Monographs, American Mathematical Society, Providence, (1989), pp. 101–170.

[S1] W. Schmid, *Homogeneous complex manifolds and representations of semisimple Lie groups*, Ph.D. dissertation, University of California, Berkeley, 1967, reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Math Surveys and Monographs, American Mathematical Society, Providence, (1989), 223–286.

[S2] W. Schmid, *L^2-Cohomology and the discrete series*, Ann. of Math. **103** (1976), 375–394.

[V1] D. A. Vogan, *The algebraic structure of the representation of semisimple Lie Groups I*, Ann. of Math. **109** (1979), 1–60.

[V2] D. A. Vogan, *Representations of Real Reductive Lie Groups*, Birkhäuser, Boston, 1981.

[Wo] H.W. Wong, *Cohomological induction in various categories and the maximal globalization conjecture* (1995), preprint.