# SELF-DUAL HI-CELLULAR STRUCTURES

A Dissertation Presented

by

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Abstract of the Dissertation
Self-Dual HI-Cellular Structures

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In this dissertation we define the concepts of *HI-cell* and of *HI-cell complex*. HI-cell complexes are structures on polyhedra which generalize the notion of a cell complex, and in which one can define a generalized *barycentric subdivision*. We use this barycentric subdivision to define the concept of *the dual complex* of a *dualizable HI-cell complex*. We then define a *self-dual HI-cell complex* and show (constructively):

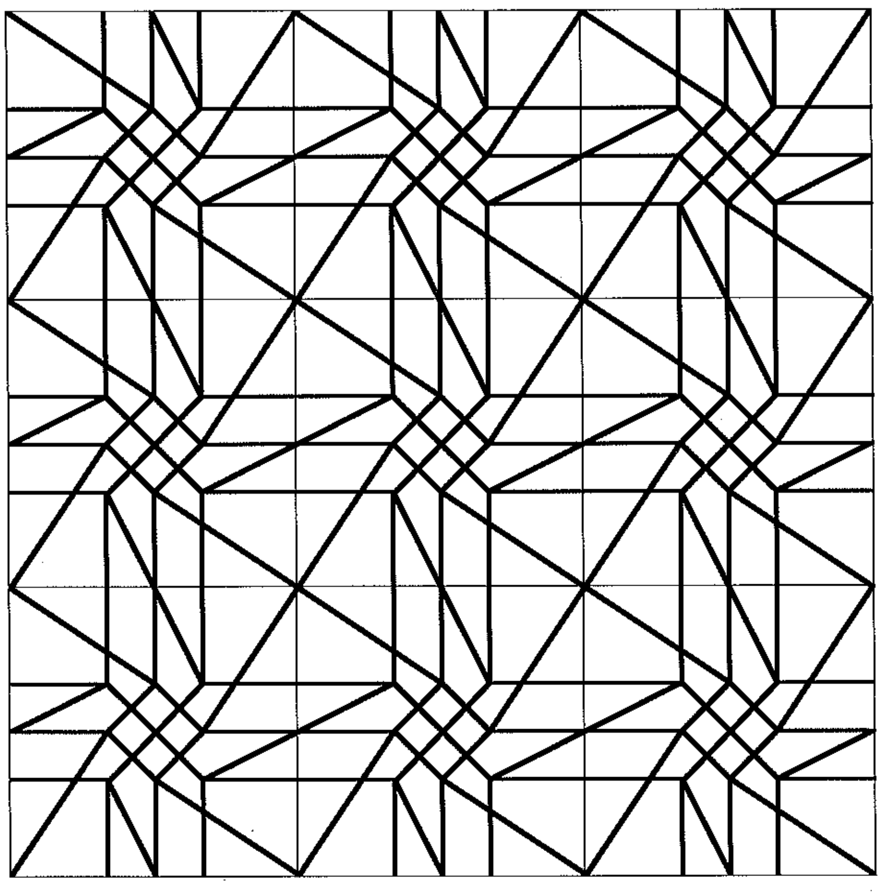
The unique self-dual HI-cellular structure (complex) on the sphere is the triangular one given by the faces of a tetrahedron. The torus possesses an infinite number of self-dual HI-cellular structures, all of which must be quadrilateral. All the surfaces of genus $g \geq 2$ possess pentagonal, hexagonal and octagonal self-dual HI-cellular structures, and these are the only "shapes" possible for all surfaces of genus $g \geq 2$.

# Dedication

To Beth, Daniel and Noah.

Without their love, support and encouragement neither my career nor this dissertation would have been possible.

# Self-Dual HI-Cellular Structures



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# TABLE OF CONTENTS

# Acknowledgments

It is with great pleasure and deepest appreciation that I acknowledge and thank those that were instrumental in the development of my mathematical career and in the completion of this dissertation.

In the distant past I was fortunate to have taken an undergraduate topology course in the fabled Moore style, which was "conducted" by Edwin Moise. He was my first contact with a mathematician and from him I learned how to learn and have fun while doing it. To him I owe my initial decision to study mathematics.

During an extended leave of absence after two initial years at Stony Brook, I met David Stone at CUNY. He is responsible for reigniting my geometric-topological flame which was all but extinguished after a long period of disuse. Not only did I learn a lot from him, but he also introduced me to the beautiful topological work of MacPherson, McCrory and Banshoff. This is the mathematics that I was searching for. I also appreciate the comments and suggestions he offered as a reader of this dissertation. Thank you David.

In the more recent past my path again intersected that of another mathematician. I was allowed to attend a seminar in Intersection Homology, at Stony Brook which was being run by Blaine Lawson. Making his acquaintance has completely changed my life. He was instrumental in my return as a student to Stony Brook; and if that were not already enough, I ended up becoming his student. To him I owe the completion of this dissertation. Without his encouragement, understanding and the sharing of his knowledge I would not have finally "sat down and written". Thank you Blaine for all of the above and for having once said: "I'm here to advise, not to dictate." This allowed freedom was priceless.

Finally I want to thank Marie-Louise Michelsohn and Anthony Phillips. As the graduate director, Marie-Louise has facilitated immensely my stay at Stony Brook and allowed me to keep my teaching job outside this school. I am also grateful for her having allowed me many times in her home in order to meet with Blaine; I'm sure some of these times were inconvenient. I also thank her for chairing my doctoral committee. Thank you Marie-Louise.

As the co-chairman of the mathematics department, Anthony Phillips has understandably a very busy schedule. I thank him for the extra time spent as a reader of my dissertation. I also would like to apologize for the amount of reading material I gave him at the last minute. Thank you Tony.

# 0 Introduction

We present here a brief outline of the contents of this dissertation.

In chapter 1 we review the concepts of *polyhedron* and related PL topics. Following Rourke and Sanderson [R-S] we define a polyhedron to be a set (space) in Euclidean space having the property that all of its points have a “linear” cone neighborhood in the set. We wanted to have such an intrinsic definition of polyhedron, since a part of this manuscript introduces a generalization of the concept of triangulation, and the spaces supporting these new constructions should be recognized as polyhedra, without us having to verify that they can also support a simplicial triangulation.

In chapter 2 we begin the main body of this work.

We define the concepts of *H-cell* and *HI-cell* (Defs. 2.12 and 2.26). These are pairs of the form $\left(C, h{:}\overline{C} \to C\right)$ where $\overline{C}$ is a convex polyhedron (i.e. a cell), and $h$ is respectively a PL (piece-wise-linear) Homeomorphism (thus the *H* in H-cell), or a PL surjection which is a Homeomorphism in the Interior of every face of $\overline{C}$ (thus the *HI* in HI-cell). Although HI-cells might collapse faces etc., we think of these objects as carrying a “memory” of their original linear structure. Thus the image of points in the interior of cells “remember” the cell-conical structure having for vertex its pre-image, and for base the boundary of the cell; i.e.

these images of interior points remember where the vertices, etc. ought to be, even if they do not exist anymore. This remembrance is what we need later in chapter 3 when we define subdivisions of HI-cell complexes.

These *HI-cell complexes* are defined also in chapter 2 (Def. 2.28). They generalize and contain the classical cell complexes. HI-cell complexes are sets of HI-cells such that when the images of two HI-cells intersect, they both recognize (in their memories) the intersection as being unions of linearly isomorphic pairs of original faces of each. (We have actually defined in 2.28 a *face recognition function.*)

Chapter 3 analyzes the possibilities of extending to HI-cell complexes the notions of *star subdivision*, *barycentric subdivision* etc.

To be able to perform star subdivisions at a point $a$ in an HI-cell complex, it turns out that one needs, in general, the existence of a *face-compatible function* which chooses which pre-image $a$ "remembers". We show in Remark 3.13 an (surprising?) example of an HI-cell complex and a point $a$ in it for which no such choice function exists. Various propositions deal with the study of conditions which are sufficient for the existence of these functions. Proposition 3.21 assures us that one can always perform *barycentric subdivisions*, which are special cases of star subdivisions. This is very fortunate since all the following work in this research is based on the existence of barycentric subdivisions.

We end this chapter with Theorem 3.29 which states that *topological* quotient spaces obtained by linearly identifying faces of a cell can be realized as polyhedra in Euclidean space and as the codomains of HI-cells.

In chapter 4 we define the concept of *self-duality* (Def. 4.14). We use as model the Euclidean-geometric duality possessed by the tetrahedron and known since antiquity: If the tetrahedron were made of clay and its vertices were pushed-in (in a controlled way) one would obtain a new tetrahedron with *triangular* faces having the old vertices in their interiors, with edges transversal to the original edges (now deformed), and with vertices in the middle of the original *triangular* faces. (Faces acquire the *"shape* "of faces in complementary dimension.)

We use the generalized notion of barycentric subdivision to define a similar concept of self-duality for HI-cell complexes.

Chapter 5 concludes the present work. We investigate the question of the existence of other surfaces besides the sphere which possess self-dual HI-cellular structures. We obtain in 5.17 necessary numerical conditions for the existence of self-duality. We conclude by showing (by actual constructions)

- That the only self-dual HI-cell complex on the sphere is the classical triangular one given by the faces of a tetrahedron. (Theorem 5.18)
- That the torus can have an infinite amount of self-dual structures, all of which must be quadrilateral. (Theorem 5.22)
- That the surfaces of genus $g \geq 2$ all possess pentagonal, (Theorem 5.27) hexagonal (Theorem 5.29) and octagonal (Theorem 5.19) self-dual structures, and these are the only "shapes" that work for all $g \geq 2$ (Remark 5.25).

# 1 Polyhedra

Most of the definitions in this chapter can be found in [**R-S**].

## Preliminary Definitions

As usual, $\mathbf{R}$ denotes the real numbers and $\mathbf{R}^n$ denotes the Euclidean space of $n$-tuples $\{x=(x_1,x_2,\ldots,x_n)\}$, $x_i \in \mathbf{R}$. However, the metric $d$ on $\mathbf{R}^n$ is given by $d(x,y)=\max\{|x_i-y_i|\}_{i=1,\cdots,n}$. Also as usual, $I$ denotes the closed interval $[0,1]\subset\mathbf{R}$. A ***map*** is a continuous function. ***Linear*** means linear in the affine sense, thus a *linear subspace* $V\subset\mathbf{R}^n$ is a translated vector subspace; equivalently, for each finite set of points $\{x_i\}\subset V$ and real numbers $\alpha_i$ with $\sum\alpha_i=1$ we have $\sum\alpha_i x_i\in V$, and a map $f{:}V\to\mathbf{R}^m$ is *linear* if $f(\sum\alpha_i x_i)=\sum\alpha_i f(x_i)$.

## Polyhedra

**1.1** Definition. Let $A,B\subset\mathbf{R}^n$. The ***join of A and B, AB***, is the subset of $\mathbf{R}^n$ given by $AB=\{\alpha a+\beta b \mid a\in A, b\in B; \alpha,\beta\in\mathbf{R}; \alpha,\beta\geq 0; \alpha+\beta=1\}$, i.e., $A\ B$ consists of all points on straight "segments" with end-points in $A$ and $B$. We also define $\emptyset B=B$, and the join $\{a\}B$ will be denoted by $aB$.

Note that two different segments $\alpha a+\beta b$ and $\gamma c+\delta d$ in $AB$ might intersect in their interiors.

**1.2** DEFINITION. The join $aB$ is called a ***cone with vertex a and base B*** if $a \notin B$, and if each point $p$ in $aB$ different from $a$ has a unique representation $p = \alpha a + \beta b$, with $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $b \in B$.

Figure 1 shows two different join presentations of the same set. One is a cone, and the other is not.

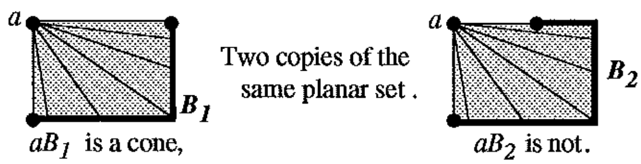
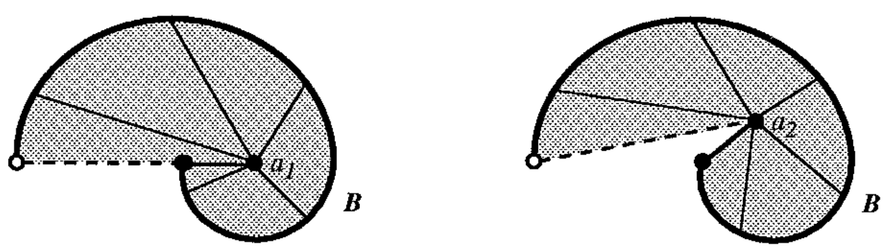



Fig. 1

Note that there are cones $a_1B$ and $a_2B$ in $\mathbf{R}^n$, having the same base $B$, and such that there is no homeomorphism $a_1B \to a_2B$, mapping $a_1 \mapsto a_2$ and which is the identity on $B$. Figure 2 below illustrates this fact, where $B$ is a spiral curve in $\mathbf{R}^2$ with an endpoint missing. Note also that the vertex $a_1$ on the left of fig. 2, does not have a "small" neighborhood having boundary homeomorphic to $B$.




Two cones with the same base but their vertices do not have homeomorphic neighborhoods.

Fig. 2

Note that if $aB$ is a cone in $\mathbf{R}^n$, there is a linear map from $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$ which maps $(x_1, \cdots, x_n, 0)$ to $(x_1, \cdots, x_n)$ and $(0, \cdots, 0, 1)$ to $a$. The restriction of this linear map to the cone $(0, \cdots, 0, 1)(B \times 0)$ is a 1-1 continuous map onto $aB$

(mapping base to base and vertex to vertex). If $B$ is compact (which is not the case in the example in Fig. 2) $aB$ is also compact (see 1.8), and thus the above restriction is a homeomorphism. Thus, if $B$ is compact, "small" neighborhoods of $a$ have boundaries homeomorphic to $B$ (to $B \times t, 0 < t < 1$). Also in this situation, any two cones $a_1B$ and $a_2B$ in $\mathbf{R}^n$ are homeomorphic via a linear homeomorphism which maps vertex to vertex, and one base to the other.

**1.3** DEFINITION. $P \subset \mathbf{R}^n$ is a ***polyhedron*** if every point $p \in P$ has a cone neighborhood $N = pL$ in $P$, with $L$ compact. $N$ is called a ***star of p in P***, and $L$ is a ***link***. To avoid confusion, we sometimes write $N = N(p,P)$ *and* $L = L(p,P)$.

**1.4** REMARK. Since $L$ is compact and $p \cap L = \varnothing$, $N$ can always be chosen to have the form $N_\varepsilon(p,P) = \{x | x \in P, d(p,x) \le \varepsilon\}$ for suitable small $\varepsilon$, and this choice of $N$ has link $L = \dot{N}_\varepsilon(p,P) = \{x | x \in P, d(p,x) = \varepsilon\}$. [**R-S**]

By taking $L = \varnothing$ we have that a singleton $\{p\}$ is a polyhedron. An open (circular) disk is also a polyhedron, while a closed disk is not, since a point on its boundary does not have a cone neighborhood.

**1.5** PROPOSITION. *Let* $P = \cup P_i$ *be a locally finite union of polyhedra* $P_i \subset \mathbf{R}^n$. *Then P is a polyhedron in* $\mathbf{R}^n$.

*Proof.* Let $p \in P$ and let $\{P_{i_j}\}$ be the (finite) set of all polyhedra in $\{P_i\}$ containing $P$. Then for each $i_j, p$ has a star neighborhood $N_{\varepsilon_{ij}}(p, P_{i_j})$ in $P_{i_j}$.

Let $\varepsilon = \min\{\varepsilon_{ij}\}$, then $N = \bigcup_{i_j}\left[N_\varepsilon(p, P_{i_j})\right]$ is a cone neighborhood of $p$ in $P$ of the form $p\left(\bigcup_{i_j}\left[\dot{N}_\varepsilon\left(p, P_{i_j}\right)\right]\right)$. □

Using a similar reasoning, we also have the following:

**1.6** PROPOSITION. *Let $P = \cap P_i$ be a finite intersection of polyhedra $P_i \subset \mathbf{R}^n$. Then P is a polyhedron in $\mathbf{R}^n$.*

□

**1.7** PROPOSITION. *Let $a_1L_1$ and $a_2L_2$ be cones in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Then $a_1L_1 \times a_2L_2$ is a cone in $\mathbf{R}^{n+m}$ with presentation:*
$a_1L_1 \times a_2L_2 = (a_1,a_2)\left[(a_1L_1 \times L_2) \cup (L_1 \times a_2L_2)\right]$.

*Proof.* Let $(x_1,x_2) \in a_1L_1 \times a_2L_2$. Then $x_1 = (1-s)a_1 + sb_1$ and $x_2 = (1-t)a_2 + tb_2$; $0 \le s,t \le 1$, $b_1 \in L_1$, $b_2 \in L_2$.

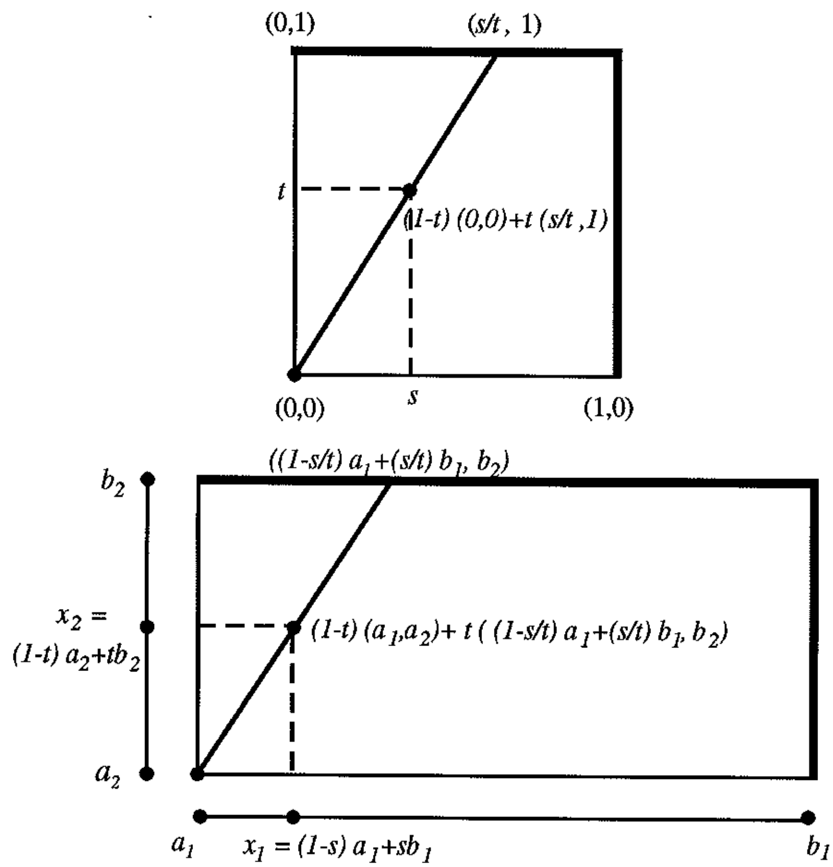
We have:

i) if $t \ge s$, then $(x_1, x_2) = (1-t)(a_1, a_2) + t\left(\frac{t-s}{t}a_1 + \frac{s}{t}b_1, b_2\right)$ (See Fig. 3).

ii) if $s \ge t$, then $(x_1, x_2) = (1-s)(a_1, a_2) + s\left(b_1, \frac{s-t}{s}a_2 + \frac{t}{s}b_2\right)$.

Since $[0,1]\times[0,1]$ has a cone presentation of the form $[0,1]\times[0,1]=(0,0)\left(I\times(0,1)\cup(1,0)\times I\right)$, i) and ii) give a bijection: $(s,t)\mapsto(x_1,x_2)$, between $[0,1]\times[0,1]$ and $a_1b_1 \times a_2b_2=(a_1,a_2)\left((a_1b_1 \times b_2)\cup(b_1 \times a_2b_2)\right)$ which preserves cone coordinates. Now, since $a_1L_1$ and $a_2L_2$ are cones, we have that any point $(x,y)\in a_1L_1\times a_2L_2$, $(x,y)\neq(a_1,a_2)$, has a unique representation as in i) or in ii).

Thus, $a_1L_1\times a_2L_2$ has a cone presentation $(a_1,a_2)\left[(a_1L_1\times L_2)\cup(L_1\times a_2L_2)\right]$. □




Case i) $t\geq s$, $\left(\left(1-\frac{s}{t}\right)a_1+\frac{s}{t}b_1\ ,\ b_2\right)\in a_1L_1\times L_2$.

Fig. 3

**1.8** LEMMA. *Let $L\subset\mathbf{R}^n$ be compact. Then the join $aL$ is compact.*

*Proof.* There is a continuous map from the compact set $L\times I\subset\mathbf{R}^{n+1}$ to $\mathbf{R}^n$ whose image is $aL$. Namely, $(b,t)\mapsto((1-t)b+ta)$. □

1.7 and 1.8 immediately give us the following:

**1.9** COROLLARY. *A finite product of polyhedra is a polyhedron.*

□

**1.10** DEFINITION. Let $a=(a_1,a_2,\ldots,a_n)\in\mathbf{R}^n$. The $\varepsilon$-neighborhood $N_\varepsilon(a,\mathbf{R}^n)=[a_1-\varepsilon,a_1+\varepsilon]\times[a_2-\varepsilon,a_2+\varepsilon]\times\cdots\times[a_n-\varepsilon,a_n+\varepsilon]$ will be called an ***n-cube centered at a***.

A ***face of*** $N_\varepsilon(a,\mathbf{R}^n)$ is a subset of $N_\varepsilon(a,\mathbf{R}^n)$ obtained by replacing each factor $[a_1-\varepsilon,a_1+\varepsilon]$ in the above product either with itself or with $a_1-\varepsilon$ or $a_1+\varepsilon$.

$[-1,1]^n = N_1((0,0,\cdots,0),\mathbf{R}^n)$ is called ***the unit n-cube*** and it is denoted by $J^n$.

Note that $J^n \neq I^n=[0,1]^n$. Thus, for example, a unit 2-cube is not equal to the (usual) unit square.

**1.11** REMARK. Cubes and faces are polyhedra by Lemma 1.9. Since $\dot{N}_\varepsilon(a,\mathbf{R}^n)$, the boundary of a cube, is the union of all the proper faces of $N_\varepsilon(a,\mathbf{R}^n)$, we have that $\dot{N}_\varepsilon(a,\mathbf{R}^n)$ is also a polyhedron by Proposition 1.5.

These facts together with Remark 1.4 and Proposition 1.6 imply that we can always assume that all star neighborhoods and all links in a polyhedron are also polyhedra.

**1.12** PROPOSITION. *Let $Y$ be a compact polyhedron and $aY$ be a cone* ($\subset \mathbf{R}^n$). *Then $aY$ is also a polyhedron.*

*Proof.* Let $x \in aY$. We must show that $x$ has a cone neighborhood $N(x,aY) = xL$ with compact link $L = L(x,aY)$.

If $x = a$ we let $N(x,aY) = aY$ and since $Y$ is compact we are done.

If $x \neq a$, we know, since $aY$ is a cone, that there is a unique $b \in Y$ such that $x$ lies in the segment $ab$. Now let $N(b,Y)$ be a cone neighborhood of $b$ in $Y$ as in the definition of polyhedron. We claim that $aN(b,Y)$ is the required cone neighborhood of $x$ in $aY$. [With link $N(b,Y) \cup aL(b,Y)$ if $x \notin Y$ (both sets in this union are compact by 1.8); and with link $aL(b,Y)$ if $x = b \in Y$.] (See Fig. 4.)

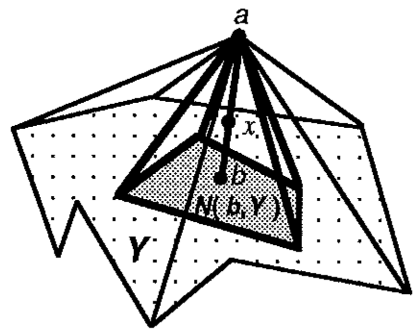
To prove the claim, we must show that any $y \in aN(b,Y)$, $y \neq x$, has a unique representation $y = tx + (1-t)c$, $0 \leq t \leq 1$, with $c \in N(b,Y) \cup aL(b,Y)$ or with $c \in aL(x,Y)$.

Notice that $N(b,Y)$ is the union of segments $bb', b' \in L(b,Y)$, and thus, $aN(b,Y)$ is the union of triangles $abb'$ which intersect each other in the segment $ab$.

If $y \in ab$ then it lies uniquely in the subsegment $ax$ or in $xb$. If $y \in ax$, we take $c = a$; if $y \in xb$, we take $c = b$.

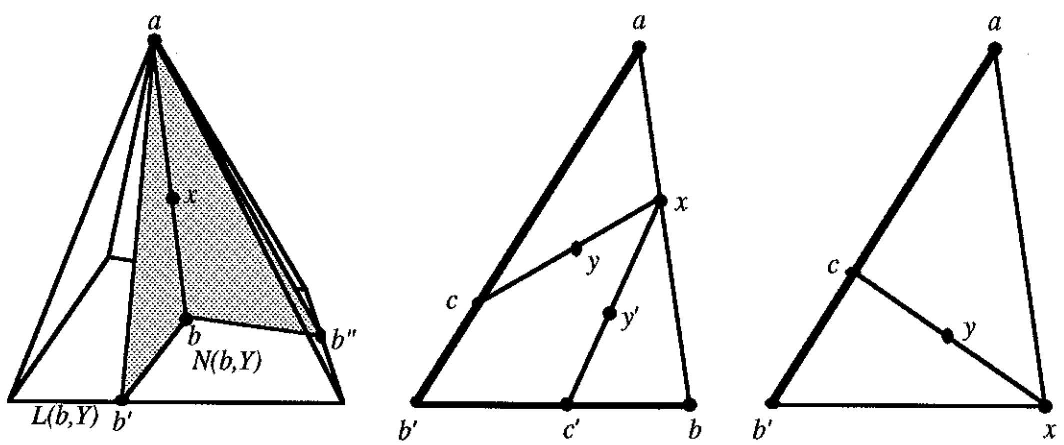
If $y \notin ab$ then it lies in a unique triangle $abb'$, and for a triangle we have that $y = tx + (1-t)c$ with $c \in bb' \cup ab'$ if $x \neq b$, or $c \in ab'$ if $x = b$. (See Fig. 5.)

Since $aN(b,Y)$ is the union of such triangles $abb'$, $N(b,Y) \cup aL(b,Y)$ is the union $\bigcup_{b'} bb' \cup ab'$, and $aL(x,Y) = \bigcup_{b'} ab'$, ($b' \in L(b,Y)$ throughout), we can conclude that $aN(b,Y)$ is the claimed cone neighborhood of $x$ in $aY$. □




$aN(b,Y)$ is a cone neighborhood of $x$ in $aY$.

Fig. 4




If $y$ is not in $ab$, it lies in a unique triangle $abb'$.

Fig. 5

## PIECE-WISE-LINEAR MAPS

**1.13** DEFINITION. Let $P$ and $Q$ be polyhedra in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. A continuous map $f: P \to Q$ is called ***piece-wise-linear*** (PL) if the graph of $f$, $\Gamma(f) = \{(x, f(x)) | x \in P\} \subset \mathbf{R}^{n+m}$, is a polyhedron in $\mathbf{R}^{n+m}$.

The following shows that a map is PL if and only if it is "locally conical".

**1.14** PROPOSITION. *$f: P \to Q$ is PL if and only if every $p \in P$ has a star neighborhood $N(p,P) = pL$ such that $f(tp + (1-t)b) = tf(p) + (1-t)f(b)$, where $b \in L$ and* $0 \le t \le 1$.

*Proof.* Firstly, assume that every $p \in P$ has a star $N = N(p,P)$ with the stated properties. Let $(p, f(p)) \in \Gamma(f)$, the graph of $f$. We claim that $N_\Gamma = \{(q, f(q)) \mid q \in N\}$ is a star of $(p, f(p))$ in $\Gamma(f)$.

Let $(r, f(r)) \in N_\Gamma, (r, f(r)) \neq (p, f(p))$. Since $N_\Gamma$ is homeomorphic to $N$ (via the projection), we have $r \neq p$. Thus, we have uniquely: $r = tp + (1-t)b, b \in L$, and so

$$(r, f(r)) = (tp + (1-t)b, tf(p) + (1-t)f(b)) = t(p, f(p)) + (1-t)(b, f(b))$$

for unique $t$ and $(b, f(b))$. And $(b, f(b))$is in the compact (homeomorphic to $L$) set $L_\Gamma = \{(c, f(c)) \mid c \in L\}$. Thus $\Gamma(f)$ is a polyhedron.

Conversely, assume that $\Gamma(f)$ is a polyhedron. Let $p \in P$. The assumption means that $(p, f(p))$ has a star, $N_\Gamma = N_\Gamma((p, f(p)), \Gamma(f))$, in $\Gamma(f)$. Let $N$ be the (homeomorphic) "standard" projection of this star into $P$, and let $q \in N$, $q \neq p$. There is a unique $t \in [0,1)$ and unique $(b, f(b)) \in N \subset \mathbf{R}^{n+m}$ such that $(q, f(q)) = t(p, f(p)) + (1-t)(b, f(b)) = (tp + (1-t)b, tf(p) + (1-t)f(b))$.

Equating coordinates, we have: $q = tp + (1-t)b$ and $f(q) = tf(p) + (1-t)f(b)$ □

**1.15** PROPOSITION. *Let $f: P \to Q$ and $g: Q \to S$ be PL maps. Then the composition $f \circ g: P \to S$ is also PL.*

*Proof.* Let $p \in P$ *and* $N_\varepsilon(f(p), Q)$ be a cubical star of $f(p)$ (showing that $g$ is PL, as in Prop. 1.14). Choose $\delta > 0$ so that $N_\delta(p, P)$ is a star of $p$ relative to $f$

and such that $f(N_\delta(p,P)) \subset N_\varepsilon(f(p),Q)$. $N_\delta(p,P)$ is also a star relative to the composition $f \circ g$ since for any $x \in N_\delta(p,P)$, $x = tp + (1-t)b$,
$b \in \text{boundary}(N_\delta(p,P))$, we have:
$g(f(x)) = g(f(tp+(1-t)b)) = g(tf(p)+(1-t)f(b)) = tg(f(p)) + (1-t)g(f(b))$. □

Note that in the above proof, $f(b)$ does not necessarily lie on the boundary (link) of $N_\varepsilon(f(p),Q)$. However, an easy computation shows that if $g$ maps a segment $ab$ conically into $\mathbf{R}^n$, i.e., $ta+(1-t)b \mapsto tg(a)+(1-t)g(b)$, then $g$ is linear in $ab$. i.e., for <u>any</u> $c,d \in ab$, we have: $sc+(1-s)d \mapsto sg(c)+(1-s)g(d)$. Thus the second equality in the above proof is justified.

**1.16** LEMMA. *Let $f: V \to \mathbf{R}^n$ be a 1-1 linear function, where $V$ is an affine subspace of $\mathbf{R}^n$, Let $P \subset V$ be a polyhedron. Then f(P) is a polyhedron.*

*Proof.* Let $q \in Q$. Let $p = f^{-1}(q)$ and $N$ be a star of $p$ in $P$. It is easy to see that linearity (see page 4) and injectivity imply that f*(N)* is a star of $q$. □

**1.17** PROPOSITION. *Let P and Q be polyhedra and $h: P \to Q$ be a PL homeomorphism. Then $h^{-1}$ is also PL.*

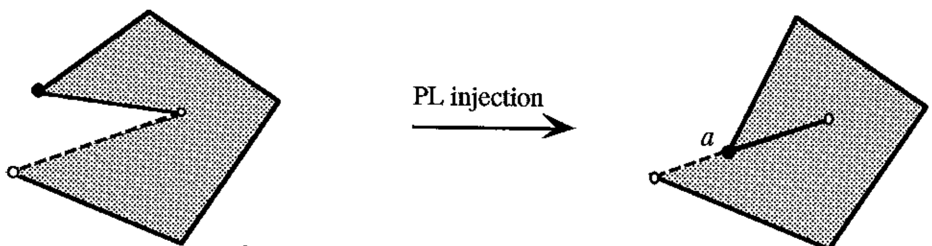
*Proof.* The involution $i: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^n, (x,y) \mapsto (y,x)$ is a linear isomorphism mapping $\Gamma(h)$ onto $\Gamma(h^{-1})$. Since $\Gamma(h)$ is a polyhedron, Lemma 1.16 implies that $\Gamma(h^{-1})$ is also a polyhedron; therefore $h^{-1}$ is PL. □

Propositions 1.15 and 1.17 imply that PL homeomorphism is an equivalence relation on polyhedra.

**1.18** DEFINITION. Polyhedra $P$ and $Q$ are ***polyhedral equivalent*** ( $P \cong Q$) if they are PL homeomorphic.

**1.19** REMARK. In contrast to Lemma 1.16, if $f$ is only PL (although still 1-1), $f(P)$ need not be a polyhedron as Fig. 6 exemplifies, but we will see later, (1.28), that if $P$ is compact then $f(P)$ is a polyhedron.

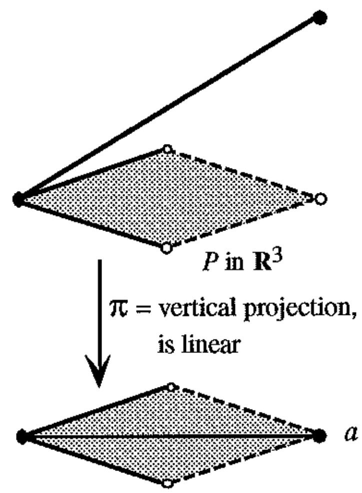
Also, if $f$ is linear but not injective, again $f(P)$ need not be a polyhedron. See Fig. 7.




Polyhedron in $\mathbf{R}^2$.

Not a polyhedron. The point $a$ does not have a star.

Fig. 6




$\pi(P)$ in $\mathbf{R}^2$ is not a polyhedron. The point $a$ does not have a star in $\pi(P)$.

Fig. 7

**1.20** DEFINITION. A set $S \subset \mathbf{R}^n$ is in ***general position*** if no $k$-dimensional linear subspace, with $k<n$ contains more than $k+1$ of the points of $S$.

A finite set $S=\{x_0, x_1, \cdots, x_l\}$ in $\mathbf{R}^n$ in general position and with $l \le n$, is also said to be an ***independent set***.

Note that a circle in the plane is in general position but it is not an independent set. The same applies to the set consisting of the 4 vertices of a square.

Also note that a set $S=\{x_0, x_1, \cdots, x_l\} \subset \mathbf{R}^n$, $l \le n$, is independent if and only if the vectors $\{x_i - x_0 | i \ne 0\}$ are linearly independent.

**1.21** DEFINITION. $\sigma \subset R^m$ is a ***simplex of dimension n***, or an ***n-simplex,*** if $\sigma$ is the repeated join $v_0 v_1 \cdots v_n$ of $n+1$ independent points. The points $v_i$ are called the ***vertices*** of $\sigma$. The simplex $\tau$ is a ***face of*** $\sigma$ if the set of vertices of $\tau$ is a non-empty subset of the set of vertices of $\sigma$.

To indicate the dimension, we will sometimes use the symbol $\sigma^n$ to denote an $n$-simplex.

**1.22** REMARK. Using induction on the number of vertices, it can be shown that any point $x \in \sigma^n$ has a unique representation as a sum:

$$x = \sum_{i=0}^{n} \alpha_i v_i \, , \; \alpha_i \geq 0 \, , \; \sum \alpha_i = 1.$$

The numbers $\alpha_i$ are called the ***barycentric coordinates of x***.

**1.23** DEFINITION. Let $\{v_0, v_1, \cdots, v_n\}$ be the vertices of an $n$-simplex $\sigma$. The ***barycenter of*** $\sigma$ is the point $\hat{\sigma} = \sum_{i=0}^{n} \frac{1}{n+1} v_i$.

**1.24** REMARK. Using proposition 1.12 inductively on the number of vertices , we immediately see that a simplex is a polyhedron.

The proofs of 1.25-1.28 below can be found in **[R-S]** (pages 12-13).

**1.25** THEOREM. *A polyhedron is a locally finite union of simplices. A compact polyhedron is a finite union of simplices.*

**1.26** COROLLARY. *Let $f: P \to Q$ be PL, then there is a locally finite union of simplices $P = \cup \sigma_i$ such that $h|_{\sigma_i}$ is linear.*

**1.27** LEMMA. *The linear image of a simplex is a polyhedron.*

**1.28** COROLLARY. *The image of a compact polyhedron under a PL map is a compact polyhedron.*

## DIMENSION OF POLYHEDRA

**1.29** DEFINITION. Let $P \subset \mathbf{R}^n$ be a polyhedron. The ***span of P***, span(*P*), is the smallest subspace of $\mathbf{R}^n$ containing *P*. i.e., $\text{span}(P) = \cap V_i$, where $V_i$ is an affine subspace of $\mathbf{R}^n$ containing *P*.

Note that the dimension (as an affine space) of $\text{span}(\sigma^m)$ is equal to *m*.

**1.30** DEFINITION. We say that a polyhedron $P \subset \mathbf{R}^m$ has ***dimension n*** ($\dim(P) = n$) if in a union $P = \cup \sigma_i$, as given by Theorem 1.25, we have $n = \max\{\text{dimension of } \sigma_i\}$.

**1.31** PROPOSITION. *<u>Dimension</u> is a well-defined concept, i.e.,* $P = \cup \sigma_i$ *and* $P = \cup \tau_j$, *will give rise to the same number for* $\dim(P)$.

*Proof.* Let $\Sigma = \{\sigma_i\}$ be a set of simplices in $\mathbf{R}^m$ such that $P = \cup \sigma_i$, and let $\mathrm{T} = \{\tau_j\}$ be another set of simplices in $\mathbf{R}^m$ such that $P = \cup \tau_j$.
Let $\dim_\Sigma(P) = \max\{\text{dimension of } \sigma_i\}$ and $\dim_\mathrm{T}(P) = \max\{\text{dimension of } \tau_j\}$.
We will show that $\dim_\Sigma(P) = \dim_\mathrm{T}(P)$.

Assume that $\dim_{\Sigma}(P)=n$, and let $\sigma^n$ be an $n$-dimensional simplex in $\Sigma$. Observe that for any $\tau_k$ in T, we have $\text{dimension}(\text{span}(\tau_k)\cap\text{span}(\sigma^n))\leq n$, and $\text{dimension}(\text{span}(\tau)\cap\text{span}(\sigma^n))=n$ only if $\text{span}(\sigma^n)\subset\text{span}(\tau_k)$. (Here, *dimension*(span($\tau_k$)$\cap$span($\sigma^n$)) etc. , means *dimension* in the (translated) vector space sense.)

Let $\left\{\tau_{j_1},\tau_{j_2},\cdots\tau_{j_s}\right\}$ be the (finite) set of all the simplices in T which intersect $\sigma^n$ and such that $\text{dimension}(\text{span}(\tau_{j_r})\cap\text{span}(\sigma^n))<n$. Since $\text{dimension}(\text{span}(\tau_{j_r})\cap\text{span}(\sigma^n))<n$, none of the sets $\tau_{j_r}\cap\sigma^n$ contain a set which is open in $\sigma^n$. Therefore, the finite union, $\cup\left(\tau_{j_r}\cap\sigma^n\right)$ also does not contain a set open in $\sigma^n$, and thus, one can find a point $p\in\sigma^n$ such that $p\notin\cup\left(\tau_{j_r}\cap\sigma^n\right)$. Since $\sigma^n\subset P=\cup\tau_j$, there is a simplex $\tau$, $\tau\in\text{T}$, such that $p\in\tau$. Thus $\tau\notin\left\{\tau_{j_1},\tau_{j_2},\cdots\tau_{j_s}\right\}$ and so, $\text{dimension}(\text{span}(\tau)\cap\text{span}(\sigma^n))=n$, which as seen above, implies that $\text{span}(\sigma^n)\subset\text{span}(\tau)$.which further implies that $\dim_{\Sigma}(P)=\text{dimension}(\text{span}(\sigma^n))\leq\text{dimension}(\text{span}(\tau))\leq\dim_{\text{T}}(P)$.

Symmetrically we have $\dim_{\text{T}}(P)\leq\dim_{\Sigma}(P)$. Thus, $\dim_{\Sigma}(P)=\dim_{\text{T}}(P)$. □

**1.32** PROPOSITION. *Let P and Q be polyhedral equivalent polyhedra* (see Def. 1.18). *Then* $\dim(P)=\dim(Q)$.

*Proof.* Let $h:P\to Q$ be a PL homeomorphism. From Corollary 1.26, we know that there is a locally finite union of simplices $P=\cup\sigma_i$ such that $h|_{\sigma_i}$ is linear for all *i*. Since h is a PL homeomorphism, $h|_{\sigma_i}$ is the restriction to $\sigma_i$ of a linear isomorphism $H:\text{span}(\sigma_i)\to H(\text{span}(\sigma_i))$. Therefore, $h(\sigma_i)=H(\sigma_i)$ is a simplex in $Q$ with $\dim(h(\sigma_i))=\dim(\sigma_i)$ for all *i*. Thus, $Q$ is a locally finite union of simplices, $Q = \cup h(\sigma_i)$, and since $\dim(h(\sigma_i)) = \dim(\sigma_i)$, we immediately conclude that $\dim(Q) = \dim(P)$. □

# 2 HI-CELL COMPLEXES

## CELLS

**2.1** DEFINITION. A polyhedron $C$ (or $C^n$) in $\mathbf{R}^m$ is called an ***n-cell*** $(n \le m)$ if $C$ is compact, convex and $\dim(\operatorname{span}(C)) = n$.

In general, without referring to dimension, we say that $C$ is a ***cell*** if $C$ is a compact convex polyhedron.

Note that the intersection, and the product of two cells is a cell, and also that the image of a cell under a linear map is again a cell, since all these operations result in polyhedra and preserve convexity.

**2.2** LEMMA. *Let $P \subset \mathbf{R}^m$ be a polyhedron such that* $\dim(\operatorname{span}(C)) = n$. *Then $P$ contains $n+1$ points in general position.*

*Proof.* Let $\{v_0, v_1, \cdots, v_k\}$ be a set of points of $P$ in general position having maximal cardinality. Assume that $k < n$. Since $\operatorname{span}\{v_0, v_1, \cdots, v_k\}$ is a subspace having dimension equal to $k$, and since $k < n$, we see that $P \not\subset \operatorname{span}\{v_0, v_1, \cdots, v_k\}$. Thus, there is a point $p \in P$ such that $p \notin \operatorname{span}\{v_0, v_1, \cdots, v_k\}$. Now the set $\{v_0, v_1, \cdots, v_k, p\}$ of points of $P$, is also in general position and it has cardinality equal to $k+1$, which contradicts the assumption that $k$ is the maximal cardinality of sets in general position. Thus, $k \ge n$, i.e., $P$ contains $n+1$ points in general position. □

**2.3** PROPOSITION. *An n-cell C has dimension equal to n.*

*Proof.* Since $\dim(\mathrm{span}(C)) = n$, using Lemma 2.2 we see that $C$ has $n+1$ points in general position. Therefore by convexity we can conclude that an $n$-cell $C$ contains an $n$-simplex $\sigma^n$. However, $C$ does not contain any simplex $\tau$ of dimension larger than $n$, otherwise we would have $\mathrm{span}(\tau) \subset \mathrm{span}(C)$, as well as $\dim(\mathrm{span}(C)) = n < \dim(\tau) = \dim(\mathrm{span}(\tau))$ which is an impossibility. Now let $P = \cup\sigma_i$, as given in Theorem 1.25 (p. 16). We also have $P = (\cup\sigma_i) \cup \sigma^n$ and since $\dim(\sigma_i) \leq n$ we see that dimension of $C$ is equal to $n$. □

Note that if $C \subset \mathbf{R}^m$ is a cell and if $V$ is a subspace in $\mathbf{R}^m$ that intersects $C$ then $V \cap C$ is a cell. In particular, if $V = L$ is a line, then $V \cap C$ is a cell of dimension 0 or 1, i.e., a point or a segment.

**2.4** DEFINITION. Let $C$ be a cell. ***The interior of C***, Int($C$), is defined to be the interior of $C$ in span($C$), and ***the boundary of C***, Bd($C$), is the frontier of $C$ in span($C$).

We recall that for $A \subset X$, $X$ a topological space, ***the frontier of A in X*** is $\mathrm{Fr}A = \mathrm{Fr}_X A = \overline{A} \cap \overline{X - A}$, where for $B \subset X$, $\overline{B}$ denotes the closure of $B$ in $X$.

Note that since an $n$-cell $C$ contains an $n$-simplex, (Proposition 2.3), the interior of $C$ is non-empty.

**2.5** DEFINITION. Let $C \subset \mathbf{R}^m$ be a cell and $x \in C$. Let $V(x,C)$ be the union of all lines $L$ in $\mathbf{R}^m$ such that $L \cap C$ is a 1-cell containing $x$ in its interior. Using the convexity of $C$, it is easily proved in [R-S] (page 27) that if $V(x,C) \neq \varnothing$ then $V(x,C)$ is a subspace in $\mathbf{R}^m$. Now let $F(x,C)$ be $V(x,C) \cap C$ if $V(x,C) \neq \varnothing$, and let $F(x,C)=x$ if $V(x,C) = \varnothing$. $F(x,C)$ are cells called ***faces of C*** and in the particular case $F(x,C)=x$, $x$ is called a ***vertex*** *of* $C$. Faces of dimension 1 are called ***edges*** and faces of codimension 1 (of dimension equal to $\dim(C)-1$) are called ***facets***.

One can easily see that for a simplex or a cube, the previous definitions of *face of a simplex* and *face of a cube* coincide with the above definition of *face* if we consider those objects as cells.

Given a set of points $\{x_0, x_1, \cdots, x_n\}$ in $\mathbf{R}^m$ and a simplex $\sigma^n$ with vertices $\{v_0, v_1, \cdots, v_n\}$ there is a unique linear map $\mathrm{X}: \sigma^n \to \mathbf{R}^m$ mapping $v_i \mapsto x_i$. Since $\sigma^n$ is a cell, its image under $\mathrm{X}$ is also a cell $C$ (see note below definition 2.1). It is easy to see that $C$ is the convex hull of the set $\{x_0, x_1, \cdots, x_n\}$. ( The *convex hull* of a set $A$ is the intersection of all the convex sets containing $A$).

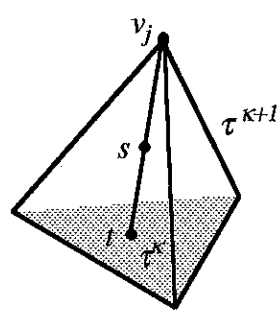
**2.6** DEFINITION. Let $\{x_0, x_1, \cdots, x_n\}$ be a set of points in $\mathbf{R}^m$ and $C$ be the cell $C = \mathrm{X}(\sigma^n)$ as described above. In this situation we say that $\{x_0, x_1, \cdots, x_n\}$ ***supports C***. More generally, if $A$ is a set and $H$ is the convex hull of $A$, we say that ***A supports H.***

**2.7 PROPOSITION.** *Let $\sigma^n$ be an n-simplex with vertices $\{v_0, v_1, \cdots, v_n\}$, and let $\{x_0, x_1, \cdots, x_n\} \subset \mathbf{R}^m$ support the cell $C = \mathrm{X}(\sigma^n)$, where $\mathrm{X} : \sigma^n \to \mathbf{R}^m$ is the linear map defined by $v_i \mapsto x_i$. Let x be a vertex of C. Then, $\mathrm{X}^{-1}(x) = v_j$ for some j, and thus $x = x_j$.*

*Proof.* Let $x$ be a vertex of $C$ and $s \in \sigma^n$ be a point in $\mathrm{X}^{-1}(x)$. We will show that $s \notin \mathrm{int}(\tau^i)$ for any face $\tau^i$ of $\sigma^n$ of dimension $i$, $1 \le i \le n$, which implies that $s$ must be a vertex of $\sigma^n$. We will show this by induction on $i$.

Firstly, we see that $s \notin \mathrm{int}(\tau^1)$, for any 1-dimensional face $\tau^1$. For if $s \in \mathrm{int}(\tau^1)$, then $s$ is in the interior of a segment $v_k v_l$; $v_k, v_l$ vertices of $\sigma^n$. Thus, $x = \mathrm{X}(s) \in \mathrm{int}(x_k x_l) \subset C$, therefore not a vertex of $C$ (see definition 2.5).

Now assume that for a certain $k < n$, we have: If $t \in \mathrm{int}(\tau^i), 1 \le i \le k$ then $y = \mathrm{X}(t)$ is not a vertex of $C$. Let $s \in \mathrm{int}(\tau^{k+1})$, then $s$ is in the interior of a segment $v_j t$, where $v_j$ is a vertex of $\sigma^n$, and $t \in \mathrm{int}(\tau^k)$, (see fig. 8). If X maps the whole segment $v_j t$ onto a point of $C$ then $x = \mathrm{X}(s) = \mathrm{X}(t)$, and by the above assumption, $x$ is not a vertex of $C$. On the other hand, if $\mathrm{X}(v_j) \ne \mathrm{X}(t)$, then $x = \mathrm{X}(s)$ is in the interior of the segment $\mathrm{X}(v_j)\mathrm{X}(t) = x_j \mathrm{X}(t)$, which is contained in $C$, and again, $x$ is not a vertex of $C$. □




$s$ is in the interior of the segment $v_j t$.

Fig. 8

The above proposition states that the set of vertices of $C$ is a subset of the set $\{x_0, x_1, \cdots, x_n\}$, but not every $x_i$ is a vertex. For example, let $\{x_0, x_1, x_2, x_3\}$ be a set of 4 points in $\mathbf{R}^2$ where $x_0, x_1, x_2$ are the vertices of a triangle and $x_3$ is a point in the interior of that triangle; $\{x_0, x_1, x_2, x_3\}$ supports the triangle, but $x_3$ is not a vertex. But in [R-S] (pages 27-30) the following is proved:

**2.8** PROPOSITION. *Let C be an n-cell, then:*

1) *C has finitely many vertices* $\{x_0, x_1, \cdots, x_k\}$ *which support C.*

2) *If F is a face of C, then F is supported by a subset of* $\{x_0, x_1, \cdots, x_k\}$ *(but not every subset supports a face).*

3) $C = \text{disjoint}\cup\{\text{Int}(F) | F < C\}$

   $\text{Bd}(C) = \text{disjoint}\cup\{\text{Int}(F) | F < C,\ F \neq C\}$.

4) *If F<D<C then F<C.*

5) *If F,D<C then* $F \cap D < C$.

6) *Let* $x \in C$*, Then C is the cone xB, where B is the union of all faces of C which do not contain x.*

Note that 2.8 1) together with the statements which precede definition 2.6 above, imply that $C$ is a cell, if and only if $C$ is the image $\text{X}(\sigma)$ of a simplex $\sigma$ under a linear map $\text{X}$, and that $\sigma$ can chosen so as to have the same number of vertices as $C$.

**2.9** LEMMA. *Let C be a cell of dimension n, and let F<C, F≠C. Then dim(F) < n.*

*Proof.* Let $x \in F$. Since $V(x,C) \subset \text{span}(C)$, see Def. 2.5, we have, $\dim(V(x,C)) \leq \dim(\text{span}(C)) = n$. If these dimensions are equal, then $V(x,C) = \text{span}(C)$ and then, $F = V(x,C) \cap C = \text{span}(C) \cap C = C$ which is a contradiction. □

**2.10** PROPOSITION. *Let C be a cell of dimension n. Then C has faces of dimension i, for* $0 \leq i \leq n$.

*Proof.* Let $x \in \text{Int(C)}$. From Proposition 2.8, we know that $C$ is the cone $C = x(\cup F), F < C, x \notin F$.

CLAIM 1. *No proper face F of C contains x.*

Since $x \in$ Int(C), there is a cubical neighborhood $N$ of dimension $n$, such that $x \in N \subset C$. Let $x \in F$, $F$ a face of $C$. Since $N$ is the union of 1-cells containing $x$ in its interior, we have that $N \subset F$ (see Def. 2.5). Thus $\dim(F) \geq n$, and now using Lemma 2.9 we conclude that $F = C$. ∇

In view of the above claim, we can conclude that $C$ is the cone with vertex $x$ and base equal to the union of all the proper faces of $C$, i.e.,

$C = x(\cup F), F < C, F \neq C$.

CLAIM 2. *Let D be a cell of dimension d in* $\mathbf{R}^d \subset \mathbf{R}^m$ *and let* $x \in \mathbf{R}^m$ *be such that xD is a cone. Then xD has dimension d+1.*

Let $(0,0,\cdots,0,1) \in \mathbf{R}^{d+1}$, the cone $(0,0,\cdots,0,1)D$ in $\mathbf{R}^{d+1}$ is a polyhedron of dimension $d+1$ since it contains a $d+1$-simplex (with maximal possible dimension) obtained as the cone $(0,\cdots,0,1)\sigma^d$, where $\sigma^d$ is a $d$-simplex in $D$.

Since there is a 1-1 linear map $T:\mathbf{R}^{d+1} \to \mathbf{R}^m$ fixing $D$ and mapping $(0,0,\cdots,0,1)$ to $x$, we see that $xD$ has dimension $d+1$. $\nabla$

Now since $C = x(\cup F), F < C, x \notin F$, we have that $C = \cup xF, F < C, F \neq C$, and since this union is finite, one of the cones $xF$ (for some face $F$) must have dimension $n$, and thus $F$ has dimension $n$-$1$.

Now applying the same method to a face $F$ of $D$, with $\dim(F) = i > 1$, we see that $F$ has a face $G$ of dimension $i$-1. But $G$ is again a face of $D$ by Proposition 2.8 4), and we thus conclude that $D$ has faces of all dimensions $d$, $0 \leq d \leq n$. □

**2.11** DEFINITION. 1-cells are called ***edges***, and 2-cells (in $\mathbf{R}^n$) are called ***polygons***, and thus 2-cells will be denoted: *triangle, quadrilateral, pentagon, n-gon* etc. in a manner analogous to planar polygons.

## H-CELLS

**2.12** DEFINITION. An ***H-cell*** is a pair $(C, h:\overline{C} \to C)$ where $C$ is a polyhedron, $\overline{C}$ is a cell and $h:\overline{C} \to C$ is a PL homeomorphism (onto $C$).

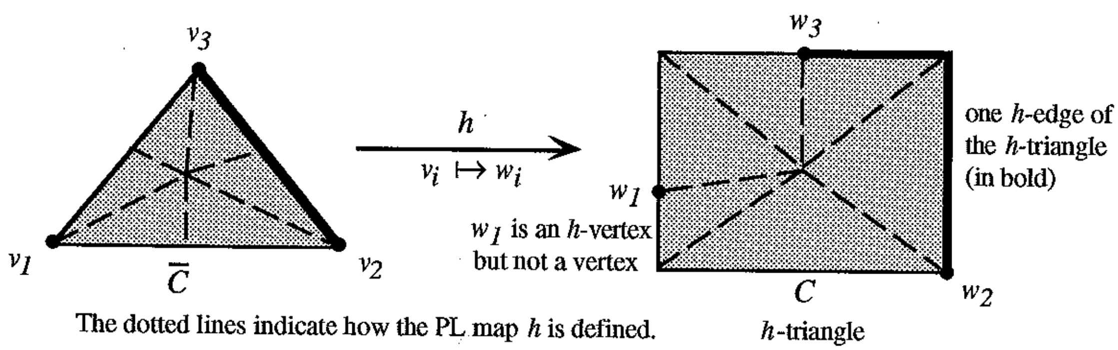
A ***face of an H-cell*** $(C, h:\overline{C} \to C)$ is a pair $(h(F)\, h|_F : F \to h(F) \subset C)$ where $F$ is a face of $\overline{C}$, and $h|_F : F \to h(F) \subset C$ is the restriction of $h$ to $F$. Thus, a face of an H-cell is again an H-cell.

Given an H-cell $(C, h:\overline{C} \to C)$ where $\overline{C}$ is an edge, triangle, n-gon, n-cube, simplex, etc. we say that $h$ is an H-edge, H-triangle, H-n-gon, H-n-cube, H-simplex, etc., and that $C$ is an $h$-edge, $h$-triangle, $h$-n-gon, $h$-n-cube, $h$-simplex, etc.

Let $\left(C, h{:}\overline{C} \to C\right)$ be an H-cell, where $\overline{C}$ is a cell of dimension $n$. We observe that Proposition 1.32 implies that $\dim(C) = n$. We thus make the following definition:

**2.13** DEFINITION. The ***dimension of an H-cell*** $\left(C, h{:}\overline{C} \to C\right)$ is defined to be the dimension of $\overline{C}$.

Note that if $\left(C, h{:}\overline{C} \to C\right)$ is an H-cell, then the number of $d$-dimensional subpolyhedra $h(F)$ of $C$, where $F$ is a face of $\overline{C}$, is equal to the number of faces of $\overline{C}$ having dimension $d$. Also note that if $C$ is itself a cell, $h(F)$ is not necessarily a face of $C$. See figure 9.



The dotted lines indicate how the PL map $h$ is defined.

Fig. 9

Note that for brevity we sometimes denote an H-cell $\left(C, h{:}\overline{C} \to C\right)$ by $(C, h)$.

## H-CELL COMPLEXES

**2.14** DEFINITION. An ***H-cell complex*** $K$ is a finite set $K=\left\{\left(C_i, h_i{:}\overline{C_i} \to C_i\right)\right\}$ of H-cells, $C_i$ in $\mathbf{R}^n$, $C_i \neq C_j$ if $i \neq j$, such that the following is satisfied:

1) If $\left(C_i, h_i{:}\overline{C}_i \to C_i\right) \in K$ and if $\left(h_i(F), h_i|_F{:}F \to h_i(F)\right)$ is a face of $\left(C_i, h_i\right)$, then there is an H-cell $\left(C_k, h_k{:}\overline{C}_k \to C_k\right) \in K$ with $C_k = h_i(F)$ and a linear homeomorphism $i_F{:}\overline{C}_k \to F \subset \overline{C}_i$ such that

$$\begin{array}{ccc} \overline{C}_k & \xrightarrow{i_F} & F \subset \overline{C}_i \\ & {\scriptstyle h_k}\searrow \quad \swarrow{\scriptstyle h_i|_F} & \\ & C_k = h_i(F) & \end{array}$$

commutes.

2) If $\left(C_i, h_i{:}\overline{C}_i \to C_i\right)$ and $\left(C_j, h_j{:}\overline{C}_j \to C_j\right)$ are in $K$ and if $C_i \cap C_j \neq \varnothing$, then there are faces $\left(C_i \cap C_j, h_i|_F{:}F \to C_i \cap C_j\right)$ and $\left(C_i \cap C_j, h_j|_G{:}G \to C_i \cap C_j\right)$ of $\left(C_i, h_i\right)$ and $\left(C_j, h_j\right)$ respectively.

(And thus there is an H-cell $\left(C_k, h_k{:}\overline{C}_k \to C_k\right) \in K$, with $C_k = C_i \cap C_j$ and linear homeomorphisms $i_F{:}\overline{C_k} \to F$ and $i_G{:}\overline{C_k} \to G$ such that the diagram

$$\begin{array}{ccccc} \overline{C}_i \supset F & \xleftarrow{i_F} & \overline{C}_k & \xrightarrow{i_G} & G \subset \overline{C}_j \\ & {\scriptstyle h_i|_F}\searrow & \downarrow{\scriptstyle h_k} & \swarrow{\scriptstyle h_j|_G} & \\ & & C_k = C_i \cap C_j & & \end{array}$$

commutes.)

**2.15** DEFINITION. A ***cell complex K*** is an H-cell complex $K=\left\{\left(C_i, h_i{:}\overline{C_i} \to C_i\right)\right\}$ where for every $i$, $C_i$ is a cell, $\overline{C}_i = C_i \times i$, $h_i{:}\overline{C_i} \to C_i$ is the projection map $\pi_i{:}C_i \times i \to C_i$, and the linear homeomorphisms $i_F{:}\overline{C_k} = C_k \times k \to F \subset C_l \times l$, are the maps $i_F = \pi_k \times (k \mapsto l)$, i.e., $(x,k) \mapsto (x,l)$.

A ***simplicial complex*** is a cell complex where every cell is a simplex.

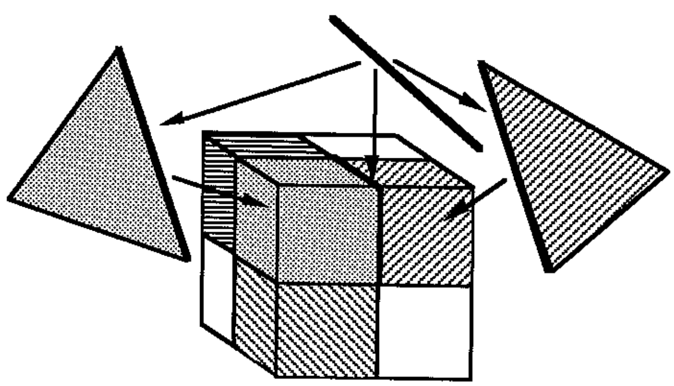
**2.16** DEFINITION. Given an H-cell complex $K = \{ (C_i, h_i : \overline{C_i} \to C_i) \}$, the ***underlying polyhedron of K***, $|K|$, is the polyhedron given by $|K| = \cup C_i$. If $P$ is a polyhedron such that $P = |K|$ for some H-cell complex (cell complex, simplicial complex) $K$, we say that $K$ is an ***H-cellular*** (*cellular, simplicial*) ***structure on P***. A simplicial structure on P is sometimes called a ***triangulation of P***.

**2.17** DEFINITION. The ***dimension of an H-cell complex K***, $\dim(K)$, is defined by $\dim(K) = \max\{\dim(C_i, h_i) | (C_i, h_i) \in K\}$.

For every integer $i$, $0 \le i \le \dim(K)$, the ***i-skeleton of K*** is the H-complex $K^i = \{(C_r, h_r : \overline{C}_r \to C_r) | (C_r, h_r) \in K, \dim(C_r, h_r) \le i\}$ and each linear homeomorphism $i_F : \overline{C_k} \to F$, $(C_k, h_k), (C_l, h_l) \in K^i$, $F < \overline{C}_l$, is the same as it is in $K$.

If $V \in K^0$, then $V$ is called a ***vertex*** of the complex $K$, and if $E \in K^1 - K^0$, $E$ is called an ***edge*** of $K$ (more properly, an *H-vertex* and an *H-edge* of $K$).

*Example* Figure 10 below, shows part of an H-cellular structure $K$ on the boundary of $J^3 \subset \mathbf{R}^3$ consisting of 8 triangles, 12 edges and 6 vertices.



An H-cell complex on the boundary of a cube.
Fig. 10

**2.18** REMARK. Observe that for any H-complex $K$, Prop. 2.10 implies that $K^i \neq \varnothing,\ 0 \leq i \leq \dim(K)$.

**2.19** PROPOSITION. *Let* $\left(C, h{:}\overline{C} \to C\right)$ *be an H-cell. The sets*

$$K = \left\{\left(h(F_i),\ h|_{F_i}{:}F_i \to h(F_i)\right) \,\middle|\, F_i \text{ is a face of } \overline{C}\right\} \quad \textit{and}$$

$$\dot{K} = \left\{\left(h(F_i),\ h|_{F_i}{:}F_i \to h(F_i)\right) \,\middle|\, F_i \text{ is a proper face of } \overline{C}\right\}$$

*are H-cell complex with* $|K| = C$ *and* $|\dot{K}| = h\left(\mathrm{Bd}\left(\overline{C}\right)\right)$.

*Proof.* To see this, we need only to verify that

$$L = \left\{\left(F_i,\ \pi_i{:}F_i \times i \to F_i\right) \,\middle|\, F_i \text{ is a face of } \overline{C}\right\} \text{ and}$$

$$\dot{L} = \left\{\left(F_i,\ \pi_i{:}F_i \times i \to F_i\right) \,\middle|\, F_i \text{ is a proper face of } \overline{C}\right\}$$

are cell complexes with $|L| = \overline{C}$ and $|\dot{L}| = \mathrm{Bd}\left(\overline{C}\right)$, and this verification follows immediately from Proposition 2.8. □

**2.20** DEFINITION. Let $K = \left\{\left(C_i, h_i{:}\overline{C_i} \to C_i\right)\right\}$ and $L = \left\{\left(D_j, g_j{:}\overline{D_j} \to D_j\right)\right\}$ be H-cell complexes, and $f{:}|K| \to |L|$ be a map. We say that $(f, K, L)$ is ***H-cellular***, or briefly, $f$ is an ***H-cellular map*** if:

1) for all $i$ we have $f_i(C_i) = D_j$ for some $j$, where $f_i$ is the restriction of $f$ to $C_i$.

2) $\bar{f}_i{:}\overline{C_i} \to \overline{D_j}$, the ***lift*** of $f_i$, is linear, where $\overline{f_i}$ is defined via the following commutative diagram:

$$\begin{array}{ccc} \overline{C_i} & \xrightarrow{\ \overline{f_i}\ } & \overline{D_j} \\ \big\downarrow{\scriptstyle h_i} & & \big\downarrow{\scriptstyle g_j} \\ C_i & \xrightarrow{\ f_i\ } & D_j \end{array}$$

($h_i$, $g_j$ are PL homeomorphisms.)

Note that Def. 2.18 implies that each $f_i$ is a PL map, and thus, $f$ is also PL.

**2.21** DEFINITION. If $K$ and $L$ are both cell complexes, then a map $f$ as in Def. 2.20 will be called a ***cellular map*** . If $K$ and $L$ are or both simplicial complexes, $f$ will be called a ***simplicial map***.

Note that if $f$ is cellular, the maps $f_i$ are now linear, since in this situation both $h_i$ and $g_j$ are linear homeomorphisms.

Note also that cellular maps, and thus H-cellular maps, are somewhat restricted as shown below:

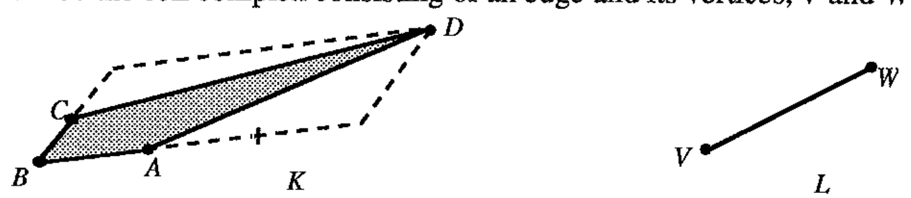
- EXAMPLE. Let $K$ be a cell complex consisting of a convex quadrilateral and all of its faces. Since any three of the vertices ($A$, $B$, $C$ and $D$ ) of $K$ span the plane containing the quadrilateral, we have:

$$(*)\ \ D = \alpha_1 A + \alpha_2 B + \alpha_3 C,\ \sum \alpha_i = 1.$$

Observe that $\alpha_i \neq 0$, for if, say $\alpha_3 = 0$, then $D = \alpha_1 A + \alpha_2 B$, $\alpha_1 + \alpha_2 = 1$ and thus $A$, $B$ and $D$ would be collinear (possibly with $D = A$ or $D = B$).

Assume that $K$ is such that for $i = 1,2,3$, we have $\alpha_i \neq 1$ in $(*)$. [Concretely, in the figure below we have $D = 3A + 2C - 4B = 3(A - B) + 2(C - B) + B$.]

Let $L$ be the cell complex consisting of an edge and its vertices, $V$ and $W$.




Claim: The only cellular maps from $K$ to $L$ are the two maps which map $|K|$ to $V$ or $|K|$ to $W$.

This is true because any linear map defined on the quadrilateral $ABCD$ is uniquely determined by the images of $A$, $B$ and $C$, and either:

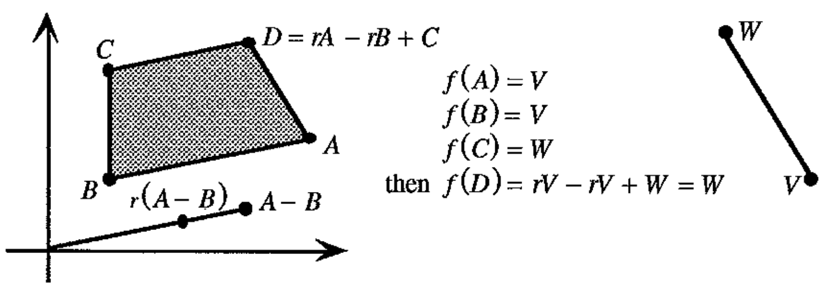
1) $A, B$ and $C$ are all mapped to the same vertex of the edge, implying that $D$ , as well as all of $|K|$, also get mapped to that same vertex,

or

2) Two of the vertices $\{A, B, C\}$ are mapped to one of the vertices of the edge, and the third is mapped to the other vertex, say $A$ and $B$ are mapped to $V$, and $C$ to $W$. In this case the image of $D$ cannot be a vertex of $L$, because for $D$ to be mapped to $V$ we need $D = \alpha_1 A + (1-\alpha_1)B + 0C$ which is impossible as observed at the beginning of this example; and for $D$ to be mapped to $W$ we need $D = \alpha_1 A - \alpha_1 B + 1C$ which is impossible by our assumption that $\alpha_i \neq 1$.

This proves the claim.

We see from the above argument, that the only quadrilaterals that can be mapped cellularly <u>onto</u> an edge, are quadrilaterals whose vertices can be labeled as $A, B, C$ and $D$, such that $D = rA - rB + 1C = r(A-B) + C,\ r \in \mathbf{R}, r \neq 0$. Thus, $D$ is on the line through $C$ which is parallel to the vector $A - B$. Therefore the quadrilateral $ABCD$ is a trapezoid. See Figure 11.




The only quadrilaterals that can be mapped linearly onto an edge are the trapezoids.

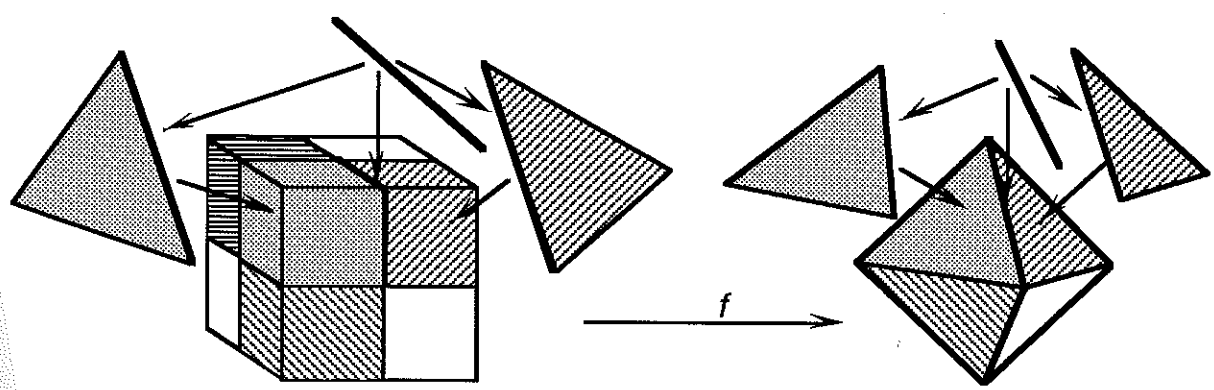
Fig. 11

We can also see that the only polygons (2-cells) that can be cellularly mapped onto an edge are the triangles and the trapezoids. For if the polygon has more than 4 vertices, at least three of them must map onto the same vertex of the edge, thus mapping the whole polygon onto that vertex.

**2.22** REMARK. Let $(f,K,L)$ be H-cellular where $f:|K|\to|L|$ is a PL homeomorphism. If $f_i(C_i)=D_j$ ( as in Def. 2.20) then $\overline{f_i^{-1}}=\left(\overline{f}_i\right)^{-1}:\overline{D}_j\to\overline{C}_i$ is linear. Therefore $\left(f^{-1},L,K\right)$ is H-cellular.

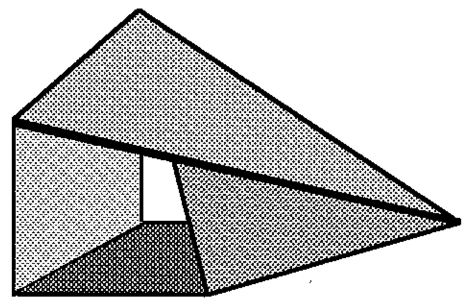
**2.23** DEFINITION. Two H-cell complexes $K$ and $L$ are ***isomorphic*** if there is a homeomorphism $f:|K|\to|L|$ such that $(f,K,L)$ is H-cellular.

*Example* The H-cell complex $K$ consisting of 8 triangles, 12 edges and 6 vertices, $|K|$ the boundary of a cube, given in Fig. 10 is isomorphic to the <u>cell</u> complex consisting of all the proper faces of an octahedron. (See Fig.12)




Isomorphic H-cell complexes.
Fig.12

**2.24** REMARK. There are H-cell complexes which are not isomorphic to cell complexes: it may be impossible to flatten all the polyhedra $C_i$ in the underlying space of an H-cell complex into cells. A simple example is furnished by the H-cell complex depicted in Fig. 13 consisting of 3 rectangular H-cells (and their faces) and whose underlying space is a Möbius band.



An H-cell complex on a Möbius band consisting of 3 quadrilateral H-cells that is not isomorphic to a cell complex.

Fig. 13

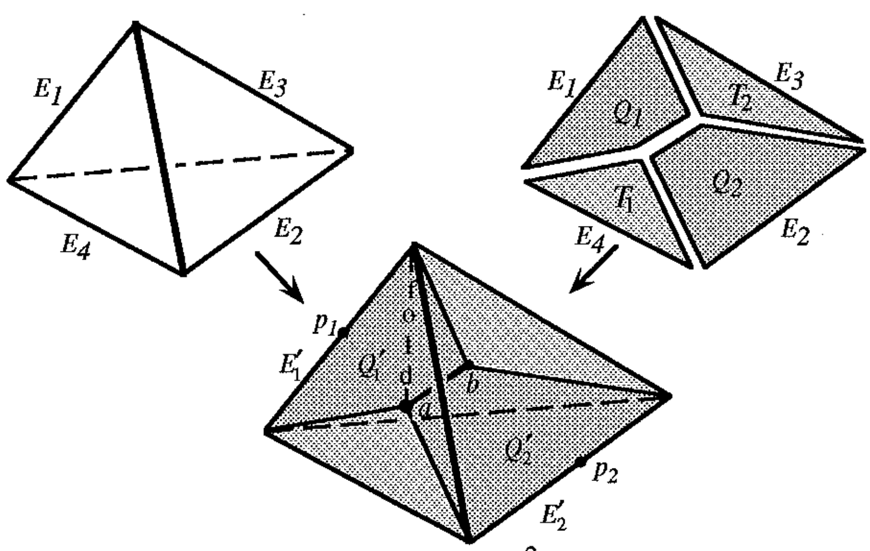
However, perhaps a more interesting example is the 3-dimensional H-cell complex $K^3$ described further below, whose underlying space is a *3-simplex*! (Thus embedable in $\mathbf{R}^3$.) It is called the *Barnette's topological diagram* and was discovered for a different purpose [**Z**, 142]. $K^3$ is not isomorphic to any cell complex whose underlying space lies in $\mathbf{R}^3$. It is however isomorphic to a cell complex whose underlying space lies in $\mathbf{R}^4$ and is a PL homeomorph of the 3-simplex.

We first construct a 2-dimensional subcomplex $K^2$ of $K^3$ as follows: ($K^2$ is sketched in Fig. 14.)

Four triangles are homeomorphically linearly-mapped onto each of the four facets of a 3-simplex in $\mathbf{R}^3$. Two more triangles, $T_1$ and $T_2$ and two quadrilaterals, $Q_1$ and $Q_2$ are injectively and piece-wise linearly mapped into $\mathbf{R}^3$. They are attached to the facets of the 3-simplex along the four edges $E_1, E_2, E_3, E_4$, and except for these edges, their images lie in the interior of the 3-simplex. $T_1$, $T_2$, $Q_1$ and $Q_2$ are also attached among themselves along 5 edges as indicated in Fig. 14.

We let $Q_i'$, $E_i'$, and $T_i'$ denote respectively the images of $Q_i$, $E_i$ and $T_i$.

Note that to map the four cells, $Q_i$, $T_i$ into the 3-simplex, we may first map $Q_2$ linearly, then map the two triangles $T_i$ also linearly; then subdivide $Q_1$ along a diagonal into two subtriangles and map both of these subtriangles linearly into the 3-simplex. $Q_1'$ is thus a folded quadrilateral in $\mathbf{R}^3$.




The subcomplex $K^2$.
It is impossible to embed both quadrilaterals $Q_1$ and $Q_2$ linearly in the 3-simplex.

Fig. 14

Observe that the collection of all of these 2-dimensional H-cells (together with their faces) form a 2-dimensional H-cell complex $K^2$, since any two H-cells that intersect, do so along a common face.

To construct $K^3$, we observe that the union of $Q_1'$, $Q_2'$, $T_1'$ and $T_2'$ separates the interior of the simplex into two regions, $I_1$ and $I_2$, with the property that $B_i$, the closure of $I_i$, can be represented as a cone whose base is the boundary of $B_i$ and whose vertex is a suitably chosen point $v_i \in I_i$. We now construct 12 conical cells using the 8 2-dimensional cells in $K^2$ (the domains of the H-cells of $K^2$). Two cones are constructed over each of the cells $T_1$, $T_2$, $Q_1$ and $Q_2$, and only one cone is constructed over each of the other four triangles.

We map the bases of these 3-dimensional cones into $|K^2|$ using the previous maps of $K^2$. We map their vertices as follows: For each pair of cones constructed over the same base, we map the vertex of one to $v_1$ and the vertex of the other to $v_2$; for each of the other four cones, we map its vertex to $v_i$ if its base is mapped to the boundary of $B_i$. We then map each whole cone into the 3-simplex in the obvious conical manner. These new H-cells (and their faces) form $K^3$.

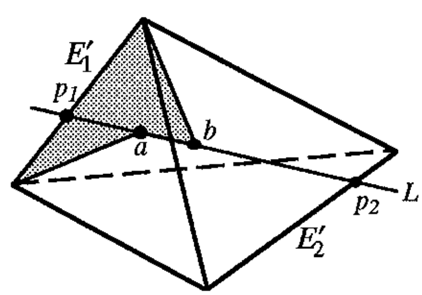
Observe that any two H-cells in $K^3$ that intersect, do so along a common face, since this was the case in $K^2$. Thus $K^3$ is an H-cell complex whose underlying space is the 3-simplex.

**2.25** PROPOSITION. *The H-cell complex $K^3$ is not isomorphic to any cell complex whose underlying space is a polyhedron in* $\mathbf{R}^3$.

*Proof.* We first note there is "only" one way to embed the boundary of a 3-simplex in $\mathbf{R}^3$ up to linear equivalence and that any (topological) homeomorphism between such two 3-simplices carries the interior (in $\mathbf{R}^3$) of one to the interior of the other. Thus any cell complex in $\mathbf{R}^3$ isomorphic to $K^3$ must have a 3-simplex as its underlying space.

(We refer again to Fig. 14 where the subcomplex $K^2$ is described.)

Assume that the quadrilaterals $Q_1$ and $Q_2$ are both mapped <u>linearly</u> into the interior of a 3-simplex. Then $Q_1'$ and $Q_2'$ are convex and span affine planes $\pi_1$ and $\pi_2$ in $\mathbf{R}^3$ respectively. Since the simplex is the join of the edges $E_1'$ and $E_2'$, the plane $\pi_1$ must intersect the interior of $E_2'$ in a point $p_2$ and likewise $\pi_2$ intersects the interior of $E_1'$ in a point $p_1$. Since the four vertices of a 3-simplex are not coplanar, we have that $\pi_1 \neq \pi_2$ and thus their intersection is a line $L$. Therefore the points $p_1, a, b$ and $p_2$ all lie in $L$ (and assume in this given order). (See Fig. 15.) But this implies that the quadrilateral $Q_i'$ is not convex since the line through its edge $ab$ intersects its edge $E_i'$ in the point $p_i$. This is a contradiction. □




$Q_1'$ is not convex. The line through the edge $ab$ intersects the edge $E_1'$.

Fig. 15

We will now construct a <u>cell</u> complex $K^{3'}$ isomorphic to $K^3$ whose underlying space is a PL homeomorph of the 3-simplex and lies in $\mathbf{R}^4$.

We first construct a cell complex $K^{2'}$ in $\mathbf{R}^3$ isomorphic to $K^2$. (See Fig. 16.) We start by linearly mapping each quadrilateral $Q_i$ onto $Q_i'' \subset \mathbf{R}^3 \times 0 \subset \mathbf{R}^4$ in such a way that the planes $\pi_i'$ spanned by $Q_i''$ are distinct.

Let $L'$ be the line through the edge $a'b'$ and let $p_i'$ be the point (if it exists) where $L'$ intersects the line through $E_i''$. We require that the quadrilaterals $Q_i$ are such that at least one intersection point $p_i'$ exists, and when both exist we must have that $p_1' \neq p_2'$. Observe that these requirements depend only on the "shape" of $Q_i$ and not on the particular linear embeddings chosen. See note at the end.

Observe that the edges $E_1''$ and $E_2''$ are not coplanar since this would imply that $p_1' = p_2'$, or else that no point $p_i'$ exists, thus violating the above requirements. We thus obtain as a corollary that the four vertices of these two edges are in general position, and therefore we can linearly embed the four triangles of the boundary of a 3-simplex into $\mathbf{R}^3$ using these four points as the vertices.

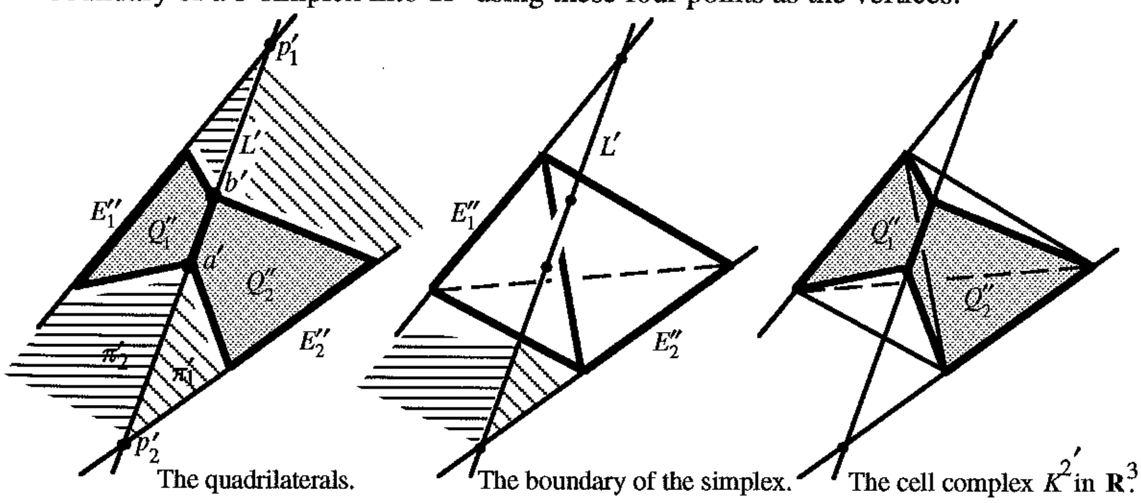



Fig.16

We finish the construction of $K^{2'}$ by mapping the triangles $T_i$ onto $T_i''$ also linearly and observe that neither $Q_i''$ nor $T_i''$ intersects the interior of the simplex as ruled out by Prop. 2.25.

To obtain the cell complex $K^{3'} \subset \mathbf{R}^4$, we now embed linearly into $\mathbf{R}^4$ the 3-dimensional cones whose bases are the 2-cells of $K^{2'}$. We do this as follows: The vertex of a cone is mapped to $(0,\cdots,0,1) \in \mathbf{R}^4$ if its vertex is mapped to $v_1$ in the previous construction of the H-cell complex $K^3$, and it is mapped to $(0,\cdots,0,-1)$ if in $K^3$ it is mapped to $v_2$.

NOTE regarding the "shape" of the quadrilaterals $Q_i$:

Observe that we can always construct the H-cell complex $K^3$ regardless of the shape of the quadrilaterals, since we are allowed to embed them piece-wise-linearly into the simplex. In particular, if we use two trapezoids whose edges $E_i$ are parallel to the edge "$ab$", the situation is even worse than in the case that we analyzed. In this case there is no cell comple isomorphic to $K^3$, because in any linear embedding of these quadrilaterals in $\mathbf{R}^n$, the edges $E_i''$ would always be coplanar, thus making it impossible to embed the four facets of the simplex.

## HI-CELLS

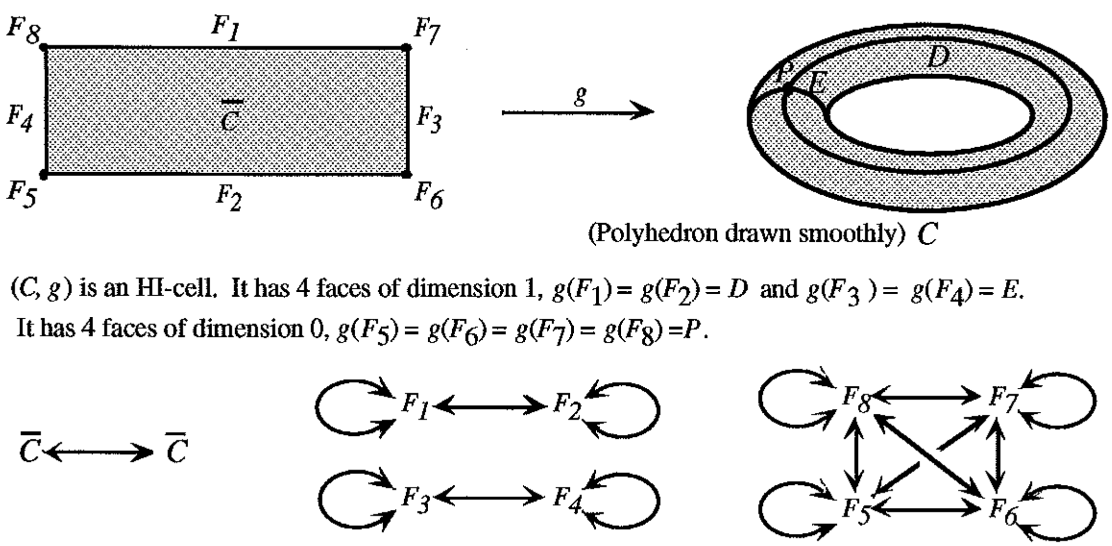
**2.26** DEFINITION. An ***HI-cell*** is a pair $\left(C, g:\overline{C}\to C\right)$ where $C$ is a polyhedron, $\overline{C}$ is a cell and $g:\overline{C}\to C$ is a PL surjective map, satisfying the following:

1) $g$ maps the interior of every face $F$ of $\overline{C}$ homeomorphically onto its image.

2) If $\left(F_i, F_j\right)$ is an ordered pair of faces of $\overline{C}$ such that $g\left(\text{int}(F_i)\right)\cap g\left(\text{int}(F_j)\right)\neq\varnothing$ then there is a <u>linear</u> homeomorphism $f_{ij}:F_i\to F_j$ such that $\left(g|_{F_j}\right)\circ f_{ij}=g|_{F_i}$, $\left(g|_{F_j}\right)\circ f_{ij}:F_i\to g(F_i)=g\left(F_j\right)$.

A ***face of an HI-cell*** $\left(C, g:\overline{C}\to C\right)$ is a pair $\left(g(F), g|_F:F\to g(F)\right)$ where $F$ is a face of $\overline{C}$, and $g|_F:F\to g(F)\subset C$ is the restriction of $g$ to $F$. Thus, a face of an HI-cell is again an HI-cell.

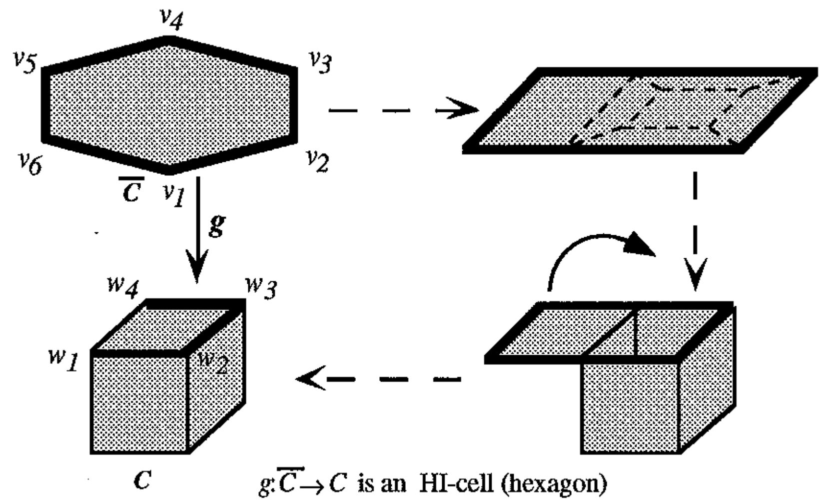
The ***dimension of an HI-cell*** $\left(C, g:\overline{C}\to C\right)$ is the dimension of $\overline{C}$.
(See figures 17 and 18 for examples of HI-cells.)




$(C, g)$ is an HI-cell. It has 4 faces of dimension 1, $g(F_1)=g(F_2)=D$ and $g(F_3)=g(F_4)=E$.
It has 4 faces of dimension 0, $g(F_5)=g(F_6)=g(F_7)=g(F_8)=P$.

Diagrams of the linear homeomorphisms $f_{ij}:F_i\longrightarrow F_j$

Fig. 17




$g:\overline{C} \to C$ is an HI-cell (hexagon)

Fig. 18

**2.27** PROPOSITION. *Let* $(F_i, F_j)$ *be a pair of faces of* $\overline{C}$ *as in 2.26 2) above. Then:*

*1) The homeomorphism* $f_{ij}: F_i \to F_j$ *is unique (thus,* $f_{ii}: F_i \to F_i = id_i: F_i \to F_i$*).*

*2)* $f_{ji}: F_j \to F_i$ *is the inverse of* $f_{ij}: F_i \to F_j$.

*Proof.* 1) Let $d = \dim(F_i) = \dim(F_j)$ (this equality is a consequence of the existence of the homeomorphism $f_{ij}$), and let $x_0, x_1, \cdots, x_d$ be $d$+1 independent points in $\mathrm{int}(F_i)$. Let $y_0, y_1, \cdots, y_d$ be the corresponding $d$+1 images (in $F_j$) of $x_0, x_1, \cdots, x_d$ under the PL homeomorphism given by the composition: $\left(g|_{\mathrm{int}(F_j)}\right)^{-1} \circ g|_{\mathrm{int}(F_i)}$. Any linear homeomorphism $\hat{f}_{ij}: F_i \to F_j$ satisfying $\left(g|_{F_j}\right) \circ \hat{f}_{ij} = g|_{F_i}$ must map $x_i \mapsto y_i, 0 \le i \le d$, and since every linear map $F_i \to F_j$ is uniquely determined by the image of $d$+1 independent points, we have that $\hat{f}_{ij} = f_{ij}$.

2) To see that $f_{ji} = f_{ij}^{-1}$, let $x \in \mathrm{int}(F_i)$. Because for $y \in F_j$ we have $g\left(f_{ji}(y)\right) = g(y)$, and for $x \in F_i$ we have $g\left(f_{ij}(x)\right) = g(x)$, we obtain the

following equalities: $g\left(f_{ji}\left(f_{ij}(x)\right)\right) = g \circ f_{ji}\left(f_{ij}(x)\right) = g\left(f_{ij}(x)\right) = g(x)$. Since $g$ is a homeomorphism on $\operatorname{int}(F_i)$, the above equality gives $f_{ji}\left(f_{ij}(x)\right) = x$ on $\operatorname{int}(F_i)$. By continuity we have: $f_{ji}\left(f_{ij}(x)\right) = x$ on all of $F_i$. □

### HI-CELL COMPLEXES

**2.28** DEFINITION. An ***HI-cell complex*** $K$ is a finite set $K = \left\{ \left(C_i, h_i : \overline{C_i} \to C_i\right) \right\}$ of HI-cells, with $C_i$ in $\mathbf{R}^n$, $C_i \neq C_j$ if $i \neq j$, such that the following is satisfied:

1) If $\left(C_i, h_i : \overline{C_i} \to C_i\right) \in K$ and if $\left(h_i(F), h_i|_F : F \to h_i(F)\right)$ is a face of $\left(C_i, h_i\right)$, then there is an HI-cell $\left(C_k, h_k : \overline{C_k} \to C_k\right) \in K$ with $C_k = h_i(F)$ and a ***face recognition*** linear homeomorphism $i_F : \overline{C_k} \to F \subset \overline{C_i}$ such that

$$\begin{array}{ccc} \overline{C_k} & \xrightarrow{i_F} & F \subset \overline{C_i} \\ & {\scriptstyle h_k}\searrow \quad \swarrow{\scriptstyle h_i|_F} & \\ & C_k = h_i(F) & \end{array}$$

commutes.

2) If $\left(C_i, h_i : \overline{C_i} \to C_i\right)$ and $\left(C_j, h_j : \overline{C_j} \to C_j\right)$ are in $K$ and if $C_i \cap C_j \neq \varnothing$, then there are faces (not necessarily all different) $F_1, F_2, \cdots, F_s$ of $\overline{C_i}$ and $G_1, G_2, \cdots, G_s$ of $\overline{C_j}$ such that

$h_i(F_r) = h_j(G_r)$ for $1 \le r \le s$, and $\bigcup_{r=1}^{s} h_i(F_r) = C_i \cap C_j$.

(And thus for each $r$ there is an HI-cell $\left(C_k, h_k : \overline{C_k} \to C_k\right) \in K$, with $C_k \subset \left(C_i \cap C_j\right)$ and linear homeomorphisms $i_{F_r} : \overline{C_k} \to F_r$ and $i_{G_r} : \overline{C_k} \to G_r$ such that the diagram

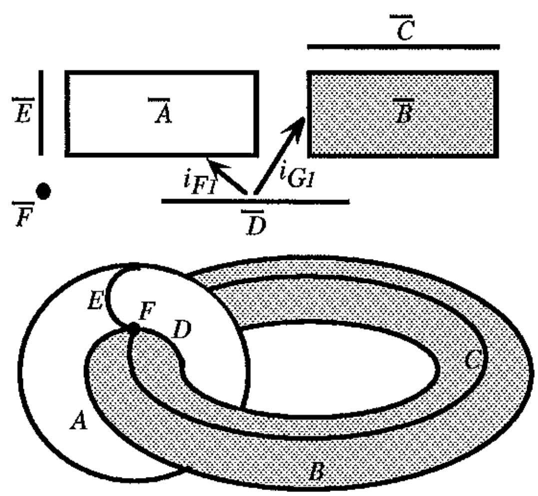
$$\begin{array}{ccccc} \overline{C_i} \supset F_r & \xleftarrow{i_{F_r}} & \overline{C_k} & \xrightarrow{i_{G_r}} & G_r \subset \overline{C_j} \\ & {\scriptstyle h_i|_{F_r}}\searrow & \downarrow{\scriptstyle h_k} & \swarrow{\scriptstyle h_j|_{G_r}} & \\ & & C_k \subset C_i \cap C_j & & \end{array}$$

commutes.)

We let the concept of ***the underlying polyhedron of K***, $|K|$, be defined as in Def. 2.16, and we will say that $K$ is an ***HI-cellular structure on*** $|K|$.

Also, we observe that the concepts ***dimension of K***, $\dim(K)$, and ***the i-skeleton of K***, $K^i$, as defined previously for an H-cell complex $K$ in Def. 2.17, are also applicable if $K$ is an HI-cell complex. We thus take Def. 2.17 as the definition of these concepts in the more general situation of HI-cell complexes.

*Example .* Figure 19 shows an HI-cell complex $K$ with one 0-dimensional HI-cell, three 1-dimensional HI-cells and two 2-dimensional HI-cells, where $|K|$ is the union of two tori along a common circle ($D$).

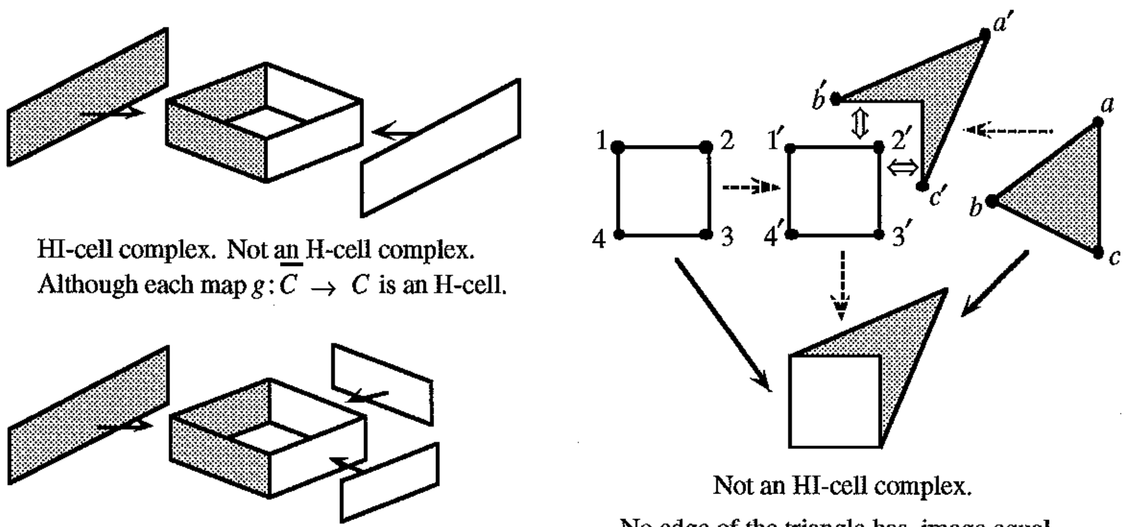



An HI-cell complex.
Fig.19

**2.29** REMARK. We observe that in an H-cell complex if $C_i \cap C_j \neq \varnothing$ then $C_i \cap C_j$ is the image of a <u>unique</u> face $F$ of $\overline{C_i}$ and of a unique face $G$ of $\overline{C_j}$, whereas in an HI-cell complex, $C_i \cap C_j$ is a <u>union</u> of the images of faces $F_r$ of $\overline{C_i}$ as well as a union of images of faces $G_r$ of $\overline{C_j}$ as above.

Figure 20 below illustrates some polyhedra which are unions of images of cells. Regarding Remark 2.29 above, note that the example on the right is not an HI-cell complex, although the intersection of the images of the triangle and of the square, is equal to a union of images of faces of the square, and also equal to the image of a face of the triangle (but not pairwise matched).

However if we had mapped the triangle onto the shaded region via a PL homeomorphism taking $a$ to $1'$, $b$ to $2'$, $c$ to $3'$, and a point $x$ in the interior of the edge $ac$ to $a'$, the polyhedron on the bottom right would now be the underlying polyhedron of an HI-cell complex.




HI-cell complex. Not an H-cell complex.
Although each map $g:\overline{C} \to C$ is an H-cell.

H-cell complex.

Not an HI-cell complex.
No edge of the triangle has image equal to the image of the edge12 of the square.

Only some of the HI-cells are shown in each example.

Fig. 20

**2.30** DEFINITION. Let $h:\overline{C} \to C$ be an (H-) HI-cell. The ***(H-) HI-interior of C***, HI-int($C$), is defined by HI-int($C$) = $h\left(\text{int}\left(\overline{C}\right)\right)$.

**2.31** PROPOSITION. *Let* $K=\left\{\left(C_i,\ h_i:\overline{C_i}\to C_i\right)\right\}_{i=1,\cdots,l}$ *be an HI-cell complex.*

*Let* $f:|K|\to\mathbf{R}^n$ *be a PL map which is injective on the HI-interior of each* $C_i$ *and let* $\left\{D_j\right\}_{j=1,\cdots,k}=\left\{f(C_i)\right\}_{i=1,\cdots,l}$, $k\le l$.

*Let f further possess the following properties:*

*1) If* $f(\text{HI-int}(C_u))\cap f(\text{HI-int}(C_v))\neq\varnothing$ *then* $f(C_u)=f(C_v)$.

*2) If for every j,* $1\le j\le k$, $\left\{C_{j1},C_{j2},\cdots,C_{jw}\right\}$, $j1<j2<\cdots<jw$, *is the set of polyhedra in* $|K|$ *mapping onto* $D_j$, *then for every pair of cells* $\left(\overline{C}_{jp},\overline{C}_{jq}\right)$ *there are linear homeomorphisms*

$i_{jp,jq}:\overline{C}_{jp}\to\overline{C}_{jq}$ *such that*

$$\begin{array}{ccc} \overline{C}_{jp} & \xrightarrow{i_{jp,jq}} & \overline{C}_{jq} \\ {\scriptstyle h_{jp}}\downarrow & & \downarrow{\scriptstyle h_{jq}} \\ C_{jp} & & C_{jq} \\ & {\scriptstyle f}\searrow\ D_j\ \swarrow{\scriptstyle f} & \end{array}$$

*commutes.*

*For all j, set* $\overline{D}_j=\overline{C}_{j1}$ *(the polyhedron with smallest index mapping onto* $D_j$*) and define* $g_j:\overline{D}_j\to D_j$ *to be the composition:* $g_j:\overline{D}_j=\overline{C}_{j1}\xrightarrow{h_{j1}}C_{j1}\xrightarrow{f}D_j$. *Then* $L=\left\{\left(D_j,g_j:\overline{D}_j\to D_j\right)\right\}$ *is an HI-cell complex.*

*Proof.* (See Figure 21, p. 48.) Firstly we must show that every pair $\left(D_j,g_j:\overline{D}_j\to D_j\right)$ is indeed an HI-cell. Part 1) of the definition of an HI-cell (Def. 2.26) requiring that $g_j$ be a PL homeomorphism on the interior of all the faces of $\overline{D}_j$ is obviously met by $g_j$.

Now for 2.26 2). If $(F_r,F_s)$ is an ordered pair of faces of $\overline{D}_j$ with $g_j(\text{int}(F_r))\cap g_j(\text{int}(F_s))\neq\varnothing$, we need to construct the linear homeomorphism $f_{rs}:F_r\to F_s$ such that $g_j\big|_{F_s}\circ f_{rs}=g_j\big|_{F_r}$. Since $g_j:\overline{D}_j\to D_j$ is in $L$, there is $h_t:\overline{C}_t\to C_t$ in K with $\overline{D}_j=\overline{C}_t$.

a) If $h_t(F_r) = h_t(F_s)$, then we have already a linear homeomorphism with the required property, namely $f_{rs}: F_r \to F_s$ required in $K$. We use this homeomorphism in $L$.

b) If $h_t(F_r) \neq h_t(F_s)$, there are (in $K$) linear homeomorphisms, $i_{F_s}: \overline{C}_u \to F_s \subset \overline{C}_t$ and $i_{F_r}: \overline{C}_v \to F_r \subset \overline{C}_t$ such that $h_t \circ i_{F_s}(\overline{C}_u) = h_t(F_s)$ and $h_t \circ i_{F_r}(\overline{C}_v) = h_t(F_r)$. We now define $f_{rs}: F_r \to F_s$ to be the composition:

$$f_{rs}: F_r \xrightarrow{(i_{F_r})^{-1}} \overline{C}_v \xrightarrow{i_{v,u}} \overline{C}_u \xrightarrow{i_{F_s}} F_s.$$

We have shown that $g_j: \overline{D}_j \to D_j$ is an HI-cell.

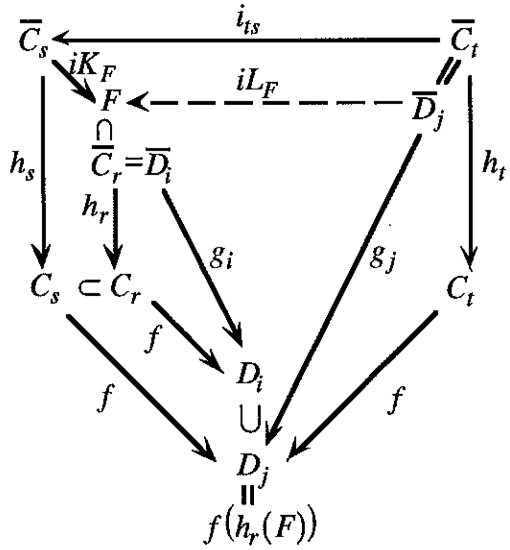
Now we will show that $L = \left\{ \left( D_j, g_j: \overline{D}_j \to D_j \right) \right\}$ is an HI-cell complex.

i) We need to define the linear homeomorphisms $i_F$ (in $L$) as in the definition of HI-cell complexes (Def. 2.28). Below, to distinguish the homeomorphism $i_F$ in $L$ from $i_F$ in $K$, we use the notations $iL_F$ and $iK_F$ respectively.

Let $\left(D_i, g_i: \overline{D}_i \to D_i\right)$ be in $L$, and $F < \overline{D}_i$, thus for some $r$, $\overline{D}_i = \overline{C}_r$ and $\left(C_r, h_r: \overline{C}_r \to C_r\right)$ is in $K$. Since $F < \overline{C}_r$ we have that for some $s$, $\left(C_s, h_s: \overline{C}_s \to C_s\right)$ is in $K$, with $C_s = h_r(F)$, and also there exists a linear homeomorphism $iK_F: \overline{C}_s \to F \subset \overline{C}_r$ such that $h_s(\overline{C}_s) = h_r\left(iK_F(\overline{C}_s)\right) = h_r(F)$.

Since $h_r(F) = h_s(\overline{C}_s) = C_s$, we have that $f(h_r(F)) = f(C_s) = D_j$ for some $j$, with $g_j: \overline{D}_j \to D_j$ in $L$. Thus there exists $\left(\overline{C}_t, h_t: \overline{C}_t \to C_t\right)$ in K with $\overline{C}_t = \overline{D}_j$ and $f(C_t) = D_j = f(C_s)$. Therefore there is a linear homeomorphism $i_{ts}: \overline{C}_t \to \overline{C}_s$ as in the diagram above in property 2) of our hypothesis. We now define our desired linear homeomorphism $iL_F: \overline{D}_j \to F \subset \overline{D}_i$ to be the composition (of linear homeomorphisms) $iL_F: \overline{D}_j = \overline{C}_t \xrightarrow{i_{ts}} \overline{C}_s \xrightarrow{iK_F} F$.

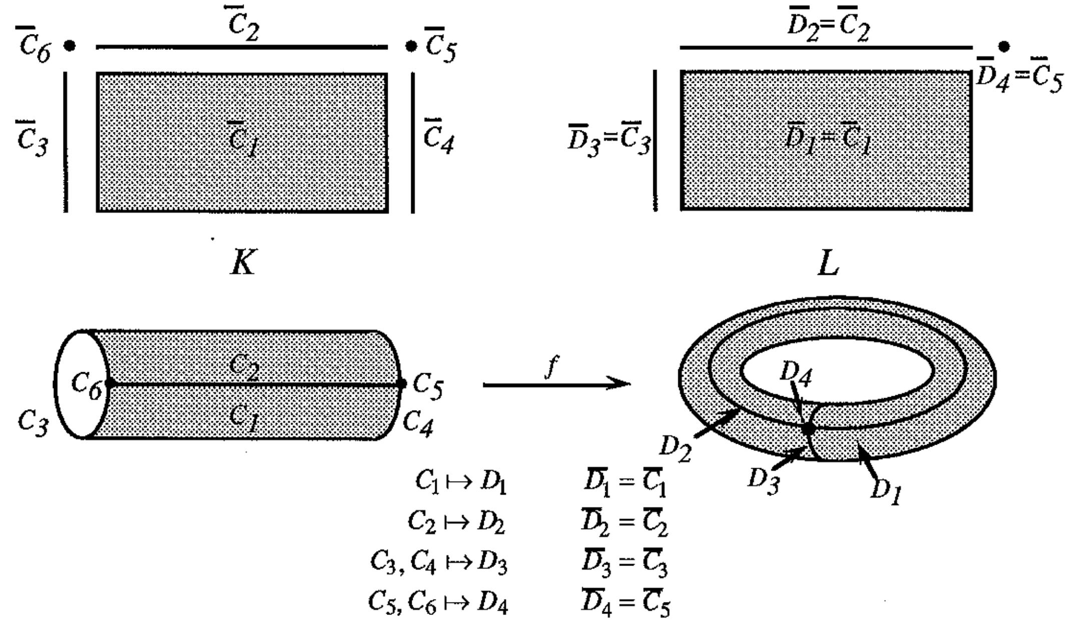
Below is a diagram of the above construction of $iL_F$:




ii) We need now to show that if $(D_i, g_i: \overline{D}_i \to D_i)$ and $(D_j, g_j: \overline{D}_j \to D_j)$ are in $L$, with $D_i \cap D_j \neq \varnothing$, then there are pairs of faces $(F_r, G_r)$, $1 \leq r \leq s$, $F_r < \overline{D}_i$ and $\overline{G}_r < D_j$, such that $h_i(F_r) = h_j(G_r)$ and $\bigcup_{r=1}^{s} h_i(F_r) = C_i \cap C_j$.

These conditions are met in $K$, and because of 1) in the statement of this proposition, we have that $D_i = D_j$ or $D_i \cap D_j$ is a union of images of proper faces of $\overline{D}_i$ and of $\overline{D}_j$. Now using this fact, and a "diagram chase" similar to the one in i), we get that these conditions are also met in $L$. □

**2.32** DEFINITION. The HI-cell complex $L$ obtained in Proposition 2.31 is called ***the HI-cell complex induced by K and f.***




$L$ is induced by $K$ and $f$ as in Proposition 2.31.
Fig. 21

Proposition 2.31 implies that if $g:\overline{C} \to C$ is an HI-cell then $C$ and $h(\mathrm{Bd}(\overline{C}))$ both have "canonical" HI-cellular structures . Formally we have the following:

**2.33** PROPOSITION. *Let $g:\overline{C} \to C$ be an HI-cell, and let $\{F_i\}$ be the set of faces of $\overline{C}$. For every polyhedron $g(F_i) \subset C$ let $\{F_{i_j}\} \subset \{F_i\}$, $i_1 < i_2 < \cdots < i_k$, be the set of faces of $\overline{C}$ such that $g(F_{i_j}) = g(F_i)$. Then*

$$K = \left\{ \left( g(F_i),\ g|_{F_{i_1}} : F_{i_1} \to g(F_i) \right) \middle| F_i \text{ is a face of } \overline{C} \right\} \quad \textit{and}$$

$$\dot{K} = \left\{ \left( g(F_i),\ g|_{F_{i_1}} : F_{i_1} \to g(F_i) \right) \middle| F_i \text{ is a proper face of } \overline{C} \right\}$$

*are HI-cell complex with $|K| = C$ and $|\dot{K}| = g(\mathrm{Bd}(\overline{C}))$.*

(Note that $F_{i_1}$, chosen above, is the face of $\overline{C}$ which has smallest index and maps onto $g(F_i)$.)

*Proof.* We know (see the proof of Proposition 2.18) that

$$L=\{(F_i,\ \pi_i{:}F_i\times i\rightarrow F_i)\mid F_i \text{ is a face of } \overline{C}\} \text{ and}$$

$$\dot{L}=\{(F_i,\ \pi_i{:}F_i\times i\rightarrow F_i)\mid F_i \text{ is a proper face of } \overline{C}\}$$

are cell complexes with $|L|=\overline{C}$ and $|\dot{L}|=\mathrm{Bd}(\overline{C})$. Now letting $g{:}\overline{C}\rightarrow C$ play the role of $f$ as in the hypothesis of Proposition 2.31 we get that

$$M=\left\{\left(g(F_i),\ F_{i_1}\times i_1\xrightarrow{\pi_{i_1}}F_{i_1}\xrightarrow{g}g(F_i)\right)\middle|\ F_i \text{ is a face of } \overline{C}\right\} \text{ and}$$

$$\dot{M}=\left\{\left(g(F_i),\ F_{i_1}\times i_1\xrightarrow{\pi_{i_1}}F_{i_1}\xrightarrow{g}g(F_i)\right)\middle|\ F_i \text{ is a face of } \overline{C}\right\}$$

are HI-cell complexes. Now replacing the maps $F_{i_1}\times i_1\xrightarrow{\pi_{i_1}}F_{i_1}\xrightarrow{g}g(F_i)$ by the maps $F_{i_1}\xrightarrow{g}g(F_i)$ we obtain the desired result, i.e.,

$$K=\left\{\left(g(F_i),\ g|_{F_{i_1}}{:}F_{i_1}\rightarrow g(F_i)\right)\middle|\ F_i \text{ is a face of } \overline{C}\right\} \quad \textit{and}$$

$$\dot{K}=\left\{\left(g(F_i),\ g|_{F_{i_1}}{:}F_{i_1}\rightarrow g(F_i)\right)\middle|\ F_i \text{ is a proper face of } \overline{C}\right\}$$

are HI-cell complex with $|K|=C$ and $|\dot{K}|=g(\mathrm{Bd}(\overline{C}))$. □

Observe that in contrast to the case of the H-cell complexes $K$ and $\dot{K}$, or more specifically $K(h)$ and $\dot{K}(h)$, constructed from an H-cell $h{:}\overline{C}\rightarrow C$ (Proposition 2.18), where we have that the number of H-cells in $K$ is equal to the number of faces of $\overline{C}$ and the number of H-cells of $\dot{K}$ is equal to the number of proper faces of $\overline{C}$, the complexes $K(g)$ and $\dot{K}(g)$, constructed from an HI-cell $g{:}\overline{C}\rightarrow C'$ as in Proposition 2.31 have in general fewer cells.

**2.34** DEFINITION. Let $K=\left\{\left(C_i, h_i{:}\overline{C}_i \to C_i\right)\right\}$ and $L=\left\{\left(D_j, g_j{:}\overline{D}_j \to D_j\right)\right\}$ be HI-cell complexes, and $f{:}|K| \to |L|$ be a map. We say that $(f,K,L)$ is ***HI-cellular***, or briefly, $f$ is an ***HI-cellular map*** if:

1) For all $i$ we have $f_i(C_i)=D_j$ for some $j$, where $f_i$ is the restriction of $f$ to $C_i$.

2) For all $i$ there is a (unique) <u>linear</u> map $\bar{f}_i{:}\overline{C}_i \to \overline{D}_j$, called ***the lift*** of $f_i$, such that the following diagram commutes:

$$\begin{array}{ccc} \overline{C}_i & \xrightarrow{\bar{f}_i} & \overline{D}_j \\ \downarrow h_i & & \downarrow g_j \\ C_i & \xrightarrow{f_i} & D_j \end{array}$$

We observe that the above definition implies that each $f_i$ is a PL map, and thus, $f$ is also PL.

**2.35** REMARK. Let $(f,K,L)$ be HI-cellular where $f{:}|K| \to |L|$ is a PL homeomorphism. If $f_i(C_i)=D_j$ (as in Def. 2.34) then $\overline{f_i^{-1}}=\left(\overline{f}_i\right)^{-1}{:}\overline{D}_j \to \overline{C}_i$ is linear. Therefore $\left(f^{-1},L,K\right)$ is HI-cellular.

**2.36** DEFINITION. Let $K$ and $L$ be HI-cell complexes. We say that $(f,K,L)$ (briefly, $f$) is an ***HI-isomorphism*** if $(f,K,L)$ is HI-cellular and $f{:}|K| \to |L|$ is a PL homeomorphism. Two HI-cell complexes $K$ and $L$ are ***isomorphic*** if there exists an HI-isomorphism $(f,K,L)$ .

# 3 Subdivisions of HI-Cell Complexes

## Definition of Subdivision

**3.1** Definition. Let $K=\{(C_i, h_i:\overline{C}_i \to C_i)\}$ and $L=\{(D_j, g_j:\overline{D}_j \to D_j)\}$ be HI-cell complexes. We say that ***L is a subdivision of K***, denoted $L \lhd K$, if:

1) $|L|=|K|$

2) For every HI-cell $(D_j, g_j:\overline{D}_j \to D_j)$ in $L$ there is an HI-cell $(C_i, h_i:\overline{C}_i \to C_i)$ in $K$ and a <u>linear</u> embedding $e_{ji}:\overline{D}_j \to \overline{C}_i$ such that the diagram:

$$\begin{array}{ccc} \overline{D}_j & \xrightarrow{e_{ji}} & \overline{C}_i \\ \downarrow g_j & & \downarrow h_i \\ D_j & \xrightarrow{\text{inclusion}} & C_i \end{array} \quad \text{commutes.}$$

## Preliminary Results

Proposition 3.8 will show how to construct a <u>cell</u> complex from a given cell $C$ and a point $a$ in $C$ by *starring at a*. This construction will then be utilized to construct (define) *stellar subdivisions* of HI-cell complexes.

We will firstly obtain some preliminary results, and rewrite Def. 2.5 below in order to recall some needed concepts and notation.

> **2.5** Definition. Let $C \subset \mathbf{R}^m$ be a cell and $x \in C$. Let $V(x,C)$ be the union of all lines $L$ in $\mathbf{R}^m$ such that $L \cap C$ is a 1-cell containing $x$ in its interior. (If $V(x,C) \neq \varnothing$ then $V(x,C)$ is a subspace in $\mathbf{R}^m$.) Now let $F(x,C)$ be $V(x,C) \cap C$ if $V(x,C) \neq \varnothing$, and let $F(x,C)=x$ if $V(x,C)=\varnothing$. $F(x,C)$ are cells called ***faces of C*** and in the particular case $F(x,C)=x$, $x$ is called a ***vertex** of C*.

Also recall that the metric $d$ on $\mathbf{R}^n$ is given by $d(x,y)=\max\{\,|x_i-y_i|\,\}_{1\le i\le n}$ and that the ***interior of*** $F(x,C)$ is the interior of $F(x,C)$ in $V(x,C)$.

**3.2** LEMMA. *Let $C$ be a cell in $\mathbf{R}^n$ and $x\in C$.* Then $x\in \operatorname{int}(F(x,C))$.

*Proof.* Without loss of generality we can assume that $x=(0,0,\cdots,0)$ and that $V(x,C)=\mathbf{R}^d\times 0\subset \mathbf{R}^d\times\mathbf{R}^{n-d}=\mathbf{R}^n$. However since our analysis below will take place solely in $V(x,C)$, any point $y\in V(x,C)$ will be understood to be a point $y=(y_1,\cdots,y_d)$ in $\mathbf{R}^d$.

Let $\left\{v_1,\cdots,v_{2^{d-1}}\right\}$ be the set of points in $\mathbf{R}^d$ that have their first coordinate equal to 1 and all the other coordinates equal to either 1 or -1. Note that because of the condition on their first coordinates, if $i\neq j$ then $v_i$ is not a scalar multiple of $v_j$. Also note that $\left\{v_1,\cdots,v_{2^{d-1}}\right\}$ and $\left\{-v_1,\cdots,-v_{2^{d-1}}\right\}$ are disjoint and that their union is the set of the $2^d$ vertices of the $d$-cube $N_1\left(x,\mathbf{R}^d\right)$.

For all $1\le i\le 2^{d-1}$ let $L_i$ denote the line in $\mathbf{R}^d$ going through $v_i$ and $-v_i$. Note that $x\in L_i$. Since every line $L$ in $\mathbf{R}^d=V(x,C)$ that passes through $x$ intersects $F(x,C)$ in a one-dimensional cell $\widehat{L}$ with $x\in\operatorname{int}\left(\widehat{L}\right)$, there is an $\varepsilon>0$ such that the 1-cells $S_i\subset L_i$, connecting $w_i=\varepsilon v_i$ to $-w_i=-\varepsilon v_i$ are all in $F(x,C)$.

Since $\{w_i\}\cup\{-w_i\}$ is the set of all the vertices of $N_\varepsilon\left(x,\mathbf{R}^d\right)$, all of which are in $F(x,C)$, and since a $d$-cube is the convex hull of its vertex set, we conclude using the convexity of $F(x,C)$ that $N_\varepsilon\left(x,\mathbf{R}^d\right)\subset F(x,C)$. We thus obtain that $x\in\operatorname{int}(F(x,C))$. □

**3.3 COROLLARY.** *Let* $F(y,C)$ *be a face of a cell* $C$ *and let* $x \in \text{int}(F(y,C))$. *Then* $F(x,C) = F(y,C)$.

*Proof.* We have that $x \in \text{int}(F(x,C))$ as well as $x \in \text{int}(F(y,C))$ and since different faces of $C$ have disjoint interiors ( Prop.2.8 3) ), we get that $F(x,C) = F(y,C)$. □

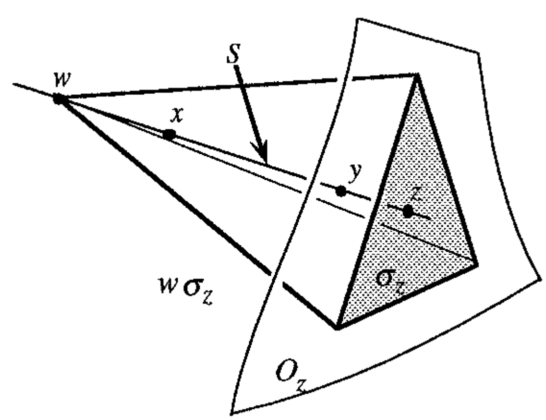
**3.4 LEMMA.** *Let x and y be two points in a cell C. Then* $F(x,C) = F(y,C)$ *if and only if there is a 1-cell S in C with the property that both x and y lie in the interior of S.*

*Proof.* Firstly we assume that $F(x,C) = F(y,C) = F$. We note from the outset that neither $x$ nor $y$ are vertices of $C$, since then $x = F(x,C) = F(y,C) = y$. From Lemma 3.2, we know that both $x$ and $y$ lie in the interior of $F$ ( and $\dim(F) \geq 1$ ). Using cubical neighborhoods $N_\varepsilon(x,F)$ and $N_\delta(y,F)$, the 1-cell $xy$, which is in $F$ by convexity of $F$, can be extended at both endpoints to a larger 1-cell $S$ in $F \subset C$ containing both $x$ and $y$ in its interior.

Conversely, assume the existence of a 1-cell $S$ with the stated properties. Since $x$ and $y$ are both in the interior of $S$ and $S \subset C$, we conclude that $S \subset F(x,C)$ as well as $S \subset F(y,C)$ (recall the definition of $F(z,C)$ ). Since $x$ is in the interior of $F(x,C)$ as well as in the interior of $S$, and since $S \subset F(x,C)$, we can choose a point $w \in \text{int}\, F(x,C) \cap S$ with the further property that $w$ is in the component of $S - \{x\}$ which does not contain $y$ (see Fig.22). Likewise we can choose a point $z \in \text{int}\, F(y,C) \cap S$ such that $z$ is in the component of $S - \{y\}$

which does not contain $x$. Note that there is an orientation (ordering) of $S$ such that the points w,x,y,z are sequentially positioned in S in the order given by the 4-tuple $(w,x,y,z)$. Also note that by Corollary 3.3 we conclude that $F(w,C)=F(x,C)$, $V(w,C)=V(x,C)$ and $F(y,C)=F(z,C)$, $V(y,C)=V(z,C)$.

Let $V(y,C)$ have dimension equal to $d$, which is $\geq 1$. Since both $z$ and $S$ are in $V(y,C)$, there is an affine subspace $O_z$ of $V(y,C)$ which contains $z$ and is normal to the line containing $S$. (Let $O_z$ equal $z$ if $d$=1). Since $z$ is in the interior of $F(y,C)$ and since $O_z$ has dimension $d$-1, we can choose a ($d$-1)-simplex $\sigma_z$ contained in $F(y,C)\cap O_z$ with the property that $z \in \text{int}(\sigma_z)$. (See Figure 22.) ( Let $\sigma_z$ equal z if $d$=1) Now observe that the $d$-simplex $w\sigma_z$ contains both $x$ and $y$ in its interior and that by convexity of $F(y,C)$, $w\sigma_z$ is contained in $F(y,C)$ and consequently also in $V(y,C)$. Since $w\sigma_z$ is a $d$-simplex and $V(y,C)$ has dimension also equal to $d$, we have that $\text{int}(w\sigma_z)$ is an open neighborhood of $y$ in $F(y,C)$. Since $x \in \text{int}(w\sigma_z)$, we finally conclude that $x$ is a point in the interior of $F(y,C)$, and thus using Corollary 3.3, we get that $F(x,C)=F(y,C)$.




$w\sigma_z$ is a neighborhood of $y$ in $F(y,C)$ containing $x$ in its interior.

Fig. 22

**3.5** PROPOSITION. *Let $C$ be a d-cell, and $\{C, F_1, F_2, \cdots, F_n\}$ be a listing of the faces of $C$ where no face is listed twice. Let $aC$ be a cone.*

*Then $aC$ is a d+1-cell and $\{C, F_1, \cdots, F_n, aC, aF_1, \cdots, aF_n, a\}$ is a listing of the set of faces of $aC$ also written without repetitions.*

*Proof.* We already know that $aC$ is a polyhedron (Prop. 1.12, p. 10) having dimension equal to $d$+1 (Prop.2.10, Claim 2.), and therefore to show that it is a cell we need only to show that $aC$ is a convex set.

Thus, let $x$ and $y$ be two points in $aC$.

If one of them, say $x$, is equal to $a$, we let $c_y$ be the ***projection from a*** of $y$ on $C$, i.e., $c_y$ is the unique point in $C$ such that $y = ta + (1-t)c_y, 0 \le t < 1$, as in Def. 1.2 of a cone (p. 5). Thus $a$ and $y$ are in the segment $ac_y$ which is contained in $aC$, and since $ac_y$ is convex, the segment $ay$ lies in $ac_y$; hence it also lies in $aC$.

If neither $x$ nor $y$ are equal to $a$, we let $c_x$ and $c_y$ be their respective projections on $C$ from $a$. Since $C$ is convex, the line segment $c_xc_y$ lies in $C$ and therefore the 2-simplex $ac_xc_y$ is contained in $aC$. Since 2-simplices are convex and $x$ and $y$ are points in $ac_xc_y$, the segment $xy$ lies in $ac_xc_y$ and thus also in $aC$.

We have thus shown that $aC$ is a cell.

We will show now that $\{C, F_1, \cdots, F_n, aC, aF_1, \cdots, aF_n, a\}$ is the set of faces of $aC$.

i) *$a$ is a vertex of $aC$.*

For if $a$ is not a vertex, then $a$ is a point in the interior of a 1-cell $S = L \cap aC$ where $L$ is a line. But then there are points $x$ and $y$ in $aC$ each belonging to different components of $S - a$ and thus their projections from $a$ to $C$, $c_x$ and $c_y$, lie also on different components of $S-a$ which further implies that $a$ is a point on the segment $c_x c_y$. Since $C$ is convex, $c_x c_y$ lies entirely in $C$ and therefore $a$ belongs to $C$. This is impossible. (See Def. 1.2, p. 5.)

ii) *Let* $x \in C$, *then* $F(x,C) = F(x,aC)$. *i.e.: If F is a face of C, then F is a face of aC.*

Without loss of generality we can assume that $C \subset \mathbf{R}^d$ and $aC \subset \mathbf{R}^{d+1}$, $d = \dim(\mathrm{C})$. We now observe that $a \notin \mathbf{R}^d$ for otherwise $aC$ would also be in $\mathbf{R}^d$ which is a dimensional impossibility. We conclude that $aC - C$ lies in the component of $\mathbf{R}^{d+1} - \mathbf{R}^d$ containing $a$.

It easily verified that every 1-cell in $\mathbf{R}^{d+1}$ containing a point of $\mathbf{R}^d$ in its interior is either wholly contained in $\mathbf{R}^d$ or it contains points from both components of $\mathbf{R}^{d+1} - \mathbf{R}^d$. Now let $x \in C$. Since $aC - C$ lies in a single component of $\mathbf{R}^{d+1} - \mathbf{R}^d$, every line $L$ in $\mathbf{R}^{d+1}$ with the property that $L \cap aC$ is a 1-cell containing $x$ in its interior must be contained in $\mathbf{R}^d$. Thus the above conclusion implies that $L \cap aC = L \cap C$. Therefore $F(x,aC)$, the face of $aC$ containing $x$ in its interior, is equal to $F(x,C)$, the face of $C$ containing $x$ in its interior.

Observe that this reasoning also applies if $F$ is a vertex of $C$.

iii) *Let* $x \in aC - C, x \neq a$, *then* $F(x,aC) = aF(c_x,C)$, *where* $c_x$ *denotes the projection from a of x into C as described above.*

First note that the 1-cell $ac_x$ is contained in $F(x,aC)$ since $x \in \mathrm{int}(ac_x)$. Now let $z \in F(c_x,C)$. By definition of $F(c_x,C)$ there exists a 1-cell $S \subset F(c_x,C)$ such that $c_x \in \mathrm{int}\, S$ and $z \in S$. Now let $z' \in az$. Since $x$ is in the interior of the 2-

simplex $aS$ and $z' \in aS$, there is a 1-cell $S' \subset aS$ such that $x \in \operatorname{int} S'$ and $z' \in S'$. Therefore $z \in F(x, aC)$, and thus all segments $az$ with $z \in F(c_x, C)$ are contained in $F(x, aC)$. We have proved $aF(c_x, C) \subset F(x, aC)$.

Now we verify that $F(x, aC) \subset aF(c_x, C)$.

We just apply the above reasoning backwards. Let $z \in F(x, aC)$. There is a 1-cell $S$ in $aC$ with $x$ in its interior and also containing $z$. Project this cell from $a$ into $C$. This projection is either the point $c_x$, in which case $z$ is a point of $ac_x$, therefore also in $F(c_x, C)$, or $S$ projects onto a 1-cell $S' \subset C$ with $c_x \in \operatorname{int} S'$ and $z_x \in S'$. Thus $z_x \in F(c_x, C)$ and since $z \in az_x$ we obtain $z \in aF(c_x, C)$.

iv) *Let* $F(x, aC) = aF(c_x, C)$ *be a face of aC as in* iii). *Let G be a face of C such that we also have* $F(x, aC) = aG$. *Then* $F(c_x, C) = G$.

Let $g \in \operatorname{int} G$ and let $y$ be a point in the interior of the 1-cell $ag$. $y$ is thus a point in the interior of $aG$, i.e., $y \in \operatorname{int} F(x, aC)$. By Corollary 3.3 we know that $F(x, aC) = F(y, aC)$. Now by Lemma 3.4 we know that there is a 1-cell $S$ in $aC$ with the property that both $x$ and $y$ lie in the interior of $S$. Let $S'$ denote the projection from $a$ of $S$ into $C$. If $S'$ is a point then $g = c_x$ and since different faces of $C$ have disjoint interiors we conclude that $G = F(c_x, C)$. If $S'$ is a 1-cell then both $g$ and $c_x$ are in the interior of $S'$, thus by Lemma 3.4 we obtain that $F(g, C) = F(c_x, C)$. By Corollary 3.3 we know that $G = F(g, C)$ and therefore, $G = F(c_x, C)$.

v) *If F is a face of C then aF is a face of aC.*

Choose $y \in \operatorname{int} F$ and $x \in \operatorname{int}(ay)$. Since $x \in aC - C, x \neq a$ we use iii) to conclude that $F(x, aC) = aF(c_x, C) = aF(y, C) = aF$. □

**3.6 LEMMA.** *Let* $K = \{(C_i, \pi_i : C_i \times i \to C_i)\}_{i=1,\cdots,n}$ be a <u>cell</u> complex and $a|K|$ be a cone. Let $D_1 = C_1, \cdots, D_n = C_n$; $D_{n+1} = aC_1, \cdots, D_{2n} = aC_n$ and $D_{2n+1} = a$. Then $aK = \{(D_j, \pi_j : D_j \times j \to D_j)\}_{j=1,\cdots,2n+1}$ is a cell complex.

*Proof.* From Proposition 3.5 we know that for $1 \le j \le 2n+1$, $D_j$ is a cell. To show that $aK = \{(D_j, \pi_j : D_j \times j \to D_j)\}_{j=1,\cdots,2n+1}$ is a cell complex we need only to show (See Definitions. 2.14 and 2.15, p. 28) that: i) if $F$ is a face of $D_i$ then $F \in \{D_j\}$ and ii) if $D_i \cap D_k \neq \varnothing$ then $D_i \cap D_k$ is a face of both $D_i$ and $D_k$.

If $D_i = C_i$, i.e. if $1 \le i \le n$ , Then i) is satisfied since $K = \{(C_i, \pi_i : C_i \times i \to C_i)\}_{i=1,\cdots,n}$ is a cell complex. If n+1$\le i \le 2n$ then $D_i = aC_{i-n}$ thus by Prop. 3.5, we have that if $F$ is a face of $D_i$, then $F$ is a face of $C_{i-n}$, or $F = aG$, where $G$ is a face of $C_{i-n}$, or $F = a$. Since $K$ is a cell complex all the faces of $C_{i-n}$ are in $\{C_i\}$. and therefore we obtain that in all these three situations $F \in \{D_j\}$. We have thus shown i).

To show ii) let $D_i \cap D_k \neq \varnothing$.

If $D_i \cap D_k \subset |K|$ then $D_i \subset |K|$ or $D_k \subset |K|$ (otherwise their intersection would be the cone on their intersection in |K| ). So, let $D_i \subset |K|$ i.e. $D_i = C_i$ and $D_k = aC_{k-n}$. Then $D_i \cap D_k = C_i \cap C_{k-n}$ and since |K| is a cell complex, $D_i \cap D_k$ is a face of both $C_i$ and $C_{k-n}$, thus by Prop. 3.5 it is also a face of both $D_i$ and $D_k$.

If $D_i \cap D_k \not\subset |K|$ then $D_i \cap D_k = \{a\}$ in which case ii) is satisfied, or $D_i \cap D_k = a(D_i \cap D_k \cap |K|)$, but $D_i \cap D_k \cap |K| = C_{i-n} \cap C_{k-n}$ so $D_i \cap D_k \cap |K|$ is a face of both $C_{i-n}$ and $C_{k-n}$. Therefore by Prop. 3.5 $D_i \cap D_k$ is a face of both $aC_{i-n} = D_i$ and $aC_{k-n} = D_k$ . □

**3.7** REMARK. Let C be a cell and $a \in C$. It is easy to see (see Props. 2.8, p.24 and 2.19, p. 30) that $B_a = \{(F_i,\ \pi_i\colon F_i \times i \to F_i) \mid F_i < C, a \notin F_i\}$ is a cell complex and that $B_a$ is a subcomplex of $\dot{K} = \{(F_i,\ \pi_i\colon F_i \times i \to C) \mid F_i \text{ a proper face of } C\}$.

**STELLAR SUBDIVISIONS**

**3.8** PROPOSITION. *Let C be a cell with faces* $C = F_1, F_2, \cdots, F_m$ *and let* $a \in C$. *Let* $E_1, \cdots, E_n$ *be the faces of C which do not contain a. Also let* $G_1 = E_1, \cdots, G_n = E_n;\ G_{n+1} = aE_1, \cdots, G_{2n} = aE_n$, and $G_{2n+1} = a$. *Then* $K_a = \{(G_j,\ \pi_j\colon G_j \times j \to G_j) \mid 1 \le j \le 2n+1\}$ *is a cell complex and* $K_a$ *is a subdivision of* $K = \{(F_i,\ \pi_i\colon F_i \times i \to F_i) \mid F_i < C\}_{1 \le i \le m}$.

*Proof.* Let $B_a$ be the cell complex as in Remark 3.7. By Prop. 2.8 6) p. 24, we have that $|K| = C = a|B_a|$. Using Lemma 3.6 we see that $K_a$ is the cell complex given by $K_a = aB_a$. Now to see that $K_a$ is a subdivision of $K$ we first observe that $|K_a| = |aB_a| = a|B_a|$ and by the above we get $|K_a| = |K|$. [We thus have 1) of Def. 3.1. ]

Now we verify 2) of Def. 3.1. For all $(G_j,\ \pi_j\colon G_j \times j \to G_j)$ in $K_a$ we define the linear embeddings $e_{j1}\colon G_j \times j \to F_1 \times 1 = C \times 1$ by $(x, j) \mapsto (x, 1)$ and one sees at once that

$$\begin{array}{ccc} D_j \times j & \xrightarrow{\ e_{j1}\ } & C_1 \times 1 \\ \pi \downarrow & & \downarrow \pi \\ D_j & \xrightarrow{\text{inclusion}} & C_1 \end{array}$$

commutes. □

**3.9** DEFINITION. Let $C$ be a cell and $a$ be a point in $C$. The cell complex $K_a$ given in Proposition 3.8 is said to be obtained by ***starring C at a***.

More generally, given an HI-cell complex $\underset{\cdot\cdot}{K} = \left\{ \left( C_i, h_i{:}\, \overline{C}_i \to C_i \right) \right\}$, a point $a \in |K|$, and a way to chose for each cell $\overline{C}_i$ with $h_i^{-1}(a) \neq \varnothing$ a unique point $\overline{a}_i \in h_i^{-1}(a) \subset \overline{C}_i$ [a choice that must be *compatible with face recognition maps* as defined in 3.10 below], we will construct in 3.15 a subdivision $L$ of $K$ by (a slight modification of) starring each such cell $\overline{C}_i$ at the chosen point $\overline{a}_i$.

Note that if $\underset{\cdot\cdot}{K} = \left\{ \left( C_i, h_i{:}\, \overline{C}_i \to C_i \right) \right\}$ is an H-<u>cell</u> complex then there is a <u>unique</u> point in $h_i^{-1}(a)$ for each $\overline{C}_i$ as above, and thus there is only one "choice allowed".

For $F$ a face of $\overline{C}_i$, recall the existence of the *face recognition* linear homeomorphism $i_F$ and of the commutative diagram below, which were given in the definition of HI-cell complexes (Def. 2.28, p. 42) :

$$\begin{array}{ccc} \overline{C}_k & \xrightarrow{\ i_F\ } & F \subset \overline{C}_i \\ & {\scriptstyle h_k}\searrow \quad \swarrow{\scriptstyle h_i|_F} & \\ & C_k = h_i(F) & \end{array}$$

**3.10** DEFINITION. Let $\underset{\cdot\cdot}{K} = \left\{ \left( C_i, h_i{:}\, \overline{C}_i \to C_i \right) \right\}_{1 \le i \le n}$ be an HI-cell complex and $a \in |K|$. Also let $\underset{\cdot\cdot}{\overline{K}}(a) = \left\{ \overline{C}_j \,\middle|\, \left( C_j, h_j{:}\, \overline{C}_j \to C_j \right) \in K,\, h_j^{-1}(a) \neq \varnothing \right\}_{1 \le j \le r}$.

A ***face-compatible pre-image choice for a*** (briefly, a *choice function for a,* a *face-compatible choice for* $a$, etc.) is a function

$$Ch^a{:}\, \underset{\cdot\cdot}{\overline{K}}(a) \to \bigcup_{1 \le j \le r} h_j^{-1}(a), \qquad \overline{C}_j \mapsto \overline{a}_j, \qquad \text{such that:}$$

1) $Ch^a\left(\overline{C}_j\right) = \overline{a}_j \in h_j^{-1}(a) \subset \overline{C}_j$

2) For every proper face $F$ of $\overline{C}_j \in \overline{K}(a)$, with $\overline{a}_j \in F$ and face recognition homeomorphism $i_F{:}\, \overline{C}_k \to F \subset \overline{C}_j$, we have: $i_F^{-1}\left(\overline{a}_j\right) = \overline{a}_k$.

Note that in Def. 3.10 we used the inverses of the maps $i_F$. Hence $\dim(\overline{C}_j) > \dim(\overline{C}_k)$.

**3.11** EXAMPLE Let $K = \{(C_i, h_i\colon \overline{C}_i \to C_i)\}_{1\le i\le n}$ be an HI-cell complex. For all $1\le i\le n$ let $F_{i1}, F_{i2}, \cdots, F_{is_i}$ denote the faces of the cell $\overline{C}_i$.

Note that a face recognition $i_{F_{jt}}\colon \overline{C}_k \to F_{jt} \subset \overline{C}_j$ maps a face $F_{kr}$ of $\overline{C}_k$ onto a face $F_{ju}$ of $\overline{C}_j$. Let $F_{kr}$ and $F_{ks}$ be two faces of a cell $\overline{C}_k$, let $i_{F_{jt}}\colon \overline{C}_k \to F_{jt} \subset \overline{C}_j$ be a face recognition and let $F_{ju}$ and $F_{jv}$ be the respective images of $F_{kr}$ and $F_{ks}$ under $i_{F_{jt}}$. Assume that (using the notation of the previous sentence) we always have that if $r<s$ then $u<v$.

Then for *any* $a \in |K|$, we can define a *face-compatible pre-image choice for a*, $Ch^a\colon \overline{A} \to \bigcup_{1\le j\le r} h_j^{-1}(a)$, as follows:

For $(C_j, h_j\colon \overline{C}_j \to C_j) \in K$ such that $h_j^{-1}(a) \neq \varnothing$ we let $\overline{a}_j = Ch^a(\overline{C}_j)$ be the unique point in $h_j^{-1}(a) \subset \overline{C}_j$ lying in the interior of the face $F_{jr}$ where $r = \min\{u \mid F_{ju} \text{ is a face of } \overline{C}_j,\ \operatorname{int}(F_{ju}) \cap h_j^{-1}(a) \neq \varnothing\}$.

**3.12** LEMMA. *Let $K = \{(C_i, h_i\colon \overline{C}_i \to C_i)\}_{1\le i\le n}$ be an HI-cell complex. Let $a \in |K|$, and let $\overline{K}(a) = \{\overline{C}_j \mid (C_j, h_j\colon \overline{C}_j \to C_j) \in K,\ h_j^{-1}(a) \neq \varnothing\}_{1\le j\le r}$. Assume that $C_1$ is the unique polyhedron in $|K|$ which contains a in its HI-interior.*

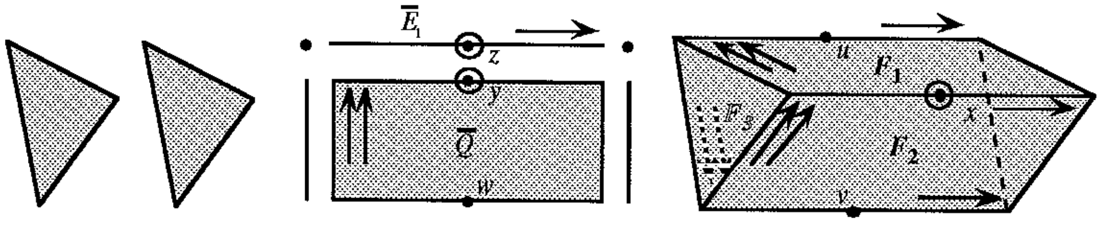
*The existence of face-compatible pre-image choice for a, $Ch^a$, is equivalent to the existence of a set $\{i_{F(r,s)}\colon \overline{C}_r \to F(r,s) \subset \overline{C}_s\}$ of face recognition maps – one for each pair $(r,s)$ with $\overline{C}_r$ and $\overline{C}_s$ "in" $\overline{K}(a)$, $\overline{C}_r$ a "face" of $\overline{C}_s$ – which have the property that $\overline{C}_1 \xrightarrow{i_{F(1,r)}} \overline{C}_r \xrightarrow{i_{F(r,s)}} \overline{C}_s$ is the face recognition $\overline{C}_1 \xrightarrow{i_{F(1,s)}} \overline{C}_s$.*

*Proof.* Given a face-compatible choice function $Ch^a$, $\overline{C}_j \mapsto \overline{a}_j$, we use 2) of Def. 3.10 and define $i_{F(k,j)}: \overline{C}_k \to \overline{C}_j$ to be the face recognition $i_F: \overline{C}_k \to F \subset \overline{C}_j$ where $F$ is a face of $\overline{C}_j$ containing $\overline{a}_j$ in its interior.

Conversely, given such a set $\left\{ i_{F(r,s)} : \overline{C}_r \to F(r,s) \subset \overline{C}_s \right\}$, we define $\overline{a}_j = Ch^a\left(\overline{C}_j\right)$ to be the point $i_{F(1,j)}(a)$. □

**3.13** REMARK. Given an HI-cell complex $K$ and a point $a \in |K|$ there might not exist a function $Ch^a$ as in 3.10. An example is furnished in Fig. 23. It depicts the covering cells (domains) in the *canonical* HI-cell complex induced by an HI-triangular prism $\left(C, h: \overline{C} \to C\right)$ (as in Prop. 2.33, p. 48). The three rectangular faces $F_i$ of the prism are all mapped by $h$ onto the same polyhedron $Q$ – a PL homeomorph of $\mathrm{Bd}\left([0,1]^2\right) \times [0,1]$. The double arrows drawn on these three faces $F_i$ show how the face recognition maps $i_{F_i}: \overline{Q} \to F_i$, $i = 1,2,3$, are defined; the arrows on three of the edges of the prism show how $\overline{E}_1$ is mapped onto them via face recognition maps. The existence of the polyhedron $C \subset \mathbf{R}^n$ and of the map $h$ will be shown in Theorem 3.29.

Let $a \in \text{HI-int}\left(E_1\right) \subset |K|$. The six points $u$, $v$, $w$, $x$, $y$, $z$ represent the pre-images of $a$. We can see that there exists no face-compatible choice function for $a$. The circled dots, $x$, $y$ and $z$ represent a futile attempt to define such a function $Ch^a$, [for example $i_{F_1}^{-1}(x) \neq y$] and any other choice would likewise fail, as one can easily verify.




The covering cells of $K$. There exists $a \in |K|$ for which there is no function $Ch^a$.

Fig. 23

**3.14** PROPOSITION. *If $K$ is an H-cell complex and $a \in |K|$ then there always exists a unique face-compatible pre-image choice function $Ch^a$ for $a$.*

*Proof.* As before let $\overline{K}(a) = \left\{ \overline{C}_j \middle| \left(C_j, h_j : \overline{C}_j \to C_j\right) \in K, h_j^{-1}(a) \neq \varnothing \right\}_{1 \le j \le r}$.

We have already noticed (just before 3.10) that in an H-cell complex $K$, there is a unique function $\overline{C}_j \mapsto \overline{a}_j \in h_j^{-1}(a)$ for $\overline{C}_j \in \overline{K}(a)$. Since the face recognition maps $i_F : \overline{C}_k \to F \subset \overline{C}_j$ commute with the maps $h_j$, we see that $i_F^{-1}\left(\overline{a}_j\right) = \overline{a}_k$. Thus the function $\overline{C}_j \mapsto \overline{a}_j \in h_j^{-1}(a)$ is a face-compatible choice function $Ch^a$. □

**3.15** CONSTRUCTION (STAR SUBDIVISION). Let $K = \left\{ \left(C_i, h_i : \overline{C}_i \to C_i\right) \right\}_{1 \le i \le n}$ be an HI-cell complex. Let $a \in |K|$ and let a face-compatible choice function $Ch^a$ be given.

We may assume that the polyhedra $C_i$ are labeled in such a way that $\{C_1, C_2, \cdots, C_m\}$ $(m \le n)$ is the set of all the polyhedra in $|K|$ which contain the point $a$, and assume further that $\dim C_1 \le \dim C_2 \le \cdots \le \dim C_m$. For each $1 \le j \le m$ let $\overline{a}_j = Ch^a\left(\overline{C}_j\right)$.

We claim that $a$ is in the HI-interior of $C_1$. We first note that $a \in \text{HI-int}\,(C_r)$ for some $r$. If we also have that $a \in C_s, s \neq r$ then $a$ cannot be in the HI-interior of $C_s$, therefore $a$ is in the HI-interior of $h_s(F)$ where $F$ is a <u>proper</u> face of $\overline{C}_s$ and thus $\dim F < \dim \overline{C}_s$. Now from 2) of the definition of HI-cell complex (Def. 2.28, p. 42) we see that $\overline{C}_r$ and $F$ are linearly homeomorphic and thus we have $\dim C_r = \dim \overline{C}_r = \dim F < \dim \overline{C}_s = \dim C_s$, and since this is a strict inequality we conclude that $C_r = C_1$. Observe that $\overline{a}_1$ is the only point in $\overline{C}_1$ that maps onto $a$.

We remark that below, we will write $h_i : \overline{C}_i \to |K|$ to denote the HI-cell $\left(C_i, h_i : \overline{C}_i \to C_i\right)$, $C_i \subset |K|$. This will simplify the notation especially in the case of the composition of such functions.

We now construct a subdivision $L$ of $K$ as follows: (See Fig. 24)
Let $\dim \overline{C}_1 = d$. We first construct a subdivision $L_d$ of the $d$-skeleton $K^d$ of $K$ by defining:

$$L_d = \left\{h_i : \overline{C}_i \to |K| \,\middle|\, \dim \overline{C}_i \leq d, i \neq 1\right\} \cup \left\{h_1 : \overline{a}_1 F_{1j} \to |K| \,\middle|\, \overline{a}_1 \notin F_{1j}\right\} \cup \left\{h_1 : \{\overline{a}_1\} \to |K|\right\}$$

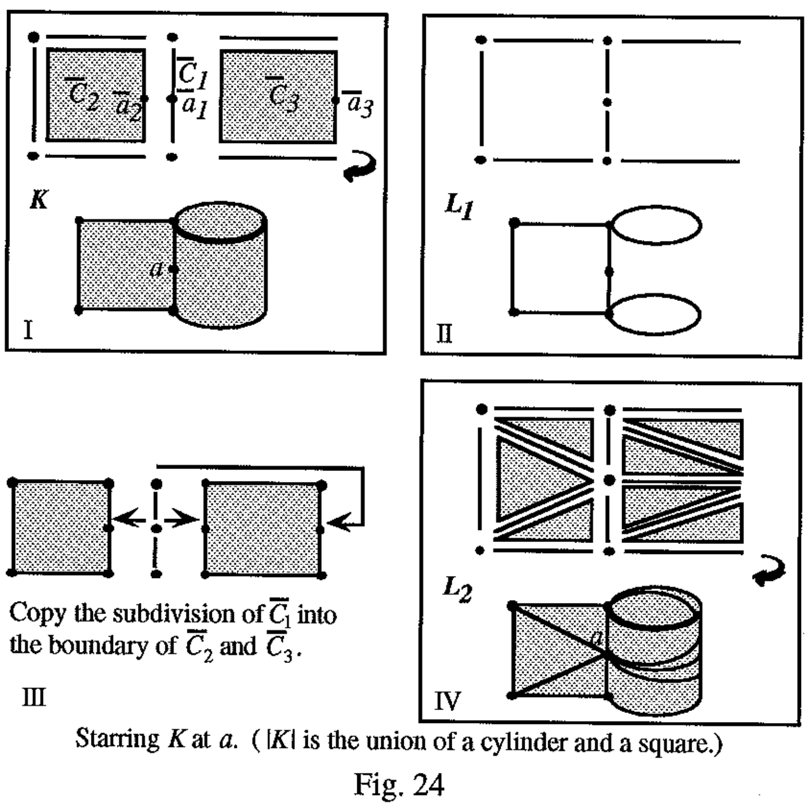
where $h_i \in K$, $F_{1j}$ is a face of $\overline{C}_1$. One can easily but tediously verify that $L_d$ is an HI-cell complex. Thus, $L_d$ can be colloquially described as the HI-cell complex obtained from $K^d$ by replacing $\overline{C}_1$ with the cell complex obtained from starring $\overline{C}_1$ at $\overline{a}_1$.

For every $d$+1 cell $\overline{C}_j$ containing points mapping onto $a$ via $h_j$, there are linear homeomorphisms (face recognitions) $i_{F_{jk}} : \overline{C}_1 \to F_{jk} \subset \overline{C}_j$ (one for each $d$-face $F_{jk}$ of $\overline{C}_j$ containing a unique point mapping onto $a$ ) which commute with $h_1$ and $h_j$. [See 1) and 2) of the definition of HI-cell complex. (Def. 2.28, p. 42). ]

Using these homeomorphisms $i_{F_{jk}}$ we "copy" the subdivision of $\overline{C}_1$ obtained above onto all the faces of $\overline{C}_j$ containing points mapping to $a$. These subdivisions together with the other proper faces of $\overline{C}_j$ form a subdivision of the boundary of $\overline{C}_j$.

We now define $L_{d+1}$ to be the HI-cell complex obtained from $L_d$ and $K^{d+1}$ by replacing each $d$+1 cell $\overline{C}_j$ containing points mapping onto $a$ , with the cells obtained by conning $\overline{a}_j$ with all the cells in the above subdivision of the boundary of $\overline{C}_j$ which do not contain $\overline{a}_j$. [It is here that we need the face-compatibility property of the choice function $Ch^a$: it assures that the previous subdivisions of the faces of $\overline{C}_j$ which contain $\overline{a}_j$ coincide with the subdivision obtained at this stage.] Finally, these new cells in $\overline{C}_j$ are mapped into $|K|$ via $h_j$.

We similarly obtain $L_{d+2}$ from $L_{d+1}$ and $K^{d+2}$. We proceed inductively in this manner until we obtain $L$.

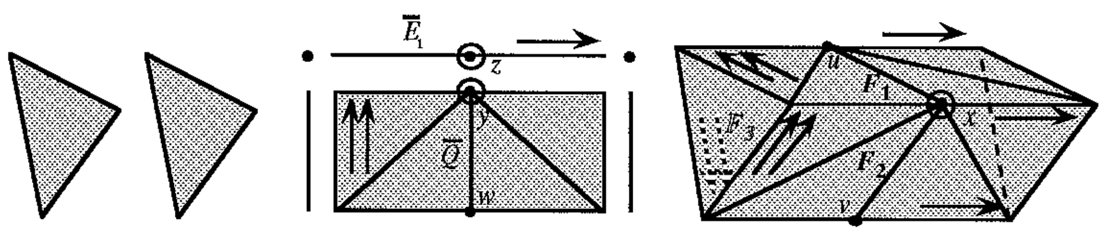



Starring $K$ at $a$. ( $|K|$ is the union of a cylinder and a square.)

Fig. 24

**3.16** DEFINITION. Given an HI-cell complex $K$, a point $a$ in $|K|$ and a face-compatible choice function for $a$, $Ch^a$, the complex $L$ constructed as in 3.15 is said to be obtained by ***starring K at a with respect to*** $Ch^a$.

**3.17** EXAMPLE In Figure 25 we show the subdivision $L_2$ obtained by starring at $a$ the 2-skeleton of the HI-cell complex $K$ described previously in example given in Remark 3.13. Note that it is impossible to extend this subdivision to a stellar subdivision of $K$ at $a$ : – The stellar subdivision of $\overline{C}$ at $x$ does not agree with the previous subdivision of $F_1$.




$\overline{C}$ (3-dimensional)

It is impossible to star $K$ (described in 3.13 and in Fig. 23) at $a$.

Fig. 25

In the above example, we obtained a stellar subdivision of the 2-dimensional skeleton of $K$ at the point $a$. This phenomenon is a property of all the 2-dimensional HI- cell complexes as will be seen shown in Corollary 3.20.

**3.18** PROPOSITION. *Let K be an H-cell complex and let* $a \in |K|$. *Then there exists a unique star subdivision of K at a. (Thus there is need to specify the choice function* $Ch^a$.*)*

*Proof.* This proposition is an immediate corollary of Proposition 3.14. □

**3.19** PROPOSITION. *Let K be a d-dimensional HI-cell complex and let* $a \in |K|$. *Let* $\overline{K}(a) = \left\{ \overline{C}_j \middle| \left(C_j, h_j : \overline{C}_j \to C_j\right) \in K, h_j^{-1}(a) \neq \varnothing \right\}_{1 \leq j \leq r}$ *and let* $F^a$ *be a function defined on* $\overline{K}(a)$ *such that* $F^a\left(\overline{C}_j\right) = \overline{a}_j \in h_j^{-1}(a)$. *(* $F^a$ *is not required to be compatible with the face-recognition functions of K. )*

Let $\left(C_1, h_1 : \overline{C}_1 \to C_1\right)$ be the HI-cell in $K$ such that $a \in$HI-int$\left(C_1\right)$.

*Then*

*1) If* $\dim\left(\overline{C}_1\right) = d$ *then* $F^a$ *is unique and is compatible with the face recognition functions of K. (In this situation there is no need to specify* $F^a$ *and we say that we can "star K at a".)*

*2) If* $\dim\left(\overline{C}_1\right) = d - 1$ *then* $F^a$ *is compatible with the face recognition functions of K. (And thus we can star K at a with respect to* $F^a$*. )*

*3) If* $\dim(K) = 2$ *and* $\dim\left(\overline{C}_1\right) = 0$ *then the "stellar subdivision of K at a with respect to* $F^a$*" exists. That is, one can successfully perform the star subdivision given in Construction 3.15 using* $F^a$ *(which is now not necessarily face-recognition compatible) in place of a face-compatible choice function* $Ch^a$.

*Proof.* In order to prove 1) and 2) recall that the existence of a set

$$(*) \qquad \left\{ i_{F(r,s)} : \overline{C}_r \to F(r,s) \subset \overline{C}_s \right\}$$

of face recognition maps – one for each pair $(r,s)$ with $\overline{C}_r$ and $\overline{C}_s$ "in" $\overline{K}(a)$, $\overline{C}_r$ a "face" of $\overline{C}_s$ – which have the property that

$$\overline{C}_1 \xrightarrow{i_{F(1,r)}} \overline{C}_r \xrightarrow{i_{F(r,s)}} \overline{C}_s \text{ is the face recognition } \overline{C}_1 \xrightarrow{i_{F(1,s)}} \overline{C}_s,$$

gives rise by Lemma 3.12 to the existence of a unique face-compatible pre-image choice function $Ch^a$ for $a$.

1) Let $\dim(\overline{C}_1) = d$, then $\overline{K}(a) = \{\overline{C}_1\}$. Thus $\{i_{F(1,1)} = \text{identity}: \overline{C}_1 \to \overline{C}_1\}$ is a set as in $(*)$ and it gives rise to $Ch^a = F^a$.

2) Let $\dim(\overline{C}_1) = d-1$ and let $\{\overline{C}_j\}_{j=2,\cdots,n}$ be the set of $d$-dimensional cells "in" $K$ which have $\overline{C}_1$ as a face, i.e. $\{\overline{C}_j\}_{j=2,\cdots,n}$ is the set of $d$-dimensional cells for which there exist face recognition maps $i_F: \overline{C}_1 \to F \subset \overline{C}_j$. Since $C_1$ is the only $d$-1 dimensional polyhedron in $|K|$ containing $a$, (as shown in Construction 3.15) we see that in this situation we have $\overline{K}(a) = \{\overline{C}_1, \overline{C}_2, \cdots, \overline{C}_n\}$.

Now for each cell in $\{\overline{C}_j\}_{j=2,\cdots,n}$ we let $F(1,j)$ be the unique $d$-1 dimensional face of $\overline{C}_j$ which contains the point $\overline{a}_j = F^a(\overline{C}_j)$ and thus the set of face recognition homeomorphisms

$$\{i_{F(1,j)}: \overline{C}_1 \to F(1,j) \subset \overline{C}_j\}_{2 \le j \le n} \cup \{\text{identity}: \overline{C}_j \to \overline{C}_j\}_{1 \le j \le n}$$

is a set with the properties of $(*)$ (at the beginning of the proof) and the face-compatible function $Ch^a$ that it induces (by Lemma 3.12) is the given function $F^a$.

3) Let $\dim(K) = 2$ and $\dim(\overline{C}_1) = 0$. We want to show that it is possible to star $K$ at $a$ with respect to the function $F^a$ given in the hypothesis, even if $F^a$ is not compatible with the face recognition functions of $K$.

Since $\dim(\overline{C}_1) = 0$, the point $a$ is a vertex of $K$, and thus the star subdivision of the 1-skeleton of $K$ at $a$, leaves it unmodified. Therefore we only get a (non trivial) subdivision of $K$ when we star each 2-cell $\overline{C}_i \in \overline{A}$ at its vertex $\overline{a}_i = F^a(\overline{C}_i)$, and again this subdivision leaves the boundary of $\overline{C}_i$ undisturbed (and thus there is no need for $F^a$ to face-recognition compatible). □

Note that in the above proposition, in contrast to 1) and 2) which apply to an HI-cell complex $K$ of any dimension, 3) does not apply in the general situation of cells $(C_1)$ of dimension $d$-2 belonging to a $d$- dimensional HI-cell complex $K$, as Example 3.17 shows.

**3.20** COROLLARY. *Let K be a 2-dimensional HI-cell complex and let $a \in |K|$. Then it is always possible to star K at a.*

*Proof.* Again let $\overline{K}(a) = \left\{ \overline{C}_j \middle| \left(C_j, h_j : \overline{C}_j \to C_j\right) \in K, h_j^{-1}(a) \neq \varnothing \right\}_{1 \le j \le r}$.

It is always possible to define a function $F^a$ (not necessary compatible with the face recognition functions of K) which assigns to each cell $\overline{C}_j \in \overline{A}$ a point $\overline{a}_j \in h_j^{-1}(a)$. Since $K$ is 2-dimensional, one of the conditions 1), 2) or 3) of Prop. 3.19 must apply. □

**DERIVED SUBDIVISIONS**

**3.21** PROPOSITION. *Let* $K = \left\{ \left(C_i, h_i : \overline{C}_i \to C_i\right) \right\}_{1 \le i \le n}$ *be a d-dimensional HI-cell complex where the HI-cells are indexed in a <u>reverse</u> order from their dimensions, i.e. if i<j then* $\dim(C_i) \ge \dim(C_j)$. *For each* $1 \le i \le n$ *let* $a_i$ *be a point in the HI-interior of* $C_i$.

*Then there is a unique subdivision of K obtained by starring sequentially at the points* $a_1, a_2, \cdots, a_n$. *(Thus no choice functions* $Ch^{a_i}$ *are required.)*

*Proof.* Since $\dim(C_1) = d$, by Proposition 3.19 1) we can star $K$ at $a_1$ in a unique manner. Let $K_1$ be the complex thus obtained. If $\dim(C_2)$ is also equal to $d = \dim(K)$, then we can also uniquely star $K_1$ at $a_2$ since $K_1$ left the HI-interior of $C_2$ unchanged ("$a_2$ is in the HI-interior of a $d$-dimensional HI-cell of the d-dimensional HI-cell complex $K_1$"). By induction we can sequentially perform star subdivisions at all points $a_i$ such that $\dim(C_i) = d$.

Note that we subdivide by starring cells sequentially in descending order of their dimensions, but when we star a point $a_i$ such that $\dim(C_i) = e$ we inductively modify "going up" the $e$+1, $e$+2, etc. skeleta of the stellar subdivision obtained from the previous point $a_{i-1}$ (see Construction 3.15).

Let $a_s$ be a point in the given sequence such that $\dim(C_s) = f < d$, and assume that we have sequentially starred at all the points $1 \le i < s$.

Let $F_1$ and $F_2$ be two $f$-dimensional faces of an $f$+1-dimensional cell $\overline{C}$ of $K$ such that both contain points mapping onto $a_s$. All previous subdivisions have left $F_1$ and $F_2$ unsubdivided (since starring at a point lying in higher dimensional HI-interior leaves $F_1$ and $F_2$ undisturbed, and the same applies to the previous subdivisions at points lying in $f$-dimensional HI-cells) and thus $F_1$ and $F_2$ are faces of different $f$+1-dimensional cells of the previous subdivision of $\overline{C}_s$ (these $f$+1 cells were first obtained when $\overline{C}_s$ was starred at the point in its interior).

Since $a_s$ is a point in the HI-interior of $C_s$ there is a unique point $\overline{a}_i$ in $\overline{C}_s$ which maps onto $a_s$. Therefore, the subdivision of the canonical HI-cell complex $h_s: \overline{C}_s \to C_s$ at $a_s$ is unambiguously defined. Now, using the underlined statement in the previous paragraph, we see that when we modify "going up" the subdivision of the $f$+1cells obtained from previous subdivisions, each $f$+1-cell receives at most one copy of the subdivision of $\overline{C}_s$. Therefore we can subdivide the $f$+1-cells (and inductively all cells of dimension greater than $f$) without needing a choice function $Ch^{a_i}$. □

**3.22** DEFINITION. Let $\underset{\cdot\cdot}{K} = \left\{ \left( C_i,\, h_i\!: \overline{C}_i \to C_i \right) \right\}_{1 \le i \le n}$ be a $d$-dimensional HI-cell complex where the HI-cells are indexed in a reverse order from their dimensions, i.e. if $i<j$ then $\dim(C_i) \geq \dim(C_j)$. For each $1 \le i \le n$ let $a_i$ be a point in the HI-interior of $C_i$.

The stellar subdivision of $K$ obtained by starring sequentially at the points $a_1, a_2, \cdots, a_n$ as in 3.21, is called a ***first derived subdivision of K*** (at $a_i$) and is denoted by $K^{(1)}$. Inductively we define an ***m-th derived subdivision***, $K^{(m)}$, by $\underset{\cdot\cdot}{K}^{(m)} = \left( K^{(m-1)} \right)^{(1)}$ (at points in the HI-interiors of the polyhedra of $\left| \underset{\cdot\cdot}{K}^{(m)} \right|$ ).

The following proposition states that in a derived subdivision as above, we can modify the sequentially ordering of the points $a_1, a_2, \cdots, a_n$ (and thus also modify the order in which the stellar subdivisions are performed) without changing the resultant derived subdivision, as long as a reverse dimensional order (as in 3.22) is maintained.

**3.23** PROPOSITION. *Let L be a derived subdivision of* $\underset{\cdot\cdot}{K} = \left\{ \left( C_i, h_i\!: \overline{C}_i \to C_i \right) \right\}$ *obtained from a sequence* $a_1, a_2, \cdots, a_n$, *(* $a_i \in \operatorname{int}(C_i)$, $i < j \Rightarrow \dim(C_i) \geq \dim(C_j)$*). Let* $b_1, b_2, \cdots, b_n$ *be a reordering of* $a_1, a_2, \cdots, a_n$ *and let* $D_1, D_2, \cdots, D_n$ *be the reordering of the polyhedra* $C_1, C_2, \cdots, C_n$ *obtained by letting* $b_i \in D_i$. *Assume that these new orderings also have the property that if* $i < j$ *then* $\dim(D_i) \geq \dim(D_j)$. *Then the subdivision* $L'$ *of K obtained from* $b_1, b_2, \cdots, b_n$ *is equal to L, i.e. L and* $L'$ *are the same HI-cell complex.*

*Proof.* Let $\dim(\overline{C}_i) = d$. Since the subdivision of $h_i : \overline{C}_i \to C_i$ obtained from starring at $a_i$ leaves the boundary-complex of $h_i$ undisturbed (as well as any other HI-cell in $K$), the subdivision of the $d$- skeleton of $K$ obtained by starring the $d$-skeleton at all the points $a_i$ in the interior of the images of the $d$-cells, is independent of the ordering of these points.

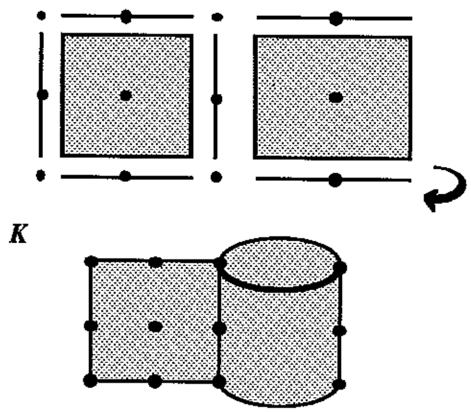
Let $\overline{C}_i$ be a "face" of a $d$+1-cell $\overline{C}_r$. Since $\overline{C}_r$ is a cone with base equal to its boundary and vertex equal to $\overline{a}_r$, when we modify "going up" the previous subdivision of $h_r : \overline{C}_r \to C_r$ by conning all the new cells lying in $\overline{C}_i$ with $\overline{a}_r$, we do not disturb the cones whose bases lie on the other $d$-faces of $\overline{C}_r$. (The subdivision of $\overline{C}_i$ left its boundary untouched.) Therefore when we modify the subdivision of the $d$+1-skeleton, the order used to subdivide the $d$-cells is not important.

Since a derived subdivision is obtained by starring sequentially at $a_1, a_2, \cdots, a_n$, $\dim(C_i) \geq \dim(C_{i+1})$, we see that any reordering $b_1, b_2, \cdots, b_n$ of this sequence having the properties stated in the hypothesis, gives rise to the same set of HI-cells. □

**3.24** DEFINITION. Let $B$ be a cell with vertices $v_1, v_2, \cdots, v_n$. Similarly to what we have done in the simplex case, we define the ***barycenter*** $\hat{B}$ of $B$ to be the point $\hat{B} = \sum_i \frac{1}{n} v_i$. If $(C, h : \overline{C} \to C)$ is an HI-cell we define the ***barycenter of C*** to be $h(\hat{\overline{C}})$, where $\hat{\overline{C}}$ is the barycenter of $\overline{C}$.

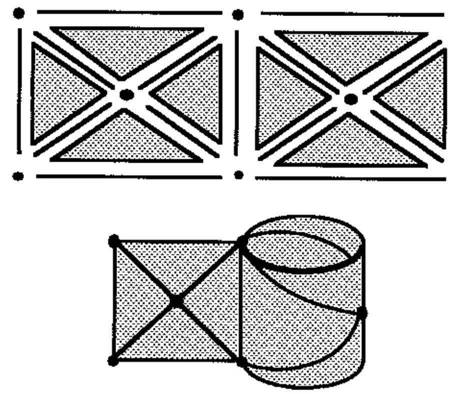
The ***barycentric subdivision of an HI-cell complex K*** is the derived subdivision $K^{(1)}$ as in 3.23 where every point $a_i$ is the barycenter of $C_i$. Also inductively, if $K^{(m-1)}$ is barycentric and if the $m$-th derived subdivision $K^{(m)}$ of $K$ is also barycentric, then $K^{(m)}$ is called the ***m-th barycentric subdivision of K***.

In Figure 26 below, we illustrate the construction of the barycentric subdivision of the complex *K* previously shown in Fig. 24 part I.



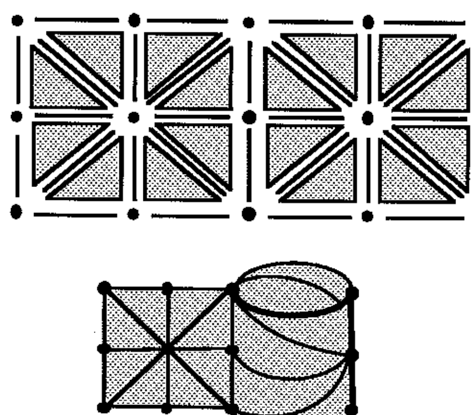

*K* and all the barycenters.

**I**



Stellar subdivision of *K* using the barycenters of the 2-dimensional cells.

**II**



Subdivision of the subdivision in II using the barycenters of the 1-dimensional cells.

**III**

A barycentric subdivision of an HI-cell complex as a stellar subdivision from "the top -down".

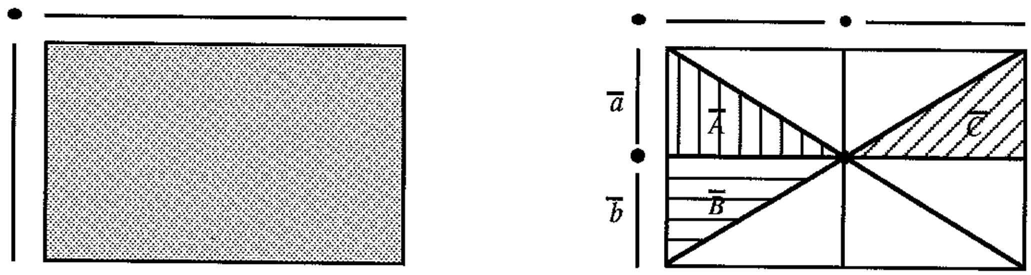
Fig. 26

**3.25** PROPOSITION. *Let $K^{(1)}$ be a derived subdivision of an HI-cell complex K. Then if $(\overline{D}_i, g_i{:}\overline{D}_i \to D_i)$ is in $K^{(1)}$ then $\overline{D}_i$ is a simplex and $g_i{:}\overline{D}_i \to D_i$ is a PL homeomorphism (on all of $\overline{D}_i$, not just on its interior).*

*Proof.* Observe that when we finally subdivide the 1-dimensional skeleton of $K$ using the points in the interiors of the 1-cells, we get 1-simplices that are mapped homeomorphically onto their images in $|K|$. (Any 1-cell mapping onto a loop is broken in two 1-cells each mapping homeomorphically onto different *halves* of the loop.) Now "going up" inductively as in Proposition 3.21, we cone each $d$-dimensional simplex $\sigma^d$ with points in the interiors of cells of dimension $d$+1 and obtain $d$+1-dimensional simplices $\sigma^{d+1}$ with the property that $\sigma^{d+1} - \sigma^d$ is contained in the interior of a $d$+1 cell $\overline{C}$ where $h{:}\overline{C} \to C$ is in a previous subdivision of $K$, and $h$ restricted to $\sigma^d$ is a homeomorphism. Since $h$ maps the interior of $\overline{C}$ homeomorphically onto its image, $h$ restricted to the entire $\sigma^{d+1}$ is also a homeomorphism. □

The above proposition shows that a first derived subdivision of an HI-cellular complex is *almost* an H-simplicial complex. Cells are simplices that are mapped homeomorphically onto their images, however the intersection of the images of two cells might not be a single *face* of both, as required in the case of H-cell complexes (see Figure 27).

However we will see in Proposition 3.26 that a second derived subdivision of an HI-cell complex is indeed an H-simplicial complex.




In the barycentric subdivision of a torus, the images of $\overline{a}$ and $\overline{b}$ intersect in two vertices and the images of triangles $\overline{A}$ and $\overline{B}$ share an edge and a vertex not in that edge, and the same applies to $\overline{A}$ and $\overline{C}$.

Fig. 27

**3.26** PROPOSITION. *The second derived subdivision $K^{(2)}$ of an HI-cell complex $K$ is a simplicial H-cell complex. (i.e. the images of the simplices intersect at most along a PL homeomorph of a common "face".)*

*Proof.* We will show the result only for one HI-cell $\left(C, h{:}\overline{C} \to C\right)$. Since in a general HI-cell complex $K$ the interiors of the cells $\overline{C_i}$ are mapped onto disjoint images, the general case follows (with some work) from the analysis of the situation for a single cell (which can have different faces mapping onto the same image in a manner analogous to the general situation).

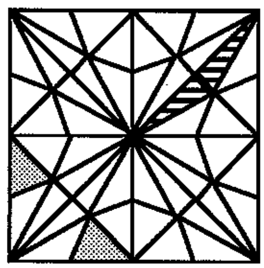
I. If $\overline{C}$ is 1-dimensional the second barycentric subdivides $\overline{C}$ into 4 edges and 4 vertices which map onto a PL image of a segment or a circle, and in either case the result is obvious.

II. If $\overline{C}$ has dimension greater than 1, let $\sigma_1$ and $\sigma_2$ be two simplices in $K^{(2)}$, the second barycentric subdivision $\overline{C}$.

II. 1) If either $\sigma_1$ or $\sigma_2$ is contained in the interior of $\overline{C}$, or if they intersect the boundary of $\overline{C}$ along simplices $\tau_1$ and $\tau_2$ belonging to proper faces $F_1$ and $F_2$ of $\overline{C}$ such that $h(F_1) \cap h(F_2) = \emptyset$, then

$$h(\sigma_1) \cap h(\sigma_2) = h\left(\sigma_1 \cap \sigma_2 \cap \operatorname{int}\left(\overline{C}\right)\right) = h(\tau)$$

where $\tau$ is a common face (possibly empty) of $\sigma_1$ and $\sigma_2$ since the theorem is true for the interior of an HI-cell. (It is true for the second barycentric subdivision of a cell (without face identifications), as it can be quickly verified, thus it remains true in the interior of an HI-cell).




(In these illustrations, simplices shaded with equal patterns, represent pairs of simplices $\sigma_1$ and $\sigma_2$, in the second barycenter subdivision of the large square, which possess the properties stated in the text. )

II. 2) If both $\sigma_1$ and $\sigma_2$ intersect the boundary of $\overline{C}$, let $F_1$ and $F_2$ be two (possibly equal) proper faces of $\overline{C}$ containing $\sigma_1 \cap \operatorname{Bd}\left(\overline{C}\right)$ and $\sigma_2 \cap \operatorname{Bd}\left(\overline{C}\right)$ respectively such that $h(F_1) = h(F_2)$. Note that $\sigma_i \cap \operatorname{Bd}\left(\overline{C}\right)$ is a simplex by construction of the barycentric subdivision, therefore $\sigma_i \cap \operatorname{Bd}\left(\overline{C}\right)$ is contained in a unique face $F_i$ of $\overline{C}$ having minimal dimension. Also recall the existence of a PL homeomorphism $f_{12} : F_1 \to F_2$ "commuting with $h$" specified in the definition of an HI-cell (Def. 2.26, p. 40).

We need to analyze two cases:

II. 2) i) $F_1 = F_2$

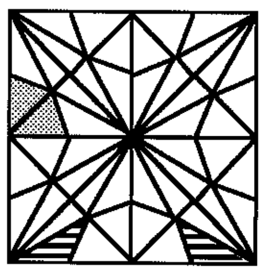
II. 2) ii) $F_1 \neq F_2$

Case II. 2) i) $F_1 = F_2$

- If either $\sigma_1$ or $\sigma_2$ intersect $F_1$ along a simplex contained in the interior of $F_1$, then $h(\sigma_1) \cap h(\sigma_2) = h(\sigma_1 \cap \sigma_2)$ and this gives the desired result.
- If both $\sigma_1$ and $\sigma_2$ intersect the boundary of $F_1$ then

$$h(\sigma_1) \cap h(\sigma_2) = h(\sigma_1 \cap F_1) \cap h(\sigma_1 \cap F_1) \cup h(\sigma_1 \cap \sigma_2)$$

By induction $h(\sigma_1 \cap F_1) \cap h(\sigma_1 \cap F_1)$ is a homeomorphic image of a simplex $\tau$ (possibly empty) in $F_1$ and thus if $h(\sigma_1 \cap \sigma_2) = \varnothing$ we are done. If $h(\sigma_1 \cap \sigma_2) \neq \varnothing$ then $\sigma_1 \cap \sigma_2$ is a cone $b(\sigma_1 \cap \sigma_2 \cap F_1)$. Therefore, $h(\sigma_1) \cap h(\sigma_2)$ is a homeomorphic image of the simplex $b\tau$.



Case II. 2) ii) $F_1 \neq F_2$

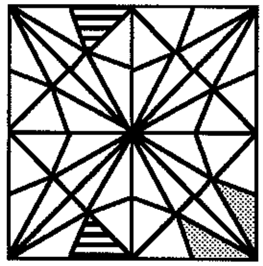
We can assume that $\sigma_i \cap \text{int}(F_i) \neq \varnothing$, otherwise we could take $F_1 = F_2$ which was analyzed above. In this situation, $\text{int}(\sigma_1) \cap \text{int}(\sigma_2) = \varnothing$, Let

$\tau_i = \sigma_i \cap F_i$. then

$$h(\sigma_1) \cap h(\sigma_2) = h(\sigma_1 \cap F_1) \cap h(\sigma_2 \cap F_2) = h(\tau_1) \cap h(\tau_2) = h(\tau_1) \cap h(f_{21}(\tau_2))$$

($f_{21}$ is the linear homeomorphism $f_{21}: F_2 \to F_1$ "commuting with $h$.")

Now since $\tau_1$ and $f_{21}(\tau_2)$ are both simplices in $F_1$, the result follows by induction.



□

Theorem 3.29, will show that the quotient space obtained by identifying linearly homeomorphic faces of a cell $\overline{C}$ can be realized as a polyhedron $C$, and that there is a map $g:\overline{C} \to C$ such that $\left(C, g:\overline{C} \to C\right)$ is an HI-cell.

Theorem 3.29 will finally demonstrate the existence of the HI-cell previously given in Remark 3.13 (p. 62) and shown in Fig. 23.

First we will make the following definitions.

**3.27** DEFINITION. Let $\overline{C}$ be a cell, and let $\mathcal{F}\overline{C}$ be the set of all the faces of $\overline{C}$. A set

$$\mathcal{F}_* = \left\{ \mathcal{F}_0 = \left\{\overline{C}\right\},\ \mathcal{F}_1 = \left\{F_{11}, F_{12}, \cdots\right\},\ \mathcal{F}_2 = \left\{F_{21}, F_{22}, \cdots\right\}, \cdots,\ \mathcal{F}_r = \left\{F_{r1}, F_{r2}, \cdots\right\} \right\}$$

of subsets of $\mathcal{F}\overline{C}$ is called ***a quotient partition of*** $\mathcal{F}\overline{C}$ if it satisfies:

1) The sets $\mathcal{F}_i$ are disjoint

2) $\cup \mathcal{F}_i = \mathcal{F}\overline{C}$

3) For each set $\mathcal{F}_i$ there exists a cell $\overline{C}_i$ and a set of linear homeomorphisms $\mathcal{L}_i = \left\{ l_{ij}: \overline{C}_i \mapsto F_{ij} \middle| F_{ij} \in \mathcal{F}_i \right\}$ (one for each $F_{ij} \in \mathcal{F}_i$)

4) If $G$ is a face of $F_{ij}$ and $H$ is a face of $F_{ik}$ ($F_{ij}, F_{ik} \in \mathcal{F}_i$) such that $l_{ik} \circ l_{ij}^{-1}(G) = H$, then there exists $u$ such that $G$ and $H$ are in the set $\mathcal{F}_u \in \mathcal{F}_*$. ($l_{ik} \circ l_{ij}^{-1}: F_{ij} \to \overline{C}_i \to F_{ik}$)

**3.28** DEFINITION. Given a quotient partition $\mathcal{F}_*$ of $\mathcal{F}\overline{C}$, ***the quotient relation*** induced by $\mathcal{F}_*$ is the equivalence relation $\sim$ on $\overline{C}$ defined by:

$$x \sim y \text{ if and only if } x \in \operatorname{int}(F_{ij}),\ y \in \operatorname{int}(F_{ik}) \text{ and } l_{ik} \circ l_{ij}^{-1}(x) = y.$$

As usual, ***the quotient space*** $\overline{C}/\sim$, is the topological space whose underlying set is the set $\{[x]_\sim\}$ of equivalence classes of $\sim$, equipped with the largest topology for which the surjection $\pi: \overline{C} \to \{[x]_\sim\}$, $x \mapsto [x]_\sim$, is continuous, i. e. $U$ is open in $\overline{C}/\sim$ if and only if $\pi^{-1}(U)$ is open in $\overline{C}$. A bijection $\pi$ as above is called a ***projection*** or a ***quotient map***.

**3.29** THEOREM. (PL Quotient Spaces) *Let $\overline{C} \subset \mathbf{R}^n$ be a cell.*
*Let* $\mathcal{F}_* = \{\mathcal{F}_0 = \{\overline{C}\},\ \mathcal{F}_1 = \{F_{11}, F_{12}, \cdots\},\ \mathcal{F}_2 = \{F_{21}, F_{22}, \cdots\}, \cdots,\ \mathcal{F}_r = \{F_{r1}, F_{r2}, \cdots\}\}$ *be a quotient partition of $\mathcal{F}\overline{C}$ and let $\sim$ be the quotient relation induced by $\mathcal{F}_*$.*

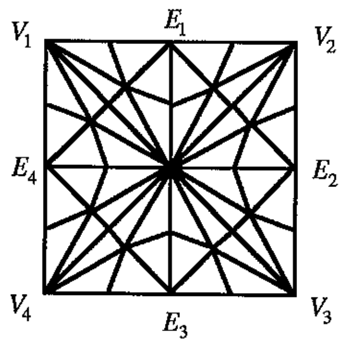
*Then there exist a polyhedron $C \subset \mathbf{R}^N$, homeomorphic to $\overline{C}/\sim$, and a PL map $g: \overline{C} \to C$ such that $(C, g: \overline{C} \to C)$ is an HI-cell.*

*Proof.* Let $K$ be the canonical cell complex on $\overline{C}$ consisting of all the faces of $\overline{C}$, and let $K^{(2)}$ be its second barycentric subdivision. $K^{(2)}$ is thus a simplicial complex.

**I. CONSTRUCTION OF *g***

Recall that for each set $\mathcal{F}_i \in \mathcal{F}_*$ there is cell $\overline{C}_i$ (homeomorphic to all the faces in $\mathcal{F}_i$) Let $\alpha_i$ denote the number of vertices in the second barycentric subdivision of $\overline{C}_i$ which are contained in the interior of $\overline{C}_i$. Let $N = \alpha_0 + \alpha_1 + \cdots + \alpha_r$ (see Fig. 28) and let $\sigma^N \subset \mathbf{R}^N$ be an $N$-dimensional simplex.

$\mathcal{F}_* = \{\ \mathcal{F}_0 = \{\overline{C}\},\ \mathcal{F}_1 = \{E_1, E_3\},\ \mathcal{F}_2 = \{E_2, E_4\},\ \mathcal{F}_3 = \{V_1, V_2, V_3, V_4\}\ \}$




$\overline{C}_0$ is a parallelogram

$\overline{C}_1$ is a 1-simplex

$\overline{C}_2$ is a 1-simplex

$\overline{C}_3$ is a point

$N = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 17 + 3 + 3 + 1 = 24$

Fig. 28

We start by defining (constructing) a surjection

$$g_0 : \left\{\text{vertices of } K^{(2)}\right\} \to \left\{\text{vertices of } \sigma^N\right\}$$

as follows:

- $g_0$ sends the $\alpha_0$ vertices in the interior of $\overline{C}$ onto (any choice of) $\alpha_0$ vertices of $\sigma^N$.
- $g_0$ sends the $\alpha_1$ vertices in the interior of $F_{11}$ ($\in \mathcal{F}_1$) onto $\alpha_1$ vertices of $\sigma^N$ not previously used in the construction of $g_0$.

  $g_0$ sends any vertex $V$ in the interior of any face $F_{1j} \in \mathcal{F}_1$ to the vertex $g_0(W)$, where $W$ is the vertex in the interior of $F_{11}$ defined by $W = l_{11} \circ l_{1j}^{-1}(V)$. $(l_{11} \circ l_{1j}^{-1} : F_{1j} \to \overline{C}_1 \to F_{11})$
- Similarly, $g_0$ sends the $\alpha_i$ vertices in the interior of the face $F_{i1}$ onto $\alpha_i$ vertices of $\sigma^N$ not previously used in the construction of $g_0$.

  $g_0$ sends the vertex $V$ in the interior of any other face $F_{ij} \in \mathcal{F}_i$ to the vertex $g_0(W)$, where $W$ is the vertex in the interior of $F_{i1}$ defined by $W = l_{i1} \circ l_{ij}^{-1}(V)$.

Now we extend $g_0$ to a function $g{:}\overline{C} \to \sigma^N$ by requiring $g$ to be linear on all the simplices of $K^{(2)}$. We let $C$ denote the image $g(\overline{C})$.

$C$ is a polyhedron in $\mathbf{R}^N$ since it is a union of simplices in $\sigma^N \subset \mathbf{R}^N$; and $g{:}\overline{C} \to C$ is a PL surjection by construction.

**II. CLAIM.** *$g$ is an injection in the interior of $\overline{C}$.*

First we observe that every simplex in $K^{(2)}$ is mapped homeomorphically onto its image: No simplex of $K^{(2)}$ contains vertices in the interiors of different faces of $\overline{C}$ of equal dimension – a property already enjoyed by simplices in the first barycentric subdivision of $\overline{C}$ – thus $g$ is one-one on the set of vertices of every simplex. (And of course, any subset of the vertices of $\sigma^N$ is in general position. Thus the image of a simplex is a simplex of equal dimension.)

Now assume that $x_1$ and $x_2$ are two points in the interior of a face $F$ of $\overline{C}$ such that $g(x_1) = g(x_2)$. Let $\sigma_1$ and $\sigma_2$ be the unique simplices in $K^{(2)}$ such that $x_i \in \text{int}(\sigma_i)$. Observe that $\sigma_1 \neq \sigma_2$ since simplices are mapped homeomorphically by $g$ onto their images, as observed above. Now, let $y = g(x_1) = g(x_2)$, and let $\tau$ be the simplex in (the canonical simplicial complex) $\sigma^N$ which contains $y$ in its interior. Because $g$ is simplicial and a homeomorphism on the simplices, we see that $g(\sigma_1) = g(\sigma_2) = \tau$ and thus $\sigma_1$ and $\sigma_2$ have the same dimension, hence none of the simplices $\sigma_i$ is a face of the other.

Note that $\sigma_1$ and $\sigma_2$ share all of their vertices which lie in the interior of $F$ otherwise $g(\sigma_1) \neq g(\sigma_2)$. Note also that this set of vertices is not empty.

We conclude from the above paragraphs that $\sigma_1$ and $\sigma_2$ intersect the boundary of $F$ along simplices $\tau_1$ and $\tau_2$ having equal dimension.

There exists a vertex $V$ of $\tau_1$ and a vertex $W$ of $\tau_2$ , $V \neq W$, such that $g(V) = g(W)$; otherwise $\sigma_1$ and $\sigma_2$ would have the same set of vertices forcing $\sigma_1 = \sigma_2$. By the construction of $g$, $V$ and $W$ must lie in the interiors of homeomorphic faces $F_{ij}$ and $F_{ik}$ respectively (they are homeomorphic since they lie in the same set $\mathcal{F}_i$). Since these faces are homeomorphic, none is a face of the other.

Now it follows from the construction of the second barycentric subdivision, that $\sigma_1$ contains vertices in the interior of the cone $bF_{ij}$, where $b$ is the barycenter of $F$, and $\sigma_2$ must contain vertices in the interior of $bF_{ik}$.

It also follows from the second barycentric subdivision, that the vertices in the interior of $F$ which belong to both $\sigma_1$ and $\sigma_2$ must lie in the cone $b\left(F_{ij} \cap F_{ik}\right)$, which does not intersect $\text{int}\left(bF_{ij}\right)$ nor $\text{int}\left(bF_{ik}\right)$.

Since we have seen that $\sigma_1$ and $\sigma_2$ must share all of their vertices which lie in the interior of $F$, the conditions set forth in the above two paragraphs cannot simultaneously be satisfied.

We thus conclude that there do not exist two points $x_1$ and $x_2$ lying in the interior of a face $F$ of $\overline{C}$ for which $g(x_1) = g(x_2)$. Therefore $g$ is one-one in the interior of every face of $\overline{C}$.

**III. CLAIM** *Let $x_1$ be a point in the boundary of $\overline{C}$ and let $x_2$ be a point in the interior of $\overline{C}$. Then $g(x_1) \neq g(x_2)$.*

Assume that $g(x_1) = g(x_2)$. Let $\sigma_i$ be the simplex of $K^{(2)}$ containing $x_i$ in its interior. As in II. $g(\sigma_1) = g(\sigma_2)$ and $\dim(\sigma_1) = \dim(\sigma_2)$.

Since all the vertices of $\sigma_1$ are in a proper face of $\overline{C}$, the vertices of $\sigma_2$ must also be in a proper face of $\overline{C}$, since both set of vertices have the same image.

However in a second barycentric subdivision, (already true for a first barycentric subdivision) there are no simplices which intersect the interior of $\overline{C}$ and have all of their vertices in the boundary of $\overline{C}$.

Therefore $g(x_1) \neq g(x_2)$.

**IV. CLAIM** *g is a homeomorphism in the interior of* $\overline{C}$. (*Onto its image.* )

Let $g^o = g|_{\text{int}\,\overline{C}} : \text{int}\,\overline{C} \to g(\text{int}\,\overline{C})$. We have from II. that $g^o$ is a bijection and it is continuous since it is a restriction of a PL map. We need to show that $(g^o)^{-1}$ is continuous. i.e. we need to show that for any open set $U$ in the interior of $\overline{C}$, we have that $g(U)$ is open in $g(\text{int}\,\overline{C})$. Thus we must show that for any point $y$ in $g(U)$ there exists an open neighborhood $B$, with $y \in B \subset g(U)$.

Let $x_1 = (g^o)^{-1}(y)$ and let $V = \overline{C} - U$, $V$ is compact thus $g(V)$ is compact.

If $y \notin g(V)$ there is $\varepsilon > 0$ such that $N_\varepsilon(y, \mathbf{R}^N) \cap g(V) = \varnothing$ and thus $y \in N_\varepsilon(y, C) \subset g(U)$. $[N_\varepsilon(y, C) = N_\varepsilon(y, \mathbf{R}^N) \cap C]$

If $y \in g(V)$ there exists a point $x_2$ in $\overline{C}$ such that $g(x_2) = g(x_1) = y$. Since $g$ is one-one in the interior of $\overline{C}$, $x_2$ is a point in the boundary of $\overline{C}$. Now by III. $g(x_2) \neq g(x_1)$, and thus $y \notin g(V)$.

**V. CLAIM** *Let* $(F_i, F_j)$ *be a pair of faces of* $\overline{C}$ *such that* $g(\text{int}(F_i)) \cap g(\text{int}(F_j)) \neq \varnothing$. *Then there exists a linear homeomorphism* $f_{ij} : F_i \to F_j$ *such that* $\left(g|_{F_j}\right) \circ f_{ij} = g|_{F_i}$.

Part 2) of Def. 3.27 (p. 78) implies that there exist $\mathcal{F}_u, \mathcal{F}_v \in \mathcal{F}_*$ such that $F_i \in \mathcal{F}_u$ and $F_j \in \mathcal{F}_v$.

If $\mathcal{F}_u \neq \mathcal{F}_v$ the image under $g$ of the set of vertices of $K^{(2)}$ lying in the interior of $F_i$, and the image of the set of those lying in the interior of $F_j$ are disjoint, by construction of $g$ (in I. ). Now, since any point in the interior of a face $F$ of $\overline{C}$ lies in the interior of a simplex which has at least one vertex in the interior of $F$, we see that $g(\mathrm{int}(F_i)) \cap g(\mathrm{int}(F_j)) = \varnothing$. This contradicts the hypothesis. Therefore, $F_i$, and $F_j$ belong to the same set $\mathcal{F}_u \in \mathcal{F}_*$, and are thus labeled as $F_i = F_{us}$ and $F_j = F_{ut}$.

We now define the linear homeomorphism $f_{ij} : F_i \to F_j$ by:

$$f_{ij} : F_i = F_{us} \xrightarrow{l_{us}^{-1}} \overline{C}_u \xrightarrow{l_{ut}} F_{ut} = F_j$$

To finally show that $\left(g|_{F_j}\right) \circ f_{ij} = g|_{F_i}$ we need only to show that

$$\left(g|_{Bd(F_i)}\right) \circ f_{ij} = g|_{Bd(F_j)}$$

and this is true because of condition 4) of Def. 3.27 (p. 78) (Def. of quotient partition) and the construction of $g$.

**VI. CLAIM.** $\left(C, g: \overline{C} \to C\right)$ *is an HI-cell.*

Parts I., IV. and V. of this theorem taken together, are precisely the defining conditions for an HI-cell. (Def. 2.26, p. 40.)

**VII. CLAIM.** *$C$ is homeomorphic to the quotient space $\overline{C}/_\sim$.*

Let $\pi: \overline{C} \to \overline{C}/_\sim$ be the quotient map $x \mapsto [x]_\sim$. Let $x \sim y$, that is: $x \in \mathrm{int}(F_{ij})$, $y \in \mathrm{int}(F_{ik})$ and $l_{ik} \circ l_{ij}^{-1}(x) = y$. Let $V$ be a vertex in the interior of $F_{ij}$ and let $W$ be the vertex in the interior of $F_{ik}$ given by $l_{ik} \circ l_{ij}^{-1}(V) = W$. By construction, the map $g: \overline{C} \to C$ is such that $g(V) = g(W)$. Therefore by part V. (where $f_{ij} = l_{ik} \circ l_{ij}^{-1}$) we obtain that $g(x) = g(y)$. Hence, $g$ is constant on each set $\pi^{-1}([x]_\sim)$. It is a basic topological fact, that in these circumstances, the function $h: \overline{C}/_\sim \to C$ defined by $h([x]_\sim) = g(x)$ is continuous and $h \circ \pi = g$.

$$\begin{array}{ccc} \overline{C} & & \\ {\scriptstyle \pi}\downarrow & \searrow^{g} & \\ \overline{C}/_{\sim} & \overset{}{\underset{h}{\dashrightarrow}} & C \end{array}$$

It is easy to see that $h$ is a bijection and since $\overline{C}/_{\sim}$ is compact we conclude that $h$ is a homeomorphism. □

# 4 Self-Dual HI-Cell Complexes

### Stars and Links in $K$

**4.1** Definition. Let $K$ be an HI-cell complex and $v \in |K|$ be a vertex; i.e. $v$ is the image of a zero-dimensional HI-cell $h{:}\overline{v} \to v$ of $K$.

The ***star of v in K***, $\mathrm{st}(v,K)$, is the set of HI-cells of $K$ defined by

$$\mathrm{st}(v,K) = \left\{ \left(C_k,\, h_k{:}\overline{C}_k \to C_k\right) \middle| \left(C_k, h_k\right) \text{ is a face of } h_i{:}\overline{C}_i \to C_i, \text{ for all } C_i \text{ with } v \in C_i \right\}$$

The ***link of v in K***, $\mathrm{lk}(v,K)$, is the subset of $\mathrm{st}(v,K)$ given by

$$\mathrm{lk}(v,K) = \left\{ \left(C_r,\, h_r{:}\overline{C}_r \to C_r\right) \middle| \left(C_r, h_r\right) \in \mathrm{st}(v,K),\ v \notin C_r \right\}.$$

When clear, we will also call the underlying polyhedra $|\mathrm{st}(v,K)|$ and $|\mathrm{lk}(v,K)|$ respectively *the star of v in K* and *the link of v in K*.

**4.2** Remark.

i) In the above definition, when we say that "$\left(C_k, h_k\right)$ is a face of $h_i{:}\overline{C}_i \to C_i$" we obviously mean that there is a face $h_{i|F}{:}F \to C_k$ of $h_i{:}\overline{C}_i \to C_i$ and that $\left(C_k,\, h_k{:}\overline{C}_k \to C_k\right)$ is the HI-cell in $K$ such that

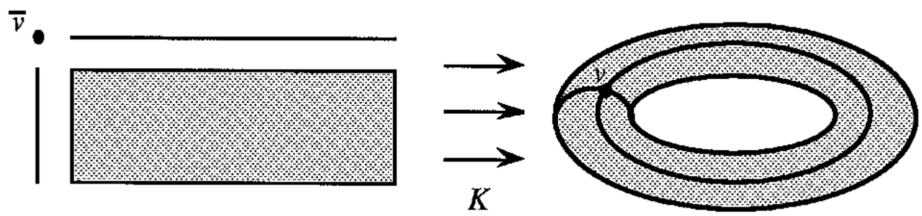
$$\begin{array}{ccc} \overline{C}_k & \xrightarrow{\ i_F\ } & F \subset \overline{C}_i \\ & {\scriptstyle h_k}\searrow \quad \swarrow{\scriptstyle h_i|_F} & \\ & C_k = h_i(F) & \end{array}$$

commutes as in the Def. 2.28 (p. 42) of an HI-cell complex.

ii) $\mathrm{st}(v,K)$ and $\mathrm{lk}(v,K)$ are subcomplexes of $K$ as it can easily be verified.

iii) Note that $|\mathrm{st}(v,K)|$ and $|\mathrm{lk}(v,K)|$ are not topological invariants of $|K|$, i.e. if $L$ is another complex with $|L| = |K|$ having the property that $v$ is also a vertex in $L$, then $|\mathrm{st}(v,K)|$ and $|\mathrm{st}(v,L)|$ are not necessarily homeomorphic and the same applies for the links.

Also, $|\mathrm{st}(v,K)|$ and $|\mathrm{lk}(v,K)|$ should not be confused with the concepts of "*a star of a point in a polyhedron*" and "*a link of a point in a polyhedron*" defined in 1.3 (p. 6) (def. of polyhedron). These stars and links as in Def. 1.3 will be from now on called ***polyhedral stars*** and ***polyhedral links*** respectively to distinguish them from ***stars and links in K***. For example, in the "usual" HI-cell complex $K$ on the torus consisting of one vertex, two edges and one rectangle (see figure 29), $|\mathrm{st}(v,K)|$ is the whole torus and $|\mathrm{lk}(v,K)|$ is empty, while a polyhedral star of any point in the torus $|K|$, is a polyhedral disk, and a polyhedral link is a circle.




st $(v, K) = K$, $|\mathrm{st}\ (v, K)|$ is the torus, $|\mathrm{lk}\ (v, K)|$ is empty.

Fig. 29

iv) Note however that if $K$ is a simplicial complex then it is a standard fact that $|\mathrm{st}(v,K)|$ and any polyhedral star of v in $|K|$ are PL-homeomorphic and the same applies for links. (See [R-S] and recall that Remark 1.11 (p. 9) states that stars and links of a point in a polyhedron can be realized as polyhedra. )

In fact, classically one proves the PL-invariance of polyhedral stars and links of a point $p$ in a polyhedron $P$ (which we had not yet done) by first proving that $P$ is the underlying space of a simplicial complex $K$ having $p$ as a vertex, and then one uses the fact that $|\mathrm{st}(v,K)|$ and any polyhedral star of $p$ in $P$ are PL-homeomorphic.

**DUAL STARS**

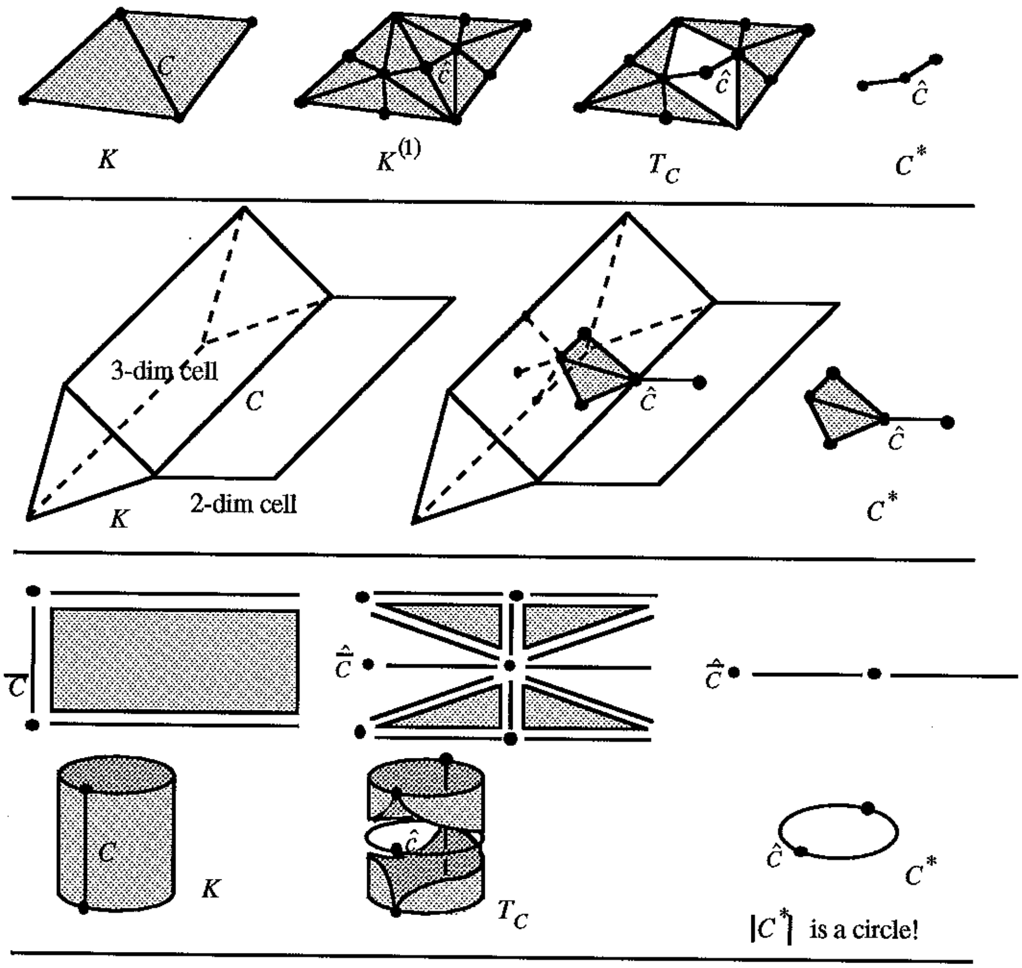
**4.3** REMARK. Let $K$ be an HI-cell complex and $\left(C, h{:}\overline{C} \to C\right) \in K$. Let $K^{(1)}$ be the first barycentric subdivision of $K$. Observe that if $\left(D, g{:}\overline{D} \to D\right) \in K^{(1)}$ then $D \cap C$ is empty or $D \cap C = g\left(\overline{F}\right)$, where $\overline{F}$ is a face of $\overline{D}$. (This follows from the definition 3.1 (p. 51) of subdivision.) We thus can refer to the dimension of $D \cap C$ as the dimension of $\overline{F}$ when $D \cap C \neq \varnothing$, and define $\dim(D \cap C) = -1$ when $D \cap C = \varnothing$.

Let $T_C = \left\{\left(D_j, h_j{:}\overline{D}_j \to D_j\right) \middle| \left(D_j, h_j\right) \in K^{(1)}, \dim\left(D_j \cap C\right) \leq 0\right\}$.

Observe that $T_C$ is a subcomplex of $K^{(1)}$: If $\left(D, g{:}\overline{D} \to D\right) \in T_C$ and $\left(F, f{:}\overline{F} \to F\right)$ is "a face of $(D, g)$" then $\dim(F \cap C) \leq \dim(D \cap C) \leq 0$; thus $(F, f) \in T_C$.

**4.4** DEFINITION. Let $K$ be an HI-cell complex and $\left(C, h{:}\overline{C} \to C\right) \in K$. The ***dual star*** $C^*$ of $C$ is $\mathrm{st}\left(\hat{C}, T_C\right)$, the star of the vertex $\hat{C}$ in $T_C$ ($\hat{C}$= barycenter of $C$). We will sometimes also call the polyhedron $\left|\mathrm{st}\left(\hat{C}, T_C\right)\right|$ the *dual star of C*.

Some examples of dual stars are shown below in figure 30.




$C^*$ is the dual star of $C$.

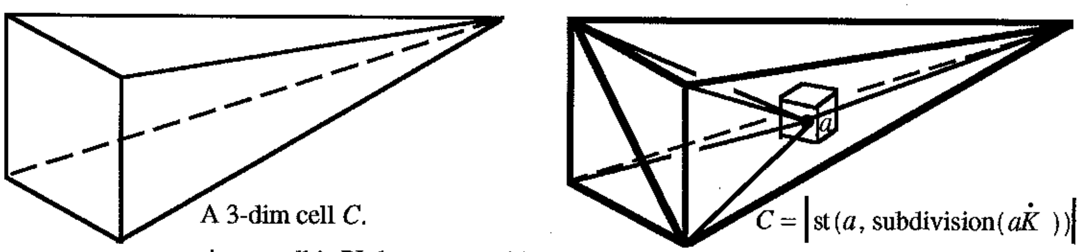
Fig. 30

## MANIFOLDS

**4.5** DEFINITION. A polyhedron $P$ is called an ***n-dimensional ball***, or **n-*ball***, if it is PL-homeomorphic to the unit n-cube $J^n = [-1,1]^n$. A polyhedron is called an ***n-sphere*** if it is PL-homeomorphic to the boundary of $J^n$.

Since an $n$-cell $C$ is a cone $a(\mathrm{Bd}(C))$, where $a \in \mathrm{Int}(C)$, we see that $C = |\mathrm{st}(a, a\dot{K})|$, where $\dot{K}$ is the cell-complex consisting of the proper faces of $C$

as in Proposition 2.19 (p. 30), and $a\dot{K}$ is the cone cell-complex obtained as in Lemma 3.6. Also, since we may assume that $\text{span}(C) = \mathbf{R}^n$, we see that any $a \in \text{Int}(C)$, has by Remark 1.4 (p. 6), a polyhedral star in $C$ (as in Def. 1.3 (p. 6)) of the form $N_\varepsilon(a,C) = \{x | x \in \mathbf{R}^n, \text{d}(a,x) \le \varepsilon\}$, which is an $n$-cube. Now subdividing $\dot{K}$ into a simplicial complex (thus also $a\dot{K}$ ) and using the fact that for a vertex $v$ in a simplicial complex $K$ we have [by iv) of the remark 4.2] that $|st(v,K)|$ and any polyhedral star of $v$ in the polyhedron $|K|$ are PL-homeomorphic, we see that $C = \left|\text{st}\left(a, \text{subdivision}\left(a\dot{K}\right)\right)\right|$ is PL-homeomorphic to $N_\varepsilon\left(a, \mathbf{R}^n\right)$.




An $n$- cell is PL homeomorphic to an $n$- cube $N_\varepsilon(a, C)$.

Fig. 31

We thus get the following:

**4.6** PROPOSITION. *Let C be an n-cell. Then C is an n-ball.*

□

**4.7** DEFINITION. A ***PL n-dimensional manifold*** (or briefly an *n-manifold*) is a polyhedron $M$ such that every point $x \in M$ has a neighborhood $U_x$ in $M$ which is PL-homeomorphic to an open set $V_x$ of $\mathbf{R}^n$.

A ***PL n-manifold with boundary*** is a polyhedron $M$ with the property that every $x \in M$ has a neighborhood $U_x$ in $M$ which is PL-homeomorphic to an open subset $V_x$ of $\mathbf{R}^n_+$, where $\mathbf{R}^n_+ = \left\{ (x_1, x_2, \cdots, x_n) \middle| x_n \geq 0 \right\}$. The boundary $\partial M$ of $M$, is the set of all the points in $M$ such that $V_x$ is not open in $\mathbf{R}^n$ (under the inclusion $\mathbf{R}^n_+ \subset \mathbf{R}^n$).

**4.8** REMARK. Let $M$ be a PL $n$-dimensional manifold (possibly with boundary), and let $U_x \subset M$ and $V_x \subset \mathbf{R}^n$ (or $V_x \subset \mathbf{R}^n_+$) be open sets as in Def. 4.7. Since each set $V_x$ is a locally finite union of closed $n$-cubes, it is also a locally finite union of $n$-simplices and thus each $V_x$ has dimension equal to $n$. Since Prop. 1.32 (p. 18) asserts that PL homeomorphic polyhedra have equal dimensions, we see that every $U_x$ has also dimension equal to $n$. By local compactness, $M$ is a locally finite union of sets $U_x$. Therefore $M$ is a locally finite union of $n$-simplices, and thus it has dimension equal to $n$.

**4.9** DEFINITION. Let $K$ be an HI-cell complex such that $|K|$ is a manifold of dimension $n$. $K$ is said to be ***dualizable*** if for every $d$-dimensional HI-cell $\left(C_i, h_i{:}\overline{C}_i \to C_i\right) \in K$ there is an $(n-d)$-dimensional HI-cell $\left(C_i^*, h_i^*{:}\overline{C_i^*} \to C_i^*\right)$ (not in $K$ but with $C_i^* \subset |K|$) called ***the dual HI-cell of*** $(C_i,h_i)$, satisfying the following:

1) $C_i^* = \left|\mathrm{st}\left(\hat{C}_i, T_{C_i}\right)\right|$, the underlying polyhedron of the HI-complex "the dual star of $C_i$." (See 4.3 and 4.4.)

2) For every HI-cell $\left(D_j, g_j{:}\overline{D}_j \to D_j\right)$ in "the dual star of $C_i$ " there is a <u>linear</u> embedding $e_{ji}{:}\overline{D}_j \to \overline{C_i^*}$ such that the diagram below commutes.

$$\begin{array}{ccc} \overline{D}_j & \xrightarrow{e_{ji}} & \overline{C_i^*} \\ {\scriptstyle g_j}\downarrow & & \downarrow{\scriptstyle h_i^*} \\ D_j & \hookrightarrow & C_i^* \end{array}$$

(i.e. the complex "the dual star of $C_i$" is a subdivision of $h_i^*$.)

3) For every proper face $\left(h_i^*(F), h_i^*|_F{:}F \to h_i^*(F)\right)$ of $\left(C_i^*, h_i^*{:}\overline{C_i^*} \to C_i^*\right)$ (the dual HI-cell of $(C_i,h_i)$) there is a unique HI-cell $\left(C_j, h_j{:}\overline{C}_j \to C_j\right)$ in $K$ with $C_i$ a proper face of $C_j$ such that "$F$ is mapped onto the *dual HI-cell of* $\left(C_j, h_j\right)$". More precisely, there is a linear homeomorphism $i_F{:}\overline{C_j^*} \to F \subset \overline{C_i^*}$ such that the diagram

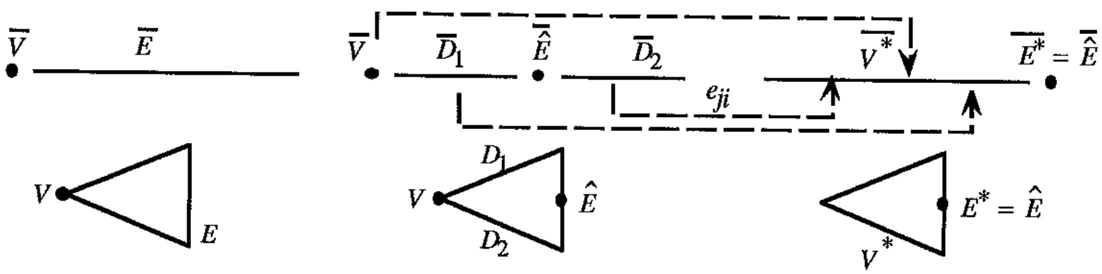
$$\begin{array}{ccc} \overline{C_j^*} & \xrightarrow{i_F} & F \subset \overline{C_i^*} \\ & {\scriptstyle h_j^*}\searrow \quad \swarrow{\scriptstyle h_i^*|_F} & \\ & C_j^* = h_i^*(F) & \end{array}$$

commutes.

Conversely, for each $C_j$ in $K$ with $C_i$ a proper face of $C_j$, there is at least one proper face $\left(h_i^*(F), h_i^*|_F{:}F \to h_i^*(F)\right)$ of $\left(C_i^*, h_i^*{:}\overline{C_i^*} \to C_i^*\right)$ "mapping onto the *dual HI-cell of* $\left(C_j, h_j\right)$".

**4.10** REMARK. The above is an *inductive* definition. Observe that in part 3) we have that $\dim(C_j) > \dim(C_i)$, and thus for 3) to make sense, we must start by verifying that the dual HI-cells of the $n$-dimensional HI-cells in $K$ satisfy the definition. (These duals are just 0-dimensional HI-cells mapping onto the barycenters of the $n$-HI-cells of $K$). Then inductively, to verify 3) for the dual HI-cells of the $d$- dimensional HI-cells of $K$, we use the (already verified) dual HI-cells of the HI-cells in $K$ of dimension greater than $d$.

In Fig. 32 we show an example of a dualizable HI-cell complex.

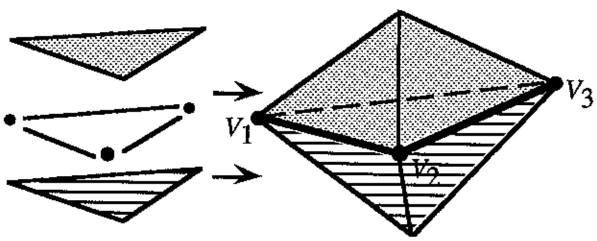



The HI-complex $K$. The dual star of $V$. The dual HI-cell of $V$ and the dual HI-cell of $E$.

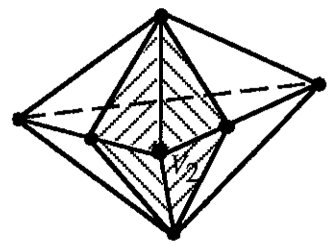
Fig. 32

**4.11** REMARK. Note that even if $K$ is a simplicial (cell) complex on a manifold, that the dual of a simplex (cell) need not be a simplex (cell).

Also note that if $K$ is a general HI-cell complex on $M$ then $K$ is not necessarily dualizable. An example is illustrated in Fig. 33, where $M$ is a (polyhedral) 2-sphere and $K$ consists of two (HI-) triangles each mapping onto a different hemisphere of $M$, together with 3 edges and 3 vertices. $K$ is an HI-cell complex (and not an H-cell complex) because the triangles intersect along 3 common edges. Note that the dual stars of the vertices are not 2-dimensional HI-cells (satisfying Def. 4.9) since by 2) and 3) of Def. 4.9, any 2-dimensional cell mapping onto the underlying polyhedron of the dual star of a vertex would have only two edges. This is impossible.




The HI-cell complex $K$.




The dual star of $V_2$.

K is not dualizable. The dual star of $V_2$ is not a 2-dimensional HI-cell: it has only two"dual- edges".

Fig. 33

We however have the following:

**4.12** PROPOSITION. *Let* $K = \{(C_i, h_i : \overline{C}_i \to C_i)\}$ *be a dualizable HI-cell complex on a manifold M. Then* $K^* = \{(C_i^*, h_i^* : \overline{C_i^*} \to C_i^*)\}$ *is an HI-cell complex on M.*

*Proof.*

- 0) We show that $|K^*| = M$.

Since it is obvious that $|K^*| \subset M$, we need only to show that $M \subset |K^*|$ and we do this by showing that any point $p$ in $M$ lies in $V^*$ for some vertex $V$ of $K$. But since $p$ lies in the HI-interior of $C_i$ for some $i$, it suffices to show that any point $\overline{p}$

in the interior of a cell $\overline{C_i}$ lies in the dual star of a vertex of $\overline{C_i}$ (in the complex consisting of all the faces of $\overline{C_i}$ ). Thus:

If $\overline{p}$ is the barycenter of $\overline{C_i}$, then $\overline{p}$ lies in the segment $\overline{p}\overline{V}$ ($\overline{V}$ any vertex of $\overline{C_i}$) and $\overline{p}\overline{V}$ is a 1-cell in the barycentric subdivision of $\overline{C_i}$ and it is contained in the dual star of $\overline{V}$. ($\overline{p}\overline{V}$ intersects $\overline{V}$ in the 0-dimensional cell $\overline{V}$.)

If $\overline{p}$ is not the barycenter of $\overline{C_i}$, $\overline{p}$ lies in a segment $bf$ where $b$ is the barycenter of $\overline{C_i}$ and $f$ is a point in a proper face $F$ of $\overline{C_i}$. By induction on the dimension of the cell, we assume that $f$ lies in the dual star $S$ (in $F$) of a vertex $\overline{V}$ of $F$. And thus, $\overline{p}$ lies in the cone $bS$ and $bS$ is contained in the dual star (in $\overline{C_i}$) of $\overline{V}$ (the cells in $bS$ are in the barycentric subdivision of $\overline{C_i}$ and are faces of cells which contain $\overline{V}$).

- 1) We see that 1) in the definition of an HI-cell complex (Def. 2.28 p. 42) [Colloquially: "For each face of an HI-cell in a complex there exists a *corresponding* HI-cell in the complex"] is satisfied by 3) of Def. 4.9.
- 2) Now we show 2) of Def. 2.28. ["Two intersecting HI-cells in a complex intersect along a union of pairwise common faces"]

Let $\left(C_i^*, h_i^*:\overline{C_i^*} \to C_i^*\right)$ and $\left(C_j^*, h_j^*:\overline{C_j^*} \to C_j^*\right)$ be in $K^*$ and $C_i^* \cap C_j^* \neq \varnothing$. Let $\left\{\left(C_r, h_r:\overline{C_r} \to C_r\right)\right\}$ be the set of HI-cells in $K$ such that both $\left(C_i, h_i:\overline{C_i} \to C_i\right)$ and $\left(C_j, h_j:\overline{C_j} \to C_j\right)$are "faces" of $\left(C_r, h_r:\overline{C_r} \to C_r\right)$ but they are not "faces" of any proper face of $\left(C_r, h_r\right)$.

[There exists at least one such HI-cell $\left(C_r, h_r:\overline{C_r} \to C_r\right)$ otherwise $C_i^*$ and $C_j^*$ would be disjoint. It is also easy to see that if $K$ is an H-cell complex that there is exactly one such H-cell $(C_r, h_r)$. (Since in this case H-cells intersect at most along one common face.)]

CLAIM: $C_i^* \cap C_j^* = \bigcup_r C_r^*$.

Similarly to 0) above, we need only to show that $\overline{C_i}^*(\overline{C}) \cap \overline{C_j}^*(\overline{C}) = \overline{C_r}^*(\overline{C})$, where $\overline{C_K}$ are all cells and $\overline{C_k}^*(\overline{C})$ denotes the dual star of $\overline{C_k}$ in (the cell complex induced by) a single $n$- dimensional cell $\overline{C}$. That is, we have the following situation: $\overline{C_i}$ and $\overline{C_j}$ are faces of $\overline{C_r}$ (and not faces of a proper face of $\overline{C_r}$) and they are all faces of an $n$-dimensional cell $\overline{C}$.

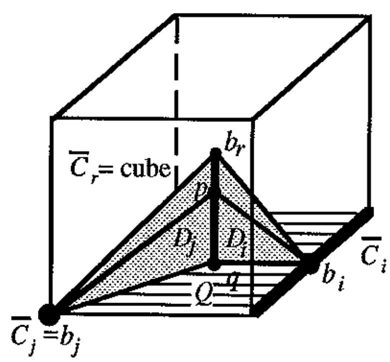
We will show that $\overline{C_i}^*(\overline{C}) \cap \overline{C_j}^*(\overline{C}) = \overline{C_r}^*(\overline{C})$ by showing that for any face $F$ of $\overline{C}$ with $\overline{C_i}$, $\overline{C_j}$ and $\overline{C_r}$ all faces of $F$, we have $\overline{C_i}^*(F) \cap \overline{C_j}^*(F) = \overline{C_r}^*(F)$. This will be shown by induction on the dimension of $F$.

Note that $\overline{C_r}$ is the unique face of $\overline{C}$ of minimal dimension containing both $\overline{C_i}$ and $\overline{C_j}$ , as observed in the parenthetical remark at the beginning of 2). We thus start the induction by showing: $\overline{C_i}^*(\overline{C_r}) \cap \overline{C_j}^*(\overline{C_r}) = \overline{C_r}^*(\overline{C_r})$, and observe that $\overline{C_r}^*(\overline{C_r}) = b_r$ where $b_r$ is the barycenter of $\overline{C_r}$.

By construction of the barycentric subdivision, it is immediate that $b_r \in \overline{C_i}^*(\overline{C_r}) \cap \overline{C_j}^*(\overline{C_r})$. Now assume that there is a point $p \neq b_r$ with $p \in \overline{C_i}^*(\overline{C_r}) \cap \overline{C_j}^*(\overline{C_r})$. There is a unique point $q$ with $q$ in a proper face $Q$ of $\overline{C_r}$ such that $p$ lies in the segment $b_r q$. Let $b_i$ and $b_j$ be the barycenters of $\overline{C_i}$ and $\overline{C_j}$ respectively. (See Fig. 34.) Since $p \in \overline{C_i}^*(\overline{C_r}) \cap \overline{C_j}^*(\overline{C_r})$ the segment $pb_i$ lies in a cell $D_i$ of $\overline{C_i}^*(\overline{C_r})$ and $pb_j$ lies in a cell $D_j$ of $\overline{C_j}^*(\overline{C_r})$. By construction of the barycentric subdivision, the cells $D_i$ and $D_j$ are cones with vertex $b_r$ and bases in $Q$. Thus the segments $qb_i$ and $qb_j$ are in $Q$, and therefore $Q$ intersects the interiors of $\overline{C_i}$ and $\overline{C_j}$, hence $\overline{C_i}$ and $\overline{C_j}$ are faces of $Q$. This is a contradiction.

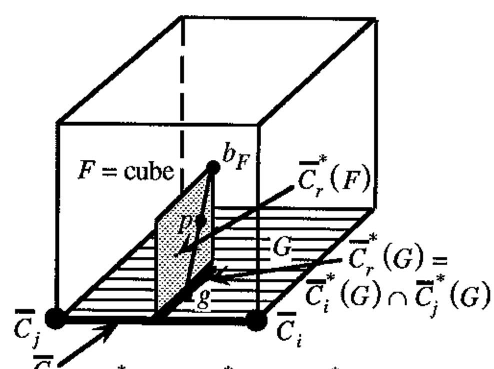
Now let $F$ be a face of $\overline{C}$ such that $\overline{C_i}$, $\overline{C_j}$ and $\overline{C_r}$ are all faces of $F$. Since $\overline{C}_r^*(F)$ does not intersect any proper face of $\overline{C}_r$, we see that $\overline{C}_r^*(F)$ is contained in both $T_{\overline{C}_i}$ and $T_{\overline{C}_j}$ (see 4.3 for definition). We thus have that the cone complexes $b_i\overline{C}_r^*(F)$ and $b_j\overline{C}_r^*(F)$ are subcomplexes of $\overline{C}_i^*(F)$ and $\overline{C}_j^*(F)$ respectively. We immediately obtain that $\overline{C}_r^*(F) \subset \left(\overline{C}_i^*(F) \cap \overline{C}_j^*(F)\right)$.

Now to show the reverse inclusion, we inductively assume that for any face $G$ of $\overline{C}$ with $\dim(G) < \dim(F)$ and with $\overline{C_i}$, $\overline{C_j}$ and $\overline{C_r}$ contained in $G$, that $\left(\overline{C}_i^*(G) \cap \overline{C}_j^*(G)\right) \subset \overline{C}_r^*(G)$. Let $p \in \left(\overline{C}_i^*(F) \cap \overline{C}_j^*(F)\right)$ and let $b_F$ denote the barycenter of $F$. $p$ lies in a segment $b_F g$ with $g$ in a proper face $G$ of $F$, and as done in the first step of the induction ($F = \overline{C}_r$), we see that $g$ lies in $\overline{C}_i^*(G)$ as well as in $\overline{C}_j^*(G)$ (See Fig. 35). By induction we obtain that $g \in \overline{C}_r^*(G)$ and therefore $b_F g$ (and thus $p$) lies in $\overline{C}_r^*(F)$. Hence, $\left(\overline{C}_i^*(F) \cap \overline{C}_j^*(F)\right) \subset \overline{C}_r^*(F)$. □



$\overline{C}_i$ and $\overline{C}_j$ are faces of $Q$.

Fig. 34



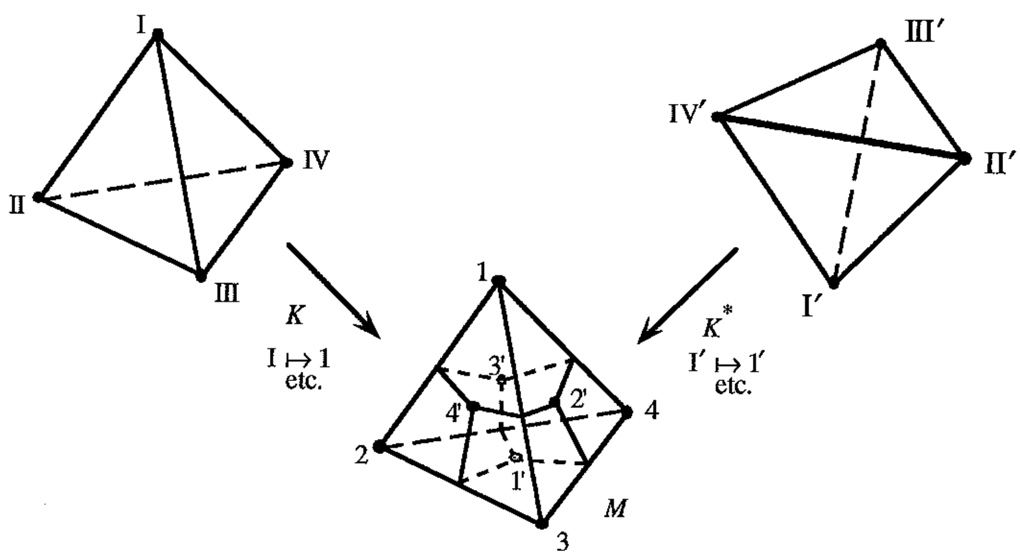
$\overline{C}_i^*(F) \cap \overline{C}_j^*(F) = \overline{C}_r^*(F)$.

Fig. 35

## SELF-DUAL HI-CELL COMPLEXES

**4.13** DEFINITION. Let $K$ be a dualizable HI-cell complex on a manifold $M$. The HI-cell complex $K^*$ on $M$ given by Proposition 4.12 is called the ***dual complex of K***.

The H-cell complex on the boundary of a cube consisting of 8 H-triangles, 12 H-edges and 6 vertices previously given in Figs. 10 and 12 (pp. 29, 33), is isomorphic to the cell complex consisting of the proper faces of an octahedron. This H-cell complex is the dual of the cell complex consisting of all the proper faces of the cube; thus it is usually said that "the octahedron is the dual of the cube". Interestingly, the dual H-cell complex of the cell complex $K$ consisting of the proper faces of a 3-simplex, is isomorphic to $K$ itself (see Fig. 36), and hence we say that "the tetrahedron is dual to itself".



The tetrahedron is dual to itself.

Fig. 36

**4.14** DEFINITION. Let $K$ be a dualizable HI-cell complex on a manifold $M$. A ***self-duality of K*** is an HI-isomorphism $\left(f, K, K^*\right)$. We say that $K$ is ***self-dual*** if there exists a self-duality of $K$.

Recalling 2.34, 2.35 and 2.36 (p.50, Def. of Isomorphic HI-cell Complexes) we observe that an HI-cell complex $K$ on an $n$-dimensional manifold $M$ is self-dual if there exists a PL-homeomorphism $f: M \to M$ such that for every $d$-dimensional HI-cell $\left(C_i . h_i : \overline{C}_i \to C_i\right)$ of $K$ we have:

1) $f_i(C_i) = C_j^*$ where $f_i$ is the restriction of $f$ to $C_i$ [and $\left(C_j . h_j : \overline{C}_j \to C_j\right)$ is an ($n$-$d$)-dimensional HI-cell of $K$.]
2) a linear homeomorphism $\bar{f}_i : \overline{C}_i \to \overline{C_j^*}$ such that the following commutes:

$$\begin{array}{ccc} \overline{C}_i & \xrightarrow{\bar{f}_i} & \overline{C_j^*} \\ \downarrow h_i & & \downarrow h_j^* \\ C_i & \xrightarrow{f_i} & C_j^* \end{array}$$

Note that the complex $K$ on the boundary of the 3-simplex given in Fig. 36 is self-dual: The PL homeomorphism $f: M \to M$ (given in the figure by $x \mapsto x'$) satisfies 1) and 2) of the definition, and is homotopic to the map given by reflection on the barycenter of the 3-simplex (the "antipodal" map).

We will define in 4.16 the notion of *structural self-duality* which is a weaker form of self-duality where the existence of the homeomorphism $f: M \to M$ (with the above stated properties) is not required; we only require the existence of local PL-homeomorphisms (in neighborhoods of the cells) which preserve the local HI-cellular *structure*.

We need first to make the following definition:

**4.15** DEFINITION. Let $K = \left\{ \left( C_i,\, h_i : \overline{C}_i \to C_i \right) \right\}$ be an HI-cell complex and $L$ a subcomplex of $K$, i.e. $L$ is an HI-cell complex and $L \subset K$. The ***HI-cellular neighborhood of L in K***, $N(L,K)$, is the subcomplex of $K$ defined by:

$$N(L,K) = \left\{ \left( C, h : \overline{C} \to C \right) \middle| (C,h) \text{ a face of } (B,g) \in K, B \cap |L| \neq \varnothing \right\}.$$

Observe that $(C_i,h_i) \in K$ and $\left(C_j^*,h_j^*\right) \in K^*$ can both be considered as subcomplexes of $K^{(2)}$, the second barycentric subdivision of $K$, and considering them as such, we let $N_i = N\left( (C_i,h_i), K^{(2)} \right)$ and $N_{j*} = N\left( \left(C_j^*,h_j^*\right), K^{(2)} \right)$. This notation will be used in the definition below.

**4.16** DEFINITION. Let $K = \left\{ \left( C_k,\, h_k : \overline{C}_k \to C_k \right) \right\}$ be a dualizable HI-cell complex on a manifold $M$ and let $K^* = \left\{ \left( C_k^*,\, h_k^* : \overline{C_k^*} \to C_k^* \right) \right\}$ be its dual complex.

A ***structural self-duality on K*** is a bijection

$$B : \left\{ \left( C_k,\, h_k : \overline{C}_k \to C_k \right) \right\} \to \left\{ \left( C_k^*,\, h_k^* : \overline{C_k^*} \to C_k^* \right) \right\}$$

such that if $\quad B\left( \left( C_i,\, h_i : \overline{C}_i \to C_i \right) \right) = \left( C_j^*,\, h_j^* : \overline{C_j^*} \to C_j^* \right) \quad$ then

1) $N_i$ and $N_{j*}$ are isomorphic HI-cell complexes (as subcomplexes of $K^{(2)}$) via a PL-homeomorphism $F_i : |N_i| \to |N_{j*}|$. (See Defs. 2.34 and 2.36, p. 50.)

2) There exists a linear homeomorphism $\bar{f}_i : \overline{C}_i \to \overline{C_j^*}$ such that the diagram

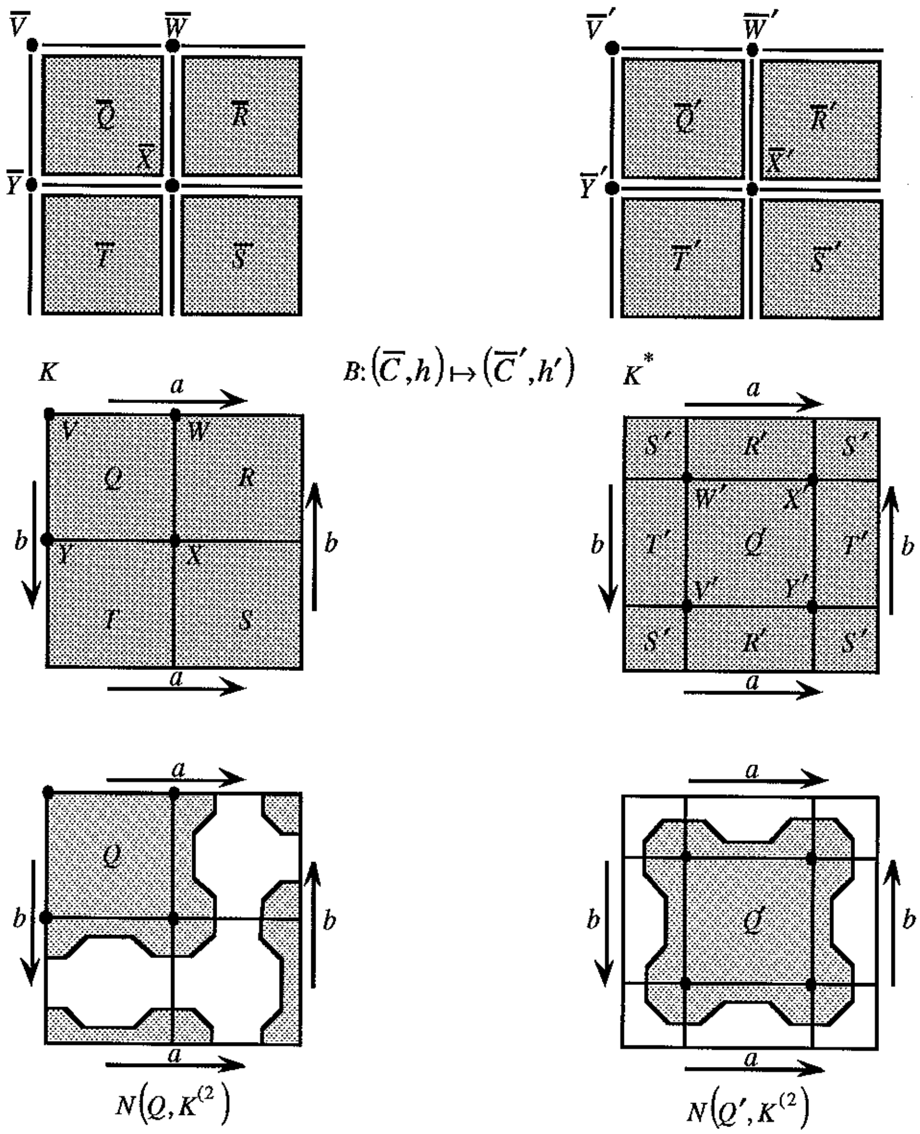
$$\begin{array}{ccc} \overline{C}_i & \xrightarrow{\bar{f}_i} & \overline{C_j^*} \\ \swarrow h_i & & h_j^* \searrow \\ C_i \subset |N_i| & \xrightarrow{F_i} & |N_{j*}| \supset C_j^* \end{array}$$

commutes.

We say that $K$ is ***structurally self-dual*** if there exists a structural self-duality on $K$.

**4.17** REMARK. Observe that a self-dual HI-cell complex is structurally self-dual: The bijection $B$ is obtained via the HI-isomorphism $\left(f,K,K^*\right)$ of Def. 4.14.

In figure 37 we give an example of a <u>structurally self-dual</u> HI-cell complex on the Klein bottle which is <u>not self-dual</u>. In this example, the image of $(C,h)$ under the bijection $B$ is denoted by $(C',h')$ (we have labeled only the vertices and the quadrilaterals).




*B* is a bijection of the HI-cells of *K* and $K^*$ respecting local cellular structures.
Fig. 37

Note that in the above example there is no homeomorphism $f: M \to M$ respecting the bijection $B$ since the quadrilateral $Q$ shares edges with $R$, $S$ and $T$, while $Q'$ only shares edges with $R'$ and $T'$. Note this situation is not unique to our indicated bijection $B$ since any quadrilateral in $K$ shares its edges with the other three, while any quadrilateral in $K^*$ shares its edges with only two other quadrilaterals.

**4.18** REMARK.

1) In the example given above in Fig. 37, and in all subsequent examples of manifolds (and other polyhedra) $M$ with a given HI-cellular structure $K$, which are obtained as quotient spaces of a given polyhedra $P$ with an HI-cellular structure $L$, we *implicitly assume* that there exists a (specific) PL map $\pi: P \to \mathbf{R}^m$ (which can be constructed by a generalization of the construction I. of theorem 3.29, p. 79) such that:

a) $\pi(P) = M$

b) for the first barycentric subdivision $L^{(1)}$ of $L$, $\pi$ possesses the properties stated for the function denoted by $f$ in the hypothesis of Proposition 2.31 (p. 45).

c) the HI-cell complex on $M$ induced by $L^{(1)}$ and $\pi$ (see Def. 2.32, p. 47.) is the first barycentric subdivision of $K$.

For example, in the HI-cell complex $K$ on the Klein bottle, depicted on the left side of Fig. 37, we are assuming that $M$ is a given polyhedral Klein bottle embedded in $\mathbf{R}^m$, $P$ is the quadrilateral depicted in the left of the middle row in

Fig.37. The (HI-) cell complex $L$ on $P$ has 9 vertices, 12 edges and 4 faces but the top and middle rows in the figure taken together, are a *representation* of $K$ (4 HI-vertices, 8 HI-edges and 4 HI-faces) on the Klein bottle $M$ (via the implicit $\pi$). (In Fig. 37 $\pi$ identifies cells of $L^{(1)}$ that "lie" in different edges labeled $a$ according to their placement in the order given by the arrows, and it does the same for cells in $b$; $\pi$ is injective in the rest of $P$. Similar conventions will be used in future depictions of polyhedra given by identifications of HI-cells in a polyhedron $P$.)

The reason for $\pi$ to identify HI-cells of $L^{(1)}$ and not of $L$ is apparent on the complex illustrated on the right side of Fig. 37: The edges of the (middle) quadrilateral do not lie along HI-cells of the depicted HI-cell complex but they lie along HI-cells of its first barycentric subdivision.

2) In Fig. 37 and in similar figures showing examples of structural self-duality (or self-duality), we take the linear map $\bar{f}_i:\overline{C_i} \to \overline{C_j^*}$ of Def. 4.12 (shown in the top row of the figure by $\bar{x} \mapsto \bar{x}'$) to be the identity $i_i:\overline{C_i} \to \overline{C_i}$, and the map $h_j^*:\overline{C_j^*} \to C_j^*$ is taken to be the composition $\overline{C_i} \xrightarrow{h_i} C_i \xrightarrow{F_i|_{C_i}} C_j^*$ (or the composition $\overline{C_i} \xrightarrow{h_i} C_i \xrightarrow{f_i} C_j^*$ in the case of self-duality). The map $F_i$ (or $f_i$) is also implicitly given, and the two bottom rows in the figure are an attempt to describe it, again via the identification map $\pi$.

# 5 Self-Duality on Surfaces

## Euler-Poincaré Characteristic

We recall below the (topological) notions of *attaching a finite number of disjoint n-disks to a topological space and* of *finite CW complex* [V. p.62]. Let $D^n = \left\{(x_i, x_2, \cdots, x_n) \in \mathbf{R}^n \middle| \sum x_i^2 \le 1\right\}$ and $S^{n-1} = \left\{(x_i, x_2, \cdots, x_n) \in \mathbf{R}^n \middle| \sum x_i^2 = 1\right\}$. $D^n$ is called the *n-disk* and $S^{n-1}$ is called the *round (n-1)-sphere*. Contrast this nomenclature with the one in Def. 4.5 and recall that in our metric $d$ on $\mathbf{R}^n$ (given on page 4), the disk $D^n$ is not the 1-neighborhood of $(0,0,\cdots,0) \in \mathbf{R}^n$. (See Def. 1.10, p. 9.)

**5.1** Definition. Let $D_1^n, D_2^n, \cdots, D_m^n$ be $m$ disjoint copies of $D^n$ and $S_1^{n-1}, S_2^{n-1}, \cdots, S_m^{n-1}$ be their respective round ($n$-1)-spheres (boundaries). Let $Y$ be a topological space and for $1 \le i \le m$ let $f_i : S_i^{n-1} \to Y$ be a continuous function. Let $\sim$ be the least equivalence relation on $D_1^n \cup D_2^n \cup \cdots \cup D_m^n \cup Y$ such that for $x \in S_i^{n-1}$ we have $x \sim f_i(x)$. Then the space $X = D_1^n \cup D_2^n \cup \cdots \cup D_m^n \cup Y / \sim$ is said to be ***obtained by attaching m n-disks to Y via*** $f_i, 1 \le i \le m$.

**5.2** Definition. A ***finite CW complex*** is a sequence $X^0 \subset X^1 \subset \cdots \subset X^n = X$ of closed subspaces of a compact Hausdorff space $X$ such that:

1) $X^0$ is a finite set of points
2) $X^k$ is obtained from $X^{k-1}$ by attaching a finite number of $k$-disks to $X^{k-1}$.

**5.3** PROPOSITION. Let $K = \left\{ \left( C_i, h_i : \overline{C}_i \to C_i \right) \right\}$ be an $n$-dimensional HI-cell complex. Then $\left|K^0\right| \subset \left|K^1\right| \subset \cdots \subset \left|K^n\right| = |K|$ is a finite CW complex where $K^i$ is the $i$-skeleton of $K$.

*Proof.* Every $\left|K^i\right|$ in the sequence $\left|K^0\right| \subset \left|K^1\right| \subset \cdots \subset \left|K^n\right| = |K| = X$ of the underlying polyhedra of the skeleta of $K$, is closed in $|K|$. Every $k$-cell $\overline{C}_i$ is PL homeomorphic to the unit $k$-cube, as shown in Prop. 4.6 (p. 90), therefore it is homeomorphic to the disk $D^k$. From the definition 2.26 (p. 40) of an HI-cell, together with the definition 2.28 (p. 42) of an HI-cell complex, we see that we can consider each polyhedron $\left|K^k\right|$ to be obtained from $\left|K^{k-1}\right|$ by attaching all the cells $\overline{C}_i$ of dimension $k$ [$\left( C_i, h_i : \overline{C}_i \to C_i \right) \in K$] via the maps $h_i$ restricted to the boundary of $\overline{C}_i$. □

Recall that for a topological space $X$, the ***i*-th *Betti number of X***, $b_i(X)$, is the rank of $H_i(X)$ [$H_i(X)$ is the $i$-th integral singular homology group of $X$] and that the ***Euler-Poincaré Characteristic of X***, $\chi(X)$, is the homotopy-invariant integer defined as $\chi(X) = \sum_i (-1)^i b_i(X)$.

**5.4** REMARK. If $X^0 \subset X^1 \subset \cdots \subset X^n = X$ is a finite CW complex obtained by attaching a number $d_i$ of disks of dimension $i$ to $X^{i-1}$ [and letting $d_0$ be the number of points in $X^0$], then it is well known [V, p. 73] that $\chi(X) = \sum_{i=0}^{n} (-1)^i d_i$.

**5.5** PROPOSITION. *Let* $K=\{(C_j, h_j:\overline{C}_j \to C_j)\}$ *be an n-dimensional HI-cell complex and let* $\alpha_i$ *denote the number of HI-cells* $(C_j, h_j:\overline{C}_j \to C_j) \in K$ *having dimension equal to i, then* $\chi(|K|)=\sum_{i=0}^{n}(-1)^i \alpha_i$.

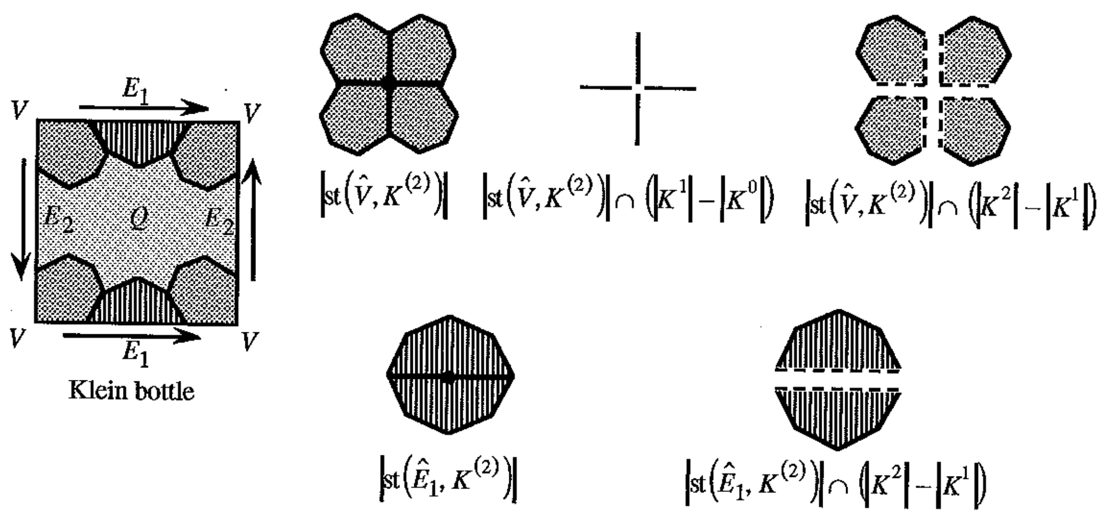
*Proof.* The result follows immediately from Prop. 5.3 and Remark 5.4. □

**VALENCE NUMBERS**

Recall that for an HI-cell complex $K$ and integer $i \geq 1$, $K^{(i)}$ denotes the *i-th barycentric subdivision of* $K$, and that for an integer $j \geq 0$, $K^j$ denotes the *j-skeleton of K.* Also recall that $\mathrm{st}(v,K)$ denotes the *star of a vertex v in K* given in 4.1 (p. 86), and that for $(C_i,h_i)$ in $K$, $\hat{C}_i$ denotes the *barycenter of* $C_i$. (3.24 p. 72).

**5.6** DEFINITION. Let $K=\{(C_i,h_i:\overline{C}_i \to C_i)\}$ be an HI-cell complex. For every $C_i \subset |K|$ and integer $m$ such that $\dim(C_i) < m \leq \dim(K)$ we define the ***m-valence number of*** $C_i$, $V_m(C_i)$, to be the number of connected components of the polyhedron $\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right| \cap \left(\left|K^m\right| - \left|K^{m-1}\right|\right)$.

**5.7** EXAMPLE In Fig. 38, the two drawings on the right side of the top row and the one on the right of the bottom row, show three polyhedra (out of possible four) of the form $\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right| \cap \left(\left|K^m\right| - \left|K^{m-1}\right|\right)$ for an HI-cell complex $K$ on the Klein bottle consisting of one vertex $V$, two HI- edges $E_1$ and $E_2$, and one HI-quadrilateral $Q$. Note for example that $V_1(V)=4$, since this is the number of components of the "cross" shown on the top of the figure, and observe that 4 is in a certain sense "the local number of edges having $V$ as a vertex" although there are only two 1-dimensional HI-cells in $K$. Similarly, $V_m(C_i)$ is the "local number of $m$-dimensional HI-cells having $C_i$ as a proper face".




$V_1(V) = 4$, $V_2(V) = 4$ and $V_2(E_1) = 2$. (Not shown, $V_2(E_2) = 2$. )

Fig. 38

**5.8** REMARK.

1) If $K$ is an H-cell complex, it can be easily proved that the $m$-valence number of $C_i$ is the number of $m$-dimensional cells of $K$ that contain $C_i$ as a proper face.

2) If $K$ is a dualizable HI-cell complex with $|K|$ an $n$-manifold and if $P$ is a vertex in $K$, then the 1-valence number of $P$, $V_1(P)$, is equal to the number of ($n$-1)-dimensional faces (facets) of the $n$-cell $\overline{V^*}$, where $\left(\overline{V^*}, h^*\right) \in K^*$ is the dual HI-cell of $(V,h)$. [See 3) of Def. 4.9, p. 92.]

3) It can be shown, by induction on the dimension, that an $n$-cell has at least $n$+1 faces of dimension $n$-1. Thus for any vertex $P$ of a dualizable HI-cell complex on an $n$-manifold, we obtain by 2) that $V_1(P) \geq n+1$.

**5.9** PROPOSITION. *Let B be a structural self-duality on an HI-cell complex K. Let* $\left(C_i, h: \overline{C_i} \to C_i\right) \in K$ *and let* $B\left(\left(C_i,\, h_i: \overline{C_i} \to C_i\right)\right) = \left(C_j^*,\, h_j^*: \overline{C_j^*} \to C_j^*\right)$.

*Then for all* $m$, *such that* $\dim(C_i) < m \leq \dim(K)$, *we have* $V_m(C_i) = V_m\left(C_j^*\right)$.

[ The valence number $V_m\left(C_j^*\right)$ is computed using the skeleta of $K^*$, i.e. $V_m\left(C_j^*\right)$ is the number of components of $\left|\mathrm{st}\left(\hat{C}_j^*, K^{(2)}\right)\right| \cap \left(\left|K^{*m}\right| - \left|K^{*m-1}\right|\right)$. ]

*Proof.* The polyhedron $\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right| \cap \left(\left|K^m\right| - \left|K^{m-1}\right|\right)$ used in the definition of $V_m(C_i)$ is the underlying space of a subcomplex of $N_i = N\left(\left(C_i, h_i\right), K^{(2)}\right)$, the HI-cellular neighborhood of $\left(C_i, h_i\right)$ in the second barycentric subdivision of K. (See Defs. 4.15 and 4.16, p. 100. )

Let $N_{j*} = N\left(\left(C_j^*, h_j^*\right), K^{(2)}\right)$. Since $B\left(\left(C_i,\, h_i: \overline{C_i} \to C_i\right)\right) = \left(C_j^*,\, h_j^*: \overline{C_j^*} \to C_j^*\right)$ and $K$ is structurally self-dual, we have (by Def. 4.16):

1) $N_i$ and $N_{j*}$ are isomorphic HI-cell complexes (as subcomplexes of $K^{(2)}$) via a PL-homeomorphism $F_i: \left|N_i\right| \to \left|N_{j*}\right|$.

2) There exists a linear homeomorphism $\bar{f}_i: \overline{C_i} \to \overline{C_j^*}$ such that the diagram

$$\begin{array}{ccccccc} & & \overline{C_i} & \xrightarrow{\bar{f}_i} & \overline{C_j^*} & & \\ & \swarrow^{h_i} & & & & \searrow^{h_j^*} & \\ C_i \subset & |N_i| & & \xrightarrow{F_i} & & |N_{j*}| & \supset C_j^* \end{array}$$

commutes.

From the definition of $K^*$ (Def. 4.9, p. 92) and Prop. 4.12 (p. 94), one sees that the homeomorphism $F_i$ maps $\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right| \cap \left(\left|K^m\right| - \left|K^{m-1}\right|\right)$ homeomorphically onto $\left|\mathrm{st}\left(\hat{C}_j^*, K^{(2)}\right)\right| \cap \left(\left|K^{*m}\right| - \left|K^{*m-1}\right|\right)$. Hence, $V_m(C_i) = V_m\left(C_j^*\right)$. □

**5.10** NOTATION. For a cell $\overline{C}$ and integer $i$, $0 \le i \le \dim(\overline{C})$, let $\Delta_i(\overline{C})$ denote the number of $i$-dimensional faces of $\overline{C}$.

**5.11** PROPOSITION. *Let $K = \{(C_i, h_i{:}\overline{C_i} \to C_i)\}$ be an HI-cell complex. Let $\alpha_d$ denote the number of d-dimensional HI-cells in K, and to emphasize the dimension, let $\left\{\left(C_j^d, h_j^d{:}\overline{C_j^d} \to C_j^d\right)\right\}_{1 \le j \le \alpha_d}$ denote the set of d-dimensional HI-cells in K. Then for integers $0 \le d < m \le \dim(K)$, the sum of the m-valences of all of the d-dimensional polyhedra $C_j^d \subset |K|$ is equal to the sum of the numbers of d-dimensional faces in all the (covering) m-dimensional cells $\overline{C_k^m}$. i.e. :*

$$\sum_{j=1}^{\alpha_d} V_m\left(C_j^d\right) = \sum_{k=1}^{\alpha_m} \Delta_d\left(\overline{C_k^m}\right).$$

*Proof.*

**I.** Let $\overline{K}$ be the (covering) <u>cell</u> complex given by $\overline{K} = \bigcup_{\text{disjoint}} \overline{C_i}$ (and all of their faces) and let $H{:}|\overline{K}| \to |K|$ be the PL map $H = \cup(h_i{:}\overline{C_i} \to C_i)$. Observe that the second barycentric subdivision of $K$, $K^{(2)}$, is the HI-cell complex induced (see Def. 2.32) by $\overline{K}^{(2)}$, the second barycentric subdivision of $\overline{K}$, and the map $H$, as an analysis of Construction 3.15 (p. 63) and Def. 3.24 (p. 72) will reveal.

**II.** Also note that since each map $h_i{:}\overline{C_i} \to C_i$ is a PL homeomorphism in the <u>interior</u> of $\overline{C_i}$, and the cells $\overline{C_i}$ are disjoint, we see that $V_m(C_i)$, i.e. the number of connected components of $\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right| \cap \left(|K^m| - |K^{m-1}|\right)$, is the number of connected components of $H^{-1}\left(\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right|\right) \cap \left(\bigcup_{k=1}^{\alpha_m} \overline{C_k^m}\right)$.

**III.** Now let $\left\{\hat{\overline{C}}_{i,s}\right\}_{1\le s\le s(i)} = H^{-1}\left(\hat{C}_i\right)$, i.e. the set of pre-images of the barycenter of $C_i$. Using I. above, we see that $H^{-1}\left(\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right|\right) \cap \left(\bigcup_{k=1}^{\alpha_m} \overline{C_k^m}\right)$ is the disjoint union: $\bigcup_{s=1}^{s(i)} \left|\mathrm{st}\left(\hat{\overline{C}}_{i,s}, \left(\overline{K}^m\right)^{(2)}\right)\right|$ where $\overline{K}^m$ denotes the second barycentric subdivision of the $m$-skeleton of $\overline{K}$.

Combining II. and III. and letting $\#(X)$ stand for "the number of connected components of $X$" we obtain:

$$(*) \quad V_m(C_i) = \#\left(H^{-1}\left(\left|\mathrm{st}\left(\hat{C}_i, K^{(2)}\right)\right|\right) \cap \left(\bigcup_{k=1}^{\alpha_m} \overline{C_k^m}\right)\right) = \#\left(\bigcup_{s=1}^{s(i)} \left|\mathrm{st}\left(\hat{\overline{C}}_{i,s}, \left(\overline{K}^m\right)^{(2)}\right)\right|\right)$$

Since $\left|\left(\overline{K}^m\right)^{(2)}\right| = \bigcup_{k=1}^{\alpha_m} \overline{C_k^m}$ and the cells $\overline{C_k^m}$ are disjoint, it is enough to prove the theorem for the case of an HI-cell complex $L$ consisting of a <u>single</u> $m$-dimensional HI-cell $\left(C^m, h : \overline{C}^m \to C^m\right)$ (together with a subset of its faces).

We thus need to prove:

**IV.** LEMMA. *Let* $\left(C^m, h : \overline{C}^m \to C^m\right)$ *be an m-dimensional HI-cell and let L be the canonical HI-cell complex on* $C^m$ *given by Prop. 2.33 (p. 48).*
*Then if* $\left\{\left(C_j^d, h_j^d : \overline{C_j^d} \to C_j^d\right)\right\}_{1\le j\le \alpha_d}$ *represents the set of d-dimensional HI-cells in L we have:* $\sum_{j=1}^{\alpha_d} V_m\left(C_j^d\right) = \Delta_d \overline{C}^m$.

[ Refer again to Fig. 38, p. 107, where it can be seen that $V_2(V) = 4 = \Delta_0\left(\overline{Q}\right)$ and $V_2(E_1) + V_2(E_2) = 4 = \Delta_1\left(\overline{Q}\right)$. ]

To see that the lemma is true we first observe that for the canonical HI-cell complex $L$ on an $m$-dimensional HI-cell $(C^m, h)$ as in the lemma, we get from $(*)$ the following result:

$$(**)\qquad \sum_{j=1}^{\alpha_d} V_m\left(C_j^d\right) = \#\left(\bigcup_{j=1}^{\alpha_d}\left(\bigcup_{s=1}^{s(j)}\left|\operatorname{st}\left(\hat{\overline{C}}_{j,s}, \left(\overline{C^m}\right)^{(2)}\right)\right|\right)\right)$$

Let $\left\{F_u^d\right\}_{u=1,\cdots,\Delta_d\left(\overline{C^m}\right)}$ denote the set of all the $d$-dimensional faces of $\overline{C^m}$, and observe that the union of all the $d$-dimensional faces of $\overline{C}^m$ maps onto the union of the polyhedra $C_j^d$. Thus, $\bigcup_{j=1}^{\alpha_d} H^{-1}\left(\hat{C}_j^d\right) = \bigcup_{j=1}^{\alpha_d}\left(\bigcup_{s=1}^{s(j)} \hat{\overline{C}}_{j,s}\right)$, the union of the pre-images of the barycenters of $C_j^d$, is equal to the set of the barycenters of the $d$-faces of $\overline{C}^m$. We thus obtain:

$$\bigcup_{j=1}^{\alpha_d}\left(\bigcup_{s=1}^{s(j)} \hat{\overline{C}}_{j,s}\right) = \bigcup_{u=1}^{\Delta_d\left(\overline{C^m}\right)} \hat{F}_u^d$$

Therefore from $(**)$ we obtain:

$$(***)\qquad \sum_{j=1}^{\alpha_d} V_m\left(C_j^d\right) = \#\left(\bigcup_{u=1}^{\Delta_d\left(\overline{C^m}\right)}\left|\operatorname{st}\left(\hat{F}_u^d, \left(\overline{C^m}\right)^{(2)}\right)\right|\right)$$

and thus, to finish the proof of the lemma we must show that the number of connected components of $\bigcup_{u=1}^{\Delta_d\left(\overline{C^m}\right)}\left|\operatorname{st}\left(\hat{F}_u^d, \left(\overline{C^m}\right)^{(2)}\right)\right|$ is equal to $\Delta_d\left(\overline{C^m}\right)$.

Since each polyhedron $\left|\operatorname{st}\left(\hat{F}_u^d, \left(\overline{C^m}\right)^{(2)}\right)\right|$ is compact and connected, (it is a finite union of simplices all of which intersects the barycenter), we need only to show:

*For* $v \neq w$, $\left|\operatorname{st}\left(\hat{F}_v^d, \left(\overline{C^m}\right)^{(2)}\right)\right|$ *and* $\left|\operatorname{st}\left(\hat{F}_w^d, \left(\overline{C^m}\right)^{(2)}\right)\right|$ *are disjoint.*

(See Fig. 38, p. 107.)

From the definition of the second barycentric subdivision we see that

$$\left|\operatorname{st}\left(\hat{F}_v^d,\left(\overline{C^m}\right)^{(2)}\right)\right| \subset \operatorname{int}\left|\operatorname{st}\left(\hat{F}_v^d,\left(\overline{C^m}\right)^{(1)}\right)\right|$$

and
$$\left|\operatorname{st}\left(\hat{F}_w^d,\left(\overline{C^m}\right)^{(2)}\right)\right| \subset \operatorname{int}\left|\operatorname{st}\left(\hat{F}_w^d,\left(\overline{C^m}\right)^{(1)}\right)\right|$$

From the definition of the first barycentric subdivision, we obtain that both of the above stars on the right intersect either only on the barycenter of $\overline{C^m}$, or along a cone $\overline{C^m}F$, where $F$ is a common face of $F_v^d$ and $F_w^d$. Thus in both cases the stars on the right intersect only on a subpolyhedron of their boundaries. Therefore the stars on the left are disjoint since they lie in the interiors of those on the right.

□

**5.12** REMARK. It is interesting to note that each number $V_m\left(C_j^d\right)$ depends on how the images of the $m$-dimensional cells are attached to $C_j^d$ (i.e. it depends on the local PL topology of certain transversal neighborhoods). However Prop. 5.11 shows that the sum $\sum_{j=1}^{\alpha_d} V_m\left(C_j^d\right)$ depends only on the disjoint set of linear isomorphism classes of the $m$-cells $\overline{C_k^m}$—not on the attaching HI-cell maps $h_k^m$.

**5.13** PROPOSITION. *Let $K=\left\{\left(C_i,\, h_i\colon \overline{C_i} \to C_i\right)\right\}$ be an HI-cell complex. Let $\alpha_d$ denote the number of d-dimensional HI-cells in K, and let $\left\{\left(C_{i_d}^d, h_{i_d}^d\colon \overline{C_{i_d}^d} \to C_{i_d}^d\right)\right\}_{1\le i_d\le \alpha_d}$ denote the set of d-dimensional HI-cells in K.*

*Then* $$\sum_{i_0=1}^{\alpha_0} V_m\left(C_{i_0}^0\right) - \sum_{i_1=1}^{\alpha_1} V_m\left(C_{i_1}^1\right) + \sum_{i_2=1}^{\alpha_2} V_m\left(C_{i_2}^2\right) - \cdots + (-1)^{m-1} \sum_{i_{m-1}=1}^{\alpha_{m-1}} V_m\left(C_{i_{m-1}}^{m-1}\right)$$

*is equal to $2\alpha_m$ when m is odd, and equal to 0 when m is even.*

*Proof.* Since by definition a cell is a convex polyhedron, it is contractible. Therefore a cell has the same homology as a point; hence its Euler-Poincaré characteristic is equal to 1.

We see then that the polyhedron $\left|\overline{K}^m\right| = \text{disjoint} \bigcup_{i_m=1}^{\alpha_m} \overline{C_k^m}$ has thus Euler-Poincaré characteristic, $\chi$, equal to $\alpha_m$.

Using Prop. 5.5 to compute $\chi$ we obtain:

$$\sum_{i_m=1}^{\alpha_m} \Delta_o\left(\overline{C_{i_m}^m}\right) - \sum_{i_m=1}^{\alpha_m} \Delta_1\left(\overline{C_{i_m}^m}\right) + \sum_{i_m=1}^{\alpha_m} \Delta_2\left(\overline{C_{i_m}^m}\right) - \cdots + (-1)^{m-1} \sum_{i_m=1}^{\alpha_m} \Delta_{m-1}\left(\overline{C_{i_m}^m}\right) + (-1)^m \alpha_m = \alpha_m$$

which by using Prop. 5.10 becomes:

$$\sum_{i_0=1}^{\alpha_0} V_m\left(C_{i_0}^0\right) - \sum_{i_1=1}^{\alpha_1} V_m\left(C_{i_1}^1\right) + \sum_{i_2=1}^{\alpha_2} V_m\left(C_{i_2}^2\right) - \cdots + (-1)^{m-1} \sum_{i_{m-1}=1}^{\alpha_{m-1}} V_m\left(C_{i_{m-1}}^{m-1}\right) + (-1)^m \alpha_m = \alpha_m$$

Thus:

$$\sum_{i_0=1}^{\alpha_0} V_m\left(C_{i_0}^0\right) - \sum_{i_1=1}^{\alpha_1} V_m\left(C_{i_1}^1\right) + \sum_{i_2=1}^{\alpha_2} V_m\left(C_{i_2}^2\right) - \cdots + (-1)^{m-1} \sum_{i_{m-1}=1}^{\alpha_{m-1}} V_m\left(C_{i_{m-1}}^{m-1}\right) = (-1)^{m+1} \alpha_m + \alpha_m$$

which gives the desired result. □

**5.14** COROLLARY. *Let $\{P_0, P_1, \cdots, P_{\alpha_0}\}$ be the set of "vertices" of an HI-cell complex K, then $\sum_{i=1}^{\alpha_0} V_1(P_i) = 2\alpha_1$, where $\alpha_1$ denotes the number of 1-dimensional HI-cells in K.*

*Proof.* This follows trivially from Prop. 5.13. □

## Homogeneous HI-Cell Complexes

**5.15** DEFINITION. Let $K$ be an HI-cell complex whose underlying polyhedron $|K|$ is an $n$-dimensional manifold. We say that $K$ is ***homogeneous*** if for any two $n$-dimensional HI-cells $\left(C_i, h_i : \overline{C}_i \to C_i\right)$ and $\left(C_j, h_j : \overline{C}_j \to C_j\right)$ in $K$ there exists a pair of maps $\left(\bar{I}_{ij}, I_{ij}\right)$, where $\bar{I}_{ij} : \overline{C}_i \to \overline{C}_j$ is a linear homeomorphism and $I_{ij} : C_i \to C_j$ is a PL homeomorphism, such that the diagram below commutes:

$$\begin{array}{ccc} \overline{C}_i & \xrightarrow{\bar{I}_{ij}} & \overline{C^*_j} \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h^*_j} \\ C_i & \xrightarrow{I_{ij}} & C^*_j \end{array}$$

Observe that the self-dual HI-cell complex on the 2-sphere given in Fig. 36 (p. 98) as well as the structurally self-dual HI-cell complex on the Klein bottle given in Fig. 37 (p. 101) are both homogeneous. Generalizing the above mentioned classical construction on the 2-sphere, we will show in this chapter that any orientable surface (an orientable 2-manifold) is the underlying polyhedron of a homogeneous self-dual HI-cell complex. We will in fact show that any orientable surface of genus 2 or larger can possess pentagonal, hexagonal and octagonal homogeneous self-dual HI-cellular structures, and that these are the only "types" of HI-polygons in any homogeneous self-dual HI-cell complex whose underlying polyhedron is any given surface of genus $g \geq 2$. (If we restrict the genus, $g$, by $g \geq k, k > 2$, then additional polygonal shapes as above exist.)

**5.16** PROPOSITION. *Let $K$ be a homogeneous structurally self-dual HI-cell complex on the n-manifold $|K|$, and let $\left(P, h: \overline{P} \to P\right)$ be a vertex in K. Then the 1-valence number of P, $V_1(P)$, is equal to the number of (n-1)-dimensional faces of any n-dimensional cell $\overline{C}$ such that $\left(C, h: \overline{C} \to C\right) \in K$.* [Thus in this situation, the function $V_1: K^0 \to \mathbf{Z}_+$ is constant. ($\mathbf{Z}_+$denotes the set of non-negative integers.)]

*Proof.* From Remark 5.8 3) we know that $V_1(P)$ is the number of $(n\text{-}1)$-dimensional faces of the $n$-cell $\overline{P^*}$, where $\left(\overline{P^*}, h^*\right)$ is the dual HI-cell of $(P, h)$.

Since $K$ is structurally self dual, by 2) of Def. 4.16 (p. 100), there exists an $n$-dimensional HI-cell $\left(C_i, h_i: \overline{C_i} \to C_i\right) \in K$ with the property that there exists a linear homeomorphism $\bar{f}_i: \overline{C_i} \to \overline{P^*}$; and thus, since $K$ is homogeneous, $\overline{P^*}$ is linearly homeomorphic to any $n$-dimensional cell $\overline{C}$ with $\left(C, h: \overline{C} \to C\right) \in K$. Since linearly homeomorphic cells are isomorphic when viewed as cell complexes (as in Prop. 2.20, p. 30), the result follows. □

## STRUCTURAL SELF-DUALITY ON SURFACES

### 5.17 NECESSARY CONDITIONS

Let $K$ be a homogeneous structurally self-dual HI-cell complex. Let $\alpha_d$ denote the number of $d$-dimensional HI-cells of $K$ and let $v$ denote the (constant) valence number $V_1(P)$, where $P$ is a vertex of $K$. (See Prop. 5.16.)

Let $\Sigma$ be a PL 2-manifold and let $\chi$ denote the Euler-Poincaré characteristic of $\Sigma$. The following are three necessary conditions required for the existence of a homogeneous structurally self-dual HI-cell complex $K$ with $|K| = \Sigma$.

**1.** $\alpha_0 - \alpha_1 + \alpha_2 = \chi$

**2.** $\alpha_0 = \alpha_2$

**3.** $v\alpha_0 = 2\alpha_1$

Note: **1.** is Prop. 5.5. (p. 106) **2.** follows from the existence of the bijection $B$ (structural self-duality) of Def. 4.16 (p. 100) **3.** follows from Corollary 5.14 and Prop. 5.16.

Performing Gaussian elimination, we get the following:

$$(\mathbf{M}_1) \qquad \begin{matrix} \alpha_1 & \alpha_1 & \alpha_2 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & \chi \\ 1 & 0 & -1 & 0 \\ v & -2 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -\chi \\ 0 & 0 & v-4 & -2\chi \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & {}^{-2\chi}\!/_{v-4} \\ 0 & 1 & 0 & {}^{-v\chi}\!/_{v-4} \\ 0 & 0 & 1 & {}^{-2\chi}\!/_{v-4} \end{array}\right)$$

We analyze first the case of the orientable surfaces.

Let $\Sigma_g$ denote a PL surface of genus $g$; i.e. $\Sigma_g$ is a "sphere with $g$ handles". Since the 2-sphere has characteristic equal to 2 and since to attach a handle we must remove two 2-dimensional cells and "glue" a cylinder which has characteristic equal to zero, along 1-spheres, also of characteristic equal to zero, we see that the Euler-Poincaré characteristic is reduced by 2 for each handle attached and thus $\chi(\Sigma_g) = 2 - 2g$. Using this, from the last matrix in $(\mathrm{M}_1)$ we get:

$$(\mathbf{M}_2) \qquad \begin{matrix} \alpha_0 & \alpha_1 & \alpha_2 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & {}^{4g-4}\!/_{v-4} \\ 0 & 1 & 0 & {}^{-v(2-2g)}\!/_{v-4} \\ 0 & 0 & 1 & {}^{4g-4}\!/_{v-4} \end{array}\right)$$

**SELF-DUALITY ON SURFACES: Genus $g$=0–The Sphere**

**5.18** THEOREM. *The only (up to HI-isomorphism) homogeneous self-dual HI-cellular structure K on the 2-sphere, $\Sigma_0$, is the one given by the canonical cell complex on the boundary of a 3-simplex. (See Fig. 36, p. 98.)*

*Proof.* We have shown the self-duality of the above complex $K$ on page 96.

Substituting $g = 0$ in the matrix $M_2$ of 5.17, we obtain for any homogeneous self-dual HI-cell complex $L$ on $\Sigma_0$ the following conditions:

$$\alpha_0 = {}^{-4}\!/_{\nu-4} \qquad \alpha_1 = {}^{-2\nu}\!/_{\nu-4} \qquad \alpha_2 = {}^{-4}\!/_{\nu-4}.$$

Since by 3) of 5.8 we have that $\nu \geq 3$, and since the numbers $\alpha_i$ are positive integers, we conclude that $\nu = 3$. Hence, we obtain: $\alpha_0 = 4 \quad \alpha_1 = 6 \quad \alpha_2 = 4$.

Since $\nu = 3$, the 2-dimensional (covering) cells of $L$ are triangles.

Since $6 = \mathbf{C}_2^4$ is the maximal number of 1-dimensional edges possible in a cell complex having four vertices, and likewise $4 = \mathbf{C}_3^4$ is the maximal number of triangles in such a complex, the complex on the boundary of a 3-simplex (being uniquely maximal) is the unique H-cell complex satisfying the above conditions.

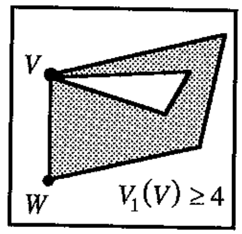
We will now show (by contradiction) that an HI-cell complex $L$ as above is an H-cell complex, thus proving the theorem. With this in mind, we assume:

**I.** *There exists a covering cell $\overline{C}$ in L such that (at least) two faces of $\overline{C}$ are identified.* Since every HI-cell in $L$ is a face of a triangle, it suffices to assume: *There exists a covering triangle $\bar{t}$ in L such that (at least) two faces of $\bar{t}$ are identified.*

The possible cases are:

**a.** *Exactly two vertices of the covering triangle $\bar{t}$ are identified but no edges of $\bar{t}$ are identified.*

Let $V$ denote the common vertex in the canonical complex $T$ induced by the HI-triangle having domain $\bar{t}$, and let $W$ denote the other vertex. Let $V_1(P,T)$ denote the 1-valence of a vertex $P$ computed in the complex $T$. Since $V_1(V,T)+V_1(W,T)=3\cdot 2=6$ by Corollary 5.14, (p. 113) and $V_1(W,T)=2$ (because the HI-triangle is a local homeomorphism near $W$) we obtain: $V_1(V,T)=4$. Hence $V_1(V)\geq 4$ [$V_1(V)$ is the valence in $L$ ]. This is impossible since $v=3$.

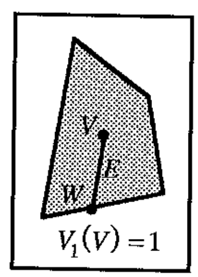



**b.** *Three vertices of the covering triangle $\bar{t}$ are identified but no edges of $\bar{t}$ are identified.*

Similarly to a. we obtain that $V_1(V)\geq 6$ and this is again a contradiction.

**c.** *Two edges of the triangle $\bar{t}$ are identified.*

Again let $T$ denote the canonical HI-cell complex on the HI-triangle $\bar{t}$. Let $E$ denote the (image of the) HI-edge of $T$ corresponding to the two identified edges of the triangle $\bar{t}$, and let $V$ denote the image of the vertex $\overline{V}$ of $\bar{t}$ belonging to the two identified edges. Since the link of $\overline{V}$ (in the second barycentric subdivision of $\bar{t}$) is mapped to a 1-sphere in the 2-sphere $\Sigma_0$, the star of $\overline{V}$ in $\bar{t}$ is mapped to a 2-ball having $V$ in its interior. Since $\Sigma_0$ is a manifold and is the underlying polyhedron of the HI-cell complex $L$, $V$ cannot be the face of any other HI-edge of $L$, nor can $E$ be a loop. Therefore $V_1(V)=1$. This contradicts $V_1(V)=v=3$.



**d.** *Three edges of the triangle $\bar{t}$ are identified.*

In this case there is only one HI-edge $(E,h:\bar{E}\to E)$ in $T$. Thus $V_2(E,T)=3$ by Proposition 5.11 (p.109) and therefore $V_2(E)\geq 3$. Since the underlying space of $L$ is the compact 2-manifold $\Sigma_0$, the 2-valence (in $L$) of any HI-edge of $L$ must be 2. This contradicts the above. (d. also follows from c.)

We conclude from the results obtained in part I. above that all the covering cells in $L$ are mapped homeomorphically onto their images.

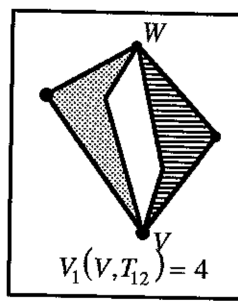
We now assume (reducing the analysis as in I. only to the triangular cells):

**II.** *Every covering triangle is mapped homeomorphically onto its image and there exist two HI-triangles $(t_1,h_1:\bar{t}_1\to t_1)$ and $(t_2,h_2:\bar{t}_2\to t_2)$ in L such that $t_1\cap t_2\neq\varnothing$ is not a common face of both $t_1$ and $t_2$.*

**a.** *$t_1\cap t_2=\{V,W\}$ where V and W are the images of two vertices $(V,h:\bar{V}\to V)$ and $(W,h:\bar{W}\to W)$ in L.*

Let $T_{12}$ be the HI-cell complex obtained by the union of the canonical complexes induced by the HI-triangles $(t_1,h_1:\bar{t}_1\to t_1)$ and $(t_2,h_2:\bar{t}_2\to t_2)$.

There are 6 HI-edges in $T_{12}$, hence the sum of the 1-valences (computed in $T_{12}$) of all the vertices in $T_{12}$ is equal to 12. Note also that there are (exactly) 4 HI-vertices in $T_{12}$ (since there are no identifications in any of the triangles). Since for a vertex $P$ in $T_{12}$ other than $V$ or $W$ we have $V_1(P,T_{12})=2$, we obtain $V_1(V,T_{12})+V_1(W,T_{12})=8$. This contradicts the fact that the 1-valence of any vertex in $L$ is equal to 3.



**b.** $t_1 \cap t_2 = \{U, V, W\}$ *where* $U$*,* $V$ *and* $W$ *are the images of three vertices as in a.*

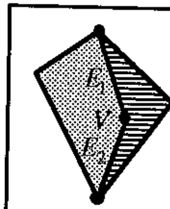
In this situation there are 6 edges and only three vertices in $T_{12}$. Thus $V_1(U,T_{12}) + V_1(V,T_{12}) + V_1(W,T_{12}) = 12$ which contradicts $v = 3$.

**c.** $t_1 \cap t_2 = \{V, E\}$ *where* $V$ *is a vertex and* $E$ *is an edge and* $V$ *and* $E$ *are disjoint.*

In this situation there are only 3 HI-vertices and 5 HI-edges in $T_{12}$. Therefore the 1-valence in $T_{12}$ of one of the vertices must be at least equal to 4. Again this is impossible.

**d.** $t_1 \cap t_2 = \{E_1, E_2\}$ *where* $E_1, E_2$ *are edges.*

Similarly to I. c. the vertex $V$ belonging to $E_1$ and to $E_2$ is in the interior of a 2-ball. Therefore it cannot be the image of a vertex of any other HI-edge of $L$. Hence $V_1(V,T_{12}) = V_1(V)$, but since $V_1(V,T_{12}) = 2$, we again obtain a contradiction.




$V_1(V,T_{12}) =$
$V_1(V) = 2$

**e.** $t_1 \cap t_2 = \{E_1, E_2, E_3\}$ *where* $E_1, E_2$*, and* $E_3$ *are edges.*

Exactly as in d. we obtain $V_1(V,T_{12}) = V_1(V) = 2$ for any vertex $V$ in $T_{12}$.

We now conclude from I. and II. that a homogenous self dual HI-cell complex $L$ on the 2-sphere must be an H-cell complex. Since we have shown at the beginning of this proof that such an H-cell complex is isomorphic to the canonical cell complex consisting of all the proper faces of a 3-dimensional simplex, we have proved the theorem. □

**SELF-DUALITY ON SURFACES: Genus $g \geq 1$**

Recall the usual topological presentation of a surface $\Sigma_g$ of genus $g$, $(g \geq 1)$ by identifying pairs of edges on the $4g$-gon as indicated in Fig. 39 for $\Sigma_2$.

Note our labels, $s_i$, $h_i$, in the $4g$-gon. They help visualize the construction of a sphere (by identifying the $s_i$) which has $2g$ holes( or $g$ handles) (via $h_i$) (See the right side of Fig. 39).

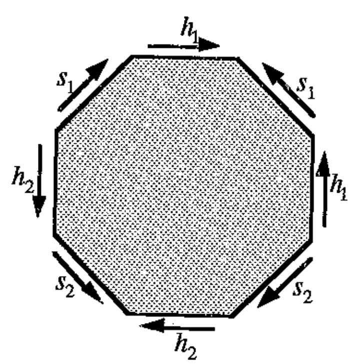
Also recall that Theorem 3.29 (p. 79) guarantees the existence of HI-cells $\left(C, h{:}\overline{C} \to C \subset \mathbf{R}^N\right)$ with $C$ homeomorphic to the quotient space obtained by identifying linearly homeomorphic faces of a cell $\overline{C}$. Thus in view of this, we take $\Sigma_g$ to be PL and the above presentation to be an HI-cell (with image $\Sigma_g$).

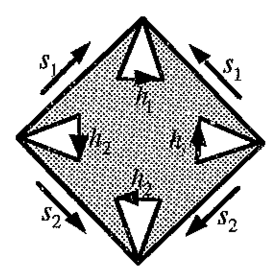
Letting $v = 4g$, in the matrix $M_2$ (page 116) we obtain the solution:

$$\alpha_0 = 1,\ \ \alpha_1 = 2g\ \text{ and }\ \alpha_2 = 1.$$

We see that this solution corresponds to the above presentation $\Sigma_g$.

Since the edges in the above surface presentation are identified in pairs, we see that there are $2g$ 1-dimensional HI-cells in the canonical HI-cell complex on $\Sigma_g$.




Sphere with two pairs of identified holes.

$\Sigma_g$ as an HI-cell having the 4g-gon as its covering cell satisfies the system of equations ($M_2$).

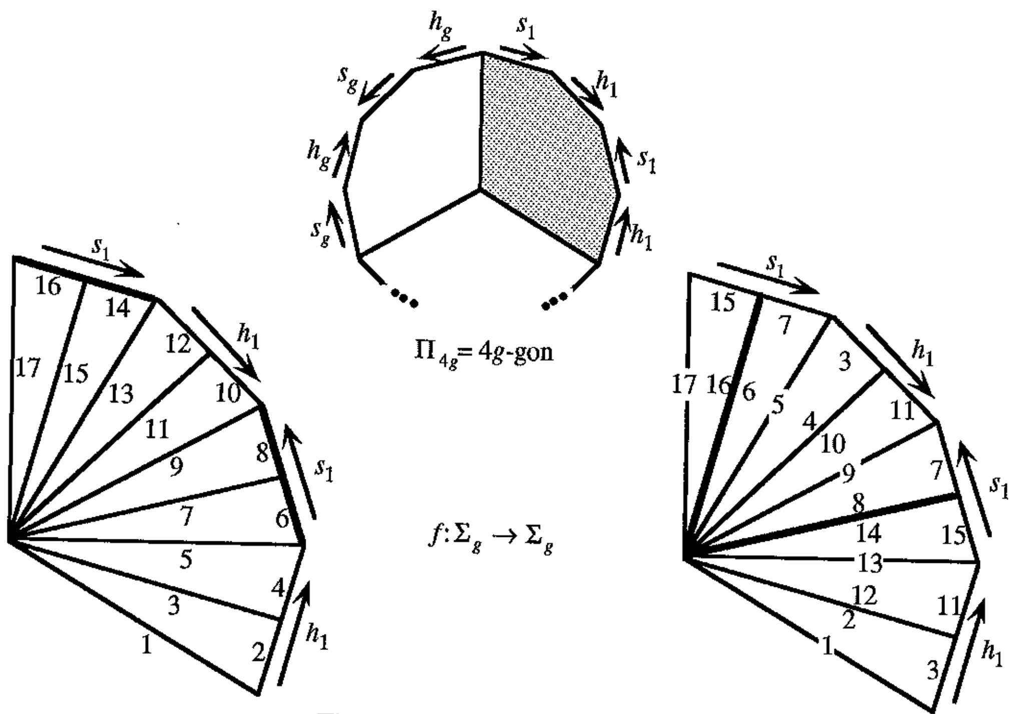
Fig. 39

**5.19** THEOREM. *For $g \geq 1$, let $\Sigma_g$ denote a PL surface of genus g and $\Pi_{4g}$ denote a regular 4g-gon. The canonical HI-cell complex K obtained from the HI-cell $(\Sigma_g, h:\Pi_{4g} \to \Sigma_g)$ (the usual presentation of $\Sigma_g$ as indicated above) is a self-dual HI-cell complex.*

*Proof.* We need to exhibit an HI-isomorphism $(f, K, K^*)$. (See Def. 2.36, p. 50.) Since $K$ has one vertex, $2g$ HI-edges and one 2-dimensional HI-cell, the required PL homeomorphism $f:|K| = \Sigma_g \to |K^*| = \Sigma_g$ must map the vertex of $K$ onto the barycenter of $\Sigma_g$, map $\Sigma_g$ onto the dual of the vertex of $K$, and map the set of $2g$ edges onto the dual set of edges. Also, $f$ must be covered by linear maps mapping the cells of $K$ onto those of $K^*$.

We show the existence of the HI-isomorphism $(f, K, K^*)$ by describing pictorially the image under $f$ of each HI-simplex in the first barycentric subdivision of $\Sigma_g$ (which are themselves images of simplices in the first barycentric subdivision of the cell $\Pi_{4g}$).

Recall that in all figures showing self-duality, we take the (covering) cells $\overline{C_i}$ in $K$ to be also the covering cells in $K^*$, and that we take the various covering linear homeomorphisms $\bar{f}_i:\overline{C_i} \to \overline{C_j^*}$ to be the identity maps $i_i:\overline{C_i} \to \overline{C_i}$, as previously remarked in 2) of Remark 4.18 (p. 103). Also recall that the simplices in the first barycentric subdivision are further subdivided by the second barycentric subdivision for the purpose of embedding $\Sigma_g$ in Euclidean space, as described in Theorem 3.29 (p. 79). However we do not show the second barycentric subdivision in Fig. 40 nor in other similar figures.




The homeomorphism $f\colon\Sigma_g\to\Sigma_g$.
Fig. 40

We partition the 4$g$-gon into $g$ regions as indicated at the top of Fig. 40. (See Remark 5.20, below, concerning $g$=1.)

[The surface $\Sigma_g$ has a presentation where the identifications $s_i h_i s_i^{-1} h_i^{-1}$ are along 4 consecutive edges of the 4$g$-gon, thus the above partition exists.]

We start by defining $f$ on the image of one of the regions of the 4$g$-gon as indicated for the shaded region in Fig. 40: An HI-edge of the first barycentric subdivision marked $x$ on the right region is the image of the HI-edge also marked $x$ on the left region. (This map is extended uniquely to the HI-triangles in the first barycentric subdivision.)

Since the image of 1 under $f$ is the same polyhedron 1, and the same is true for 17 (however $f$ reverses their orientations), we can, and therefore will, define $f$

on the other regions of the $4g$-gon in the same manner as it was defined on the first.

Please verify that this gives an HI-isomorphism between $K$ and $K^*$.

[ For example, the HI-edge $a_1$ (in bold) of $K$, represented by $6 \cup 8$ (or $16 \cup 14$) is mapped by $f$ onto its dual HI-cell, which is represented as the union of the two bold edges on the right side. ] □

**5.20** REMARK. We observe that the proof of theorem 5.19 is also valid for the torus, $\Sigma_1$. In this situation, the shaded part in Fig. 40 is a parallelogram (square) and the edges labeled 1 and 17 are a unique edge, 1, in the interior of the square and $f$ defined in 5.19 still has the desired properties.

Also note that this self-duality on the torus is not the one obtained by translating the integral square lattice on the plane by the vector $\left(\frac{1}{2},\frac{1}{2}\right)$.

From Theorems 5.18 and 5.19 we obtain the following immediate result:

**5.21** COROLLARY. *Every PL orientable surface is the underlying polyhedron of a homogeneous self-dual HI-cell complex.*

*Proof.* Every orientable surface is a surface of genus $g \geq 0$. □

We will now reanalyze the matrices in ($M_1$) and ($M_2$) (which are copied below), and will obtain additional homogeneous self-dual HI-cellular structures for all the orientable surfaces of genus $g \geq 1$, different from those given in 5.18 and 5.19.

$$
(\mathbf{M}_1) \qquad
\begin{array}{ccc} \alpha_1 & \alpha_1 & \alpha_2 \end{array}
$$

$$
\left(\begin{array}{ccc|c} 1 & -1 & 1 & \chi \\ 1 & 0 & -1 & 0 \\ \nu & -2 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -\chi \\ 0 & 0 & \nu-4 & -2\chi \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & {}^{-2\chi}\!/_{\nu-4} \\ 0 & 1 & 0 & {}^{-\nu\chi}\!/_{\nu-4} \\ 0 & 0 & 1 & {}^{-2\chi}\!/_{\nu-4} \end{array}\right)
$$

$$
(\mathbf{M}_2) \qquad
\begin{array}{ccc} \alpha_0 & \alpha_1 & \alpha_2 \end{array}
$$

$$
\left(\begin{array}{ccc|c} 1 & 0 & 0 & {}^{4g-4}\!/_{\nu-4} \\ 0 & 1 & 0 & {}^{-\nu(2-2g)}\!/_{\nu-4} \\ 0 & 0 & 1 & {}^{4g-4}\!/_{\nu-4} \end{array}\right)
$$

**5.22** THEOREM. *There are an infinite amount of homogeneous self-dual HI-cellular on the torus $\Sigma_1$, and the 2-dimensional cells of any such structure must all be quadrilaterals.*

*Proof.* The Euler-Poincaré characteristic of the torus is equal to zero, and from the bottom row of the central matrix in $M_1$, we see that if $\chi = 0$ the system of equations $M_1$ can only be solved if $\nu = 4$, and in this case we have an infinite amount of solutions all having the form

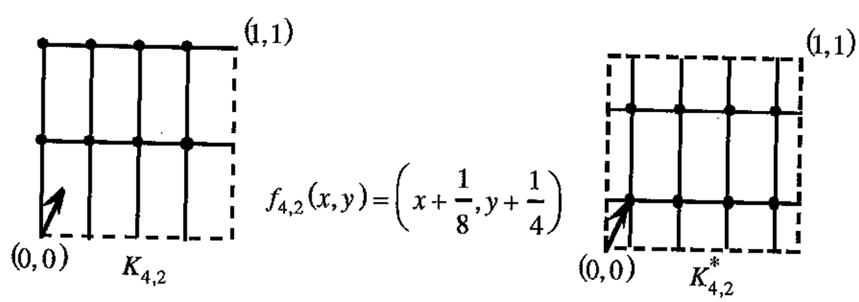
**(T)** $\alpha_0 = \alpha_2 \qquad \alpha_1 = 2\alpha_2 \qquad \alpha_2 \in \mathbf{Z}_+$

Note that $\nu = 4$ implies that the HI-2-cells in any homogeneous self-dual complex on the torus must be quadrilaterals.

The HI-cell complex on the torus obtained in Theorem 5.19 satisfies the numerical conditions (T) with $\alpha_2 = 1$. For any positive integer $k > 1$ we will now sketch the existence of self-dual HI-cell complexes on the torus with $\alpha_2 = k$.

Let $\Sigma_0$ be realized as the polyhedron $\Sigma_0 = \mathrm{Bd}\left([-1,1]^2\right) \times \mathrm{Bd}\left([-1,1]^2\right)$ in $\mathbf{R}^4$. For $k, m, n \in \mathbf{Z}_+$ such that $mn = k, m \geq n$, we let $p_{m,n} : \Sigma_0 \to \Sigma_0$ denote "the" $k$-fold PL covering space, given by a PL analogue of the topological $k$-fold covering $p: S^1 \times S^1 \to S^1 \times S^1$, $(z_1, z_1) \mapsto \left(z_1^m, z_1^n\right)$ where $z_i$ are complex numbers with $|z_i| = 1$. The self-dual HI-cell complex on $\Sigma_0$ having one HI-vertex, two HI-edges and one HI-square given in Theorem 5.19 (p. 122), can be "lifted" via the $k$-fold covering map $p_{m,n}$ to an HI-cell complex $K_{m,n}$ on $\Sigma_0$ having $k$ HI-vertices, $2k$ HI-edges and $k$ $HI$-squares. (See Fig. 40.)

To show the existence of an HI-isomorphism (a self-duality) between $K_{m,n}$ and its dual, we now view $\Sigma_0$ as being obtained by the usual edge-identifications of a unit square belonging to the usual square planar lattice whose vertices are all points having integer coordinates. We define the desired HI-isomorphism $\left(f_{m,n}, K_{m,n}, K_{m,n}^*\right)$ (on the above planar lattice) by letting it be the translation by the vector $\left(\frac{1}{2m}, \frac{1}{2n}\right)$, i. e. $f_{m,n}(x,y) = \left(x + \frac{1}{2m}, y + \frac{1}{2n}\right)$. □



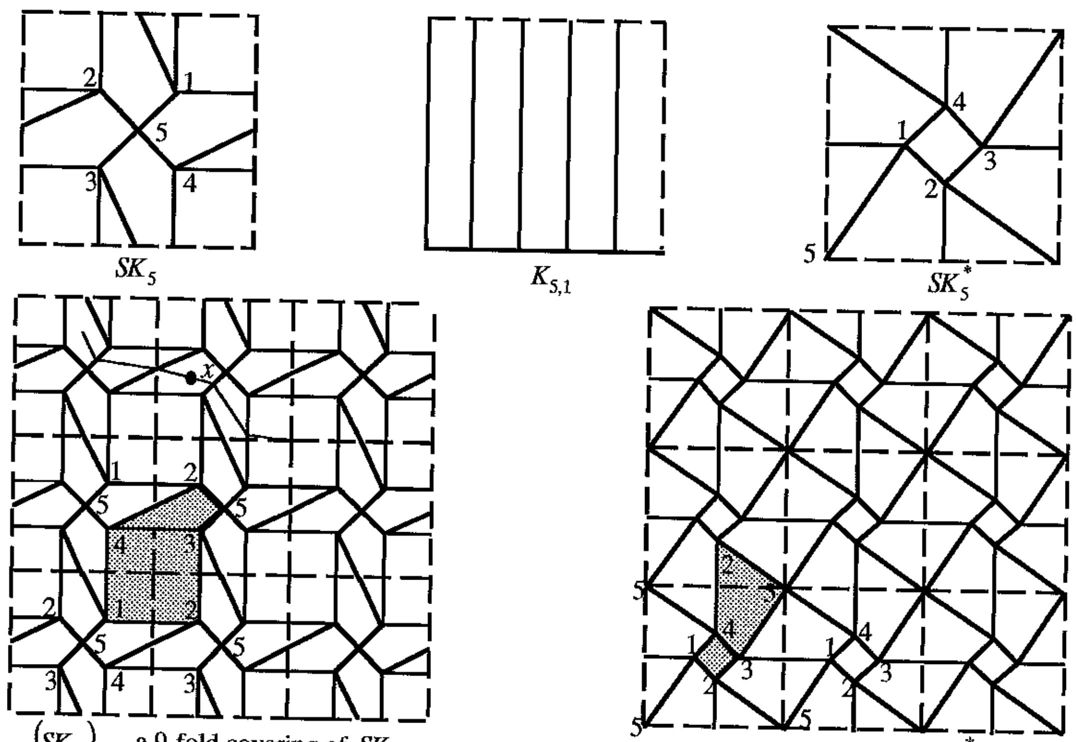

An HI-isomorphism between $K_{4,2}$ and $K_{4,2}^*$.

Fig. 41

Note that if both $m, n \geq 3$, $K_{m,n}$ can be realized as a <u>cell</u> complex on $\mathrm{Bd}(\Pi_m) \times \mathrm{Bd}(\Pi_n)$ where $\Pi_l$ is a regular $l$-gon in $\mathbf{R}^2$.

The above theorem shows the existence of infinitely many homogeneous self-dual HI-square structures $K_{m,n}$ on the torus. The following example however, shows the existence of other homogeneous self-dual HI-square structures on the torus which are not HI-isomorphic to any $K_{m,n}$.

5.23 EXAMPLE In Fig. 42 we show a (very symmetric) self-dual HI-cellular structure, $SK_5$, on the torus consisting of 5 HI-squares, and also its dual $SK_5^*$. We also show their pullback complexes, $(SK_5)_{3,3}$ and $(SK_5^*)_{3,3}$, on the torus via a 9-fold covering, and show also the complex $K_{5,1}$. All HI-quadrilaterals depicted are to be taken as PL images of squares (parallelograms).




$(SK_5)_{3,3}$, a 9-fold covering of $SK_5$. A 9-fold covering of $SK_5^*$.

$SK_5$ is not HI-isomorphic to $K_{5,1}$. $(SK_5)_{3,3}$ is not HI-isomorphic to any $K_{m,n}$.

Fig. 42

We have indicated, by numbering the vertices, an HI-isomorphism between $SK_5$ and $SK_5^*$. One can perhaps better follow this isomorphism by seeing it in the covering spaces, and for this, we have shaded in $(SK_5)_{3,3}$ the pullback of two HI-squares in $SK_5$, and have also shaded in $(SK_5^*)_{3,3}$ the pullbacks of their images.

Now note that since 5 is a prime number, the only self-dual structure $K_{m,n}$, as in 5.22, having 5 quadrilaterals is $K_{5,1}$. Note that in $K_{5,1}$ two edges of each covering square are identified, while in $SK_5$ this is not the case. Thus we have shown:

- *$SK_5$ is not HI-isomorphic to any $K_{m,n}$.*

Note that $(SK_5)_{3,3}$ is also self-dual (and can be taken as <u>cellular</u>) and has 45 vertices , 90 HI-edges and 45 HI-squares.

We also have:

- *$(SK_5)_{3,3}$ is also not HI-isomorphic to any of $K_{45,1}$, $K_{15,3}$ or $K_{9,5}$.*

To verify this fact, we observe that through the barycenter $x$ of an HI-square in $(SK_5)_{3,3}$ there are two different maximal paths which connect "linearly" the barycenter of any square which they intersect to the barycenters of opposite edges of that square; i.e. each path is the union the PL images of linear segments each connecting the barycenter of a <u>covering</u> square to the barycenters of two non-intersecting edges of that square. A section of one such path is shown in Fig. 42.

One can verify that <u>both</u> of these paths through <u>any</u> such barycenter $x$ intersect 15 quadrilaterals, and this number is an invariant of $x$ in the HI-isomorphism class of $(SK_5)_{3,3}$. However in $K_{m,n}$ through any barycenter, one of the paths as above intersect $m$ quadrilaterals and the other intersect $n$, hence none of $K_{45,1}$, $K_{15,3}$ or $K_{9,5}$ is HI-isomorphic to $(SK_5)_{3,3}$.

**5.24** THEOREM. *There exist infinitely many self-dual HI-square structures on the torus which are not isomorphic to any $K_{m,n}$.*

*Proof.* Using a notation similar to the one in Example 5.23, we let $(SK_5)_{t,t}$ denote the $t \times t$ covering of $SK_5$.

One can prove, by induction on $t$, that through a barycenter of a square in $(SK_5)_{t,t}$, there always exist exactly <u>two different</u> paths defined as in the previous example, both of which pass through $5t$ squares (we only need the result that both paths are different, and pass through the <u>same number</u> of squares).

This shows that $(SK_5)_{t,t}$ is not HI-isomorphic to any $K_{m,n}$, for if so, $mn$ would have to be a perfect square as well as $mn = 5t^2$. This is impossible. □

We will now change our point of view slightly: Instead of continuing the search to find self-dual structures that "fit" a given genus, we will investigate which homogeneous self-dual structures all of which use the same regular $n$-gon can "fit" all the surfaces of genus $g \geq 2$. (We have already analyzed in depth the cases of genus 0 and 1.)

We recall again the numerical conditions necessary for the existence of (structural) self duality which were given in ($M_2$):

$$\alpha_0 = \frac{4(g-1)}{\nu-4} \qquad \alpha_1 = \frac{2\nu(g-1)}{\nu-4} \qquad \alpha_2 = \frac{4(g-1)}{\nu-4}.$$

We make the following observations:

**5.25** REMARK.

1) For all $g \geq 2$, the only possible values for $\nu$ that will result in positive integer values for all $\alpha_i$ are $\nu = 5$, $\nu = 6$ and $\nu = 8$.

2) For $\nu = 5$, we get $\alpha_0 = 4(g-1)$ $\alpha_1 = 10(g-1)$ $\alpha_2 = 4(g-1)$

3) For $\nu = 6$, we get $\alpha_0 = 2(g-1)$ $\alpha_1 = 6(g-1)$ $\alpha_2 = 2(g-1)$

4) For $\nu = 8$, we get $\alpha_0 = (g-1)$ $\alpha_1 = 4(g-1)$ $\alpha_2 = (g-1)$

The Euler-Poincaré characteristic $\chi(\Sigma_g)$ of a surface of genus $g$ satisfies

$$\chi(\Sigma_g) = 2 - 2g = -2(g-1) = \chi(\Sigma_2) \cdot (g-1)$$

For $g \geq 2$, the similarity between 2), 3) and 4) of the above remark and the above equality can be understood geometrically by observing that $\Sigma_g$ is a $(g-1)$-fold covering of $\Sigma_2$ (for various covering maps). This observation is the cornerstone for the last investigations of this work. We will show that for $g \geq 2$ there exist self-dual pentagonal, hexagonal and octagonal HI-cell complexes on $\Sigma_g$.

We will construct on $\Sigma_2$ self-dual HI-cellular structures with:

- 4 HI-pentagons, 10 HI-edges and 4 HI-vertices.
- 2 HI-hexagons, 6 HI-edges and 2 HI-vertices.
- 1 HI-octagon, 4 HI-edges and 1 HI-vertex. (Already shown in Theorem 5.19, p. 122.)

Then for $g \geq 3$, we pull back these structures onto $\Sigma_g$, via $(g-1)$-fold coverings of $\Sigma_2$. However, we have:

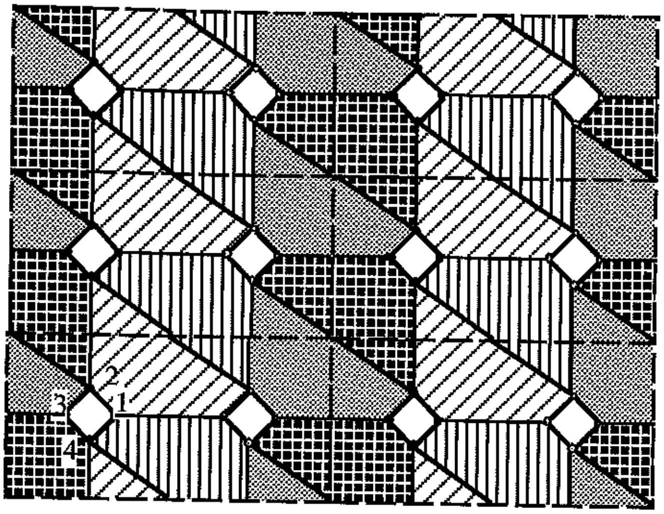
**5.26** PROPOSITION. *There are no homogeneous self-dual H-cell complexes on* $\Sigma_2$.

*Proof.* From 5.25 2), 3) and 4) we observe that for $g$=2, any such HI-pentagonal structure has only 4 vertices, any hexagonal has only 2 HI-vertices and any octagonal has only one HI-vertex. Thus every 2-dimensional covering cell of any such structure has some of its vertices identified. □

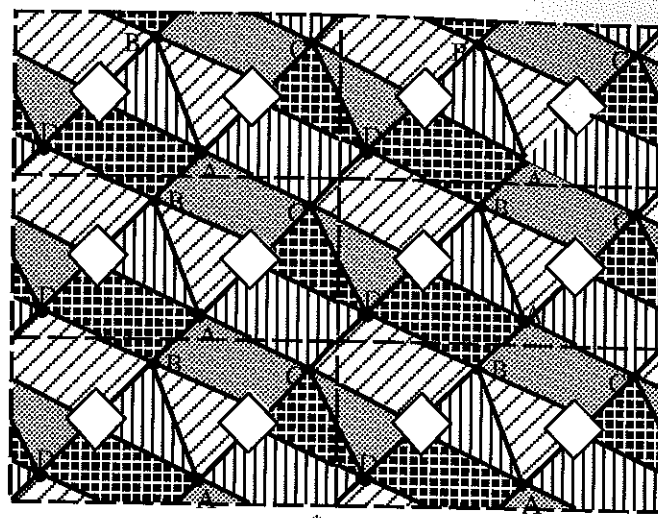
**5.27** THEOREM. *There exist self-dual HI-pentagonal complexes on all the surfaces* $\Sigma_g$ *of genus* $g \geq 2$.

*Proof.* In Fig. 43 below, we show such a complex $\mathcal{P}$ and its dual on $\Sigma_2$. We represent $\Sigma_2$ as a torus with two quadrilaterals taken out (unshaded "holes") and show the identifications along the boundaries of these holes by numbering identified vertices with the same integers (note that we must rotate the "handles" by 90 degrees to identify the edges of the holes).

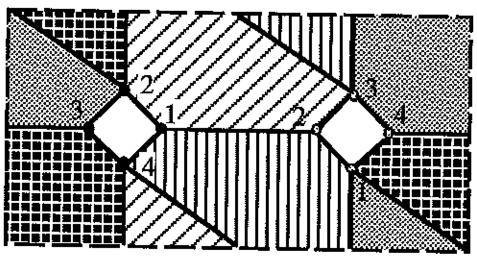
We can thus "lay" various copies of $\Sigma_2$ flatly on the plane and use the $m \times n$ coverings of the torus given in Theorem 5.22 (p. 125) to obtain coverings of $\Sigma_2$. The shading on the pentagons show the HI-isomorphisms. Vertex 1 is mapped to A, 2 to B etc. □



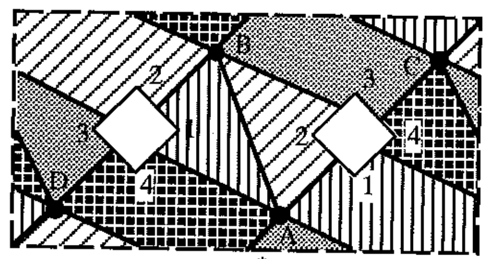
$\mathcal{P}_{(2,3)}$ on $\Sigma_7$



$\mathcal{P}^*_{(2,3)}$



$\mathcal{P}$ on $\Sigma_2$, the torus with 2 holes.



$\mathcal{P}^*$

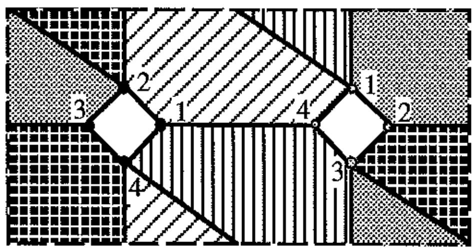
Fig. 43

Note that the identifications on edges of the holes occur after a $90^0$ rotation.

**5.28** REMARK.

1) Note that one could start with pentagons on a torus with 12 holes as given on Fig. 43 without it being a covering of $\Sigma_2$. One could then identify different pairs of holes and still obtain self-dual structures on $\Sigma_7$ not HI-isomorphic to $\mathcal{P}_{(2,3)}$ or to $\mathcal{P}_{(1,6)}$. (To make the pictures fit, note that in the above theorem, in contrast to Theorem 5.22, in the pairs $(m,n)$ representing the covering maps we have $m \leq n$.)

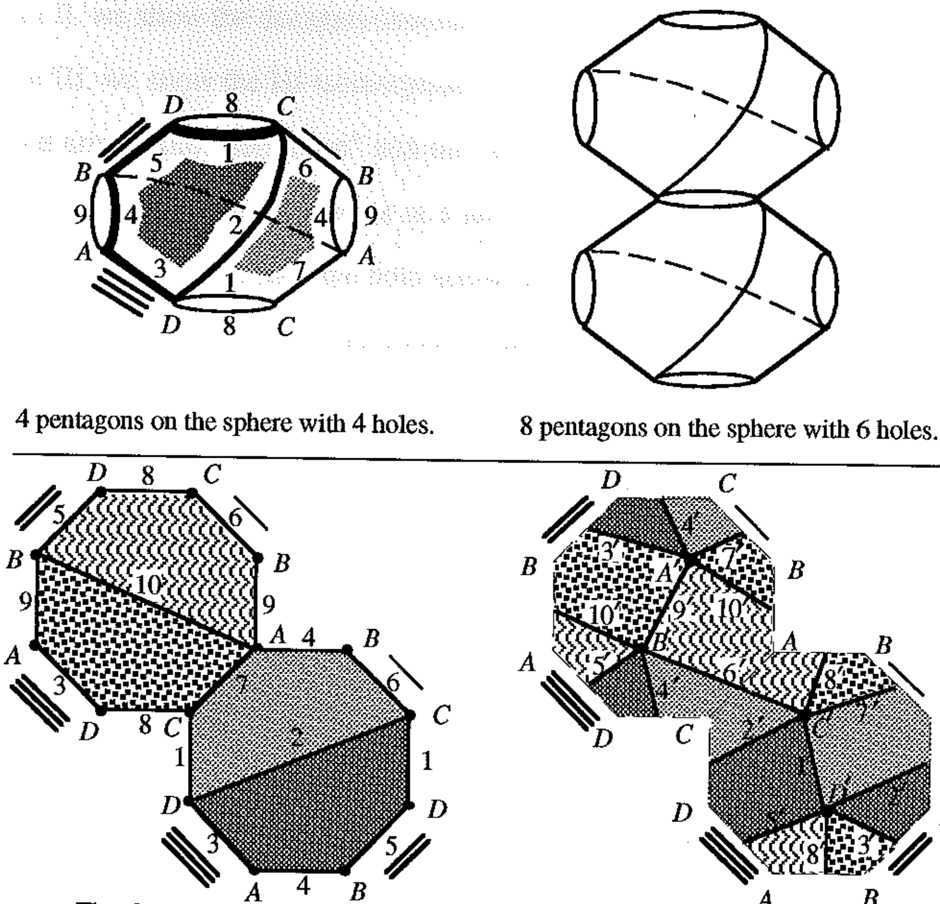
2) One can obtain a different self-dual structure on $\Sigma_2$ by identifying the vertices along the boundaries of the holes in a different manner (rotate now 90 degrees in the opposite way from before). We show it in Fig. 44, and we have verified its self-duality. Note that in the structure given in Fig. 43 only two vertices of each covering pentagon were identified, but in the structure below two pairs of vertices of each pentagon are identified: For example, the diagonally shaded HI-pentagon has only 1, 2 and 4 as its HI-vertices.




The HI-pentagons only have 3 HI-vertices.

Fig. 44

We show in Fig 45 a complex similar to the one in Fig. 43. However in Fig 45 we represent $\Sigma_2$ as a sphere with 4 holes, in 3-dimensional space. This might be easier to visualize than the previous planar lattices. One also can visualize pullbacks of this complex onto the surfaces of higher genus by assembling such spheres. For example, one could stack them vertically to obtain a $1 \times n$ covering space.



4 pentagons on the sphere with 4 holes. 8 pentagons on the sphere with 6 holes.

The above sphere opened up. The dual complex.

Fig. 45

**5.29** THEOREM. *There exist self-dual HI-hexagonal complexes on all the surfaces* $\Sigma_g$ *of genus* $g \geq 2$.

*Proof.* We present one such structure in Fig. 46.

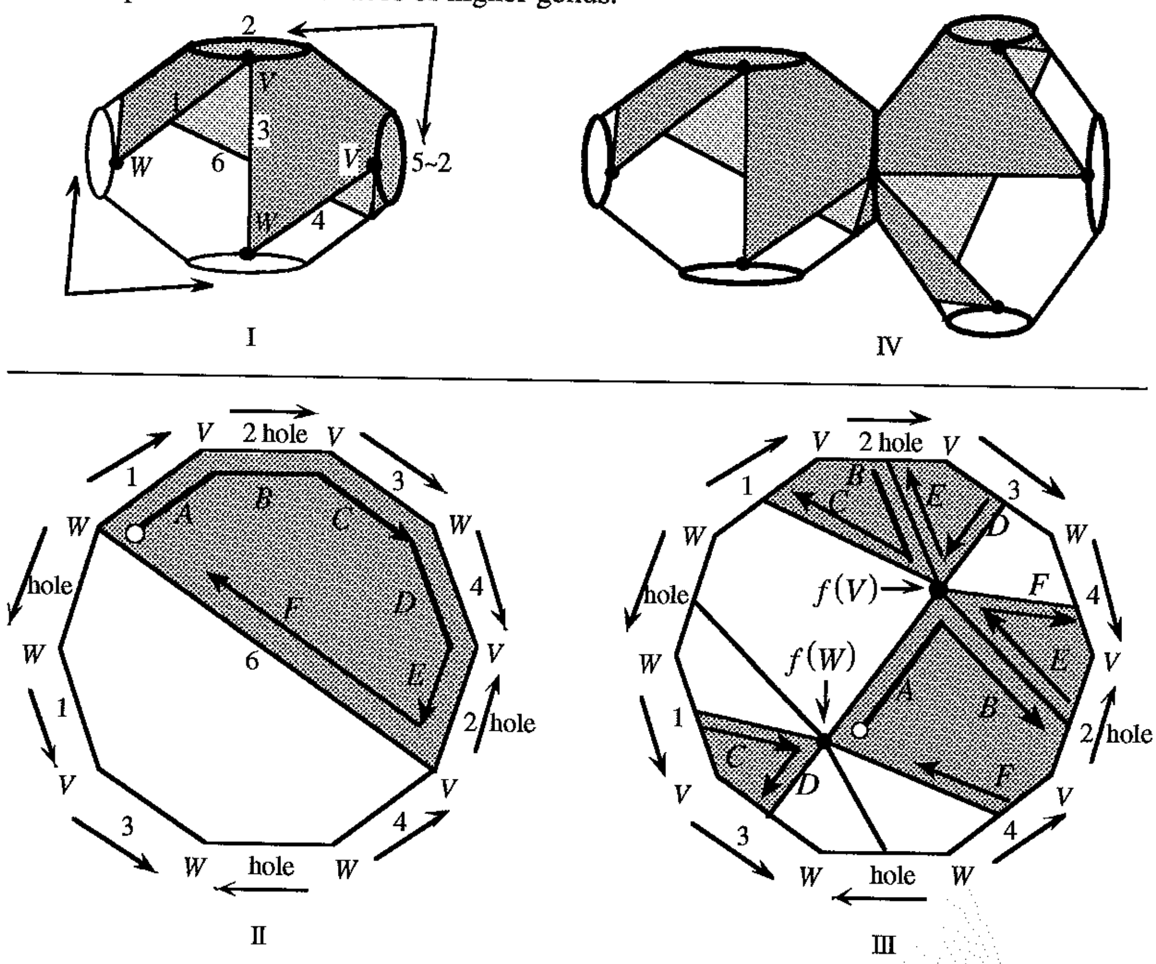
In I. we show two hexagons (one is "transparent") on the sphere with two holes, as well as their identifications.

Notice that the induced complex in $\Sigma_2$ has 2 HI-hexagons, 6 HI-edges and 2 HI-vertices. Note also that the 1-valence of each HI-vertex is indeed equal to 6.

In II. we show the decagon obtained by cutting I. along edges 1, 3 and 4.

In III. we show the HI-isomorphism $f$ establishing the self-duality. We show in bold arrows the "definition" and the verification of the required isomorphism. Note for example, that the images of $B$ and $E$ coincide, as they should, and note as well that the order in which the bold arrows transverse the vertices gets preserved.

In IV. we show how to concatenate copies of I. to obtain self-dual hexagonal complexes on the surfaces of higher genus.




A self-dual HI-hexagonal structure on $\Sigma_2$.

Fig. 46

We end this work by commenting that we have already shown a self-dual octagonal HI- structure in $\Sigma_2$ and it is a simple matter to lift it to all the orientable surfaces of genus $g \geq 3$. We have thus obtained:

**5.30** THEOREM. *All the orientable surfaces of genus $g \geq 2$, admit homogeneous self-dual, pentagonal, hexagonal and octagonal, HI-cellular structures.*

*Conversely, these are the only "types "of HI-cell complexes that can have all the surfaces of genus $g \geq 2$, as their underlying polyhedra.*

□

# References

[R-S] C. P. Rourke and B. J. Sanderson:
*Introduction to Piecewise-Linear Topology,* Springer-Verlag (1982)

[V] James W. Vick:
*Homology Theory*, Academic Press (1973)

[Z] Günther M. Ziegler:
*Lectures on Polytopes,* Springer-Verlag (1995)