# A Rigorous Definition of Fiberwise Quantum Cohomology and Equivariant Quantum Cohomology

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# Abstract of the Dissertation

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In this thesis we give a mathematically rigorous definition of the fiberwise quantum cohomology and the equivariant quantum cohomology in the semi-positive symplectic case. We prove that the quantum multiplication is associative and has several properties such as direct product, restriction, and induction. We also compute several examples.

To my grandparents

# Contents

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# Chapter 1

# Introduction

The notion of quantum cohomology was first proposed by the physicist Vafa [V] and its mathematical foundation was established by Ruan and Tian [RT] for semi-positive symplectic manifolds. However, the computation of quantum cohomology is a difficult task in general since it does not have the functorial property of behaving well under pull-back. If $X$ and $Y$ are two semi-positive symplectic manifolds, and $f : X \to Y$ is a continuous map, then there is a ring homomorphism $f^* : H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ for ordinary cohomology. However for quantum cohomology, there does not exist such homomorphism preserving quantum multiplication. The quantum cohomology of projective spaces has been known for quite some time. The quantum cohomology of Grassmannians was computed by Witten [W] and by Seibert and Tian [ST]. The notion of equivariant quantum cohomology was first proposed by Givental and Kim in a heuristic manner [GK], and they conjectured several properties of equivariant quantum cohomology, namely direct product, restriction and induction. Assuming these properties, they computed the quantum cohomol-

ogy of complete flag manifolds rigorously. Later, another account was given by Astashkevich and Sadov [AS]. They also introduced the notion of fiberwise quantum cohomology heuristically (called vertical qunatum cohomology in [AS]), and computed the quantum cohomology of partial flag manifolds again assuming the conjectured properties. Kim did a similar thing for partial flag manifolds [K1]. The advantage of fiberwise quantum cohomology and equivariant quantum cohomology is that the functorial pull-back property of ordinary cohomology is partially restored for them, i.e. restriction property. It is exactly this property and several other functorial properties that enables Givental et al. to complete the computation. Ciocan-Fontanine computed the Gromov-Witten invariants of complete flag manifolds using the canonical complex structure, so he could give a presentation of the quantum cohomology of complete flag manifolds rigorously [C]. Other examples of quantum cohomology were worked out by Batyrev for toric manifolds [B].

In this work we will give a mathematically rigorous definition of fiberwise quantum cohomology and equivariant quantum cohomology in the semi-positive symplectic case. Assume that $\mathcal{X} \to Y$ is a smooth fiber bundle, $Y$ is a closed smooth manifold, and the fiber is a closed manifold. Further assume that $\mathcal{X}$ admits a differential two form $\omega$ whose restriction to each fiber $V_z$ is a semi-positive symplectic form. The main results of this paper are: 1) We will define the fiberwise Gromov-Witten mixed invariants (called fiberwise mixed invariants below) of arbitrary genus for $\mathcal{X} \to Y$ using perturbed pseudo-holomorphic maps, then we prove several properties of the fiberwise mixed invariants such as direct product, restriction, induction and the so-

called composition law; 2) Using the fiberwise mixed invariants, we will define fiberwise quantum cohomology and equivariant quantum cohomology. The properties alluted to above namely direct product, restriction, induction and associativity of fiberwise quantum cohomology and equivariant quantum cohomology will then follow from the corresponding properties of the fiberwise mixed invariants. So we put the computation in [GK] , [AS] and [K1] on a solid foundation; 3) As examples, we will compute the fiberwise quantum cohomology of several fiber bundles with fiber a flag manifold. In this paper, we basically will follow the approach of Ruan and Tian to quantum cohomology in [RT]. The main part of this paper will be devoted to a proof of the composition law of the fiberwise mixed invariants. In the proof, most analytic results are adopted from [RT].

During the preparation of this paper, the author learned of B. Kim's paper "On equivariant quantum cohomology "[K2], in which he defined equivariant quantum cohomology for flag manifolds only. His approach was different from ours; he used Kontsevich's moduli space of stable maps to define the equivariant Gromov-Witten classes.

# Chapter 2

# A Compactness Theorem and Several Transversality Results

## 2.1 A compactness theorem

**Definition 2.1.1** *i) An almost complex structure $J$ on a symplectic manifold $(V,\omega)$ is said to be $\omega$-tamed if $\omega(\xi, J\xi) > 0, \forall \xi \in TV, \xi \neq 0$, and if further $\omega(J\xi, J\eta) = \omega(\xi,\eta), \forall \xi,\eta \in TV$, we call $J$ $\omega$-compatible.*

*ii) A symplectic manifold $(V,\omega)$ is called semi-positive if for any $\omega$-compatible almost complex structure $J$ on $V$, every $J$-holomorphic curve $u : \mathbb{C}P^1 \to V$ has non-negative first Chern number. i.e. $\int_{\mathbb{C}P^1} u^* c_1(TV) \geq 0$. If this inequality is always strictly positive, we call $(V,\omega)$ positive symplectic manifold.*

The following is the basic setup we will work with in this section.

**Assumption** $(\star)$ *Let $\mathcal{X} \xrightarrow{p} Y$ be a smooth fiber bundle over a closed smooth manifold $Y$ whose fiber is closed manifold $V$ of dimension $2n$. Assume that $\mathcal{X}$ admits a differential two form $\omega$ with bounded $H^{l,q}$ Sobolev norm ($l \geq 10, q >$*

1), *whose restriction to each fiber* $V_z$ *is a symplectic form. Here* $V_z$ *is the fiber over a point* $z \in Y$. *i.e.,* $\omega$ *is a fiberwise symplectic form on* $\mathcal{X}$. *Assume further that* $(V_z, \omega|_{V_z})$ *is a semi-positive symplectic manifold for each* $z$ *in* $Y$.

To define the fiberwise quantum cohomology for $\mathcal{X} \to Y$, we first need to define the fiberwise mixed invariants. Suppose that $\Sigma$ is a stable curve of genus $g$ with $m$ irreducible components $\Sigma_i$ and $k$ marked points. Assume that $A_i \in H_2(V; \mathbb{Z})$ are $m$ homology classes. The fiberwise mixed invariant can be considered as a multilinear map

$$\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)} : (\prod_1^k H^*(\mathcal{X}, \mathbb{Q})) \times (\prod_1^l H^*(\mathcal{X}, \mathbb{Q})) \to H^*(Y, \mathbb{Q}).$$

To evaluate the image of this map on the element $\gamma \in H_*(Y, \mathbb{Q}))$, a key observation is that we should use the pull-back pseudo-manifold fiber bundle $\mathcal{X}_Z \to Z$ where $F : Z \to Y$ is a pseudo-manifold representative of $\gamma$ and the pseudo-manifold $\mathcal{X}_Z$ is the fiber product $Z \times_F \mathcal{X}$. The advantage of this definition is that it enables us to prove the restriction property easily. Its price is that $Z$ is only a pseudo-manifold, so that we need to carefully treat the moduli space of each stratum of $Z$.

A compact pseudo-manifold in $Y$ is a stratified space $Z$ together with a continuous map $F : Z \to Y$ satisfying: each stratum is a smooth manifold without boundary, each lower stratum is at least codimension 2, and $F$ is smooth on each stratum. Note that any rational homology class in $Y$ can be represented by a finite dimensional compact pseudo-manifold, and any two

such pseudo-manifolds representing the same homology class form the boundary of a compact pseudo-manifold (and so are cobordant in the usual sense). Fix a finite dimensional compact pseudo-manifold $F : Z \to Y$, and denote by $\mathcal{X}_Z$ the fiber product $Z \times_F \mathcal{X}$. Then $\mathcal{X}_Z \to Z$ is a pseudo-manifold fibration with fiber $V$. $\omega$ induces a two form on $\mathcal{X}_Z$ by the natural map $\mathcal{X}_Z \xrightarrow{\tilde{F}} \mathcal{X}$, which is still denoted by $\omega$. It is a fiberwise symplectic form on $\mathcal{X}_Z$. Note that $\omega$ is continuous and is smooth on each stratum. $(\mathcal{X}_Z, \omega, Z, V)$ is the basic object we will study below.

**Remark 2.1.1** *We can define a trivialization of $\mathcal{X}_Z$ as follows. For any $z_\infty \in Z$, choose a neighborhood $W$ of $F(z_\infty)$ in $Y$, such that $\mathcal{X}|_W \cong W \times V$. Choose a neighborhood $U$ of $z_\infty$ in $Z$ such that $F(U) \subset W$, then $\mathcal{X}_Z|_U$ has a induced trivialization $\mathcal{X}_Z|_U \cong U \times V$, where $U$ may not be smooth. Note that for any two different trivializations of $\mathcal{X}_Z$ over $U_1$ and $U_2$, the transition function $f_{12} : U_1 \cap U_2 \to Diff(V)$ factors through as $U_1 \cap U_2 \xrightarrow{F} W_1 \cap W_2 \xrightarrow{h_{12}} Diff(V)$, where $h_{12}$ is a smooth transition function between two trivializations of $\mathcal{X}$ over $W_1$ and $W_2$. This property is important for defining various convergences below. This trivialization of $\mathcal{X}_Z$ is crucial when we study the structure of moduli space. We fix a such trivialization for $\mathcal{X}_Z$ from now on.*

Define a vector bundle $E \xrightarrow{\pi} \mathcal{X}_Z$ whose fiber $E_x$ is $T_x V_z$ for each $x \in \mathcal{X}_Z, z = p(x)$. A complex structure $J$ on $E$ is called a fiberwise almost complex structure on $\mathcal{X}_Z$. A complex structure $J$ on $E$ is said to be tamed by $\omega$ if $\omega(\xi, J\xi) > 0, \forall \xi \in E, \xi \neq 0$. If further $\omega(J\xi, J\eta) = \omega(\xi, \eta), \forall \xi, \eta \in E$, we call the complex structure $J$ on $E$ $\omega$-compatible. We now show that there is a

fiberwise almost complex structure on $\mathcal{X}_Z$ compatible with $\omega$.

**Lemma 2.1.1** *Under Assumption($\star$), the vector bundle $E$ admits a continuous complex structure which has bounded $H^{l,q}$ norm on each stratum, and is compatible with $\omega$. Furthermore for any $z_\infty \in Z$, $J$ satisfies*

**Condition 1.** $\| J|_{z_r \times V} - J|_{z_\infty \times V} \|_{C^4(V)} \to 0$ *as* $z_r \to z_\infty, z_r \in U$. *Here we use the trivialization in Remark 2.1.1 and view $J|_*$ as a section of $Hom(TV, TV)$.*

**Proof:** Since the vector bundle $E$ is a pseudo-manifold smooth in fiber directions, we can find a continuous Riemannian metric $g$ on $E$, such that $g$ is smooth on each stratum and $g$ satisfies $\| g|_{z_r \times V} - g|_{z_\infty \times V} \|_{C^4(V)} \to 0$ as $z_r \to z_\infty, z_r \in U$. Here we use the trivialization in Remark 2.1.1 and view $g|_*$ as a section of $Hom(TV \times TV, \mathbb{R})$. Then there is an unique automorphism $A$ of $E$ such that $\omega(\xi, \eta) = g(A\xi, \eta), \forall \xi, \eta \in E_x, x \in \mathcal{X}_Z$. Since $\omega$ is skew symmetric, $-A^2$ is positive definite with respect to $g$. It is easy to check that $J = (-A^2)^{-\frac{1}{2}} A$ is the required complex structure on $E$. □

Note that since the deformation of the complex structure on the bundle $E$ is un-obstructed, the tangent space at $J$ consists of all the sections $s$ of the bundle $End(E)$ satisfying $s \circ J + J \circ s = 0$. Let $\mathcal{J}^v_{l,q}(\mathcal{X}_Z, \omega)$ be the space of $\omega$-tamed continuous complex structures on $E$ with bounded $H^{l,q}$ norm on each stratum satisfying Condition 1. Here and in the following ' $v$' means fiberwise. Although $J$ is not smooth, $\mathcal{J}^v_{l,q}(\mathcal{X}_Z, \omega)$ is a smooth Banach manifold.

Next we will define the so-called fiberwise perturbed $J$-holomorphic map. Recall that a $k$-point genus $g$ stable curve $C = (\Sigma; x_1, \cdots, x_k)$ is a reduced

connected curve $\Sigma$ whose singularities are only double points (it may have no singularities at all), plus $k$ distinct smooth points $x_1, \cdots, x_k$ in $\Sigma$, such that every smooth rational component of $\Sigma$ contains at least three points from $x_1, \cdots, x_k$ or double points of $\Sigma$, and every elliptic curve component contains at least one such points. Such a $\Sigma$ is called an admissible curve.

Assume that $\Sigma$ has $m$ components $\Sigma_1, \cdots, \Sigma_m$ (where $\Sigma_i$ may have double points). Let $J$ be a fiberwise almost complex structure on $\mathcal{X}_Z$, and $\pi_j$ be the projection from $\Sigma_i \times \mathcal{X}_Z$ to its j-th factor $(j = 1, 2)$. A fiberwise inhomogeneous term $\nu$ over $\Sigma \times \mathcal{X}_Z$ is a set of sections $\nu_1, \cdots, \nu_m$, where each $\nu_i$ is a section of $Hom(\pi_1^* T\Sigma_i, \pi_2^* E)$ satisfying

i)$\nu_i$ is continuous everywhere and differentiable on each stratum;

ii) **Condition 2.** Use the notation of Remark 2.1.1, $\forall z_\infty \in Z, \| \nu_i|_{\Sigma_i \times z_r \times V} - \nu_i|_{\Sigma_i \times z_\infty \times V} \|_{C^4(\Sigma_i \times V)} \to 0$, as $z_r \to z_\infty, z_r \in U$;

iii) The $J$-anti-linear condition: $\nu_i(j_{\Sigma_i}(v) = -J(\nu_i(v))$, where $j_{\Sigma_i}$ is the complex structure on $\Sigma_i$, $v$ is any vector in $T\Sigma_i$.

**Definition 2.1.2** *Let $\Sigma$ be an admissible curve, $J$ a fiberwise almost complex structure on $\mathcal{X}_Z$, and $\nu$ a fiberwise inhomogeneous term on $\Sigma \times \mathcal{X}_Z$. A fiberwise $(J, \nu)$-perturbed holomorphic map from $\Sigma$ into $\mathcal{X}_Z$ is a continuous map $f : \Sigma \to \mathcal{X}_Z$ which is differentiable at the smooth points of $\Sigma$, such that $f(\Sigma) \subset V_z$ for some $z \in Z$, and $f$ satisfies the inhomogeneous Cauchy-Riemann equation on each component $\Sigma_i (i = 1, \cdots, m)$. Let $f_i = f|_{\Sigma_i}$. Then*

$$(\bar{\partial}_J f_i)(x) = \nu_i(x, f_i(x)), \forall x \in \Sigma_i \setminus Sing(\Sigma_i),\ i = 1, \cdots, m,$$

*where $\bar{\partial}_J$ denotes the differential operator $d + J \circ d \circ j_{\Sigma_i}$. We denote the space of all fiberwise $(J,\nu)$ holomorphic map $f$ from $\Sigma$ into $\mathcal{X}_Z$ satisfying $[f_{i*}(\Sigma_i)] = A_i, i = 1, \cdots, m$ by $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$. Here $A_1, \cdots, A_m$ are $m$ homology classes in $H_2(V, \mathbb{Z})$.*

We will use $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ to define the fiberwise mixed invariants later. This moduli space is similar to the moduli space used to prove that the Gromov-Witten mixed invariants are independent of the choice of the almost complex structure (see [RT]). In fact that moduli space is the same as our moduli space for the case where the bundle is $V \times [0,1] \to [0,1]$. The further generality lies in that the bundle $\mathcal{X}_Z \to Z$ may be nontrivial and that $Z$ is only a pseudo-manifold. Now we begin establishing a compactness theorem about the structure of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ which is required for the definition of the fiberwise mixed invariants.

Recall that a degeneration of admissible curves is a holomorphic fibration $\tilde{\pi} : S \to \Delta \subset \mathbb{C}^l$ satisfying: (1) $S$ is a $l+1$-dimensional complex variety with normal crossings; (2) All fibers of $\tilde{\pi}$ are admissible. Denote by $j_S$ the complex structure on $S$, $J \in \mathcal{J}^v_{l,q}(\mathcal{X}_Z, \omega)$ a $\omega$-tamed fiberwise almost complex structure on $\mathcal{X}_Z$. Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be the projection maps from $S \times \mathcal{X}_Z$ to $S$ and $\mathcal{X}_Z$ respectively. An inhomogeneous term $\tilde{\nu}$ on $S \times \mathcal{X}_Z$ is defined to be a section of $Hom(\tilde{\pi}_1^* TS, \tilde{\pi}_2^* E) \to S \times \mathcal{X}_Z$ satisfying

i) $\tilde{\nu}$ is continuous and differentiable on each stratum;

ii) **Condition 2′.** Using the notation of Remark 2.1.1, $\forall z_\infty \in Z$, $\| \tilde{\nu}|_{S \times z_r \times V} - \tilde{\nu}|_{S \times z_\infty \times V} \|_{C^4(S \times V)} \to 0$, as $z_r \to z_\infty$, $z_r \in U$;

iii) The anti-$J$-linear condition: $J \circ \tilde{\nu} = -\tilde{\nu} \circ j_S$.

The setting for the compactness theorem is: fix a degeneration of admissible curves $\tilde{\pi} : \mathcal{S} \to \Delta$, assume that $\tilde{\omega}$ is the induced fiberwise continuous symplectic form on $\mathcal{X}_Z \times \Delta \to Z \times \Delta$ with bounded $H^{k,p}$ norm on each stratum, $\tilde{J} \in \mathcal{J}^v_{l,q}(\mathcal{X}_Z \times \Delta, \tilde{\omega})$ is a fiberwise almost complex structure, and $\tilde{\nu}_1, \cdots, \tilde{\nu}_m$ are a family of fiberwise inhomogeneous terms on $S$. Let $\{t_r\}$ be a sequence in $\Delta$ converging to the origin as $r \to \infty$, set $\omega^{(r)} = \tilde{\omega}|_{\mathcal{X}_Z \times t_r}$, $J^{(r)} = \tilde{J}|_{\mathcal{X}_Z \times t_r}$, $\Sigma^{(r)} = \tilde{\pi}^{-1}(t_r)$. Further assume that each $\Sigma^{(r)}$ consists of $m$ components $(\Sigma^{(r)}_1, \cdots, \Sigma^{(r)}_m)$, and for any fixed $i$, $\Sigma^{(r)}_i$ are diffemorphic to each other for any $r$. Set $\nu^{(r)} = (\nu^{(r)}_1, \cdots, \nu^{(r)}_m)$, where $\nu^{(r)}_i = \tilde{\nu}_i|_{\Sigma^{(r)}_i \times \mathcal{X}_Z}$, $\nu^{(r)}$ is the fiberwise inhomogeneous term on $\Sigma^{(r)}$. Note that $\Sigma^{(r)}_i$ converges to the i-th component $\Sigma^{(\infty)}_i$ of an admissible curve $\Sigma^{(\infty)} = \tilde{\pi}^{-1}(0)$, and $\nu^{(r)}_i$ converges to the i-th component $\nu^{(\infty)}_i = \tilde{\nu}_i|_{\Sigma^{(\infty)}_i \times \mathcal{X}_Z}$ of a fiberwise inhomogeneous term $\nu^{(\infty)} = (\nu^{(\infty)}_1, \cdots, \nu^{(\infty)}_m)$ on $\Sigma^{(\infty)} \times \mathcal{X}_Z$ in $C^4$-topology in the following sense: there are continuous maps $\tau^{(r)}_i : \Sigma^{(r)}_i \to \Sigma^{(\infty)}_i$ and compact sets $K^{(r)}_i$ in $\Sigma^{(\infty)}_i$ satisfying : (1) $K^{(r)}_i \subset K^{(r+1)}_i, \cup K^{(r)}_i = \Sigma^{(\infty)}_i \backslash \{\text{double points}\}$; (2) $\tau^{(r)}_i$ restricts to a diffeomorphism from ${\tau^{(r)}_i}^{-1}(K^{(r)}_i)$ onto $K^{(r)}_i$; (3) For each index $i$, index $s$, and each stratum $Z_\alpha \subset Z$, both $\| j_{\Sigma^{(\infty)}_i} \circ d\tau^{(r)}_i - d\tau^{(r)}_i \circ j_{\Sigma^{(r)}_i} \|_{C^4((\tau^{(r)}_i)^{-1}(K^{(s)}_i))}$ and $\| \nu^{(\infty)}_i \circ d\tau^{(r)}_i - \nu^{(r)}_i \|_{C^4((\tau^{(r)}_i)^{-1}(K^{(s)}_i) \times \mathcal{X}_{Z_\alpha})}$ converge to zero as $r \to \infty$.

Given $f^{(r)} \in \mathcal{M}^v_{(A_1, \cdots, A_m)}(\Sigma^{(r)}, \mathcal{X}_Z, J^{(r)}, \nu^{(r)})$, we say that $f^{(r)}$ converges to $f^{(\infty)} : \Sigma^{(\infty)} \to \mathcal{X}_Z$ if $\| f^{(\infty)}_i \circ \tau^{(r)}_i - f^{(r)}_i \|_{C^3, loc} \to 0$ as $r \to 0$. Here we need the trivialization of Remark 2.1.1 to define $C^3, loc$-norm, and we also need a trivialization of $V$ to define $f^{(\infty)}_i \circ \tau^{(r)}_i - f^{(r)}_i$.

**Proposition 2.1.1 (Compactness Theorem)** *Let* $f^{(r)}$ *be in* $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma^{(r)}, \mathcal{X}_Z, J^{(r)}, \nu^{(r)})$. *Then there is an admissible curve* $\tilde{\Sigma}$, *which is the union of the smooth resolution* $\tilde{\Sigma}^{(\infty)}$ *of* $\Sigma^{(\infty)}$ *and finitely many smooth rational curves such that a subsequence of* $\{f^{(r)}\}_1^\infty$ *converges to a* $(J^{(\infty)}, \nu)$*-perturbed holomorphic map* $f$ *on* $\tilde{\Sigma}$, *where the fiberwise inhomogeneous term* $\nu$ *coincide with* $\nu^{(\infty)}$ *on* $\tilde{\Sigma}^{(\infty)}$ *and vanishes on those rational curves, and moreover we have* $[f_*(\tilde{\Sigma})] = A_1 + \cdots + A_m$ *and* $f(\Sigma) \subset V_z$, *for some* $z \in Z$.

**Proof:** Without loss of generality we may assume that $\Sigma^{(r)}$ is smooth, since otherwise we can consider the $\{\Sigma_i^{(r)}\}_1^\infty$ family or its resolution for each fixed $i$. In the rest part of proof, we will use $A$ instead of $A_1, \cdots, A_m$. We also make a reduction to eliminate the fiberwise inhomogeneous term $\nu^{(r)}$. Setting $\mathcal{W} = S \times \mathcal{X}_Z$, this has a fiber bundle structure $\mathcal{W} \xrightarrow{\tilde{p}} Z$, $\tilde{p}(s,x) = p(x)$, and we introduce a fiberwise almost complex structure $J^{(r)}_{\mathcal{W}} \in \mathcal{J}^v_{l,q}(\mathcal{W}, \omega)$ on $\mathcal{W}$ as follows. Given the vector bundle $E_{\mathcal{W}} \to \mathcal{W}$ whose fiber is $E_{\mathcal{W}}|_{(s,x)} = T_sS \times E_x$, if $(u,v)$ is a vector in $E_{\mathcal{W}}|_{(s,x)}$, define

$$J^{(r)}_{\mathcal{W}}(u,v) = (j_S(u), J^{(r)}(v) + \nu^{(r)}(j_S(u))).$$

It is easy to check that this is a fiberwise almost complex structure on $\mathcal{W}$ tamed for sufficiently small $\nu^{(r)}$ by the fiberwise symplectic form $\omega_{\mathcal{W}}$ which is the sum of the pull back of $\omega_S$ and $\omega$ to $\mathcal{W}$. Moreover if we take the graph of $f^{(r)}$, $F^{(r)} : \Sigma^{(r)} \to \mathcal{W}, F^{(r)}(x) = (x, f^{(r)}(x))$ where $x \in \Sigma^{(r)}$, then $F^{(r)}$ is a $J^{(r)}_{\mathcal{W}}$-holomorphic map. So, it suffices to show that a subsequence of $\{F^{(r)}\}_1^\infty$ converges to a holomorphic map $F : \tilde{\Sigma} \to \mathcal{W}$ with $[F_*(\Sigma)] = [F^{(r)}_*(\Sigma^{(r)})]$,

where $\tilde{\Sigma}$ is given in Proposition 2.1.1. Thus we have reduced the perturbed case to the non-perturbed case $\nu^{(r)} = 0$. From now on we assume that each $f^{(r)}$ is $J^{(r)}$-holomorphic. Now we fix a $J^{(\infty)}$-compatible metric $h$ on $E$, which satisfies

i) $h$ is continuous, and smooth on each smooth stratum $\mathcal{X}_Z$;

ii) **Condition 3**. Using the notation of Remark 2.1.1, $\forall z_\infty \in U$, $\| h|_{z_r \times V} - h|_{z_\infty \times V} \|_{C^4(V)} \to 0$, as $z_r \to z_\infty, z_r \in U$.

Because $J^{(r)}$ depends continuously on $t_r$, there is a positive constant $c$ such that $\omega^{(r)}(\xi, J^{(r)}\xi) \geq c \cdot h(\xi, \xi)$, for all $z$ and sufficiently large $r$, where $\xi \in E_z$.

**Lemma 2.1.2** *There is a uniformly constant $C_A$ which depends only on $A$ and $c$, such that for any $f^{(r)} \in \mathcal{M}_A^v(\Sigma^{(r)}, \mathcal{X}_Z, J^{(r)}, 0)$, we have*

$$\int_{\Sigma_{(r)}} |df^{(r)}|^2_{\mu_r, h} d\mu_r \leq C_A$$

*where the volume form $d\mu_r$ is defined using any conformal metric $\mu_r$ on $\Sigma^{(r)}$.*

**Proof:** Fixing $\Sigma = \Sigma^{(r)}$, $f = f^{(r)}, \omega = \omega^{(r)}, J = J^{(r)}, \mu = \mu_r, f(\Sigma) \subset V_z$, we then have

$$\int_\Sigma f^* \omega|_{V_z} = \int_{f(\Sigma)} \omega|_{V_z} = \int_A \omega|_{V_z}.$$

Since $\omega|_{V_z}$ is a continuous function of $t^{(r)}$ and $z$, the above integral is bounded by some constant independent of $t^{(r)}$ and $z$.

At each point $x \in V_z$, choose a local unitary frame $e_1, \cdots, e_n \in T_x^{1,0}V_z$ with respect to a metric $h$, and denote by $\{e_i^*\}_1^n$ the dual basis of $\{e_i\}_1^n$. The symplectic form can now be written as

$$\omega|_{V_z} = \omega_{ij}^{(2,0)}(t_r, z)e_i^* \wedge e_j^* + \omega_{ij}^{(1,1)}(t_r, z)e_i^* \wedge \bar{e_j^*} + \omega_{ij}^{(0,2)}(t_r, z)\bar{e_i^*} \wedge \bar{e_j^*}$$

Let $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$ be a local orthonormal basis of $(\Sigma, \mu)$, such that

$$j(\frac{\partial}{\partial u_1}) = \frac{\partial}{\partial u_2}, \quad j(\frac{\partial}{\partial u_2}) = -\frac{\partial}{\partial u_1},$$

then the Cauchy-Riemann equation $(df + J \circ df \circ j)(\frac{\partial}{\partial u_1}) = 0$ can be written locally as

$$f_1^i = -\sqrt{-1}f_2^i, \quad f_1^{\bar{i}} = \sqrt{-1}f_2^{\bar{i}},$$

where $df(\frac{\partial}{\partial u_1}) = f_1^i e_i + f_1^{\bar{i}}\bar{e_i}$ and $df(\frac{\partial}{\partial u_2}) = f_2^i e_i + f_2^{\bar{i}}\bar{e_i}$. From these equations we deduce that

$$\begin{aligned} f^*\omega|_{V_z} &= \omega_{ij}^{(2,0)}(t_r, z)f^*(e_i^* \wedge e_j^*) + \omega_{ij}^{(1,1)}(t_r, z)f^*(e_i^* \wedge \bar{e_j^*}) \\ &\quad + \omega_{ij}^{(0,2)}(t_r, z)f^*(\bar{e_i^*} \wedge \bar{e_j^*}) \\ &= -\sqrt{-1}\omega_{ij}^{1,1}(t_r, z)(f_1^i f_1^{\bar{j}} + f_2^i f_2^{\bar{j}})du_1 \wedge du_2 \end{aligned}$$

The fact that $J|_{V_z}$ is a $\omega|_{V_z}$-tamed almost complex structure implies that (see the choice of metric $h$ above)

$$\begin{aligned}
&\omega|_{V_z}(df(\frac{\partial}{\partial u_1}), J\circ df(\frac{\partial}{\partial u_1})) + \omega|_{V_z}(df(\frac{\partial}{\partial u_2}), J\circ df(\frac{\partial}{\partial u_2}))\\
&\geq c\cdot(|df(\frac{\partial}{\partial u_1})|_h^2 + |df(\frac{\partial}{\partial u_2})|_h^2)\\
&\doteq c|df|_h^2.
\end{aligned}$$

On the other hand, direct computation shows that

$$\begin{aligned}
&\omega|_{V_z}(df(\tfrac{\partial}{\partial u_1}), J\circ df(\tfrac{\partial}{\partial u_1})) + \omega|_{V_z}(df(\tfrac{\partial}{\partial u_2}), J\circ df(\tfrac{\partial}{\partial u_2}))\\
&\quad = -\sqrt{-1}\omega_{ij}^{1,1}(t_r, z)(f_1^i\bar{f_1^j} + f_2^i\bar{f_2^j})du_1\wedge du_2,
\end{aligned}$$

and so the lemma follows.

**Lemma 2.1.3** *There are $\epsilon_0 > 0$ and $C > 0$ which depend only on $A, \tilde{J}$, and $\tilde{\omega}$, such that for any $J$-holomorphic map $f^{(r)} \in \mathcal{M}_A^v(\Sigma^{(r)}, \mathcal{X}_Z, J^{(r)}, 0)$, $f^{(r)} : \Sigma^{(r)} \to V_{z_r}$ for some $z_r \in Z$, and any conformal metric $\mu_r$ on $\Sigma^{(r)}$ with curvature bounded by 1, if $\int_{B_{2R}(x)} |df^{(r)}|_{\mu_r}^2 dvol_{\mu_r} \leq \epsilon_0$ and the injectivity radius at $x$ is not less than $2R$, where $x \in \Sigma^{(r)}$, $R > 0$, we have*

$$\sup_{B_R(x)} |df^{(r)}|_{\mu_r}^2 \leq \frac{C}{R^2}$$

*where $B_R(x)$ is the geodesic ball centered at $x$ and with radius $R$. Consequently $\| f^{(r)} \|_{C^4(B_R(x))} \leq C_4$ for some constant $C_4$ independent of $z_r$.*

**Proof:** This is essentially Theorem 2.3 in [PW]. Here we briefly elaborate on the establishment of the Bochner type formula. Fix a connection on bundle $E$ which is compatible with the metric $h$ and the complex structure $J^{(\infty)}$.

Suppose that in local trivialization in $V_z$, $f^{(r)} = (f_1^{(r)}, \cdots, f_{2n}^{(r)})$ and $B_{2R}(x)$ has coordinates $u_1, u_2$. By direct computation, we have following formula

$$\begin{aligned}
&\Delta(df^{(r)}) = \nabla^*\nabla(df^{(r)}) \\
&= \text{polynomials of } \frac{\partial f_i^{(r)}}{\partial u_\alpha} \text{ up to degree } 3 + \text{ linear combinations of} \\
&\frac{\partial^2 f_i^{(r)}}{\partial u_\alpha\, \partial u_\beta} \cdot \frac{\partial f_j^{(r)}}{\partial u_\gamma},
\end{aligned}$$

where the coefficients of the polynomial involve $\tilde{J}$, the metric $h$, the connection $\nabla$, and their derivatives in the direction of $V_{z_r}$.

Then

$$\begin{aligned}
&\Delta_{\mu_r}(|df^{(r)}|^2_{\nu_r,h}) = 2\langle\nabla^*\nabla(df^{(r)}), df^{(r)}\rangle - 2\langle\nabla(df^{(r)}), \nabla(df^{(r)})\rangle \\
&= \tilde{C}_1|df^{(r)}|^2_{\mu_r,h} + \tilde{C}_2|df^{(r)}|^4_{\mu_r,h} - \langle\nabla(df^{(r)}), \nabla(df^{(r)})\rangle \\
&\quad + \text{linear combinations of } \frac{\partial^2 f_i^{(r)}}{\partial u_\alpha\, \partial u_\beta} \cdot \frac{\partial f_j^{(r)}}{\partial u_\gamma} \cdot \frac{\partial f_k^{(r)}}{\partial u_\delta} \\
&\leq C_1|df^{(r)}|^2_{\mu_r,h} + C_2|df^{(r)}|^4_{\mu_r,h} + \epsilon \cdot \langle\nabla(df^{(r)}), \nabla(df^{(r)})\rangle \\
&\quad + C(\epsilon)|df^{(r)}|^4_{\mu_r,h} - \langle\nabla(df^{(r)}), \nabla(df^{(r)})\rangle,
\end{aligned}$$

where we use the Cauchy-Schwartz inequality with $\epsilon \leq 1$ in the last step. Note that since $J^{(r)}, \omega^{(r)}, h, \nabla$, and their derivatives in the $V_{z_r}$ directions all depend continuously on the parameters $t_r$ and $z_r$, we can choose constants $C_1, C_2, C(\epsilon)$ to be independent of $t_r, z_r$. So we have

$$\Delta_{\mu_r}(|df^{(r)}|^2_{\mu_r,h}) \leq C_1|df^{(r)}|^2_{\mu_r,h} + C_2|df^{(r)}|^4_{\mu_r,h},$$

and the lemma is proved.

Using the trivilization in Remark 2.1.1 to identify $V_{z_r}$ with $V_z$ and seeing that the image of $f^{(r)}$ lies in $V_z$, the rest of proof becomes similar to the proof of Proposition 3.1 in [RT], and so we omit it. Observe that the image of the limit $f$ is contained in $V_z$. □

The above proposition tell us that the limit map in the compactification of the fiberwise moduli space are still those that could possibly appear in the non-fiberwise case (the so-called Gromov-Uhlenbeck compactification). So if we could establish a transversality theorem in the fiberwise case, counting the dimension of boundary will go through as in the non-fiberwise case. It is this transversality theorem we now turn to.

## 2.2 Several transversality results

**Definition 2.2.1** *Let $\Sigma, J$, and $\nu$ be as in Definition 2.1.2. A fiberwise $\Sigma$-cusp $(J,\nu)$-perturbed map $f$ is a continuous map from $\Sigma'$ to $V_z$ for some $z \in Z$, where the domain $\Sigma'$ of $f$ is obtained by joining a chain of $S^2$'s at some double points of $\Sigma$ to separate the two components at the double points, and then attaching some trees of $S^2$'s. The restriction of $f$ to $\Sigma$ is a $(J,\nu)$-perturbed holomorphic map and its restriction to the $S^2$'s is $J$-holomorphic. We call components of $\Sigma$ principal components and the others bubble components. A fiberwise $\Sigma$-cusp curve is an equivalence class of cusp maps modulo the parameterization groups of bubbles.*

By Proposition 2.1.1, we can compactify $\mathcal{M}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ by adding all possible fiberwise $\Sigma$-cusp curves with total homology class $A = A_1 + \cdots + A_m$. We divide the set of fiberwise cusp curves by some equivalence relations: (i) Some of the bubble components may be multiple covering maps, in this case we simply forget the multiplicity and take the reduced map onto its image; (ii) Adjacent or consecutive bubbles have the same image, and these we collapse into one bubble. Note that the resulting curves are still fiberwise cusp curves with possibly different total homology class. Let $\bar{\mathcal{M}}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ be the quotient of the fiberwise cusp curve compactification of $\mathcal{M}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ by above equivalence relation.

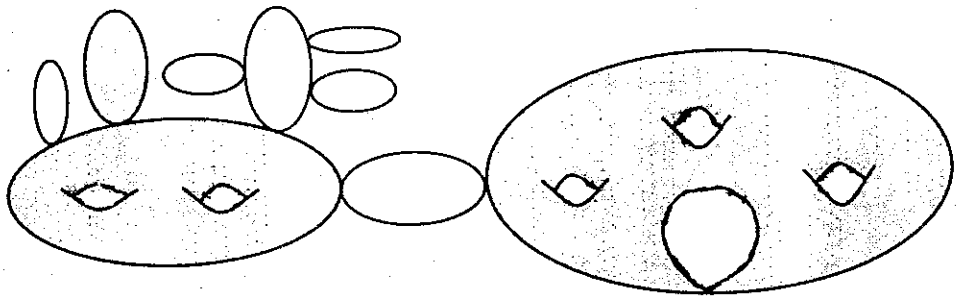

Figure 2.1: Domain of a cusp curve

**Proposition 2.2.1** *Under Assumption ($\star$), given any compact finite dimensional pseudo-submanifold $Z$ of $Y$, for a generic $(J, \nu)$, $\mathcal{M}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ is an oriented pseudo-manifold of real dimension $2C_1(V) \cdot A + 2n(1 - g(\Sigma)) + \dim_{\mathbf{R}} Z$, and $\bar{\mathcal{M}}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu) \setminus \mathcal{M}^{\nu}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ consists of finite many pieces (called strata) and each stratum is branchedly covered by a smooth manifold of codimension at least 2. Here $A = A_1 + \cdots + A_m$ and*

*$g(\Sigma)$ is the genus of the stable curve. More precisely, $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_Z,J,\nu)$ consists of smooth stratum $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ of dimension $2C_1(V)\cdot A+2n(1-g(\Sigma))+\dim_{\mathbb{R}}Z^\alpha$ where $Z^\alpha$ run through all smooth stratum of $Z$. $\bar{\mathcal{M}}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_Z,J,\nu)\setminus\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_Z,J,\nu)$ consists of all $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)\setminus\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ and each $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)\setminus\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ consists of finitely many pieces, each of which is branch-covered by a smooth manifold of codimension at least 2 in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$. Here $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ stands for the partial compactification of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$. By partial compactification, we mean that we do not include limits of family of curves $f^{(r)}$ in $\mathcal{M}^v_{A_1,\cdots,A_m}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ with image in $V_{z_r}$, where the limit point $z_\infty$ is not in $Z^\alpha$.*

**Proof:** By Proposition 2.1.1, all we need to show is: for a generic $(J,\nu)$, for each stratum $Z^\alpha$ of $Z$, $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ is a smooth manifold of dimension $2C_1(V)\cdot A+2n(1-g(\Sigma))+\dim_{\mathbb{R}}Z^\alpha$, and $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)\setminus\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ consists of finite many pieces, each piece branch-covered by a smooth manifold of codimension at least 2 in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$. We divide the proof into three steps.

**Step 1.** The stratum decomposition. A stratum in $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)\setminus\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ is a set of fiberwise cusp curves satisfying : (1) They have domain of the same homeomorphic type; (2) Each connected component has a fixed homology class; (3) We specify those components which have the same image even though they may not be adjacent to each other. We denote by $D$ a set with these three datas: a homeomorphic type of the domain of the

fiberwise cusp curves with intersection points, a homology class associated to each component, and a specification of components with the same image.

**Definition 2.2.2** *Let $D$ be given as above, $\Sigma_1, \cdots, \Sigma_m$ be the principal components $\Sigma$, and $B_1, \cdots, B_k$ be the bubble components of $D$. $D$ is called $A$-admissible if there are positive integers $b_1, \cdots, b_k$ such that $A = \sum_1^m [P_i] + \sum_1^m b_j[B_j]$, where $[P_1], \cdots, [P_m], [B_1], \cdots, [B_k]$ denote the homology classes associated with $\Sigma_1, \cdots, \Sigma_m, B_1, \cdots, B_k$. We say that $D$ is $(J, \nu)$-effective if every principal component can be represented by a $(J, \nu)$-map and every bubble component can be represented by a $J$-holomorphic map.*

Denote by $\mathcal{D}_{A,\Sigma}^{J,\nu}$ the set of all $A$-admissible, $(J, \nu)$-effective $D$'s. It is proved in [RT] that $\mathcal{D}_{A,\Sigma}^{J,\nu}$ is a finite set. For each $D \in \mathcal{D}_{A,\Sigma}^{J,\nu}$, let $\mathcal{M}_D^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ be the stratum in $\bar{\mathcal{M}}^{v,p}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \setminus \mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ specified by $D$. We perform one more reduction by identifying the domains of those components which have the same image, and change the homology class accordingly. Suppose that the resulting new domain and homology class of each component are specified by $\bar{D}$. Denote by $\mathcal{M}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ the space of cusp $(J, \nu)$-maps whose domain and the homology class of each components are specified by $\bar{D}$. Then for each $f \in \mathcal{M}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$, different bubble components have different image and each bubble map is somewhere injective. However the image of a principal component may be only one point. We identify $\mathcal{M}_D^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ with $\mathcal{M}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ below.

Again let $\Sigma_1, \cdots, \Sigma_m$ be the principal components, $\tilde{\Sigma}_1, \cdots, \tilde{\Sigma}_m$ be their smooth resolution($\Sigma_i$ may have double points), and $B_1, \cdots, B_k$ be the bub-

ble components. Now we begin to construct a smooth branched covering of $\mathcal{M}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$. Let $\Sigma_{\bar{D}}$ be the domain of the maps in $\mathcal{M}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$, and consider

$$\tilde{\mathcal{M}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) = \{f : \Sigma_{\bar{D}} \to \mathcal{X}_{Z^\alpha} |\ f \text{ is a continuous fiberwise map,}$$
$$f|_{\Sigma_i} \in \mathcal{M}^v_{[P_i]}(\Sigma_i, \mathcal{X}_{Z^\alpha}, J, \nu_i), f|_{B_i} \in \tilde{\mathcal{M}}^{v_i*}_{[B_j]}(S^2, \mathcal{X}_{Z^\alpha}, J, 0),$$
$$Im(f_{B_j}) \neq Im(f_{B_{j'}}), \text{if } j \neq j'\},$$

where $\tilde{\mathcal{M}}^{v_i*}_{[B_j]}(S^2, \mathcal{X}_{Z^\alpha}, J, 0) \subset \mathcal{M}^v_{[B_j]}(S^2, \mathcal{X}_{Z^\alpha}, J, 0)$ is the subspace of non-multiple covering maps. The action of the reparametrization group on each bubble component induces a $G^k$ action on $\tilde{\mathcal{M}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$, $G = PSL_2$, so $\mathcal{M}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) = \tilde{\mathcal{M}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)/G^k$. We first construct a covering of $\tilde{\mathcal{M}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$. Let $x^i_1, \cdots, x^i_{p_i} \in \Sigma_i$ be $p_i$ double points or marked points which are bubbling points. Let $h_i$ be the number of intersection points on $\Sigma_i$ and $h^j$ be the number of intersection points on the bubble component $B_j$. Consider the evaluation map

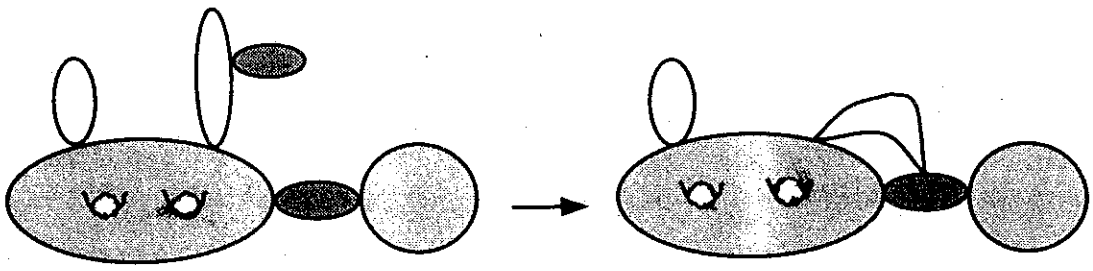

Figure 2.2: Creating a cycle in the domain

$$e_{\bar{D}} : \prod_i (\mathcal{M}^v_{[P_i]}(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, J, \nu_i) \times (\tilde{\Sigma}_i)^{h_i - p_i}) \times \prod_j (\tilde{\mathcal{M}}^{v,*}_{[B_j]}(S^2, \mathcal{X}_{Z^\alpha}, J, 0) \times (S^2)^{h^j})$$
$$\longrightarrow \prod_i \mathcal{X}^{h_i}_{Z^\alpha} \times \prod_j \mathcal{X}^{h^j}_{Z^\alpha}$$

where $e_{\bar{D}}$ is defined as follows:

For each $\tilde{\Sigma}_i$, we define

$$e_{P_i} : \mathcal{M}^v_{[P_i]}(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, J, \nu_i) \times (\tilde{\Sigma}_i)^{h_i - p_i} \longrightarrow \mathcal{X}^{h_i}_{Z^\alpha}$$
$$e_{P_i}(f, \tilde{x}_1, \cdots, \tilde{x}_{h_i - p_i}) = (f(x^i_1), \cdots, f(x^i_{p_i}), f(\tilde{x}_1), \cdots, f(\tilde{x}_{h_i - p_i})).$$

For each $B_j$, we define

$$e_{B_j} : \tilde{\mathcal{M}}^{v,*}_{[B_j]}(S^2, \mathcal{X}_{Z^\alpha}, J, 0) \times (S^2)^{h^j} \longrightarrow \mathcal{X}^{h^j}_{Z^\alpha}$$
$$e_{B_j}(f, y_1, \cdots, y_{h^j}) = (f(y_1), \cdots, f(y_{h^j})).$$

Then $e_{\bar{D}} = \prod_i e_{P_i} \times \prod_j e_{B_j}$. Recall that if $M, N$ are submanifolds of a manifold $X$, then $M \cap N$ can be interpreted as $(M \times N) \cap \Delta$, where $\Delta$ is the diagonal of $X \times X$. So we can realize the intersection pattern specified by $\bar{D}$ by constructing a "diagonal" in the product. Let $z_1, \cdots, z_{t_{\bar{D}}}$ be all the intersection points. For each $z_s$, let

$$I_s = \{\tilde{\Sigma}_{i_1}, \cdots, \tilde{\Sigma}_{i_q}, B_{j_1}, \cdots, B_{j_r}\}$$

be the set of components which intersect at $z_s$. Now we construct a fiberwise product $\mathcal{X}^v_{Z^\alpha, s}$ of $\mathcal{X}_{Z^\alpha}$ such that its diagonal describes the intersection

at $z_s$. This is done as follows: We allocate one or two factors from each of $\mathcal{X}_{Z^\alpha}^{h_{i_1}}, \cdots, \mathcal{X}_{Z^\alpha}^{h_{i_q}}$, according to whether or not $z_s$ is a self-intersection point of the corresponding principal component. We also allocate one factor from each of the $\mathcal{X}_{Z^\alpha}^{h^{j_k}}, k = 1, \cdots, r$. Here $\mathcal{X}_Z^{h_i}$ and $\mathcal{X}_Z^{h^j}$ are the images of $e_{P_i}$ and $e_{B_j}$, respectively. Then we take the fiberwise product of these factors over $Z^\alpha$, denote it by $\mathcal{X}_{Z^\alpha,s}^v$, and denote its diagonal by $\Delta_{Z^\alpha,s}^v$. Then $\Delta_{Z^\alpha,\bar{D}}^v = \Delta_{Z^\alpha,1}^v \times \cdots \times \Delta_{Z^\alpha,t_{\bar{D}}}^v \subset \mathcal{X}_{Z^\alpha}^{\sum h_i + \sum h^j}$ is the "diagonal" realizing the intersection pattern between components of $\bar{D}$. Let $\pi$ be the natural projection from

$$\prod_i (\mathcal{M}_{[P_i]}^v(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, J, \nu_i) \times (\tilde{\Sigma}_i)^{h_i - p_i}) \times \prod_j (\tilde{\mathcal{M}}_{[B_j]}^{v,*}(S^2, \mathcal{X}_{Z^\alpha}, J, 0) \times (S^2)^{h^j})$$

onto

$$\prod_i \mathcal{M}_{[P_i]}^v(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, J, \nu_i) \times \prod_j \tilde{\mathcal{M}}_{[B_j]}^{v,*}(S^2, \mathcal{X}_{Z^\alpha}, J, 0)$$

Then $\tilde{\mathcal{M}}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \subset \pi(e_{\bar{D}}^{-1}(\Delta_{Z^\alpha,\bar{D}}^v))$. But they might not be equal because we require that bubble components have different images. Define $\tilde{\mathcal{N}}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ to be $e_{\bar{D}}^{-1}(\Delta_{Z^\alpha,\bar{D}}^v) \cap \pi^{-1}(\tilde{\mathcal{M}}_{\bar{D}}^v\ (\Sigma, \mathcal{X}_{Z^\alpha},\ J,\ \nu))$, and $\mathcal{N}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ to be $\tilde{\mathcal{N}}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)/G^{\sum h^j}$. Then $\pi : \mathcal{N}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \to \mathcal{M}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is the required branched covering.

**Step 2.** We will show that for generic $(J, \nu)$, for each stratum $Z^\alpha$ in $Z$, $\mathcal{M}_{(A_1,\cdots,A_m)}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is a smooth oriented manifold of real dimension $2C_1(V) \cdot A + 2n(1-g(\Sigma)) + \dim_{\mathbb{R}} Z^\alpha$, and $\mathcal{N}_{\bar{D}}^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is a smooth manifold of real dimension

$$\sum(2C_1(V)\cdot[P_i]+2n(1-g_i))+\sum(2C_1(V)\cdot[B_j]+2n-6)$$
$$+2h_{\bar{D}}-2u_\Sigma-2s_D-2n(h_{\bar{D}}-t_{\bar{D}})+\dim_{\mathbb{R}} Z^\alpha,$$

where $g_i$ is the genus of $\Sigma_i$, $u_\Sigma$ is twice of the number of double points of $\Sigma$, $s_D$ is the number of marked points which are bubbling points, $t_{\bar{D}}$ is the number of intersection points of $\bar{D}$ and $h_{\bar{D}}=\sum h_i+\sum h^j$.

The basic idea of the proof is due to D. McDuff [M1]. Fix $k\geq 4$, $q>p$, $l\geq k+1$. Let $\mathcal{P}_{k-1,p}(\mathcal{X}_Z,\omega)\to\mathcal{J}^v_{l,q}(\mathcal{X}_Z,\omega)$ be a vector bundle whose fiber over $J\in\mathcal{J}^v_{l,q}(\mathcal{X}_Z,\omega)$ is the space of all inhomogeneous terms defined by $J$. $\mathcal{P}_{k-1,p}(\mathcal{X}_Z,\omega)$ is a smooth Banach manifold. Define

$$Map^v_{k,p}(\tilde{\Sigma}_i,\mathcal{X}_{Z^\alpha},[P_i])=\{f_i:\tilde{\Sigma}_i\to\mathcal{X}_{Z^\alpha}|\ f_i \text{ is a fiber map}$$
$$\text{with bounded } H^{k,p} \text{ norm, } [f_i(\tilde{\Sigma}_i)]=[P_i]\}$$
$$Map^v_{k,p}(S^2,\mathcal{X}_{Z^\alpha},[B_j])=\{f^j:S^2\to\mathcal{X}_{Z^\alpha}|\ f^j \text{ is a fiber map}$$
$$\text{with bounded } H^{k,p} \text{ norm, } [f^j(S^2)]=[B_j]\}.$$

(Here we require all the maps from $S^2$ to $\mathcal{X}_{Z^\alpha}$ to be somewhere injective).

Let $\Gamma_{k-1,p}(T\tilde{\Sigma}_i,e^*_{P_i}(E|_{\mathcal{X}_{Z^\alpha}}))$ and $\Gamma_{k-1,p}(TS^2,e^*_{B_j}(E|_{\mathcal{X}_{Z^\alpha}}))$ be vector bundles over $\prod_i Map^v_{k,p}(\tilde{\Sigma}_i,\mathcal{X}_{Z^\alpha},[P_i])\times\prod_j Map^v_{k,p}(S^2,\mathcal{X}_{Z^\alpha},[B_j])\times\mathcal{P}_{k-1,p}(\mathcal{X}_Z,\omega)$, whose fibers over a point $(f_i,f^j,J,\nu)$ are sections of $\Omega^{0,1}_{\Sigma_i}(f_i^*(E))$ and $\Omega^{0,1}_{S^2}(f^{j*}(E))$, respectively, with bounded $H^{k-1,p}$ norm. Then we obtain an infinite dimensional bundle with a section $S$ which is a partial differential operator,

$$\prod_i \Gamma_{k-1,p}(T\tilde{\Sigma}_i, e^*_{P_i}(E)) \times \prod_j \Gamma_{k-1,p}(TS^2, e^*_{B_j}(E))$$

$$\longrightarrow \prod_i Map^v_{k,p}(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, [P_i]) \times \prod_j Map^v_{k,p}(S^2, \mathcal{X}_{Z^\alpha}, [B_j]) \times \mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$$

$$S(f_i, f^j, J, \nu) = (\bar{\partial}_J f_i - \nu_i|_{\Sigma_i},\ \bar{\partial}_J f^j)$$

**Lemma 2.2.1** *The zero section of the above bundle is regular for the section $S$. i.e.,*

$$\tilde{\mathcal{M}}^v_{([P_i],[B_j],\mathcal{P})}(\mathcal{X}_{Z^\alpha}) = \{(f_i, f^j, J, \nu) |\ S(f_i, f^j, J, \nu) = 0\}$$

*is a Banach submanifold of* $\prod_i Map^v_{k,p}(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, [P_i]) \times \prod_j Map^v_{k,p}(S^2, \mathcal{X}_{Z^\alpha}, [B_j])$ $\times \mathcal{P}_{k-1,p}\ (\mathcal{X}_Z, \omega)$.

**Proof:** For any point $z \in Z^\alpha$, choose a neighborhood $U$ of $z$ in $Z^\alpha$, such that if we fix a trivialization $\mathcal{X}_{Z^\alpha}|_U = p^{-1}(U) \cong U \times V$ and $Tp^{-1}(U) \cong TU \times TV$, then the tangent space at $(f_i, f^j, J, \nu)$ is

$$T_{f_i} Map^v_{k,p}(\tilde{\Sigma}_i, \mathcal{X}_{Z^\alpha}, [P_i]) = \Gamma_{k,p}(\tilde{\Sigma}_i, f_i^* TV_z) \times T_z Z^\alpha$$

$$T_{f^j} Map^v_{k,p}(S^2, \mathcal{X}_{Z^\alpha}, [B^j]) = \Gamma_{k,p}(S^2, f^{j*} TV_z) \times T_z Z^\alpha$$

$$T_J \mathcal{J}^v_{l,q}(\mathcal{X}_Z, \omega) = \Gamma_{l,q}(End(E, J)).$$

(Here $\pi \circ f_i = z$, $\pi \circ f^j = z$ and $End(E, J)$ is the set of all the endmorphism of $E$ which anti-commutes with $J$). So

$$T_{(J,\nu)}\mathcal{P}_{k-1,p}(\mathcal{X}_Z,\omega) = \Gamma_{l,q}(End(E,J)) \times \prod_i \Gamma_{k-1,p}(\pi_1^* T\tilde{\Sigma}_i, \pi_2^* E)$$

$$T_{(f_i,0)}\Gamma_{k-1,p}(T\tilde{\Sigma}_i, e^*_{P_i}(E)) = \Gamma_{k,p}(\tilde{\Sigma}_i, f_i^* TV_z) \times T_z Z^\alpha \times \Omega^{0,1}_{(\tilde{\Sigma}_i,k-1,p)}(f_i^* TV_z)$$

$$T_{(f^j,0)}\Gamma_{k-1,p}(TS^2, e^*_{B_j}(E)) = \Gamma_{k,p}(S^2, f^{j*} TV_z) \times T_z Z^\alpha \times \Omega^{0,1}_{(S^2,k-1,p)}(f^{j*} TV_z)$$

Note: the fiber directions of the above two tangent spaces are $\Omega^{0,1}_{(\Sigma_i,k-1,p)}(f_i^*$ $TV_z)$ and $\Omega^{0,1}_{(S^2,k-1,p)}$ $(f^{j*}TV_z)$, respectively. Let us compute the projection of the linearization of $S$ to the fiber direction at $(f_i, f^j, J, \nu)$ satisfying $S(f_i, f^j, J, \nu)$ $= 0$.

$$\prod_i \Gamma_{k,p}(\tilde{\Sigma}_i, f_i^* TV_z) \times \prod_j \Gamma_{k,p}(S^2, f^{j*} TV_z) \times T_z Z^\alpha \times \Gamma_{l,q}(End(E,J))$$
$$\times \prod_i \Gamma_{k-1,p}(\pi_1^* T\tilde{\Sigma}_i, \pi_2^* E) \xrightarrow{DS(f_i,f^j,J,\nu)}$$
$$\prod_i \Gamma_{k,p}(\tilde{\Sigma}_i, f_i^* TV_z) \times \prod_j \Gamma_{k,p}(S^2, f^{j*} TV_z) \times T_z Z^\alpha \times \prod_i \Omega^{0,1}_{(\tilde{\Sigma}_i,k-1,p)}(f_i^* TV_z)$$
$$\times \prod_j \Omega^{0,1}_{(S^2,k-1,p)}(f^{j*} TV_z) \xrightarrow{P}$$
$$\prod_i \Omega^{0,1}_{(\tilde{\Sigma}_i,k-1,p)}(f_i^* TV_z) \times \prod_j \Omega^{0,1}_{(S^2,k-1,p)}(f^{j*} TV_z).$$

We fix a Hermitian metric $h$ on $E$ which is compatible with $J$, also fix a connection $\nabla$ on $E$ which is compatible with this metric and the fiberwise almost complex structure. Further suppose that for any $z \in Z$, $\nabla|_{V_z}$ has torsion $\frac{1}{4}N_{J|_{V_z}}$, where $N_{J|_{V_z}}$ is the Nijenhuis torsion tensor of $(V_z, J|_{V_z})$. The construction of such a connection is the same as the construction of such a connection in the non-fiberwise case. $\nabla$ defines a parallel translations $P_{i,t}$ and $P_t^j$ along path $\{f_{it}\}_{i=1}^m$ and $\{f_t^j\}_{j=1}^k$, respectively, on the bundle $E$, which will be used to define $DS$. We now can write down $Pr \circ DS$:

$$Pr \circ DS(f_i, f^j, J, \nu)(\xi, \eta, \gamma, w, \delta)$$

$$= Pr \circ DS(\xi, 0, 0, 0, 0) + Pr \circ DS(0, \eta, 0, 0, 0) + Pr \circ DS(0, 0, \gamma, 0, 0)$$

$$+ Pr \circ DS(0, 0, 0, w, 0) + Pr \circ DS(0, 0, 0, 0, \delta)$$

$$Pr \circ DS(\xi, 0, 0, 0, 0) = ((\nabla \xi_i + J|_{V_z} \circ \nabla \xi_i \circ j_{\Sigma_i} + \frac{1}{4} N_{J|_{V_z}}(\partial_{J|_{V_z}}(f_i), \xi_i)_{i=1}^m, 0)$$

$$Pr \circ DS(0, \eta, 0, 0, 0) = (0, (\nabla \eta^j + J|_{V_z} \circ \nabla \eta^j \circ j_{S^2}$$

$$+ \frac{1}{4} N_{J|_{V_z}}(\partial_{J|_{V_z}}(f^j), \eta^j))_{j=1}^k)$$

$$Pr \circ DS(0, 0, 0, w, 0) = ((\frac{1}{2} w(f_i) \circ df_i \circ j_{\Sigma_i})_{i=1}^m, (\frac{1}{2} w(f^j) \circ df^j \circ j_{S^2})_{j=1}^k)$$

where $\xi = (\xi_1, \cdots, \xi_m), \eta = (\eta^1, \cdots, \eta^k)$ and $\delta = (\delta_1, \cdots, \delta_m)$.

The computation of these formulas is exactly the same as in the non-fiberwise case since we may choose the families of maps varying in the directions of $\xi$ or $\eta$ to lie in the same fiber $V_z$(see [M1]).

To compute $Pr \circ DS(0, 0, \gamma, 0, 0)$, we choose families of maps, $f_{i,t} = f_i, f_t^j = f^j, J_t = J$, $z_t$ satisfying $\frac{dz_t}{dt}|_{t=0} = \gamma$, and $\nu_t = \nu$. Here we suppose $\mathrm{Im}(f_t) \subset V_{z_t}$, so

$$Pr \circ DS(0, 0, \gamma, 0, 0)$$

$$= ((\frac{d}{dt} P_{i,t}(df_{i,t} + J|_{V_{z_t}} \circ df_{i,t} \circ j_{\Sigma_i} - \nu_i|_{\Sigma_i}))_{i=1}^m,$$

$$(\frac{d}{dt} P_t^j(df_t^j + J|_{V_{z_t}} \circ df_t^j \circ j_{S^2}))_{j=1}^k)$$

$$= ((\nabla_\gamma(df_i + J|_{V_{z_t}} \circ df_i \circ j_{\Sigma_i} - \nu_i|_{\Sigma_i}))_{i=1}^m, (\nabla_\gamma(df^j + J|_{V_{z_t}} \circ df^j \circ j_{S^2}))_{j=1}^k)$$

$$= ((\frac{d}{dt}(df_i + J|_{V_{z_t}} \circ df_i \circ j_{\Sigma_i} - \nu_i|_{\Sigma_i}) + \Gamma_{\star\star}^\star(df_i + J|_{V_{z_t}} \circ df_i \circ j_{\Sigma_i} - \nu_i|_{\Sigma_i}))_{i=1}^m,$$

$$e_{\bar{D}} : \tilde{\mathcal{M}}^{v}_{([P_i],[B_j],\mathcal{P})}(\mathcal{X}_{Z^\alpha}) \times \prod_i \tilde{\Sigma}_i^{h_i - p_i} \times \prod_j (S^2)^{h^j} \longrightarrow \mathcal{X}_{Z^\alpha}^{\sum h_i + \sum h^j}$$

is transversal to $\Delta^{v}_{Z^\alpha,\bar{D}}$. Let $(v_1, \cdots, v_{t_{\bar{D}}})$ be a point in $\Delta^{v}_{Z^\alpha,\bar{D}}$, $I_s = \{\tilde{\Sigma}_{i_1}, \cdots, \tilde{\Sigma}_{i_p}, B_{j_1}, \cdots, B_{j_r}\}$, and

$$(f_i, f^j, J, \nu, x_1, \cdots, x_{\tilde{m}}, y_1, \cdots, y_k) \in e_{\bar{D}}^{-1}(v_1, \cdots, v_{t_{\bar{D}}}).$$

So $f_{i_1}(x_{i_1}) = \cdots = f_{i_p}(x_{i_p}) = f^{j_1}(y_{j_1}) = \cdots = f^{j_r}(y_{j_r}) = v_s$. By the Hartman-Winter Lemma ([JS], Lemma 2.6.1), there are only finitely many accumulation points of $Imf^j \cap Imf^{j'}$, since we require that $Imf^j \neq Imf^{j'}$ for $j \neq j'$. Therefore, for each $f^{j_s}$, we can choose a small disc $D_{j_s}$ around $y_{j_s}$ such that there is a smaller disc $D'_{j_s}$ satisfying that the annulus $D_{j_s} - D'_{j_s}$ does not intersect other bubble components. By the work of D. McDuff [M1], given any tangent vector $X \in T_{f^{j_s}(y_{j_s})}V_z$, where $f(\Sigma) \subset V_z$, for some $z \in Z^\alpha$, there is a perturbation of the almost complex structure $J|_{V_z}$ to $(J|_{V_z})_t$ on $V_z$ in a small neighborhood of $Imf^{j_s}|_{D_{j_s}-D'_{j_s}}$, and also a perturbation $f_t^{j_s}$ of $f^{j_s}$ on $D_{j_s}$ such that $f_t^{j_s}$ is $(J|_{V_z})_t$ holomorphic, $f_0^{j_s} = f^{j_s}$, and $\frac{df_t^{j_s}}{dt}(y_{j_s})|_{t=0} = X$. We can patch $f_t^{j_s}$ with $f^{j_s}|_{S^2-D_{j_s}}$ to get $f_t^{j_s}$ defined on the whole $S^2$ such that $f_t^{j_s}|_{D_{j_s}-D'_{j_s}}$ does not intersect other bubble components either. Note that $(J|_{V_z})_t$ can be extended to a family of complex structure $J_t$ on $E$, i.e. we can choose perturbation $(J|_{V_z})_t$ which comes from $\mathcal{P}^{v}_{k-1,p}(\mathcal{X}_Z, \omega)$. Notice that other bubbles are also $J_t$-holomorphic and $\frac{df_t^{j_s}}{dt}(y_{j_s})|_{t=0} = X$.

For a principal component $f_{i_s}$, the argument is easier since we can perturb the inhomogeneous term. We can just choose an arbitrary perturbation $f_{i_s,t}$

on a small disc $D_{i_s}$ around $x_{i_s}$ such that $\frac{df_{i_s,t}}{dt}(x_{i_s})|_{t=0} = X$, and then simply perturb $\nu_i|_{V_z}$ to $(\nu_i|_{V_z})_t$ while keeping $(\nu_{i_s}|_{V_z})_t = \bar{\partial}_J f_{i_s,t}$ on the graph of $f_{i_s,t}$. Then $f_{i_s,t}$ satisfies an inhomogeneous Cauchy-Riemann equation with inhomogeneous term $(\nu_{i_s}|_{V_z})_t$. Note that $(\nu_{i_s}|_{V_z})_t$ can be extended to a family of elements in $\mathcal{P}^v_{k-1,p}(\mathcal{X}_Z, \omega)$. Applying this argument to each point $v_s$ and every point in $e_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}})$, we have shown that the map $e_{\bar{D}}$ is transversal to $\Delta^v_{Z^\alpha,\bar{D}}$. Therefore $e_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}})$ is a smooth Banach manifold. Moreover we have the following Fredholm map

$$e_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}}) \xrightarrow{\tilde{\pi}} \mathcal{P}^v_{k-1,p}(\mathcal{X}_Z, \omega).$$

By the Sard-Smale Theorem, for a generic element $(J, \nu)$ of $\mathcal{P}^v_{k-1,p}(\mathcal{X}_Z, \omega)$, its preimage $\tilde{\pi}^{-1}(J, \nu)$ is a smooth manifold (it is crucial here that $\mathcal{P}^v_{k-1,p}(\mathcal{X}_Z, \omega)$ has smooth structure). This implies that $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is a smooth manifold when we choose $\bar{D}$ with no bubbles $f^j$, and in general $\tilde{\mathcal{N}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is smooth manifold. Since $G^{\sum h^j}$ acts freely on $\tilde{\mathcal{N}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$, $\mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) = \tilde{\mathcal{N}}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)/G^{\sum h^j}$ is smooth. Note that $G^{\sum h^j}$ acts freely on $e_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}})$, so we can define a smooth manifold $E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}}) = e_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}})/G^{\sum h^j}$, a space that will be used below.

**Step 3.** A routine counting dimension argument gives the dimension formula (see [RT]). We can see that the codimension of the boundary of the moduli space is at least two in the following way. Notice that for most fibers of the fibration $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \to Z^\alpha$, the boundary part of a fiber is of codimension two in the fiber as we know from the non-fiberwise case. Since

$\bar{\mathcal{M}}^{v,p}_{A_1,\cdots,A_m}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ has the required smooth structure, its boundary has codimension at least two. □

**Remark 2.2.1** *Since the moduli space has a canonically defined orientation in the non-fiberwise case, this orientation and the orientation of $Z^\alpha$ give a natural orientation of the moduli space $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ in the fiberwise case.*

**Proposition 2.2.2** *For a generic $(J,\nu) \in \mathcal{P}^v_{k-1,p}(\mathcal{X}_Z, \omega)$, and $D \in \mathcal{D}^{J,\nu}_{A,\Sigma}$,*

$$\dim \mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \leq 2C_1(V)\cdot A + 2n(1-g(\Sigma)) - 2k_{\bar{D}} - 2s_D + \dim Z^\alpha$$

*where $k_{\bar{D}}$ is the number of bubble components of $D$ (not $\bar{D}$), and $s_D$ is the number of marked points which are also bubbling points.*

**Proof:** Similar to Proposition 4.14 in [RT]. We omit it. □

We want to prove another transversality result which will be used in defining the fiberwise mixed invariants later. Let $\{U_i\}_{i=1}^c, \{W_j\}_{j=1}^d$ be two (possibly non-compact) families of smooth manifolds, and $L_i : U_i \to \mathcal{X}_Z, M_j : W_j \to \mathcal{X}_Z$ be smooth maps. We require that the images all lie in one stratum, say $\mathcal{X}_{Z^\alpha}$. For each data $\bar{D}$, we know that $\mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ is a smooth manifold for generic $(J,\nu)$. Consider the intersections of the components of $\mathrm{Im}f$ and $\mathrm{Im}(M_j)$, where $f \in \mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ consists of principal components $(f_i)_{i=1}^m$ and bubble components $(f^j)_{j=1}^k$. Without loss of generality, we may assume that $Im(M_1), \cdots, Im(M_p)$ intersect the principal components $\mathrm{Im}f_{i_s}$ of $f$, $1 \leq s \leq p$, and $ImM_{p+1}, \cdots, M_d$ intersect the bubble components $\mathrm{Im}f^{j_s}$ of $f$,

$p+1 \leq s \leq d$, we call this an intersection pattern, and denote it by $T$. If $M_i$ intersect more than one component, we simply choose one of them. Let $C = (\Sigma; x_1, \cdots, x_c)$ be a stable curve, and $X = (x_1, \cdots, x_c)$ be the set of marked points. We consider evaluation map

$$
\begin{aligned}
& e_{X,T} \times L \times M : \mathcal{N}_{\bar{D}}^{\nu}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) \times \prod_{s=1}^{p} \hat{\Sigma}_{i_s} \times (S^2)^{d-p} \times U \times W \\
& \longrightarrow \mathcal{X}_{Z^\alpha}^{c+d} \times \mathcal{X}_{Z^\alpha}^{c+d} \\
& e_{X,T}(f_i, f^j; y_1, \cdots, y_p; y_{p+1}, \cdots, y_d) \\
& = (f(x_1), \cdots, f(x_c); f_{i_1}(y_1), \cdots, f_{i_p}(y_p); f^{j_{p+1}}(y_{p+1}), \cdots, f^{j_d}(y_d))
\end{aligned}
$$

where $U = \prod_i U_i$, $W = \prod_j W_j$, $L = \prod_i L_i$, $M = \prod_j M_j$, $G = PGL_2$, $\Delta_T^\alpha \subset \mathcal{X}_{Z^\alpha}^{c+d} \times \mathcal{X}_{Z^\alpha}^{c+d}$ is the diagonal corresponding to the intersection pattern $T$, and $\Delta_T^\alpha$ is defined similar to $\Delta_{Z^\alpha,\bar{D}}^{\nu}$. Note that $(f; \cdots)$ is in $(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha)$ if and only if $f(x_i) \in \mathrm{Im}(L_i)$, $\mathrm{Im} f_{i_s} \cap \mathrm{Im}(M_s) \neq \emptyset, 1 \leq s \leq p$, and $\mathrm{Im} f^{j_s} \cap \mathrm{Im}(M_s) \neq \emptyset$, $p+1 \leq s \leq d$.

**Proposition 2.2.3** *For a generic $(J, \nu) \in \mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$, any $D \in \mathcal{D}_{A,\Sigma}^{J,\nu}$, $(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha)$ is a smooth manifold of dimension*

$$
\begin{aligned}
& \dim \mathcal{N}_{\bar{D}}^{\nu}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) + 2d - codim L - codim M \\
\leq\ & 2C_1(V) \cdot A + 2n(1 - g(\Sigma)) + \dim Z^\alpha \\
& -2k_{\bar{D}} - 2s_D + 2d - codim L - codim M
\end{aligned}
$$

*where $s_D, k_{\bar{D}}$ are defined as in Proposition 2.2.2, $codim L_i = \dim \mathcal{X}_{Z^\alpha} - \dim U_i$, $codim C = \Sigma_i codim C_i$, $codim M_j = \dim \mathcal{X}_{Z^\alpha} - \dim W_j$, and $codim M = \Sigma_j codim M_j$.*

*When the proposition is true, we say that* $\mathcal{N}_{\bar{D}}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$ *is transversal to* $L \times M$ *for* $(X, T)$*, and such* $(J, \nu)$ *is good.*

**Proof:** We use the notation $E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha, \bar{D}})$ as defined at the end of the proof of Proposition 2.2.1. All we need to show is that

$$e_{(X,T,\mathcal{P})} \times L \times M : E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}}) \times \prod_{s=1}^{p} \tilde{\Sigma}_{i_s} \times (S^2)^{d-p} \times U \times W \longrightarrow \mathcal{X}_{Z^\alpha}^{c+d} \times \mathcal{X}_{Z^\alpha}^{c+d}$$

is transversal to $\Delta_T^\alpha$, where map $e_{(X,T,\mathcal{P})}$ is defined similarly to $E_{X,T}$. Since $E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}}) \to \mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$ is a Fredholm map, according to the Sard-Smale Theorem, for a generic $(J, \nu)$, $e_{X,T}$ is transversal to $\Delta_T^\alpha$.

To prove $e_{(X,T,\mathcal{P})} \times L \times M$ is transversal to $\Delta_T^\alpha$, assume that $(f_i, f^j, J, \nu, y_k, u_i, w_j) \in (e_{(X,T,\mathcal{P})} \times L \times M)^{-1}(\Delta_T^\alpha)$, $f(x_i) = L_i(u_i)$, $f_{i_s}(y_s) = M_s(w_s)$, $1 \le s \le p$, $f^{j_s}(y_s) = M_s(w_s)$, $p+1 \le s \le d$, we need to prove that for any $X_i \in T_{f(x_i)}\mathcal{X}_{Z^\alpha}, Y_s \in T_{f_{i_s}(y_s)}\mathcal{X}_{Z^\alpha}, 1 \le s \le p$, and $Y_s \in T_{f^{j_s}(y_s)}\mathcal{X}_{Z^\alpha}, p+1 \le s \le d$, we can find a family of curves $(f_{i,t}, f_t^j, J_t, \nu_t, y_{s,t}) \in E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}}) \times \prod_{s=1}^{p} \Sigma_{i_s} \times_{G^{d-p}} (S^2)^{d-p}$ such that $\frac{df_t(x_i)}{dt}|_{t=0} = X_i$, $\frac{df_{i_s,t}(y_{s,t})}{dt}|_{t=0} = Y_s, 1 \le s \le p$, and $\frac{df_t^{j_s}(y_{s,t})}{dt}|_{t=0} = Y_s, p+1 \le s \le d$. We achieve this by perturbing $(f_i, f^j, J, \nu, y_k)$ several times, each time satisfying one of the requirements satisfied without affecting the other. The perturbation goes as follows: suppose we want to make $\frac{df_{i_1,t}(y_{1,t})}{dt}|_{t=0} = Y_1$. If $Y_1$ is a fiber direction in $T_{f_{i_1}(y_1)}\mathcal{X}_{Z^\alpha}$, we then only need to perturb the curve within the fiber $V_z$, the perturbation being similar to the one in the proof of Proposition 2.2.1. The only new feature is that if $y_1$ is the double or cusp point of the domain of $f$, the perturbation has to keep the intersection pattern of the cusp curve, since otherwise the perturbed family

will not be in $E_{\bar{D}}^{-1}(\Delta^v_{Z^\alpha,\bar{D}})$. This can be done as follows: choose a curve $c_t$ in $V_z$ with $c_0 = f_{i_1}(y_1)$ and $\frac{dc_t}{dt}|_{t=0} = Y_1$. For any principal component intersecting at $f_{i_1}(y_1)$, say $f_i$, by perturbing the fiberwise inhomogeneous term, we can find $f_{i,t}$ which is a local perturbation of $f_i$ around $y_1$, such that $f_{i,t}(y_1) = c_t$. For any bubble component intersecting at $f_{i_1}(y_1)$, say $f^j$, by the work of D. McDuff[M2], we can actually perturb $J$ in the same way as in Proposition 2.2.1, with $f^j_t(y_1) = c_t$. Now the different components of the map $f_t$ intersect at the point $c_t$. Once again we can extend the perturbed $(J|_{V_z})_t, (\nu|_{V_z})_t$ to $(J_t, \nu_t) \in \mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$.

If $Y_1$ is not a fiber direction, we choose a trivialization of $\mathcal{X}_{Z^\alpha}$ around $z \in Z^\alpha$ (see Remark 2.1.1). We only study the case that $Y_1$ is a horizontal vector under the trivialization, since this is sufficient for proving the surjectivity. Choose a curve $z_t$ in $Z^\alpha$ with $z_0 = z, \frac{dz_t}{dt}|_{t=0} = Y_1$, we perturb $J$ and $\nu$ in the following way: $J_t|_{V_{z_t}} \cong J|_{V_z}, \nu_t|_{V_{z_t}} \cong \nu|_{V_z}$, so take $f_t = f$ but view that $\mathrm{Im} f_t$ lies in the fiber $V_{z_t}$. Now $f_{i_1,t}(y_{1,t}) = (z_t,$ fiber part), the horizontal part of $\frac{df_{i_1,t}(y_{1,t})}{dt}|_{t=0}$ is $Y_1$ and this perturbation keeps the intersection pattern of the cusp curve. So $e_{(X,T,\mathcal{P})} \times L \times M$ is transversal to $\Delta^\alpha_T$.

Dimension counting being routine, the inequality follows from Proposition 2.2.2, and the proposition is proved. □

**Remark 2.2.2** *i) If $\tilde{Z}$ is a submanifold of $Z^\alpha$, assume that $(J_0, \nu_0)$ is good for $\mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_{\tilde{Z}}, J_0, \nu_0)$, and consider the subspace $\tilde{\mathcal{P}}_{k-1,p}(\mathcal{X}_Z, \omega) \subset \mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$ which consists of $(J, \nu)$'s whose restriction to $\mathcal{X}_{\tilde{Z}}$ equals $(J_0, \nu_0)$. Then using this space instead of $\mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$ in the proofs of Proposition 2.2.1 and 2.2.2,*

*one can show that for a generic* $(J,\nu)$ *of* $\tilde{\mathcal{P}}_{k-1,p}(\mathcal{X}_Z,\omega)$, $\mathcal{N}^v_{\tilde{D}}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$ *is smooth and transversal to* $L\times M$, *since we may perturb* $(J,\nu)$ *only outside of* $\mathcal{X}_{\tilde{Z}}$.

*ii) If* $j_t$ *is a smooth family of complex structure on* $\Sigma$, $t\in[0,1]$, *denote by* $\Sigma_t$ $\Sigma$ *equipped with* $j_t$, *then one can consider the space* $\cup_t\mathcal{N}^v_{\tilde{D}}(\Sigma_t,\mathcal{X}_{Z^\alpha},J,\nu)$. *For a generic* $(J,\nu)\in\mathcal{P}_{k-1,p}(\mathcal{X}_Z,\omega)$, *this space is smooth and transversal to* $L\times M$. *The reason is: in the proof of Proposition 2.2.1 and 2.2.2, replacing those Map and* $E^{-1}_{\tilde{D}}(\Delta^v_{Z^\alpha,\tilde{D}})$ *spaces by their union over* $t$, $t\in[0,1]$, *the remaining argument is the same.*

The following proposition about the transversality of components of fiberwise cusp curves is important for the gluing argument in Chapter 4. Adopting the notation of Proposition 2.2.1, consider the moduli space $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)$. It consists of fiberwise $(J,\nu)$ perturbed holomorphic maps which have no bubbles and whose components intersect each other at double points. We would like to show that the subset where at least two components intersect non-transversally is of codimension 2. Without loss of generality, we can assume that $\Sigma$ has only two components $\Sigma_1$ and $\Sigma_2$. Let $y_1$ and $y_2$ be the distinguished points on $\Sigma_1$ and $\Sigma_2$ corresponding to the intersection. We allow a self-intersection by setting $\Sigma_1=\Sigma_2$ and using their smooth resolution below. We then define

$$e_{y_1,y_2}:\mathcal{M}^v_{(A_1,A_2)}(\Sigma,\mathcal{X}_{Z^\alpha},J,\nu)\to Hom(T_{y_1}\Sigma_1,E|_{\mathcal{X}_{Z^\alpha}})\oplus Hom(T_{y_2}\Sigma_2,E|_{\mathcal{X}_{Z^\alpha}})$$

$$e_{y_1,y_2}(f)=(df_1(y_1),df_2(y_2))$$

where $f_1 \in \mathcal{M}_{A_1}(\Sigma_1, \mathcal{X}_{Z^\alpha}, J, \nu_1)$, $f_2 \in \mathcal{M}_{A_2}(\Sigma_2, \mathcal{X}_{Z^\alpha}, J, \nu_2)$. A generic element in $Hom(T_{y_1}\Sigma_1,\ E|_{\mathcal{X}_{Z^\alpha}}) \oplus Hom(T_{y_2}\Sigma_2, E|_{\mathcal{X}_{Z^\alpha}})$ is a smooth fibration over $\mathcal{X}_{Z^\alpha}$ of dimension $10n + \dim Z^\alpha$ and maximal rank 4. If $f_1$ and $f_2$ do not intersect transversally, then $f$'s image will have lower rank. The set of homomorphisms of lower ranks is a union of manifolds $R_i$ consisting of homomorphisms of rank $i$ (i=0,1,2,3). $\dim R_0 = 2n + \dim Z^\alpha, \dim R_i = 2(i+1)n + \dim Z^\alpha + 4 - i$, i=1,2,3. Thus their codimensions are $8n, 6n-3, 4n-2, 2n-1$.

**Proposition 2.2.4** *For a generic $(J, \nu)$, $e_{y_1,y_2}$ is transversal to $R_i$. Hence $e^{-1}_{y_1,y_2}(R_i)$ has codimension at least $2n-1$ in $\mathcal{M}^v_{(A_1,A_2)}(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu)$, i= 0,1,2,3.*

**Proof:** Use the notation $E^{-1}_{\bar{D}}(\Delta^v_{Z^\alpha,\bar{D}})$ as defined at the end of the proof of Proposition 2.2.1. Now choose data $\bar{D}$ with no bubble components and denote $E^{-1}_{\bar{D}}(\Delta^v_{Z^\alpha,\bar{D}})$ by $\mathcal{M}^v_{(A_1,A_2)}(\Sigma, \mathcal{X}_{Z^\alpha}, \mathcal{P})$. Consider the map

$$E_{y_1,y_2} : \mathcal{M}^v_{(A_1,A_2)}(\Sigma, \mathcal{X}_{Z^\alpha}, \mathcal{P}) \to Hom(T_{y_1}\Sigma_1, E|_{\mathcal{X}_{Z^\alpha}}) \oplus Hom(T_{y_2}\Sigma_2, E|_{\mathcal{X}_{Z^\alpha}})$$

$$E_{y_1,y_2}(f, J, \nu) = (df_1(y_1), df_2(y_2)),$$

we can show that $E_{y_1,y_2}$ is transversal to $R_i$, i=0,1,2,3. The proof, which is similar to the proof of Proposition 2.2.2, is omitted(see Theorem 5.10 in [RT]).

□

# Chapter 3

# Fiberwise Gromov-Witten Mixed Invariants

We now begin the definition of fiberwise mixed invariants, our aim being to use it to define the fiberwise quantum cohomology. Let $C = (\Sigma; x_1, \cdots, x_c)$ be a stable curve of genus $g$, $\Sigma = (\Sigma_1, \cdots, \Sigma_m)$, $X = (x_1, \cdots, x_c)$, and $A_1, \cdots, A_m$ be homology classes in $H_2(V; \mathbb{Z})$, $A = A_1 + \cdots + A_m$. Without loss of generality, assume that $x_1, \cdots, x_{i_1}$ lie in $\Sigma_1$, $x_{i_1+1}, \cdots, x_{i_2}$ lie in $\Sigma_2$, $\cdots$, and $x_{i_{m-1}+1}, \cdots, x_{i_m}$ lie in $\Sigma_m$. Denote this position pattern by $P_1$. Take a bundle $\mathcal{X} \xrightarrow{p} Y$ satisfying Assumption $(\star)$. Let $\{\alpha_i\}_1^c$, $\{\beta_j\}_1^d$ be rational cohomology classes of $\mathcal{X}$, and $\gamma$ be the rational homology class of $Y$ satisfying

$$\sum_1^c \deg \alpha_i + \sum_1^d (\deg \beta_j - 2) = 2C_1(V) \cdot A + 2n(1-g) + \deg(\gamma). \qquad (3.1)$$

Here we work with rational coefficients for homology and cohomology, but we could use real or complex coefficients. In Chapter 6 we will actually use complex coefficients. Integer coefficients can not be used because an integral homology class may have no pseudo-manifold representative.

Let $F : Z \to Y$ be a pseudo-manifold representative of a homology class $\gamma$. Using the notation after Assumption $(\star)$, $\tilde{F}^*(\alpha_i)$ and $\tilde{F}^*(\beta_j)$ are then cohomology classes in the pseudo-manifold $\mathcal{X}_Z$. Denote their Poincare duals by $PD(\tilde{F}^*(\alpha_i))$ and $PD(\tilde{F}^*(\beta_j))$ respectively. We can represent $PD(\tilde{F}^*(\alpha_i))$ and $PD(\tilde{F}^*(\beta_j))$ by pseudo-manifolds $U_i$ and $W_j$ in $\mathcal{X}_Z$ in the following way: there are continuous maps

$$L_i : U_i \to \mathcal{X}_Z, \quad M_j : W_j \to \mathcal{X}_Z$$

such that on each stratum, say $U_i^s$ or $W_j^t$ , both $L_i$ and $M_j$ are smooth and each image $L_i(U_i^s)$ or $M_j(W_j^t)$ lies entirely in some stratum of $\mathcal{X}_Z$, say $\mathcal{X}_{Z^{\alpha_{i,s}}}$ or $\mathcal{X}_{Z^{\beta_{j,t}}}$. Furthermore they satisfy the following general position condition.

**General position condition:**

$$\dim \mathcal{X}_{Z^{\alpha_{i,s}}} - \dim U_i^s \geq \dim \mathcal{X}_Z - \dim U_i,$$

$$\dim \mathcal{X}_{Z^{\beta_{j,t}}} - \dim W_j^t \geq \dim \mathcal{X}_Z - \dim W_j.$$

Note that if $Z$ is smooth, then this definition coincides with the ordinary definition of the pseudo-manifold representative of a rational homology class.

Denote $L = \prod_i L_i, M = \prod_j M_j$, $L \times M : \prod_i U_i \times \prod_j W_j \to \mathcal{X}_Z^{c+d}$. Denote by Im the image of a map, fix an intersection pattern T: $\mathrm{Im} f_{s_j}$ intersects $\mathrm{Im} W_j$, $j = 1, \cdots, d$. Then define the evaluation map

$$e_{(\Sigma,P,J,\nu)} : \mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu) \times \prod_{j=1}^d \Sigma_{s_j} \longrightarrow \mathcal{X}_Z^{c+d}$$

$$(f; y_1, \cdots, y_d) \longrightarrow (f(x_1), \cdots, f(x_c); f_{s_1}(y_1), \cdots, f_{s_d}(y_d)).$$

Note that $e_{(\Sigma,P,J,\nu)}$ is a smooth map. It follows from our assumption on the degree of $\alpha_i$ and $\beta_j$ that the image of $e_{(\Sigma,P,J,\nu)}$ and $L \times M$ have complementary dimensions in $\mathcal{X}_Z^{c+d}$. Moreover, we have

**Proposition 3.1** *Under assumption (3.1), for a generic fiberwise almost complex structure $J$ and a generic inhomogeneous term $\nu$ in $\mathcal{P}_{k-1,p}(\mathcal{X}_Z, \omega)$, we have*

*i) There is no sequence $\{f_r\}_1^\infty$ in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ with $f_r(x_i)$ converges to a point in $L_i(U_i)$ as $r \to \infty$ , $f_{r,s_j}(\Sigma_{s_j}) \cap M_j(W_j) \neq \emptyset$ and at least one of the following is true:*

*a) for some $i$, $f_r(x_i)$ converges to a point in $L_i(U_i \setminus U_i^0)$,*

*b) for some $j$, $f_{r,s_j}(\Sigma_{s_j}) \cap M_j(W_j \setminus W_j^0) \neq \emptyset$.*

*Here $U_i^0$ and $W_j^0$ denote the top stratums in $U_i$ and $W_j$ respectively.*

*ii)There is no sequence $\{f_r\}_1^\infty$ in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ with $f_r(x_i)$ converges to a point in $L_i(U_i)$ as $r \to \infty$, $f_{r,s_j}(\Sigma_{s_j}) \cap M_j(W_j) \neq \emptyset$ and the limit $f$ in not in $\mathcal{M}^v_{(A_1,\cdots,A_m)}$ $(\Sigma, \mathcal{X}_{Z^0}, J, \nu)$. Here $Z^0$ denotes the top stratum in $Z$.*

*iii) $e_{(\Sigma,P,J,\nu)}(\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^0}, J, \nu))$ and $L \times M(U^0 \times W^0)$ intersect transversally at finitely many points, i.e., there are finitely many $(f; y_1, \cdots, y_d) \in \mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^0}, J, \nu) \times \prod_{j=1}^d \Sigma_{s_j}$, $(u, w) \in U^0 \times W^0$, such that $e_{(\Sigma,P,J,\nu)}(f; y_1, \cdots, y_d) = (L(u), M(w))$, and at each intersection point, the image of the tangent space $T_{(u,w)}U^0 \times W^0$ under $L \times M$ is transverse to the image of the tangent space $T_{(f;y_1,\cdots,y_d)}(\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^0}, J, \nu) \times \prod_{j=1}^d \Sigma_{s_j})$ under the evaluation map.*

**Proof:** i) If this fails, by Proposition 2.1.1 (here we adopt the notation there), there is a cusp curve $f \in \mathcal{N}^v_{\bar{D}}(\Sigma, \mathcal{X}_Z, J, \nu)$ for some $\bar{D}$ satisfying: (1) $f(\tilde{\Sigma}) \cap$

$Im(M_j) \neq \emptyset$, $1 \leq j \leq d$; (2) for each marked point $x_i, i = 1, \cdots, c$, either $f(x_i) \in Im(L_i)$ or a bubble occurs at $x_i$. In the second case, $f(x_i)$ may not be in $Im(L_i)$, but $Im(L_i)$ will intersect a bubble tree coming out of $x_i$. Note that $Im(M_j)$ may intersect a bubble instead of the principal component of $f$. Therefore we see that $f$ could have fewer marked points and the number of homology classes corresponding to the unmarked part increases. Let $X' \subset X$ be the subset of marked points which are not bubble points. Suppose that $X' = (x_1, \cdots, x_p)$, then there are at least $c-p$ bubbles. $c+d-p$ is the number of pseudomanifolds in $PD(\tilde{F}^*(\alpha_i)), PD(\tilde{F}^*(\beta_j))$ which intersect Im$(f)$. Let $f(x_i) \in L_i(U_i)$ for $i = 1, \cdots, p$, Im$(f) \cap L_i(U_i) \neq \emptyset$, for $i = p+1, \cdots, c$, and Im$(f) \cap M_j(W_j) \neq \emptyset$ for $j = 1, \cdots d$. Suppose that $f$ intersects these pseudo-manifolds in a intersection pattern $T$, implying

$$(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha) \neq \emptyset$$

where Im$(f) \subset \mathcal{X}_{Z^\alpha}$ for some $\alpha$ and $e_{X,T}$ is defined before Proposition 2.2.3. Proposition 2.2.3 says that $(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha)$ is a smooth manifold, we have an estimate of its dimension

$$\begin{aligned}
& \dim(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha) \\
= \; & \dim \mathcal{N}_D^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) + 2(c+d-p) - \text{codim} U_i^{s(\alpha)} - \text{codim} W_j^{t(\alpha)} \\
= \; & 2C_1(V) \cdot A + 2n(1-g) - 2(c-p) + 2(c+d-p) - \text{codim} U_i^{s(\alpha)} \\
& - \text{codim} W_j^{t(\alpha)} + \dim Z^\alpha,
\end{aligned}$$

where $U_i^{s(\alpha)}$ stands for all the stratum in $U_i$ whose image lies in $\mathcal{X}_{Z^\alpha}$ and $W_j^{t(\alpha)}$ has a similar meaning. If $\alpha = 0$, by the assumption we have $\mathrm{codim} U_i^{s(\alpha)} \geq \mathrm{codim} U_i + 2$ or $\mathrm{codim} W_j^{t(\alpha)} \geq \mathrm{codim} W_j + 2$, so

$$\begin{aligned} & \dim(e_{X,T} \times L \times M)^{-1}(\Delta_T^0) \\ \leq\ & 2C_1(V) \cdot A + 2n(1-g) + 2d - \mathrm{codim} U_i - \mathrm{codim} W_j - 2 + \dim Z^0 \\ =\ & -2. \end{aligned}$$

If $\alpha \neq 0$, then the general position condition and $\dim Z^\alpha \leq \dim Z^0 - 2$ imply

$$\begin{aligned} & \dim(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha) \\ \leq\ & 2C_1(V) \cdot A + 2n(1-g) + 2d - \mathrm{codim} U_i - \mathrm{codim} W_j + \dim Z^0 - 2 \\ =\ & -2. \end{aligned}$$

So in either case, we get a contradiction. i) is proved.

ii) The first half is the same as the proof of i), the only modification is the dimension counting.

$$\begin{aligned} & \dim(e_{X,T} \times L \times M)^{-1}(\Delta_T^\alpha) \\ =\ & \dim \mathcal{N}_D^v(\Sigma, \mathcal{X}_{Z^\alpha}, J, \nu) + 2(c+d-p) - \mathrm{codim} U_i^{s(\alpha)} - \mathrm{codim} W_j^{t(\alpha)} \\ \leq\ & 2C_1(V) \cdot A + 2n(1-g) - 2 - 2(c-p) + 2(c+d-p) - \mathrm{codim} U_i^{s(\alpha)} \\ & -\mathrm{codim} W_j^{t(\alpha)} + \dim Z^\alpha \\ \leq\ & -2, \end{aligned}$$

where we use that $D$ has a bubble component to get $-2$ in the second to last step.

This a contradiction. ii) is proved.

iii) Considering the restriction of $e_{(\Sigma,P,J,\nu)}$ to $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^0},J,\nu)\times\prod_{j=1}^d\Sigma_{s_j}$ and the restriction of $L\times M$ to $U^0\times W^0$, iii) follows from Proposition 2.2.3. □

**Remark 3.1** *i) Note that if each of* $\dim Z, \dim U_i$, *and* $\dim W_j$ *is one dimension higher than it is in Proposition 3.1. It is still correct, except that we should modify iii) as follows: their intersection is a one dimensional smooth manifold in* $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^0},J,\nu)\times U^0\times W^0$. *The reason is: in the proof of the above proposition, if the dimensions of* $Z, U_i$, *and* $W_j$ *are raised by 1, the result of dimension counting in i) and ii) is -1, which still gives a contradiction.*

*ii) Assume* $\dim Z, \dim U_i$, *and* $\dim W_j$ *are one dimension higher than it is in Proposition 3.1, and* $Z, U_i, W_j$ *have submanifolds* $\tilde{Z}, \tilde{U}_i, \tilde{W}_j$, *respectively. According to Remark 2.2.2 i), Proposition 3.1 is still correct even if we fixed* $(J,\nu)$*'s value on* $\tilde{Z}, \tilde{U}_i, \tilde{W}_j$ *to be good. If* $\tilde{Z}, \tilde{U}_i, \tilde{W}_j$ *are boundaries, the one dimensional manifold in i) has boundary, this essentially giving a cobordism.*

*iii) Suppose that* $Z$ *satisfies (3.1), if we consider the moduli space appearing in Remark 2.2.2 ii), Proposition 3.1 is still correct. The image of the moduli space* $\cup_i\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_i,\mathcal{X}_{Z^0},J,\nu)\times\prod_{j=1}^d\Sigma_{s_j}$ *will only intersect the image of* $U^0\times W^0$, *and the intersection will be transverse giving a smooth one dimensional manifold.*

Now we define the fiberwise mixed invariants as follows. Denote by $P :=$

$(P_1, T)$ the pair: the position pattern $P_1$ and intersection pattern $T$ and denote $e_{X,T}$ by $e_{(\Sigma,P,J,\nu)}$. Fix a pair $(J,\nu)$ such that $e_{(\Sigma,P,J,\nu)}$ and $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$ satisfy all the properties in Proposition 3.1. Again we will call such $(J,\nu)$ good. First we associate a multiplicity $m(f)$ with each $f$ in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_Z, J, \nu)$. We define $m(f)$ to be zero if either $f(x_i)$ is not in $L_i(U_i)$ for some $i$, or $f_{s_j}(\Sigma_{s_j})$ does not intersect with one of the $M_j(W_j)$ for some $j$. If $f$ is given as in Proposition 3.1, there are finitely many $(y_{t1}, \cdots, y_{td})(1 \leq t \leq l)$ such that $f_{s_j}(y_{tj}) \in M_j(W_j^0)$. Putting $\epsilon(f,t)$ to be $\pm 1$, the sign being determined by the orientation of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{Z^0}, J, \nu) \times \prod_{j=1}^d \Sigma_{s_j}$ at $(f; y_{t1}, \cdots, y_{td})$, the orientation of $U^0 \times W^0$ at $(u,w)$, the orientation of $\mathcal{X}_{Z^0}^{c+d}$ at $(f(x_1), \cdots, f(x_c),$ $f_{s_1}(y_{t1}), \cdots, f_{s_d}(y_{td}))$, and the Jacobians of the maps $e_{(\Sigma,P,J,\nu)}$ and $L \times M$. We define

$$m(f) = \sum_{t=1}^{l} \epsilon(f,t).$$

Finally we define the mixed invariant

$$\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma) = \sum m(f).$$

For convenience, define

$$\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma) = 0,$$

in case that

$$\sum_1^c \deg(\alpha_i) + \sum_1^d (\deg(\beta_j) - 2) \neq 2C_1(V) \cdot A + 2n(1-g) + \deg(\gamma).$$

This defines a map

$$\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)} : (\prod H^*(\mathcal{X};\mathbb{Q})) \times (\prod H^*(\mathcal{X};\mathbb{Q})) \to H^*(Y;\mathbb{Q}).$$

Note that when $Y$ is a point, $\mathcal{X} = V$, the fiberwise mixed invariants coincide with the mixed invariants defined by Ruan and Tian in [RT].

The following proposition assures that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is indeed a symplectic invariant.

**Proposition 3.2** *$\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of $(J,\nu)$; the marked points $x_1,\cdots,x_c$ of the same position pattern $P_1$ in $\Sigma$; the conformal structure on $\Sigma$; the choice of pseudo-manifolds $(L_i,U_i),(M_j,W_j),(F,Z)$ representing $\alpha_i,\beta_j,\gamma$ $(i=1,\cdots,c; j=1,\cdots,d)$. Furthermore, the invariant depends only on the semi-positive deformation class of $\omega$.*

**Proof:** In the proof we adopt the notation used in Proposition 3.1, and divide the proof into several parts.

i) We show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of good $(J,\nu)$. Suppose $(J_0,\nu_0),(J_1,\nu_1)$ are two choices, denote $\tilde{Z} = Z\times[0,1]$, define $\mathcal{X}_{\tilde{Z}} \to \tilde{Z}$ to be a fiber bundle with fiber $V$ induced from $\mathcal{X}_Z$, the fiberwise symplectic form $\tilde{\omega}$ on $\mathcal{X}_{\tilde{Z}}$ being also the induced one. Note that $\mathcal{X}_{\tilde{Z}} \to \tilde{Z}$ satisfies Assumption $(\star)$. Define $\mathcal{P}^v_{k-1,p}(\mathcal{X}_{\tilde{Z}},\tilde{\omega})$ similarly to $\mathcal{P}^v_{k-1,p}(\mathcal{X}_Z,\omega)$, but with a boundary condition: for any $(\tilde{J},\tilde{\nu}) \in \mathcal{P}^v_{k-1,p}(\mathcal{X}_{\tilde{Z}},\tilde{\omega})$, $(\tilde{J},\tilde{\nu})|_{\mathcal{X}_{Z\times 0}} \cong (J_0,\nu_0)$, $(\tilde{J},\tilde{\nu})|_{\mathcal{X}_{Z\times 1}} \cong (J_1,\nu_1)$. Then choose $\tilde{L}_i = L_i \times id$,

$\tilde{M}_j = M_j \times id$, $\tilde{U}_i = U_i \times [0,1]$, $\tilde{W}_j = W_j \times [0,1]$. From Remark 3.1 ii), it follows that for a generic $(\tilde{J}, \tilde{\nu})$, $(e_{(\Sigma,P,\tilde{J},\tilde{\nu})} \times \tilde{L} \times \tilde{M})^{-1}(\tilde{\Delta}_T^0)$ will give a one dimensional cobordism between $(e_{(\Sigma,P,J_0,\nu_0)} \times L \times M)^{-1}(\Delta_T^0)$ and $(e_{(\Sigma,P,J_1,\nu_1)} \times L \times M)^{-1}(\Delta_T^0)$ in the space $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X}_{\tilde{Z}^0}, \tilde{J}, \tilde{\nu}) \times \prod_{j=1}^d \Sigma_{s_j} \times \tilde{U}^0 \times \tilde{W}^0$. This cobordism tells us that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of $(J,\nu)$.

ii) We show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of the conformal structure $j$ on $\Sigma$. Suppose that $j_0, j_1$ are two conformal structure on $\Sigma$ and connect them by a family $j_t, t \in [0,1]$. From Remark 3.1 iii), for a generic $(J,\nu)$, the image of the moduli space $\cup_t \mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_t, \mathcal{X}_{Z^0}, J, \nu) \times \prod_{j=1}^d \Sigma_{s_j}$ will only intersect the image of $U^0 \times W^0$, the intersection will be transverse and give a smooth one dimensional manifold. But this one dimensional manifold is the cobordism between $(e_{((\Sigma,j_0),P,J,\nu)} \times L \times M)^{-1}(\Delta_T^0)$ and $(e_{((\Sigma,j_1),P,J,\nu)} \times L \times M)^{-1}(\Delta_T^0)$. This cobordism tells us that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of the conformal structure $j$ on $\Sigma$.

iii) We show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of the marked point set $X$ within the same position pattern $P_1$ (see the definition at the beginning of this chapter). If $X'$ is another set of marked points, choose a continuous map $\phi : \Sigma \to \Sigma$ isotropic to identity, mapping $X$ to $X'$, and being a diffeomorphism on each component $\Sigma_i, i = 1,\cdots,m$. Let $\Sigma'$ to be $\Sigma$ equipped with the pullback conformal structure $\phi^* j$, then obviously $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\ \beta_1,\cdots,\beta_d)(\gamma)$ defined using the stable curve $C' = (\Sigma', X')$ is the same as the one defined using the stable curve

$C = (\Sigma, X)$. Combining this with part ii) above, the independence of the choice $X$ is proved.

iv) We show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ is independent of the choice of pseudo-manifold representatives $(F,Z)$, $(L_i,U_i)$, $(M_j,W_j)$. Suppose that $(F_0,Z_0)$, $(L_{i0},U_{i0})$, $(M_{j0},W_{j0})$ and $(F_1,Z_1)$, $(L_{i1},U_{i1})$, $(M_{j1},W_{j1})$ are two choices, both of them satisfying the general position condition. First we can find a cobordism of pseudo-manifolds $\tilde{F} : \tilde{Z} \to Y$ with $\partial\tilde{F} = F_0 \cup F_1, \partial\tilde{Z} = Z_0 \cup Z_1$, and also construct a fiber bundle $\tilde{p} : \mathcal{X}_{\tilde{Z}} \to \tilde{Z}$ with fiber $V$ satisfying the Assumption $(\star)$. Secondly we can find a cobordism of pseudo-manifolds $\tilde{L}_i : \tilde{U}_i \to \mathcal{X}_{\tilde{Z}}, \tilde{M}_j : \tilde{W}_j \to \mathcal{X}_{\tilde{Z}}$, satisfying the pseudo-manifold representative condition listed at the beginning of this chapter(in particular the general position condition), and $\partial(\tilde{L}_i,\tilde{U}_i) = (L_{i0},U_{i0}) \cup (L_{i1},U_{i1}), \partial(\tilde{M}_j,\tilde{W}_j) = (M_{j0},W_{j0}) \cup (M_{j1},W_{j1})$. Applying Remark 3.1 ii) to $(\tilde{F},\tilde{Z}),(\tilde{L}_i,\tilde{U}_i),(\tilde{M}_j,\tilde{W}_j)$, we get a smooth one dimensional manifold which is a cobordism between $(e_{(\Sigma,P,J_0,\nu_0)} \times L_0 \times M_0)^{-1}(\Delta^0_{T,0})$ and $(e_{(\Sigma,P,J_1,\nu_1)} \times L_1 \times M_1)^{-1}(\Delta^1_{T,1})$ in the space $\mathcal{X}_{\tilde{Z}}$. This cobordism tells us that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)\ (\gamma)$ is independent of the choice of $(F,Z),(L_i,U_i)$, and $(M_j,W_j)$.

v) We show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma)$ depends only on the semi-positive deformation class of $\omega$. The reason is that $(J,\nu)$ does not depend on the symplectic form $\omega$ as long as it is $\omega$-tamed, and tameness is an open condition. □

**Remark 3.2** *To define* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c\ |\beta_1,\cdots,\beta_d) \in H^*(Y,\mathbb{Q})$, *we only need to choose a basis* $\gamma_1,\cdots,\gamma_q$ *of* $H^*(Y,\mathbb{Q})$, *and evaluate* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$

$(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma_i)$. *Note that for some appropriate choice of* $\gamma_1, \cdots, \gamma_q$, *we can represent all* $\gamma_i$ *by submanifolds* $Z_i$ *in* $Y$. *This means that we can choose* $Z$ *to be a manifold from the very beginning, and assume its cobordism to be smooth too. Had we proceeded in this way, our arguments could be much simplified. Actually in the definition of fiberwise quantum cohomology and equivariant quantum cohomology,* $Z$ *is always smooth.*

Next we collect a few simple properties of the fiberwise mixed invariants, leaving the composition law for the next chapter.

**Proposition 3.3** *The fiberwise mixed invariant* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma)$ *is multilinear in* $\alpha_i$, $\beta_j$ *and* $\gamma$. *Furthermore, we have*

*i) The fiberwise mixed invariant* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma)$ *is zero if the " virtual" dimension* $2C_1(V) \cdot A + 2n(1-g) + \deg \gamma < 0$.

*ii)* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)$ *is zero if one of the* $\beta_j$ *is of degree greater than* $2n-2$.

*iii) If* $c + 2g \geq 4$ *and* $\alpha_c$ *is the fundamental class* $\mathcal{X}$, *then* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$ $(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma)$ *equals* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$ $(\alpha_1, \cdots, \alpha_{c-1} | \beta_1, \cdots, \beta_d)(\gamma)$.

*iv)* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma)$ *equals* $D \cdot \Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_{d-1})(\gamma)$, *if* $\beta_d$ *is of degree 2 and* $D = A \cdot \beta_d$ *is the intersection number, where we regard* $\beta_d$ *as the induced class on* $V$.

*v) In case* $A = 0$, $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)$ *equals zero if* $d > 0$ *and equals the intersection number* $\int_V \alpha_1 \cap \cdots \cap \alpha_k$ *if* $d = 0$.

*vi) Let* $\delta$ *be a class in* $H^*(Y, \mathbb{Q})$. *Then* $p^*\delta$ *is a class in* $H^*(\mathcal{X}, \mathbb{Q})$, *and* $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(p^*\delta, \alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d) = \delta \wedge \Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1,$

$\cdots, \beta_d)$.

**Proof:** For i), ii), iii), iv), the proof, similar to the proof of Proposition 2.5 in [RT], is omitted.

v) (2.1) implies $\sum_1^c \deg \alpha_i + \sum_1^d (\deg \beta_j - 2) = 2n(1-g) + \deg(\gamma)$. Fix a pseudo-manifold representative $F : Z \to Y$, then the moduli space is $\mathcal{X}_Z$ if we choose the fiberwise inhomogeneous term to be zero. If we choose representatives $(L_i, U_i), (M_j, W_j)$ of $\alpha_i, \beta_j$ intersecting transversally in $\mathcal{X}_Z$, by definition $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}\ (\alpha_1, \cdots, \alpha_c|\ \beta_1, \cdots, \beta_d)\ (\gamma)$ is the intersection number of $L_1(U_1) \cap \cdots \cap L_c(U_c) \cap M_1(W_1) \cap \cdots M_d(W_d)$. If $d > 0$, this intersection is empty because of the degree requirement. If $d = 0$, this intersection number is the same as

$$\begin{aligned} & \tilde{F}^*\alpha_1 \wedge \cdots \wedge \tilde{F}^*\alpha_c(\mathcal{X}_Z) \\ = & \int_V \tilde{F}^*\alpha_1 \wedge \cdots \wedge \tilde{F}^*\alpha_c(Z) \\ = & \int_V \alpha_1 \wedge \cdots \alpha_c(F_*Z) \\ = & \int_V \alpha_1 \wedge \cdots \alpha_c(\gamma) \end{aligned}$$

where $\tilde{F} : \mathcal{X}_Z \to \mathcal{X}$ is the natural map induced from $F : Z \to Y$.

vi) Suppose that $F : Z \to Y$ is a pseudo-manifold representative of a class $\gamma \in H^*(Y, \mathbb{Q})$, then we have the diagram

$$\begin{array}{ccc} \mathcal{X}_Z & \xrightarrow{\tilde{F}} & \mathcal{X} \\ p_1 \downarrow & & \downarrow p \\ Z & \xrightarrow{F} & Y \end{array}$$

Note that since $\tilde{F}^*p^*\delta = p_1^*F^*\delta$, if the Poincare dual of $F^*\delta$ is represented by a pseudo-manifold $K$, then we can represent $\tilde{F}^*p^*\delta$ by the pseudo-manifold $(L_0, p_1^{-1}(K))$. So the intersection of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\ \mathcal{X}_Z,\ J, \nu)$ with $(L_0, p_1^{-1}(K)), (L_i, U_i), (M_j, W_j)$ in $\mathcal{X}_{Z^0}^{c+d+1}$ is the same as the intersection of $\mathcal{M}^v_{(A_1,\cdots,A_m)}\ (\Sigma,\ \mathcal{X}_K,\ J, \nu)$ with $(L_i, U_i), (M_j, W_j)$ in $\mathcal{X}_{K^0}^{c+d}$. This exactly means that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}\ (p^*\delta, \alpha_1, \cdots,\ \alpha_c \mid \beta_1, \cdots, \beta_d)\ (\delta.\gamma) = \delta \wedge \Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$ $(\alpha_1, \cdots, \alpha_c \mid \beta_1, \cdots, \beta_d)(\gamma)$. The proposition is proved. □

**Proposition 3.4 (Direct Product)** *Let $\mathcal{X}_1 \to Y_1, \mathcal{X}_2 \to Y_2$ be two fiber bundles with fibers $V_1, V_2$ and fiberwise symplectic forms $\omega_1, \omega_2$, respectively, and assume both satisfy the Assumption (⋆). Then $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \to Y_1 \times Y_2$ is a fiber bundle with fiber $V_1 \times V_2$ and fiberwise symplectic form $\omega = \omega_1 \oplus \omega_2$, also satisfying the Assumption (⋆), and*

$$\begin{aligned}\Phi^{\mathcal{X},v}_{((A_1,A_1'),\cdots,(A_m,A_m'),\omega,\Sigma,P)}&(\alpha_1 \otimes \alpha_1', \cdots, \alpha_c \otimes \alpha_c' | \beta_1 \otimes \beta_1, \cdots, \beta_d \otimes \beta_d)(\gamma \otimes \gamma') \\ &= \Phi^{\mathcal{X}_1,v}_{(A_1,\cdots,A_m,\omega_1,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)(\gamma) \\ &\cdot \Phi^{\mathcal{X}_2,v}_{(A_1',\cdots,A_m',\omega_2,\Sigma,P)}(\alpha_1', \cdots, \alpha_c' | \beta_1', \cdots, \beta_d')(\gamma'), \qquad (3.2)\end{aligned}$$

*where $\Phi^{\mathcal{X},v}, \Phi^{\mathcal{X}_1,v}$, and $\Phi^{\mathcal{X}_2,v}$ denote the fiberwise mixed invariants on $\mathcal{X}, \mathcal{X}_1$, and $\mathcal{X}_2$ respectively.*

**Proof:** This follows directly from the definition of the fiberwise mixed invariants. □

**Proposition 3.5 (Restriction)** *Let $\mathcal{X} \to Y$ be a fiber bundle with fiber $V$ satisfying Assumption (⋆), and $h : Y_1 \to Y$ be a smooth map. Denote by*

*$\mathcal{X}_{Y_1} = \mathcal{X} \times_f Y_1$ to be the induced fiber bundle with fiber $V$ and the induced fiberwise symplectic form $\omega$. Then $\mathcal{X}_{Y_1} \to Y_1$ satisfies Assumption ($\star$) and we have a natural map $H : \mathcal{X}_{Y_1} \to \mathcal{X}$. Furthermore*

$$\begin{aligned}&\Phi^{\mathcal{X}_{Y_1},v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(H^*\alpha_1,\cdots,H^*\alpha_c|H^*\beta_1,\cdots,H^*\beta_d)\\&\quad = h^*\Phi^{\mathcal{X},v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d).\end{aligned} \tag{3.3}$$

**Proof:** We only need to show that for any $\gamma \in H^*(Y_1,\mathbb{Q})$,

$$\begin{aligned}&\Phi^{\mathcal{X}_{Y_1},v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(H^*\alpha_1,\cdots,H^*\alpha_c|H^*\beta_1,\cdots,H^*\beta_d)(\gamma)\\&\quad = \Phi^{\mathcal{X},v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(h_*\gamma)\end{aligned}$$

Choose a pseudo-manifold representative $(F,Z)$ of $\gamma$ in $Y_1$, and representatives $(L_i,U_i),(M_j,W_j)$ in $\mathcal{X}_Z$ of $H^*\alpha_i, H^*\beta_j$. Then the pseudo-manifold representative of $h_*\gamma$ is $(h\circ F,Z)$, and the pseudo-manifold representatives of $\alpha_i$ and $\beta_j$ in $\mathcal{X}_Z$ can be chosen to be $(L_i,U_i)$ and $(M_j,W_j)$, respectively. With these choices, both sides of the above equality amount to counting the number of intersections of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{Z^0},J,\nu)$ with $U^0\times W^0$ in $\mathcal{X}^{c+d}_{Z^0}$. Clearly they are the same. The proposition is proved. □

**Proposition 3.6 (Induction)** *Suppose that the fiber bundle $Y \to B$ has fiber $V_2$ with fiberwise symplectic form $\omega_2$ satisfying Assumption ($\star$). Suppose also that the fiber bundle $\mathcal{X} \to Y$ has fiber $V_1$ with fiberwise symplectic form $\omega$*

*satisfying Assumption (⋆). Further assume that $\omega$ is also a fiberwise symplectic form satisfying Assumption (⋆) for the induced fiber bundle $\mathcal{X} \to B$ with fiber $V$. Let $A_1, \cdots, A_m$ be in $H_2(V_1, \mathbb{Z})$. Then*

$$\begin{aligned} &\Phi^{\mathcal{X},B,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d) \\ &= \textstyle\int_{V_2} \Phi^{\mathcal{X},Y,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d) \end{aligned} \tag{3.4}$$

*where $\Phi^{\mathcal{X},B,v}$ and $\Phi^{\mathcal{X},Y,v}$ are the fiberwise mixed invariants for the fiber bundle $\mathcal{X} \to B$ and $\mathcal{X} \to Y$ respectively, and the map $\int_{V_2} : H^*(Y,\mathbb{Q}) \to H^{*-\dim V_2}(B,\mathbb{Q})$ is the Gysin map which can be defined by integration along the fiber direction.*

**Proof:** For any $\gamma \in H_*(B,\mathbb{Q})$, choose its pseudo-manifold representative $F : Z \to B$. Denote $Y \times_F Z$ by $\tilde{Z}$. Then $\tilde{F} : \tilde{Z} \to Y$ is a pseudo-manifold representing a homology class $\tilde{\gamma} \in H_*(Y,\mathbb{Q})$. All we need to show is that

$$\begin{aligned} &\Phi^{\mathcal{X},B,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\gamma) \\ &= \Phi^{\mathcal{X},Y,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1,\cdots,\alpha_c|\beta_1,\cdots,\beta_d)(\tilde{\gamma}). \end{aligned}$$

But $\mathcal{X}_Z \to Z$ is a bundle with fiber $V$, and $\mathcal{X}_{\tilde{Z}} \to \tilde{Z}$ is a bundle with fiber $V_1$. Obviously there is a natural isomorphism $\mathcal{X}_Z \cong \mathcal{X}_{\tilde{Z}}$, so we can choose the pseudo-manifolds representing $\alpha_i, \beta_j$ in $\mathcal{X}_Z$ and $\mathcal{X}_{\tilde{Z}}$ to be the same. To prove above identity, we need to find a good $(J,\nu)$ for $\mathcal{X}_Z$ and a good $(\tilde{J},\tilde{\nu})$ for $\mathcal{X}_{\tilde{Z}}$ such that $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_Z,J,\nu)$ is naturally identified with $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{\tilde{Z}},\tilde{J},\tilde{\nu})$ and they define the same evaluation map.

We achieve this goal in two steps. In step 1, we will show that there are a family of $(J,\nu)$ for $\mathcal{X}_Z \to Z$ such that any fiberwise $(J,\nu)$ holomorphic map actually lies in the fiber $V_1$ of the fiber bundle $V \to V_2$, i.e., it is actually a fiberwise $(\tilde{J},\tilde{\nu})$ holomorphic map for $\mathcal{X}_{\tilde{Z}}$. So we have an isomorphism between $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_Z,J,\nu)$ and $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{\tilde{Z}},\tilde{J},\tilde{\nu})$. In step 2, we will show that there are good $(J,\nu)$ and $(\tilde{J},\tilde{\nu})$ among the above mentioned family.

Step1. Let $E, E_1$, and $E_2$ be vector bundle over $\mathcal{X}_Z, \mathcal{X}_{\tilde{Z}}$, and $\tilde{Z}$ respectively as defined after Remark 2.1.1. Note that $E_1$ is a subbundle of $E$. Fix a splitting $E = E_1 \oplus E_1^c$, then we have diagram with $\pi_{1*}$ being an isomorphism on the fiber.

$$\begin{array}{ccc} E_1^c & \xrightarrow{\pi_{1*}} & E_2 \\ \downarrow & & \downarrow \\ \mathcal{X}_Z & \xrightarrow{\pi_1} & \tilde{Z} \end{array}$$

We define a family of $(J,\nu)$ on $E$ as follows. They are of the form $(\tilde{J},\tilde{\nu})|_{E_1} \oplus (J_2,0)|_{E_1^c}$, where $(\tilde{J},\tilde{\nu}) \in \mathcal{P}_{k-1,p}(E_1,\omega)$ and $J_2$ is the pull-back of fiberwise almost complex structure $\tilde{J}_2 \in \mathcal{P}_{k-1,p}(E_2,\omega_2)$ by $\pi_{1*}$. Note that $J$ is tamed by $\omega+\epsilon\pi_1^*\omega_2$ for small $\epsilon$. We show now that for such $(J,\nu)$, if $f$ is a fiberwise $(J,\nu)$ holomorphic curve whose image is in fiber $V$, then $\pi_1 \circ f =$ constant where $V \xrightarrow{\pi_1} V_2$ is a bundle map, i.e., it is in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma,\mathcal{X}_{\tilde{Z}},\tilde{J},\tilde{\nu})$. By definition, $df + J \circ df \circ j_\Sigma = (\tilde{\nu},0)$. Projecting it to $E_2$, we get $\pi_{1*}df + \pi_{1*}J \circ df \circ j_\Sigma = \pi_{1*}(\tilde{\nu},0)$. Since $\pi_{1*}J = J_2\pi_{1*}$, we get $d(\pi_1 \circ f) + J_2 \circ d(\pi_1 \circ f) \circ j_\Sigma = 0$, i.e., $\pi_1 \circ f$ is a $J_2$ holomorphic map in $V_2$. But since $A \in H_2(V_1,\mathbb{Z})$,

$$0 = \int_A \pi_1^*\omega_2 = \int_{f(\Sigma)} \pi_1^*\omega_2 = \int_\Sigma f^*\pi_1^*\omega_2 = \int_\Sigma (\pi_1 \circ f)^*\omega_2,$$

on the other hand $\int_\Sigma (\pi_1 \circ f)^* \omega_2 > 0$ if $\pi_1 \circ f$ is not a constant.

Step 2. The existence of a good $(J, \nu)$ can be proved in the following way. We can choose a subspace of $Map_{(A_1,\cdots,A_m)}(\Sigma, \mathcal{X})$ consisting of maps with $f(\Sigma) \subset V_1 \times pt$, and choose a subspace of $\mathcal{P}^v_{k-1,p}(\mathcal{X}, \omega)$ consisting of all $(J, \nu)$ described above. Now the argument of Proposition 2.2.3. works within these two subspaces.

The proposition is proved. □

**Remark 3.3** *Assume that $(\mathcal{X}, \omega)$ itself is a symplectic manifold, and $\omega$ is also a fiberwise symplectic form for a bundle $\mathcal{X} \to Y$. Note that $\Phi^{\mathcal{X},pt,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$ $(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)$ is equal to the non-fiberwise mixed invariant $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}$ $(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)$ if $A_1, \cdots, A_m \in H_2(V; \mathbb{Z})$. Choosing $B = pt$ in the above proposition, the non-fiberwise mixed invariant and the fiberwise mixed invariant are related by*

$$\begin{aligned} &\int_Y \Phi^{\mathcal{X},Y,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d) \\ &= \Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d). \end{aligned} \tag{3.5}$$

*In this circumstance, using the above identity, all the properties including the composition law of $\Phi^{\mathcal{X},Y,v}_{(A_1,\cdots,A_m,\omega,\Sigma,P)}(\alpha_1, \cdots, \alpha_c | \beta_1, \cdots, \beta_d)$ follow from the corresponding properties of the non-fiberwise mixed invariants. This fact was already observed by Astashkevich and Sadov in [AS].*

# Chapter 4

# Composition Law of Fiberwise Gromov-Witten Mixed Invariants

## 4.1 Gluing of $J$-Holomorphic Maps

In this section, we will apply the implicit function theorem to the study of deformation theory of perturbed $J$-holomorphic maps from a singular curve.

Recall that a degeneration of stable curves is a holomorphic fibration $\pi : S \to \Delta \subset \mathbb{C}$ with sections $\sigma_1, \cdots, \sigma_c$ satisfying : (1) For each $t \in \Delta, t \neq 0$, the fiber $\Sigma_t = \pi^{-1}(t)$ is smooth; (2) For each $t$, $C_t = (\Sigma_t; \sigma_1(t), \cdots, \sigma_c(t))$ is a $c$-point stable curve.

Adopting the notation of Chapter 2, suppose that $\mathcal{X} \xrightarrow{p} Y$ is a fiber bundle with fiber $V$ satisfying Assumption $(\star)$, $F : Z \to Y$ is a pseudo-manifold, and $Z^\alpha$ is a stratum in $Z$. For any point $z_0 \in Z^\alpha$, we can find a neighborhood $W_\alpha \subset Z^\alpha$ of $z_0$ such that there is a trivialization $\mathcal{X}_Z|_{W^\alpha} \cong W^\alpha \times V$. Denoting $\mathcal{X}_Z|_{W^\alpha}$ by $\mathcal{X}_{W^\alpha}$, we equip $W^\alpha$ with Euclidean metric $ds_0^2$ and trivial connection

$\nabla_1$. Let $h_E$ be a Hermitian metric on bundle $E$, $\nabla$ be the connection on $E$ compatible with $h_E$ and $J$, with torsion $\frac{1}{4}N_J$; $N_J$ is again the Nijenhuis tensor. Therefore $h_\alpha = ds_0^2 + h_E$ is a metric on $W^\alpha \times V$.

Let $\nu$ be an inhomogeneous term on $S \times \mathcal{X}_Z$, and $\nu_t$ be the restriction of $\nu$ to $\Sigma_t \times \mathcal{X}_Z$. Consider the moduli space of fiberwise $(J, \nu_t)$-perturbed holomorphic maps in $\mathcal{X}_{Z^\alpha}$,

$$\begin{aligned}\mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^\alpha}, J, \nu_t) = \quad & \{f : \Sigma_t \to \mathcal{X}_{Z^\alpha} |\ df + J \circ df \circ j_t = \nu_t, \\ & [f(\Sigma_t)] = A,\ p \circ f(\Sigma_t) = pt \in Z^\alpha\}\end{aligned}$$

where $j_t$ is the complex structure on $\Sigma_t (t \neq 0)$, $A$ is a fixed homology class in $H_2(V, \mathbb{Z})$.

Assume that $\Sigma_0$ has m-components $\Sigma_{01}, \cdots, \Sigma_{0m}$, and suppose that the position pattern of $\sigma_1(0), \cdots, \sigma_c(0)$ in $\Sigma_0$ is $P_1$. Fix a partition of $A = A_1 + \cdots A_m, A_i \in H_2(V, \mathbb{Z})$, which is effective, i.e., $\mathcal{M}_{A_i}^v(\Sigma_{0i}, \mathcal{X}_{Z^\alpha}, J, \nu_{0i})$ is not empty for each $i$. From Proposition 2.2.1, the moduli space $\mathcal{M}_{(A_1,\cdots,A_m)}^v(\Sigma_0, \mathcal{X}_{Z^\alpha}, J, \nu_0)$ is a smooth manifold and for generic $t \neq 0$, $\mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^\alpha}, J, \nu_t)$ is also smooth manifold with dimension

$$\dim \mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^\alpha}, J, \nu_t) = 2C_1(V) \cdot A + 2n(1-g) + \dim_{\mathbb{R}} Z^\alpha$$

where $2n$ is the real dimension of $V$ and $g$ is the genus of $\Sigma_t$.

**Theorem 4.1.1** *Let $f_0$ be any map in $\mathcal{M}_{(A_1,\cdots,A_m)}^v(\Sigma_0, \mathcal{X}_{Z^\alpha}, J, \nu_0)$ which intersects transversally at each double point. Assume $f_0(\Sigma_0) \subset V_{z_0}$. Then there is a*

*continuous family of injective maps $T_t$ from $\tilde{W}$ into $\mathcal{M}^v_A(\Sigma_t, \mathcal{X}_{Z^\alpha}, J, \nu_t)$, where $t$ is small and $\tilde{W}$ is a neighborhood of $f_0$ in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{W^\alpha}, J, \nu_0)$ such that (1) for any $f$ in $\tilde{W}$, as $t$ goes to zero, $T_t(f)$ converges to $f$ in $C^0$-topology on $\Sigma_0$ and in $C^3$-topology outside the singular set of $\Sigma_0$; (2) there are $\epsilon, \delta > 0$ satisfying: if $f'$ is in $\mathcal{M}^v_A(\Sigma_t, \mathcal{X}_{W^\alpha}, J, \nu_t)$ and $d_{\mathcal{X}_{W^\alpha}}(f'(x), f_0(y)) \leq \epsilon$ whenever $x \in \Sigma_t$, $y \in \Sigma_0$, $d_S(x, y) \leq \delta$, where $d_{\mathcal{X}_{W^\alpha}}$ and $d_S$ are the distance functions of metric $h_\alpha$ on $\mathcal{X}_{W^\alpha}$ and a Kähler metric $\mu$ on $S$,respectively, then $f'$ is in $T_t(\tilde{W})$. Moreover, for generic $t$, $T_t$ is an orientation-preserving smooth map from $\tilde{W}$ into smooth manifold $\mathcal{M}^v_A(\Sigma_t, \mathcal{X}_{Z^\alpha}, J, \nu_t)$.*

**Proof:** First, we make a reduction to the case with zero fiberwise inhomogeneous term as we did in Section 2.1. Adopting the notation there, for any deformation of $f_0$, $f_t : \Sigma_t \to \mathcal{X}_{Z^\alpha}$, its graph map $F_t : \Sigma_t \to S \times \mathcal{X}_{Z^\alpha}$ is a deformation of the graph map of $f_0$. $F_0 : \Sigma_0 \to S \times \mathcal{X}_{Z^\alpha}$. Let $\pi_i$ be the projection map from $S \times \mathcal{X}_{Z^\alpha}$ onto the i-th factor (i=1,2), then $\pi_1 \circ F_t = id$. Conversely if $\tilde{F}_t$ is a small deformation of $F_0$, then $\pi_1 \circ \tilde{F}_t$ is indeed a biholomorphism of $\Sigma_t$, and $\tilde{f}_t = \pi_2 \circ \tilde{F}_t \circ (\pi_1 \circ \tilde{F}_t)^{-1}$ is a $(J, \nu_t)$-perturbed holomorphic map, a deformation of $f_0$. This shows that to study the deformation of $f_0$ is equivalent to study the deformation of $F_0$. So we may assume $\nu = 0$. Note that although $S \times \mathcal{X}_{Z^\alpha}$ may not be compact, the object under study lies in a compact region.

In the following proof, we will use $C$ to denote a constant independent of $t$ and any $f$ near $f_0$. The actual value of $C$ may vary in different places.

Let $f \in \tilde{W}$, $f(\Sigma_0) \subset V_z$, and $z \in W^\alpha$. We will first construct an approximate $J$-holomorphic map from $\Sigma_t$ into $V_z$ for each $t$. Let $p$ be any double point

of $\Sigma_0$, and $U_p$ be a small neighborhood in $S$ containing $p$. $U_p$ has coordinate $(z_{p1}, z_{p2})$ such that

$$U_p \cap \Sigma_t = \{(z_{p1}, z_{p2}) |\ z_{p1} \cdot z_{p2} = t, |z_{p1}| < 1, |z_{p2}| < 1\}.$$

Choose a coordinate system $u_1, \cdots, u_{\tilde{n}}; y_1, \cdots, y_{2n}$ of $\mathcal{X}_{W^\alpha}$ near $f(p)$ ($\tilde{n} = \dim W$), such that the coordinate of $f(p)$ is zero and

$$J(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial y_{n+i}} + O(|y| + |u|),$$
$$J(\frac{\partial}{\partial y_{n+i}}) = -\frac{\partial}{\partial y_i} + O(|y| + |u|),$$

where $i = 1, \cdots, n$, $|y| = \sqrt{\sum_{i=1}^{2n} |y_i|^2}$, and $|u| = \sqrt{\sum_{i=1}^{\tilde{n}} |u_i|^2}$. Note that there are two connected components in $U_p \cap \Sigma_0$,

$$U_{p1} = \{(z_{p1}, 0) |\ |z_{p1}| < 1\} \text{ and } U_{p2} = \{(0, z_{p2}) |\ |z_{p2}| < 1\}.$$

Let $f_{pi} = f|_{U_{pi}}$, i=1, 2. Then we have the following expansions:

$$f_{pi}(z_{pi}) = \tilde{f}_{pi}(z_{pi}) + \text{terms of degree greater than 1},$$

where $\tilde{f}_{pi}$ are homogeneous polynomials in $z_{pi}$ of degree 1. We identify a neighborhood of $f(p)$ in $\mathcal{X}_{W^\alpha}$ with open set in $W^\alpha \times \mathbb{C}^n$ by putting $w_i = y_i + \sqrt{-1} y_{n+i}$, $i = 1, \cdots, n$. By assumption, at each double point, $f_0|_{U_{p1}}$ and $f_0|_{U_{p2}}$ intersect transversally. If we choose $\tilde{W}$ small enough, we may assume that for any $f \in \tilde{W}$ $f|_{U_{p1}}$ and $f|_{U_{p2}}$ also intersect transversally. Then by choosing $y_1, \cdots, y_{2n}$ properly, we have

$$f_{p1}((z_{p1},0)) = (z_{p1},0,\cdots,0) + O(|z_{p1}|^2) \in 0 \times \mathbb{C}^n,$$
$$f_{p2}((0,z_{p2})) = (0,z_{p2},0,\cdots,0) + O(|z_{p2}|^2) \in 0 \times \mathbb{C}^n.$$

By changing local coordinates $y_1,\cdots,y_{2n}$, we may further assume that

$$f_{p1}((z_{p1},0)) = (z_{p1},0,\cdots,0), \tag{4.1}$$
$$f_{p2}((0,z_{p1})) = (0,z_{p2},0,\cdots,0). \tag{4.2}$$

Now we begin to construct an approximate $J$-holomorphic map $f_t^{app}$ : $\Sigma_t \to V_z$ for each small $t$. Let $\phi_t$ be a smooth family of diffeomorphism from $\Sigma_0'$ into $\Sigma_t$, where $\Sigma_0'$ is the nonsingular part of $\Sigma_0$, such that $\phi_0 = id$, and

$$\| \phi_t - id \|_{C^5(\Sigma_0 \setminus U')} \leq C_{U'} \cdot t \tag{4.3}$$

for any small neighborhood $U'$ of the singular set $Sing(\Sigma_0)$ in $\Sigma_0$. Here $C_{U'}$ is a constant depending only on $U'$, and the norm is taken with respect to metric $\mu$ on $S$ by viewing both $\phi_t$ and $id$ as maps into $S$. Then for any $p$ in $Sing(\Sigma_0)$, we have

$$\| \tilde{f}|_{\Sigma_t \cap U_p} - f \circ \phi_t^{-1} \|_{C^4(\Sigma_t \cap \{(z_{p1},z_{p2}) |\ \frac{1}{2} \leq |z_{pi}| \leq 1, i=1 \text{ or } 2\})} \leq C \cdot |t| \tag{4.4}$$

where $\tilde{f}$ is the map: $(z_{p1},z_{p2}) \in U_p \to (z_{p1},z_{p2},0,\cdots,0) \in 0 \times \mathbb{C}^n$. Note that

$$\tilde{f}|_{U_{p1}} = f_{p1}; \quad \tilde{f}|_{U_{p2}} = f_{p2}. \tag{4.5}$$

By (4.4),(4.5), there is a homotopy $F_t$ on $\Sigma_t \cap (\cup_{p\in Sing(\Sigma_0)}\{(z_{p1}, z_{p2}) \mid \frac{1}{2} \leq |z_{pi}| \leq 1, i = 1 \text{ or } 2\})$ satisfying, for any $p \in Sing(\Sigma_0)$

$$\| F_t - f \circ \phi_t^{-1} \|_{C^4(\Sigma_t \cap \{(z_{p1},z_{p2}) \mid \frac{1}{2}\leq|z_{pi}|\leq 1, i=1 \text{ or } 2\})} \leq C \cdot |t|, \tag{4.6}$$

$$F_t = \tilde{f} \text{ on } \Sigma_t \cap \{(z_{p1}, z_{p2}) \mid \frac{1}{2} \leq |z_{pi}| \leq \frac{7}{10}, i = 1 \text{ or } 2\}, \tag{4.7}$$

$$F_t = f \circ \phi_t^{-1} \text{ on } \Sigma_t \cap \{(z_{p1}, z_{p2}) \mid \frac{9}{10} \leq |z_{pi}| \leq 1, i = 1 \text{ or } 2\}. \tag{4.8}$$

We define

$$\begin{aligned} &f_t^{app}(x) = f \circ \phi_t^{-1}(x) \text{ for } x \in \Sigma_t \setminus \cup_{p\in Sing(\Sigma_0)} U_p; \\ &f_t^{app}(z_{p1}, z_{p2}) = F_t(z_{p1}, z_{p2}) \text{ if } \frac{1}{2} \leq |z_{pi}| \leq 1, \ i = 1 \text{ or } 2; \\ &f_t^{app}(z_{p1}, z_{p2}) = (z_{p1}, z_{p2}, 0, \cdots, 0) \text{ if } |z_{pi}| \leq \frac{1}{2}, \ i = 1 \text{ and } 2. \end{aligned} \tag{4.9}$$

Let $\mu$ be a Kahler metric on $S$ which equals $dz_{p1} \otimes d\bar{z}_{p1} + dz_{p2} \otimes d\bar{z}_{p2}$ on $U_p$ for each $p \in Sing(\Sigma_0)$. Let $\rho$ be a smooth function on $S \setminus Sing(\Sigma_0)$ satisfying $0 < \rho \leq 3$ and

$$\rho|_{U_p}(z_{p1}, z_{p2}) = \sqrt{|z_{p1}|^2 + |z_{p2}|^2} \text{ for each } p \in Sing(\Sigma_0). \tag{4.10}$$

Define a new metric $\mu_c$ on $S$, $\mu_c = \rho^{-2}\mu$. Note that

$$\mu_c|_{\Sigma_t \cap U_p} = \frac{|dz_{pi}|^2}{|z_{pi}|^2},$$

i.e., for $t$ small, $\mu_c|_{\Sigma_t \cap U_p}$ is a cylinderlike metric. The following lemma can be easily proved from the definition of $f_t^{app}$, $\mu$, and $\mu_c$ by local computation.

**Lemma 4.1.1** *Let $\nabla$ be the connection defined at the beginning of this chapter. Denote by $D$ the covariant derivative induced from $\mu$ and $\nabla$, and $D_c$ the covariant derivative induced from $\mu_c$ and $\nabla$. Then for $1 \leq k \leq 5$,*

$$|D^k f_t^{app}|_{\mu,h_E}(x) \leq C_k(1 + \frac{|t|}{\rho(x)^{k+1}}), \tag{4.11}$$

$$|D_c^k f_t^{app}|_{\mu_c,h_E}(x) \leq C_k'(1 + \frac{|t|}{\rho(x)}) \tag{4.12}$$

*where $|\ |_{\mu,h_E}$ and $|\ |_{\mu_c,h_E}$ denote the norm defined by $\mu, h_E$ and $\mu_c, h_E$ respectively.*

Let $J_0$ be the standard complex structure on $\mathbb{C}^n$. By definition, for each $p$ in $Sing(\Sigma_0)$,

$$df_t^{app} + J_0 \circ df_t^{app} \circ j_t = 0 \text{ on } \Sigma_t \cap \{(z_{p1}, z_{p2}) \mid |z_{pi}| \leq \frac{1}{2}, i = 1 \text{ and } 2\}$$

where $j_t = j_{\Sigma_t}$. Put

$$v_t(x) = (df_t^{app} + J \circ df_t^{app} \circ j_t)(x). \tag{4.13}$$

Then

$$v_t(x) = (J - J_0) \circ df_t^{app} \circ j_t(x), \text{ on } \Sigma_t \cap \{(z_{p1}, z_{p2}) \mid |z_{pi}| \leq \frac{1}{2}, i = 1 \text{ and } 2\}.$$

Since $f$ is $J$-holomorphic on $\Sigma_0 \cap \{(z_{p1}, z_{p2}) \mid |z_{pi}| \leq \frac{1}{2}, i = 1 \text{ and } 2\}$, we have

$$J(z_{p1}, 0, \cdots, 0)(f_* \frac{\partial}{\partial z_{p1}}) = J_0(z_{p1}, 0, \cdots, 0)(f_* \frac{\partial}{\partial z_{p1}}),$$
$$J(0, z_{p2}, 0, \cdots, 0)(f_* \frac{\partial}{\partial z_{p2}}) = J_0(0, z_{p2}, 0, \cdots, 0)(f_* \frac{\partial}{\partial z_{p2}}).$$

Then we can derive from last lemma,

**Lemma 4.1.2** *For* $1 \leq k \leq 4$, *then*

$$|D^k v_t|_{\mu,h_E}(x) \leq C_k \cdot \frac{|t|}{\rho(x)^{k+1}}, \tag{4.14}$$

$$|D_c^k v_t|_{\mu_c,h_E}(x) \leq C_k' \cdot |t|. \tag{4.15}$$

We want to perturb $f_t^{app}$ into a $J$-holomorphic map from $\Sigma_t$ into $\mathcal{X}_{W^\alpha}$. Fixing the trivialization $\mathcal{X}_{W^\alpha} \cong W^\alpha \times V$ as before, we can represent $f_t^{app}$ as $(z, f_{t1}^{app})$. Identifying $W^\alpha$ with an open subset of $\mathbb{R}^{\tilde{n}}$, $z$ with 0 and $W^\alpha$ with a neighborhood with $T_z W^\alpha$, we can define a modified exponential map

$$exp^* : \ T_z W^\alpha \times \Gamma(\Sigma_t, f_t^{app,*} T(z \times V)) \to W^\alpha \times V,$$

$$(u_{t1}, u_{t2}) \to (u_{t1}, exp^*_{f_t^{app}}(u_{t1}, u_{t2}))$$

as follows. Suppose $v^i$ is coordinate of $W^\alpha$, $y^j$ is coordinate around $f_{t1}^{app}(x)$ in $V$, $x \in \Sigma_t$. Then $exp^*_{f_t^{app}(x)}\ (u_{t1},\ u_{t2}(f_t^{app}(x)))$ is defined to be $y(1)$, where $y(s)$ is the solution of

$$\begin{cases} \frac{d^2 y^j}{ds^2} + \Gamma^j_{kl}(s u_{t1}, y) \frac{dy^k}{ds} \frac{dy^l}{ds} = 0, \\ y(0) = f_{t1}^{app}(x), \\ \frac{dy}{ds}|_{s=0} = u_{t2}(f_{t1}^{app}(x)). \end{cases}$$

Here $\Gamma^j_{kl}(su_{t1}, y)$ is the Christoffel symbol of connection $\nabla$ on $E$. By definition the image of map $exp^*$ lies in the fiber $u_{t1} \times V$. We do not use exponential map defined by metric $h_\alpha$, because its image does not lie in a fiber. Note that when we choose $u_{t1} = 0$, $exp^*$ is exponential map on $z \times V$, defined by metric $h_E|_{z\times V}$. So by the implicit function theorem and the property that solution of ordinary differential equation depends smoothly on parameters, we get that for small $u_{t1}$, $exp^*_{f_t^{app}(x)}$ is a diffeomorphism on a small ball in $f_t^{app,*}T_{f_t^{app}(x)}(z \times V)$. Let $f_t$ be a map from $\Sigma_t$ into $W^\alpha \times V$, we can represent it as $(z_t, f_{t1})$. If $f_t$ is sufficiently close to $f_t^{app} = (z, f_{t1}^{app})$, then we can write

$$(z_t, f_{t1}(x)) = (u_{t1}, exp^*_{f_t^{app}(x)}(u_{t1}, u_{t2}(f_{t1}^{app}(x)))),$$

where $u_{t1}$ is a vector in $T_zW^\alpha$ and $u_{t2}$ is a vector field of $f_t^{app,*}T(z \times V)$ on $\Sigma_t$. We need to find $(u_{t1}, u_{t2})$ such that $f_t$ is $J$-holomorphic.

For any $(u_{t1}, u_{t2})$ in $T_zW^\alpha \times f_t^{app,*}T(z \times V)$, we denote by $\pi_t(u_{t1}, u_{t2}; x)$ the parallel transport from $T_{f_t(x)}(z_t \times V)$ to $T_{f_t^{app}(x)}(z \times V)$ with respect to $\nabla$ along the path $(su_{t1}, exp^*_{f_t^{app}(x)}\ (su_{t1},\ su_{t2}(f_{t1}^{app}(x))))_{1\le s\le 1}$. Since $\nabla$ is $J$-compatible, we have

$$J_z \circ \pi_t(u_{t1}, u_{t2}; x) = \pi_t(u_{t1}, u_{t2}; x) \circ J_{z_t}. \tag{4.16}$$

Let $\Lambda^{0,1}(f_t^{app,*}T(z \times V))$ be the vector bundle over $\Sigma_t$ of all anti-$(J_z, j_t)$-linear homomorphism from $T\Sigma_t$ to $f_t^{app,*}T(z \times V)$. Denote by $\Omega^{0,1}(f_t^{app,*}T(z \times V))$ and $\Omega(f_t^{app,*}T(z \times V))$ the space of sections of $\Lambda^{0,1}(f_t^{app*}T(z \times V))$ and $f_t^{app,*}T(z \times V)$ respectively.

Define

$$\Phi_t : \ T_z W^\alpha \times \Omega(f_t^{app,*}T(z \times V)) \to \Omega^{0,1}(f_t^{app,*}T(z \times V)), \qquad (4.17)$$
$$(u_{t1}, u_{t2}) \to \pi_t(u_{t1}, u_{t2}; \cdot) \circ (df_t + J_{z_t} \circ df_t \circ j_t).$$

Because of (4.16), this map is well-defined. Let

$$L_t^{v,app}(\sigma) = D\Phi_t(0)\sigma, \ t \in \Delta. \qquad (4.18)$$

**Lemma 4.1.3** *For any $\sigma = (\sigma_1, \sigma_2) \in T_z W^\alpha \times \Omega(f_t^{app,*}T(z \times V))$ and any $e \in T\Sigma_t$, we have*

$$L_t^{v,app}(\sigma_1, \sigma_2)(e) = \nabla_e \sigma_2 + J_z \circ \nabla_{j_t e}\sigma_2$$
$$+\frac{1}{4}N_{j_z}(\partial_{J_z} f_t^{app}, \sigma_2) + \frac{dJ_{z_s}}{ds}|_{s=0} \circ df_t^{app} \circ j_t e + \Gamma(\sigma_1, v_t), \qquad (4.19)$$

*where $z_s = s\sigma_1$ a path in $W^\alpha$, $\Gamma(\cdot,\cdot)$ is the Christoffel symbol of $\nabla$, and $v_t$ is defined in (4.13), $t \neq 0$. Here we identify $\Gamma(f_t^{app,*}T(z \times V))$ with $\Gamma(f_t^{app}(\Sigma_t), T(z \times V))$, so we may view $\nabla$ as a covariant derivative on $\Gamma(f_t^{app,*}$ $T(z \times V))$.*

**Proof:** The computation, which is the same as we did in Chapter 2, Section 2, is omitted. □

Let $L_t^{v,app*} : \Omega^{0,1}(f_t^{app,*}T(z \times V)) \to T_z W^\alpha \times \Omega(f_t^{app,*}T(z \times V))$ be the adjoint of $L_t^{v,app}$ with respect to the metric $ds_0^2$, $h_E$ and the metric $\mu_c$ on $\Sigma_t$. Let $(e, j_t e)$ be any local unitary basis of $T\Sigma_t$ with respect to $\mu_c$, $t \neq 0$.

**Lemma 4.1.4** *Assume that $\langle\cdot,\cdot\rangle$ is the induced metric on $f_t^{app,*}T(z\times V)$ by $h_E$. For any section $\xi\in\Omega^{0,1}(f_t^{app,*}T(z\times V))$, we then have*

$$L_t^{v,app*}(\xi)=(2\langle\xi(e),(dJ,\cdot)\circ df_t^{app}(j_te)+\Gamma(\cdot,v_t)\rangle^*,$$
$$-2\nabla_e\xi(e)-2\nabla_{j_te}\xi(j_te)+\frac{1}{2}\langle\xi(e),N_{J_z}(\partial_{J_z}f_t^{app},\cdot)(e)\rangle^*). \quad (4.20)$$

*Here $\langle\xi(e),(dJ,\cdot)\circ df_t^{app}(j_te)+\Gamma(\cdot,v_t)\rangle^*$ is defined by the relation that its inner product with $\sigma_1\in T_zW^\alpha$ is $\int_{\Sigma_t}\langle\xi(e),\frac{dJ_{z_s}}{ds}|_{s=0}\circ df_t^{app}(j_te)+\Gamma(\sigma_1,v_t)(e)\rangle d\mu_c$, and $\langle\xi(e),N_J(\partial_{J_z}f_t^{app},\cdot)\ (e)\rangle^*$ is defined by the relation that its inner product with $\sigma_2\in f_t^{app,*}T(z\times V)$ is $\langle\xi(e),N_J(\partial_{J_z}f_t^{app},\sigma_2)(e)\rangle$.*

**Proof:** Let $e_1=e$, $e_2=j_te$, then the dual basis $(e_1^*,e_2^*)$ satisfies $e_2^*=-j_te_1^*$. Set

$$\xi=\xi_1e_1^*+\xi_2e^*,\ \ \xi_i\in f_t^{app,*}T(z\times V),\ i=1,2.$$

Since $\xi$ is anti-$(J_z,j_t)$-linear, we have

$$\xi_2=-J_z\xi_1.$$

So for any anti-$(J_z,j_t)$-linear vector $X\in\Omega^{0,1}(f_t^{app,*}T(z\times V))$, we have $\langle\xi,X\rangle=2\langle\xi_1,X(e_1)\rangle$. Using this, then

$$\int_{\Sigma_t}\langle L_t^{v,app*}\xi,\sigma\rangle_{\mu_c,h_E}d\mu_c$$
$$=\int_{\Sigma_t}\langle\xi,L_t^{v,app}(\sigma_1,\sigma_2)\rangle d\mu_c$$

$$
\begin{aligned}
&= 2\int_{\Sigma_t}\langle\xi_1, L_t^{v,app}(\sigma_1,\sigma_2)(e_1)\rangle d\mu_c \\
&= 2\int_{\Sigma_t}\langle\xi_1, \nabla_{e_1}\sigma_2 + J_z\circ\nabla_{e_2}\sigma_2\rangle d\mu_c + \frac{1}{2}\int_{\Sigma_t}\langle\xi_1, N_{J_z}(\partial_{J_z}f_t^{app},\sigma_2)(e_1)\rangle d\mu_c \\
&\quad +2\int_{\Sigma_t}\langle\xi_1, \frac{\partial J_{z_s}}{ds}|_{s=0}\circ df_t^{app}(e_2) + \Gamma(\sigma_1, v_t)(e_1)\rangle d\mu_c \\
&= -2\int_{\Sigma_t}\langle\nabla_{e_1}\xi_1 + \nabla_{e_2}\xi_2, \sigma_2\rangle d\mu_c + \frac{1}{2}\int_{\Sigma_t}\langle\xi_1, N_{J_z}(\partial_{J_z}f_t^{app},\sigma_2)(e_1)\rangle d\mu_c \\
&\quad +2\int_{\Sigma_t}(\xi_1, \frac{\partial J_{z_s}}{ds}|_{s=0}\circ df_t^{app}(e_2) + \Gamma(\sigma_1, v_t)(e_1)\rangle d\mu_c.
\end{aligned}
$$

The lemma follows. □

**Lemma 4.1.5** *Let $\xi \in \Omega^{0,1}(f_t^{app,*}T(z\times V))$, and $r(\rho)$ be a positive function in $\rho$. For any $0<\epsilon<1$, there is a $C_\epsilon$ such that*

$$
\begin{aligned}
\int_{\Sigma_t} r(\rho)^2|L_t^{v,app*}\xi|^2_{\mu_c,h_E}d\mu_c \;\geq\; (2-\epsilon)\int_{\Sigma_t} r(\rho)^2|\nabla\xi|^2_{\mu_c,h_E}d\mu_c \\
-C_\epsilon\cdot\int_{\Sigma_t}(r(\rho)^2|df_t^{app}|^2_{\mu_c,h_E}\cdot|\xi|^2_{\mu_c,h_E} + |r'(\rho)|^2\cdot|\xi|^2_{\mu_c,h_E})d\mu_c. \quad (4.21)
\end{aligned}
$$

*$C_\epsilon$ depends only on $\epsilon$, the curvature tensor of $h_E$, $\mu_c$ and the fiberwise almost complex structure, $t\neq 0$.*

**Proof:** Observe that $\langle\xi(e), (dJ,\cdot)\circ df_t^{app}(j_te)+\Gamma(\cdot,v_t)\rangle^*$ and $\langle\xi(e), N_{J_z}(\partial_{J_z}f_t^{app},\cdot)(e)\rangle^*$ are zero order operators of $\xi$. The computation, which is similar to the computation in the remark after Lemma 6.4 in [RT], is omitted. □

We will apply the implicit function theorem to construct the map $T_t$ in Theorm 4.1.1. First we need to establish the lower bound of the spectrum of $\Box_t^{v,app} = L_t^{v,app}\circ L_t^{v,app*}$.

**Lemma 4.1.6** *Let $D$ be a disk in $\mathbb{C}$ and $\mu_c$ be the cylindrical metric on $D \setminus \{0\}$, i.e., $\mu_c = \frac{|dz|^2}{|z|^2}$. Suppose that $f_D : D \to z \times V$, where $z \in W^\alpha$, is a $J$-holomorphic map, $L^v_D$ is the linearization of Cauchy-Riemann equation at $f_D$. Assume that $\xi$ is a $C^2$-smooth section of $\Omega^{0,1}(f_D^* T(z \times V))$ over $D \setminus \{0\}$ satisfying $L^{v*}_D \xi = 0$ on $D \setminus \{0\}$, where $L^{v*}_D$ is the adjoint of $L^v_D$ with respect to $\mu_c$. Assume further that*

$$\int_{D\setminus\{0\}} (|\nabla \xi|^2_{\mu_c, h_E} + |z|^2 \cdot |\xi|^2_{\mu_c, h_E}) d\mu_c \ < \ \infty.$$

*Then limit $lim_{w\to 0}\xi(w)$ exists and is a vector in $T_{f_D(0)}(z \times V)$. Such a limit is called the residue of $\xi$ at $z = 0$.*

**Proof:** From Lemma 4.1.4, we know that $L^{v*}_D \xi = (L^{v*}_{D1}\xi, L^{v*}_{D2}\xi) \in T_z W^\alpha \times \Omega^0(f_D^* T(z \times V))$. Observe that $L^{v*}_{D2}$ is exactly the operator $L_0^*$ in [RT]. Then the lemma follows from Lemma 6.5 in [RT]. □

Let $L^v_0$ and $L^v$ be the linearization of the Cauchy-Riemann equation at $f_0$ and $f$ respectively, and let $L^{v*}_0$ and $L^{v*}$ be the adjoints of $L^v_0$ and $L^v$ on $\Sigma_0 \setminus Sing(\Sigma_0)$. We denote by $\mathrm{Ker} L^{v*}_0$ the set of those sections $\xi$ of $\Omega^{0,1}(f_0^* T(z_0 \times V))$ over $\Sigma_0 \setminus Sing(\Sigma_0)$ satisfying $L^{v*}_0 \xi = 0$ outside $Sing(\Sigma_0)$,

$$\int_{\Sigma_0} |\nabla \xi|^2_{\mu_c, h_E} + \rho^2 \cdot |\xi|^2_{\mu_c, h_E} d\mu_c \ < \ \infty,$$

and for every node $p$ in $Sing(\Sigma_0)$

$$lim_{z\to 0, z\in U_{p1}} \xi_z + lim_{z\to 0, z\in U_{p2}} \xi_z \ = \ 0.$$

Here $U_p$ is any small neighborhood of $p$ with two irreducible components $U_{p1}$, $U_{p2}$. We define $\mathrm{Ker}L^{v*}$ similarly.

**Proposition 4.1.1** *For a generic $(J,\nu)$, $KerL_0^{v*}$ is trivial, and for any $f \in \tilde{W}$, $kerL^{v*}$ is trivial also.*

**Proof:** Fact 1. Write $L_0^{v*} = (L_{01}^{v*}, L_{02}^{v*})$, and $\mathrm{Ker}L_0^{v*} \subset \mathrm{Ker}L_{02}^{v*}$. According to Lemma 6.1 in [RT], $\mathrm{Ker}L_{02}^{v*} = 0$, so $\mathrm{Ker}L_0^{v*}$ is trivial.

Fact 2. Again $\Sigma_{0i}$ are components of $\Sigma_0$. Let $\Sigma_{0i}^0 = \Sigma_{0i} \setminus Sing(\Sigma_0)$ and denote by $L_0^{v*}|_{\Sigma_{0i}^0}$ and $L^{v*}|_{\Sigma_{0i}^0}$ the restriction of $L_0^{v*}$ and $L^{v*}$ to $\Omega^{0,1}(\Sigma_{0i}^0, f_0^*T(z_0 \times V))$ and $\Omega^{0,1}(\Sigma^0 0i, f^*T(z \times V))$ respectively. Since for elliptic operators, $\dim \mathrm{Ker}L^{v*}|_{\Sigma_{0i}^0}$ is upper semi-continuous function of $f$, $\dim \mathrm{Ker}L^{v*}|_{\Sigma_{0i}^0} \leq \dim \mathrm{Ker}L_0^{v*}|_{\Sigma_{0i}^0}$. Here we probably need to shrink $\tilde{W}$ a little bit.

Fact 3. Consider

$$
\begin{aligned}
&R_0 : \prod_{i=1}^{m} \mathrm{Ker}(L_0^{v*}|_{\Sigma_{0i}^0}) \to \prod_{p \in Sing(\Sigma_0)} T_{f_0(p)}(z_0 \times V) \times T_{f_0(p)}(z_0 \times V) \\
&R_0(\xi) = \prod_{p \in Sing(\Sigma_0)} (lim_{z \to p} \xi_{\Sigma_{0i}^0}, lim_{z \to p} \xi_{\Sigma_{0i'}^0}), \\
&\Delta_0 = \prod_{p \in Sing(\Sigma_0)} \{(u,-u) \mid u \in T_{f_0(p)}(z_0 \times V)\} \\
&R : \prod_{i=1}^{m} \mathrm{Ker}(L^{v*}|_{\Sigma_{0i}^0}) \to \prod_{p \in Sing(\Sigma_0)} T_{f(p)}(z \times V) \times T_{f(p)}(z \times V) \\
&R(\xi) = \prod_{p \in Sing(\Sigma_0)} (lim_{z \to p} \xi_{\Sigma_{0i}^0}, lim_{z \to p} \xi_{\Sigma_{0i'}^0}), \\
&\Delta = \prod_{p \in Sing(\Sigma_0)} \{(u,-u) \mid u \in T_{f(p)}(z \times V)\}.
\end{aligned}
$$

On each component $\Sigma_{0i}^0$, $f$ converges to $f_0$ in $C^4$-topology, and $\mathrm{Ker}L^{v*}|_{\Sigma_{0i}^0}$ converges to $\mathrm{Ker}L_0^{v*}|_{\Sigma_{0i}^0}$ in $C^2$-topology, i.e., map $R$ is $C^2$-close to $R_0$. From

the proof of Proposition 6.1 in [RT], map $R_0$ is transversal to $\Delta_0$ for generic $J$, $R$ map is transversal to $\Delta$. Ker$L^{v*} = 0$ follows from above three facts and a simple dimension counting. □

**Lemma 4.1.7** *There is a constant $C > 0$, independent of $t$, such that for $t$ sufficiently small, the first eigenvalue $\lambda_1(\Box_t^{v,app})$ of $\Box_t^{v,app}$ is bounded from below by $\frac{C}{(log|t|)^2}$.*

**Proof:** Write $L_t^{v,app} = L_{t1}^{v,app} + L_{t2}^{v,app}$, where $L_{t1}^{v,app} : T_z W^\alpha \to \Omega^{0,1}(\Sigma_t, f_t^{app,*}T(z \times V))$ and $L_{t,app}^{v2} : \Omega(\Sigma_t, f_t^{app,*}T(z \times V)) \to \Omega^{0,1}(\Sigma_t, f_t^{app,*}T(z \times V))$. Then $L_t^{v,app*} = (L_{t1}^{v,app*}, L_{t2}^{v,app*})$, and $\Box_t^{v,app} = L_{t1}^{v,app} \circ L_{t1}^{v.app*} + L_{t2}^{v,app} \circ L_{t2}^{v,app*}$. Note that $L_{t2}^{v,app} \circ L_{t2}^{v,app*}$ is exactly the operator $L_t \circ L_t^* = \Box_t$ in [RT]. Lemma 6.6 in [RT] then implies that the first eigenvalue $\lambda_1(L_t \circ L_t^*) \geq \frac{C}{(log|t|)^2}$. Note in the proof of Lemma 6.6 [RT], we need the second half of Proposition 4.1.1. Since operator $L_{t1}^{v,app} \circ L_{t1}^{v,app*}$ is non-negative, and the spetrum of $L_{t2}^{v,app} \circ L_{t2}^{v,app*}$ are all eigenvalues, $\lambda_1(\Box_t^{v,app}) \geq \frac{C}{(log|t|)^2}$ follows from functional analysis. □

Let $\Phi_t : T_z W^\alpha \times \Omega(f_t^{app,*}T(z \times V)) \to \Omega^{0,1}(f_t^{app,*}T(z \times V))$ be the map defined in (4.17), define

$$\Psi_t(\xi) = \Phi_t \circ L_t^{v,app*} : \Omega^{0,1}(f_t^{app,*}T(z \times V)) \to \Omega^{0,1}(f_t^{app,*}T(z \times V)) \quad (4.22)$$

To find a $J$-holomorphic $f_t$, it suffices to show that $\Psi_t(\xi)$ has a zero solution $\xi$.

**Lemma 4.1.8** $\Psi_t(\xi)$ *has the following expansion,*

$$\Psi_t(0) = v_t, \tag{4.23}$$

$$\Psi_t(\xi) = v_t + L_t^{v,app} \circ L_t^{v,app*}\xi + H_t(\xi), \tag{4.24}$$

*where $v_t$ is defined in (4.13). Furthermore, $H_t(\xi)$ satisfies*

$$\| H_t(\xi_1) - H_t(\xi_2)\|_{0,\frac{1}{2}} \leq C(\| \xi_1 \|_{2,\frac{1}{2}} + \| \xi_2 \|_{2,\frac{1}{2}}) \cdot \| \xi_1 - \xi_2 \|_{2,\frac{1}{2}}, \tag{4.25}$$

$$\| H_t(\xi) \|_{0,\frac{1}{2}} \leq C \cdot \| \xi \|_{1,\frac{1}{2}} \cdot \| \xi \|_{2,\frac{1}{2}} \tag{4.26}$$

*where we denote by $\| \cdot \|_{k,\frac{1}{2}}$ the $C^{k,\frac{1}{2}}$-Hölder norm defined by the metric $h_\alpha$ on $W^\alpha \times V$ and the metric $\mu_c$ on $\Sigma_t$.*

**Proof:** Consider the expansion of $\Phi_t(\sigma_1, \sigma_2)$,

$$\Phi_t(\sigma_1, \sigma_2) = v_t + L_{t1}^{v,app}\sigma_1 + L_{t2}^{v,app}\sigma_2 + G(\sigma_1, \sigma_2),$$

where $G(\sigma_1, \sigma_2)$ is the nonlinear part.

**Claim:**

$$G(\sigma_1, \sigma_2)(e) = L_1(\sigma_1, \sigma_2) \circ \nabla_e \sigma_2 + L_2(\sigma_1, \sigma_2) \circ \nabla_{j_t e}\sigma_2$$

$$+Q_1(\sigma_1, \sigma_2) \circ df_t^{app}(e) + Q_2(\sigma_1, \sigma_2) \circ df_t^{app}(j_t e),$$

$$\| L_i(\sigma_1, \sigma_2) \|_{0,\frac{1}{2}} \leq C \cdot (\| \sigma_1 \|_0 + \| \sigma_2 \|_{1,\frac{1}{2}}), i = 1, 2,$$

$$\| Q_i(\sigma_1, \sigma_2) \|_{0,\frac{1}{2}} \leq C \cdot (\| \sigma_1 \|_0^2 + \| \sigma_2 \|_{1,\frac{1}{2}}^2), i = 1, 2,$$

$$\| G(\sigma_1, \sigma_2) - G(\sigma_1', \sigma_2') \|_{0,\frac{1}{2}} \leq C \cdot (\| (\sigma_1, \sigma_2) \|_{1,\frac{1}{2}} + \| (\sigma_1', \sigma_2') \|_{1,\frac{1}{2}})$$

$$\cdot \| (\sigma_1, \sigma_2) - (\sigma_1', \sigma_2') \|_{1,\frac{1}{2}},$$

$$\| G(\sigma_1, \sigma_2) \|_{0,\frac{1}{2}} \leq C \cdot \| (\sigma_1, \sigma_2) \|_{0,\frac{1}{2}} \cdot \| (\sigma_1, \sigma_2) \|_{1,\frac{1}{2}},$$

*where e is a vector in $T\Sigma_t$, $L_1$ and $L_2$ are linear first order differential operator in $(\sigma_1, \sigma_2)$, $Q_1(\sigma_1, \sigma_2)$ and $Q_2(\sigma_1, \sigma_2)$ are quadratic operators in $(\sigma_1, \sigma_2, \nabla\sigma_2)$.*

To prove the claim, view $W^\alpha \times V$ as a symplectic manifold with standard symplectic form on $W^\alpha \subset \mathbb{R}^{\bar{n}}$ (if $W^\alpha$ is odd dimension, we replace it by $W^\alpha \times \mathbb{R}$). Then $\Phi_t(\sigma_1, \sigma_2)$ is operator $\bar{\partial}$ at $f_t^{app}$ for symplectic manifold $W^\alpha \times V$. The claim follows from the calculation done by A. Floer ([F], section 3). Note although the constant $C$ depends on $f_t^{app}$, since $\| f_t^{app} \|_{5,\frac{1}{2}}$ is bounded, we may assume constant $C$ does not depend on $f_t^{app}$ at all.

By choosing $\sigma_1 = L_{t1}^{v,app*}\xi$, $\sigma_2 = L_{t2}^{v,app*}\xi$ in the claim, note $\| \sigma_1 \|_0 \leq C \cdot \| \xi \|_{1,0}$, $\| \sigma_2 \|_{1,\frac{1}{2}} \leq C \cdot \| \xi \|_{2,\frac{1}{2}}$, all the estimates of the lemma then follow from the claim. □

Define $\Gamma^{k,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)) = \{\xi \in \Omega^{0,1}(f_t^{app,*} T(z \times V)) \mid \| \xi \|_{k,\frac{1}{2}} < \infty\}$. Then

$$\Psi_t : \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)) \to \Gamma^{0,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)),$$
$$\Box_t^{v,app} : \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)) \to \Gamma^{0,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)).$$

**Lemma 4.1.9** *Assume $\xi \in \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V))$, $\zeta = \Box_t^{v,app}\xi$ on $\Sigma_t$. Then*

$$\| \xi \|_{2,\frac{1}{2}} \leq C \cdot (-log|t|)^{\frac{5}{2}} \| \zeta \|_{0,\frac{1}{2}} . \tag{4.27}$$

**Proof:** Note that

$$
\begin{aligned}
&Vol(\Sigma_t, \mu_c) \leq C \cdot (-log|t|),\\
&\int_{\Sigma_t} |\xi|_0^2 d\mu_c \leq C \cdot \parallel \xi \parallel_0^2 \cdot (-log|t|),\\
&\int_{\Sigma_t} |L_t^{v,app*}\xi|^2 d\mu_c = \int_{\Sigma_t} \langle \xi, \Box_t^{v,app}\xi \rangle d\mu_c\\
&= \int_{\Sigma_t} \langle \xi, \zeta \rangle d\mu_c\\
&\leq C \cdot \parallel \xi \parallel_0 (-log|t|)^{\frac{1}{2}} \cdot (\int_{\Sigma_t} |\zeta|_0^2 d\mu_c)^{\frac{1}{2}}.
\end{aligned}
$$

Lemma 4.1.7 implies that

$$
\int_{\Sigma_t} \langle \xi, \Box_t^{v,app}\xi \rangle d\mu_c \geq \lambda_1(\Box_t^{v,app}) \cdot \int_{\Sigma_t} |\xi|_0^2 d\mu_c.
$$

Then

$$
\int_{\Sigma_t} |\xi|_0^2 d\mu_c \leq C \cdot \parallel \zeta \parallel_0^2 \cdot (-log|t|)^5.
$$

Then (4.27) follows from above inequality and the standard elliptic estimate. □

**Proposition 4.1.2** *There is a $t_0 > 0$ such that for any $0 < |t| < |t_0|$, there is a unique $\xi \in \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V))$ satisfying $\parallel \xi \parallel_{2,\frac{1}{2}} \leq \sqrt{|t|}$ and $\Psi_t(\xi) = 0$ on $\Sigma_t$, i.e., $f_t = exp^*_{f_t^{app}}(L_t^{v,app*}\xi)$ is a J-holomorphic map from $\Sigma_t$ into $\tilde{z} \times V$ for some $\tilde{z} \in W^\alpha$. Denote this solution by $f_t^{sol}$ in the following.*

**Proof:** Let $B_{\sqrt{|t|}}(0)$ be the ball in $\Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V))$ with radius $\sqrt{|t|}$ and the center at origin. Then solving equation $\Psi_t(\xi) = 0$ for some $\xi \in B_{\sqrt{|t|}}(0)$ is equivalent to solving equation

$$\xi = (\Box_t^{v,app})^{-1}(-v_t + H_t(\xi)),$$

where $(\Box_t^{v,app})^{-1}$ is the inverse of $\Box_t^{v,app}$. Note that

$$(\Box_t^{v,app})^{-1}(-v_t + H_t(\cdot)) : \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)) \to \Gamma^{2,\frac{1}{2}}(\Lambda^{0,1} f_t^{app,*} T(z \times V)).$$

First we show $(\Box_t^{v,app})^{-1}(-v_t + H_t(\cdot)) : B_{\sqrt{|t|}}(0) \to B_{\sqrt{|t|}}(0)$. By lemma 4.1.8,

$$\begin{aligned}
& \| (\Box_t^{v,app})^{-1}(-v_t + H_t(\xi)) \|_{2,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot \| v_t \|_{0,\frac{1}{2}} + C \cdot (-log|t|)^{\frac{5}{2}} \cdot \| H_t(\xi) \|_{0,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot |t| + C \cdot (-log|t|)^{\frac{5}{2}} \cdot \| \xi \|_{1,\frac{1}{2}} \cdot \| \xi \|_{2,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot \sqrt{|t|} \cdot \sqrt{|t|} \\
\leq\ & \sqrt{|t|}.
\end{aligned}$$

when $|t|$ is small enough.

Secondly we show $(\Box_t^{v,app})^{-1}(-v_t + H_t(\cdot))$ is a contracting map.

$$\begin{aligned}
& \| (\Box_t^{v,app})^{-1}(-v_t + H_t(\xi_1)) - (\Box_t^{v,app})^{-1}(-v_t + H_t(\xi_2)) \|_{2,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot \| H_t(\xi_1) - H_t(\xi_2) \|_{0,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot (\| \xi_1 \|_{2,\frac{1}{2}} + \| \xi_2 \|_{2,\frac{1}{2}}) \cdot \| \xi_1 - \xi_2 \|_{2,\frac{1}{2}} \\
\leq\ & C \cdot (-log|t|)^{\frac{5}{2}} \cdot \sqrt{|t|} \cdot \| \xi_1 - \xi_2 \|_{2,\frac{1}{2}} \cdot
\end{aligned}$$

When $|t|$ is small enough, it is a contraction map. So by the Banach contracting mapping theorem, $(\Box_t^{v,app})^{-1}(-v_t + H_t(\cdot))$ has an unique fixed point. The proposition is proved. □

Assume that $f_0 \in \mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{W^\alpha}, J, 0)$. By the transversality result in Section 2.2, the tangent space of $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{W^\alpha}, J, 0)$ at $f_0$ is naturally identified with the kernel $\mathrm{Ker}L_0^v$, where $L_0^v$ is the linearization of the Cauchy-Riemann equation at $f_0$, i.e., a tangent vector at $f_0$ is a continuous section $u = (u_1, u_2) \in T_{z_0}W^\alpha \times \Omega(\Sigma_0, f_0^* T(z_0 \times V)$ satisfying $L_0^v u = 0$ on $\Sigma_0 \setminus Sing(\Sigma_0)$. This implies that there is a local diffeomorphism from a neighborhood of 0 in $\mathrm{Ker}L_0^v$ into $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{W^\alpha}, J, 0)$. We assume that $\tilde{W}$ is contained in the image of such a neighborhood, we may assume that for any $f_a$ and $f_b$ in $\tilde{W}$,

$$\| f_a - f_b \|_{C^4(\Sigma_0)} \leq C \cdot \| f_a - f_b \|_{C^0(\Sigma_0)} .$$

Given any $f$ in $\tilde{W}$, there is a unique section $u_{ft}^{app} = (u_{ft1}^{app}, u_{ft2}^{app})$ in $T_{z_0}W^\alpha \times \Omega(\Sigma_0, f_{0t}^{app,*} T(z_0 \times V))$ such that

$$f_t^{app}(x) = exp^*_{f_{0t}^{app}(x)}(u_{ft}^{app}(f_{0t}^{app}(x))),\ x \in \Sigma_t$$

where $f_{0t}^{app}(x)$ is the approximate solution constructed from $f_0$. A straightforward computation shows

$$|\nabla u_{ft2}^{app}|_{\mu_c, h_E}(x) \leq C \cdot \rho(x),$$

and for any $f_a$, $f_b$ in $\tilde{W}$

$$\| u^{app}_{f_at} - u^{app}_{f_bt} \|_{C^4(\Sigma_t)} \leq C \cdot \| f_a - f_b \|_{C^0(\Sigma_0)} .$$

We define $T_t : \tilde{W} \to \mathcal{M}^v_A(\Sigma_t, \mathcal{X}_{W^\alpha}, J, 0)$ by assigning $f^{sol}_t$ in Proposition 4.1.2 to each $f$ in $\tilde{W}$. Clearly, $T_t(f)$ converges to $f$ as $t$ goes to zero. It is easy to see that $T_t$ is smooth. We want to examine the invertibility of the differential of $T_t$ at any point $f$ in $\tilde{W}$. For simplicity, we do it at $f_0$.

**Lemma 4.1.10** *Let $\pi_t(u^{app}_{ft}, x)$ be the parallel transport along the path $\{exp^*_{f^{app}_{0t}(x)}(su^{app}_{ft}(f^{app}_{0t}(x)))\}_{0\leq s\leq 1}$. Then there is a uniform constant $C > 0$ such that*

$$\| v_{0t} - \pi_t(u^{app}_{ft}, x)v_t \|_{C^0} \leq C \cdot \sqrt{|t|} \cdot \| f - f_0 \|_{C^0(\Sigma_0)} \tag{4.28}$$

*where $v_{0t} = df^{app}_{0t} + J_{z_0} \circ df^{app}_{0t} \circ j_t$.*

**Proof:** Choose a diffeomorphism $\phi$ from a neighborhood of $f^{app}_{0t}(\Sigma_t)$ onto a neighborhood of $f^{app}_t(\Sigma_t)$ satisfying (1) $\| \phi - id \|_{C^4} \leq \| f_0 - f \|_{C^5}$; (2) for each node $p$ in $Sing(\Sigma_0)$, let $U_p$, $z_{p1}$ and $z_{p2}$ be as in the begining of this chapter, then

$$f^{app}_{0t}(z_{p1}, z_{p2}) = \phi^{-1} \circ f^{app}_t(z_{p1}, z_{p2}) = (z_0, z_{p1}, z_{p2}, 0, \cdots, 0) \ \in z_0 \times V$$

for all $(z_{p1}, z_{p2})$ in $U_p$.

When $x \in \Sigma_t \setminus \{\cup_{p\in Sing\Sigma_0} U_p\}$, then by the definition of $f^{app}_{0t}$ and $f^{app}_t$, we have

$$f^{app}_{0t}(x) = f_0 \circ \phi_t^{-1}(x), \quad f^{app}_t(x) = f \circ \phi_t^{-1}(x),$$

where $\phi_t^{-1}$ is the diffemorphism used in the construction of approximated solution. Since both $f$ and $f_0$ are $J$-holomorphic, we have

$$v_{0t}(x) = df_{0t}^{app} + J \circ df_{0t}^{app} \circ j_t = J_{z_0} \circ df_0 \circ (\phi_{t*}^{-1} \circ j_t - j_0 \circ \phi_{t*}^{-1}),$$
$$v_t(x) = J_z \circ df \circ (\phi_{t*}^{-1} \circ j_t - j_0 \circ \phi_{t*}^{-1}).$$

It follows that

$$\begin{aligned} & |v_{0t}(x) - \phi_*^{-1} v_t(x)| \\ = \; & |(J_{z_0} \circ df_0 - \phi_*^{-1} \circ J_z \circ df) \cdot (\phi_{t*}^{-1} \circ j_t - j_0 \circ \phi_{t*}^{-1})|(x) \\ \leq \; & C \cdot |f - f_0|_{C^1} \cdot |t|. \end{aligned}$$

When $x$ is in $U_p$ where $p$ is a double point of $\Sigma_0$. Then $\phi^{-1} \circ f_t^{app}(z_{p1}, z_{p2}) = f_{0t}^{app}(z_{p1}, z_{p2}) = (z_0, z_{p1}, z_{p2}, 0, \cdots, 0)$ implies that

$$\phi_*^{-1} v_t(x) = (\phi_*^{-1} \circ J_z \circ \phi_* - J_0) \circ d\phi^{-1} \circ f_t^{app} \circ j_t),$$

where $J_0$ is the standard complex structure on $\mathbb{C}^n$.

Since $\phi^{-1} \circ f$ is $\phi_*^{-1} \circ J \circ \phi_*$-holomorphic, we have

$$\phi_*^{-1} \circ J_z \circ \phi_*|_{f(z_{p1},0)} = J_0 = \phi_*^{-1} \circ J_z \circ \phi_*|_{f(0,z_{p2})}.$$

In $\mathbb{C}^n$, we view $\phi_*^{-1} \circ J_z \circ \phi_*|_{f(z_{p1},0)}$ as an almost complex structure at $f(z_{p1}, z_{p2})$. We may assume that $|z_{p1}| \geq |z_{p2}|$ at $x$. It follows that

$$\phi_*^{-1}v_t(x) = (\phi_*^{-1} \circ J_z \circ \phi_*|_{f(z_{p1},z_{p2})} - \phi_*^{-1} \circ J_z \circ \phi_*|_{f(z_{p1},0)}) \circ df_{0t}^{app} \circ j_t.$$

Similarly we also have

$$v_{0t}(x) = (J_{z_0}|_{f_0(z_{p1},z_{p2})} - J_{z_0}|_{f_0(z_{p1},0)}) \circ df_{0t}^{app} \circ j_t.$$

By the mean value theorem we deduce on $U_p$

$$\begin{aligned} & |v_{0t}(x) - \phi_*^{-1}v_t(x)|_{\mu_c} \\ = \; & Sup_{0\le\epsilon\le 1} \parallel \nabla(J - \phi_*^{-1} \circ J \circ \phi_*)|_{f(z_{p1},z_{p2})} \cdot |z_{p1}| \cdot |df_{0t}^{app}|(x) \\ \le \; & C \cdot \parallel f - f_0 \parallel_{c^0(\Sigma_0)} \cdot \sqrt{t}. \end{aligned}$$

It follows from the definition of approximate solution that

$$|v_{0t}(x) - v_t(x)| \le C \cdot \parallel f - f_0 \parallel_{C^0(\Sigma_0)} \cdot \sqrt{|t|}. \tag{4.29}$$

The lemma follows from combining this inequality, $|v_t|_{C^0} \le C \cdot |t|$, and $\parallel \pi_t^{-1}(u_{ft}^{app}, \cdot) \circ \phi_*^{-1} - id \parallel_{C^2} \le C \cdot \parallel f - f_0 \parallel_{C^0}$ . □

Let $\xi_1$, $\xi_2$ be the sections in Proposition 4.1.2, such that $T_t(f_0) = exp^*_{f_{0t}^{app}}(L_{0t}^{v,app*}\xi_1)$, and $T_t(f) = exp^*_{f_t^{app}}(L_t^{v,app*}\xi_2)$. Using the fact that $T_t(f_0)$, $T_t(f)$ are $J$-holomorphic, we have

$$0 = v_{0t} + L_{0t}^{v,app} \circ L_{0t}^{v,app*}(\xi_1) + H_{0t}(\xi_1), \tag{4.30}$$

$$0 = \pi_t(u_{ft}^{app}, \cdot)(v_t + L_t^{v,app} \circ L_t^{v,app*}(\xi_2) + H_{0t}(\xi_2)), \tag{4.31}$$

$$\| \xi_1 \|_{1,\frac{1}{2}} \leq \sqrt{|t|}, \quad \| \xi_2 \|_{1,\frac{1}{2}} \leq \sqrt{|t|}.$$

Note that

$$\begin{aligned} & \| (\pi_t(u_{ft}^{app}, \cdot) \circ L_t^{v,app} \circ L_t^{v,app*} - L_{0t}^{v,app} \circ L_{0t}^{v,app*} \circ \pi_t(u_{ft}^{app}, \cdot))(\xi_2) \|_{0,\frac{1}{2}} \\ \leq \; & C \cdot \sqrt{|t|} \cdot \| f - f_0 \|_{C^0}, \end{aligned} \tag{4.32}$$

$$\begin{aligned} & \| (\pi_t(u_{ft}^{app}, \cdot) H_t(\xi_2) - H_{0t}(\pi_t(u_{ft}^{app}, \cdot)(\xi_2)) \|_{0,\frac{1}{2}} \\ \leq \; & C \cdot \sqrt{|t|} \cdot \| f - f_0 \|_{C^0} \, . \end{aligned} \tag{4.33}$$

Substracting (4.30) from (4.31), we obtain

$$\begin{aligned} & \pi_t(u_{ft}^{app}, \cdot) v_t - v_{0t} \\ = \; & \Box_t^v(\xi_1 - \pi_t(u_{ft}^{app}, \cdot)\xi_2) + H_{0t}(\xi_1) - H_{0t}(\pi_t(u_{ft}^{app}, \cdot)\xi_2) + B. \end{aligned} \tag{4.34}$$

(4.32) and (4.33) imply

$$\| B \|_{0,\frac{1}{2}} \leq C \cdot \sqrt{|t|} \cdot \| f - f_0 \|_{C^0}, \tag{4.35}$$

Applying Lemma 4.1.9 to (4.34), and using Lemma 4.1.10 and (4.35), we get

$$\| \xi_1 - \pi_t(u_{ft}^{app}, \cdot)\xi_2 \|_{2,\frac{1}{2}} \leq C \cdot \sqrt{|t|} \cdot (-log|t|)^{\frac{5}{2}} \cdot \| f - f_0 \|_{C^0} \, .$$

Therefore the map $T_t$ is injective near $f_0$. Moreover, if $D_{f_0}T_t$ denotes the derivative of $T_t$ at $f_0$, then for any $u$ in $\mathrm{Ker}L_0^v$, we have

$$(1 - C \cdot |t|^{\frac{1}{4}}) \parallel u \parallel_0 \leq \parallel D_{f_0} T_t(u) \parallel_0 \leq (1 + C \cdot |t|^{\frac{1}{4}}) \parallel u \parallel_0 . \qquad (4.36)$$

Denote by $L_{0t}^{v,sol}$ the linearization of the Cauchy-Riemann equation at $f_{0t}^{sol}$.

**Lemma 4.1.11** *Let $KerL_{0t}^{v,sol}$ be the set of all solutions for $L_{0t}^{v,sol}u = 0$. Then $KerL_{0t}^{v,sol}$ converges uniformly to $KerL_0^v$ as $t$ goes to zero. In particular, the dimension of $KerL_{0t}^{v,sol}$ stays as a constant for $t$ small enough.*

**Proof:** The proof of the first part is similar to the proof of Lemma 6.9 in [RT]. Note that $\mathrm{Ker}L_{0t}^{v,sol} \subset T_{\tilde{z}}W^{\alpha} \times \Omega(f_{0t}^{sol*}T(\tilde{z} \times V))$, one can show that $\dim \mathrm{Ker}L_{0t}^{v,sol}$ is a upper semi-continuous at $t = 0$ as in Proposition 4.1.1. Then (4.36) implies it is actually continuous. □

**Corollary 4.1.1** *The derivative $D_{f_0}T_t$ is an isomorphism between $KerL_0^v$ and $KerL_{0t}^{v,sol}$ satisfying (4.36).*

**Remark 4.1.1** *Note that in non-fiberwise case $T_t$ is an orientation preserving map, and the $T_{\tilde{z}}W^{\alpha}$ direction in $KerL_{0t}^{v,sol}$ has fixed orientation. So in fiberwise case $T_t$ is again an orientation preserving map.*

It remains to show the second part of the Theorem 4.1.1. Let $f'$ be given as in Theorem 4.1.1. For any $f$ in $\tilde{W}$, then there is a unique vector field $u_{f_t^{sol}} = (u_{f_t^{sol}1}, u_{f_t^{sol}2})$ such that $f'(x) = exp^*_{f_t^{sol}(x)}(u_{f_t^{sol}}(f_t^{sol}(x)))$ and $\parallel u_{f_t^{sol}} \parallel_0 \leq \epsilon'$, where $\epsilon'$ is small and depends only on $\tilde{W}$ and $\epsilon$ in Theorem 4.1.1. We want to show that $f'$ coincides with one of $f_t^{sol}$.

**Lemma 4.1.12** *Let $\rho$ be the function in Lemma 4.1.5, $F$ be either $f_t^{sol}$ or $f'$. Then there is a uniform constant $\lambda < 1$ such that*

$$\int_{-\frac{1}{2}\cdot log|t| \geq -log\rho \geq R} |dF|^2_{\mu_c, h_E} d\mu_c \leq 4\lambda^R. \tag{4.37}$$

*Consequently, for some uniform $\beta_0 > 0$,*

$$|dF|^2_{\mu_c, h_E}(x) \leq C \cdot \rho(x)^{8\beta_0}. \tag{4.38}$$

**Proof:** This is exactly Lemma 6.10 in [RT]. □

**Lemma 4.1.13** *If $\epsilon$, $|t|$ are sufficiently small, there is a $f$ in $\tilde{W}$ such that*

$$\| u_f \|_{1,0} \leq C \cdot |t|^{\beta_0} \tag{4.39}$$

*where $\beta_0$ is given in (4.38).*

**Proof:** By cutting $\Sigma_t$ along the loops in $\{x \in \Sigma_t \mid \rho(x) = \sqrt{2|t|}\}$ and gluing disks to the boundary components of the resulting surface, we obtain $m$ surfaces $\Sigma_{it}$, $i = 1, \cdots, m$. Let $\tilde{\Sigma}_t$ be the disjoint union of those surfaces. We can naturally embed $\Sigma_t \setminus \{x \in \Sigma_t \mid \rho(x) = \sqrt{2|t|}\}$ into $\tilde{\Sigma}_t$ as a submanifold. Then we can extend the conformal structure $j_t$ on $\Sigma_t$ to be a natural conformal structure $\tilde{j}_t$ on $\tilde{\Sigma}_t$.

It follows from Lemma 4.1.12 that when $2 \cdot \sqrt{|t|} \leq \rho(x) \leq 10 \cdot \sqrt{|t|}$,

$$|df'|_{\mu_c, h_E}(x) \leq C \cdot |t|^{2\beta_0}. \tag{4.40}$$

Therefore we can extend $f'$ to be a map $\tilde{f}$ from $\tilde{\Sigma}_t$ into $z \times V$ satisfying

$$\| \tilde{v} \|_{0,\frac{1}{2}} \leq C \cdot |t|^{2\beta_0},$$

where $\tilde{v} = d\tilde{f} + J_z \circ d\tilde{f} \circ \tilde{j}_t$. We denote by $\tilde{L}^v$ the linearization of Cauchy-Riemann equation at $\tilde{f}$, by $\tilde{L}^{v*}$ its adjoint. Then by the same arguement as in the proof of Lemma 4.1.7, one can show that the first eigenvalue of $\tilde{L}^v \circ \tilde{L}^{v*}$ is no less than $C \cdot (-log|t|)^{-2}$, where $C$ is independent of $f'$ and $t$. By applying the implicit function theorem in a similar manner to the proof of Proposition 4.1.2, we can find $\xi \in T_z W^\alpha \times \Omega^{0,1}(\tilde{f}^* T(z \times V)$ such that $\tilde{f}^{sol} = exp^*_{\tilde{f}}(\tilde{L}^{v*}\xi)$ is $J$-holomorphic. Moreover, if $|t|$ is sufficiently small, we have

$$\| \xi \|_{2,\frac{1}{2}} \leq C \cdot |t|^{\frac{3}{2}\beta_0}.$$

So the distance between $Im(f')$ and $Im(\tilde{f}^{sol})$ is less than $C \cdot |t|^{\frac{3}{2}\beta_0}$. The map $\tilde{f}^{sol}$ may not be in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{Z^\alpha}, J, \nu_0)$. However, using (4.40) and the fact that $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{Z^\alpha}, J, \nu_0)$ is smooth at $f_0$, one can show that $\tilde{f}^{sol}$ lies in a $C \cdot |t|^{\beta_0}$ neighborhood of some $f_t$ in $\tilde{W}$, as long as both $|t|$ and $\epsilon$ are sufficiently small. The lemma is proved. □

Without loss of generality, we assume that $f = f_0$ in Lemma 4.1.13. Let $\xi$ be the unique solution of $L^{v,sol}_{0t} \circ L^{v,sol*}_{0t} \xi = L^{v,sol}_{0t} u_{f_t^{sol}}$. Multiplying this equation by $\xi$ and integrating by parts, we obtain

$$\int_{\Sigma_t} |L^{v,sol*}_{0t}\xi|^2 d\mu_c \leq \sqrt{\int_{\Sigma_t} |L^{v,sol}_{0t} u_{f_t^{sol}}|^2 d\mu_c \cdot \int_{\Sigma_t} |\xi|^2 d\mu_c}. \qquad (4.41)$$

By the same arguement as in the proof of Lemma 4.1.9, one can show

$$\int_{\Sigma_t} |\xi|^2 d\mu_c \leq C \cdot (-log|t|)^{\frac{5}{2}} \cdot \int_{\Sigma_t} |L_{0t}^{v,sol*}\xi|^2 d\mu_c. \tag{4.42}$$

Combine with (4.39) and (4.41), we get for $t$ sufficiently small,

$$\int_{\Sigma_t} |\xi|^2 d\mu_c \leq C \cdot |t|^{\frac{\beta_0}{2}}.$$

Since $L_{0t}^{v,sol} \circ L_{0t}^{v,sol*}\xi = L_{0t}^{v,sol} u_{f_t^{sol}} = O(\| u_{f_t^{sol}} \|_{1,0})$, by the standard elliptic estimate we have

$$\| \xi \|_{1,\frac{1}{2}} \leq C \cdot |t|^{\frac{\beta_0}{2}}.$$

Consequently

$$\| u_{f_t^{sol}} - L_{0t}^{v,sol*}\xi \|_{1,\frac{1}{2}} \leq C \cdot |t|^{\frac{\beta_0}{2}}. \tag{4.43}$$

Now we want to find a new $f_1$ in $\tilde{W}$ which is very close to $f = f_0$, such that $u_{f_{1t}^{sol}} = L_{1t}^{v,sol*}\xi_1$ for some $\xi_1 \in \Omega^{0,1}(\Sigma_t, f_{1t}^{sol}(W \times V))$, where $L_{1t}^{v,sol}$ is linearization of $\bar{\partial}$ operator at $f_{1t}^{sol}$ . Using equation $f' = exp^*_{f_{1t}^{sol}}(u_{f_{1t}^{sol}}) = exp^*_{f_t^{sol}}(u_{f_t^{sol}})$, we can define a map $S_t$ from a neighborhood of $f_{0t}^{sol}$ into $\mathrm{Ker}L_{0t}^{sol}$ at $f_{0t}^{sol}$,

$$S_t(f_1) = \pi(f_0, f_1)(u_{f_{1t}^{sol}} - L_{1t}^{v,sol*}\xi_1)$$

where $L_{1t}^{v,sol} u_{f_1^{sol}} = L_{1t}^{v,sol} \circ L_{1t}^{v,sol*}\xi_1$, $\pi(f_0, f_1)$ denotes the isomorphism from $\mathrm{Ker}L_{1t}^{v,sol}$ at $f_{1t}^{sol}$ to $\mathrm{Ker}L_{0t}^{sol}$ at $f_{0t}$ which depends smoothly on $f_1$. Clearly $S_t$ is a smooth map. By the same arguement as in the proof of (4.36), one can show

$$(1 - C \cdot |t|^{\frac{\beta_0}{4}}) \parallel u \parallel_0 \leq \parallel D_f S_t(u) \parallel_0 \leq (1 + C \cdot |t|^{\frac{\beta_0}{4}}) \parallel u \parallel_0 . \qquad (4.44)$$

Then by the implicit function theorem, there is a $f_1$ such that $S_t(f_1) = 0$, and $\parallel u_{f_{1t}^{sol}} \parallel_{1,\frac{1}{2}} \leq C \cdot |t|^{\frac{\beta_0}{8}}$. For simlicity, we may assume that $f_1$ coincides with $f_0$, then $u_{f_{0t}^{sol}} = L_{0t}^{v,sol*}\xi$. Since both $f'$ and $f_0$ are $J$-holomorphic, it follows from Lemma 4.1.8 that

$$\Box_{0t}^{v}\xi = -H_t(\xi).$$

Multiplying this equation by $\xi$ and integrating by parts, one can deduce

$$\int_{\Sigma_t} |L_{0t}^{v,sol*}\xi|^2 d\mu_c \leq C \cdot \parallel L_{0t}^{v,sol*}\xi \parallel_1 \cdot \int_{\Sigma_t} |\xi|^2 d\mu_c. \qquad (4.45)$$

However $\parallel L_{0t}^{v,sol*}\xi \parallel_1 \leq C \cdot |t|^{\frac{\beta_0}{8}}$, so for $t$ small enough, (4.45) is impossible unless $u_{f_t^{sol}} = L_t^{v,sol*}\xi = 0$, i.e., $f' = f_{0t}$. The theorem is proved. □

**Remark 4.1.2** Essentially Theorem 4.1.1 is a corollary of corresponding gluing theorem in non-fiberwise case. This can be seen as follows: view $W^\alpha \times V$ as a symplectic manifold, now fiberwise moduli space $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{W^\alpha}, J, 0)$ and $\mathcal{M}^v_A(\Sigma_t, \mathcal{X}_{W^\alpha}, J, 0)$ is a subspace of $\mathcal{M}_{(A_1,\cdots,A_m)}(\Sigma_0, W^\alpha \times V, J, 0)$ and $\mathcal{M}_A$ $(\Sigma_t, W^\alpha \times V, J, 0)$. The key difference is that in the non-fiberwise case, the deformation space of $J$-holomorphic map $f$ is parametrized by subspace of $\Omega(f^*T(W^\alpha \times V))$ whose $TW^\alpha$ component may not be a constant. Fortunately, by modifying exponential map and applying the requirement that

$TW^\alpha$-component is a constant to non-fiberwise deformation theory, each deformed map is a fiberwise $J$-holomorphic map, which gives exactly fiberwise gluing theorem.

## 4.2 Composition Law of Fiberwise Gromov-Witten Mixed Invariants

Let $(L_i, U_i)_{i=1}^k$ and $(M_j, W_j)_{j=1}^l$ be pseudo-manifolds in $\mathcal{X}_Z$ satisfying the general position condition. Assume

$$\begin{aligned} & \sum_{i=1}^{k}(2n + \dim Z - \dim U_i) + \sum_{j=1}^{l}(2n + \dim Z - \dim W_j - 2) \\ = \; & 2C_1(V) \cdot A + 2n(1-g) + \dim Z, \end{aligned} \tag{4.46}$$

where $g$ is the genus of a smooth Riemann surface $\Sigma_t$.

Adopting the notations at the beginning of this chapter, define the evaluation map for $t \neq 0$

$$\begin{aligned} & e_t : \mathcal{M}_A^v(\Sigma_t, \mathcal{X}_Z, J, \nu_t) \times (\Sigma_t)^l \to \mathcal{X}_Z^{k+l} \\ & e_t(f; y_1, \cdots, y_l) \; = \; (f(\sigma_1(t)), \cdots, f(\sigma_k(t)); f(y_1), \cdots, f(y_l)). \end{aligned} \tag{4.47}$$

Recall that for a generic $(J, \nu)$ and for generic nonzero $t$, the image $Im(e_t)$ intersects the product $\prod_{i=1}^k U_i \times \prod_{j=1}^l W_j$ transversally. Considering an intersection point $(f_s; y_{s1}, \cdots, y_{sl})$, we can assign a sign to $(f_s; y_{s1}, \cdots, y_{sl})$ by using the orientation of $\mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^0}, J, \nu_t)$, $(\mathcal{X}_{Z^0})^{k+l}$ and $\prod_{i=1}^k U_i^0 \times \prod_{j=1}^l W_j^0$, where

$Z^0$, $U_i^0$, and $W_j^0$ mean the top stratums. For $t = 0$, after fix a position pattern $P_1$ of $(\sigma_1(0), \cdots, \sigma_k(0))$ in $\Sigma_0$ and an intersection pattern $T$, again for generic $(J, \nu_0)$, the map $e_0$ intersects $\prod_{i=1}^k U_i^0 \times \prod_{j=1}^l W_j^0$ transversally at finite many points in $\mathcal{X}_{Z^0}^{k+l}$. Let $(f_0; y_{01}, \cdots, y_{0l})$ be one of such intersection points. Theorem 4.1.1 gives

**Corollary 4.2.1** *Let $(J, \nu)$ be generic and $(f_0; y_{01}, \cdots, y_{0l})$ be as the above which intersects transversally. Then there are $\epsilon$, $\delta > 0$ such that for $t$ sufficiently small, there is a unique $(f_t; y_{t1}, \cdots, y_{tl})$ in the space $\mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^0}, J, \nu_t) \times (\Sigma_t)^{k+l}$ satisfying: (1) $d_S(y_{tj}, y_{0j}) \leq \epsilon$, and $d_{\mathcal{X}_Z}(f_0(x), f_t(y)) \leq \epsilon$ whenever $d_S(x, y) \leq \delta$. Moreover the sign associated to $(f_t; y_{t1}, \cdots, y_{tl})$ is the same as the sign associated to $(f_0; y_{01}, \cdots, y_{0l})$ for those generic $t$ such that $\mathcal{M}_A^v(\Sigma_t, \mathcal{X}_{Z^0}, J, \nu_t)$ is a smooth manifold.*

In Chapter 3 we have defined the fiberwise mixed invariants $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma_0,P)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma)$, denote $\Phi^v_{(A,\omega,\Sigma_t,P)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma)$ by $\Phi^v_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma)$, where $P$ stands for the trivial position pattern and the trivial intersection pattern since $\Sigma_t$ is smooth.

**Theorem 4.2.1 ( Composition law )** *Suppose that fiber bundle $\mathcal{X} \to Y$ satisfies Assumption ($\star$). Then for any rational cohomology classes $\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_l \in H^*(\mathcal{X}, \mathbb{Q})$ and rational homology class $\gamma \in H_*(Y, \mathbb{Q})$,*

$$\begin{aligned} & \Phi^v_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma) \\ = & \sum_{A=A_1+\cdots A_m} \sum_{T} \Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma_0,(P_1,T))}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma), \end{aligned} \quad (4.48)$$

*for any fixed position pattern* $P_1$.

**Proof:** Fix a degeneration $\pi : S \to \Delta$ of a $k$-point genus $g$ smooth stable curves, such that the central fiber is a stable curve $C$ and other fibers are smooth genus $g$ Riemann surfaces with $k$-marked points, where $\Delta \subset \mathbb{C}$ is a unit disc. Suppose that for $t \neq 0$, $(f_t; y_{t1}, \cdots, y_{tl})$ is an intersection point, then $f_t(\sigma_i(t)) \in Im(L_i)$, and $f_t(\Sigma_t) \cap Im(M_j) \neq \emptyset$. By Proposition 2.1.1., and taking a subsequence if necessary, we may assume that $f_t$ converges to $f_0$ in $\bar{\mathcal{M}}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_Z, J, \nu_0)$ for some partition $A_1, \cdots, A_m$ of $A$ and some intersection pattern $T$ as $t$ goes to zero. By Proposition 4.1.1, if $(J, \nu_0)$ is generic, $f_0$ is actually in $\mathcal{M}^v_{(A_1,\cdots,A_m)}(\Sigma_0, \mathcal{X}_{Z^0}, J, \nu_0)$, and by Proposition 2.2.4, we may assume that $f_0$ intersects transversally at every double points. Let $y_j^0$ be the limit of $y_{tj}$, clearly $f_0(\sigma_i(0)) \in Im(L_i)$, $f_0(y_j^0) \in Im(M_j)$, therefore $(f_0; y_1^0, \cdots, y_l^0)$ is an intersection point. Theorem follows from Corollary 4.2.1. □

Next we want to show that $\Phi^v_{(A_1,\cdots,A_m,\omega,\Sigma_0,P)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma)$ could be explicitly calculated in terms of the fiberwise mixed invariant of each component of $\Sigma_0$ and the contribution from the double points. There are two kinds of double points on a stable curve: intersection of two different components or self-intersection of one component. In each case we will give a formula.

**Case 1.** Suppose that $\Sigma_0$ has two components $\Sigma_{(01)}$ and $\Sigma_{(02)}$ of genus $g_1$ and $g_2$ respectively, satisfying: (1) $\Sigma_{(01)}$ and $\Sigma_{(02)}$ intersect at a double point $p$, (2) $\Sigma_{(01)}$ contains $m$ marked points $x_1, \cdots, x_m$ and $\Sigma_{(02)}$ contains the rest

of marked points. This is the position pattern $P_1$. Let $C = (\Sigma_0; x_1, \cdots, x_k)$. Then $\Phi^v_{(A_1,A_2,\omega,C,P)}$ can be calculated as follows.

Assume that $Z$ is a pseudo-manifold representative of $\gamma$. Let $[H^\alpha_Z]$ be a basis of $H^*(\mathcal{X}_Z, \mathbb{Q})$. Consider the diagonal $\Delta_Z$ in $\mathcal{X}_Z \times \mathcal{X}_Z$. By the Kunneth formula, we could write the Poincare dual of $\Delta$ as $PD(\Delta_Z) = \sum_{\alpha,\beta} \eta_{(Z,\alpha,\beta)} \cdot H^\alpha_Z \otimes H^\beta_Z$, where $\eta_{(Z,\alpha,\beta)}$ is the intersection matrix of $H^*(\mathcal{X}_Z, \mathbb{Q})$.

**Theorem 4.2.2** *Let the intersection pattern $T$ be that the image of $\Sigma_{(01)}$ intersects $Im(M_j)$, $j = 1, \cdots, l'$ and that the image of $\Sigma_{(02)}$ intersects $Im(M_j)$, $j = l'+1, \cdots, l$. Let $A_1$ and $A_2$ be two homology classes in $H_2(V, \mathbb{Z})$. Then*

$$\begin{aligned}
&\Phi^v_{(A_1,A_2,\omega,C,P)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma) \\
= &\sum_{\tau,\delta} \eta_{(Z,\tau,\delta)} \cdot \Phi^v_{(A_1,\omega,g_1)}(\alpha_1, \cdots, \alpha_m, [H^\tau_Z] \mid \beta_1, \cdots, \beta_{l'})(\gamma) \\
&\cdot \Phi^v_{(A_2,\omega,g_2)}(\alpha_{m+1}, \cdots, \alpha_k, [H^\delta_Z] \mid \beta_{l'+1}, \cdots, \beta_l)(\gamma), \qquad (4.49)
\end{aligned}$$

$$\begin{aligned}
&\Phi^v_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma) \\
= &\sum_{B_1+B_2=A} \sum_{j=1}^{l} \sum_{\sigma} \sum_{\tau,\delta} \frac{1}{j! \cdot (l-j)!} \eta_{(Z,\tau,\delta)} \\
&\cdot \Phi^v_{(A_1,\omega,g_1)}(\alpha_1, \cdots, \alpha_m, [H^\tau_Z] \mid \beta_{\sigma(1)}, \cdots, \beta_{\sigma(j)})(\gamma) \\
&\cdot \Phi^v_{(A_2,\omega,g_2)}(\alpha_{m+1}, \cdots, \alpha_k, [H^\delta_Z] \mid \beta_{\sigma(j+1)}, \cdots, \beta_{\sigma(l)})(\gamma), \qquad (4.50)
\end{aligned}$$

*where $B_1$ and $B_2$ are any two homology classes in $H_2(V, \mathbb{Z})$ and $\sigma$ are permutations.*

**Proof:** The proof follows from applying the argument of Theorem 7.3 in [RT] to moduli space $\mathcal{M}^v_{(A_1,A_2)}(\Sigma_0, \mathcal{X}_Z, J, \nu_0)$. □

**Case 2.** Suppose that $\Sigma_0$ is a genus $g-1$ curve with one self-intersection point. Let $C = (\Sigma_0; x_1, \cdots, x_k)$.

**Theorem 4.2.3** *We have*

$$\begin{aligned} & \Phi^v_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k \mid \beta_1, \cdots, \beta_l)(\gamma) \\ = & \sum_{\tau,\delta} \eta_{(Z,\tau,\delta)} \cdot \Phi^v_{(A,\omega,g-1)}(\alpha_1, \cdots, \alpha_k, [H^\tau_Z], [H^\delta_Z] \mid \beta_1, \cdots, \beta_l)(\gamma). \quad (4.51) \end{aligned}$$

**Proof:** The proof follows from applying the argument of Theorem 7.5 in [RT] to moduli space $\mathcal{M}^v_A(\Sigma_0, \mathcal{X}_Z, J, \nu_0)$. □

# Chapter 5

# Fiberwise Quantum Cohomology and Equivariant Quantum Cohomology

Suppose that $\mathcal{X} \xrightarrow{p} Y$ is a fiber bundle with fiber $V$ satisfying Assumption $(\star)$. Let $A_1, \cdots, A_m$ be an integral basis of $H_2(V, \mathbb{Z})$. Any homology class $A \in H_2(V, \mathbb{Z})$ can then be written as $A = d_1 \cdot A_1 + \cdots + d_m \cdot A_m$. For simplicity we assume that for any genus zero $J$-holomorphic map $f$ of $V$, $f_*(\mathbb{C}P_1) = d_1 \cdot A_1 + \cdots + d_m \cdot A_m$ with $d_i \geq 0$, $i = 1, \cdots, m$.

Let

$$\begin{aligned} & \tilde{\Phi}^v_\omega(\alpha_1, \cdots, \alpha_k) \\ = & \sum_{A \in H_2(V,\mathbb{Z})} \Phi^v_{(A,\omega,0)}(\alpha_1, \cdots, \alpha_k \mid \ )([Y]) q_1^{d_1} \cdots q_m^{d_m} \in \mathbb{C}[[q_1, \cdots, q_m]], \end{aligned} \tag{5.1}$$

where $\alpha_i \in H^*(\mathcal{X}, \mathbb{C})$, $i = 1, \cdots, k$, $q_1, \cdots, q_m$ are parameters, and $\mathbb{C}[[q_1, \cdots, q_m]]$ is the set of formal power series of $q_i$.

We define fiberwise quantum multiplication

$$\begin{aligned} * : \quad & H^*(\mathcal{X}, \mathbb{C})[[q_1, \cdots, q_m]] \times H^*(\mathcal{X}, \mathbb{C})[[q_1, \cdots, q_m]] \\ & \to H^*(\mathcal{X}, \mathbb{C})[[q_1, \cdots, q_m]] \end{aligned} \tag{5.2}$$

by

$$\alpha * \beta(B) = \tilde{\Phi}^v_\omega(\alpha, \beta, PD(B)), \tag{5.3}$$

where $\alpha, \beta \in H^*(\mathcal{X}, \mathbb{C})$, $B \in H_*(\mathcal{X}, \mathbb{C})$, and $PD(B)$ is the Poincare dual of $B$. We extend $*$ linearly to all of $H^*(\mathcal{X}, \mathbb{C})[[q_1, \cdots, q_m]]$.

We call $H^*(\mathcal{X}, \mathbb{C})[[q_1, \cdots, q_m]]$ with fiberwise quantum multiplication $*$ the fiberwise quantum cohomology associated to the fiber bundle $\mathcal{X} \xrightarrow{p} Y$, denoted by $QH^*(\mathcal{X}, Y, \mathbb{C})$. Note that when $Y$ is a point, the fiberwise quantum cohomology becomes quantum cohomology of $V$.

**Theorem 5.1** *For any fiber bundle $\mathcal{X} \xrightarrow{p} Y$ with fiber $V_1$ and fiberwise symplectic form $\omega$ satisfying Assumption ($\star$), there is a well defined fiberwise quantum cohomology. It has the following properties.*

*i) It is a $H^*(Y, \mathbb{C})[[q_1, \cdots, q_m]]$ module. This module structure is the same as that of $H^*(\mathcal{X}, \mathbb{C})$ as a $H^*(Y, \mathbb{C})$-module.*

*ii) It is graded commutative:*

$$\alpha * \beta = (-1)^{\deg \alpha \cdot \deg \beta} \cdot \beta * \alpha. \tag{5.4}$$

*iii)* **(Direct Product)** *For any two fiber bundles $\mathcal{X}_1 \to Y_1$ and $\mathcal{X}_2 \to Y_2$ satisfying Assumption ($\star$). There is an isomorphism*

$$QH^*(\mathcal{X}_1 \times \mathcal{X}_2, Y_1 \times Y_2, \mathbb{C}) \cong QH^*(\mathcal{X}_1, Y_1, \mathbb{C}) \times QH^*(\mathcal{X}_2, Y_2, \mathbb{C}) \qquad (5.5)$$

*as a* $H^*(Y_1 \times Y_2, \mathbb{C})[[q_1, \cdots, q_{m_1+m_2}]]$*-module. Here* $q_1, \cdots, q_{m_1}$ *are parameters for* $QH^*(\mathcal{X}_1, Y_1, \mathbb{C})$ *and* $q_{m_1+1}, \cdots, q_{m_1+m_2}$ *are the parameters for* $QH^*(\mathcal{X}_2, Y_2, \mathbb{C})$

*iv)* **(Restriction)** *Let* $Y_1$ *be a closed finite dimensional manifold and* $h : Y_1 \to Y$ *be a smooth map. Set* $\mathcal{X}_{Y_1} = \mathcal{X} \times_h Y_1$*, the induced fiber bundle with fiber* $V$ *and induced fiberwise symplectic form* $\omega$*. Then there are a natural map* $H : \mathcal{X}_{Y_1} \to \mathcal{X}$ *and a* $H^*(Y, \mathbb{C})[[q_1, \cdots, q_m]]$*-module homomorphism*

$$H^* : QH^*(\mathcal{X}, Y, \mathbb{C}) \to QH^*(\mathcal{X}_{Y_1}, Y_1, \mathbb{C}). \qquad (5.6)$$

*Here we assume that* $QH^*(\mathcal{X}_{Y_1}, Y_1, \mathbb{C})$ *has the induced* $H^*(Y, \mathbb{C})[[q_1, \cdots, q_m]]$*-module structure from* $h^* : H^*(Y, \mathbb{C}) \to H^*(Y_1, \mathbb{C})$*.*

*v)* **(Induction)** *Suppose that fiber bundle* $Y \to B$ *has fiber* $V_2$ *with fiberwise symplectic form* $\omega_2$ *satisfying Assumption* $(\star)$*. Assume that* $\omega$ *is also a fiberwise symplectic form satisfying Assumption* $(\star)$ *for the induced fiber bundle* $\mathcal{X} \to B$ *with fiber* $V$*. Assume further that* $H_2(V, \mathbb{Z})$ *has an integral basis* $B_1, \cdots, B_{m_1+m_2}$ *such that* $B_{m_1+1}, \cdots, B_{m_1+m_2}$ *are the integral basis of* $H_2(V_2, \mathbb{Z})$*. Then there is an isomorphism*

$$QH^*(\mathcal{X}, B, \mathbb{C})/\{(q_{m_1+1}, \cdots, q_{m_1+m_2}) \cdot H^*(\mathcal{X}, B, \mathbb{C})\} \cong QH^*(\mathcal{X}, Y, \mathbb{C}) \quad (5.7)$$

*as a* $H^*(B, \mathbb{C})[[q_1, \cdots, q_{m_1}]]$*-module.*

**Proof:** The proof follows directly from the corresponding properties of the fiberwise mixed invariants. □

**Theorem 5.2** *For any fiber bundle $\mathcal{X} \xrightarrow{p} Y$ satisfying Assumption ($\star$), fiberwise quantum cohomology $QH^*(\mathcal{X}, Y, \mathbb{C})$ is associative.*

**Proof:** Suppose $\alpha_1, \alpha_2, \alpha_3 \in H^*(\mathcal{X}, \mathbb{C})$, $H^a$ is a basis of $H^*(\mathcal{X}, \mathbb{C})$ and $(\eta_{a,b})$ is the intersection matrix, then

$$\begin{aligned}
&(\alpha_1 * \alpha_2) * \alpha_3 \\
= \;& (\eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_1, \alpha_2, H^{a_1}) \cdot H^{b_1}) * \alpha_3 \\
= \;& \eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_1, \alpha_2, H^{a_1}) \cdot \tilde{\Phi}^v_\omega(H^{b_1}, \alpha_3, H^{a_2}) \cdot \eta_{a_2,b_2} \cdot H^{b_2}.
\end{aligned}$$

$$\begin{aligned}
&\alpha_1 * (\alpha_2 * \alpha_3) \\
= \;& \alpha_1 * (\eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_2, \alpha_3, H^{a_1}) \cdot H^{b_1}) \\
= \;& \eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_1, H^{b_1}, H^{a_2}) \cdot \tilde{\Phi}^v_\omega(\alpha_2, \alpha_3, H^{a_1}) \cdot \eta_{a_2,b_2} \cdot H^{b_2}.
\end{aligned}$$

It follows from Theorem 4.2.2 that

$$\begin{aligned}
&\eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_1, \alpha_2, H^{a_1}) \cdot \tilde{\Phi}^v_\omega(H^{b_1}, \alpha_3, H^{a_2}) \\
= \;& \tilde{\Phi}^v_\omega(\alpha_1, \alpha_2, \alpha_3, H^{a_2}) \\
= \;& \eta_{a_1,b_1} \cdot \tilde{\Phi}^v_\omega(\alpha_1, H^{b_1}, H^{a_2}) \cdot \tilde{\Phi}^v_\omega(\alpha_2, \alpha_3, H^{a_1}).
\end{aligned}$$

So

$$(\alpha_1 * \alpha_2) * \alpha_3 = \alpha_1 * (\alpha_2 * \alpha_3).$$

The theorem is proved. □

Suppose that Lie group $G$ acts on a semi-positive symplectic manifold $(V,\omega)$, keeping $\omega$ invariant. Denote by $EG \to BG$ the universal $G$ bundle. Let $Y = BG$ and $\mathcal{X} = V \times_G EG$. However we can not apply the fiberwise quantum multiplication directly to this case because $Y$ is infinite dimensional. Note that by the definition equivariant cohomology $H^*_G(V,\mathbb{C}) = H^*(\mathcal{X},\mathbb{C})$. To define the equivariant quantum multiplication $\alpha * \beta$ for $\alpha$ and $\beta \in H^*_G(V,\mathbb{C})$, we use finite dimensional approximation of $Y$. We choose a $N$ dimensional approximation $BG_N$ of $BG$ such that $H^i(BG_N,\mathbb{C})$ are isomorphic to $H^i(BG,\mathbb{C})$ for $i \leq deg\alpha + \deg\beta + 1$. Then $\mathcal{X}_N = V \times_G BG_N$ has same cohomology as $\mathcal{X}$ up to degree at least $\deg\alpha + \deg\beta + 1$. We define $\alpha * \beta$ to be $(\alpha * \beta)_N$, defined using $\mathcal{X}_N \to BG_N$.

We need to show this multiplication is well-defined. Suppose $N'$ is another such integer. We need to show $(\alpha * \beta)_N(B) = (\alpha * \beta)_{N'}(B)$, for any $B \in H_*(\mathcal{X},\mathbb{C})$. Since

$$(\alpha * \beta)_N(B) = \sum_{A \in H_2(V,\mathbb{Z})} \Phi^{\mathcal{X}_N,v}_{(A,\omega,0)}(\alpha,\beta,PD(B))([BG_N])q_1^{d_1}\cdots q_m^{d_m},$$
$$(\alpha * \beta)_{N'}(B) = \sum_{A \in H_2(V,\mathbb{Z})} \Phi^{\mathcal{X}_{N'},v}_{(A,\omega,0)}(\alpha,\beta,PD(B))([BG_{N'}])q_1^{d_1}\cdots q_m^{d_m},$$

where $\Phi^{\mathcal{X}_N,v}_{(A,\omega,0)}$ and $\Phi^{\mathcal{X}_{N'},v}_{(A,\omega,0)}$ are the fiberwise mixed invariants defined by $\mathcal{X}_N$ and $\mathcal{X}_{N'}$ respectively. So all we need to show is

$$\Phi^{\mathcal{X}_N,v}_{(A,\omega,0)}(\alpha,\beta,PD(B))([BG_N]) = \Phi^{\mathcal{X}_{N'},v}_{(A,\omega,0)}(\alpha,\beta,PD(B))([BGN'])$$

Without loss of generality, assume that $N < N'$. Choosing $BG_N \subset BG_{N'} \subset BG$ to be submanifold, then $\mathcal{X}_N \subset \mathcal{X}_{N'}$ is also a submanifold. From the choice of $N$, we can choose pseudo-manifold representatives $(L_1, U_1), (L_2, U_2)$ of $\alpha, \beta$ in $\mathcal{X}_{N'}$ such that their images, intersecting with $\mathcal{X}_N$, give the pseudo-manifold representatives of $\alpha$ and $\beta$ in $\mathcal{X}_N$. $(L_3, U_3)$ is the pseudo-manifold representative of $B$ in $\mathcal{X}_N$. Under the inclusion $\mathcal{X}_N \subset \mathcal{X}_{N'}$, we view $(L_3, U_3)$ as a representative of $B$ in $\mathcal{X}_{N'}$. From Remark 2.2.2 i), we know that there is a generic $(J, \nu)$ which is good for $\mathcal{X}_{N'}$ and its restriction to $\mathcal{X}_N$ is also good. $\Phi^{\mathcal{X}_{N'},\nu}_{(A,\omega,0)}(\alpha, \beta, \delta)(BG_{N'})$ is defined as counting the number of the intersection of $\mathcal{M}^{\nu}_{A}(S^2, \mathcal{X}_{N'}, J, \nu) \times (S^2)^3$ with $(L_1, U_1), (L_2, U_2), (L_3, U_3)$ under evaluation map. The intersections happen only inside $\mathcal{X}_N$ because of the choice of $(L_3, U_3)$. Also note that $\mathcal{M}^{\nu}_{A}(S^2, \mathcal{X}_N, J, \nu)$ is a subspace of $\mathcal{M}^{\nu}_{A}(S^2, \mathcal{X}_{N'}, J, \nu)$. So the intersections are exactly the same as the intersections of $\mathcal{M}^{\nu}_{A}(S^2, \mathcal{X}_N, J, \nu) \times (S^2)^3$ with $(L_1, U_1), (L_2, U_2), (L_3, U_3)$ under evaluation map. The number of the latter intersections is by definition $\Phi^{\mathcal{X}_N,\nu}_{(A,\omega,0)}(\alpha, \beta, PD(B))$ $(BG_N)$. So well-definedness is proved.

When $Y$ is infinite dimensional, for a fiber bundle $\mathcal{X} \to Y$ satisfying Assumption $(\star)$, we can define the fiberwise quantum cohomology $QH^*(\mathcal{X}, Y, \mathbb{C})$ similarly. We will use this definition in Section 5.2.

We call $H^*_G(V, \mathbb{C})[[q_1, \cdots, q_m]]$ with equivariant quantum multiplication equivariant quantum cohomology, denoted by $QH^*_G(V, \mathbb{C})$. It is a $q$-deformation of equivariant cohomology. The following two theorems follow from Theorem 5.1 and 5.2.

**Theorem 5.3** *Suppose that Lie group $G$ acts on a semi-positive symplectic manifold $(V,\omega)$, keeping $\omega$ invariant. Then there is a well-defined equivariant quantum cohomology $QH^*_G(V,\mathbb{C})$. It has the following properties.*

*i) $QH^*_G(V,\mathbb{C})$ is a $H^*(BG,\mathbb{C})[[q_1,\cdots,q_m]]$-module. This module structure is the same as the $H^*(BG;\mathbb{C})$-module structure of $H^*_G(V;\mathbb{C})$.*

*ii) $QH^*_G(V,\mathbb{C})$ is graded commutative:*

$$\alpha * \beta = (-1)^{\deg\alpha+\deg\beta}\beta * \alpha. \tag{5.8}$$

*iii)***(Direct Product)** *Suppose that Lie groups $G_i$ act on the semi-positive symplectic manifolds $(V_i,\omega_i)$, keeping $\omega_i$ invariant, $i = 1, 2$. Then*

$$QH^*_{G_1\times G_2}(V_1\times V_2,\mathbb{C}) \cong QH^*_{G_1}(V_1,\mathbb{C})\otimes QH^*_{G_2}(V_2,\mathbb{C}), \tag{5.9}$$

*where $\omega_1\times\omega_2$ is the symplectic form on $V_1\times V_2$ and $G_1\times G_2$ acts on $V_1\times V_2$ by product action.*

*iv)* **(Restriction)** *Suppose $\tilde{G}\subset G$ is a normal Lie subgroup. Then there is a homomorphism*

$$H^*: QH^*_G(V,\mathbb{C}) \to QH^*_{\tilde{G}}(V,\mathbb{C}) \tag{5.10}$$

*as a $H^*(BG)[[q_1,\cdots,q_m]]$-module. Here we choose $E\tilde{G} = EG$ which induces a map $B\tilde{G}\to BG$ and the $H^*(BG)[[q_1,\cdots,q_m]]$-module structure of $QH^*_{\tilde{G}}(V,\mathbb{C})$ is induced from $h^*: H^*(BG,\mathbb{C})\to H^*(B\tilde{G},\mathbb{C})$.*

*v)* **(Induction)** *Let $\tilde{G}\subset G$ be a normal Lie subgroup with $G/\tilde{G}$ being simply connected. Suppose that $G/\tilde{G}$ is a semi-positive symplectic manifold. Assume*

*that $\tilde{G}$ acts on a semi-positive symplectic manifold $(\tilde{V}, \tilde{\omega})$, keeping $\tilde{\omega}$ invariant. Let $V = \tilde{V} \times_{\tilde{G}} G$. Assume that $\tilde{\omega}$ can be extended to a semi-positive symplectic form on $V$ invariant under $G$ action. Assume further that $H_2(V, \mathbb{Z})$ has basis $B_1, \cdots, B_{m_1+m_2}$ with $B_{m_1+1}, \cdots, B_{m_1+m_2}$ being basis of $G/\tilde{G}$. Then*

$$QH_G^*(V; \mathbb{C})/\{(q_{m_1+1}, \cdots, q_{m_1+m_2})QH_G^*(V, \mathbb{C})\} \cong QH_{\tilde{G}}^*(\tilde{V}, \mathbb{C}). \qquad (5.11)$$

**Theorem 5.4** *Equivariant quantum cohomology is associative.*

# Chapter 6

# Several Examples

## 6.1 Classical cohomology rings

Let $n$ be a natural number and $k_1, \cdots, k_s$ be a partition of $n$, i.e., $k_1 + \cdots + k_s = n$ ($k_1, \cdots, k_s$ are natural numbers). Recall that a partial flag manifold $F_{(k_1,\cdots,k_s)}$ is defined to be the set of all flags in $\mathbb{C}^n$, $0 \subset C_1 \subset \cdots \subset C_s = \mathbb{C}^n$ with $\dim C_i = k_1 + \cdots + k_i$, $i = 1, \cdots, s$. There are s complex vector bundle $L_i$ of rank $k_i$ over $F_{(k_1,\cdots,k_s)}$ whose fiber at above flag is $C_i/C_{i-1}$. Here we use convention $C_0 = 0$. Denote by $\mathbb{C}^n$ the trivial bundle of rank $n$. These bundles satisfy the following relation

$$L_1 \oplus \cdots \oplus L_s = \mathbb{C}^n. \tag{6.1}$$

**Lemma 6.1.1** *The cohomology ring $H^*(F_{(k_1,\cdots,k_s)}, \mathbb{C})$ is isomorphic to*

$$\mathbb{C}[u_1^1, \cdots, u_{k_1}^1; \cdots; u_1^s, \cdots, u_{k_s}^s]/\{\sigma_1, \cdots, \sigma_n\}$$

*where $u_j^i$ are parameters which we assign with a degree $2j$ and $\sigma_j$ is degree $2j$ part of the expansion of $(1+u_1^1+\cdots,u_{k_1}^1)\cdots(1+u_1^s+\cdots+u_{k_s}^s)$. Under the isomorphism $u_j^i$ is mapped to the j-th Chern class $c_j(L_i)$. Furthermore the first Chern class of $F_{(k_1,\cdots,k_s)}$ is $\sum_{j=1}^{s-1}(k_j+k_{j+1})p_j$ where $p_j=\sum_{k=j}^{s-1}u_1^{k+1}$.*

**Proof:** This result is well-known. For convenience, we sketch a proof.

From (6.1), there is a natural map which maps $u_j^i$ to $c_j(L_i)$,

$$\mathbb{C}[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s]/\{\sigma_1,\cdots,\sigma_n\}\xrightarrow{p}H^*(F_{(k_1,\cdots,k_s)},\mathbb{C})$$

First we show that $p$ is surjective by showing that $H^*(F_{(k_1,\cdots,k_s)};\mathbb{C})$ is generated by $c_j(L_i)$. We argue by induction. When $s=2$, $F_{k_1,k_2}$ is a Grassmannian. The statement is true. Suppose that it is true for $s-1$ case. Then consider fiberation

$$F_{(k_1,\cdots,k_s)}\xrightarrow{\pi}G(n-k_s,n)$$

with a fiber $F_{(k_1,\cdots,k_{s-1})}(\mathbb{C}^{n-k_s})$ where $\pi$ maps flag $0\subset C_1\subset\cdots\subset C_s$ to $C_{s-1}$. Note it is easily to see the restriction of $c_j(L_i)$ for $1\le i\le s-1$ to each fiber is the Chern class of those "$L_i$"s of the fiber, by assumption they generate the cohomology of the fiber. The Chern classes of similarly constructed rank $k_s$ bundle $\tilde{L}_2$ on $G(n-k_s,n)$ generate the cohomology of $G(n-k_s,n)$, and they are mapped to $c_j^s$ under $\pi^*$. So the Leray-Hirsch Theorem for fibration applies. $H^*(F_{(k_1,\cdots,k_s)},\mathbb{C})$ is generated by $c_j(L_i)$'s.

Secondly, we show that $p$ is an isomorphism. The Poincare series $P_t(H^*(F_{(k_1,\cdots,k_s)},\mathbb{C}))$ is the product of Poincare series of $H^*(F_{(k_1,\cdots,k_{s-1})},\mathbb{C})$ and $H^*(G(n-k_s,n),\mathbb{C})$, a simple induction gives

$$P_t(H^*(F_{(k_1,\cdots,k_s)},\mathbb{C})) = \frac{\prod_{i=1}^{n}(1-t^{2i})}{\prod_{j=1}^{s}\prod_{i=1}^{k_j}(1-t^{2i})}, \tag{6.2}$$

which is exactly the Poincare series of $\mathbb{C}[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s]/\{\sigma_1,\cdots,\sigma_n\}$ (see [BT], section 23). So $p$ is an isomorphism.

Denote by $W_j$ the vector bundle of rank $\sum_{i=j}^{s-1} k_{i+1}$ over $F_{(k_1,\cdots,k_s)}$ with fiber at flag $0\subset C_1\subset\cdots\subset C_s$ to be $\mathbb{C}^n/C_j$. Note that $W_j\cong\oplus_{i=j}^{s-1}L_{i+1}$ and

$$TF_{(k_1,\cdots,k_s)} = \oplus_{j=1}^{s-1} Hom(L_j,W_j).$$

It follows

$$\begin{aligned}
c_1(F_{(k_1,\cdots,k_s)}) &= \sum_{j=1}^{s-1} c_1(Hom(L_j,W_j)) \\
&= \sum_{j=1}^{s-1} c_1(L_j^*\otimes W_j) \\
&= \sum_{j=1}^{s-1}[-(k_{j+1}+\cdots+k_s)c_1(L_j)+k_jc_1(W_j)] \\
&= \sum_{j=1}^{s-1}[-(k_{j+1}+\cdots+k_s)u_1^j+k_j(u_1^{j+1}+\cdots+u_1^s)] \\
&= k_2p_1+\cdots+k_sp_{s-1}+k_1p_1+\cdots+k_{s-1}p_{s-1} \\
&= \sum_{j=1}^{s-1}(k_j+k_{j+1})p_j.
\end{aligned}$$

Where we use the fact $-(u_1^1 + \cdots + u_1^j) = p_j$ in the second to last step. This fact follows from $u_1^1 + \cdots + u_1^s = 0$. □

Suppose that $E \to X$ is a complex vector bundle of rank $n$. The partial flag manifold $F_{(k_1,\cdots,k_s)}(E)$ over $X$ can be constructed as follows. Let $\{U_\alpha\}$ be a covering of $X$ such that $E$ can be trivialized on each $U_\alpha$, $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$, with transition functions $(\phi_{\alpha\beta}, A_{\alpha\beta})$, where $\phi_{\alpha\beta} : U_\alpha \to U_\beta$ and $A_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$. Then patching together $U_\alpha \times F_{(k_1,\cdots,k_s)}$ using the induced action of $A_{\alpha\beta}$ on $F_{(k_1,\cdots,k_s)}$, we get $F_{(k_1,\cdots,k_s)}(E)$. Note that $F_{(k_1,\cdots,k_s)}(E) \xrightarrow{p} X$ is a fiber bundle with fiber $F_{(k_1,\cdots,k_s)}$. Similarly we can define vector bundle $L_i(E)$ over $F_{(k_1,\cdots,k_s)}(E)$. It is not hard to see

$$L_1(E) \oplus \cdots L_s(E) \cong p^*E. \tag{6.3}$$

**Lemma 6.1.2** *The cohomology ring $H^*(F_{(k_1,\cdots,k_s)}(E), \mathbb{C})$ is isomorphic to*

$$H^*(X; \mathbb{C})[u_1^1, \cdots, u_{k_1}^1; \cdots; u_1^s, \cdots, u_{k_s}^s]/\{\sigma_1 - c_1(p^*E), \cdots, \sigma_n - c_n(p^*E)\}$$

*as a $H^*(X; \mathbb{C})$-module. Here parameter $u_j^i$ is mapped to $c_j(L_i(E))$ and $\sigma_j$ is defined as in Lemma 6.1.1.*

**Proof:** Consider fiberation $F_{(k_1,\cdots,k_s)}(E) \xrightarrow{p} X$ which is a fiber bundle with fiber $F_{(k_1,\cdots,k_s)}$. Note that the restriction of $c_j(L_i(E))$ to each fiber $F_{(k_1,\cdots,k_s)}$ gives $c_j(L_i)$. The lemma follows from the Leray-Hirsch theorem for fiberation and (6.3). □

Note that the standard action of $U(n)$ on $\mathbb{C}^n$ induces an action of $U(n)$ on $F_{(k_1,\cdots,k_s)}$ and a fiberwise action on $F_{(k_1,\cdots,k_s)}(E)$. So we can make the following construction. Suppose $G = U(l_1) \times \cdots \times U(l_t) \subset U(n)$ is a subgroup with the induced action on $F_{(k_1,\cdots,k_s)}$ and $F_{(k_1,\cdots,k_s)}(E)$. Then

$$F_{(k_1,\cdots,k_s)}(E) \times_G EG \xrightarrow{p} X \times BG$$

is a fiber bundle with a fiber $F_{(k_1,\cdots,k_s)}$. Here $EG \to BG$ is the universal principal $G$ bundle. Construct the $L_i(E,G)$ bundle over $F_{(k_1,\cdots,k_s)}(E) \times_G EG$ as before. There is a bundle $\tilde{E} = E \times_{U(n)} EG$ of rank $n$ over $X \times BG$ whose restriction to each $X$ slice is $E$ and whose restriction to each $BG$ slice is $EG \times_G \mathbb{C}^n$. Denote by $c_i(G)$ the i-th Chern class of universal $G$ bundle $EG \times_G \mathbb{C}^n$. Then $c_i(\tilde{E})$ can be expressed in terms of $c_j(E)$ and $c_k(G)$, $i,j,k = 1,\cdots,n$, and

$$L_1(E,G) \oplus \cdots \oplus L_s(E,G) \cong p^*\tilde{E}. \tag{6.4}$$

**Lemma 6.1.3** *The cohomology ring $H^*(F_{(k_1,\cdots,k_s)}(E) \times_G EG, \mathbb{C})$ is isomorphic to*

$$H^*(X;\mathbb{C}) \times H^*(BG;\mathbb{C})[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s]$$
$$/\{\sigma_1 - c_1(p^*\tilde{E}),\cdots,\sigma_n - c_n(p^*\tilde{E})\}$$

*as a $H^*(X,\mathbb{C}) \times H^*(BG,\mathbb{C})$-module. Here parameter $u_j^i$ is mapped to $c_j(L_i(E,G))$, and $\sigma_j$ is defined as in Lemma 6.1.1.*

**Proof:** The proof , which is similar to the proof of lemma 6.1.2, is omitted. □

Note that if $X$ is a point and $G = U(n)$ in Lemma 6.1.3, we get the equivariant cohomology of flag manifold

$$\begin{aligned} & H^*_{U(n)}(F_{(k_1,\cdots,k_s)}, \mathbb{C}) \\ = \; & C[u^1_1, \cdots, u^1_{k_1}; \cdots; u^s_1, \cdots, u^s_{k_s}; c_1(U(n)), \cdots, c_n(U(n))] \\ & /\{\sigma_1 - c_1(U(n)), \cdots, \sigma_n - c_n(U(n))\}. \end{aligned}$$

Two special cases are:

**Case 1.** Consider the diagonal embedding $U(l) \to U(l) \times \cdots \times U(l) \subset U(n)$, $t$ factors $U(l)$. This embedding induces an action of $U(l)$ on $\mathbb{C}^n$; denote this action by $\rho_l$. Then the cohomology ring $H^*(F_{(k_1,\cdots,k_s)} \times_{\rho_l} EU(l), \mathbb{C})$ is isomorphic to

$$\mathbb{C}[u^1_1, \cdots, u^1_{k_1}; \cdots; u^s_1, \cdots, u^s_{k_s}; c_1(U(l)), \cdots, c_l(U(l))]/\{\sigma_1 - \tilde{c}_1, \cdots, \sigma_n - \tilde{c}_n\}$$

where $u^i_j$ and $\sigma_j$ are the same as in Lemma 5.1.1 and $\tilde{c}_j$ is given by $1+\tilde{c}_1+\cdots+\tilde{c}_n$ $= [1 + c_1(U(l)) + \cdots + c_l(U(l))]^t$.

**Case 2.** Consider a embedding $U(1) \to U(1) \times \cdots \times U(1)$ which send $e^{2\pi\sqrt{-1}\theta}$ to $(e^{2\pi\sqrt{-1}r_1\theta}, \cdots, e^{2\pi\sqrt{-1}r_n\theta})$ with $r_i \in \mathbb{Z}$. This embedding induces an action of $U(1)$ on $\mathbb{C}^n$; denote this action by $\tilde{\rho}$. Then the cohomology ring $H^*(F_{(k_1,\cdots,k_s)} \times_{\tilde{\rho}}$ $EU(1); \mathbb{C})$ is isomorphic to

$$\mathbb{C}[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s;c_1]/\{\sigma_1-\tilde{c}_1,\cdots,\sigma_n-\tilde{c}_n\}$$

where $u_j^i$ and $\sigma_j$ are the same as in Lemma 5.1.1 and $\tilde{c}_j$ is given by $1+\tilde{c}_1+\cdots\tilde{c}_n=(1+r_1c_1)\cdots(1+r_nc_1)$.

Note that if $s=2$ and $k_1=1$. Under above action we get, after eliminating one parameter,

$$H^*_{U(1)}(\mathbb{C}P^n)\cong\mathbb{C}[x,c_1]/\{(x-r_1c_1)\cdots(x-r_nc_1)\}.$$

## 6.2 Fiberwise quantum cohomology of $F_{(k_1,\cdots,k_s)}(E)\times_G EG\to X\times BG$

Suppose $\mathcal{X}\to Y$ is a fiber bundle satisfying Assumption $(\star)$ and fiber $V$ is a positive symplectic manifold. Assume further that the cohomology ring of $\mathcal{X}$ can be presented by a ring isomorphism

$$H^*(Y)[x_1,\cdots,x_t]/\{f_1,\cdots,f_s\}\xrightarrow{p}H^*(\mathcal{X},\mathbb{C}),$$

where we view the left side as graded commutative polynomial algebra of $x_1,\cdots,x_t$. $p(x_1),\cdots,p(x_t)$ are homogeneous generators of $H^*(\mathcal{X};\mathbb{C})$, and $f_1,\cdots,f_s$ are homogeneous polynomials of $x_1,\cdots,x_t$.

**Proposition 6.2.1** *If we give a appropriate degree to $q_j$, then there are homogeneous polynomials $g_i(x_1,\cdots,x_t;q_1,\cdots,q_m)$ such that $g_i(x_1,\cdots,x_t;0,\cdots,0)=0$ and $QH^*(\mathcal{X},Y,\mathbb{C})$ can be presented by a ring isomorphism*

Note that the standard action of $U(n)$ on $\mathbb{C}^n$ induces an action of $U(n)$ on $F_{(k_1,\cdots,k_s)}$ and a fiberwise action on $F_{(k_1,\cdots,k_s)}(E)$. So we can make the following construction. Suppose $G = U(l_1) \times \cdots \times U(l_t) \subset U(n)$ is a subgroup with the induced action on $F_{(k_1,\cdots,k_s)}$ and $F_{(k_1,\cdots,k_s)}(E)$. Then

$$F_{(k_1,\cdots,k_s)}(E) \times_G EG \xrightarrow{p} X \times BG$$

is a fiber bundle with a fiber $F_{(k_1,\cdots,k_s)}$. Here $EG \to BG$ is the universal principal $G$ bundle. Construct the $L_i(E,G)$ bundle over $F_{(k_1,\cdots,k_s)}(E) \times_G EG$ as before. There is a bundle $\tilde{E} = E \times_{U(n)} EG$ of rank $n$ over $X \times BG$ whose restriction to each $X$ slice is $E$ and whose restriction to each $BG$ slice is $EG \times_G \mathbb{C}^n$. Denote by $c_i(G)$ the i-th Chern class of universal $G$ bundle $EG \times_G \mathbb{C}^n$. Then $c_i(\tilde{E})$ can be expressed in terms of $c_j(E)$ and $c_k(G)$, $i,j,k = 1,\cdots,n$, and

$$L_1(E,G) \oplus \cdots \oplus L_s(E,G) \cong p^*\tilde{E}. \tag{6.4}$$

**Lemma 6.1.3** *The cohomology ring $H^*(F_{(k_1,\cdots,k_s)}(E) \times_G EG, \mathbb{C})$ is isomorphic to*

$$H^*(X;\mathbb{C}) \times H^*(BG;\mathbb{C})[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s]$$
$$/\{\sigma_1 - c_1(p^*\tilde{E}),\cdots,\sigma_n - c_n(p^*\tilde{E})\}$$

*as a $H^*(X,\mathbb{C}) \times H^*(BG,\mathbb{C})$-module. Here parameter $u_j^i$ is mapped to $c_j(L_i(E,G))$, and $\sigma_j$ is defined as in Lemma 6.1.1.*

**Proof:** The proof , which is similar to the proof of lemma 6.1.2, is omitted. □

Note that if $X$ is a point and $G = U(n)$ in Lemma 6.1.3, we get the equivariant cohomology of flag manifold

$$\begin{aligned} & H^*_{U(n)}(F_{(k_1,\cdots,k_s)}, \mathbb{C}) \\ = \; & C[u^1_1, \cdots, u^1_{k_1}; \cdots; u^s_1, \cdots, u^s_{k_s}; c_1(U(n)), \cdots, c_n(U(n))] \\ & /\{\sigma_1 - c_1(U(n)), \cdots, \sigma_n - c_n(U(n))\}. \end{aligned}$$

Two special cases are:

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where $u^i_j$ and $\sigma_j$ are the same as in Lemma 5.1.1 and $\tilde{c}_j$ is given by $1+\tilde{c}_1+\cdots+\tilde{c}_n$ $= [1 + c_1(U(l)) + \cdots + c_l(U(l))]^t$.

**Case 2.** Consider a embedding $U(1) \to U(1) \times \cdots \times U(1)$ which send $e^{2\pi\sqrt{-1}\theta}$ to $(e^{2\pi\sqrt{-1}r_1\theta}, \cdots, e^{2\pi\sqrt{-1}r_n\theta})$ with $r_i \in \mathbb{Z}$. This embedding induces an action of $U(1)$ on $\mathbb{C}^n$; denote this action by $\tilde{\rho}$. Then the cohomology ring $H^*(F_{(k_1,\cdots,k_s)} \times_{\tilde{\rho}}$ $EU(1); \mathbb{C})$ is isomorphic to

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where $u_j^i$ and $\sigma_j$ are the same as in Lemma 5.1.1 and $\tilde{c}_j$ is given by $1+\tilde{c}_1+\cdots\tilde{c}_n=(1+r_1c_1)\cdots(1+r_nc_1)$.

Note that if $s=2$ and $k_1=1$. Under above action we get, after eliminating one parameter,

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Suppose $\mathcal{X}\to Y$ is a fiber bundle satisfying Assumption $(\star)$ and fiber $V$ is a positive symplectic manifold. Assume further that the cohomology ring of $\mathcal{X}$ can be presented by a ring isomorphism

$$H^*(Y)[x_1,\cdots,x_t]/\{f_1,\cdots,f_s\}\xrightarrow{p}H^*(\mathcal{X},\mathbb{C}),$$

where we view the left side as graded commutative polynomial algebra of $x_1,\cdots,x_t$. $p(x_1),\cdots,p(x_t)$ are homogeneous generators of $H^*(\mathcal{X};\mathbb{C})$, and $f_1,\cdots,f_s$ are homogeneous polynomials of $x_1,\cdots,x_t$.

**Proposition 6.2.1** *If we give a appropriate degree to $q_j$, then there are homogeneous polynomials $g_i(x_1,\cdots,x_t;q_1,\cdots,q_m)$ such that $g_i(x_1,\cdots,x_t;0,\cdots,0)=0$ and $QH^*(\mathcal{X},Y,\mathbb{C})$ can be presented by a ring isomorphism*

$$H^*(Y)[x_1,\cdots,x_t;q_1,\cdots,q_m]/\{f_1-g_1,\cdots,f_s-g_s\} \xrightarrow{\tilde{p}} QH^*(\mathcal{X},Y,\mathbb{C}).$$

**Proof:** Let $I$ be an ideal in $\mathbb{C}[[q_1,\cdots,q_m]]$ generated by $q_1,\cdots,q_m$. Since $V$ is a positive symplectic manifold, for any $\alpha$, $\beta \in H^*(\mathcal{X};\mathbb{C})$, we have

$$\alpha * \beta = \alpha \wedge \beta + h(\alpha,\beta;q_1,\cdots,q_m)$$

where $h$ is a polynomial in $q_1,\cdots,q_m$ and $h(\alpha,\beta;0,\cdots,0)=0$. So

$$\begin{aligned} QH^*(\mathcal{X},Y,\mathbb{C}) = \quad & \text{span } \{p(x_1),\cdots,p(x_t)\} \text{ use } * \text{ multiplication} \\ & + I \cdot QH^*(\mathcal{X},Y,\mathbb{C}). \end{aligned}$$

By the Nakayama Lemma, it follows

$$QH^*(\mathcal{X},Y,\mathbb{C}) = \text{ span } (p(x_1),\cdots,p(x_t)) \text{ use } * \text{ multiplication.} \qquad (6.5)$$

If $f_i(x_1,\cdots,x_t) = \sum a^i_{j_1,\cdots,j_t} \cdot x_1^{j_1},\cdots,x_t^{j_t}$, denote product $p(x_i)*\cdots*p(x_i)$ of $j$ factors by $p(x_i)^{*j}$, consider

$$\begin{aligned} & \sum a^i_{j_1,\cdots,j_t} \cdot p(x_1)^{*j_1} * \cdots * p(x_t)^{*j_t} \\ = \quad & \sum a^i_{j_1,\cdots,j_t} \cdot p(x_1)^{j_1} \wedge \cdots \wedge p(x_t)^{j_t} \\ & + \sum b^i_{j_1,\cdots,j_t}(q_1,\cdots,q_m) \cdot p(x_1)^{j_1} \wedge \cdots \wedge p(x_t)^{j_t} \end{aligned}$$

with $b^i_{j_1,\cdots,j_t}(0,\cdots,0)=0$. By (6.5), we may assume

$$p(x_1)^{j_1}\wedge\cdots\wedge p(x_t)^{j_t}=\sum c^{j_1,\cdots,j_t}_{i_1,\cdots,i_t}\cdot p(x_1)^{*i_1}*\cdots*p(x_t)^{*i_t}.$$

We get

$$\begin{aligned}&\sum a^i_{j_1,\cdots,j_t}\cdot p(x_1)^{*j_1}*\cdots*p(x_t)^{*j_t}\\ =\;&\sum b^i_{j_1,\cdots,j_t}(q_1,\cdots,q_m)\cdot c^{j_1,\cdots,j_t}_{i_1,\cdots,i_t}\cdot p(x_1)^{*i_1}*\cdots*p(x_t)^{*i_t}.\end{aligned}$$

Let $g_i=\sum b^i_{j_1,\cdots,j_t}(q_1,\cdots,q_m)\cdot c^{j_1,\cdots,j_t}_{i_1,\cdots,i_t}\cdot x_1^{i_1}\cdots x_t^{i_t}$. We see that $f_i-g_i$ is mapped to zero under map $\tilde{p}$. $\tilde{p}$ is a ring homomorphism. Since $p$ is an isomorphism, $\tilde{p}$ is an isomorphism modulo ideal $I$. The fact that $\tilde{p}$ is an isomorphism follows from the Nakayama Lemma. The proposition is proved. □

Since we have mathematically defined the equivariant quantum cohomology and established the properties of restriction and induction, the following results from [AS] and [GK] are theorems(see proof in [GK] and [AS]). Some simple matrix manipulation will reduce the matrix in [AS] to the following simpler form.

**Theorem 6.2.1 (Astashkevich-Sadov)** *The equivariant quantum cohomology of the flag manifold $QH^*_{U(n)}(F_{(k_1,\cdots,k_s)},\mathbb{C})$ can be presented as*

$$\mathbb{C}[u^1_1,\cdots,u^1_{k_1};\cdots;u^s_1,\cdots,u^s_{k_s};q_1,\cdots,q_{s-1};c_1,\cdots,c_n]/\{\sigma_1(q)-c_1,\cdots,\sigma_n(q)-c_n\},$$

*where $c_i$ is the i-th universal Chern class $c_i(U(n))$, the degree of $q_i$ is $k_i + k_{i+1}$, the degree of $u_j^i$ is 2j, and $\sigma_i(q)$ is defined to be the degree $2i$ part of the expansion of* $\det(A)$. *The matrix A is defined to be*

$$\begin{pmatrix} 1+u_1^1+\cdots+u_{k_1}^1 & q_1 & 0 & \cdots & 0 \\ -1 & 1+u_1^2+\cdots+u_{k_1}^2 & q_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & q_{s-1} \\ 0 & 0 & 0 & \cdots & 1+u_1^s+\cdots+u_{k_s}^s \end{pmatrix}.$$

As a special case, when $s = n$, $k_1 = \cdots = k_n = 1$, we have

**Theorem 6.2.2 (Givental-Kim)** *The equivariant quantum cohomology of the flag manifold* $QH^*_{U(n)}(F_{(1,\cdots,1)}, \mathbb{C})$ *can be presented as*

$$\mathbb{C}[u_1, \cdots, u_n; q_1, \cdots, q_{n-1}; c_1, \cdots, c_n]/\{\sigma_1(q) - c_1, \cdots, \sigma_n(q) - c_n\},$$

*where $c_i$ is the i-th universal Chern class, the degree of $q_i$ is 2, the degree of $u_j$ is 2j, and $\sigma_i(q)$ is defined to be the degree $2i$ part of the expansion of* $\det(A)$. *The matrix A is defined to be*

$$\begin{pmatrix} 1+u_1 & q_1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1+u_2 & q_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1+u_{n-1} & q_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1+u_n \end{pmatrix}.$$

These result can be generalized to

**Proposition 6.2.2** *Let $G = U(l_1) \times \cdots \times U(l_t)$, the fiberwise quantum cohomology $QH^*(F_{(k_1,\cdots,k_s)}\ (E) \times_G EG,\ X \times BG, \mathbb{C})$ can be presented as*

$$H^*(X,\mathbb{C}) \otimes H^*(BG,\mathbb{C})[u_1^1,\cdots,u_{k_1}^1;\cdots;u_1^s,\cdots,u_{k_s}^s;q_1,\cdots,q_{s-1}]$$
$$/\{\sigma_1(q) - c_1(p^*\tilde{E}),\cdots,\sigma_n(q) - c_n(p^*\tilde{E})\},$$

*where $\sigma_i(q)$ is defined as in Theorem 6.2.1.*

**Proof:** By the Proposition 6.2.1, we only need to show that relations $\sigma_i(q) - c_i(p^*\tilde{E}) = 0, i = 1,\cdots n$, hold in $QH^*(F_{(k_1,\cdots,k_s)}\ (E) \times_G EG,\ X \times BG, \mathbb{C})$. Choosing a $N$ large enough such that $H^i(BG) \cong H^i((BG)_N)$ for $i = 1,\cdots,2n+$

2. Consider the classifying map $h : X \times (BG)_N \to B(U(n))$ of bundle $\tilde{E}$. $h$ pulls back the universal flag manifold over $B(U(n))$ to $F_{(k_1,\cdots,k_s)}(E) \times_G (EG)_N$. From Theorem 6.2.1 and the restriction property of the fiberwise quantum cohomology, we conclude that the relation $\sigma_i(q) - c_i(p^*\tilde{E}) = 0$ hold for $i = 1, \cdots, n$. □

Two special cases are:

**Case 1.** The fiberwise quantum cohomology $QH^*(F_{(k_1,\cdots,k_s)} \times_{\rho_l} EU(l), BU(l), \mathbb{C})$ can be presented as

$$\mathbb{C}[u_1^1, \cdots, u_{k_1}^1; \cdots; u_1^s, \cdots, u_{k_s}^s; c_1, \cdots, c_l; q_1, \cdots, q_{s-1}]$$
$$/\{\sigma_1(q) - \tilde{c}_1, \cdots, \sigma_n(q) - \tilde{c}_n\},$$

where $\sigma_i(q)$ is defined as in Theorem 6.2.1 and $\tilde{c}_i$ is defined as in Case 1 in Section 6.1.

**Case 2.** The fiberwise quantum cohomology $QH^*(F_{(k_1,\cdots,k_s)} \times_{\tilde{\rho}} EU(1), BU(1), \mathbb{C})$ can be presented as

$$\mathbb{C}[u_1^1, \cdots, u_{k_1}^1; \cdots; u_1^s, \cdots, u_{k_s}^s; c_1; q_1, \cdots, q_{s-1}]/\{\sigma_1(q) - \tilde{c}_1, \cdots, \sigma_n(q) - \tilde{c}_n\},$$

where $\sigma_i(q)$ is defined as in Theorem 6.2.1 and $\tilde{c}_i$ and $\tilde{\rho}$ are defined as in Case 2 in Section 6.1.

When $s = 2$, $k_1 = 1$, and $\tilde{\rho}$ is the action of $U(1)$ on $\mathbb{C}^n$, we have

$$QH^*_{U(1)}(\mathbb{C}P^{n-1}; \mathbb{C}) \cong \mathbb{C}[x, c_1, q]/\{(x - r_1c_1)\cdots(x - r_nc_1) - q^n\}.$$

where degree of $q$ is 2.

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