Connected Sums of Self-Dual Orbifolds

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Abstract of the Dissertation

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We give conditions to construct families of self-dual metrics on the type I and type II connected sums of self-dual orbifolds. The conditions are given by the vanishing of certain cohomology groups. This is the generalization of the works of Donaldson-Friedman [15] and LeBrun-Singer [44]. It is achieved by generalizing the concept of complex orbifolds to that of V-analytic spaces, and studying the corresponding deformation problems.

To my parents.

Contents

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Chapter 1

Introduction

We will prove the existence of self-dual metrics on the connected sums of self-dual orbifolds with some natural conditions. This is a generalization of the results of Donaldson-Friedman [15] and LeBrun-Singer[44]. Such a result gives a new way to construct examples of self-dual metrics. It also shows that the type of degenerations of self-dual conformal classes observed first by LeBrun [41] is general.

Orbifolds are natural generalizations of manifolds introduced by Satake [57], which he called V-manifolds. They are topological spaces modeled on quotient spaces of Euclidean spaces by discrete group actions. The simplest examples of orbifolds are the orbit spaces of manifolds by finite group actions. Natural examples are given by the orbit spaces of manifolds by proper Lie group actions with finite isotropy groups; indeed every orbifold can be realized as the quotient of its 'frame bundle' by the natural action of the general linear group [32]. It is natural to generalize the fundamental theories of topology and geometry of manifolds to orbifolds, for example, deRham theory [57], Hodge theory [7], Gauss-Bonnet theorem [58], index theory [32], theory of instanton over 4-orbifolds [20], etc. However, the motivation of this thesis is that sometimes the study of manifolds requires the study of orbifolds. For example, the limit of Einstein metrics on 4-manifolds can collapse the manifolds into orbifolds joined at the singular points [1]. It was first noted by LeBrun [41] that similar phenomenon happens when we study the moduli of self-dual conformal classes.

Self-duality is a special property of geometry in dimension 4. For an oriented Riemannian manifold (M, g), one can define the Hodge star operator

$$* : \wedge^i(M) \to \wedge^{4-i}(M).$$

In particular, $*$ maps $\wedge^2(M)$ into itself, and $*^2 = 1$ on $\wedge^2(M)$. Hence we have a decomposition

$$\wedge^2(M) = \wedge^2_+(M) \oplus \wedge^2_-(M),$$

such that $*|_{\wedge^2_\pm(M)} = \pm id$. This simple fact has remarkable applications in mathematics and physics, e.g., Donaldson theory [16], Seiberg-Witten theory [66], gravity [19], etc. We will be concerned with Riemannian 4-manifolds or orbifolds with self-dual Weyl curvatures. We call them self-dual manifolds or orbifolds. They are the important ingredient of Penrose's twistor program. Self-dual Einstein manifolds are very important in both physics and mathematics [3]. When we study the compactification of the moduli space of self-dual metrics, we are led to the connected sum of manifolds or orbifolds.

The connected sum construction of manifolds can be generalized to orbifolds in two different ways. One can either form the connected sum of two orbifolds through two regular points , which is called a type I connected sum; or more importantly, one can construct the connected sum of two orbifolds across two compatible isolated singular points, which is called a type II connected sum. The latter was defined by LeBrun-Singer [44], which they called the generalized connected sum of orbifolds. The fact that (multiple) Type II connected sums produce manifolds makes it very useful. When we have two manifolds or orbifolds with some structure, we want to know when the connected sum admits the same structure. The solution of such a problem not only provides a way to construct new manifolds with the structure, but also is an important ingredient in the study of the moduli spaces of this structure. In the study of self-dual conformal classes, Donaldson-Friedman's theory gives conditions for the existence of self-dual conformal classes on the connected sums of self-dual 4-manifolds. It was generalized to the case of reflection orbifolds by LeBrun and Singer [44]. This thesis deals with general orbifolds.

When Atiyah-Hitchin-Singer [6] used Penrose's idea to study the instantons over self-dual manifolds, the only known examples were examples conformally flat manifolds, Fubini-Study metric on $\mathbb{CP}_2$, and $K3$ metrics [29]. Poon [53] found self-dual metrics on $2\mathbb{CP}_2$. Generalizations to $n\mathbb{CP}_2$ were made in different ways by Floer [21], Donaldson-Friedman [15] and LeBrun [41]. The last one was later generalized in [40], [43], [34] to construct scalar flat Kähler surfaces, which are special cases of self-dual manifolds. LeBrun's metrics were related to the Donaldson-Friedman construction by Kim-Pontercovo [35] and Pedersen-Poon [51] using different special properties of the metrics. Though we now have the recent theorem of Taubes [63], it is still interesting to give new constructions of self-dual metrics. It is important to prove and generalize the following conjectures of LeBrun [41]:

Conjecture 1. *Donaldson-Friedman's theory can be generalized to the connected sums of self-dual orbifolds.*

Conjecture 2. *LeBrun metrics on* $n\mathbb{CP}_2$ *can be constructed from the connected sum of a LeBrun orbifold and a Gibbons-Hawking orbifold across the orbifold points via the generalized Donaldson-Friedman theory.*

LeBrun-Singer [44] proved the Conjecture 1 for the special case of reflection orbifolds, and showed that the generalized connected sum of two Eguchi-Hanson orbifolds gives us the self-dual metrics on $2\mathbb{CP}_2$. The latter is the Conjecture 2 when $n = 2$. This thesis will prove these conjectures in the general case.

Donaldson and Friedman [15] used the twistor spaces of two self-dual manifolds to construct an analytic space with normal crossing singularity, which is called the singular twistor space . Using a suitable deformation of this singular twistor space to smooth out the singularity, they then got the twistor spaces of self-dual conformal classes on the connected sum of the two manifolds by Penrose correspondence. The generalization to orbifold case is straightforward except for one difficulty. The twistor spaces of self-dual orbifolds are complex orbifolds, i.e. they are locally the quotients of twistor spaces by finite group actions. If one makes the natural modification of the above construction of the singular twistor space, one gets a space which is locally the quotients of analytic spaces with normally crossing singularities by finite group actions, except in the case of reflection orbifolds. We call such a space a V-analytic space, or V-space for short. Since we want to keep the local quotient structure, the natural deformation of the singular twistor space should then be the deformation in the category of V-spaces, which we call a V-deformation. We define and study V-deformations as a generalization of deformation theory of singular analytic spaces.

The rest of the thesis is arranged as follows: Chapter 2 describes the twistor theory of the self-dual orbifolds, connected sums of orbifolds and the construction of the singular twistor space; Chapter 3 studies V-deformation theory, which is applied in Chapter 4 to prove LeBrun Conjecture 1; Chapter 5 proves LeBrun Conjecture 2 ; Chapter 6 contains the computations of some examples.

Chapter 2

Twistor Spaces of Self-Dual Orbifolds and Connected Sums

2.1 Twistor Spaces

For details of twistor spaces introduced by Penrose, see [6]. Let (M^4, g) be a smooth oriented Riemannian four-manifold. The Hodge star operator

$$* : \wedge^2(TM) \to \wedge^2(TM)$$

is conformally invariant, and $*^2 = 1$. We then have a decomposition

$$\wedge^2(TM) = \wedge^2_+(TM) \oplus \wedge^2_-(TM),$$

where $\wedge^2_+$ and $\wedge^2_-$ are the subbundles of self-dual and anti-self-dual 2-forms respectively. Notice that the orientation on TM induces natural orientations on $\wedge^2_+$ and $\wedge^2_-$ respectively. Now the Riemann curvature tensor R, viewed as an element in $S^2(\wedge^2(TM))$, has a corresponding decomposition

$$R = \begin{pmatrix} W_+ + \frac{s}{12}I & B \\ & \\ B^t & W_- + \frac{s}{12}I \end{pmatrix},$$

where s is the scalar curvature, W_+ and W_- are the self-dual and anti-self-dual Weyl tensor respectively, B corresponds to the trace-free Ricci tensor.

Definition. The oriented Riemannian manifold (M^4, g) is called *self-dual* if $W_- = 0$. Notice that this is a conformally invariant condition ([9], 1.159(c)).

For any $x \in M$, and any almost complex structure J_x on T_xM giving the reversed orientation, $\omega_x \in \Lambda^2(T_xM)$ defined by $\omega_x(X,Y) = g(J_xX, Y)$, for $X, Y \in T_xM$, lies in $\Lambda_-(T_xM)$ and had length $\sqrt{2}$. Let $Z = S_{\sqrt{2}}(\Lambda_-(TM))$ be the S^2-bundle

$$\begin{array}{ccc} S^2 & \longrightarrow & Z(M) \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

of vectors of length $\sqrt{2}$ in $\Lambda_-(TM)$. The fibers $F_x = \pi^{-1}(x)$ are called the twistor fibers. The Levi-Civita connection induces a connection on $\Lambda_-(TM)$, which induces the following decomposition

$$TZ \cong TF \oplus \pi^*TM,$$

where TF denotes the tangent bundle along the fibers. Then at each point $z \in F_x \subset Z$, we define an almost complex structure $J : TZ \to TZ$ by taking the almost complex structure corresponding to z on $\pi^* T_x M$ and the standard one on the fiber F_x, which is S^2 with Riemannian metric and orientation.

Definition. The *twistor space* Z of (M,g) is the manifold $S_{\sqrt{2}}(\Lambda_-(TM))$ with the almost complex structure J defined above.

Since the antipodal map on S^2 is anti-holomorphic, the map $\sigma : Z \to Z$ defined by $\sigma(\omega_x) = -\omega_x$ for $\omega_x \in S_{\sqrt{2}}(\Lambda_-(T_x M))$ is a fix-point free involution such that

$$\sigma_* J = -J\sigma_*.$$

It is clear that the twistor fibers are fixed by σ.

Theorem. (Penrose Construction [52], [6])

1) The twistor space (M,J) is a complex manifold if and only if $W_- = 0$, i.e. (M,g) is self-dual.

2) When the condition of 1) is satisfied, $\sigma : Z \to Z$ is anti-holomorphic. The twistor fibers are complex submanifolds of Z, biholomorphic to $\mathbb{CP}_1$ with normal bundles isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

There is a converse to the above Penrose construction [6]:

Theorem.. *Let Z be a complex 3-manifold with the following properties:*

1) Z has a free anti-holomorphic involution σ;

2) Z has a σ-invariant foliation by $\mathbb{CP}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Then there is a smooth oriented four-manifold M with a self-dual conformal class such that $Z(M)$ is Z.

An alternative description of the twistor space is very useful. Without loss of generality, assume that M admits a $Spin^c$ structure [38]. This is the case when M^4 is compact and oriented [33]. Let V^+ and V^- be the spinor bundles. Then $Z \cong \mathsf{P}(V^-)$ and the twistor fibers $F_x = \pi^{-1}(x) \cong \mathsf{P}(V_x^-)$ has normal bundles $\pi^* V_x^+ \otimes \mathcal{O}_{F_x}(1)$.

2.2 Orbifolds

The notion of a smooth orbifold [57] is a generalization of that of a manifold and that of the orbit space of a manifold by a finite group action. We recall the the definition of orbifold by Kawasaki [32]. The original terminology for orbifold was V-manifold in [57]. Since it is convenient to use terminology like V-bundles, we use V-manifold in the following definition, which is interchangeable with orbifold.

Denote by $\mathcal{M}$ the category of connected smooth manifolds and open embeddings (we call an embedding open if its image is open). A subcategory $\mathcal{M}_S$ (the category of manifolds with finite symmetries) is defined as follows. The objects of $\mathcal{M}_S$ are the classes of pairs (M, G), where M is a connected smooth manifold, and G is a finite group acting effectively on M. Let (M, G), (M', G') be two such objects, morphism $\{\phi\} : (M, G) \to (M', G')$ is a family of open inclusions $\phi : M \to M'$ satisfying:

1. For each $\phi \in \{\phi\}$, there is a group homomorphism $\lambda_\phi : G \to G'$ that makes ϕ be λ_ϕ-equivariant.

2. If $g'\phi(M) \cap \phi(M) \neq \emptyset$ for some $g' \in G'$, then $g' \in Im\lambda_\phi$.

3. G' acts on the set $\{\phi\}$ simply transitively, where the action is defined by $(g' \cdot \phi)(x) = g' \cdot \phi(x)$, for $x \in M$ and $g' \in G'$.

The morphism $\{\phi\}$ induces a unique open embedding of orbit spaces $i_\phi : M/G \to M'/G'$. If $\mathcal{T}$ is the category of connected topological spaces and open embeddings, then we have a functor $\mathcal{L} : \mathcal{M}_S \to \mathcal{T}$ defined by $\mathcal{L}(M,G) = M/G$ and $\mathcal{L}(\{\phi\}) = i_\phi$. In fact, condition 1 implies the existence of maps $i_\phi : M/G \to M'/G'$, condition 2 implies that these maps are open inclusions, condition 3 guarantees that they are all the same.

Definition. Let X be a paracompact Hausdorff space and $\mathcal{U}$ be a covering of X by connected open subsets, satisfying the following condition: for any $x \in U \cap V$, $U, V \in \mathcal{U}$, there is $U' \in \mathcal{U}$ such that $x \in U' \subset U \cap V$. Let $\mathcal{T}(\mathcal{U})$ be the subcategory of $\mathcal{T}$ consisting of all the elements of $\mathcal{U}$ and the inclusions. Then a *V-manifold structure* is a functor $\mathcal{V} : \mathcal{T}(\mathcal{U}) \to \mathcal{M}_S$ such that $\mathcal{L} \circ \mathcal{V} = I_{\mathcal{T}(U)}$ (the identity functor).

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$ satisfying the same condition as $\mathcal{U}$, then there is a V-manifold structure $\mathcal{V}' : \mathcal{T}(\mathcal{U}') \to \mathcal{M}_S$ such that $\mathcal{V} \cup \mathcal{V}' : \mathcal{T}(\mathcal{U} \cup \mathcal{U}') \to \mathcal{M}_S$ is a V-manifold structure. We regard $\mathcal{V}$ and $\mathcal{V}'$ as equivalent, and consider the equivalence classes of V-manifold structures. So we may choose $\mathcal{U}$ to be arbitrarily fine such that the open sets in $\mathcal{U}$ form a basis of the underlying topology.

Let $(X, \mathcal{U}, \mathcal{V})$ be a V-manifold, for each $U \in \mathcal{U}$, denote $\mathcal{V}(U) = (\tilde{U}, G_U)$, we have a system $\pi_U : (\tilde{U}, G_U) \to U$ giving identification $U = \tilde{U}/G_U$. It is called a uniformization chart or simply a chart of the V-manifold and all of them together form a uniformization system. Using the uniformization system, we can give a Satake-type definition of V-manifold. G_U is called the local uniformization group of U. For each $x \in X$, $G_x = \lim_{x \in U \in \mathcal{U}} G_U$ is called the local uniformization group of X at x.

We can define complex orbifold by replacing $\mathcal{M}$ with the category of complex manifolds with finite symmetry. The morphisms are understood as to preserve the holomorphic structures. Other structures on manifolds can be generalized to orbifolds similarly, e.g. Riemannian metrics, conformal classes, orientations, bundles, etc. In particular, we can define the notion of a self-dual orbifold. The following are some examples of self-dual orbifolds.

Example. 2.1. (Global quotients) Let M' be an oriented 4-manifold with an orientation preserving action by a finite group Γ, then $M = M'/\Gamma$ has the structure of an oriented orbifolds. If $[g']$ is a Γ-invariant self-dual conformal class on M', then it gives a self-dual conformal structure on M, for example, oriented Euclidean 4-space modulo the action of a discrete subgroup Γ of $SO(4)$. In particular, if the cyclic group $\mathbb{Z}_n$ acts on $\mathbb{R}^4$ in the following way:

$$(z_1, z_2) \mapsto (\lambda z_1, \lambda^q z_2),$$

where we identify $\mathbb{R}^4$ with $\mathbb{C}^2$, and λ is a primary root of unity of order n, q is an integer. We denote the corresponding orbifold by $\mathbb{C}^2/\mathbb{Z}_n(1,q)$ or $\mathbb{R}^4/\mathbb{Z}_n(1,q)$. An orbifold singular point of this form is called of type $\mathbb{Z}_n(1,q)$. It is easy to see that such an orbifold has isolated singularity if and only if $(n,q) = 1$.

$Z(\mathbb{C}^2)$ can be identified [9] with the total space of the vector bundle $\mathcal{O}(1)\oplus$

$\mathcal{O}(1) \to \mathbb{CP}_1$. In fact, V^- can be identified with $\mathbb{C}^2$ by fixing an element of V^+. Let λ_1, λ_2 be the linear coordinates on V^- and z_1, z_2 be the coordinates on $\mathbb{C}^2$, then

$$Z(\mathbb{C}^2) = \{((z_1, z_2), [\lambda_1 : \lambda_2]) \in \mathbb{C}^2 \times \mathbb{P}(V^-)\}.$$

The map define by

$$((z_1, z_2), [\lambda_1 : \lambda_2]) \mapsto (\mu = \lambda_1/\lambda_2, w_1 = \mu z_1 + z_2, w_2 = -\mu \bar{z}_2 + \bar{z}_1)$$

when $\lambda_2 \neq 0$, and by

$$((z_1, z_2), [\lambda_1 : \lambda_2]) \mapsto (\nu = \lambda_2/\lambda_1, \tilde{w}_1 = \nu z_1 + z_2, \tilde{w}_2 = -\nu \bar{z}_2 + \bar{z}_1)$$

when $\lambda_1 \neq 0$, gives the identification

$$Z(\mathbb{C}^2) \cong \mathcal{O}(1) \oplus \mathcal{O}(1),$$

where μ, ν are local coordinates on $\mathbb{CP}_1$, (w_1, w_2) and $(\tilde{w}_1, \tilde{w}_2)$ are local coordinates of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in the fiber direction. The action of $\mathbb{Z}_n(1, q)$ on $Z(\mathbb{C}^2)$ is given by:

$$\lambda \cdot (\mu, w_1, w_2) = (\lambda^{q-1}\mu, \lambda^q w_1, \lambda^{-1} w_2),$$

$$\lambda \cdot (\nu, \tilde{w}_1, \tilde{w}_2) = (\lambda^{1-q}\nu, \lambda \tilde{w}_1, \lambda^{-q} \tilde{w}_2).$$

In particular, if $q = 1$, then the action on the singular fiber is trivial.

Example. 2.2. (ALE orbifolds [37], [26], [29])These are one-point conformal compactifications of the the ALE spaces. If the fundamental group at infinity is $\mathbb{Z}_n$, these are type $\mathbb{Z}_n(1, -1)$ examples with respect to the reversed orientation.

Example. 2.3. (LeBrun orbifolds [40]) LeBrun constructed scalar-flat Kähler metrics on the total spaces of the line bundles $\mathcal{O}(-n) \to \mathbb{CP}^1$ with $SU(2)$ symmetry. The one-point conformal compactifications gives us type $\mathbb{Z}_n(1,1)$ examples.

Example. 2.4. (Galicki-Lawson orbifolds [25] and Joyce orbifolds [31]) Using the quaternionic Kähler quotient construction, Galicki and Lawson constructed lots of example of self-dual Einstein orbifolds. In particular, they show that these orbifolds are not global quotients. Joyce constructed many self-dual orbifolds using his quaternionic quotient construction, which generalizes the above.

Example. 2.5. (Self-dual reflection orbifolds [44]) Reflection orbifolds are 4-orbifolds which has only singularity of the form $\mathbb{R}^4/\pm 1$.

Using these compact self-dual orbifolds as building blocks in the connected sum construction, we can get a lot of self-dual manifolds/orbifolds from the main result of this thesis. Notice that most of the above orbifolds are not global quotients as Riemannian orbifolds. It is interesting nevertheless to observe that a lot of them are pseudo-free orbifolds [20], hence are the quotients of smooth manifold by $\mathbb{Z}_n$.

Proposition. (Kawasaki [32]) *Every orientable orbifold M^n is the orbit space of a smooth $SO(n)$-action on a smooth manifold PM.*

In fact [45], we can choose a Riemannian metric on M, and take PM to be the principal $SO(n)$ V-bundle of M. It is easy to see that PM is smooth.

Proposition. (Liang [45]) *Let M be a compact orientable orbifold with only Abelian local uniformization groups, and assume that the singular set of M has codimension > 1. Then M is the orbit space of a smooth fixed-point free S^1-action on a closed manifold N.*

Definition. (Fintushel-Stern [20]) A *pseudo-free S^1-action* is a smooth S^1-action on a smooth $(2n+1)$-manifold such that the action is free except for finitely many exceptional orbits with isotropy $\mathbb{Z}_{a_1}, \cdots, \mathbb{Z}_{a_n}$, where $a_1, \cdots, a_n$ are pairwise relatively prime. The *total isotropy* is the product $\alpha = a_1 \cdots a_n$. A *pseudo-free orbifold* $X = Q^5/S^1$ is the quotient of a smooth 5-manifold Q^5 by a pseudo-free S^1-action.

Proposition. (Fintushel-Stern [20]) *Let X be a pseudo-free orbifold with singular points $\{x_1, \cdots, x_n\}$ and total isotropy α. Then there is a smooth closed 4-manifold $M(X)$ with a smooth $\mathbb{Z}_\alpha$-action and a branched cover*

$$\lambda : M(X) \to X$$

branched over $F \cup \{x_1, \cdots, x_n\}$, where F is a surface contained in $X - \{x_1, \cdots, x_n\}$.

For the examples of self-dual orbifolds discussed earlier, the Riemannian metrics have the singular points as the orbifolds. Hence cannot use the above branched covering to study them.

We now follow [44] to define the twistor space of a self-dual orbifold. Locally on each unformization chart (U', Γ) on a self-dual orbifold M, we have a local twistor space $Z(U')$ for U'. But Γ acts by conformal maps, so the action lifts to $Z(U')$ to an action by biholomorphic maps. We can then patch up U'/Γ, and get a complex 3-orbifold $Z = Z(M)$. The twistor fibrations $Z(U') \to U'$ induce a map $\pi : Z \to M$, which we call V-twistor fibration. We call $F_x = \pi^{-1}(x) \cong \mathbb{CP}_1/\Gamma_x$ twistor fibers. It is clear that the singularities of Z are contained in the twistor fibers over the singular points in M. The theorem of Penrose construction can be easily formulated for the orbifold case, since its proof is of local nature. Similarly, the theorem on inverse Penrose construction can be also generalized to the orbifold case.

2.3 Connected Sums

Let (M_1, g_1), (M_2, g_2) be two oriented Riemannian 4-manifolds, and $x_i \in M_i$ two points. For $i = 1, 2$, we can identify a neighborhood U_i of each point x_i with a neighborhood V_i of $0 \in T_i = T_{x_i}M_i$ by geodesic normal coordinates. Choose an orientation reversing linear isometry $A : T_1 \to T_2$, when ϵ is a small enough positive parameter, the map $f_{A,\epsilon} : T_1 \to T_2$ defined by

$$v \mapsto \epsilon \frac{A(v)}{|v|^2}$$

induces an orientation-reversing diffeomorphism between a pair of annular regions in U_i. The connected sum $M_1 \# M_2$ is formed by removing small balls around x_i and gluing these annuli.

If M_1, M_2 are two orbifolds, and x_1, x_2 are smooth points of them respectively, the above construction still applies to give the connected sum $M_1 \# M_2$.

We call it the type I connected sum.

LeBrun-Singer [44] gave the following generalization: let p_i be isolated singular orbifold points of M_i, with the same local group Γ, let $\pi_i : (U_i', \Gamma) \to U_i$ be the uniformization charts centered at p_i. Let $T_i = T_0 U_i'$. If there is an orientation reversing isometry $A : T_1 \to T_2$ which intertwines the action of Γ, that is to say, we have a commutative diagram:

$$\begin{array}{ccc} \Gamma \times T_1 & \xrightarrow{(id_\Gamma, A)} & \Gamma \times T_2 \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow[A]{} & T_2, \end{array}$$

where the vertical arrows give the group actions. We can then use the map $f_{A,\epsilon}$ to construct $U_1' \# U_2'$. But now Γ acts freely on $U_1' \# U_2'$, we can pass to the quotient. The resulting space is called the Γ-connected sum of M_1 and M_2 across two orbifold points p_1, p_2, and denoted by $M_1 \#_\Gamma M_2$ We call it a type II connected sum.

Donaldson-Friedman [15] gave twistor interpretation of connected sum of two self-dual manifolds. This was generalized by LeBrun-Singer in the following way: if in the above construction M_i are self-dual orbifolds, let $\tilde{Z}(U_i')$ be the blowing-up of $Z(U_i')$ along $\pi'^{-1}(0)$, let Q_i' be the exceptional divisor, the Γ-action lift to $\tilde{Z}(U_i')$, A induces an identification of Q_1' and Q_2', the Γ-actions on Q_i' are compatible with this identification, so it acts on $\tilde{Z}(U_1') \cup_{Q_i'} \tilde{Z}(U_2')$, passing to the quotient, we then form a singular V-space Z_0. Besides possible orbifold singularity from other orbifold points, Z_0 has V-semistable normal crossing singularity along Q_i'/Γ, that is to say, near Q_i'/Γ, Z_0 is the quotient by Γ of an analytic set with only normal crossing singularity.

Example. (Flat models) $\mathbb{Z}_n(1,q)$-action and $\mathbb{Z}_n(1,-q)$-action on $\mathbb{R}^4$ are intertwined by $A : \mathbb{R}^4 \to \mathbb{R}^4$, $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4)$, if we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by $(x_1, x_2, x_3, x_4) \mapsto (x_1 + ix_2, x_3 + ix_4)$. The blowing-up space $\tilde{Z}(\mathbb{R}^4)$ of $Z(\mathbb{R}^4)$ along $\pi^{-1}(0)$ can be identified with the total space of the line bundle $\mathcal{O}(1,-1) \to \mathbb{CP}_1 \times \mathbb{CP}_1$. The map A induces a map $\mathbb{CP}_1 \times \mathbb{CP}_1 \to \mathbb{CP}_1, (x,y) \mapsto (y,x)$ and a quadratic form $q : \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \to \mathcal{O}$. We can use the explicit formula in Example 2.1 to write down the $\mathbb{Z}_n$-actions on $\mathcal{O}(1,-1)$ and $\mathcal{O}(-1,1)$, and check that q is $\mathbb{Z}_n$-invariant. Now $Z_0' = \tilde{Z}(\mathbb{R}^4) \cup_{\mathbb{CP}_1 \times \mathbb{CP}_1} \tilde{Z}(\mathbb{R}^4) = \{(x,y) \in \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \mid q(x,y) = 0\}$ has a smoothing by $Z_t' = \{q(x,y) = t\}$, so $Z_0 = Z_0'/\mathbb{Z}_n$ has a smoothing by $Z_t = Z_t'/\mathbb{Z}_n$, which is the twistor spaces for $(\mathbb{R}^4/\mathbb{Z}_n(1,q)) \#_{\mathbb{Z}_n} (\mathbb{R}^4/\mathbb{Z}(1,-q))$.

We will develop the deformation theory for general singular twistor space constructed above modeled on this example. This has been done by LeBrun-Singer [44] in the case of reflection orbifolds, they got a theory similar to that of Donaldson-Friedman and some interesting examples.

Chapter 3

Deformation Theory

We define in section 3.1 the V-analytic spaces and V-deformations. Section 3.2 reviews the deformation theory of differential graded Lie algebras, which is used in Sections 3.3 and 3.4 to study V-deformations. We extend the results to the relative case in Section 3.5.

3.1 V-Analytic Spaces

We define V-analytic space similar to the definition of V-manifold (orbifold) by Kawasaki [32] as in Section 2.2. For reader's convenience, we give the details here. Let $\mathcal{M}$ denote the category of open analytic sets and open embeddings (we call an embedding open if its image is open). A subcategory $\mathcal{M}_S$ (the category of open analytic sets with finite symmetries) is defined as follows. The objects of $\mathcal{M}_S$ are the classes of pairs (G, M), where M is an open connected analytic set, and G is a finite group acting effectively on M by analytic maps. Let (G, M), (G', M') be two such objects, morphism

$\{\phi\} : (G, M) \to (G', M')$ is a family of open embeddings $\phi : M \to M'$ satisfying:

1. For each $\phi \in \{\phi\}$, there is a group homomorphism $\lambda_\phi : G \to G'$ that makes ϕ be λ_ϕ-equivariant.

2. If $g'\phi(M) \cap \phi(M) \neq \emptyset$ for some $g' \in G'$, then $g' \in Im\lambda_\phi$.

3. G' acts on the set $\{\phi\}$ simply transitively, where the action is defined by $(g' \cdot \phi)(x) = g' \cdot \phi(x)$, for $x \in M$ and $g' \in G'$.

The morphism $\{\phi\}$ induces a unique open embedding of orbit spaces $i_\phi : M/G \to M'/G'$. If $\mathcal{T}$ is the category of connected topological spaces and open embeddings, then we have a functor $\mathcal{L} : \mathcal{M}_S \to \mathcal{T}$ defined by $\mathcal{L}(G, M) = M/G$ and $\mathcal{L}(\{\phi\}) = i_\phi$. In fact, condition 1 implies the existence of maps $i_\phi : M/G \to M'/G'$, condition 2 implies that these maps are open embeddings, condition 3 guarantees that they are all the same. Notice that the effectiveness of the group actions and equivariance of the embeddings imply that the homomorphisms $\{\lambda_\phi\}$ are injective.

Definition. Let X be a paracompact Hausdorff space and $\mathcal{U}$ be a covering of X by connected open subsets, satisfying the following condition: for any $x \in U \cap V$, $U, V \in \mathcal{U}$, there is $U' \in \mathcal{U}$ such that $x \in U' \subset U \cap V$. Let $\mathcal{T}(\mathcal{U})$ be the subcategory of $\mathcal{T}$ consisting of all the elements of $\mathcal{U}$ and the inclusions. Then a *V-analytic space structure* on X is a functor $\mathcal{V} : \mathcal{T}(\mathcal{U}) \to \mathcal{M}_S$ such that $\mathcal{L} \circ \mathcal{V} = I_{\mathcal{T}(U)}$ (the identity functor).

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$ satisfying the same condition as $\mathcal{U}$, then there is a V-space structure $\mathcal{V}' : \mathcal{T}(\mathcal{U}') \to \mathcal{M}_S$ such that $\mathcal{V} \cup \mathcal{V}' : \mathcal{T}(\mathcal{U} \cup \mathcal{U}') \to \mathcal{M}_S$ is a V-space structure. We regard $\mathcal{V}$ and $\mathcal{V}'$ as equivalent, and consider the equivalence classes of V-space structures, so we may choose $\mathcal{U}$ to be arbitrarily fine such that the open sets in $\mathcal{U}$ form a basis of the underlying topology.

Let $(X, \mathcal{U}, \mathcal{V})$ be a V-analytic space, for each $U \in \mathcal{U}$, denote $\mathcal{V}(U) = (G_U, \tilde{U})$, we have a given identification $U = \tilde{U}/G_U$. We call $\pi_U : (G_U, \tilde{U}) \to U$ a uniformization chart or simply a chart of the V-space and all of them together form a uniformization system. Using the uniformization system, we can give a Satake-type definition of V-spaces. G_U is called the local uniformization group of U. For each $x \in X$, $G_{x,X} = \lim_{x \in U \in \mathcal{U}} G_U$ is called the local uniformization group of X at x; when there is no risk of confusion, we denote it simply by G_x.

Given two charts $\pi_1 : (G_{U_1}, \tilde{U}_1) \to U_1$ and $\pi_2 : (G_{U_2}, \tilde{U}_2) \to U_2$, the transition between them is given by charts $\pi_V : (G_V, \tilde{V}) \to V$ such that $V \subset U_1 \cap U_2$. But we have the following

Lemma. *If $\pi_1 : (G_{U_1}, \tilde{U}_1) \to U_1$ and $\pi_2 : (G_{U_2}, \tilde{U}_2) \to U_2$ are two charts with $G_{U_1} \cong G_{U_2}$ Abelian (for simplicity of notation, we will denote them simply by G) and $U_1 \cap U_2$ is simply connected. Then we have a G-equivariant isomorphism:*

$$\Phi : \pi_1^{-1}(U_1 \cap U_2) \to \pi_2^{-1}(U_1 \cap U_2).$$

Proof of Lemma. Cover $U_1 \cap U_2$ by $\{V_\alpha\}$ such that there is a chart $\pi_\alpha : (G_\alpha, \tilde{V}_\alpha) \to V_\alpha$, then $\{\pi_i^{-1}(V_\alpha)\}$ covers $\pi_i^{-1}(U_1 \cap U_2)$. Since $V_\alpha \subset U_i$, we have an injection of group homomorphism $G_\alpha \to G$, and the following commutative diagram

$$\begin{array}{ccc} \tilde{V}_\alpha & \xrightarrow{f_{\alpha i}} & \tilde{U}_i \\ \downarrow & & \downarrow \\ V_\alpha & \longrightarrow & U_i, \end{array}$$

where $f_{\alpha i}$ is a G_α-equivariant open embedding. Set $A_{\alpha i} = Imf_{\alpha i}$, then there is a G_α-equivariant isomorphism $\psi_\alpha : A_{\alpha 1} \to A_{\alpha 2}$. Now $\pi_i^{-1}(V_\alpha) = \cup_{g \in G} g \cdot A_{\alpha i}$, ψ_α can be extended to G-equivariant map $\phi_\alpha : \pi_1^{-1}(V_\alpha) \to \pi_2^{-1}(V_\alpha)$. $\{\phi_\alpha\}$ may not glue together nicely. But on $\pi_1^{-1}(U_\alpha \cap U_\beta), \phi_\alpha = g_{\alpha\beta} \cdot \phi_\beta$ for some $g_{\alpha\beta} \in G$, and

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1},$$

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \in G,$$

so $\{g_{\alpha\beta}\}$ defines a cohomology class in $H^1(\{V_\alpha\}, G)$. Since $U_1 \cap U_2$ is simply connected, we can choose $\{V_\alpha\}$ such that $H^1(\{V_\alpha\}, G) = 0$, then there exist $\{h_\alpha\} \in G$ such that $g_{\alpha\beta} = h_\alpha^{-1} h_\beta$. So $\phi_\alpha = h_\alpha^{-1} h_\beta \phi_\beta$, hence $h_\alpha \phi_\alpha = h_\beta \phi_\beta$. Set $\Phi_\alpha = h_\alpha \phi_\alpha$, then $\Phi_\alpha = \Phi_\beta$, hence $\{\Phi_\alpha\}$ define a G-equivariant isomorphism $\Phi : \pi_1^{-1}(U_1 \cap U_2) \to \pi_2^{-1}(U_1 \cap U_2)$. Q.E.D.

By a result of H.Cartan [11], the quotient space of an analytic set by a finite group of analytic automorphisms is also an analytic set. Hence V-analytic spaces are analytic spaces if we forget the V-space structures.

Using the uniformization charts, we can easily to define V-analytic sub-space, V-analytic maps between V-analytic spaces, etc. For simplicity, we consider here only the map $f : X \to Y$ from a V-analytic space X to an ordinary analytic space Y. f is called V-analytic if for any $x \in X$, there is a uniformization chart $\pi_U : \tilde{U} \to U$ such that $f \cdot \pi_U : \tilde{U} \to Y$ is analytic and G_U-invariant. It is easy to see that a V-analytic map is itself analytic when we forget the V-analytic structure. A V-analytic map f is called V-flat if all the maps $f \cdot \pi_U$ above are flat maps. We refer to [17] for definition of flat maps and the relationship between flatness and deformation.

Definition. Let X_0 be a V-analytic space. A *V-deformation* of X_0 is a V-flat V-analytic map $p : \mathcal{X} \to S$ from a V-analytic space $\mathcal{X}$ to an ordinary analytic space S such that X_0 is isomorphic as a V-analytic space to $p^{-1}(o)$ for a distinguished point $o \in S$.

In other words, $\forall x \in \mathcal{X}$, there is a uniformization chart $\tilde{\pi}_{\mathcal{U}} : (G_{\mathcal{U}}, \tilde{\mathcal{U}}) \to \mathcal{U}$ of $\mathcal{X}$ near x, such that the composition $p \circ \pi_{\mathcal{U}} : \tilde{\mathcal{U}} \to S$ is a flat morphism; and if $x \in p^{-1}(o) \cong X_0$, there is a uniformization chart $\pi_U : (G_U, \tilde{U}) \to U$ near x in X, and a uniformization chart $\pi_{\mathcal{U}} : (G_{\mathcal{U}}, \tilde{\mathcal{U}}) \to \mathcal{U}$ such that $U = \mathcal{U} \cap p^{-1}(o), G_U = G_{\mathcal{U}}$, and there is a G-equivariant embedding $\phi : \tilde{U} \to \tilde{\mathcal{U}}$ such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\phi} & \tilde{\mathcal{U}} \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{U}}} \\ U & \longrightarrow & \mathcal{U}, \end{array}$$

where the lower horizontal map is the inclusion. We call such a uniformization chart $\pi_u : (G_U, U) \to U$ of X an extendable chart with respect to $\mathcal{X}$.

It is easy to see that if $\pi_U : (G_U, \tilde{U}) \to U$ is an extendable chart of X, $\pi_V : (G_V, \tilde{V}) \to V$ another chart with $V \subset U$, then it is also extendable. From this, we know that $G_{x,X} = G_{x,\mathcal{X}}$ for $x \in X$.

Proposition. *Let $p : \mathcal{X} \to S$ be a V-deformation of a V-analytic space $X_0 \cong p^{-1}(o)$ for some $o \in S$. If $Q \subset X$ is a compact V-subspace, such that there is a chart $\pi : (G, \tilde{U}) \to U \supset Q$ in X with G Abelian. Then there is a chart $\varpi : (G, \tilde{\mathcal{U}}) \to \mathcal{U}$ in $\mathcal{X}$ which restrict to the above chart in X.*

Proof. Cover Q by extendable charts $\{\pi_\alpha : (G_\alpha, \tilde{U}_\alpha) \to U_\alpha\}$ such that $U_\alpha \subset U$, and if $\{\varpi_\alpha : (G_\alpha, \tilde{\mathcal{U}}_\alpha) \to \mathcal{U}_\alpha\}$ are their extensions, $U_\alpha \cap U_\beta$ and $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ are simply connected.

Since $U_\alpha \subset U$, G_α is a subgroup of G, $\pi^{-1}(U_\alpha)$ is isomorphic to $|G/G_\alpha|$ copies of $\tilde{U}_\alpha$. Hence there is a G-action on $\coprod_{i \in G/G_\alpha} \tilde{U}_\alpha$, which restrict to the G_α-action on each copy of $\tilde{U}_\alpha$. This action extends to a G-action on $\tilde{\mathcal{V}}_\alpha = \coprod_{i \in G/G_\alpha} \tilde{\mathcal{U}}_\alpha$, with $\tilde{\mathcal{V}}_\alpha / G = \mathcal{U}_\alpha$. So we have charts $\varpi_\alpha : (G, \tilde{\mathcal{V}}_\alpha) \to \mathcal{U}_\alpha$ which restrict to the charts $\pi|_{\pi^{-1}(U_\alpha)} : (G, \pi^{-1}(U_\alpha)) \to U_\alpha$. By the above lemma, there are transition maps $\Phi_{\alpha\beta}$ and $\phi_{\alpha\beta}$ for them respectively, and $\Phi_{\alpha\beta}$ restrict to $\phi_{\alpha\beta}$. Now $\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} \in G$, but we have $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$ since $\pi^{-1}(U_\alpha \cap U_\beta)$ glue to give $\tilde{U}$. Hence $\Phi_{\alpha\beta}\Phi_{\beta\gamma}\Phi_{\gamma\alpha} = 1$. So we can use $\Phi_{\alpha\beta}$ to glue $\varpi_\alpha : (G, \tilde{\mathcal{V}}_\alpha) \to \tilde{\mathcal{V}}_\alpha$ to a chart $\varpi : (G, \tilde{\mathcal{U}}) \to \mathcal{U}$ in $\mathcal{X}$ which restrict to the above chart on X. Q.E.D.

Corollary. *If $X = \tilde{X}/G$ is a compact V-space with the natural V-space structure, and $\pi : \mathcal{X} \to S$ a V-deformation of X, then there is a deformation $\tilde{\pi} : \tilde{\mathcal{X}} \to S$ with G-action, which covers the deformation of X.*

A V-deformation is clearly a deformation of a V-space in the category of V-spaces. But we will see that it also gives an ordinary deformation when we forget the V-structure. We need some simple algebraic fact, which we list without proof .

Let A be a commutative ring with unit, M an A-module, and G a finite group acting on M preserving the module structure (G doesn't act on A). Let M^G be the set of G-invariant elements of M, then

(1) M^G is also an A-module, and there is an averaging homomorphism of A-modules $a : M \to M^G$, defined by $m \mapsto \frac{1}{|G|}\Sigma_{g\in G}g(m)$. If $i : M^G \to M$ is the inclusion (it is also a homomorphism of A-modules), then $a \cdot i = 1$.

(2) If M_1, M_2 are A-modules and $h : M_1 \to M_2$ is an A-module homomorphism commuting with the group actions of G as module automorphisms, then we have the following commutative diagram of A-module homomorphisms:

$$\begin{array}{ccc} M_1^G & \longrightarrow & M_2^G \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ M_1 & \longrightarrow & M_2 \\ {\scriptstyle a_1}\downarrow & & {\scriptstyle a_2}\downarrow \\ M_1^G & \longrightarrow & M_2^G. \end{array}$$

(3) If we have a sequence of G-equivariant homomorphisms of A-modules

$$M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$$

and $\beta\alpha = 0$, then the commutative diagram:

$$\begin{array}{ccccc} M_1^G & \xrightarrow{\alpha^G} & M_2^G & \xrightarrow{\beta^G} & M_3^G \\ \downarrow & & \downarrow & & \downarrow \\ M_1 & \xrightarrow{\alpha} & M_2 & \xrightarrow{\beta} & M_3 \\ \downarrow & & \downarrow & & \downarrow \\ M_1^G & \xrightarrow{\alpha^G} & M_2^G & \xrightarrow{\beta^G} & M_3^G \end{array}$$

identifies $ker\beta^G/im\alpha^G$ with $(ker\beta/im\alpha)^G$. In particular, if $M_1 \to M_2 \to M_3$ is exact, $M_1^G \to M_2^G \to M_3^G$ is also exact.

Lemma. *Let A be a commutative ring with unit, M be a flat A-module, G a finite group acting on M as automorphisms of A-module. If M^G is the set of elements in M fixed by G, then M^G is also a flat A-module.*

Proof of Lemma. It is easy to check that M^G is an A-module. Now if E is an A-module, let G act on the second factor of $E \otimes M$, then $E \otimes M^G \to (E \times M)^G$ is an inclusion, since for any $\sum_i e_i \otimes m_i \in (E \otimes M)^G$, we have

$$\begin{aligned} \sum_i e_i \times m_i &= \frac{1}{|G|}\sum_{g\in G} g(\sum_i e_i \otimes m_i) \\ &= \sum_i e_i \otimes \left(\frac{1}{|G|}\sum_{g\in G} g(m_i)\right) \in E \otimes M^G, \end{aligned}$$

hence $E \otimes M^G = (E \otimes M)^G$. Now if

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

is an exact sequence of A-modules, since M is A-flat,

$$0 \to E_1 \otimes M \to E_2 \otimes M \to E_3 \otimes M \to 0$$

is also exact. Using (3) above, it is easy to see that it induces an exact sequence of A-modules:

$$0 \to (E_1 \otimes M)^G \to (E_2 \otimes M)^G \to (E_3 \otimes M)^G \to 0.$$

Under the identification $(E_i \otimes M)^G = E_i \otimes M^G$, it gives an exact sequence:

$$0 \to E_1 \otimes M^G \to E_2 \otimes M^G \to E_3 \otimes M^G \to 0.$$

But it is easy to see that this is just the same as tensoring the original sequence with M^G, so M^G is a flat A-module. Q.E.D.

As a corollary, we have

Lemma. *If $\tilde{\pi} : \tilde{\mathcal{X}} \to S$ is a flat morphism of analytic spaces and G is a finite group of automorphisms of $\tilde{\mathcal{X}}$, and $\tilde{\pi}$ is G-invariant, then the induced morphism $\pi : \mathcal{X} = \tilde{\mathcal{X}}/G \to S$ is also flat.*

From this it is clear that a V-deformation is also an ordinary deformation if we ignore the V-space structure.

If $\pi : \mathcal{X} \to S$ is a V-deformation, $f : T \to S$ an analytic map between ordinary analytic spaces, then we can form a fibered product

$$\mathcal{X} \times_S T = \{(x,t) \in \mathcal{X} \times T | \pi(x) = f(t)\},$$

which is easily seen to have an induced V-space structure. Let $f^*(\pi)$ be the natural projection $\mathcal{X} \times_S T \to T$, then $f^*(\pi) : \mathcal{X} \times_S T \to T$ is also a V-deformation, we call it the induced V-deformation by f and we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{X} \times_S T & \longrightarrow & \mathcal{X} \\ {\scriptstyle f^*(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & S. \end{array}$$

A V-deformation $\pi : \mathcal{X} \to S$ is called versal for $X_0 = \pi^{-1}(o)$ if any other V-deformations are locally induced by maps to S; it is universal if all such maps are unique.

3.2 Algebraic Deformation Theory

We start with the general philosophy of P. Deligne [47]: " In characteristic zero a deformation problem is controlled by a differential graded Lie algebra." Some results in this section is taken from [27] with certain changes of notations. Throughout this section, k will be a fixed field of characteristic zero.

Differential Graded Lie Algebra

A *graded Lie algebra* over k is a graded k-vector space $L = \oplus_{i \geq 0} L^i$ with bilinear maps $[\cdot, \cdot] : L^i \times L^j \to L^{i+j}$ satisfying the (graded) skew-symmetry:

$$[x, y] = -(-1)^{m \cdot n}[y, x],$$

and the (graded) Jacobi identity:

$$[x, [y, z]] = [[x, y], z] + (-1)^{m \cdot n}[y, [x, z]],$$

for $x \in L^m, y \in L^n$.

A *derivation of degree* l of a graded Lie algebra L is given by k-linear maps $d : L^I \to L^{i+l}$ such that $d[x, y] = [dx, y] + (-1)^{l \cdot m}[x, dy]$ for $x \in L^m$. Hence for any $x \in L^l$, the k-linear maps $[x, \cdot] : L^i \to L^{i+l}$ is a derivation of degree l by the graded Jacobi identity. Let $Der_l(L)$ be the set of derivations of L of degree l and $Der(L) = \oplus Der_l(L)$. If $d_1 \in Der_l(L), d_2 \in Der_m(L)$, set $[d_1, d_2] = d_1 \circ d_2 - (-1)^{l \cdot m} d_2 \circ d_1$, then $[d_1, d_2] \in Der_{l+m}(L)$. It is easy to check that $[\cdot, \cdot]$ defined above gives $Der(L)$ an induced structure of graded Lie algebra.

Remark. In the above definitions, we can replace k by any commutative k-algebra, so we also have graded Lie algebras over commutative k-algebras.

A *differential graded Lie algebra* (DG Lie algebra) is a graded Lie algebra with a differential d, i.e. d is a derivation of degree 1 and $d^2 = 0$. For a DG Lie algebra $(L, d, [\cdot, \cdot])$, define $H^*(L) = H^*(L, d) = Ker d / Im d$. It is easy to see that $H^*(L)$ has an induced structure of graded Lie algebra. Let $x_1 \in H^i(L), x_2 \in H^j(L)$, if $y_1 \in L^i, y_2 \in L^j$ are their representatives respectively, then $d[y_1, y_2] = [dy_1, y_2] + (-1)^i [y_1, dy_2] = 0$. it is direct to see that the class of $[y_1, y_2]$ in $H^{i+j}(L)$ is independent of the choices of y_1, y_2, we denote it by $[x_1, x_2]$, then this operation gives $H^*(L)$ a structure of graded Lie algebra. A construction [55] generalizing the above gives the Lie-Massey bracket $[x_1 \cdots x_n]$ for $x_1, \cdots, x_n \in H^*(L)$. If $(L, d, [\cdot, \cdot])$ is a DG Lie algebra, for $u \in Der_m(L), v \in Der_n(L)$, let $[u, v]' = u \circ v - (-1)^{m \cdot n} v \circ u, d'u = d \circ u - (-1)^m u \circ d = [d, u]$, then $(Der(L), d', [\cdot, \cdot]')$ is also a DG Lie algebra. It is easy to see that the map $L \rightarrow Der(L), x \rightarrow [x, \cdot]$ is a homomorphism of DG Lie algebras. By abuse of notation, we denote its image also by L.

Deformation of DG Lie algebra

Let S be a commutative local noetherian k-algebra with unit element and maximal ideal m, $(L, d, [\cdot, \cdot])$ a DG Lie algebra over k. Consider $L \otimes_k S$. We can extend d and $[\cdot, \cdot]$ by defining $d(x \otimes s) = (dx) \otimes s$ and $[x \otimes s, y \otimes t] = [x, y] \otimes (st)$

for $x, y \in L, s, t \in S$, then $(L \otimes S, d, [\cdot,\cdot])$ is a DG Lie algebra over S. *A deformation of d over S* is a differential $D \in Der_1(L \otimes_k S)$ such that $D^2 = 0$ and $\omega := D - d \in L_1 \otimes m$.

Given any deformation D, from $D^2 = 0$ we deduce the following fundamental equation:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

We call it the *Maurer-Cartan equation* since it has the same form as the equation satisfied by the Maurer-Cartan form on a Lie group.

Let $(L_0 \otimes S)^\times$ be the set of invertible elements in $L_0 \otimes S$, then it is a group. Two deformations D_1, D_2 are said to be equivalent if there is $g \in (L_0 \otimes S)^\times$ such that $g \cdot D_1 = D_2 \cdot g$. Let $D_i = d + \omega_i$, then $\omega_1 = g^{-1}\omega_2 g + g^{-1}dg$. This manifests the following 'gauge invariance' of the Maurer-Cartan equation: if ω is a solution, so is $g^{-1}\omega g + g^{-1}dg$ for any $g \in (L_0 \otimes S)^\times$.

Let T be another commutative local k-algebra with maximal ideal m_T. If $f : S \to T$ is a k-algebra homomorphism and $D : L \otimes_k S \to L \otimes_k S$ is a deformation of d over S, then $L \otimes_k T \cong (L \otimes_k S) \otimes_S T$. Let $(f^*D)(z \otimes_S t) = (Dz) \otimes_S t$, then $(f^*D)^2 = 0$, and $f^*D - d = f^*(D - d) \in L_1 \otimes m_T$, hence f^*D is a deformation of d over T, it is called the deformation induced by f. A deformation $D : L \otimes S \to L \otimes S$ is called a *versal deformation* if any equivalence class of deformations is induced by a k-algebra homomorphism.

Deformation and Cohomology

We now examine the relationship between deformations and cohomology of DG Lie algebra.

Let $\mathcal{D} = k[t]/(t^2)$, then a deformation D of d over $\mathcal{D}$ is called an infinitesimal deformation (or a first order formal deformation). Let $T^1(L)$ be the set of equivalence classes of infinitesimal deformations of d. Then $T^1(L)$ is in one-to-one correspondence with $H^1(L)$. In fact, for any class in $H^1(L)$, let $u_1 \in L_1$ be a representative of it, write $D = d+tu_1$, then $du_1 = 0$ implies $D^2 = 0$, i. e. D is an infinitesimal deformation of d. If $u_1 + dv$ is another representative, for some $v \in L_0$. Now $(1+tv)^{-1} = 1-tv$, hence $(1+tv)^{-1}D(1+tv) = d+t(u_1+dv)$. So we get a map $H^1(L) \to T^1(L)$, it is direct to check it is an identification. Another way to interpretate this identification is the following: let $\mathcal{O} = \mathcal{O}_{H^1(L),0}$, and m be the maximal ideal of $0 \in H^1(L)$, then there is a deformation D over $S_1 = \mathcal{O}/m^2$, such that for any deformation D' over a local k-algebra T with maximal ideal m_T such that $m_T^2 = 0$, there is a homomorphism $f : S_1 \to T$, such that $f^*D = D'$.

More generally, given any deformation $\omega \in L_1 \otimes m$, it gives an element in $L_1 \otimes (m/m^2)$, Maurer-Cartan equation implies that this gives an element in $H^1(L) \otimes (m/m^2)$. Since $m/m^2 \cong (T^0S)^*$, it induces a linear map: $\rho : T^0S \to H^1(L)$. It is called the *Kodaira-Spencer map* of the deformation D.

Given an infinitesimal deformation $d + tu_1$ of d over $k\{t\}/(t^2)$, we want to extend it to the second order: $d + tu_1 + t^2u_2$. Maurer-Cartan equation implies:

$$du_2 + \frac{1}{2}[u_1, u_1] = 0.$$

Let x be the class of u_1 in $H^1(L)$, then this equation has solution iff $[x, x] = 0 \in H^2(L)$. So we have a quadratic map $f_1 : H^1(L) \to H^2(L)$ describing the first obstruction. The equation of next order is

$$du_3 + [u_1, u_2] = 0.$$

It is easy to see that $d[u_1, u_2] = 0$, so it has solution if and only if f the class of $[u_1, u_2]$ is zero in $H^2(L)$. Notice now this class depends on the choices of u_1, u_2. By induction, we see that the higher obstructions all lie in $H^2(L)$. Lie-Massey bracket can be used to describe these obstructions.

Existence of Formal Versal Deformation

From the above discussion on inductive extension of deformation, we see that it is hard to construct a versal deformation for each $n \in \mathbb{N}$, in the category of local k-algebras S with $m_s^n = 0$, where m_S is the maximal ideal of S.

A local k-algebra S with maximal ideal m_S is called an Artin algebra if $m_S^n = 0$ for some $n \in \mathbb{N}$.

To establish the existence of versal deformations, it is useful to use a language due to Grothendieck [59]. Let $\mathcal{C}$ be the category of Artin local k-algebras, a covariant functor F from $\mathcal{C}$ to the category $\mathcal{S}$ of sets is called pro-presentable if it has the following form

$$F(A) \cong Hom_{alg}(R, A), A \in \mathcal{C}$$

where R is a complete local k-algebra with maximal ideal m such that $R/m^n \in \mathcal{C}$ for all $n \geq 1$ (R is a complete local k-algebra if $R \cong \overset{\lim}{\leftarrow} R/M^n$, see [5]). Schlessinger [59] gave a criteria for pro-presentabilty of such functors.

Now if $dim H^1(L) < \infty$, let $C^1(L)$ be a complement to the 1-boundaries $B^1(L) \subset L_1$, then the Maurer-Cartan equation tells us that the set of equivalence classes of deformations over an Artin local k-algebra S with maximal ideal m_S is given by

$$Y_L(A) = \{\eta \in C^1(L) \otimes m_S | d\eta + \frac{1}{2}[\eta, \eta] = 0\}.$$

Goldman and Millson used the Schlessinger criteria to prove the following

Theorem. ([27], thm. 1.1) *The functor $Y_L(A)$ is pro-presentable; that is, there is a complete local k-algebra R_L and a deformation D of (L, d) over R_L such that for any deformation D' over an Artin local k-algebra S, there is a homomorphism $f : R_L \to S$ such that $D' \cong f^*D$.*

Using a construction due to Deligne, they also proved

Theorem. ([27], thm. 4.1) *The isomorphism class of R_L doesn't depend on the choice of $C^1(L)$.*

Remark. Lie-Massey brackets [55], [60] provide an algorithm to construct R_L.

Analytic DG Lie Algebra and Kuranishi Theory

The above result does not give us a versal deformation over the analytic algebra. We need more structures on L. A *Banach DG Lie algebra* is defined to be a DG Lie algebra $(L, d, [\cdot,\cdot])$ with norms $|\cdot|_i$ on L_i such that

1. $(L_i, |\cdot|_i)$ is a Banach space;

2. $d^i : L_i \to L_{i+1}$ is continuous;

3. $|\cdot|_i : L_1 \otimes L_1 \to L_2$ is continuous

for all i. An *analytic DG Lie algebra* is a Banach DG Lie algebra with finite dimensional cohomology in degrees 0 and 1, and continuous splittings of the following two short exact sequences:

$$0 \to Z^j(L) \to L_j \xrightarrow{d} B^{j+1}(L) \to 0,$$

$$0 \to B^j(L) \to Z^j(L) \to H^j(L) \to 0.$$

Under the slittings, let $\mathcal{B}^{j+1}$ be the image of $B^{j+1}(L)$ in L_j, and $\mathcal{H}^j$ the image of $H^j(L)$ in $Z^j(L)$. Then $\mathcal{B}^{j+1}$ and $\mathcal{H}^j$ are closed subspaces of L_j. We have the following decompositions:

$$Z^j(L) = B^j(L) \oplus \mathcal{H}^j$$

$$L_j = Z^j(L) \oplus \mathcal{B}^{j+1} = B^j(L) \oplus \mathcal{H}^j \oplus \mathcal{B}^{j+1}$$

and the projection of L_j to all three factors are continuous.

Let δ be the compositions of the following maps:

$$L_{j+1} \to B^{j+1} \to \mathcal{B}^{j+1} \to L_j,$$

where the first map is the projection, the second is given by the splitting and the last one is the inclusion. Then it is easy to see that $\delta^2 = 0$ and

$$d \circ \delta + \delta \circ d = I - \mathbb{H}$$

where $\mathbb{H}$ is the projection from $L_j \to \mathcal{H}^j$. This is very similar to Hodge theory for elliptic operators.

Define the following polynomial map of degree 2 (for polynomial maps on Banach spaces, see [17]) $F : L_1 \to L_1$ by $\xi \mapsto \xi + \frac{1}{2}\delta[\xi, \xi]$. Since $\delta F(\xi) = \delta\xi$, F maps $\mathcal{B}^2 \oplus \mathcal{H}^1 = Ker\delta$ to itself. By implicit function theorem, it is easy to see that

$$F : \mathcal{B}^2 \oplus \mathcal{H}^1 \to \mathcal{B}^2 \oplus \mathcal{H}^1$$

is a local analytic isomorphism near 0.

The *Kuranishi map* $K : U \to \mathcal{H}^2$ is defined by

$$K(\eta) = \mathsf{H}([F^{-1}(\eta), F^{-1}(\eta)]),$$

where $U \subset \mathcal{H}^1$ is a neighborhood of 0. Since K is a composition of analytic maps, it is also analytic. So the *Kuranishi space* $\mathcal{K}_L = K^{-1}(0)$ is an analytic space[œ[Bœ[ce in the finite dimensional space $\mathcal{H}^1 \cong H^1(L)$.

Theorem. (Kuranishi) *Let* $Y = \{\xi \in \mathcal{B}^2 \oplus \mathcal{H}^1 | d\xi + \frac{1}{2}[\xi, \xi] = 0\}$, *then* F *induces a homeomorphism from a neighborhood of* 0 *in* Y *to a neighborhood of* 0 *in* $\mathcal{K}_L$.

Developing the formal Kuranishi theory, Goldman and Millson then prove the following (cf. [27], thm. 3.3, cor. 3.10):

Theorem. *The functor* $A \to Y_L(A)$ *on the category of Artin local* k*-algebras is represented by the analytic germ* $(\mathcal{K}_L, 0)$. *And* R_L *is the completion of the local ring* $\mathcal{O} = \mathcal{O}_{\mathcal{K}_L,0}$.

The versal deformation $\mathcal{D}$ over $\mathcal{O}$ is also versal for deformations over complete local k-algebras. In fact, for any deformation D over a complete local

k-algebra S with maximal ideal m, for $n > 1$, S/m^n is an Artin local k-algebra and $\mathcal{D}$ induces a deformation D_n over S/m^n. So there is a homomorphism $f_n : \mathcal{O} \to S/m^n$ such that $f^*\mathcal{D} = D_n$, passing to the limit, we get a homomorphism $f : \mathcal{O} \to S$ such that $f^*\mathcal{D} = D$. Hence Kuranishi theory gives us a formal versal deformation.

Artin's Lemma

Let $\mathcal{O}$ be an analytic algebra, and $\mathcal{D}$ a deformation over $\mathcal{O}$ which is formally versal. We will see that it is a versal deformation in the category of analytic algebras. Now let k be a field of characteristic zero with a non-trivial valuation. A convergent power series is defined to have non-zero radius of convergence. Let $k\{x_1, \cdots, x_n\}$ and $k[[x_1, \cdots, x_n]]$ be the sets of convergent and formal power series respectively, then they are both Noetherian local k-algebras, and $k[[x_1, \cdots, x_n]]$ is the completion of $k\{x_1, \cdots, x_n\}$ with respect to its maximal ideal $< x_1, \cdots, x_n >$. Then we have the following

Artin's Lemma [2] *Let $f(x,y) = (f_1(x,y), \cdots, f_m(x,y))$ be convergent power series in the variables $x = (x_1, \cdots x_n)$ and $y = (y_1, \cdots, y_N)$. Suppose that $\bar{y}(x) = (\bar{y}_1(x), \cdots \bar{y}_N(x))$ are formal power series without constant term satisfying $f(x, \bar{y}) = 0$. Then any integer c, there exists a convergent power series $y = y(x)$ such that $f(x, y(x)) =)$ and $y(x) \equiv \bar{y}(x) \bmod m^c$. Here m is the maximal ideal of $k[[x]]$.*

A useful generalization is the following

Lemma. [65] *Let* $\mathcal{U}_1, \cdots, \mathcal{U}_m, \mathcal{V}_1, \cdots, \mathcal{V}_N$ *be proper ideals in* $\mathbb{C}\{x\}$. *Let*

$$f_i(x,y) \in \mathbb{C}\{x,y\}, \quad i = 1, \cdots, m.$$

Let $u_1, \cdots u_N \in \mathbb{C}\{x\}$. *Suppose that for each* k *we have* $y_\nu^{(k)} \in \mathbb{C}\{x\}$ *with no constant term such that*

$$f_i(x, y_\nu^{(k)}(x)) \in \mathcal{U}_i + \mathcal{M}^{k+1}, \quad y_\nu^{(k)} \cong u_\nu \quad (\text{mod } (\mathcal{V} + \mathcal{M}^{k+1})),$$

where $\mathcal{M}$ *is the maximal ideal of* $\mathbb{C}[x]$. *Then there are* $y_\nu \in \mathbb{C}\{x\}$ *with no constant term such that*

$$f_i(x, y(x)) \in \mathcal{U}_i, \quad y_\nu \cong u_\nu \quad (\text{mod } \mathcal{V}_\nu).$$

Now let $\mathcal{O}$ be an analytic algebra and $\mathcal{D}$ a deformation which is versal for Artin local k-algebras. Given any deformation D over an analytic algebra S, we need to find a homomorphism $f : \mathcal{O} \to S$ such that $f^*\mathcal{D} = D$. This can be formulated as a problem of solving a family of analytic equations as in the above lemma. It has solution in any finite order by the versality of $\mathcal{D}$. Applying the above lemma, we get the following

Theorem. *Let* $\mathcal{D}$ *be a deformation over an analytic algebra* $\mathcal{O}$. *If it is formally versal, then it is versal in the category of analytic algebras.*

Strategy

In Kodaira-Spencer-Kuranishi theory of deformations of complex manifolds, the DG Lie algebra is given by $(\Omega^{0,*}(TM), d, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is given by Nijenhuis [24]. Put an Hermitian metric on TM and use the Hodge theory, we can get an analytic DG Lie algebra. It remains then only to see that the deformations of complex structures correspond to the deformations of this analytic DG Lie algebra. The deformations of compact complex analytic spaces are more involved, since local deformation of germs of analytic spaces is not trivial. For a complex manifold, we cover it by charts, a deformation is given by the deformation of the way they are glued, i.e. the deformation of the transition functions. For a compact analytic space, we cover it by local model spaces, which are zero sets of analytic functions on complex linear spaces. It is a nontrivial fact that the deformation of these model spaces can be embedded in the product of the base space and the same complex linear spaces, hence are related to deformations of the locally resolutions of their structure sheaves in $\mathbb{C}^n$. The global deformation can described by gluing all the local deformations of the local resolutions, from this we can construct a DG Lie algebra [50]. The analytic DG Lie algebra structure is constructed using a privileged covering [18].

Our approach to V-deformation theory is then to modify the above. We first study the corresponding model spaces, then show that under mild conditions, a V-analytic spaces can be covered by the models spaces in a suitable way, so that we can construct a DG Lie algebra to describe its deformation.

3.3 Local Theory

We first study the equivariant deformation of an analytic algebra, then relate it to the V-deformation of the germ of V-spaces. We follow closely the approach of Palamodov [49] and Donin [14].

Invariant Tate Resolution and Associoted DG Lie Algebra

Let R be a commutative Noetherian ring with unit element, $\mathcal{I}$ an ideal of R. Then a *Tate resolution* [61] of $A := R/\mathcal{I}$ is an exact sequence:

$$\cdots \xrightarrow{s} R_{-2} \xrightarrow{s} R_{-1} \xrightarrow{s} R_0 \xrightarrow{\epsilon} A \to 0,$$

where $R_0 = R$, $\epsilon : R_0 = R \to R/\mathcal{I} = A$ is the induced ring homomorphism, $\{R_n\}$ are free R-modules such that

1. $R_* = \oplus R_n$ is a free graded commutative R-algebra: $R_m \cdot R_n \subset R_{m+n}$ and $x \cdot y = (-1)^{m \cdot n} y \cdot x$ for $x \in R_m, y \in R_n$ and $x^2 = 0$ for x having odd degree;

2. s is a differential of degree 1 on R_*: $s^2 = 0$ and $s(x \cdot y) = s(x) \cdot y + (-1)^m x \cdot s(y)$ for $x \in R_m, y \in R_n$.

Tate gave a construction of a Tate resolution of $R/\mathcal{I}$ for any ideal $\mathcal{I} \subset R$. Using the Noetherian property of R, we can assume that we have only finitely many generators at each degree.

Given a Tate resolution R_* of $A = R/\mathcal{I}$, let $Der_l(R_*)$ be the set of R_0-derivations of degree l, i.e. $u : R_n \to R_{n+l}$ such that $u(x \cdot y) = u(x) \cdot y + (-1)^m x \cdot u(y)$ for $x \in R_m, y \in R_n$. Define $[u, v] = u \circ v - (-1)^{l \cdot m} v \circ u$ for

$u \in Der_l(R_*), v \in Der_m(R_*)$. Let $d = [s, \cdot]$, $Der(R_*) = \oplus Der_l(R_*)$, it is then easy to check that $(Der(L), d, [\cdot, \cdot])$ is a DG Lie algebra. $H^*(Der(R_*), d)$ with its Lie-Massey bracket independent of the choice of Tate resolution [55], we denote it by $H^*(A)$.

Let G be a finite group acting on R as ring automorphisms, $\mathcal{I}$ a G-invariant ideal of R. Then we can construct a Tate resolution of $A = R/\mathcal{I}$

$$\cdots \xrightarrow{s} R_{-2} \xrightarrow{s} R_{-1} \xrightarrow{s} R_0 \xrightarrow{\epsilon} A \to 0,$$

such that the G-action extends to R_*, preserving all the generators and commuting with s. G acts naturally on $Der(R_*)$, preserving the DG Lie algebra structure, let $Der(R_*)^G$ be the set of G-invariant elements of $Der(R_*)$, then $(Der(R_*)^G, d, [\cdot, \cdot])$ is a DG Lie subalgebra of $(Der(R_*), d[\cdot, \cdot])$. Using an averaging process, we can see that $H^*(Der(R)^G, d) \cong H^*(Der(R_*), d)^G$.

Tyurina Resolution

Let $\mathbb{C}\{z_1, \cdots, z_n\}$ denote the algebra of convergent power series in n complex variables. It is called a regular analytic algebra. If $\mathcal{I}$ is an ideals of $\mathbb{C}\{z_1, \cdots, z_n\}$, $A = \mathbb{C}\{z_1, \cdots, z_n\}/\mathcal{I}$ is called an analytic algebra. It is a commutative algebra with unit. A *Tyurina resolution* [50] of an analytic algebra A is a Tate resolution of A:

$$\cdots \xrightarrow{s} R_{-2} \xrightarrow{s} R_{-1} \xrightarrow{s} R_0 \xrightarrow{\epsilon} A \to 0,$$

such that R_0 is a regular analytic algebra over $\mathbb{C}$. $T^i(A) = H^i(Der(R_*), d)$ is called the i-th *tangent cohomology* of A. It is independent of the choice of Tyurina resolution and concentrated on non-negative degrees.

Every analytic algebra then has a Tyurina resolution, and different Tyurina resolutions of the same analytic algebra are homotopic [50]. Furthermore, for any morphism of analytic algebras $A \to B$ and Tyurina resolutions $R(A)$ and $R(B)$, there is a morphism of Tyurina resolutions $R(A) \to R(B)$ such that we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{s_A} & R(A)_{-2} & \xrightarrow{s_A} & R(A)_{-1} & \xrightarrow{s_A} & R(A)_0 & \xrightarrow{p_A} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{s_B} & R(B)_{-2} & \xrightarrow{s_B} & R(B)_{-1} & \xrightarrow{s_B} & R(B)_0 & \xrightarrow{p_B} & B & \longrightarrow & 0.
\end{array}$$

Now a deformation of A is defined to be a homomorphism of analytic algebras $f : S \to \mathcal{A}$ which makes $\mathcal{A}$ a flat S-module and such that $\mathcal{A} \otimes_S \mathbb{C} \cong A$, where $\mathbb{C}$ is made an S-module by the canonical epimorphism $S \to \mathbb{C}$, see [28]. A deformation of A can be given by a deformation of s in the Tyurina resolution [60]:

$$\cdots \xrightarrow{\tilde{s}} R_{-2} \otimes S \xrightarrow{\tilde{s}} R_{-1} \otimes S \xrightarrow{\tilde{s}} R_0 \otimes S$$

such that $\tilde{s}^2 = 0$, $\tilde{s} \in Der_1(R) \otimes S$. Let $D = [\tilde{S}, \cdot]$, then D is a deformation of d of $Der(R_*)$ over S. Hence, when $dim_{\mathbb{C}} T^1(A) < \infty$, we have a formal versal deformation of A.

Now let G be a finite group of automorphisms of A. By a result of H.Cartan [11], there is a regular analytic algebra $R_0 = \mathbb{C}\{z_1, \cdots, z_n\}$ with an G-action induced by a linear group action on $\mathbb{C}^n$, and a G-equivariant epimorphism $p : R_0 \to A$. $\mathcal{I} = kerp$ is G-invariant, we can take a G-invariant finite set$\{f_i\}$ of its generators (take any finite set of generator, add all the transformations under G), we then form the Koszul complex $R^1 = K(R_0, f)$, which is a graded differential R_0 algebra with generators $e_1, \cdots, e_{r_1}$ of degree -1, and a differential s_1 defined on generators by $s_1(e_i) = f_i$. If the germ corresponding to A is a complete intersection, $H^n(R^1) = 0$ for $n \leq -1$, and $H^0(R^1) \cong A$. If not, $H^{-1}(R^1) \neq 0$, we add generators $e_j^{(2)}$ to kill the cohomology, and use the induction to kill all the other cohomologies in the same way. We then get a Tyurina resolution of A (this is a modified Tate construction). Now G acts naturally on R_0, commuting with the differentials in the resolution. So we get a G-invariant Tyurina resolution for A. Similarly we can see that any two G-invariant Tyurina resolutions are G-equivariantly homotopic.

Given a G-invariant Tyurina resolution $R(A)$:

$$\cdots \xrightarrow{s} R_{-2} \xrightarrow{s} R_{-1} \xrightarrow{s} R_0 \xrightarrow{p} A \longrightarrow 0,$$

let G act on $Der(R)$ by $u \mapsto gug^{-1}$. Then $Der_k(A)^G$ is the set of $u : R_n(A) \to R_{n+k}(A)$ such that

$$\begin{array}{ccc} R_n(A) & \xrightarrow{u} & R_{n+k}(A) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ R_n(A) & \xrightarrow{u} & R_{n+k}(A) \end{array}$$

is commutative for all $g \in G$. It is easy to see that $[Der_i(A)^G, Der_j(A)^G] \subset Der_{i+j}(A)^G$ and $dDer_i(A)^G \subset Der_{i+1}(A)^G$, hence $(Der(A)^G, d)$ is a DG Lie subalgebra, define

$$T^{*G}(A) := H^*(Der(A)^G, d) \cong H^*(Der(A), d)^G.$$

It is independent of the choice of the G-invariant Tyurina resolution, hence is an invariant of A. $T^{*G}(A)$ has an induced structure of graded Lie algebra, it is isomorphic to the graded Lie subalgebra $T^*(A)^G$ of $T^*(A)$.

While $T^{0G}(A)$ corresponds to G-invariant derivations of A, $T^{1G}(A)$ and $T^{2G}(A)$ govern the G-equivariant deformations of A, which are deformations $f : S \to \mathcal{A}$ of A admitting G-action on $\mathcal{A}$ which restricts to that on A, and f is G-invariant. Then $S \to \mathcal{A}^G$ is also flat, so it gives us a deformation of A^G. Now if $dim_{\mathbb{C}} T^{1G}(A) < \infty$, there is a formal versal G-equivariant deformation of A.

Donin related the deformations of the germs of complex spaces to the deformation of analytic algebra described above, through Proposition 1.1, 1.2, 1.3 of [14]. In the same fashion, we will relate the V-deformation of the germ of V-spaces to the equivariant deformation of analytic algebras described above.

The definition of V-deformation of a germ of V-spaces is similar to that of a V-space, it is actually simpler, since we need just to study the equivariant deformation of a germ of complex spaces.

Proposition 1. *Let $\pi : \mathcal{X} \to S$ be a morphism of analytic spaces with distinguished points. A finite group G acts on $\mathcal{X}$ analytically, preserving the fibers with respect to S. If $\phi : X_0 = \pi^{-1}(0) \to \mathbb{C}^n$ is an embedding, such that G acts linearly on $\mathbb{C}^n$ and ϕ is G-equivariant. Then ϕ can be extended to an embedding $\Phi : \mathcal{X} \to \mathbb{C}^n \times S$ such that G acts linearly on the first factor of $\mathbb{C}^n \times S$ as above and trivially on the second one. Furthermore, Φ can be taken to be G-equivariant and such that the following diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathbb{C}^n \times S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_2} \\ S & \xrightarrow[id]{} & S \end{array}$$

is commutative, where $p_2 : \mathbb{C}^n \times S \to S$ *is the projection onto the second factor.*

Proof. By Proposition 1.1 of [14], there is an embedding $\psi : \mathcal{X} \to \mathbb{C}^n \times S$ extending $\phi : X_0 \to \mathbb{C}^n$. Let $\psi(x) = (\alpha(x), \pi(x))$, define $\Phi : \mathcal{X} \to \mathbb{C}^n \times S$ by

$$\Phi(x) = \left(\frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha(gx), \pi(x) \right).$$

For any $h \in G$,

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha(ghx) &= \frac{1}{|G|} \sum_{g \in G} h \cdot (gh)^{-1}\alpha(ghx) \\ &= h \cdot \frac{1}{|G|} \sum_{g \in G} g^{-1}\alpha(gx), \end{aligned}$$

hence $\Phi(hx) = h \cdot \Phi(x)$, Φ is an embedding extending ϕ. Q.E.D.

Proposition 2. *Let D be the germ of neighborhoods of zero in $\mathbb{C}^n$, X a germ of complex subspaces, and $\mathcal{X}, \mathcal{X}'$ complex subspaces in $D \times S$ which induces X in the distinguished fiber $D \times o$. A finite group G acts analytically on $X, \mathcal{X}, \mathcal{X}'$, and linearly on $\mathbb{C}^n$, such that the inclusions $X \hookrightarrow \mathcal{X}(\mathcal{X}') \hookrightarrow D \times S$ are G-equivariant. Then each fiber-preserving G-equivariant morphism $\psi : \mathcal{X} \to \mathcal{X}'$ which is identity on X can be extended to a fiber-preserving G-equivariant map $\Psi : D \times S \to D \times S$ which is identity on $D \times o$.*

Proof. By Proposition 1.2 of [14], there exist a fiber-preserving map $\beta : D \times S \to D \times S$ extending ψ. Let

$$\Psi(v, s) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\phi(gv, s),$$

then for any $h \in G$, we have

$$\begin{aligned}\Psi(hv,s) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi(ghv,s)\\ &= \frac{1}{|G|}\sum_{g\in G} h(gh)^{-1}\phi(ghv,s) = h\cdot\Psi(v,s),\end{aligned}$$

then Ψ is the G-equivariant map extending ϕ. Q.E.D.

Now by [11], a germ of complex space X with finite group action by G can be identified with a G-invariant germ of complex subspaces in the germ D of neighborhoods of zero in $\mathbb{C}^n$ acting linearly by G. We fix such an inclusion. Then any G-equivariant deformation of X over S can be considered as a G-invariant complex subspace of $D \times S$ invariant under the natural G-action on $D \times S$, such the distinguished fiber in $D \times o$ is identified with X together with the G-action, and the natural projection $\mathcal{X} \hookrightarrow D \times S \to S$ is flat. If $\mathcal{X}'$ is another such equivariant deformation, then $\mathcal{X}$ and $\mathcal{X}'$ are equivalent as equivariant deformations if and only if there exists a fiber-preserving equivariant isomorphism $D \times S \to D \times S$ which is an identity in the distinguished fiber and induces an isomorphism of $\mathcal{X}$ and $\mathcal{X}'$. This follows from Proposition 1 and Proposition 2.

Similar to the construction of G-invariant Tyurina resolution of an analytic algebra, we can construct a G-invariant Tyurina resolution of $\mathcal{O}_X$ over D

$$\mathcal{R} : \cdots \to \mathcal{R}_{-2} \to \mathcal{R}_{-1} \to \mathcal{R}_0 \to \mathcal{O}_X \to 0.$$

Using Proposition 1.3 of [14] and a similar averaging process, we can prove

Proposition 3. *a) Let $\pi : \mathcal{X} \to S$ be a G-equivariant deformation of the germ X given in $D \times S$, then the G-invariant resolution $\mathcal{R}$ of $\mathcal{O}_X$ can be extended to a G-invariant free resolution of $\mathcal{O}_{\mathcal{X}}$ over $D \times S$.*

b) Conversely, if one is given a germ S and a G-invariant complex of free sheaves $\mathcal{C}$ over $D \times S$ which restricts to $\mathcal{R}$ with the group action, then $\mathcal{C}$ is a free resolution of the structure sheaf of some G-equivariant deformation of the germ X with base S.

From the above propositions, it is easy to see that a G-equivariant deformation of the germ X can be considered as G-invariant deformation of a G-invariant Tyurina resolution of its structure sheaf.

The privileged neighborhood theorem [17] gives us a way to get the required Banach space structures on $Der(\mathcal{R})^G$ and the splittings. Applying the general machinery, we get the following

Theorem. *Let X be a germ of complex spaces, invariant under a finite group action by G, such that $l = dim_{\mathbf{C}} T^{1G}(X) < \infty$, where $T^{1G}(X) = T^{1G}(A)$, A is the analytic algebra corresponding to X. Then there exists a versal effective family of G-equivariant deformation of X:*

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ o & \longrightarrow & S \end{array}$$

such that $dim_{\mathbf{C}} T_o S = l$.

3.4 Global Theory

V-Polyhedral Covering

Let X be V-analytic space, $\pi_U : (G_U, \tilde{U}) \to U$ a uniformization chart of X, such that there is a G_U-equivariant embedding $\phi : \tilde{U} \to \mathbb{C}^n$ with values in a neighborhood V of the unit polydisc D^n in $\mathbb{C}^n$, where G_U acts linearly on $\mathbb{C}^n$ preserving D^n and V. Set $\tilde{P} = \phi^{-1}(D^n), P = \pi_U(\tilde{P})$. P is then called a V-polyhedron in X.

By a result of H. Cartan [11], any analytic finite group action can be locally linearized. If the finite group is Abelian, we can assume the action is given by diagonal matrices, hence preserves the unit polydisc. Hence if G_x is Abelian for any $x \in X$, X is covered by V-polyhedra.

In general, given any covering $\mathcal{U} = \{U_\alpha | \alpha \in \mathcal{A}\}$ of a topological space X, the nerve $\mathcal{N}$ of $\mathcal{U}$ is defined to be the set of finite subsets $A \subset \mathcal{A}$ such that $U_A := \cap_{\alpha \in A} U_\alpha$ is not empty. $\mathcal{N}$ has the structure of a simplicial complex, with A above as a simplex of dimension $|A| - 1$, where $|A|$ is the number of elements in A. Inclusions of simplices are clearly induced by the inclusions of finite subsets of $\mathcal{A}$. A simplicial complex can be considered as a category, whose objects are the simplices, and morphisms are the inclusions of simplices.

Now let $\mathcal{P} = \{P_\alpha | \alpha \in \mathcal{A}\}$ be a covering of a V-analytic space X by V-polyhedra $P_\alpha = \phi_\alpha^{-1}(D^{n_\alpha})$ given by uniformization charts $\pi_\alpha : (G_\alpha, \tilde{U}_\alpha) \to U_\alpha$ and G_α-equivariant embedding $\phi_\alpha : \tilde{U}_\alpha \to V_\alpha \subset \mathbb{C}^{n_\alpha}$ with G_α acting linearly on $\mathbb{C}^{n_\alpha}$ preserving the unit polydisc D^{n_α} and its neighborhood V_α. Consider the nerve $\mathcal{N}$ of $\mathcal{P}$. For $A \in \mathcal{N}$, set $P_A = \cap_{\alpha \in A} P_\alpha, U_A = \cap_{\alpha \in A} U_\alpha, n_A = \sum_{\alpha \in A} n_\alpha$.

Taking U_α small enough, we can assume that there is a uniformization chart $\pi_A : (G_A, \tilde{U}_A) \to U_A$, then the composition of maps

$$\phi_A : \tilde{U}_A \to \prod_{\alpha\in A} \tilde{U}_\alpha \xrightarrow{\prod \phi_\alpha} \prod_{\alpha\in A} \mathbb{C}^{n_\alpha} \cong \mathbb{C}^{n_A}$$

is a G_A-equivariant embedding of $\tilde{U}_A$ into a G_A-invariant neighborhood of D^{n_A} in $\mathbb{C}^{n_A}$, and this gives P_A the structure of a V-polyhedron.

Definition. A *V-polyhedral covering* of a V-analytic space X is a covering $\mathcal{P} = \{P_\alpha\}$ by V-polyhedra such that for any $A \in \mathcal{N}$ nerve of $\mathcal{P}$, P_A has the structure of V-polyhedron as above.

Remark. A V-polyhedral covering is clearly a functor from the category defined by $\mathcal{N}$ to the category of V-polyhedra of X.

A V-analytic space X with G_x Abelian for all $x \in X$ admits a V-polyhedral covering.

V-Resolutions

Let $\mathcal{N}$ be the nerve of a covering viewed as a category, a covariant functor $F : \mathcal{N} \to \mathcal{F}$ is by definition the following assignments: to each $A \in \mathcal{N}$, an object $F(A) \in \mathcal{F}$; to each inclusion $A \hookrightarrow B$, a morphism $F_A^B : F(A) \to F(B)$ such that $F_A^A = id$ and if $A, B, C \in \mathcal{N}, A \hookrightarrow B \hookrightarrow C$, $f_B^C \circ f_A^B = f_A^C$. Contravariant functors can be defined in a similar way.

Let G acts linearly on $\mathbb{C}^n$, $\tilde{Z}$ a G-invariant closed analytic subspace in a G-invariant open set $U \subset \mathbb{C}^n$, $Z = \tilde{Z}/G$ is called a model V-analytic space. We have an epimorphism of sheaves of analytic algebras: $\mathcal{O}_{\mathbb{C}^n}|U \to \mathcal{O}_{\tilde{Z}}$. Set

$\mathcal{R}_0 = \mathcal{O}_{\mathbb{C}^n}|U$, a complex of $\mathcal{R}_0$-sheaves:

$$\cdots \to \mathcal{R}_{-2} \xrightarrow{s} \mathcal{R}_{-1} \xrightarrow{s} \mathcal{R}_0$$

on U is called a *V-resolution* of the model V-space X if it has the following properties:

1. it gives a Tyurina resolution of $\mathcal{O}_{\tilde{Z},\tilde{x}}$ when restricted to each $\tilde{x} \in \tilde{Z}$;

2. there is a subset e of sections over U of $\mathcal{R}_*$ which defines in each fiber a distinguished set of generators for the resolution;

3. the G-action on $\mathcal{R}_0$ induced from the action of G on $\mathbb{C}^n$ can be extended as a group of automorphisms of sheaf of differential graded algebras fixing e.

For each model complex V-space we can construct a V-resolution for the open V-subspace $Z \cap (U'/G)$ if $\bar{U}'$ is compact in U. We can prove this by repeating the construction of invariant Tyurina resolution in the preceding section with obvious changes.

The V-resolution of model V-spaces form a category: if X_1, X_2 are two model V-spaces given by $X_i = \tilde{X}_i/G_i$, for $\tilde{X}_i \subset U_i \subset \mathbb{C}^{n_i}$, and G_i acts on $\mathbb{C}^{n_i}$ linearly preserving U_i and $\tilde{X}_i$, and $\mathcal{R}^{(i)}$ are G_i-invariant two Tyurina resolutions with generator sets e_i fixed by G_i. Then a morphism consists of the following:

1. an injective group homomorphism $h: G_2 \to G_1$;

2. a coordinate projection $p: \mathbb{C}^{n_2} \to \mathbb{C}^{n_1}$, G_2-equivariant with respect to h;

3. a G_2-equivariant map of sheaves $\psi : p^*(\mathcal{R}^{(1)}) \to \mathcal{R}^{(2)}$ such that ψ commutes with the differentials, $\psi(e_1) \subset e_2$ (ψ maps generators to generators, hence preserves the degree) and $\psi_0 : p^*(\mathcal{R}_0^{(1)}) \to \mathcal{R}_0^{(2)}$ is induced by $p^* : \mathcal{O}_{\mathbb{C}^{n_1}} \to \mathcal{O}_{\mathbb{C}^{n_2}}$.

Definition. Given a V-polyhedral covering $\mathcal{P} = \{P_\alpha\}$ of a V-space X with nerve $\mathcal{N}$, a *V-resolution* $\mathcal{R}$ of X on $\mathcal{P}$ is a contravariant functor from $\mathcal{N}$ to the category described above. If $\mathcal{R}$ and $\mathcal{R}'$ are two V-resolutions on $\mathcal{P}$, then in degree zero, they coincide. A *morphism* from $\mathcal{R}$ to $\mathcal{R}'$ is a covariant functor from $\mathcal{N}$ to the category of sheaves of graded differential algebras with symmetry that is identity at degree 0.

Proposition. *For any complex V-space X with polyhedral covering $\mathcal{P}$, there is a V-resolution $\mathcal{R}$ on $\mathcal{P}$. If $\mathcal{R}$ and $\mathcal{R}'$ are two V-resolutions on $\mathcal{P}$, then there is at least one morphism $m : \mathcal{R}' \to \mathcal{R}$.*

This can be proved by induction on $dim A$ for $A \in \mathcal{N}$ as in the proof of the corresponding results for ordinary analytic spaces ([50], th. 2.1, th. 2.2).

Relative V-Resolution

Let $p : \mathcal{X} \to S$ be a V-deformation of a V-space over an ordinary analytic space S. Let $\varpi : (G, \tilde{\mathcal{U}}) \to \tilde{U}$ be a chart on $\mathcal{X}$, and $\Phi : \tilde{\mathcal{U}} \to \mathbb{C}^n \times S$ a G-equivariant embedding, where G acts linearly on $\mathbb{C}^n$ and trivially on S. If G preserves the unit polydisc D^n, $\varpi(\Phi^{-1}(D^n \times S))$ is called a relative V-polyhedron in $\mathcal{X}$.

A relative V-polyhedral covering $\mathcal{P}$ of $\mathcal{X}$ and its nerve $\mathcal{N}$ can be defined similar to the absolute case. We note that if $o \in S$ is a distinguished point, $\mathcal{P}_0 = \mathcal{P} \cap X_0$ is a V-polyhedral covering of $X_0 = p^{-1}(o)$, we call $\mathcal{P}_0$ a V-polyhedron of X_0 extendable in $\mathcal{X}$. Use proposition 1 of the preceding section, it is easy to prove the following

Proposition. *Let $p : \mathcal{X} \to S$ be a V-deformation of $X_0 = p^{-1}(o)$ for a distinguished point $o \in S$. Then any V-polyhedral covering of X_0 is extendable in $\mathcal{X}$.*

We can define a *relative resolution* of $\mathcal{X}$ similar the absolute one but with the following modifications: for any $A \in \mathcal{N}$, $\mathbb{C}^{n_A}$ is replaced by $\mathbb{C}^{n_A} \times S$, $\mathcal{O}_{\mathbb{C}^{n_A}}$ by $\mathcal{O}_{\mathbb{C}^{n_A} \times Y}$, $D_A = D^{n_A}$ by $D^{n_A} \times Y$. Let $\mathcal{P}$ be a relative V-polyhedral covering on $\mathcal{X}$, $\mathcal{P}_0$ the induced V-polyhedral covering on X_0, and $\mathcal{R}$ a relative V-resolution on $\mathcal{P}$, then $\mathcal{R}_0 = \mathcal{R}|_{X_0}$ is a resolution of X_0 on $\mathcal{P}_0$. $\mathcal{R}_0$ is called an extendable V-resolution.

Proposition. *Let $p : \mathcal{X} \to S$ be a V-deformation of $X_0 = p^{-1}(o)$ for a distinguished point $o \in S$, $\mathcal{P}$ a relative V-polyhedral covering of $\mathcal{X}$ which restrict to a V-polyhedral covering $\mathcal{P}_0$ of X_0. Then every V-resolution of X_0 on $\mathcal{P}_0$ can be extended to a relative V-relative resolution of $\mathcal{X}$ on $\mathcal{P}$.*

V-Tangent Sheaves, Complex and Cohomology

Let X be a V-analytic space, $\mathcal{P}$ a V-polyhedral covering of X with nerve $\mathcal{N}$ and $\mathcal{R}$ a V-resolution on $\mathcal{P}$. For each k, $\mathcal{R}$ defines a sheaf τ_X^{kV} whose stalk at each point $x \in X$ is given by $T^k(\mathcal{O}_{\tilde{U},0})^{G_x}$, where $\pi_x : (G_x, \tilde{U}) \to U$ is a chart centered at x. $\tau_X^{*V} = \oplus \tau_X^{kV}$ is a sheaf of graded Lie algebras. Since $T^*(A)^G$ with its graded Lie algebra structure for a G-invariant analytic algebra A is independent of the choice of the G-invariant Tyurina resolution, τ_X^{*V} is independent of the choice of the V-resolution $\mathcal{R}$ and the V-polyhedral covering $\mathcal{P}$, we call it the *V-tangent sheaf* of the V-space X.

A derivation of $\mathcal{R}$ of degree k is given by $u = \{u_A\}_{A \in \mathcal{N}}$ such that for each $A \in \mathcal{N}$, $u_A : \mathcal{R}(A) \to \mathcal{R}(A)$ is G_A-equivariant mapping of degree k of graded sheaves, and commutes with the differentials s_A, (we write $u_A \in Der_k(\mathcal{R}(A))^{G_A}$, and for $A \subset B, A, B \in \mathcal{N}$, $u_B \circ r_A^B = r_A^B \circ (p_A^B)^* u_A$. Let $T^{*V}(\mathcal{R})$ be the set of all derivations of $\mathcal{R}$, it is then a vector space graded by degrees. The collection of differentials $s = \{s_A\}$ is by definition an element of $T^{1V}(\mathcal{R})$.

If u, v are two derivations, then it is easy to define their composition $u \circ v$ by $u_A \circ v_A$. Set $[u, v] = u \circ v - (-1)^{degu \cdot degv} v \circ u$. Then

$$[u, v] = -(-1)^{degu \cdot degv}[v, u],$$

$$[u, [v, w]] = [[u, v], w] + (-1)^{degu \cdot degv}[v, [u, w]].$$

Hence $T^{*V}(\mathcal{R})$ is a graded Lie algebra. Let $d = [s, \cdot]$, then $d^2 = 0$ and

$$d[u, v] = [du, v] + (-1)^{degu}[u, dv].$$

So $(T^{*V}(\mathcal{R}), d, [\cdot, \cdot])$ is a graded differential Lie algebra. It is called the *V-tangent complex* of X defined by $\mathcal{R}$.

Let $T_X^{*V} = H^*(T^{*V}(\mathcal{R}), d)$. It is a general fact that the cohomology of a graded differential Lie algebra has an induced structure of graded Lie algebra. It can be checked that T_X^{*V} with its induced structure of graded Lie algebra is independent of the choice of the V-resolution $\mathcal{R}$ and the V-polyhedral covering $\mathcal{P}$.

In general, a double complex leads to two spectral sequences. Consider the double complex given by $\{Der(\mathcal{R}(A))^{G_A}\}$, we get a spectral sequence

$$E_2^{p,q} = H^p(X, \tau_X^{qV}) \Rightarrow T_X^{*V}.$$

This can be used to compute the global V-tangent cohomology from the local V-tangent sheaves. In particular, we have $T_X^{0V} = H^0(X, \tau_X^{0V})$ and the following exact sequence

$$\begin{aligned} 0 \to H^1(X, \tau^{0V}) &\to T_X^{1V} \to H^0(X, \tau^{1V}) \to H^2(X, \tau^{0V}) \\ &\to T_X^{2V} \to H^1(X, \tau^{1V}) \oplus H^0(X, \tau^{2V}). \end{aligned}$$

Relative V-Tangent Complex

Let $p : \mathcal{X} \to S$ be a V-deformation of $X_0 = p^{-1}(o)$ for $o \in S$, $\mathcal{P}$ a relative V-polyhedral covering of $\mathcal{X}$ which restrict to a V-polyhedral covering $\mathcal{P}_0$ on X_0, $\mathcal{R}$ a relative V-resolution of $\mathcal{X}$ on $\mathcal{P}$ which restricts to a relative V-resolution $\mathcal{R}_0$ of X_0 on $\mathcal{P}_0$. $\mathcal{R}$ then defines a relative V-tangent sheaf $\tau_{\mathcal{X}/S}^{*V}$ on $\mathcal{X}$, it is independent of the choice of $\mathcal{R}$ and $\mathcal{P}$. It restricts to $\tau_{X_0}^{*V}$ on X_0.

For any open set U of S, put $X_U = p^{-1}(U)$, consider the relative V-resolution $\mathcal{R}_U = \mathcal{R} \otimes \mathcal{O}_U$ of X_U. Define $T^{*V}(\mathcal{R}_U)$ to be the graded vector space of derivations of $\mathcal{R}_U$ which commutes with multiplications by elements in $\Gamma(U, \mathcal{O}_S)$. Define the differential d and $[\cdot,\cdot]$ in $T^{*V}(\mathcal{R}_U)$ similar to the absolute case. Thus $(T^{*V}(\mathcal{R}_U), d, [\cdot,\cdot])$ is DG $\Gamma(U, \mathcal{O}_S)$-Lie algebra. Consider the germ of neighborhoods of $o \in S$, we get a deformation of $T^{*V}_{X_0}$ over $\mathcal{O}_{S,o}$. Conversely, such a deformation induces a V-deformation of X_0 over S. So the V-deformation of the V-space X_0 corresponds to a deformation of its V-tangent complex, which is a DG Lie-algebra. So we can use the general machinery. In particular, a V-deformation defines a *V-Kodaira-Spencer map* . The deformation theory of DG Lie algebra enables us to construct a deformation of $T^{*V}_{X_0}$ over a complete algebra $\hat{A}$, which is formally versal. Forster-Knorr's method method can be used to prove that $\hat{A}$ is the completion if an analytic algebra A. Let S be the analytic space corresponding to A, then we get a versal V-deformation of X_0 over S. So we have

Theorem. *Every compact V-space X has a versal V-deformation $p : \mathcal{X} \to (S, o)$ such that the V-Kodaira-Spencer map $T_oS \to T^{1V}_X$ is an isomorphism. In particular, when $T^{2V}_X = 0$, S can be taken to be the germ of the neighborhood of 0 in T^{1V}_X.*

3.5 Deformations of Relative V-Spaces and V-Maps

It is easy to generalize the above discussion to the cases of relative V-spaces and V-maps. They are similar to the generalization of ordinary compact analytic spaces to the deformations of relative spaces and analytic maps [50].

3.5.1 V-Deformations of Relative V-Spaces

By a *relative V-space* we mean a V-space X together with a V-map f into a fixed V-space Y. We denote it by $X/_VY$. A V-deformation of $X/_VY$ with base (S, o) is a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F} & Y \times S \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr} \\ S & \xrightarrow[id]{} & S, \end{array}$$

where $p : \mathcal{X} \to S$ is a V-deformation, together with an isomorphism $i : X \to p^{-1}(o)$ such that $F \circ i = f$. Isomorphism of V-deformations and V-deformation induced by an analytic map $\phi : T \to S$ can be defined in the natural way.

Two particular cases are worth mentioning. The first is the case where Y is a point, which i just the case of the (absolute) V-deformation. The second is the case where $f : X \to Y$ is an embedding, which gives us the deformation of a V-subspace of an fixed V-space.

The V-deformation theory of relative V-spaces can be described by the relative V-tangent cohomology $T^{*V}_{X/_VY}$, which can be defined using relative V-polyhedral covering. It can be computed by the following long exact sequence

$$0 \to T^{0V}_{X/_VY} \to T^{0V}_{X} \to T^{0V}_{Y,\mathcal{O}_X} \to T^{1V}_{X/_VY} \to \cdots,$$

where $T^{*V}_{Y,\mathcal{O}_X}$ is the V-tangent cohomology of Y with coefficients in the $\mathcal{O}_Y$-sheaf $\mathcal{O}_X$.

3.5.2 V-Deformations of V-Maps

Let $f : X \to Y$ be a V-map between two compact V-spaces X and Y. A *V-deformation of the V-map* f over a base consists of a triple consisting of a V-deformation $p : \mathcal{X} \to S$ with isomorphism $i : X \to p^{-1}(o)$, a V-deformation $q : \mathcal{Y} \to S$ with isomorphism $j : Y \to q^{-1}(o)$, and a V-map $F : \mathcal{X} \to \mathcal{Y}$ such that $qF = p$ and $Fi = jf$, i.e., we have the following commutative diagrams:

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathcal{X} & \xrightarrow{p} & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle id} \\ Y & \xrightarrow[j]{} & \mathcal{Y} & \xrightarrow[q]{} & S. \end{array}$$

The other concepts in a deformation theory such as isomorphisms of V-deformations can be defined in the natural way.

The V-deformation theory of V-,aps can be described by the V-tangent cohomology T_f^{*V} of the V-map f. It can be computed by the following long exact sequence:

$$0 \to T^{0v}_{X/_V Y} \to T^{0v}_f \to T^{0V}_Y \to T^{1V}_{X/_V Y} \to \cdots .$$

This long exact sequence is related to the long exact sequence for relative space by the following commutative diagrams:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & T^{iv}_{X/_V Y} & \longrightarrow & T^{iv}_f & \longrightarrow & T^{iV}_Y & \longrightarrow & T^{(i+1)V}_{X/_V Y} & \longrightarrow & \cdots \\ & & {\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\ \cdots & \longrightarrow & T^{iV}_{X/_V Y} & \longrightarrow & T^{0V}_X & \longrightarrow & T^{0V}_{Y,\mathcal{O}_X} & \longrightarrow & T^{1V}_{X/_V Y} & \longrightarrow & \cdots, \end{array}$$

where the horizontal lines are long exact sequences. It is an easy exercise in diagram chasing to prove the following

Lemma. *If we have the following commutative diagrams*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^i & \longrightarrow & B^i & \longrightarrow & C^i & \longrightarrow & A^{i+1} & \longrightarrow & \cdots \\ & & {\scriptstyle id}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle id}\downarrow & & \\ \cdots & \longrightarrow & A^i & \longrightarrow & D^i & \longrightarrow & E^i & \longrightarrow & A^{i+1} & \longrightarrow & \cdots. \end{array}$$

Then we have an induced long exact sequence

$$\cdots \to B^i \to C^i \oplus D^i \to E^i \to B^{i+1} \to \cdots.$$

Hence we have a long exact sequence

$$\cdots \to T_f^{iV} \to T_X^{iV} \oplus T_Y^{iV} \to T_{Y,\mathcal{O}_X}^{iV} \to T_f^{(i+1)V} \to \cdots.$$

It is the generalization of the corresponding long exact sequence in [54].

Chapter 4

Connected Sums of Self-Dual Orbifolds

The V-deformation theory of the last chapter is used to study the connected sums of self-dual orbifolds. We generalize Friedman's deformation theory [23] of spaces with normal crossing singularities in two different ways.

4.1 Type I connected Sum

V-Space with Normal Crossing Singularity

Assume that X_1, X_2 are two V-spaces with Abelian local uniformization groups, and that for $i = 1, 2$, X_i are smooth near smooth hypersurfaces D_i, such that there is an isomorphism $f : D_1 \to D_2$. Let $X_0 = X_1 \cup_f X_2$, $f_i : X_i \to X_0$ the natural inclusions, and D the common image of D_i in X_0. We call X_0 a V-space with normal crossing singularity at D. Let $i : D \to X_0$ be the inclusion, $q : X' = X_1 \coprod X_2 \to X_0$ the natural map formed from f_1 and f_2, and ν_i the normal bundles of D_i in X_i. It is easy to see that

$$\tau_{X_0}^{1V} = f_{1*}(\tau_{X_1}^{1V}) \oplus f_{2*}(\tau_{X_2}^{1V}) \oplus i_*(\nu_1 \otimes \nu_2),$$

$$\tau_{X_0}^{2V} = f_{1*}(\tau_{X_1}^{2V}) \oplus f_{2*}(\tau_{X_2}^{2V}),$$

and there is an exact sequence of sheaves on X_0:

$$0 \to \tau_{X_0}^{0V} \to q_*\tau_{X',D_1\cup D_2}^{0V} \to i_*\tau_D^0 \to 0,$$

where $\tau_{X',D_1\cup D_2}^{0V}$ is the sheaf of V-tangent vector fields tangent to D_1 and D_2. Since X_i is smooth near D_i, the support of $i_*(\nu_1\otimes\nu_2)$ is away from the supports of $f_{i*}(\tau_{X_i}^{1V})$. So from the exact sequence for a general V-space X:

$$\begin{aligned} 0 \quad &\to \quad H^1(X,\tau_X^{0V}) \to T_X^{1V} \to H^0(X,\tau_X^{1V}) \to H^2(X,\tau_X^{0V}) \\ &\to \quad T_X^{2V} \to H^1(X,\tau_X^{1V}) \oplus H^0(X,\tau_X^{2V}), \end{aligned}$$

we get the following exact sequence:

$$\begin{aligned} 0 \quad &\to \quad H^1(X_0,\tau_{X_0}^{0V}) \to T_{X_0}^{1V} \\ &\to \quad H^0(X_1,\tau_{X_1}^{1V}) \oplus H^0(X_2,\tau_{X_2}^{1V}) \oplus H^0(D,\nu_1\otimes\nu_2) \to H^2(X_0,\tau_{X_0}^{0V}) \\ &\to \quad T_{X_0}^{2V} \to H^1(X_1,\tau_{X_1}^{1V}) \oplus H^1(X_2,\tau_{X_2}^{1V}) \oplus H^1(D,\nu_1\otimes\nu_2) \\ &\qquad\qquad\qquad \oplus H^0(X_1,\tau_{X_1}^{2V}) \oplus H^0(X_2,\tau_{X_2}^{2V}). \end{aligned}$$

We call D a *semistable normal crossing singularity* of X_0 if $\nu_1\otimes\nu_2 \cong \mathcal{O}_D$.

Proposition. *Let X_0 be as above and semi-stable, $\pi : \mathcal{X} \to \Delta(\subset \mathbb{C})$ a V-deformation of X_0 such that the V-Kodaira-Spencer class $\rho^V(\partial/\partial t)$ generates $\nu_1\otimes\nu_2 \cong \mathcal{O}_D$ at all points of D. Then $\mathcal{X}$ is smooth near D.*

Since the proof of the corresponding result in the ordinary analytic space context by Friedman (Proposition (2.5) of [23]) is of local nature, it also proves the above proposition.

Application to Type I Connected Sum

We can now apply the above results to study the type I connected sum of two self-dual orbifolds with only Abelian local uniformization groups. Recall that if X_1, X_2 are two orbifolds, $x_1 \in X_1, x_2 \in X_2$ two smooth points, then the connected sum across x_1, x_2 is called a type I connected sum of X_1 and X_2 and we denote it by $X_1 \# X_2$. If for $i = 1, 2$, X_i are self-dual orbifolds, let $\pi_i : Z_i \to X_i$ be its twistor space, $L_i = \pi_i^{-1}(x_i)$. Blow up Z_i along L_i to get $\tilde{Z}_i$ and the exceptional divisor Q_i. Then $\tilde{Z}_i$ are complex orbifolds smooth near Q_i. Now an orientation reversing isometry $\phi : T_{x_1}X_1 \to T_{x_2}X_2$ induces an isomorphism $f : Q_1 \to Q_2$. Then $Z_0 = \tilde{Z}_1 \cup_f \tilde{Z}_2$ is a V-space with normal crossing singularity along Q, which is the common image of Q_1, Q_2 in Z_0. Following Donaldson-Friedman [15], we make the following

Definition. A *standard I deformation* of Z_0 consists of the following:

1. a V-deformation $\pi : \mathcal{Z} \to S$, where S is a neighborhood of 0 in $\mathbb{C}^n$ and $\pi^{-1}(0) \cong Z_0$;

2. antiholomorphic involutions σ on $\mathcal{Z}$ and τ on S compatible under p such that $\sigma|_{\pi^{-1}(0)}$ is the natural one on Z_0.

In addition, we assume that near any point of $Q \subset Z_0 \subset \mathcal{Z}$, there are local coordinates $z_1, z_2, z_3, z_4, t_2, \cdots, t_n$ on $\mathcal{Z}$ such that $\pi(z_1, z_2, z_3, z_4, t_2, \cdots, t_n) = (z_1 z_2, t_2, \cdots, t_n)$.

Theorem. *If $\pi : \mathcal{Z} \to S$ is a standard I deformation of a V-space Z_0 as above, then for sufficiently small vector s in the fixed locus S^σ of σ, not lying* *in the hyperplane* $\{t_1 = 0\}$, $\pi^{-1}(s)$ *is the twistor space of a self-dual conformal structure on the orbifold* $X_1 \# X_2$.

Proof. The proof of Theorem 4.1 of [15] works with the following modification: near the fibers coming from orbifold points, Z_0 is locally the quotient of a twistor space by finite group action, since $\mathcal{Z}$ is a V-deformation of Z_0, these twistor fibers have neighborhoods in $\mathcal{Z}$ which are quotient by finite group actions. Q.E.D.

To study the existence of standard deformation of $Z_0 = \tilde{Z}_1 \cup_f \tilde{Z}_2$, notice first that since $\tilde{Z}_i$ are orbifolds, $\tau^{1V}_{\tilde{Z}_i} = \tau^{2V}_{\tilde{Z}_i} = 0$, for $i = 1, 2$, so the exact sequence gives

$$\begin{aligned} 0 \to H^1(Z_0, \tau^{0V}_{Z_0}) &\to T^{1V}_{Z_0} \to H^0(Q, \mathcal{O}_Q) = \mathbb{C} \to H^2(Z_0, \tau^{0V}_{Z_0}) \\ &\to T^{2V}_{Z_0} \to H^1(Q, \mathcal{O}_Q) = 0. \end{aligned}$$

So we need to compute $H^i(Z_0, \tau^{0V}_{Z_0})$. Since $\tilde{Z}_i$ is smooth near Q_i, a similar computation as in [15] yields the following

Proposition. *For* $Z_0 = \tilde{Z}_1 \cup_f \tilde{Z}_2$ *as above, we have*

$$H^2(Z_0, \tau^{0V}_{Z_0}) = H^2(Z_1, \tau^{0V}_{Z_1}) \oplus H^2(Z_2, \tau^{0V}_{Z_2}),$$

and an exact sequence:

$$\begin{aligned} 0 &\to T^{0V}_{Z_0} \to H^0(Z_1, \tau^{0V}_{Z_1}) \oplus H^0(Z_2, \tau^{0V}_{Z_2}) \to H^0(Q, TQ) \\ &\to H^1(Z_0, \tau^{0V}_{Z_0}) \to H^1(Z_1, \tau^{0V}_{Z_1}) \oplus H^1(Z_0, \tau^{0V}_{Z_0}) \to 0. \end{aligned}$$

Hence if $H^2(Z_1, \tau_{Z_1}^{0V}) = H^2(Z_2, \tau_{Z_2}^{0V}) = 0$, we have $H^2(Z_0, \tau_{Z_0}^{0V}) = 0$, and the following exact sequence:

$$T^{1V}(Z_0) \to H^0(Q, \mathcal{O}_Q) = \mathbb{C} \to 0 \to T^{2V}(Z_0) \to 0,$$

So $T^{2V}(Z_0) = 0$, therefore every direction in $T^{1V}(Z_0)$ can be realized by a V-deformation of Z_0. In particular, since $T^{1V}(Z_0) \to H^0(Q, \mathcal{O}_Q) = \mathbb{C}$ is now surjective, we can choose a direction in $T^{1V}(Z_0)$ whose restriction to Q is nowhere zero, any one-dimensional V-deformation having it as V-Kodaira-Spencer class then gives a smoothing of Z_0 near Q. To take care of the real structure, notice that the trick in Section 6.1 of [15] can be extended to the V-space case. Hence we have the following

Theorem. *For any two compact self-dual orbifolds X_1, X_2 with only Abelian local uniformization group, let Z_1 and Z_2 be their twistor spaces respectively, and x_1, x_2 two smooth points of X_1, X_2 respectively, construct $Z_0 = \tilde{Z}_1 \cup_f \tilde{Z}_2$ as above. If $H^2(Z_1, \tau_{Z_1}^{0V}) = H^2(Z_2, \tau_{Z_2}^{0V}) = 0$, then Z_0 admits a standard I deformation, hence the type I connected sum of X_1 and X_2 admits a self-dual orbifold metric.*

4.2 Type II Connected Sum

V-Normal Crossing Singularity

Now let Y_1, Y_2 be two compact V-spaces, again with only Abelian local uniformization groups, $E_i \subset Y_i$ compact subset, such that there are uniformization charts $\pi_i : \tilde{U}_i \to U_i \cong \tilde{U}_i/G$ near E_i, where $\tilde{U}_i$ are smooth and the finite group G acts on $\tilde{U}_i$ and freely outside the smooth hypersurfaces $\tilde{E}_i = \pi_i^{-1}(E_i)$. If there is a G-equivariant isomorphism $\tilde{\phi} : \tilde{E}_1 \to \tilde{E}_2$ which induces an isomorphism $\phi : E_1 \to E_2$, then $Y_0 = Y_1 \cup_\phi Y_2$ is also a V-space. Let $h_i : Y_i \to Y_0$ be the natural inclusion, and E the image of E_i, $p : Y' = Y_1 \coprod Y_2 \to Y_0$ the obvious map formed from h_1 and h_2, $j : E \to Y_0$ the inclusion. We call Y_0 a V-space with V-normal singularity at E. If $\tilde{\nu}_i$ is the normal bundle of $\tilde{E}_i$ in $\tilde{U}_i$, then G acts holomorphically on $\tilde{\nu}_i \to \tilde{E}_i$. If we identify $\tilde{E}_1, \tilde{E}_2$ with their common image in $\tilde{U}_1 \cup_{\tilde{\phi}} \tilde{U}_2$ and regard $\tilde{\nu}_1, \tilde{\nu}_2$ as line bundle on $\tilde{E}$, then G acts holomorphically on $\tilde{\nu}_1 \otimes \tilde{\nu}_2$.

In general, assume that G is a finite group acting on X, $\pi : X \to X/G$ be the natural projection, and $\mathcal{F}$ is a sheaf on X, such that there is a G-action on $\mathcal{F}$ covering that on X. Then the assignment that for any open set $U \subset X/G$, gives $U \mapsto \Gamma(\pi^{-1}(U), \mathcal{F})^G$ is a presheaf. It induces a sheaf on X/G which we denote by $\pi_*^G \mathcal{F}$. Now let $\pi : \tilde{U}_1 \cup_{\tilde{\phi}} \tilde{U}_2 \to U_1 \cup_\phi U_2$ be the natural map constructed by gluing π_1 and π_2, and let $\mathcal{E} = \pi_*^G(\tilde{\nu}_1 \otimes \tilde{\nu}_2)$, then it is easy to see that:

$$\tau_{Y_0}^{1V} = h_{1*}\tau_{Y_1}^{1V} \oplus h_{2*}\tau_{Y_2}^{1V} \oplus \mathcal{E},$$

$$\tau_{Y_0}^{2V} = h_{1*}\tau_{Y_1}^{2V} \oplus h_{2*}\tau_{Y_2}^{2V},$$

and we have an exact sequence:

$$0 \to \tau_{Y_0}^{0} \to p_*\tau_{Y', E_1 \cup E_2}^{0V} \to j_*\tau_E^{0V} \to 0.$$

Hence we have the following exact sequence:

$$0 \;\to\; H^1(Y_0, \tau_{Y_0}^{0V}) \to T_{Y_0}^{1V} \to H^0(Y_1, \tau_{Y_1}^{1V}) \oplus H^0(Y_2, \tau_{Y_2}^{1V}) \oplus H^0(E, \mathcal{E})$$

$$\rightarrow \quad H^2(Y_0, \tau^{0V}_{Y_0}) \rightarrow T^{2V}_{Y_0}$$

$$\rightarrow \quad H^1(Y_1, \tau^{1V}_{Y_1}) \oplus H^1(Y_2, \tau^{1V}_{Y_2}) \oplus H^1(E, \mathcal{E}) \oplus H^0(Y_0, \tau^{2V}_{Y_0}).$$

In particular, we have a natural map: $T^{1V}(Y_0) \rightarrow H^0(E, \mathcal{E})$. If $\tilde{\nu}_1 \otimes \tilde{\nu}_2 \cong \mathcal{O}_{\tilde{E}}$ and G-action on it is trivial, then we say that Y_0 has *semistable V-normal crossing singularity* at E. In this case, we have $\mathcal{E} \cong \mathcal{O}_E$.

Proposition. *Let Y_0 be a V-space with semistable V-normal crossing singularity at E as above, and $\pi : \mathcal{Y} \rightarrow \Delta(\subset \mathbb{C})$ a V-deformation of $X_0 \cong \pi^{-1}(0)$ such that the V-Kodaira-Spencer class $\rho^V(\partial/\partial t)$ is mapped to a nonzero element in $H^0(E, \mathcal{E})$. Then there is a neighborhood $\mathcal{U}$ of E in $\mathcal{Y}$, a smooth complex manifold $\tilde{\mathcal{U}}$ with G-action, and a G-invariant map $p : \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ which identifies $\mathcal{U}$ with $\tilde{\mathcal{U}}/G$, such that $(p \circ \pi)^{-1}(0) = \tilde{U}$ and the G-action restricts to that on $\tilde{U}$.*

Proof. Since $\pi : \mathcal{X} \rightarrow \Delta$ is a V-deformation, there is a chart $p : (\tilde{\mathcal{U}}, G) \rightarrow \mathcal{U}$ of $\mathcal{Y}$ near E which extends the chart near E in Y_0, then $\tilde{\mathcal{U}} \rightarrow \Delta$ is a deformation of $\tilde{U}$ whose Kodaira-Spencer map generates $\tau^1_{\tilde{U}}$ everywhere, so by the result of Friedman [23], $\tilde{\mathcal{U}}$ is smooth. Q.E.D.

Type II Connected Sum

We now use the above discussion to study the type II connected sum of two self-dual orbifolds. Recall that if X_1, X_2 are two orbifolds, $x_1 \in X_1$, $x_2 \in X_2$ two isolated orbifold points with the same (finite) local transformation group G. If $f_i : (G, \tilde{U}_i) \rightarrow U_i$ are uniformization charts centered at x_i, $f^{-1}(x_i) = \tilde{x}_i$

and if there is an equivariant orientation reversing map $\phi : T_{\tilde{x}_1}\tilde{U}_1 \to T_{\tilde{x}_2}\tilde{U}_2$, then type II connected sum can be formed by gluing $\tilde{U}_1$ with $\tilde{U}_2$ then modulo G. Denote if by $X_1\#_G X_2$. If X_1, X_2 are self-dual orbifolds, the twistor space of X_i can be defined by taking the twistor space over U_i to be $Z(\tilde{U}_i)/G$. Let $\pi_i : Z_i \to X_i$ be the twistor projection over X_i, $L_i = \pi_i^{-1}(x_i)$, we can 'blow up' Z_i in the following way. Let $\tilde{\pi}_i : Z(\tilde{U}_i) \to \tilde{U}_i$ be the twistor projection over $\tilde{U}_i$, we blow up $Z(\tilde{U}_i)$ along $\tilde{L}_i = \tilde{\pi}_i^{-1}(\tilde{x}_i)$ to get $\tilde{Z}(\tilde{U}_i)$ with exception set $\tilde{Q}_i$. Modulo G, we can constructed the blown-up twistor space $\tilde{Z}_i$ with exceptional set Q_i. The G-equivariant orientation reversing isometry induces a G-equivariant isomorphism $\tilde{\psi} : \tilde{Q}_1 \to \tilde{Q}_2$ which descends to an isomorphism $\psi : Q_1 \to Q_2$. It is easy to see that $Z_0 = \tilde{Z}_1 \cup_\psi \tilde{Z}_2$ is a V-space with semistable normal crossing singularity at Q, which is the common image of the inclusions of Q_1, Q_2 in Z_0. Again following Donaldson-Friedman [15], we make the following

Definition. A standard II deformation of Z_0 above consists of:

1. a V-deformation $\pi : \mathcal{X} \to S$ where S is a neighborhood of 0 in $\mathbb{C}^n$, such that $\pi^{-1}(0) \cong Z_0$;

2. antiholomorphic involutions $\sigma : \mathcal{Z} \to \mathcal{Z}$ and $\tau : S \to S$ which are compatible under π, such that $\sigma|_{\pi^{-1}(0)}$ is the natural antiholomorphic involution on Z_0.

In addition, we assume that near any point of $Q \subset Z_0 \subset \mathcal{Z}$, there are local coordinates $(z_1, z_2, z_3, z_4, t_2, \cdots, t_n)$ in a uniformization chart such that $\tilde{\pi}(z_1, z_2, z_3, z_4, t_2, \cdots, t_n) = (z_1 z_2, t_2, \cdots, t_n)$.

Similar to the proof of the theorem in the preceding section, Donaldson-Friedman's proof with the obvious modification can be used to prove the following

Theorem. *If $\pi : \mathcal{Z} \to S$ is a standard II deformation of a V-space Z_0 constructed above, then for sufficiently small vector s in the fixed locus S^σ of the involution σ, not lying in the hypersurface $\{t_1 = 0\}$, the fiber $\pi^{-1}(s)$ is the twistor space of a self-dual conformal structure on $X_1 \#_G X_2$.*

To study the existence of standard II deformation, notice that since $\tilde{Z}_1, \tilde{Z}_2$ are orbifolds, we have $\tau^{1V}_{\tilde{Z}_i} = \tau^{2V}_{\tilde{Z}_i} = 0$. So we have the following exact sequence

$$\begin{aligned} 0 \;\to\; & H^1(Z_0, \tau^{0V}_{Z_0}) \to T^{1V}(Z_0) \to H^0(Q, \mathcal{O}_Q) = \mathbb{C} \\ \to\; & H^2(Z_0, \tau^{0V}_{Z_0}) \to T^{2V}(Z_0) \to H^1(Q, \mathcal{O}_Q) = 0. \end{aligned}$$

To compute $H^i(Z_0, \tau^{0V}_{Z_0})$, we use the following exact sequence:

$$0 \to \tau^{0V}_{Z_0} \to p_* \tau^{0V}_{Z_0', Q_1 \cup Q_2} \to j_* \tau^{0V}_Q \to 0$$

to get a long exact sequence:

$$\begin{aligned} 0 \;\to\; & H^0(Z_0, \tau^{0V}_{Z_0}) \to H^0(\tilde{Z}_1, \tau^{0V}_{\tilde{Z}_1, Q_1}) \oplus H^0(\tilde{Z}_2, \tau^{0V}_{\tilde{Z}_2, Q_2}) \to H^0(Q, \tau^{0V}_Q) \\ \to\; & H^1(Z_0, \tau^{0V}_{Z_0}) \to H^1(\tilde{Z}_1, \tau^{0V}_{\tilde{Z}_1, Q_1}) \oplus H^1(\tilde{Z}_2, \tau^{0V}_{\tilde{Z}_2, Q_2}) \to H^1(Q, \tau^{0V}_Q) \\ \to\; & H^2(Z_0, \tau^{0V}_{Z_0}) \to H^2(\tilde{Z}_1, \tau^{0V}_{\tilde{Z}_1, Q_1}) \oplus H^2(\tilde{Z}_2, \tau^{0V}_{\tilde{Z}_2, Q_2}) \to H^2(Q, \tau^{0V}_Q). \end{aligned}$$

To compute $H^j(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i, Q_i})$, we use the exact sequence:

$$0 \to \tau^{0V}_{\tilde{Z}_i, Q_i} \to \tau^{0V}_{\tilde{Z}_i} \to \nu^V_i \to 0$$

to get the following long exact sequence:

$$\begin{aligned}
0 &\rightarrow H^0(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i,Q_i}) \rightarrow H^0(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i}) \rightarrow H^0(Q_i, \nu_i^V) \\
&\rightarrow H^1(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i,Q_i}) \rightarrow H^1(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i}) \rightarrow H^1(Q_i, \nu_i^V) \\
&\rightarrow H^2(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i,Q_i}) \rightarrow H^2(\tilde{Z}_i, \tau^{0V}_{\tilde{Z}_i}) \rightarrow H^2(Q_i, \nu_i^V).
\end{aligned}$$

So we need to compute $H^j(Q, \tau_Q^{oV})$ and $H^j(Q_i, \nu_i^V)$. This can be done using the Dolbeault theorem on orbifolds [7]:

$$H^j(Q, \nu_j^V) \cong H^j_{\bar{\partial}}(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(-1,1))^G = 0$$

for $j = 1, 2$. Similarly $H^j(Q, \tau_Q^{0V}) = 0$ for $j = 1, 2$. We then have:

$$H^2(Z_0, \tau^{0V}_{Z_0}) \cong H^2(Z_1, \tau^{0V}_{Z_1}) \oplus H^2(Z_2, \tau^{0V}_{Z_2}).$$

So when $H^2(Z_1, \tau^{0V}_{Z_1}) = H^2(Z_2, \tau^{0V}_{Z_2}) = 0$, we have $H^2(Z_0, \tau^{0V}_{Z_0}) = 0$, and therefore the following exact sequence:

$$T^{1V}(Z_0) \rightarrow H^0(Q, \mathcal{O}_Q) = \mathbb{C} \rightarrow 0 \rightarrow T^{2V}(Z_0) \rightarrow 0.$$

Therefore, $T^{2V}(Z_0) = 0$, and given any non-zero section of $\mathcal{E}$, it is the restriction of some vector $v \in T^{1V}(Z_0)$. Since there is no obstruction, there is a V-deformation with V-Kodaira-Spencer class v, this gives us the desired smoothing. We again can use Donaldson-Friedman's trick in Section 6 of [15] to prove the following

Theorem. *Let X_1, X_2 be two self-dual orbifolds with twistor spaces Z_1 and Z_2, $x_1 \in X_1$ is an orbifold point of type $\mathbb{Z}_n(1, -q)$ and $x_2 \in X_2$ of type $\mathbb{Z}_n(1, q)$,* *construct* Z_0 *as above. If* $H^2(Z_1, \tau_{Z_1}^{0V}) = H^2(Z_2, \tau_{Z_2}^{0V}) = 0$*, and all local uniformization groups of* X_1*,* X_2 *are Abelian, then* Z_0 *admits a standard II deformation, hence the type II connected sum of* X_1 *and* X_2 *across* x_1, x_2 *admits a self-dual orbifold metric.*

Remark. It is easy to see that the results of the last two sections can be generalized to self-connected sums, multiple connected sums, or a combination of type I and type II connected sums, etc.

Chapter 5

Explicit Degenerations of LeBrun Metrics

In this chapter, we prove that the degeneration of LeBrun's metrics on $n\mathbb{CP}_2$ into a connected sum of a LeBrun orbifold and a Gibbons-Hawking orbifold can be described in (generalized) Donaldson-Friedman picture using explicit description of the twistor spaces. It is interesting to notice that corresponding to the rescaling process to manifest the Gibbons-Hawking orbifold, we need a weighted blow-up construction in the family of twistor spaces.

5.1 Weighted Projective Varieties

To compactify the twistor spaces of the ALE spaces, we need weighted projective spaces and weighted blow-up. We reformulate some materials in [13] where they were presented in the language of scheme.

5.1.1 Weighted Projective Spaces

Let $K = (k_0, \cdots, k_n)$ be an ordered set of positive integers, consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} - \{0\}$ defined by

$$t \cdot (z_0, \cdots, z_n) = (z_0 t^{k_0}, \cdots, z_n t^{k_n}).$$

Since every k_i is positive, all the orbits of the action are closed subset of $\mathbb{C}^{n+1} - \{0\}$, so we can take the quotient. Let $\mathbb{CP}_n(K) = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$ be the complex analytic quotient space. It is called the *complex weighted projective space* with weight K. It is compact since it can be identified with $S^{2n+1}/S^1(k_0, \cdots, k_n)$, where $S^1(k_0, \cdots, k_n)$ means S^1 action on S^{2n+1} is induced by the above action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} - \{0\}$. Alternatively, $\mathbb{CP}_n(K)$ can be identified with the symplectic quotient of $\mathbb{C}^{n+1}$ by $S^1(k_0, \cdots, k_n)$-action.

Let $[z_: \cdots : z_n]_K$ denote the K-homogeneous coordinates, we then have a natural map:

$$\mathbb{CP}_n \to \mathbb{CP}_n(K)$$

$$[u_0 : \cdots : u_n] \mapsto [u_0^{k_0} : \cdots : u_n^{k_n}]_K$$

which gives an isomorphism:

$$\mathbb{CP}_n(K) \cong \mathbb{CP}_n/(\mathbb{Z}_{k_0} \times \cdots \times \mathbb{Z}_{k_n}).$$

And so $\mathbb{CP}_n(K)$ is an orbifold. Let $U_i = \{[z_0 : \cdots : z_n]_K \mid z_i \neq 0\}$, then we have a map:

$$U_i \to \mathbb{C}^n/\mathbb{Z}_{k_i}(k_0, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n),$$

$$[z_0 : \cdots : z_n] \mapsto \left(\frac{z_0}{(z_i^{1/k_i})^{k_0}}, \cdots, \frac{z_{i-1}}{(z_i^{1/k_i})^{k_{i-1}}}, \frac{z_{i+1}}{(z_i^{1/k_i})^{k_{i+1}}}, \cdots, \frac{z_n}{(z_i^{1/k_i})^{k_n}} \right)$$

where Z_i^{1/k_i} is any k_i-th root of z_i, and $\mathbb{Z}_{k_i}(k_0, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n)$ means that $\mathbb{Z}_{k_i}$ acts on $\mathbb{C}^n$ by:

$$e^{2\pi\sqrt{-1}/k_i} \cdot (z_0, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n) \mapsto$$
$$(e^{2\pi\sqrt{-1}k_o/k_i} z_0, \cdots, e^{2\pi\sqrt{-1}k_{i-1}/k_i} z_{i-1}, e^{2\pi\sqrt{-1}k_{i+1}/k_i} z_{i+1}, \cdots, e^{2\pi\sqrt{-1}k_n/k_i} z_n).$$

Hence each U_i is a global quotient, for $i = 0, \cdots, n$. By suitable choice of the weights, we see that $\mathbb{CP}_n(K)$ may have singularities of different types, but nevertheless, it is a global quotient.

It is easy to see that if d is a positive integer, and $dK = (dk_0, \cdots dk_n)$, then $\mathbb{CP}_n(dK) \cong \mathbb{CP}_n(K)$. In fact, $[z_0 : \cdots : z_n]_{dK} \mapsto [z_0 : \cdots : z_n]_K$ is an isomorphism. So we can divide the greatest common divisor of $k_0, \cdots, k_n$, and assume that $\gcd(k_0, \cdots, k_n) = 1$. Then we have the following

Proposition. (Delorme [13]) *Let* $K = (k_0, \cdots, k_n)$, *and* $\gcd(k_0, \cdots, k_n) = 1$, *let*

$$d_i = \gcd(k_0, \cdots, k_{i-1}, k_{i+1}, \cdots, k_n),$$

$$m_i = lcm(d_0, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n),$$

$$m = lcm(d_0, \cdots, d_n),$$

$$l_i = k_i/m_i, L = (l_0, \cdots, l_n).$$

Then $\mathbb{CP}_n(K) \cong \mathbb{CP}_n(L)$.

Proof. Since $\gcd(k_0, \cdots, k_n) = 1$, we have $(k_j, d_j) = 1$, and $(d_j, d_l) = 1$, for $j \neq l$. Hence $m_i = d_0 \cdots d_{i-1} d_{i+1} \cdots d_n$ and $m = d_0, \cdots, d_n$. Let $P =$

$(d_0k_0, \cdots, d_nk_n)$, define a map:

$$\mathbb{CP}_n(K) \to \mathbb{CP}_n(P),$$

$$[z_0 : \cdots : z_n]_K \mapsto [z_0^{d_0} : \cdots : z_n^{d_n}]_P.$$

It is clearly surjective. To see it is also injective, notice that $[z_0, \cdots, z_n]_K$ and $[t_0^{m_0} z_0, \cdots, t_n^{m_n} z_n]_K$ have the same image, where for $i = 0, \cdots, n$, $t_i = exp(2\pi\sqrt{-1}/d_i)$, and m_i is an arbitrary integer. Let $t = exp(2\pi\sqrt{-1}/m)$. To see that the above two points are the same in $\mathbb{CP}_n(K)$, it suffices to find an integer p such that

$$t^{k_i} = t_i,$$

for each i, i.e., it suffices to solve the following system of congruence equations:

$$l_j p \equiv m_j (\bmod d_j), \quad j = 0, \cdots, n.$$

Since $(l_j, d_j) = 1$, there exists r_j, such that $k_j r_j \equiv 1(\bmod d_j)$. So we just need to solve the following system:

$$p \equiv l_j r_j (\bmod d_j), \quad j = 0, \cdots, n.$$

But now $(d_j, d_l) = 1$, for $j \neq l$, this always has solution by Chinese Remainder Theorem. So we have an isomorphism:

$$\mathbb{CP}_n(k_0, \cdots, k_n) \cong \mathbb{CP}_n(k_0d_0, \cdots, k_nd_n).$$

But $\gcd(k_0d_0, \cdots, k_nd_n) = lcm(d_0, \cdots, d_n) = m$, since $k_jd_j/m = k_j/m_j$, we have

$$\mathbb{CP}_n(k_0d_0, \cdots, k_nd_n) \cong \mathbb{CP}_n(k_0/m_0, \cdots, k_n/m_n).$$

Q.E.D.

Here are some examples: $\mathbb{CP}_2(2,3,6) \cong \mathbb{CP}_2$, $\mathbb{CP}_n(k,\cdots,k,1) \cong \mathbb{CP}_n$, in particular, $\mathbb{CP}_3(2,2,2,1) \cong \mathbb{CP}_3$.

5.1.2 Weighted Projective Varieties

A polynomial $f(z_0,\cdots,z_n)$ is called *quasi-homogeneous* of degree d with weight $K=(k_0,\cdots,k_n)$ if

$$f(t^{k_0}z_0,\cdots,t^{k_n}z_n) = t^d f(z_0,\cdots,z_n), t \in \mathbb{C},$$

If $f_1,\cdots,f_m$ are quasi-homogeneous polynomials with the same weight K of degrees $d_1,\cdots,d_m$ respectively, they define a weighted projective variety

$$\mathbb{P}_K\{f_0 = \cdots = f_m = 0\} \subset \mathbb{CP}_n(k_0,\cdots,k_n).$$

It is called *quasismooth* in $\mathbb{CP}_n(K)$ if the affine cone $\{f_1 = \cdots f_m = 0\}$ in $\mathbb{C}^{n+1}$ is smooth outside 0.

Using local charts as in Section 5.1.1, it is easy to see that a quasismooth weighted projective variety has an induced orbifold structure from that of the weighted projective space [13].

For example, $xy - z^n$ is a quasihomogeneous polynomial of degree $2n$ if (x,y,z) have weights $(n,n,2)$. We have a map

$$\mathbb{CP}_1 \to \mathbb{CP}_2(n,n,2),$$

$$[z_1 : z_2] \mapsto [z_1^n : z_2^n : z_1 z_2]_{(n,n,2)}$$

which identifies $\mathbb{P}_{(n,n,2)}\{xy = z^n\}$ in $\mathbb{CP}_2(n,n,2)$ with $\mathbb{CP}_1/\mathbb{Z}_n$, hence it is a global quotient. We can also see this from the orbifold structure of $\mathbb{CP}_2(n,n,2)$.

Consider the weighted hypersurface:

$$X = \mathbb{P}_{(n,n,2,1)}\{xy = (z - w^2 a_1) \cdots (z - w^2 a_n)\} \subset \mathbb{CP}_3(n,n,2,1),$$

where (x, y, z, w) have weight $(n, n, 2, 1)$, and $a_1, \cdots, a_n$ are fixed complex numbers. X contains the rational curve

$$C = \mathbb{P}_{(n,n,2,1)}\{xy = z^n, w = 0\} \cong \mathbb{P}_{(n,n,2)}\{xy = z^n\},$$

we now want to show that near C, X is a global quotient by $\mathbb{Z}_n$.

Let $U_x = \{[x : y : z : w]_{(n,n,2,1)} | x \neq 0\}$, and $U_y = \{[x : y : z : w]_{(n,n,2,1)} | y \neq 0\}$, we have local charts over $X \cap U_x$ and $X \cap U_y$ given by:

$$\phi_x : \mathbb{C}^2 \to X \cap U_x, (z_1, w_1) \mapsto [1 : (z_1 - w_1^2 a_1) \cdots (z_1 - w_1^2 a_n) : z_1 : w_1]_{(n,n,2,1)}$$

$$\phi_y : \mathbb{C}^2 \to X \cap U_y, (z_2, w_2) \mapsto [(z_2 - w_2^2 a_1) \cdots (z_2 - w_2^2 a_n) : 1 : z_2 : w_2]_{(n,n,2,1)}.$$

There are small neighborhood U of $(\mathbb{C} - \{0\}) \times \{0\} \subset \mathbb{C}^2$, such that

$$f(z, w) = ((z - w^2 a_1) \cdots (z - w^2 a_n))^{1/n}$$

is well-defined and $f(z, 0) = z$. Let $V = f(U)$. Choose small neighborhoods W_x, W_y of $(0, 0)$ in $\mathbb{C}^2$, glue $U \cup W_x$ with $V \cup W_y$ to get a space Y by the map

$$U \to V, (z_1, w_1) \mapsto \left(\frac{z_1}{f(z_1, w_1)^2}, \frac{w_1}{f(z_1, w_1)} \right),$$

notice that the $\mathbb{Z}_n(2, 1)$-action on $U \cup W_x$ is glued with $\mathbb{Z}_n(-2, -1)$-action on $V \cup W_y$ to give us a global action of $\mathbb{Z}_n$ on Y, whose quotient is clearly a neighborhood of C in X, and C is the global quotient of a rational curve (when $w_1 = w_2 = 0$) with normal bundle $\mathcal{O}(1)$ in Y.

Under the identification of the weighted projective spaces in Section 5.1.1, the weighted projective varieties can also be identified with each other. For example, under

$$\mathbb{CP}_3(2,2,2,1) \cong \mathbb{CP}_3$$

$$[x,y,z,\epsilon]_{(2,2,2,1)} \mapsto [x,y,z,\epsilon^2]$$

we have an induced identification of $\mathbb{P}_{(2,2,2,1)}\{xy = (z-\epsilon^2 a_1)(z-\epsilon^2 a_2)\}$ in $\mathbb{CP}_3(2,2,2,1)$ with $\mathbb{P}\{xy = (z-ta_1)(z-ta_2)\}$ in $\mathbb{CP}_3$, where $a_1, a_2 \in \mathbb{C}$. It is easy to see that the rational curve $\mathbb{P}\{xy = z^n\}$ has normal bundle $\mathcal{O}(2)$ in $\mathbb{P}\{xy = (z-ta_1)(z-ta_2)\}$.

5.1.3 Weighted Blow-Up

Consider the space:

$$\hat{\mathbb{C}}^{n+1}_K = \{((z_0,\cdots,z_n), \quad [w_0:\cdots:w_n]) \in \mathbb{C}^{n+1}\times\mathbb{CP}_n(K) \mid z_j = t^{k_j}w_j, \quad t\in\mathbb{C},\ \ j=0,\cdots,n\}.$$

We have a natural projection $\pi : \hat{\mathbb{C}}^{n+1}_K \to \mathbb{C}^{n+1}$. For $v \in \mathbb{C}-\{0\}$, $\pi^{-1}(v)$ is just a point, but $\pi^{-1}(0) \cong \mathbb{CP}(K)$. We call $\hat{\mathbb{C}}^{n+1}_K$ the weighted blow-up of $\mathbb{C}^{n+1}$ at 0 with weight K, we also say that we blow up $\mathbb{C}^{n+1}$ at 0 with weight K, and $E = \pi^{-1}(0)$ is called the exceptional set.

$\hat{\mathbb{C}}^{n+1}_K$ is an orbifold. In fact, if $\hat{\mathbb{C}}^{n+1}$ is the usual blow-up of $\mathbb{C}^{n+1}$ at 0, we have a map:

$$\hat{\mathbb{C}}^{n+1} \to \hat{\mathbb{C}}^{n+1}_K,$$

$$((x_0,\cdots,x_n),[y_0:\cdots:y_n])\mapsto((x_0^{k_0},\cdots,x_n^{k_n}),[y_0^{k_0}:\cdots:y_n^{k_n}]_K),$$

this gives us an identification:

$$\hat{\mathbb{C}}_K^{n+1}\cong\hat{\mathbb{C}}^{n+1}/(\mathbb{Z}_{k_0}\times\cdots\mathbb{Z}_{k_n}).$$

If $U_i=\{[w_0:\cdots:w_n]_K\mid w_i\neq 0\}$, $\mathcal{U}_i=\pi^{-1}(U_i)$, we have:

$$\mathcal{U}_i\to\mathbb{C}^{n+1}/\mathbb{Z}_{k_i}(0,k_0,\cdots,k_{i-1},k_{i+1},\cdots,k_n),$$

$$\begin{aligned}((x_0,\cdots,x_n),[y_0:\cdots:y_n]) \quad\mapsto\quad & (x_i,\frac{x_0}{(x_i^{1/k_i})^{k_0}},\cdots,\frac{x_{i-1}}{(x_i^{1/k_i})^{k_{i-1}}},\\ & \frac{x_{i+1}}{(x_i^{1/k_i})^{k_{i+1}}},\cdots\frac{x_n}{(x_i^{1/k_i})^{k_n}}),\end{aligned}$$

hence each $\mathcal{U}_i$ is an orbifold, for $i=0,\ldots,n$.

Similar to the discussion in Proposition 5.1.1, we have the following:

$$\hat{\mathbb{C}}_{dK}^{n+1}\cong\hat{\mathbb{C}}_K^{n+1},$$

$$\hat{\mathbb{C}}_L^{n+1}\cong\hat{\mathbb{C}}_K^{n+1}/(\mathbb{Z}_{d_0}\times\cdots\times\mathbb{Z}_{d_n}),$$

where $K,L,d_0,\cdots,d_n$ are as in Proposition 5.1.1. In particular, we have

$$\hat{\mathbb{C}}_{(2,2,2,1)}^4/\mathbb{Z}_2\cong\hat{\mathbb{C}}^4.$$

If $X\subset\mathbb{C}^{n+1}$ is an affine variety, $\hat{X}_K=\overline{X-\{0\}}\subset\hat{\mathbb{C}}_K^{n+1}$ is called the strict transform of X in $\hat{\mathbb{C}}_K^{n+1}$. $E_X=\hat{X}\cap E$ is called the exceptional set in $\hat{X}_K$.

Consider the map $\mathbb{C}^2\to\mathbb{C}^3$ defined by

$$(z_1,z_2)\mapsto(x,y,z)=(z_1^n,z_2^n,z_1z_2),$$

the image is $X = \{xy = z^n\}$, it induces a map $\hat{\mathbb{C}}^2 \to \hat{\mathbb{C}}^3_{(n,n,2)}$, defined by

$$((z_1, z_2), [w_1, w_2]) \mapsto ((z_1^n, z_2^n, z_1 z_2), [w_1^n, w_2^n, w_1 w_2]_{(n,n,2)}).$$

This gives us an identification:

$$\hat{\mathbb{C}}^2/\mathbb{Z}_n(1,-1) \cong \hat{X}_{(n,n,2)},$$

where $\mathbb{Z}_n(1,-1)$ means the $\mathbb{Z}_n$-action on $\hat{\mathbb{C}}^2$ is induced by the following $\mathbb{Z}_n$-action on $\mathbb{C}^2$:

$$e^{2\pi\sqrt{-1}/n} \cdot (z_1, z_2) = (e^{2\pi\sqrt{-1}/n} z_1, e^{-2\pi\sqrt{-1}/n} z_2),$$

hence the exception set E_X in $\hat{X}_{(n,n,2)}$ is a rational curve, and $\hat{X}$ is a global quotient of $\mathcal{O}(-1) \to \mathbb{CP}_1$ by $\mathbb{Z}_n$-action.

We now consider the following affine variety:

$$Y = \{xy = (z - w^2 a_1) \cdots (z - w^2 a_n)\},$$

where $(x, y, z, w) \in \mathbb{C}^4$, and $a_1, \cdots, a_n$ are fixed complex numbers. Blow up Y at $0 \in \mathbb{C}^4$ with weights $(n, n, 2, 1)$, then

$$E_Y = \mathbb{P}_{(n,n,2,1)}\{xy = (z - w^2 a_1) \cdots (z - w^2 a_n)\}$$

is the weighted projective variety we described in last subsection. We have a natural projection $\pi : Y \to \mathbb{C}, \pi(x, y, z, w) = w$ and an induced projection $\hat{\pi} : \hat{Y}_{(n,n,2,1)} \to \mathbb{C}$, let $Y_w = \pi^{-1}(w), \hat{Y}_w = \hat{\pi}^{-1}(w)$, then $\hat{Y}_w = Y_w$ for $w \neq 0$, and $\hat{Y}_0 = E_Y \cup \hat{Y}_{0K}$, where $\hat{Y}_{0K}$ is the strict transform of Y_0, it is easy to see that $Y_0 \cong X$ and $\hat{Y}_{0K} \cong \hat{X}_{(n,n,2)}$. Notice that

$$C = E_Y \cap \hat{Y}_{0K} = \mathbb{P}_{(n,n,2,1)}\{xy = z^n, w = 0\} \cong \mathbb{P}_{(n,n,2)}\{xy = z^n\}.$$

Near C, we have local charts on $\hat{Y}$,

$$\phi_x : \mathbb{C}^3 \to \hat{Y}, (x_1, z_1, w_1) \mapsto (x_1^n, (z_1 - w_1^2 a_1) \cdots (z_1 - w_1^2 a_n), x_1^2 z_1, x_1 w_1),$$

$$\phi_y : \mathbb{C}^3 \to \hat{Y}, (y_2, z_2, w_2) \mapsto ((z_2 - w_2^2 a_1) \cdots (z_2 - w_2^2 a_n), y_2^n, x_2^2 z_2, y_2 w_2).$$

We can use the same trick as in last subsection to see that near C, $\hat{Y}$ is a global $\mathbb{Z}_N$-quotient of a smoothing of some normal crossing singularity, each fiber of the smoothing is preserved by the $\mathbb{Z}_n$-action.

5.2 Twistor Spaces

We review in this section the explicit description of the twistor spaces of some self-dual spaces.

5.2.1 LeBrun Metrics and LeBrun Orbifolds

We begin with the twistor spaces of the self-dual LeBrun metrics on $n\mathbb{CP}_2$ constructed by the hyperbolic ansatz [41]. Let $\bar{p} = (p_1, p_2, \cdots, p_n)$ be an unordered set of arbitrary points in $\mathcal{H}^3$, not necessarily all distinct. Each point p_i corresponds to a polynomial $P_i \in \Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1,1))$. If $([z_1 : z_2], [w_1, w_2])$ are homogeneous coordinates on $\mathbb{CP}_1$, then P_i can be written as $a_i z_1 w_1 + b_i z_1 w_2 + c_i z_2 w_1 + d_i z_2 w_2$, P_i is real in the sense that $\bar{a}_i = a_i, b_i = \bar{c}_i$, and $\bar{d}_i = d_i$.

We define an algebraic variety

$$\bar{Z}(\bar{p}) \subset \mathbb{P}(\mathcal{O}(n-1,1) \oplus \mathcal{O}(1,n-1) \oplus \mathcal{O})$$

by the equation

$$xy = t^2 \prod_{j=1}^{n} P_j,$$

where $x \in \mathcal{O}(n-1,1), y \in \mathcal{O}(1,n-1), t \in \mathcal{O}$.

$\bar{Z}(\bar{p})$ contains two surfaces $x = t = 0$ and $y = t = 0$, the canonical projection identifies them with the base $\mathbb{CP}_1 \times \mathbb{CP}_1$. These surfaces don't contain any singular points of $\bar{Z}(\bar{p})$ their normal bundles in $\bar{Z}(\bar{p})$ are respectively $\mathcal{O}(1-n,-1)$ and $\mathcal{O}(-1,1-n)$, so they can be blown down to rational curves with normal bundles $\mathcal{O}(1-n) \oplus \mathcal{O}(1-n)$. Denote the resulting space by $\tilde{Z}(\bar{p})$.

$\tilde{Z}(\bar{p})$ has some singular points, we will consider two special cases. First assume that all $\{p_i\}$ are distinct, and no three lie on a common hyperbolic geodesic in $\mathcal{H}^3$. Let C_i be the curve in $\mathbb{CP}_1 \times \mathbb{CP}_1$ corresponding to the polynomial P_i, then the above assumption is equivalent to the condition that no three of these curves pass through a common point. Then the points $x = y = 0, t \neq 0, P_j = P_k$ for some $j \neq k$ are easily seen to be the only (isolated) singular points on $\tilde{Z}(\bar{p})$, we can then make suitable small resolution of them and the resulting smooth threefold is the twistor space of some self-dual LeBrun metric on $n\mathbb{CP}_2$ constructed by the hyperbolic ansatz with centers $p_1, \cdots p_n$. For details, the reader is referred to [41].

We may also consider the case when $p_1 = p_2 = \cdots = p_n = p$ for some point p, corresponding to a polynomial P, which defines a curve C in $\mathbb{CP}_1 \times \mathbb{CP}_1$. The singular set in $\tilde{Z}(\bar{p})$ is then:

$$\mathcal{C} = \{x = y = P = 0, t \neq 0\}.$$

In local coordinates near $\mathcal{C}$, $\tilde{Z}(\bar{p})$ has the form $xy = z^n$, so we have a family of

A_{n-1} singularities parameterized by $C \cong \mathbb{CP}_1$. We can easily check that $\tilde{Z}(\bar{p})$ is the (singular) twistor space of the self-dual LeBrun orbifold obtained by the one-point conformal compactification of the scalar-flat Kähler metric on the total space of the line bundle $\mathcal{O}(-n) \to \mathbb{CP}_1$ for $n \geq 2$ constructed in [40].

5.2.2 Gibbons-Hawking Spaces and Orbifolds

A Riemannian manifold (M^4, g) is called asymptotically locally Euclidean (ALE) if there is a compact set $K \subset M$, such that $M - K$ is diffeomorphic to the quotient of the complement V of the unit ball in $\mathbb{R}^4$ by a discrete group Γ and the lift of g to V is asymptotically Euclidean, and Γ acts on V as isometries with respect to this metric. We call Γ the fundamental group of M at infinity. Recall that (M^4, g) is called hyperkähler if it admits three complex structures I, J, K compatible with the metric which gives three closed Kähler forms, and such that $IJ = K$. Hyperkähler manifolds are Ricci-flat, hence scalar-flat. In dimension four, it implies that (M^4, g) is self-dual with respect to the orientation reverse to the canonical one defined by the complex structures [39], [12], hence it has a complex twistor space.

Hyperkähler ALE spaces with $\Gamma \subset SU(2) \subset SO(4)$ have been classified by Kronheimer [37]. The special cases of $\Gamma = \mathbb{Z}_n \subset SU(2)$ were studied earlier by Gibbons-Hawking [26] and Hitchin [29], we call them Gibbons-Hawking spaces. The twistor spaces of them were explicitly described by Hitchin [29], they are similar to those in Section 5.2.1.

Let $\bar{q} = (q_1, \cdots, q_n)$ be an unordered set of points in $\mathbb{R}^3$, they correspond to polynomials $Q_i \in \Gamma(\mathbb{CP}_1, \mathcal{O}(2))$. Consider the variety

$$\tilde{W} = \{x \in \mathcal{O}(n), y \in \mathcal{O}(n), z \in \mathcal{O}(2) \mid xy = (z - Q_1)\cdots(z - Q_n)\}.$$

When $q_1, \cdots, q_n$ are in general positions, $\tilde{W}$ has isolated singular points at:

$$x = y = z - Q_j = z - Q_k = 0, j \neq k.$$

They are in the fibers where $Q_j = Q_k$, we can use suitable small resolution to get the the twistor spaces of ALE metrics constructed by Gibbons-Hawking ansatz [29].

The case when $q_1 = \cdots = q_n = q$ is not interesting. If Q is the corresponding polynomial, we can use a coordinate change $z - Q \to z$ to identify $\tilde{W}$ with $\{xy = z^n\} \subset \mathcal{O}(n) \oplus \mathcal{O}(n) \oplus \mathcal{O}(2)$. But this is just $\mathcal{O}(1) \oplus \mathcal{O}(1)/\mathbb{Z}_n$, so it corresponds to the flat orbifold metric on $\mathbb{R}^4/\mathbb{Z}_n$.

If (M^4, g) is an ALE space with finite fundamental group Γ at infinity, let $\bar{M}$ be the one-point compactification of M, $\bar{M}$ is then an orbifold with a single singular point at infinity. The ALE condition allows us to extend g to an orbifold metric $\bar{g}$ on $\bar{M}$ after a conformal change. Since the self-dual condition is conformally invariant, if (M^4, g) is also hyperkähler, $(\bar{M}, \bar{g})$ is then a self-dual orbifold. The twistor space of $(\bar{M}, \bar{g})$ is then a compactification of the twistor space $Z(M)$ of (M, g).

For Gibbons-Hawking orbifolds, the blown-up twistor spaces are easy to describe. Consider the following variety:

$$\tilde{V}_n = \mathbb{P}_{(n,n,2,1)}\{x, y \in \mathcal{O}(n), z \in \mathcal{O}(2), w \in \mathcal{O} | xy = (z - w^2 Q_1)\cdots(z - w^2 Q_n)\},$$

where (x, y, z, w) have weights $(n, n, 2, 1)$. The open subset of $\tilde{V}_n$ where $w \neq 0$ is exactly the space

$$\tilde{W} = \{x, y \in \mathcal{O}(n), z \in \mathcal{O}(2) | xy = (z - Q_1) \cdots (z - Q_n)\}$$

discussed above; at infinity where $w = 0$, we have a quadric:

$$\begin{aligned} Q &= \mathbb{P}_{(n,n,2,1)}\{x, y \in \mathcal{O}(n), z \in \mathcal{O}(2), w \in \mathcal{O} | xy = z^n, w = 0\} \\ &\cong \mathbb{P}_{(n,n,2)}\{x, y \in \mathcal{O}(n), z \in \mathcal{O}(2) | xy = z^n\} \\ &\cong (\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1))/\mathbb{Z}_n = (\mathbb{CP}_1 \times \mathbb{CP}_1)/\mathbb{Z}_n, \end{aligned}$$

where the first factor is from the fiber, and the second from the base. Using the trick in Section 5.1.1, it is easy to see that near Q, $\tilde{V}_n$ is a global quotient by $\mathbb{Z}_n$, and Q is covered by a quadric with normal bundle $\mathcal{O}(1, -1)$. So after resolving the singularities suitably, we get the blown-up twistor space of the Gibbons-Hawking orbifold, we can blow down the second factor in the quadric to get the twistor space. It can be constructed from a weighted projective variety by small resolutions [4].

When $n = 2$, we can use the identification $\mathbb{CP}_3(2, 2, 2, 1) \cong \mathbb{CP}_3$ to see that

$$\tilde{V}_2 \cong \mathbb{P}\{x, y, z \in \mathcal{O}(2), t \in \mathcal{O} | xy = (z - tQ_1)(z - tQ_2)\},$$

after the coordinate change $z' = z - \frac{t}{2}(Q_1 + Q_2)$, $Q_1' = -Q_2' = \frac{1}{2}(Q_2 - Q_1)$, can be identified with

$$\mathbb{P}\{x, y, z \in \mathcal{O}(2), t \in \mathcal{O} | xy = z^2 - t^2 a^2\}$$

which is used by LeBrun and Singer in [44]. Under this identification, Q is a quadric with normal bundle $\mathcal{O}(2, -2)$.

5.3 Explicit Degeneration: $n = 2$ Case

Consider the following variety:

$$\tilde{\mathcal{Z}} = \mathbb{P}\{xy = t^2(P + \epsilon P_1) \cdot (P + \epsilon P_2)\}$$

in $\mathbb{P}(\mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}) \times \mathbb{C} \to \mathbb{CP}_1 \times \mathbb{CP}_1$, where $x, y \in \mathcal{O}(1,1)$, $t \in \mathcal{O}$, $\epsilon \in \mathbb{C}$ are variables, P, P_1, P_2 are fixed sections $\mathcal{O}(1,1)$. Without loss of generality, we can assume that $P_1 = -P_2$. We have a natural projection $\Pi : \tilde{\mathcal{Z}} \to \mathbb{C}$, let $\tilde{\mathcal{Z}}_\epsilon = \Pi^{-1}(\epsilon)$.

The open subset $\{t \neq 0\}$ in $\tilde{\mathcal{Z}}$ can be identified with

$$\{xy = (P + \epsilon P_1)(P - \epsilon P_1)\} \subset (\mathcal{O}(1,1) \oplus \mathcal{O}(1,1)) \times \mathbb{C},$$

it contains all the singular points of $\tilde{\mathcal{Z}}$:

$$\{x = y = P + \epsilon P_1 = P - \epsilon P_1 = 0\} = \{x = y = P = \epsilon P_1 = 0\}.$$

When $\epsilon \neq 0$, $\tilde{\mathcal{Z}}$ has a conjugate pair of singular points at $P = P_1 = 0$; when $\epsilon = 0$, the singular set becomes $\{P = 0\}$. To simultaneously resolve the singularity, we blow up $\tilde{\mathcal{Z}}$ along the rational curve $\{x = y = P = \epsilon = 0\}$ in $(\mathcal{O}(1,1) \oplus \mathcal{O}(1,1)) \times \mathbb{C}$. In suitable local coordinates, the singularity has the following form:

$$xy = (z + w)(z - w),$$

it is independent of ϵ, so they can be simultaneously resolved, and since they appear in conjugate pairs with respect to the natural real structure on $\tilde{\mathcal{Z}}$, we make suitable choices as in [41] to guarantee that the real structure lifts to a real structure on the resolved space. Similar to the discussion in Section 5.2, we also notice that we can simultaneously blow down $\{x = t = 0\}$ and $\{y = t = 0\}$ in $\tilde{\mathcal{Z}}$ to get rational curves $\mathbb{CP}_1$ with normal bundle $\mathcal{O}(1,1)$. Denote the space we get after the above two operations by $\mathcal{Z}$.

The projection $\Pi : \tilde{\mathcal{Z}} \to \mathbb{C}$ gives a projection $\pi : \mathcal{Z} \to \mathbb{C}$, let $Z_\epsilon = \pi^{-1}(\epsilon)$. By construction and Section 5.2, for each real $\epsilon \neq 0$, Z_ϵ is the twistor space of some self-dual metric on $2\mathbb{CP}_2$. We will now examine Z_0.

First notice that $Z_0 = E \cup \hat{\hat{Z}}_0$, where E is the exceptional set of the blow up and $\hat{\hat{Z}}_0$ is the strict transform of $\Pi^{-1}(0) = \{xy = P^2, \epsilon = 0\}$. Since $P, P_1 \in \Gamma(\mathbb{CP}_1 \times \mathbb{CP}_{,}\mathcal{O}(1,1))$, $x, y \in \mathcal{O}(1,1)$, $E = \mathbb{P}\{\bar{x}\bar{y} = (z + \bar{\epsilon}\bar{P}_1)(z - \bar{\epsilon}\bar{P}_1)\} \subset \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O})$, where $\bar{x}, \bar{y}, z \in \mathcal{O}(2)$, $\bar{\epsilon} \in \mathcal{O}$ are variables, $\bar{P}_1$ is the restiriction of P_1 to the rational curve $\{P = 0\} \subset \mathbb{CP}_1 \times \mathbb{CP}_1$, which can be identified with a section of $\mathcal{O}(2)$. So from Section 5.2, E is the blown-up twistor space of Eguchi-Hanson orbifold.

Secondly, $Q := E \cap \hat{\hat{Z}}_0 = \mathbb{P}\{\bar{x}\bar{y} = z^2, \bar{\epsilon} = 0\}$ is a quadric $\mathbb{CP}_1 \times \mathbb{CP}_1$, where the first factor is in the fiber of $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O})$, and the second factor is the base. Again from Section 5.2, we have $\nu_{Q,E} \cong \mathcal{O}(2, -2)$, since Q is the blown-up singular twistor fiber in E.

Thirdly, $\hat{\hat{Z}}_0$ can be identified with the strict transform of $\mathbb{P}\{xy = t^2P^2\}$ when we blow up $X = \mathbb{P}(\mathcal{O}(1,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O})$ along the rational curve $C = \mathbb{P}\{x = y = P = 0, t \neq 0\}$, we denote the blow-up of X by $\hat{X}$, and the exceptional set by Y. Now

$$\nu_{C,X} \cong (\mathcal{O}(1,1) \oplus \mathcal{O}(1,1)) \mid_C \oplus (T(\mathbb{CP}_1 \times \mathbb{CP}_1) \mid_C / TC \cong \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2),$$

so the exceptional set $Y \cong \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)) \cong \mathbb{CP}_2 \times \mathbb{CP}_1)$, and $\nu_{Y,\hat{X}} \cong \mathcal{O}(-1,2)$. Since we have the following commutative diagram of embeddings:

$$\begin{array}{ccc} Q & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \hat{\tilde{Z}}_0 & \longrightarrow & X \end{array}$$

we know that $T\hat{\tilde{Z}}_0 \mid_Q /TQ \to (T\hat{X} \mid_Y /TY) \mid_Q$ is injective bundle homomorphism of these two line bundles, hence $\nu_{Q,\hat{\tilde{Z}}_0} \cong \nu_{Y,\hat{X}}|_Q \cong \mathcal{O}(-2,2)$. Hence $\hat{\tilde{Z}}_0$ is also the blown-up twistor space of Eguchi-Hanson orbifold.

So Z_0 is the singular space with normal crossing singularity defined by LeBrun-Singer [44], it is then straightforward to check that $\pi : \tilde{Z} \to \mathbb{C}$ is a standard deformation of Z_0. Since when ϵ is real and $\epsilon \to 0$, $Q - \epsilon Q_1 \to Q$ and $Q - \epsilon Q_2 \to Q$, so the above shows explicitly, in the picture of Donaldson-Friedman-LeBrun-Singer, the degeneration of the self-dual Poon-LeBrun metrics on $2\mathbb{CP}_2$ to two Eguchi-Hanson metrics when the two points used in the hyperbolic ansatz approach a common point in a suitable way.

5.4 Explicit Degeneration: $n > 2$ Case

We now study the general case of $n > 2$. It is similar to the discussion of Section 5.3.

Consider the variety:

$$\tilde{\mathcal{Z}} = \mathbb{P}\{xy = t^2(P - \epsilon^2 P_1) \cdots (P - \epsilon^2 P_n)\},$$

where $x \in \mathcal{O}(n-1,1)$, $y \in \mathcal{O}(1, n-1)$ and $t \in \mathcal{O}, \epsilon \in \mathbb{C}$ are variables,

$P, P_1, \cdots, P_n$ are fixed real sections of $\mathcal{O}(1,1)$ over $\mathbb{CP}_1 \times \mathbb{CP}_1$. There is a natural projection: $\Pi : \tilde{\mathcal{Z}} \to \mathbb{C}$ by mapping (x, y, t, ϵ) to ϵ, let $\tilde{Z}_\epsilon = \Pi^{-1}(\epsilon)$.

The open subset of $\tilde{\mathcal{Z}}$ where $t \neq 0$ can be identified with

$$\{xy = (P - \epsilon^2 P_1) \cdots (P - \epsilon^2 P_n)\} \subset (\mathcal{O}(n-1,1) \oplus \mathcal{O}(1, n-1)) \times \mathbb{C}.$$

It contains all the singular points of $\tilde{\mathcal{Z}}$, which form the following sets:

$$l_{jk} = \{x = y = P - \epsilon^2 P_j = P - \epsilon^2 P_k = 0\}.$$

For $\epsilon \neq 0$, $\tilde{Z}_\epsilon$ has conjugate pairs of singular points at the places where $P - \epsilon^2 P_j = P - \epsilon^2 P_k = 0$; when $\epsilon = 0$, the whole rational line $\{P = 0\} \in \mathbb{CP}_1 \times \mathbb{CP}_1$ is the singular set of $\tilde{Z}_0$. To simultaneously resolve the singularity, we blow up $\tilde{\mathcal{Z}}$ along the rational curve $\{x = y = P = \epsilon = 0\}$ in $(\mathcal{O}(n-1,1) \oplus \mathcal{O}(1, 1-n) \oplus \mathcal{O}) \times \mathbb{C}$ with weights $(n, n, 2, 1)$ for (x, y, P, ϵ). We denote the blow-up space of $\tilde{\mathcal{Z}}$ by $\hat{\mathcal{Z}}$, we then have an induced projection: $\hat{\Pi} : \hat{\mathcal{Z}} \to \tilde{\mathcal{Z}} \to \mathbb{C}$. Let $\hat{Z}_\epsilon = \hat{\Pi}^{-1}(\epsilon)$. We then have

$$\hat{Z}_0 = \hat{\Pi}^{-1}(0) = \hat{\tilde{Z}}_0 \cup E,$$

where $\hat{\tilde{Z}}_0$ is the strict transform of $\tilde{Z}_0$ under the weighted blow-up, E is the exceptional set in $\hat{\mathcal{Z}}$. Now

$$E \cong \mathbb{P}_{(n,n,2,1)}\{\bar{x}\bar{y} = (z - \bar{\epsilon}^2 \bar{P}_1) \cdots (z - \bar{\epsilon}^2 \bar{P}_n)\},$$

where $\bar{x}, \bar{y} \in \mathcal{O}(n)$, $z \in \mathcal{O}(2)$ are variables, $\bar{P}_1, \cdots, \bar{P}_n$ are the restrictions of $P_1, \cdots P_n$ to the rational curve $P = 0$ in $\mathbb{CP}_1 \times \mathbb{CP}_1$. Hence E is the blown-up twistor space of a Gibbons-Hawking orbifold along the singular twistor line by

Section 5.2. Moreover,

$$\begin{aligned} Q &= \hat{\hat{Z}}_0 \cap E \\ &= \mathbb{P}(n,n,2,1)\{\bar{x}\bar{y} = (z - \bar{\epsilon}^2 \bar{P}_1)\cdots(z - \bar{\epsilon}^2 \bar{P}_n), \epsilon = 0\} \\ &\cong \mathbb{P}_{(n,n,2)}\{\bar{x}\bar{y} = z^n\} \\ &\cong \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbb{CP}_1 \times \mathbb{CP}_1, \end{aligned}$$

where the first factor is from the fiber, the second from the base.

One can identify $\hat{\hat{Z}}_0$ with the strict transform of $\mathbb{P}\{xy = t^2 P^n\}$ when we blow up $X = \mathbb{P}(\mathcal{O}(n-1,1) \oplus \mathcal{O}(1,n-1) \oplus \mathcal{O})$ along the rational curve $C = \mathbb{P}\{x = y = P = 0, t \neq 0\}$ with weights $(n,n,2)$ for (x,y,P). From Section 5.2, $\hat{\hat{Z}}_0$ is the blow-up twistor space of the LeBrun orbifold along the singular twistor line.

Now the singular set of $\hat{\mathcal{Z}}$, in some suitable local coordinates, can be put into the form:

$$xy = (z - f_1(z,w))\cdots(z - f_n(z,w)),$$

it is independent of ϵ, so we can simultaneously resolve the singularity, notice that on each $\hat{Z}_\epsilon$, the singular points appear in pairs, we use the techniques in [41] to make suitable resolution such that the real structure on $\tilde{\mathcal{O}}$ lifts to the resolved space $\check{\mathcal{Z}}$. Again the surfaces $x = t = 0$ and $y = t = 0$ have normal bundles $\mathcal{O}(-1, 1-n)$ and $\mathcal{O}(1-n,-1)$ respectively in $\hat{Z}_\epsilon$. They can be blown down to rational curves with normal bundle $\mathcal{O}(1-n) \oplus \mathcal{O}(1-n)$, denote the space we get by $\mathcal{Z}$.

There is an induced projection $\pi : \tilde{Z} \to \mathbb{C}$, then $Z_0 = \pi^{-1}(0)$ is the singular space with $V-$normal crossing singularity defined by LeBrun-Singer

[44]. One can easily see that $\pi : \tilde{Z} \to \mathbb{C}$ is a standard V-deformation of Z_0. Since when $\epsilon \to 0$, $P - \epsilon^2 P_1, \cdots, P - \epsilon^2 P_n$ also go to P, so the above gives us the explicit degeneration of self-dual LeBrun metrics on $n\mathbb{CP}_2$ in the picture of Donaldson-Friedman-LeBrun-Singer when the points in the hyperbolic ansatz approach a common point in a suitable way. This was conjectured by LeBrun [41].

Notice that there is a $\mathbb{Z}_2$-symmetry on $\tilde{Z}$ by mapping ϵ to $-\epsilon$, when n is even, the $\mathbb{Z}_2$-action on Z_0 is trivial, and the quotient by $\mathbb{Z}_2$ also gives us a standard deformation, when $n = 2$, this is what we discussed in the last section.

Chapter 6

Examples

We will give some examples of self-dual orbifolds with $H^2(Z, \tau_Z^{0V}) = 0$.

6.1 LeBrun Orbifolds

Recall that we can get the twistor space Z of a LeBrun orbifold from

$$\tilde{Z} = \mathbb{P}\{x \in \mathcal{O}(n-1,1), y \in \mathcal{O}(1,n-1), t \in \mathcal{O} | xy = t^2 P^n\}$$

by blowing down two surfaces $x = t = 0$ and $y = t = 0$ to rational curves L_1, L_2 with normal bundles $\nu_{L_1} \cong \nu_{L_2} \cong \mathcal{O}(1-n) \oplus \mathcal{O}(1-n)$. Now from the short exact sequence:

$$0 \to \tau_{Z,L_1 \cup L_2}^{0V} \to \tau_Z^{0V} \to \nu_{L_1} \oplus \nu_{L_2} \to 0$$

we have a long exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^0(\tau_{Z,L_1 \cup L_2}^{0V}) \to H^0(\tau_Z^{0V}) \to H^0(\nu_{L_1} \oplus \nu_{L_2}) \\ \to \quad & H^1(\tau_{Z,L_1 \cup L_2}^{0V}) \to H^1(\tau_Z^{0V}) \to H^1(\nu_{L_1} \oplus \nu_{L_2}) \\ \to \quad & H^2(\tau_{Z,L_1 \cup L_2}^{0V}) \to H^2(\tau_Z^{0V}) \to H^2(\nu_{L_1} \oplus \nu_{L_2}) \cong 0. \end{aligned}$$

In particular, $H^2(\tau^{0V}_{Z,L_1\cup L_2}) \to H^2(\tau^{0V}_Z)$ is surjective. By Leray spectral sequence for the blowing down map $\tilde{Z} \to Z$, we have $H^i(\tau^{0V}_{Z,L_1\cup L_2}) \cong H^i(\tau^{0V}_{\tilde{Z}})$ So to show that $H^2(Z,\tau^{0V}_Z) \cong 0$, it suffices to show that $H^2(\tau^{0V}_{\tilde{Z}}) \cong 0$.

We now apply Leray spectral sequence for the natural projection $p : \tilde{Z} = \mathsf{P}\{xy = t^2P^n\} \to \mathbb{CP}_1 \times \mathbb{CP}_1$. At points where $P \neq 0$, the fibers are smooth quadrics ($\cong \mathbb{CP}_1$) and $\tau^{0V}_{\tilde{Z}}|_{\mathbb{CP}_1} \cong \mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}$. Hence $H^i(\mathbb{CP}_1, \tau^{0V}_{\tilde{Z}}) = 0$ for $i > 0$. At points where $P = 0$, the fibers have normal crossing singularities. They are singular quadrics isomorphic to $\mathbb{CP}_1 \vee \mathbb{CP}_1$, and $\tau^{0V}_{\tilde{Z}}|_{\mathbb{CP}_1\vee\mathbb{CP}_1} \cong \tau^0_{\mathbb{CP}_1\vee\mathbb{CP}_1} \oplus \mathcal{O}$, using the normalization of $\mathbb{CP}_1 \vee \mathbb{CP}_1$ as in [15], it is easy to show that $H^i(\mathbb{CP}_1 \vee \mathbb{CP}_1, \tau^0_{\mathbb{CP}_1\vee\mathbb{CP}_1}) = H^i(\mathbb{CP}_1 \vee \mathbb{CP}_1, \mathcal{O}) = 0$, hence $H^i(\mathbb{CP}_1 \vee \mathbb{CP}_1, \tau^{0V}_{\tilde{Z}}|_{\mathbb{CP}_1\vee\mathbb{CP}_1}) = 0$ for $i = 1, 2$. Hence $p^i_*(\tau^{0V}_{\tilde{Z}}) = 0$ for $i = 1, 2$. So $H^2(\tilde{Z}, \tau^{0V}_{\tilde{Z}}) \cong H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, p_*\tau^{0V}_{\tilde{Z}})$. From the following exact sequence

$$0 \to \mathcal{V} \to \tau^{0V}_{\tilde{Z}} \to p^*\tau_{\mathbb{CP}_1\times\mathbb{CP}_1,C} \to 0$$

where $\mathcal{V}$ is the sheaf of holomorphic tangents along the fibers, and C is the curve in $\mathbb{CP}_1 \times \mathbb{CP}_1$ defined by $P = 0$. Since the above argument can be used to prove that $p^i_*\mathcal{V} = 0$ for $i = 1, 2$, we have the following exact sequence

$$0 \to p_*\mathcal{V} \to p_*\tau^{0V}_{\tilde{Z}} \to \tau_{\mathbb{CP}_1\times\mathbb{CP}_1,C} \to 0.$$

We can use the following exact sequence

$$0 \to \tau_{\mathbb{CP}_1\times\mathbb{CP}_1,C} \to \tau_{\mathbb{CP}_1\times\mathbb{CP}_1} \to \nu_C(\cong \mathcal{O}(2)) \to 0$$

to see that $H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, \tau_{\mathbb{CP}_1\times\mathbb{CP}_1,C}) = 0$, hence

$$H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, p_*\mathcal{V}) \to H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, p_*\tau^{0V}_{\tilde{Z}})$$

is surjective. Now on $\mathbb{P}\{xy = at^2\}$, when $a \neq 0$, $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, 2at\frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, 2at\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}$ span all the holomorphic vector fields; when $a = 0$, $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial t}, x\frac{\partial}{\partial t}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ span all the holomorphic vector fields. so we have the following exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{O}\{x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, 2P^n t\frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, 2P^n t\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}\} \\ &\rightarrow p_*\mathcal{V} \rightarrow \mathcal{O}_C\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\} \rightarrow 0. \end{aligned}$$

From this we get $H^2(p_*\mathcal{V}) = 0$, hence from the above results, we have the following

Proposition. *If Z is the twistor space of a LeBrun orbifold, then we have* $H^2(Z, \tau_Z^{0V}) = 0$.

6.2 Weitzenböck Formula

The deformation of self-dual conformal classes near $[g]$ on a closed oriented 4-manifold M can be described by the following complex:

$$\Gamma(TM) \xrightarrow{\delta^*} \Gamma(S_0^2 T^* M) \xrightarrow{DW_-} \Gamma(S_0^2 \Lambda_-^2),$$

where DW_- is the linearization of W_-. It is an elliptic complex with index $\frac{1}{2}(15e(M) - \tau(M))$, if we denote the cohomology of this complex by $H_g^*(M)$, then there is an analytic map $\psi : U \rightarrow H_g^2$ with $T_0\psi = 0$, where U in a neighborhood of $0 \in H_g^1$, such that $\psi^{-1}(0)$ is isomorphic to a neighborhood of $[g]$ in the moduli space of self-dual conformal structures [19], [36]. Furthermore, Penrose transform gives us isomorphisms $H^i(Z,TZ) \cong H_g^i \otimes \mathbb{C}$ for $i = 0,1,2$ and $H^3(Z,TZ) = 0$, where Z is the twistor space of $(M,[g])$.

All these can be generalized to orbifolds. The reason is that most of the arguments used to get the above results consist of two parts. The first is of of global nature, which uses results on Banach spaces, having nothing to do with the orbifold structure. the second is of local nature, near orbifold points, we can use the uniformization system and consider invariant objects.

V-bundles over orbifolds were defined in [7], we can then consider elliptic operators on the spaces of sections of V-bundles. They are Fredholm operators and the index theorems are given by Kawasaki [32]. By abuse of notation, we use the same notations to denote natural V-bundles over orbifolds, the corresponding differential operators and cohomologies. Then the moduli space of orbifolds self-dual conformal structures can be locally be described by H_g^1, H_g^2 as above. The Penrose transform can also be performed on self-dual orbifolds. This is because Dolbeault theorem holds for complex orbifolds. Following [36], we start with the Dolbeault complex

$$\Gamma TZ \xrightarrow{\bar\partial} \Gamma(\Omega^{0,1} \otimes TZ) \xrightarrow{\bar\partial} \Gamma(\Omega^{0,2} \otimes TZ) \xrightarrow{\bar\partial} \Gamma(\Omega^{0,3} \otimes TZ)$$

use the exact sequence $0 \to T_\pi TZ \to N \to 0$, where T_π is the bundle of vertical tangent vectors, and Salamon's cohomology lemma [56] once, we get Floer's exact sequences for orbifolds [21], use Salamon's lemma again, $H^*(Z,TZ)$ is identified with the cohomology of some complex:

$$\Gamma(TM) \xrightarrow{D_1} \Gamma(S_0^2 T^*M) \xrightarrow{D_2} \Gamma(S_0^2 \Lambda_-^2).$$

The identification of D_1, D_2 can be done using local uniformization systems. By [21], they are given by linearizations of the group action and anti-self-dual Weyl tensor. To sum up, we have the following

Proposition. *Let (M,g) be a self-dual orbifolds, Z its twistor space, TZ the tangent V-bundle of Z, then*

$$H^i(Z,\tau_Z^{0V}) = H^i(Z,TZ) = H_g^i$$

for $i=0,1,2$ and $H^3(Z,\tau_Z^{0V})=0$.

If $(DW_-)^*$ is the formal adjoint of DW_2, the elliptic operator theory over orbifolds can be used to see that $H_g^2 \cong Ker(DW_-)^*$. Using spinor formalism, $DW_- : S_0^2T^*M \to S_0^2\Lambda_-^2$ is given by the following compositions [21]:

$$DW_- : S^2V_+ \otimes S^2V_- \xrightarrow{D_1} V_- \otimes V_+ \otimes S^2V_- \xrightarrow{D_2} V_- \otimes V_- \otimes S^2V_- \xrightarrow{\pi_1} S^4V_-,$$

where $V_\pm$ are spinor bundles, D_i is the Dirac operator on the i-th factor and π_1 is symmetrization map. Here we don't require M to be a spin manifold, even though V_+, V_- may not be defined globally, the tensor product we need is defined globally nevertheless, and we use uniformization systems near orbifold points. It is easy to see that the formal adjoint of DW_- is given by the composition of the following maps:

$$(DW_-)^* : S^4V_- \xrightarrow{D_2^*} V_- \otimes V_+ \otimes S^2V_- \xrightarrow{D_1^*} V_+ \otimes V_+ \otimes S^2V_- \xrightarrow{\pi_2} S^2V_+ \otimes S^2V_-,$$

where π_2 is the symmetrization map, hence

$$(DW_-)\cdot(DW_-)^* = \pi_1 D_2 D_1 \pi_2 D_1^* D_2^*.$$

By direct computations,

$$\begin{aligned}
& D_2^*(v_a \otimes v_b \otimes s^{ab}) \\
= & (\nabla^i v_a) \otimes (e_i v_b) \otimes s^{ab} + v_a \otimes (e_i \nabla^i v_b) \otimes s^{ab} + v_a \otimes (e_i v_b) \otimes (\nabla^i s^{ab}),
\end{aligned}$$

$$\begin{aligned}
& D_1^* D_2^*(v_a \otimes (e_j v_b) \otimes s^{ab}) \\
= & (e_j \nabla^j \nabla^i v_a) \otimes (e_i v_b) \otimes s^{ab} + (e_j \nabla^i v_a) \otimes \nabla^j (e_i v_b) \otimes s^{ab} \\
& +(e_j \nabla^i v_a) \otimes (e_i v_b) \otimes \nabla^j s^{ab} + (e_j \nabla^j v_a) \otimes (e_i \nabla^i v_b) \otimes s^{ab} \\
& +(e_j v_a) \otimes \nabla^j (e_i \nabla^i v_b) \otimes s^{ab} + (e_j v_a) \otimes (e_i \nabla^i v_b) \otimes \nabla^j s^{ab} \\
& +(e_j \nabla^j v_a) \otimes (e_i v_b) \otimes (\nabla^i s^{ab}) + (e_j v_a) \otimes \nabla^j (e_i v_b) \otimes (\nabla^i s^{ab}) \\
& +(e_j v_a) \otimes (e_i v_b) \otimes (\nabla^j \nabla^i s^{ab}) \\
= & (e_j \nabla^i v_a) \otimes (e_i \nabla^j v_b) \otimes s^{ab} + (e_j \nabla^j v_a) \otimes (e_i \nabla^i v_b) \otimes s^{ab} \\
& +(e_j \nabla^j v_a) \otimes (e_i v_b) \otimes \nabla^j s^{ab} + (e_i v_b) \otimes (e_j \nabla^i v_a) \otimes \nabla^j s^{ab} \\
& +(e_j v_a) \otimes (e_i \nabla^i v_b) \otimes \nabla^j s^{ab} + (e_i \nabla^i v_b) \otimes (e_j v_a) \otimes \nabla^j s^{ab} \\
& +(e_i v_a) \otimes (e_j \nabla^i \nabla^j v_b) \otimes s^{ab} + e_j (\nabla^i \nabla^j - R_{V_-}^{ij}) v_b \otimes (e_i v_a) \otimes s^{ab} \\
& +\frac{1}{2}(e_i v_a \otimes e_j v_b + e_j v_b \otimes e_i v_a) \otimes \nabla^i \nabla^j s^{ab} \\
& +\frac{1}{2} e_i v_a \otimes e_j v_b \otimes R_{S^2 V_-}^{ij} s^{ab},
\end{aligned}$$

where $R_{V_-}, R_{S^2 V_-}$ are the curvatures of the corresponding bundles, for simplicity of notations, we will only use R^{ij}. It is easy to see that the only terms which is possibly not symmetric are:

$$\begin{aligned}
& (e_i v_a) \otimes e_j R_{V_-}^{ij} v_b \otimes s^{ab} + \frac{1}{2}(e_i v_a) \otimes (e_j v_b) \otimes R_{S^2 V_-}^{ij} s^{ab} \\
= & \omega_{+i}(I \otimes R^{+i})(v_a \otimes v_b) \otimes s^{ab} + \omega_{-i}(I \otimes R^{-i})(v_a \otimes v_b) \otimes s^{ab} \\
& +\frac{1}{2}\omega_{+i}(v_a \otimes v_b) \otimes R^{+i} s^{ab} + \frac{1}{2}\omega_{-i}(v_a \otimes v_b) \otimes R^{-i} s^{ab},
\end{aligned}$$

where $R^{\pm 1} = R^{12} \pm R^{34}$, etc. $\omega_{+i}(I \otimes R^{+i})(v_a \otimes v_b) \otimes s^{ab}$ lies in $S^2V_+ \otimes S^2V_-$ because by Schur's lemma, an invariant map $S^2V_+ \otimes (V_- \otimes V_-) \to V_+ \otimes V_+$ has image in S^2V_+; $\omega_{+i}(v_a \otimes v_b) \otimes R^{+i}s^{ab}$ is zero since any invariant map $S^2V_+ \otimes S^2V_- V_+ \otimes V_+$ is zero; $\omega_{-i}(I \otimes R^{-i})(v_a \otimes v_b) \otimes s^{ab} = \frac{s}{12}\omega_{-i}(I \otimes \omega_{-i})(v_a \otimes v_b) \otimes s^a b = 0$ since any invaraint map $S^4V_- \to V_+ \otimes V_+ \otimes S^2V_-$ is zero; similarly, $\omega_{-i}(v_a \otimes v_b) \otimes R^{-i}s^{ab} = \frac{s}{12}\omega_{-i}(v_a \otimes v_b) \otimes \omega^{-i}s^{ab} = 0$. Hence $\pi_2 D_2^* D_1^* = D_2^* D_1^*$.

A similar computaion gives:

$$
\begin{aligned}
& D_1^* D_2^* - D_2^* D_1^* (v_a \otimes v_b \otimes s^{ab}) \\
= \;& e_i R^{ij} v_a \otimes v_a \otimes e_j v_b \otimes s^{ab} + e_i v_a \otimes e_j R^{ij} v_b \otimes s^{ab} \\
+ \;& \omega_{+i}(v_a \otimes v_b) \otimes R^{+i}s^{ab} + \omega_{-i}(v_a \otimes v_b) \otimes R^{-i}s^{ab} \\
= \;& \omega_{+i}R^{+i}(v_a \otimes v_b) \otimes s^{ab} + \omega_{-i}R^{-i}(v_a \otimes v_b) \otimes s^{ab} \\
+ \;& \omega_{+i}(v_a \otimes v_b) \otimes R^{+i}s^{ab} + \omega_{-i}(v_a \otimes v_b) \otimes R^{-i}s^{ab} \\
= \;& 0,
\end{aligned}
$$

by using Schur's lemma as above. Hence $D_2^* D_1^* = D_1^* D_2^*$. Taking formal adjoint, we get $\pi_1 D_2 D_1 = \pi_1 D_1 D_2$. Hence

$$(DW_-)(DW_-)^* = \pi_1 D_2 D_1 D_1^* D_2^* = \pi_1 D_1 D_2 D_1^* D_2^*.$$

The same type of computation gives:

$$
\begin{aligned}
& (D_1^* D_2 - D_2 D_1^*)(v_a \otimes u_b \otimes s^{ab}) \\
= \;& e_i R^{ij} v_a \otimes (e_j u_b) \otimes s^{ab} + (e_i v_a) \otimes e_j R^{ij} u_b \otimes s^{ab} + (e_i v_a) \otimes (e_j u_b) \otimes R^{ij} s^{ab} \\
= \;& \omega_{+i}(R^{+i} \otimes I)(v_a \otimes v_b) \otimes s^{ab} + \omega_{-i}(I \otimes R^{-i})(v_a \otimes v_b) \otimes s^{ab} \\
+ \;& \omega_{-i}(R^{-i} \otimes I)(v_a \otimes v_b) \otimes s^{ab} + \omega_{+i}(I \otimes R^{+i})(v_a \otimes v_b) \otimes s^{ab}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \omega_{+i}(v_a \otimes v_b) \otimes R^{+i} s^{ab} + \omega_{-i}(v_a \otimes v_b) \otimes R^{-i} s^{ab} \\
&= \quad \omega_{+i}(v_a \otimes v_b) \otimes R^{+i} s^{ab} - \frac{s}{12}(u_b \otimes v_a \otimes s^{ab}) \\
&= \quad \phi(v_a \otimes u_b \otimes s^{ab}) - \frac{s}{12}\sigma_{12}(u_b \otimes v_a \otimes s^{ab}),
\end{aligned}
$$

where ϕ depends linearly on traceless Ricci curvature and σ_{12} is the linear map which switch the the fisrt and second factors of the tensor product. Now since

$$
D_1 D_1^* = D_2 D_2^* = \nabla^* \nabla + \omega_{-i} R^{-i} = \nabla^* \nabla + \frac{s}{2},
$$

we have

$$
\begin{aligned}
& (DW_-)(DW_-)^* = \pi_1 D_1 D_2 D_1^* D_2^* \\
= \quad & \pi_1 D_1 (D_1^* D_2^* - \phi + \frac{s}{12}\sigma_{12}) D_2^* \\
= \quad & (\nabla^* \nabla + \frac{s}{2})^2 - \pi_1 D_1 \phi D_2^* + \pi_1 \frac{s}{12} \sigma_{12} D_2 D_2^* \\
= \quad & (\nabla^* \nabla)^2 + \frac{13s}{12}(\nabla^* \nabla) + \frac{7s^2}{24} - \pi_1 D_1 \phi D_2^*
\end{aligned}
$$

So when (M, g) is Einstein, we have the following:

$$
(DW_-)(DW_-)^* = (\nabla^* \nabla)^2 + \frac{13s}{12}(\nabla^* \nabla) + \frac{7s^2}{24}.
$$

Prosposition. *Let (M^4, g) be an oriented compact Riemannian 4-manifold or orbifold. If (M, g) is self-dual Einstein with positive scalar curvature, we have $Ker(DW_-)^* = 0$ hence $H^2(Z, TZ) = 0$; if (M, g) is self-dual Ricci-flat, any element of $Ker(DW_-)^*$ is parallel. If (M, g) is self-dual Einstein with negative scalar curvture but $-\frac{s}{2}$ and $-\frac{7s}{12}$ are not eigenvalues of $\nabla^* \nabla$, we also have $Ker(DW_-)^* = 0$, hence $H^2(Z, TZ) = 0$.*

To get examples, notice that self-dual Einstein manifolds/orbifolds are quaternionic Kähler with reversed orientation, hence can be constructed by quaternionic Kähler reduction [25]; self-dual Ricci-flat manifolds/orbifolds are hyperkähler with reversed orientation, they can be constructed by Gibbons-Hawking ansatz.

6.3 Quaternionic Kähler Reduction

A manifold (M, g) of dimension $4n$ $(n > 1)$ is quaternionic Kähler if its holonomy group is contained in $Sp_n Sp_1 \subset SO(4n)$. To describe the quaternionic Kähler reduction, it is better to use the following equivalent picture: let $\mathcal{G} \subset Hom(TX, TX)$ be a 3-dimensional subbundle such that near each point we have local smooth sections J_1, J_2, J_3 such that

$$J_i \circ J_j = -\delta_{ij} Id + \epsilon_{ijk} J_k$$

and $g(J_i V, J_i W) = g(V, W)$. Using the metric, we get an isometric embedding $\mathcal{G} \subset \Lambda^2 T^* M$ by $J \mapsto \omega_J$, where $\omega_J(V, W) = g(JV, W)$. Let $\Omega = \sum_{i=1}^{3} \omega_{J_i} \wedge \omega_{J_i}$, then Ω is globally defined and $\Omega^n = (2n+1)! dvol_g$. Then (M, g) is quaternionic Kähler if and only if $\nabla \Omega = 0$, where ∇ is the Levi-Civita connection. We call Ω the fundamental 4-form of (M, g). It becomes self-dual if we reverse the orientation.

Regard the Riemann curvature tensor as a symmetric endomorphism $R : \Lambda^2 \to \Lambda^2$, if (M, g) is quaternionic Kähler, we have $R|_{\mathcal{G}} = \lambda Id_{\mathcal{G}}$ for some constant λ. So an oriented Riemannian 4-manifolds is defined to be quaternionic Kähler if it satisfies the above property. In this case $\mathcal{G} = \Lambda_+$, $R|_{\Lambda_+} = (\frac{s}{12}Id + W_+ \oplus Z)$, so (M^4, g) is quaternionic Kähler if and only if it is anti-self-dual Einstein.

Let (M, g) be a quaternionic Kähler manifold of dimension > 4, then $\nabla\Omega = 0$ implies $d\Omega = 0$. If a Lie group H acts on M by isometries and preserves Ω, Galicki-Lawson [25] gave an analogue of Marsden-Weinstein reduction. Let $\mathcal{H}$ denote the Lie algebra of H, the group action induces a homomorphism $\mathcal{H} \to Vect(M)$, where $Vect(M)$ is the space of vector fields on M. Denote the image of $V \in \mathcal{H}$ by $\tilde{V}$. When the scalar curvature of the quaternionic Kähler manifold is nonzero, there is a moment map

$$f : \mathcal{H} \to \Gamma(\mathcal{G}) \subset \Lambda^2$$

defined uniquely by $\nabla f_V = \sum_j (i_{\tilde{V}}\omega_i) \otimes \omega_i$. f is H-equivariant and can be thought of as a section of $\mathcal{H}^* \otimes \mathcal{G} \subset \Gamma(\Omega^2(\mathcal{H}^*))$. Let $Z_H = \{x \in X | f(x) = 0\}$. If H acts freely on Z_H, then Z_H/H equipped with the submersed metric (the one which makes $Z_H \to Z_H/H$ a Riemannian submersion) is a quaternionic Kähler manifold. If $H \cong S^1$ and H acts locally freely on Z_H (i.e. all the isotropy groups are finite), Z_H/H is then a quaternionic Kähler orbifold.

Let (q_0, q_1, q_2) be the linear coordinates on the quaternionic vector space $\mathbb{H}^3$ with multiplications from the right. The quaternionic projective space $\mathbb{HP}_2 = (\mathbb{H}^3 - \{0\}/\mathbb{H}^*$ has the canonical symmetric space metric which is quaternionic Kähler and Sp_3-invariant. Notice that all isometries of $\mathbb{HP}_2$ preserve Ω. Let n_0, n_1, n_2 be three natural numbers with $(n_0, n_1, n_2) = 1$, consider the following action of S^1 on $\mathbb{HP}_2$ defined by

$$e^{it} \cdot (q_0, q_1, q_2) = (e^{n_0 it} q_0, e^{n_1 it} q_1, e^{n_2 it} q_2).$$

Then the zero set of the moment map is

$$Z = \{(q_0, q_1, q_2) \in \mathbb{HP}_2 | n_0 \bar{q}_0 i q_0 + n_1 \bar{q}_1 i q_1 + n_2 \bar{q}_2 i q_2 = 0\}.$$

Proposition. *We have the following identification of orbifolds:*

$$Z/S^1 \cong \mathbb{CP}_{n_1+n_2, n_2+n_0, n_0+n_1}.$$

Proof. Let $\sqrt{n_l} = u_l + v_l j$, then the equation is equivalent to

$$|u_0|^2 + |u_1|^2 + |u_2|^2 = |v_0|^2 + |v_1|^2 + |v_2|^2,$$

$$u_0 \bar{v}_0 + u_1 \bar{v}_1 + u_2 \bar{v}_2 = 0.$$

The action of $\mathbb{H}^*$ on these homogeneous coordinates are give by

$$(u_l, v_l) \mapsto (u_l x - v_l \bar{y}, u_l y + v_l \bar{x}),$$

where $x + yj \in \mathbb{H}^*$. Hence we have

$$Z \cong \{(u, v) \in \mathbb{C}^3 \times \mathbb{C}^3 | |u| = |v| = 1, u \perp v\}/SU(2).$$

This can be further identified with S^5 by

$$(u, v) \mapsto (z_0, z_1, z_2) = (u_1 v_2 - u_2 v_1, u_2 v_0, u_0 v_2, u_0 v_1 - u_1 v_0).$$

It is direct to see this map has image in S^5 and it induces a diffeomorphism $Z \cong S^5$. Now the S^1-action on Z is identified with the following action on S^5:

$$e^{it} \cdot (z_0, z_1, z_2) = (e^{i(n_1+n_2)t} z_0, e^{i(n_2+n_0)t} z_1, e^{i(n_1+n_2)t} z_2).$$

Hence $Z/S^1 \cong \mathbb{CP}_{n_1+n_2, n_2+n_0, n_0+n_1}$. Q.E.D.

Notice that the action of S^1 on Z is locally free, so we get many examples of weighted projective spaces with self-dual Einstein metrics of positive scalar curvature. When $(n_1 + n_2, n_2 + n_0) = (n_2 + n_0, n_0 + n_1) = (n_0 + n_1, n_1 + n_2)$, $\mathbb{CP}_{n_1+n_2, n_2+n_0, n_0+n_1}$ has only isolated orbifold points admitting self-dual Einstein metrics. In particular, we can take $(n_0, n_1, n_2) = (1, n, n+1)$ for $n > 1$, or $(n_0, n_1, n_2) = (2, 2n+1, 2n+3)$ for $n \geq 1$, etc.

6.4 Gibbons-Hawking Orbifolds

Gibbons-Hawking ansatz can be used to construct examples of hyperkähler manifolds or orbifolds. Let $q_1, \cdots, q_k$ be k distinctive points in $\mathbb{R}^3$ with Euclidean metric ds^2, and $n_1, \cdots, n_k$ positive integers with $n = n_1 + \cdots + n_k$. Define

$$V = \frac{1}{2} \sum_{j=1}^{k} \frac{n_j}{|x - q_j|}$$

on $\mathbb{R}^3$. If $*$ is the Hodge star operator on $\mathbb{R}^3$, then the integral of $\frac{1}{2\pi} * dV$ on the boundaries of small balls centered at q_j is n_j, since such spheres generates $H_2(\mathbb{R}^3 - \{q_j\}, \mathbb{Z})$, we have $[\frac{1}{2\pi} * dV] \in H^2(\mathbb{R}^3 - \{q_j\}, \mathbb{Z}) \hookrightarrow H^2_{dR}(\mathbb{R}^3 - \{q_j\}, \mathbb{R})$. By Chern-Weil theory, there is a principal S^1-bundle $\pi_0 : M_0 \to \mathbb{R}^3 - \{q_j\}$

with connection 1-form $\omega \in \Omega^1(M_0)$ such that

$$\pi_0^*(*dV) = d\omega.$$

The connection form ω is unique up to gauge transformations, since $\mathbb{R}^3 - \{q_j\}$ is simply connected. We define the following metric on M_0:

$$g = \frac{1}{V}\omega \odot \omega + V(\pi_0^* ds^2).$$

It is a Ricci-flat Kähler metric and anti-self-dual with respect to the orientation defined by the complex structure. We construct a space M by adding a point p_j for each q_j, then M is an orbifold with singular points p_j of type $\mathbb{Z}_{n_j}(1,1)$ if $n_j > 1$. Using a similar argument as in [41], we can see the metric extends to an orbifold metric on M, it is an ALE hyperkähler orbifold metric, so if $\bar{M}$ is the one-point compactification of M by addding a point ∞ at infinity, the $\bar{M}$ is an orbifold with an extra orbifold point ∞ of type $\mathbb{Z}_n(1,-1)$, and the conformal class of the metric g on $M - 0$ extends to a self-dual conformal structure on $\bar{M}$ with respect to the reversed orientation.

Proposition. *Let $\bar{Z}$ be the twistor space of $\bar{M}$ with the above conformal structure and $T\bar{Z}$ be its orbifold tangent bundle, then $H^2(\bar{Z}, T\bar{Z}) = 0$.* *Proof.* By

Penrose transform, we need to prove $Ker(D\bar{W}_-)^* = 0$, by conformal invariance, the elements of $KerD(\bar{W})$ correspond to the solutions of $(DW_-)^*\alpha = 0$ with the following decay condition:

$$|\alpha| = O(r^{-4}), |\nabla\alpha| = O(r^{-5}), |\nabla^8\nabla\alpha| = O(r^{-6}).$$

Now the Weitzenböck formula for hyperkähler metric is $(DW_-)(DW_-)^* = (\nabla^*\nabla)^2$, hence

$$\int_M |\nabla^*\nabla\alpha|^2 = \int_M ((DW_-)(DW_-)^*\alpha, \alpha) = 0.$$

This implies $|\nabla^*\nabla\alpha|^2 = 0$, so

$$\int_M |\nabla\alpha|^2 = \int_M (\nabla^*\nabla\alpha, \alpha) = 0.$$

Hence $\nabla\alpha = 0$, so α is parallel, but the decay condition force it to vanish. □

If we denote by $X(n)$ the self-dual LeBrun orbifold constructed by compactify the total space of $\mathcal{O}(-n) \to \mathbb{CP}_1$, and by $Y(n; n_1, \cdots, n_k)$ the self-dual orbifold constructed above. For each $n_j > 1$, we can remove the corresponding singular point by gluing with $Y(n_j; n_{j,1}, \cdots, n_{j,k_j})$, repeat this arbitrarily until we get only $Y(m; 1, \cdots, 1)$. This tree of orbifolds has compatible singularities indicated by the tree, and at the root, $Y(n; n_1, \cdots, n_k)$ has a singular point compatible with that of $X(n)$. Since the obstruction groups vanish for all of these orbifolds, the multiple connected sum admits a family of self-dual metrics. It is easy to see the multiple connected sum manifold is $n\mathbb{CP}_2$, so this shows that the degeneration of self-dual metrics on it can be very complicated.

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