On Teichmüller and Bers fiber spaces

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Abstract of the Dissertation
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This paper consists of three parts. In the first two, we concentrate on establishing relationships between Bers fiber spaces (which were introduced by L. Bers in the early 70's) and Teichmüller spaces. Our main results are extension of some results in Bers [8], Earle-Kra [19] [20], and Kra [28].

First, we assume that Γ is a finitely generated Fuchsian group of the first kind which contains elliptic elements. The question as to whether or not the Bers fiber space $F(\Gamma)$ is biholomorphically equivalent to a Teichmüller space was posed by Bers in the early 70's, and was initially investigated by C. J. Earle and I.

Kra in 1974. Let $F(g,n;\nu_1,\ldots,\nu_n)$ be the Bers fiber space $F(\Gamma)$ for Γ of signature $(g,n;\nu_1,\ldots,\nu_n)$, and $T(g',n')$ the Teichmüller space $T(\Gamma)$ for Γ of type (g',n'). Earle and Kra proved that, with the exclusion of finitely many signatures, $F(g,n;\nu_1,\ldots,\nu_n)$ is not equivalent to any Teichmüller space. On the other hand, it is well-known that there are at least five pairs $((g,n;\nu_1,\ldots,\nu_n),\ (g',n'))$ such that $F(g,n;\nu_1,\ldots,\nu_n)$ is equivalent to $T(g',n')$. After the work of Earle and Kra, there remain 39 unknown pairs. By analyzing the work of Earle and Kra, one can reduce the problem to the study of fixed point set of some specific periodic automorphisms in both $F(g,n;\nu_1,\ldots,\nu_n)$ and $T(g',n')$, which involves a careful investigation of the hyperelliptic loci in Teichmüller space. This observation, together with topological constructions of some special periodic self-maps of Riemann surfaces developed by Magnus [31], enables us to settle 27 cases (out of the 39 cases mentioned above). We point out here that our methods do not work for the remaining 12 cases, because for certain special pairs $((g,n;\nu_1,\ldots,\nu_n),\ (g',n'))$ all possible automorphisms acting on $F(g,n;\nu_1,\ldots,\nu_n)$ do act on $T(g',n')$ as well. We also reprove the theorem of Earle and Kra. As an interesting application, we establish a complete list of pairs $((g,n;\nu_1,\ldots,\nu_n),\ (g,n+1))$ for which there is a biholomorphic map between $F(g,n;\nu_1,\ldots,\nu_n)$ and $T(g,n+1)$.

When Γ is torsion free, a similar question was initially studied by Bers in 1973. He showed that there exist biholomorphic maps of $F(g,n;\infty,\ldots,\infty)$ onto $T(g,n+1)$. Kra has asked if the biholomorphic maps constructed by Bers can be extended to the boundary (see the introduction for the definition of boundary). We prove that those biholomorphic maps cannot be continuously extended to the boundary of $F(g,n;\infty,\ldots,\infty)$ in the case of dim $F(g,n;\infty,\ldots,\infty) \geq 2$.

We return to the case that Γ may have torsion, and continue to study the relationships between Teichmüller spaces and Bers fiber spaces. As a result, we give a complete solution to Kra's problem described above; that is, we prove that any biholomorphic map between these two moduli spaces (if exists) admits no homeomorphic extension to the boundary provided that dim $F(g,n;\nu_1,\ldots,\nu_n) \geq 2$. Note that in the case of dimension one, the question is exactly a famous conjecture of Bers which states that the Bers embedding of the Teichmüller space $T(0,4)$ is a Jordan domain.

Finally, we establish several results on fiber-preserving biholomorphic maps among Bers fiber spaces and on biholomorphic maps among Teichmüller curves. Those results are natural generalizations of several important theorems in Teichmüller spaces to Bers fiber spaces and Teichmüller curves; the proofs are included in this paper since they do not appear in the literature.

To my parents, my wife Ping, and my son Jimmy

Contents

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Introduction and statement of results

Let Γ be a finitely generated Fuchsian group of the first kind which is of type (g, n) and acts on the upper half plane U. The Teichmüller space $T(\Gamma)$ is the space of complex structures on the orbifold U/Γ modulo the equivalence relation, where two complex structures are equivalent if there is an isometry between them which is isotopic to the identity. Associated to each point $x \in T(\Gamma)$ is a certain Jordan domain D_x depending holomorphically on x. The Bers fiber space $F(\Gamma)$ over $T(\Gamma)$ is the set of points (x, z) with $x \in T(\Gamma)$ and $z \in D_x$.

The Teichmüller space $T(\Gamma)$, with $g \geq 0$, $n \geq 0$, and $3g - 3 + n > 0$, admits a representation as a simply connected domain in $\mathbb{C}^{3g-3+n}$. The Bers fiber space $F(\Gamma)$ is a simply connected domain in $\mathbb{C}^{3g-2+n}$.

In this paper, we consider the following general problem: find all biholomorphic maps (we will use the term "isomorphisms" throughout this paper) among various moduli spaces of Riemann surfaces. Let Γ' be another finitely generated Fuchsian group of the first kind, and let $\sigma = (g, n; \nu_1, \ldots, \nu_n)$, $\sigma' = (g', n'; \nu'_1, \ldots, \nu'_{n'})$ denote the signatures of Γ and Γ', respectively, where ν_i, ν'_j are integers or ∞ with $2 \leq \nu_1 \leq \cdots \leq \nu_n \leq \infty$ and $2 \leq \nu'_1 \leq \cdots \leq \nu'_{n'} \leq \infty$. It is well-known that as complex manifolds $F(\Gamma)$ depends on the signature of Γ, whereas $T(\Gamma')$ depends only on the type of Γ'. Let $F(g, n; \nu_1, \ldots, \nu_n)$ be the Bers fiber space $F(\Gamma)$ for Γ of signature $(g, n; \nu_1, \ldots, \nu_n)$, and $T(g', n')$ the

Teichmüller space $T(\Gamma')$ for Γ' of type (g', n'). In the attempt to establish all isomorphisms among various moduli spaces, we encounter in particular the following several problems:

(A1) Enumerate all pairs $((g, n; \nu_1, \ldots, \nu_n), (g', n'))$ for which there is an isomorphism between $F(g, n; \nu_1, \ldots, \nu_n)$ and $T(g', n')$.

(A2) Describe all possible isomorphisms between Teichmüller spaces and Bers fiber spaces.

(B1) Do the isomorphisms in (A2) admit homeomorphic extensions?

(B2) Do the isomorphisms in (A2) admit continuous extensions?

(C1) Enumerate all pairs $((g, n; \nu_1, \ldots, \nu_n), (g', n'; \nu'_1, \ldots, \nu'_{n'}))$ for which there is a fiber-preserving isomorphism between $F(g, n; \nu_1, \ldots, \nu_n)$ and $F(g', n'; \nu'_1, \ldots, \nu'_{n'})$.

(C2) Describe all fiber-preserving isomorphisms among Bers fiber spaces.

The original motivation for considering these questions comes from a paper of Royden [43], in which he proved that all automorphisms of $T(g, 0)$ with $g \geq 3$ are induced by self-maps of a surface of genus g. Later, Earle and Kra [19] generalized this result to all cases of analytically finite Riemann surfaces. On the other hand, in his paper [42], Patterson gave a complete solution to the problem of finding all isomorphisms among Teichmüller spaces. Since then, important progress concerning the isomorphisms between Bers fiber spaces and Teichmüller spaces has been made by Bers [8] and Earle-Kra [19]. In this paper, we continue to study isomorphisms of various moduli spaces of Riemann surfaces; our main results, which extend the work of Bers [8], Earle-Kra [19] [20], and Kra [28], give partial solutions to the above general problems.

The major part of this paper is an investigation of relationships between Bers fiber spaces and Teichmüller spaces. Our discussion of this topic is divided into three parts. The first part deals with the case that Γ contains elliptic elements, and give a partial solution to problems (A1) and (A2); in the second part, we answer question (B2) in the case that Γ is torsion free; in the third part, we return to general cases, and give a complete solution to problem (B1).

In the first part, Bers has asked wether $F(\Gamma)$ is isomorphic to $T(\Gamma')$ for some group Γ'. The study of this question (called Bers' question in the sequel) was initiated by Earle and Kra [19]. They proved that in most cases the answer to Bers' question is "no". More precisely, the statement of their result is the following:

Theorem 0.1 *Suppose that Γ contains elliptic elements and $F(\Gamma)$ is isomorphic to $T(\Gamma')$ for some group Γ'. If the types of Γ and Γ' are (g,n) and (g',n'), respectively, then the pair $((g,n),(g',n'))$ is among the entries of the table:*

$((0,3),(0,4))$,	$((0,3),(1,1))$,	$((0,4),(0,5))$,	$((0,4),(1,2))$,
$((1,1),(1,2))$,	$((1,1),(0,5))$,	$((0,5),(1,3))$,	$((0,5),(0,6))$,
$((0,5),(2,0))$,	$((1,2),(1,3))$,	$((1,2),(0,6))$,	$((1,2),(2,0))$,
$((0,6),(1,4))$,	$((0,6),(2,1))$,	$((0,7),(2,2))$,	$((0,8),(3,0))$.

Table (A)

Moreover, every elliptic element of Γ has order 2, unless Γ is of type $(0,3)$.

The remaining question is: what happens when the pair $((g,n),(g',n'))$ lies in Table (A)?

There are, of course, some obvious isomorphisms between the Bers fiber space $F(g,n;\nu_1,\ldots,\nu_n)$ and the Teichmüller space $T(g',n')$ when the pair $((g,n),(g',n'))$ lies in Table (A). To enumerate all well-known isomorphisms, we first note that if Γ is of type $(0,3)$, then $T(\Gamma)$ is a single point. So $F(0,3;\nu_1,\nu_2,\nu_3)$ is a disc for ν_1, ν_2, $\nu_3 \in \{2,3,\cdots\}\cup\{\infty\}$ with $\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}<1$. This leads to the isomorphism:

$$F(0,3;\nu_1,\nu_2,\nu_3) \cong T(0,4) \cong T(1,1). \tag{0.1}$$

Recall that a surface with signature $(g,n;\nu_1,\ldots,\nu_n)$ is a 2-orbifold with genus g and n distinguished points $x_1,\ldots,x_n$ whose ramification numbers are $\nu_1,\ldots,\nu_n$, respectively. When some x_i is a puncture, the corresponding ν_i is set to be "∞". Next, we observe that a Riemann surface X of type (2,0), (1,2), or (1,1) always admits a hyperelliptic involution. If we denote by J the hyperelliptic involution, then $X/<J>$ is an orbifold of signature $(0,6;2,\ldots,2)$ if X is of type (2,0), an orbifold of signature $(0,5;2,\ldots,2,\infty)$ if X is of signature $(1,2;\infty,\infty)$, and an orbifold of signature $(0,4;2,2,2,\infty)$ if X is of signature $(1,1;\infty)$. In addition, any Riemann surface X of signature $(0,4;\infty,\ldots,\infty)$ admits three conformal involutions (Earle-Kra [19]). To explain this fact, we take an arbitrary Riemann surface X of signature $(0,4;\infty,\ldots,\infty)$, and let $x_1,\ldots,x_4$ denote the punctures. Then there are three elliptic Möbius transformations (of order 2) J_1, J_2 and J_3 defined on X, where J_1 maps x_1 to x_3, x_2 to x_4, J_2 maps x_1 to x_4, x_2 to x_3, and J_3 maps x_1 to x_2, x_3 to x_4. Note that the three quotient spaces (orbifolds) $X/<J_1>$, $X/<J_2>$ and $X/<J_3>$ are of signature $(0,4;2,2,\infty,\infty)$.

From the above observation it is easy to see that $F(0,6;2,\ldots,2) \cong F(2,0;\_)$, $F(0,5;2,\ldots,2,\infty) \cong F(1,2;\infty,\infty)$, and $F(0,4;2,2,2,\infty) \cong F(0,4;2,2,\infty,\infty) \cong F(0,4;\infty,\ldots,\infty)$. On the other hand, by Bers' isomorphism theorem [8], we have $F(2,0;\_) \cong T(2,1)$, $F(0,4;\infty,\ldots,\infty) \cong T(0,5) \cong T(1,2)$, and $F(1,2;\infty,\infty) \cong T(1,3)$. Thus, we obtain three other isomorphisms:

$$F(0,6;2,2,2,2,2,2) \cong T(2,1), \tag{0.2}$$

$$F(0,5;2,2,2,2,\infty) \cong T(1,3), \tag{0.3}$$

$$F(0,4;2,2,2,\infty) \cong T(1,2) \cong T(0,5) \cong F(0,4;2,2,\infty,\infty). \tag{0.4}$$

We observe that there are countably many quasiconformal equivalence classes of Fuchsian groups Γ of type $(0,3)$ with distinct signatures $(0,3;\nu_1,\nu_2,\nu_3)$ such that $F(\Gamma)$ is isomorphic to $T(0,4)$, and therefore also to $T(1,1)$, where $\nu_1,\nu_2,\nu_3 \in \{2,3,\cdots\} \cup \{\infty\}$. We also see that there are at least three quasiconformal equivalence classes of Fuchsian groups Γ of type $(0,4)$ with distinct signatures such that $F(\Gamma)$ is isomorphic to $T(0,5)$, thus also to $T(1,2)$. (0.4) provides two such examples, and another example is given by the equivalence: $F(0,4;\infty,\ldots,\infty) \cong T(0,5)$.

The Bers question will be completely answered by solving the following conjecture which was posed in 1974:

Conjecture (Earle-Kra [19]). *If* Γ *contains elliptic elements, then* (0.1)—(0.4) *exhaust all possible isomorphisms between Bers fiber spaces and Teichmüller spaces.*

As we see, Theorem 0.1 is an important step towards this conjecture. What is left unanswered is a finite number of cases. As a matter of fact, if Theorem 0.1 is combined with (0.1)–(0.4), we can immediately find that there remain 39 unknown cases, which are exhibited in the following Table (B):

signature $(g, n; \nu_1, \dots, \nu_n)$	type (g', n')	# of cases
$(0, 8; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{8-m})$, $0 < m \leq 8$,	$(3, 0)$	8
$(0, 7; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{7-m})$, $0 < m \leq 7$,	$(2, 2)$	7
$(0, 6; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{6-m})$, $0 < m < 6$,	$(2, 1)$	5
$(0, 6; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{6-m})$, $0 < m < 6$,	$(1, 4)$	5
$(0, 5; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{5-m})$, $0 < m \leq 5$, $m \neq 4$,	$(0, 6)$ or $(2, 0)$	4
$(0, 5; \underbrace{2, \dots, 2}_{m}, \underbrace{\infty, \dots, \infty}_{5-m})$, $0 < m \leq 5$, $m \neq 4$,	$(1, 3)$	4
$(1, 2; 2, m)$, $m = 2$ or $m = \infty$	$(1, 3)$	2
$(1, 2; 2, m)$, $m = 2$ or $m = \infty$	$(0, 6)$ or $(2, 0)$	2
$(0, 4; 2, \infty, \infty, \infty)$	$(0, 5)$ or $(1, 2)$	1
$(1, 1; 2)$	$(0, 5)$ or $(1, 2)$	1

Table (B)

The main body of Chapter 2 of our paper is a contribution to the conjecture of Earle and Kra. What we attempt to do is to eliminate most entries of Table (B). More precisely, we obtain the following theorem:

Theorem 0.2 *Suppose that Γ has torsion and $F(\Gamma)$ is isomorphic to $T(\Gamma')$ for some group Γ'. If Γ has signature $(g,n;\nu_1,\ldots,\nu_n)$ and Γ' has type (g',n'), then the pair $((g,n;\nu_1,\ldots,\nu_n),(g',n'))$ is among the entries of the following Table (C):*

signature $(g,n;\nu_1,\ldots,\nu_n)$ of Γ	type (g',n') of Γ'
$(0,8;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{8-m}),\quad m=3,6,$	$(3,0)$
$(0,7;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{7-m}),\quad m=2,4,6,$	$(2,2)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m}),\quad m\neq 5,$	$(2,1)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m}),\quad m=3,4,$	$(1,4)$
$(0,5;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{5-m}),\quad m=2,4,$	$(1,3)$
$(0,4;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{4-m}),\quad m=2,3,$	$(1,2)$ or $(0,5)$
$(0,3;\nu_1,\nu_2,\nu_3)$, $\nu_1,\nu_2,\nu_3\in\{2,3,\cdots\}\cup\{\infty\}$ with $\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}<1$	$(1,1)$ or $(0,4)$

Table (C)

In the second part (when Γ is torsion free), the study of relationships between Bers fiber spaces and Teichmüller spaces was initiated by Bers in 1973. Assume that $\dot{\Gamma}$ is another torsion free finitely generated Fuchsian group of the first kind with type $(g, n+1)$. In [8] Bers proved that there are biholomorphic maps (called Bers' isomorphisms throughout this paper) between $F(\Gamma)$ and $T(\dot{\Gamma})$.

Let $B_2(L, \Gamma)$ denote the Banach space consisting of all holomorphic functions ϕ defined on the lower half plane L such that

$$\sup \{|z - \bar{z}|^2 |\phi(z)|;\ z \in L\} < \infty,$$

and

$$(\phi \circ \gamma)(z)(\gamma')^2(z) = \phi(z) \text{ for all } \gamma \in \Gamma \text{ and } z \in L.$$

The Teichmüller space $T(\Gamma)$ (resp. $T(\dot{\Gamma})$) can be embedded into the complex Banach space $B_2(L, \Gamma)$ (resp. $B_2(L, \dot{\Gamma})$). In this sense, the topological boundary of $T(\dot{\Gamma})$ is defined. On the other hand, as a simply connected domain of $B_2(L, \Gamma) \times \mathbb{C}$, the Bers fiber space $F(\Gamma)$ has a natural boundary. Kra asked whether or not the Bers isomorphisms can be extended to the boundary (in the sense as described above). We prove that the answer is "no" in the case when Γ is not of type $(0, 3)$. Precisely we have:

Theorem 0.3 *Suppose that dim $T(\Gamma) \geq 1$. Then no biholomorphic map of $F(\Gamma)$ onto $T(\dot{\Gamma})$ admits a continuous extension to the boundary of $F(\Gamma)$.*

For the proof, we first identify some special elements of the group Aut $F(\Gamma)$ of holomorphic automorphisms with certain elements of the Teichmüller modular group Mod $\dot{\Gamma}$ by invoking an important result of Kra [28], then we apply the Thurston-Bers classification theorem for modular transformations.

Some interesting results on iterates of modular transformations on the Bers embedding of $T(\dot{\Gamma})$ are also used in our proof.

Notice that the method used in proving Theorem 0.3 does not work in the case of dim $T(\Gamma) = 0$. In this special case, the corresponding Teichmüller space is $T(0,4)$ or $T(1,1)$. By squeezing any curve which is homotopic neither to a null curve nor to a puncture on a surface of type $(0,4)$ or $(1,1)$, we obtain two or one thrice punctured spheres on which there are "no moduli". As a matter of fact, in the case of dimension one the problem under consideration is linked to Bers' conjecture which asserts that the Bers embedding of $T(0,4)$ is a Jordan domain.

In the third part, we return to the case that Γ is an arbitrary finitely generated Fuchsian group of the first kind, and continue to study the relationships between Teichmüller spaces and Bers fiber spaces. As we mentioned above, the Bers conjecture states that for $\nu_1, \nu_2, \nu_3 \in \{2, 3, \cdots\} \cup \{\infty\}$ with $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1$, the isomorphism (0.1) admits a homeomorphic extension. A natural question to be asked is: what happens if Γ is any other Fuchsian group? Theorem 3 answers this question when Γ is torsion free. The following result gives a complete solution to problem (B1):

Theorem 0.3' *Let Γ, Γ' be arbitrary finitely generated Fuchsian groups of the first kind. Assume that Γ is not of type $(0,3)$ and that Γ' is torsion free. Then either there is no isomorphism between $F(\Gamma)$ and $T(\Gamma')$, or any isomorphism between these two moduli spaces admits no homeomorphic extension to the* *boundary of* $F(\Gamma)$.

* * * * *

Besides the above results on relationships between Bers fiber spaces and Teichmüller spaces, we also investigate relationships among Bers fiber spaces and among Teichmüller curves. Assume that Γ and Γ' are finitely generated Fuchsian groups of the first kind which may contain elliptic elements. Again, let $\sigma = (g, n; \nu_1, \ldots, \nu_n)$ and $\sigma' = (g', n'; \nu_1, \ldots, \nu'_{n'})$ be the signatures of Γ and Γ'. If $\sigma = \sigma'$, then by a theorem of Bers [8], there exists an isomorphism, which is called a Bers allowable mapping, of $F(\Gamma)$ onto $F(\Gamma')$ (see §1.3 for more details). The following result gives a partial solution to problems (C1) and (C2):

Theorem 0.4 *Let* Γ *be a finitely generated Fuchsian group of the first kind whose signature is* σ. *Assume that dim* $T(\Gamma) \geq 2$, *and that* σ *is not* $(2, 0; \_)$, $(0, 6; 2, \ldots, 2)$, $(1, 2; \infty, \infty)$, *or* $(0, 5; 2, \ldots, 2, \infty)$. *Also assume that* Γ *contains at least one parabolic element if* $g \leq 1$. *Let* Γ' *be a Fuchsian group of signature* σ'. *Then there is a fiber-preserving isomorphism* $\varphi\colon F(\Gamma) \to F(\Gamma')$ *if and only if* $\sigma = \sigma'$ *and* φ *is a Bers allowable mapping.*

Remark. Among the entries of Table (C), we know that for some pairs $((g, n; \nu_1, \ldots, \nu_n), (g', n'))$ there is an isomorphism of $F(g, n; \nu_1, \ldots, \nu_n)$ onto $T(g', n')$. See (0.1)–(0.4). However, it is not known whether or not there is an isomorphism of $F(g, n; \nu_1, \ldots, \nu_n)$ onto $T(g', n')$ for any other pairs in Table (C). Theorem 0.4 gives us information that if, for example, there is an isomorphism of $F(0,6;2,\infty,\ldots,\infty)$ onto $T(2,1)$, then we must have a non fiber-preserving isomorphism of $F(0,6;2,\infty,\ldots,\infty)$ onto $F(0,6;2,\ldots,2)$.

Our approach to Theorem 0.4 relies on the following fact: every holomorphic automorphism of a Bers fiber space $F(\Gamma)$ which keeps each fiber invariant is an element of Γ, provided that Γ satisfies the same condition as in Theorem 0.4. The proof of this fact is based on the results of Hubbard [26], Earle-Kra [19] [20] which assert that the set of (global) holomorphic sections of each Teichmüller curve is finite. (Again, there are a few exceptions, see Theorem 0.4 above or §4.1 for more details.)

As usual, let $(g,n;\nu_1,\ldots,\nu_n)$ denote the signature of Γ. Since the action of Γ on $F(\Gamma)$ is holomorphic and keeps each fiber invariant, we form the quotient space $V(\Gamma)=F(\Gamma)/\Gamma$ and the projection $\pi_0\colon V(\Gamma)\to T(\Gamma)$. $V(\Gamma)$ is called the Teichmüller curve and is denoted by $V(g,n;\nu_1,\ldots,\nu_n)$; it is a complex manifold (see Earle-Kra [20]). Relationships among Teichmüller curves are also investigated in this paper; we will discuss the following two problems:

(D1) Find all pairs $((g,n;\nu_1,\ldots,\nu_n),(g',n';\nu'_1,\ldots,\nu'_{n'}))$ for which there is an isomorphism between $V(g,n;\nu_1,\ldots,\nu_n)$ and $V(g',n';\nu'_1,\ldots,\nu'_{n'})$,

(D2) Describe all possible isomorphisms among Teichmüller curves.

A complete classification theorem (Theorem 4.4.1) for Teichmüller curves, which is well known to experts but does not appear in the literature, is proved in §4.5. We also examine all possible isomorphisms among Teichmüller curves (Theorem 4.4.2). The results obtained are viewed as natural generalizations of

Theorem 1.2.1, Theorem 1.2.2 and Theorem 1.2.3; their proofs involve generalizations of Duma's theorem [16] to Teichmüller curves of orbifolds (Proposition 4.4.3 and Proposition 4.4.4).

Aforementioned, Theorem 1.2.3 states that the group of holomorphic automorphisms of $T(\Gamma)$ for a torsion free group Γ coincides with the Teichmüller modular group Mod Γ (there are a few exceptions, see §1.2 for details). Instead of dealing with biholomorphic maps, we can consider a holomorphic surjective map $\tau : T(g,n) \to T(g,0)$, $g \geq 2$, and ask whether or not τ is always a forgetful map (see §4.5 for the definition). The answer to this general question is unknown. On the other hand, Earle-Kra [20] proved that τ can be lifted to a holomorphic map ζ of $V(g,n)$ onto $V(g,0)$ if τ is the forgetful map (where $V(g,n)$ is the n-pointed Teichmüller curve, see §4.1 for the definition). We shall show in this paper that the converse remains true under certain conditions on the surjective map $\tau : T(g,n) \to T(g,0)$; that is, we will prove that a lift ζ of τ exists if and only if τ is the forgetful map.

This paper is organized as follows. In Chapter 1, we briefly review some basic definitions and some fundamental theorems (without proofs) in the theory of Teichmüller space which we will use in the subsequent chapters.

Chapter 2 consists of eight sections. In §2.1, an equivalent version of Theorem 0.2 is stated (Theorem 2.1.1 and Theorem 2.1.2) so that Theorem 0.2 can be treated step by step. In §2.2 we investigate some periodic homeomorphisms of surfaces and construct some elliptic modular transformations. We also study the structure of the hyperelliptic loci in the Teichmüller space

$T(2,1)$ which is useful in proving Theorem 2.1.2. In §2.3, we give another proof of a theorem of Earle-Kra [19], which is Theorem 0.1 in this paper. In §2.4, we prove Theorem 2.1.2. In §2.5, a preparation for proving Theorem 2.1.1 is given. Some new methods of studying the conjecture of Earle and Kra are developed. These methods will be applied repeatedly in §2.6, §2.7 and §2.8. We also study an extension problem (detailed discussion is in the section), and give a partial solution to the problem. §2.6, §2.7, and §2.8 are devoted to the proof of Theorem 2.1.1.

In Chapter 3, we prove two non-extendibility theorems for the Bers isomorphism as well as its generalization. We also discuss properties of iterates of hyperbolic modular transformations on Teichmüller space by appealing to results of Bers [10] and Gallo [22].

In Chapter 4, we study fiber-preserving isomorphisms among Bers fiber spaces and isomorphisms among Teichmüller curves. We also study the forgetful maps of Teichmüller curves. In §4.2 – §4.3 we investigate fiber-preserving isomorphisms among Bers fiber spaces and prove Theorem 0.4. In §4.4 we study isomorphisms among Teichmüller curves. The problem as to whether or not these isomorphisms are "geometric" is also discussed; a partial solution is given. §4.5 is devoted to proofs of these results.

Chapter 1

On Teichmüller and Bers fiber spaces

In this chapter we review some definitions and some well known properties of the theory of Teichmüller spaces. More details can be found in [19], [20], [23] and [39]. All results quoted here will play important roles in our study.

1.1 Teichmüller spaces

The theory of Teichmüller space is based on the fundamental work of Ahlfors-Bers [4].

Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal map. Then f solves the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) f_z(z) \tag{1.1}$$

in $\mathbb{C}$, where $\mu(z) = f_{\bar{z}}(z)/f_z(z)$ is a measurable function whose L^∞-norm in $\mathbb{C}$ is less than one. Let D be a domain in $\mathbb{C}$, and let $L^\infty(D)$ denote the complex Banach space consisting of all measurable functions μ defined in D

with $\| \mu \|_\infty = \text{ess sup } \{|\mu(z)|;\ z \in D\}$. Let $M(D)$ denote the open unit ball in $L^\infty(D)$.

Theorem 1.1.1 (Ahlfors-Bers [4]) *For each $\mu \in M(\mathbb{C})$, there exists a unique quasiconformal map f^μ of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself which solves the equation (1.1) and fixes 0, 1, ∞. Moreover, for any fixed point z in $\mathbb{C}$, the map*

$$\mu \mapsto f^\mu(z)$$

is a holomorphic function on $M(\mathbb{C})$.

Throughout this paper all quasiconformal maps which fix 0, 1, ∞ are called *normalized quasiconformal maps.*

Let Γ be a finitely generated Fuchsian group of the first kind operating on the upper half plane $U = \{z \in \mathbb{C};\ \text{Im } z > 0\}$. Let $L^\infty(U, \Gamma)$ denote the subspace of $L^\infty(U)$ consisting of all $\mu \in L^\infty(U)$ with

$$(\mu \circ \gamma) \cdot \overline{\gamma'}/\gamma' = \mu \quad \text{a.e. in } U \quad \text{for all } \gamma \in \Gamma. \tag{1.2}$$

Let $M(\Gamma)$ denote the intersection $M(U) \cap L^\infty(U, \Gamma)$.

To each $\mu \in M(\Gamma)$, we can associate an element $\hat{\mu} \in M(\mathbb{C})$ defined by

$$\hat{\mu}(z) = \begin{cases} \mu(z) & \text{if } z \in U; \\ 0 & \text{if } z \in L = \{z \in \mathbb{C};\ \text{Im } z < 0\}. \end{cases}$$

By using Theorem 1.1.1, one sees at once that there is a unique normalized quasiconformal self-map w^μ of $\mathbb{C}$, which solves the equation (1.1) with respect to $\hat{\mu}$. The restriction $w^{\mu}|_L$ is a conformal map. However, the domain $w^{\mu}(U)$ may become very complicated.

Similarly, we construct

$$\tilde{\mu}(z) = \begin{cases} \mu(z) & \text{if } z \in U; \\ \overline{\mu(\bar{z})} & \text{if } z \in L = \{z \in \mathbb{C};\ \operatorname{Im} z < 0\}. \end{cases}$$

By Theorem 1.1.1 again, we can find a normalized quasiconformal self-map, denoted by w_{μ}, of U which solves equation (1.1) with respect to $\tilde{\mu}$. Note that $w_{\mu}(U) = U$ and for each fixed $z \in U$, the map $\mu \mapsto w_{\mu}(z)$ is real analytic.

A basic fact is that $\mu \in M(\Gamma)$ if and only if $w_{\mu}\Gamma(w_{\mu})^{-1}$ is again a (finitely generated) Fuchsian group of the first kind.

Two elements μ, $\mu' \in M(\Gamma)$ are called *equivalent* (write $\mu \sim \mu'$) if $w^{\mu} = w^{\mu'}$ on $\hat{\mathbb{R}} = \partial U$. The equivalence class of μ defined in this way is denoted by $[\mu]$. The *Teichmüller space* $T(\Gamma)$ of Γ is the space of equivalence classes $[\mu]$ for $\mu \in M(\Gamma)$. By Theorem 1.1.1, we can associate to any $\mu \in M(\Gamma)$ a normalized quasiconformal self-map w_{μ} of U which conjugates Γ to another Fuchsian group. If we denote by $Q(\Gamma)$ the group of all quasiconformal self-maps w of U with $w\Gamma w^{-1}$ a Fuchsian group, then the complex dilatation μ of $w \in Q(\Gamma)$ (defined by $\mu(z) = \frac{w_{\bar{z}}(z)}{w_z(z)}$ for $z \in U$) belongs to $M(\Gamma)$. We see that $M(\Gamma)$ is in 1-1 correspondence with the set

$$Q_n(\Gamma) = \{w \in Q(\Gamma);\ w \text{ is a normalized quasiconformal self-map of } U\}.$$

Our definition of the Teichmüller space $T(\Gamma)$ of Γ shows that $T(\{1\})$ is the group $Q_n(\{1\})/Q_0(\{1\})$, where $\{1\}$ denotes the trivial group and $Q_0(\{1\})$ is the normal subgroup of $Q_n(\{1\})$ consisting of those $w \in Q_n(\{1\})$ which restrict to the identity map on $\hat{\mathbb{R}}$. It is easy to see that $T(\Gamma)$ is the image of $Q_n(\Gamma)$ under the projection $Q_n(\Gamma) \subset Q_n(\{1\}) \hookrightarrow T(\{1\})$.

A fundamental theorem of Ahlfors [3] (see also Bers [5]) states that the Teichmüller space $T(\Gamma)$ has a unique complex structure so that the natural projection $\Phi\colon M(\Gamma) \to T(\Gamma)$ (by sending μ to $[\mu]$) is holomorphic. $T(\Gamma)$, with the complex structure defined above, is a complex manifold. The complex dimension of $T(\Gamma)$ is $3g - 3 + n$, where g is the genus of U/Γ and n is the number of the distinguished points of U/Γ. (In this thesis, by a distinguished point we mean a point on the closed surface $\overline{U/\Gamma}$ which is either a puncture of U/Γ, or a branch point on U/Γ coming from the fixed point of an elliptic element of Γ.) An important theorem of Teichmüller [45] [46] asserts that $T(\Gamma)$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$. This means in particular that $T(\Gamma)$ is a cell.

The *Teichmüller distance* between two points $[\mu]$ and $[\mu'] \in T(\Gamma)$ is defined by

$$< [\mu], [\mu'] >= \frac{1}{2} \log \inf K(w),$$

where w runs over those quasiconformal self-maps w' in $Q(\Gamma)$ for which w' agrees with $w_\mu \circ (w_{\mu'})^{-1}$ on $\mathbb{R}$ and $K(w)$ is the maximal dilatation of w. The Teichmüller distance is differentiable (see Earle [17]). It is also well known that $(T(\Gamma), <,>)$ is a complete metric space.

As we mentioned, since $T(\Gamma)$ is a complex manifold, the *Kobayashi pseudo-*

metric on $T(\Gamma)$ can be defined as the largest pseudo-metric d so that

$$d(f(z_1), f(z_2)) \leq \rho_U(z_1, z_2),$$

for all holomorphic maps f of U into $T(\Gamma)$ and for all $z_1, z_2 \in U$, where ρ_U is the Poincaré distance in U given by

$$\rho_U(z_1, z_2) = \log \frac{|z_1 - \bar{z}_2| + |z_1 - z_2|}{|z_1 - \bar{z}_2| - |z_1 - z_2|}.$$

A theorem of Royden [43] asserts that the Kobayashi metric on $T(\Gamma)$ coincides with the Teichmüller metric.

The Teichmüller space $T(\Gamma)$, with its complex structure, can be realized as a bounded domain of a complex Banach space $B_2(L, \Gamma)$ (which is called *Bers' embedding* of $T(\Gamma)$) defined as follows: the elements of $B_2(L, \Gamma)$ are holomorphic functions $\phi(z)$ defined in the lower half plane L, with the following two properties:

(1) $(\phi \circ \gamma)(z)(\gamma')^2(z) = \phi(z)$, for all $\gamma \in \Gamma$ and all $z \in L$;

(2) $\| \phi \|_2 = \sup \{|z - \bar{z}|^2 |\phi(z)|;\ z \in L\} < \infty$.

It is known that $B_2(L, \Gamma)$ is a closed linear subspace of $B_2(L, \Gamma_0)$ for any Fuchsian group $\Gamma_0 \subset \Gamma$ and that dim $B_2(L, \Gamma) < \infty$ if and only if Γ is a finitely generated Fuchsian group of the first kind. For each $\mu \in M(\Gamma)$, by Theorem 1.1.1, there is a normalized quasiconformal self-map w^μ of $\mathbb{C}$ which solves (1.1) and satisfies the property that $w^\mu|_L$ is a conformal map. Let $\{w^\mu|_L, z\}$ denote the Schwarzian derivative of $w^\mu|_L$. It is not difficult to see that $\{w^\mu|_L, z\}$ is an element of $B_2(L, \Gamma)$ and that $\{w^\mu|_L, z\}$ depends only on the equivalence class $[\mu]$ of $\mu \in M(\Gamma)$. We thus obtain a well-defined map

$$T(\Gamma) \ni [\mu] \longmapsto \{w^\mu|_L, z\} \in B_2(L, \Gamma). \tag{1.3}$$

By Nehari's theorem [40], we have $\| \{w^\mu|_L, z\} \|_2 \leq 3/2$. Further, it is shown in [12], [39] that the map (1.3) is a holomorphic bijection of $T(\Gamma)$ onto a holomorphically convex domain in $B_2(L, \Gamma)$ containing the open ball of radius 1/2.

If we identify $T(\Gamma)$ with its image under the Bers embedding described above, the boundary of $T(\Gamma)$ is naturally defined. It was Bers who pioneered the study of the boundary of Teichmüller space. Any boundary point of a Teichmüller space is characterized as a Kleinian group of special kind which is called a *b*-group. Systematic investigations of boundaries of Teichmüller spaces can be found in Bers [6], Maskit [33], Abikoff [1] and McMullen [37].

1.2 Holomorphic maps of Teichmüller spaces

Let Γ be as in §1.1, and let U_Γ denote all points z in U which are not fixed by any elliptic element of Γ. Then, the surface U_Γ/Γ is a compact Riemann surface of genus g with n points removed. The pair (g, n) is called the *type* of Γ. Let $x_1, \ldots, x_n$ be the n punctures of U_Γ/Γ. The ramification number ν_i of x_i is the order of the subgroup of Γ fixing any point of U that lies over x_i if such a point exists. x_i is called *parabolic* if no such a point exists. In the case when x_i is parabolic, ν_i is set to be "∞". The collection

$$(g, n; \nu_1, \ldots, \nu_n) \tag{1.4}$$

is called the *signature* of Γ, where ν_i, $i = 1, \ldots, n$ are arranged so that $2 \leq \nu_1 \leq \cdots \leq \nu_n \leq \infty$. It is a known fact that there is a Fuchsian group Γ with signature (1.4) if and only if

$$2g - 2 + \sum_{j=1}^{n}(1 - \frac{1}{\nu_j}) > 0.$$

Our discussion is based on the following result:

Theorem 1.2.1 (Bers-Greenberg [11]) *Let Γ, Γ' be finitely generated Fuchsian groups of the first kind. Assume that Γ and Γ' have the same type. Then the Teichmüller spaces $T(\Gamma)$ and $T(\Gamma')$ are biholomorphically equivalent.*

Remark. There are three different proofs of Theorem 1.2.1. See Earle-Kra [19], Marden [32] and Gardiner [23].

Theorem 1.2.1 tells us that $T(\Gamma)$ depends only on the type of Γ. For this reason, we usually denote by $T(g,n)$ the Teichmüller space $T(\Gamma)$ for a Γ of type (g,n).

For a moment, let Γ be of signature $(2.0; \_)$. Then U/Γ admits a hyperelliptic involution J which leaves precisely 6 points (which are called *Weierstrass points*) fixed. Lifting J to U we obtain a $\tilde{J} \in \mathrm{PSL}(2,\mathbf{R})$ such that $\tilde{J}$ and Γ generate a Fuchsian group Γ_0. It is clear that Γ is the normal subgroup of Γ_0 with index 2 and the signature of Γ_0 is $(0,6;2,\ldots,2)$. Note also that dim $T(\Gamma_0) =$ dim $T(\Gamma) = 3$. This implies that $T(\Gamma_0) \cong T(\Gamma)$. Similar phenomena occur when Γ is of signature $(1,2;\infty,\infty)$ or $(1,1;\infty)$. We obtain

$$T(0,6) \cong T(2,0), \quad T(0,5) \cong T(1,2), \quad T(0,4) \cong T(1,1). \tag{1.5}$$

A natural question arises as to whether or not there are some other isomorphisms between Teichmüller spaces. This problem is solved by the following result:

Theorem 1.2.2 (Patterson [42]) *(1.5) exhausts all isomorphisms between Teichmüller spaces with distinct types.*

Remark. Earle-Kra [18] gives another proof of this theorem by using Royden's technique developed in [43].

Given a Fuchsian group Γ, an automorphism θ of Γ is called *geometric* if there is $w \in Q(\Gamma)$ such that $\theta(\gamma) = w \circ \gamma \circ (w)^{-1}$ for all $\gamma \in \Gamma$. Let $w' \in Q(\Gamma)$. Then $w' \circ \gamma \circ (w')^{-1} = w \circ \gamma \circ (w)^{-1}$ for all $\gamma \in \Gamma$ (that is, w and w' induce the same geometric isomorphism θ) if and only if $w|_{\mathbf{R}} = w'|_{\mathbf{R}}$.

The *Teichmüller modular group* Mod Γ is defined as the group of geometric automorphisms (denoted by mod Γ) modulo the normal subgroup of inner automorphisms. The action of $\chi \in$ Mod Γ on $T(\Gamma)$ is defined as follows: Let χ be the image of θ under the quotient map $q_0 :$ mod $\Gamma \to$ Mod Γ, and let $w \in Q(\Gamma)$ be chosen so that $\theta(\gamma) = w \circ \gamma \circ (w)^{-1}$ for $\gamma \in \Gamma$. For any $\mu \in M(\Gamma)$, the Beltrami coefficient μ' of the map $w_\mu \circ w^{-1}$ is given by the formula:

$$\mu'(z) = \frac{\nu(z) + (\mu \circ w^{-1})(z) \cdot \overline{(\frac{\partial w^{-1}}{\partial z})}/(\frac{\partial w^{-1}}{\partial z})}{1 + \bar{\nu}(z) \cdot (\mu \circ w^{-1})(z) \cdot \overline{(\frac{\partial w^{-1}}{\partial z})}/(\frac{\partial w^{-1}}{\partial z})}, \tag{1.6}$$

where ν is the Beltrami coefficient of w^{-1}.

It is easy to check that μ' belongs to $M(\Gamma)$ and $[\mu'] \in T(\Gamma)$ depends on χ but not on θ and w. Hence, (1.6) induces an action of χ on $T(\Gamma)$ (defined by carrying $[\mu]$ to $[\mu']$) as a holomorphic automorphism. Thus, the group Mod Γ acts on $T(\Gamma)$ as a group of biholomorphic maps. Note that the action of Mod Γ is not always faithful, by which we mean that distinct elements of Mod

Γ do not necessarily induce distinct holomorphic automorphisms of $T(\Gamma)$. To see this, let Aut $(T(g,n))$ denote the group of holomorphic automorphisms of $T(g,n)$. Then

$$\text{Aut } (T(2,0)) \cong \text{Mod } (2,0)/\mathbb{Z}_2 \cong \text{Mod } (0,6),$$

$$\text{Aut } (T(1,2)) \cong \text{Mod } (1,2)/\mathbb{Z}_2 \cong \text{Mod } (0,5),$$

$$\text{Aut } (T(1,1)) \cong \text{Aut } (T(0,4)) \cong \text{PSL}(2,\mathbb{R}),$$

$$\text{Aut } (T(0,3)) = \{id\},$$

where Mod (g,n) is the Teichmüller modular group Mod Γ for Γ a torsion free group with type (g,n), and $\mathbb{Z}_2$ stands for the subgroup of Mod $(2,0)$ (resp. Mod $(1,2)$) determined by the hyperelliptic involution on a surface of type $(2,0)$ (resp. a surface of type $(1,2)$). However, we have the following important result:

Theorem 1.2.3 (Royden [43], Earle-Kra [19]) *Let Γ be a torsion free finitely generated Fuchsian group of the first kind with type (g,n). Then the full group of holomorphic automorphisms of $T(\Gamma)$ is isomorphic to Mod Γ except when (g,n) is $(0,3),(0,4),(1,1),(1,2)$, or $(2,0)$, and the above table exhausts all the exceptional cases.*

We still assume that the group Γ may contain elliptic elements. Obviously, the group of holomorphic automorphisms of $T(\Gamma)$ depends only on the type (g,n) of Γ and not on its signature. However, Mod Γ really depends on the signature of Γ; namely Mod Γ may be a proper subgroup of Mod Γ' for Γ' a torsion free group with type (g, n). Mod Γ is isomorphic to Mod Γ' if and only if all distinguished points on a surface are either punctures, or branch points which have the same ramification number.

1.3 Bers fiber spaces

The *Bers fiber space* over $T(\Gamma)$, denoted by $F(\Gamma)$, is a subset of $T(\Gamma) \times \mathbb{C}$ consisting of pairs $([\mu], z)$, where $[\mu] \in T(\Gamma)$, and $z \in w^{\mu}(U)$. Let $\pi: F(\Gamma) \to T(\Gamma)$ denote the natural (holomorphic) projection onto the first factor. From the definition and certain well-known facts, we see that the fiber of π over a point $[\mu] \in T(\Gamma)$ is the quasidisk $w^{\mu}(U)$ which depends only on the equivalence class of μ. $F(\Gamma)$, as an open connected and simply connected subset of $\mathbb{C}^{3g-2+n}$, becomes a complex Banach manifold modeled on $B_2(L, \Gamma) \bigoplus \mathbb{C}$, and the projection π is a holomorphic submersion.

Suppose that Γ and Γ' are two Fuchsian groups with the same signature. Then, there exists $w \in Q(\Gamma)$ such that $w\Gamma w^{-1} = \Gamma'$. By a theorem of Bers [8], w induces an isomorphism $[w]_*$ of $F(\Gamma)$ onto $F(\Gamma')$. More precisely, the isomorphism $[w]_*$ can be described by sending every point $([\mu], z) \in F(\Gamma)$ to the point $([\nu], w^{\nu} \circ w \circ (w^{\mu})^{-1}(z)) \in F(\Gamma')$, where $\nu \in M(\Gamma')$ is the Beltrami coefficient of $w^{\mu} \circ w^{-1}$. It is easy to check that $[w]_*$ is a fiber-preserving isomorphism. An isomorphism defined in this way is called a *Bers allowable mapping*.

Although $T(\Gamma)$ depends only on the type of Γ (Theorem 1.2.1), $F(\Gamma)$

depends on the signature of Γ. For the convenience of the reader, we provide some examples to illustrate this phenomenon. Let $F(g,n;\nu_1,\ldots,\nu_n)$ denote the Bers fiber space $F(\Gamma)$ for a group Γ of signature $(g,n;\nu_1,\ldots,\nu_n)$.

Examples. If the signature $(g,n;\nu_1,\ldots,\nu_n)$ of Γ has the property that $2 < \nu_i < \infty$ for some i, $i \in \{1,\ldots,n\}$, then Theorem 0.1 (Earle-Kra) asserts that $F(g,n;\nu_1,\ldots,\nu_n)$ cannot be isomorphic to any Teichmüller space. On the other hand, the Bers isomorphism theorem [8] states that $F(g,n;\infty,\ldots,\infty)$ is isomorphic to the Teichmüller space $T(g,n+1)$. Thus, $F(g,n;\nu_1,\ldots,\nu_n)$ is not isomorphic to $F(g,n;\infty,\ldots,\infty)$ if $2 < \nu_i < \infty$ for some $i \in \{1,2,\ldots,n\}$.

Another interesting example is given implicitly by Theorem 0.1 and Theorem 0.2. From (0.4) in the introduction, we know that $F(0,4;2,2,\infty,\infty) \cong F(0,4;2,2,2,\infty) \cong F(0,4;\infty,\infty,\infty,\infty) \cong T(0,5) \cong T(1,2)$. On the other hand, Theorem 0.2 asserts that $F(0,4;2,\infty,\infty,\infty)$ is not isomorphic to $T(0,5) \cong T(1,2)$. From Theorem 0.1, we see that $T(0,5)$ and $T(1,2)$ are the only two possible Teichmüller spaces to which $F(\Gamma)$ is isomorphic for Γ of type $(0,4)$ or $(1,1)$. We conclude that $F(0,4;2,\infty,\infty,\infty)$ is isomorphic to neither $F(0,4;2,2,\infty,\infty)$ nor $F(0,4;2,2,2,\infty)$, while $F(0,4;2,2,\infty,\infty)$ and $F(0,4;2,2,2,\infty)$ are isomorphic to each other. This example tells us that two Bers fiber spaces $F(\Gamma)$ and $F(\Gamma')$ may or may not be isomorphic to each other even if Γ and Γ' have the same type, both contain only elliptic elements of the same order, but their signatures are distinct.

An unsolved Problem. Describe all isomorphisms between Bers fiber spaces.

Suppose now that we are given a Bers fiber space $F(\Gamma)$. As we see in §1.2, the *modular group* mod Γ is defined as the group of geometric automorphisms of Γ. An element $\theta \in \text{mod}\,\Gamma$ acts biholomorphically on $F(\Gamma)$ as follows: let θ be represented by $w \in Q(\Gamma)$, then

$$\theta([\mu], z) = ([\nu], \hat{z}), \tag{1.7}$$

where ν is the Beltrami coefficient of the map $w^\mu \circ w^{-1}$, and $\hat{z} = w^\nu \circ w \circ (w^\mu)^{-1}(z)$. The action of mod Γ on $F(\Gamma)$ is called *effective* if for any non-trivial $\theta \in \text{mod}\,\Gamma$, there is an $x \in F(\Gamma)$ with $\theta(x) \neq x$. Since mod Γ acts effectively on $F(\Gamma)$ (Theorem 6 of Bers [8]), we usually identify the group mod Γ with its action on $F(\Gamma)$.

Theorem 1.3.1 (Bers [8]) *The modular group mod* Γ *acts as a group of biholomorphic automorphisms on* $F(\Gamma)$ *inducing the Mod* Γ *action on the base Teichmüller space* $T(\Gamma)$.

Since Γ is centerless, Γ is isomorphic to the group of inner automorphisms of Γ by associating to each $\gamma' \in \Gamma$ the automorphism $\gamma \mapsto \gamma' \circ \gamma \circ \gamma'^{-1}$; namely Γ acts by conjugation as automorphisms of Γ. It follows that Γ is isomorphic to a normal subgroup of mod Γ which we denote by Γ also. In particular, since mod Γ is identified with its action on $F(\Gamma)$, Γ acts on $F(\Gamma)$ as a group of holomorphic automorphisms; the action of $\gamma \in \Gamma$ on $F(\Gamma)$ is given by

$$\gamma([\mu], z) = ([\mu], \gamma^\mu(z)) = ([\mu], w^\mu \circ \gamma \circ (w^\mu)^{-1}(z)), \text{ for } \gamma \in \Gamma. \tag{1.8}$$

By definition, the Teichmüller modular group Mod Γ defined earlier is the factor group mod Γ/Γ, where Γ is again viewed as a group of inner automorphisms of Γ, as we discussed above. If $\theta \in \text{mod}\ \Gamma$, and χ is the image of θ in $\text{Mod}\ \Gamma$ via the natural quotient homomorphism $q_0\colon \text{mod}\ \Gamma \to \text{Mod}\ \Gamma$, then it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\ \theta\ } & F(\Gamma) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ T(\Gamma) & \xrightarrow[\ \chi\ ]{} & T(\Gamma) \end{array}$$

Note that the action of every element $\theta \in \text{mod}\ \Gamma$ on $F(\Gamma)$ is biholomorphic, properly discontinuous, fiber-preserving, and effective. Moreover, every element of Γ can be viewed as a holomorphic automorphism of $F(\Gamma)$ which leaves invariant each fiber of π.

We assume, for the moment, that Γ contains some elliptic element e, and that $z_0 \in U$ is the fixed point of e. For any $[\mu] \in T(\Gamma)$, there is only one fixed point $z_\mu = w^\mu(z_0)$ of $e^\mu = w^\mu \circ e \circ (w^\mu)^{-1}$ in the quasidisc $w^\mu(U)$. This implies that the map

$$s : T(\Gamma) \to F(\Gamma)$$

defined by sending $[\mu]$ to $([\mu], z_\mu) \in F(\Gamma)$ is a section of $\pi : F(\Gamma) \to T(\Gamma)$. Since w^μ depends holomorphically on $\mu \in M(\Gamma)$ by Theorem 1.1.1, the section s defined above is a holomorphic section. The image $s(T(\Gamma))$ under the map s is a complex manifold which is "nicely" embedded in $F(\Gamma)$ and is isomorphic to $T(\Gamma)$. This phenomenon was first discussed in Earle-Kra [19]. In this way, every elliptic element of Γ produces a holomorphic section of π. These sections are usually called *canonical sections* of π. If Γ is torsion free, then, in general, there is no global holomorphic section of $\pi: F(\Gamma) \to T(\Gamma)$. Fortunately, instead there exists a biholomorphic equivalence (which is called a *Bers isomorphism* in the literature) between $F(\Gamma)$ and $T(g, n+1)$. In this regard, the theory of Bers fiber space $F(\Gamma)$ for a torsion free group Γ with type (g, n) is parallel to the theory of Teichmüller space $T(g, n+1)$. For more details on this approach, the reader is referred to Kra [28]. In Chapter 3, we will discuss in more detail this important isomorphism.

Chapter 2

On a conjecture of Earle and Kra

The Bers fiber space $F(\Gamma)$ for Γ a torsion free finitely generated Fuchsian group of the first kind can be identified with a Teichmüller space. If Γ has torsion, Theorem 0.1 asserts that in most cases $F(\Gamma)$ is not isomorphic to any Teichmüller space. There are altogether 39 exceptional cases which remain to be settled. See Table (B) in the introduction. In this chapter, we are mainly concerned with the conjecture of Earle and Kra [19], and rule out 27 cases from 39 previously unknown cases mentioned above.

2.1 Restatement of Theorem 0.2

Let Γ be a finitely generated Fuchsian group of the first kind which contains elliptic elements. Assume that Γ is of type (g, n). The major task of this chapter is to prove the following theorem:

Theorem 2.1.1 *(1) Assume that* $(g,n) = (0,8)$ *and that the signature of* Γ *is neither*

$$(0,8;2,2,2,2,2,2,\infty,\infty)$$

$$\textit{nor} \quad (0,8;2,2,2,\infty,\infty,\infty,\infty,\infty),$$

then $F(\Gamma)$ *is not isomorphic to* $T(3,0)$. *Hence* $F(\Gamma)$ *is not isomorphic to any Teichmüller space.*

(2) Assume that $(g,n) = (0,7)$ *and that the signature of* Γ *is not in the following list:*

$$(0,7;2,2,2,2,2,2,\infty)$$

$$(0,7;2,2,2,2,\infty,\infty,\infty)$$

$$(0,7;2,2,\infty,\infty,\infty,\infty,\infty),$$

then $F(\Gamma)$ *is not isomorphic to* $T(2,2)$, *and thus* $F(\Gamma)$ *is not isomorphic to any Teichmüller space.*

(3) Assume that $(g,n) = (0,6)$. *If the signature of* Γ *is neither*

$$(0,6;2,2,2,2,\infty,\infty)$$

$$\textit{nor} \quad (0,6;2,2,2,\infty,\infty,\infty),$$

then $F(\Gamma)$ *is not isomorphic to* $T(1,4)$.

(4) Assume that $(g,n) = (0,5)$ *or* $(1,2)$. *Then* $F(\Gamma)$ *is not isomorphic to* $T(1,3)$ *(hence it is not isomorphic to any Teichmüller space) if one of the following conditions is satisfied:*

(i) Γ *is of type* $(0,5)$*, and the signature is neither*

$$(0,5;2,2,2,2,\infty)$$
$$\text{nor} \quad (0,5;2,2,\infty,\infty,\infty),$$

(ii) Γ *is of type* $(1,2)$*, and contains elliptic elements.*

(5) Assume that $(g,n) = (0,5)$ *or* $(1,2)$*, and that* Γ *contains elliptic elements. Then* $F(\Gamma)$ *is not isomorphic to* $T(0,6)$ *nor* $T(2,0)$.

(6) Assume that $(g,n) = (0,4)$ *or* $(1,1)$*. Then* $F(\Gamma)$ *is not isomorphic to* $T(0,5)$ *nor* $T(1,2)$ *(hence it is not isomorphic to any Teichmüller space) if one of the following conditions is satisfied:*

(i) Γ *is of type* $(0,4)$*, contains elliptic elements, and the signature is neither*

$$(0,4;2,2,2,\infty)$$
$$\text{nor} \quad (0,4;2,2,\infty,\infty),$$

(ii) Γ *is of type* $(1,1)$ *and contains elliptic elements.*

Remark. Note that the above theorem removes 26 entries from Table (B) in the introduction.

The basic idea which was used to prove Theorem 0.1 is the following. First, we think of Γ as a group of holomorphic automorphisms of $F(\Gamma)$ (see §1.3 for the definitions). Assume that the pair $((g,n),(g',n'))$ does not lie in Table (A) in the introduction, where (g,n) and (g',n') are the types of Γ and

Γ', respectively. Then a cyclic subgroup $\mathcal{G}$ of Γ (with prime order) can be chosen so that $\mathcal{G}$ acts on $F(\Gamma)$ as a group of holomorphic automorphisms, but $\mathcal{G}$ cannot act on $T(\Gamma')$ as a group of holomorphic automorphisms. In addition, $\mathcal{G}$ acts trivially on the image of the holomorphic section of $\pi : F(\Gamma) \to T(\Gamma)$ corresponding to the elliptic generator of $\mathcal{G}$.

In contrast, our method is slightly different from the above. We construct a cyclic group $\mathcal{G}'$ (with prime order too) of holomorphic automorphisms of $F(\Gamma)$ satisfying the condition that $\mathcal{G}'$ is not a subgroup of Γ, but it is still fiber-preserving and leaves invariant the image of a special holomorphic section of $\pi : F(\Gamma) \to T(\Gamma)$. This construction depends essentially on the signature of Γ. We will check that if the signature of Γ satisfies the condition of Theorem 2.1.1, then $\mathcal{G}'$ cannot act as holomorphic automorphisms on the corresponding Teichmüller space.

It should be indicated that the method developed in this chapter has its own limitation. The reasons that Earle-Kra's conjecture can not be completely solved are:

(1) Either the cyclic group $\mathcal{G}'$ we constructed really acts on the corresponding Teichmüller space; this situation occurs, for example, when the pair $((g, n; \nu_1, \ldots, \nu_n), (g', n'))$ lies in the 3rd row of Table (B) in the introduction; or

(2) Due to lack of knowledge on conformal automorphisms of certain special Riemann surfaces, we do not see whether or not the group $\mathcal{G}'$ acts on the corresponding Teichmüller space.

In the attempt to handle certain difficult situations which are not included in the statement of Theorem 2.1.1, a new method will be introduced which enable one to settle one more case; that is, we have the following result.

Theorem 2.1.2 *The Bers fiber space* $F(0,6;2,2,2,2,2,\infty)$ *is not isomorphic to the Teichmüller space* $T(2,1)$.

Remark. It follows from Theorem 0.1, Theorem 2.1.2, and Theorem 2.1.1 (3) that the Bers fiber space $F(0,6;2,2,2,2,2,\infty)$ cannot be isomorphic to any Teichmüller space. We see here that the two spaces $F(0,6;2,2,2,2,2,\infty)$ and $F(0,6;2,2,2,2,2,2)$ are essentially distinct. Note also that this theorem gives us one more example for the assertion that $F(\Gamma)$ depends on the signature of Γ.

If Γ is of signature $(0,6;2,\infty,\infty,\infty,\infty,\infty)$ or $(0,6;2,2,\infty,\infty,\infty,\infty)$, by Theorem 2.1.1 (3), $F(\Gamma)$ is not isomorphic to $T(1,4)$. However, there is no known proof asserting that $F(\Gamma)$ is not isomorphic to $T(2,1)$. On the other hand, Theorem 0.1 states that $F(\Gamma)$ is not isomorphic to any other Teichmüller space distinct from $T(2,1)$, and that $T(2,1)$ is the only Teichmüller space to which $F(0,6;2,2,2,2,2,2)$ is isomorphic. See (0.2) in the introduction.

If Γ is of signature $(0,5;2,2,\infty,\infty,\infty)$, then by Theorem 2.1.1 (4), $F(\Gamma)$ is not isomorphic to $T(0,6)$ (or $T(2,0)$), but we still do not know whether $F(\Gamma)$ is isomorphic to $T(1,3)$. We also see that $T(1,3)$ is the only Teichmüller space to which $F(0,5;2,2,2,2,\infty)$ can be isomorphic (see (0.3) in the introduction).

In spite of the complexity, Theorem 2.1.1 (5), (6), and (4 (ii)) give a complete solution to Bers' question in some low dimension situations. More precisely, the Earle-Kra conjecture is true if the pair $((g,n),(g',n'))$ lies in the following table:

$((0,4),(0,5))$	$((0,4),(1,2))$	$((1,1),(1,2))$
$((1,1),(0,5))$	$((1,2),(1,3))$	$((1,2),(0,6))$
$((1,2),(2,0))$	$((0,5),(0,6))$	$((0,5),(2,0))$

As we mentioned earlier, our work still leaves 12 cases which remain open. They are listed in the following Table (C'):

signature $(g,n;\nu_1,\ldots,\nu_n)$ of Γ	type (g',n') of Γ'
$(0,8;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{8-m})$, $m=3,6$,	$(3,0)$
$(0,7;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{7-m})$, $m=2,4,6$,	$(2,2)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})$, $m=1,\ldots,4$,	$(2,1)$
$(0,6;\underbrace{2,\ldots,2}_{m},\underbrace{\infty,\ldots,\infty}_{6-m})$, $m=3,4$,	$(1,4)$
$(0,5;2,2,\infty,\infty,\infty)$	$(1,3)$

Table (C')

Although Theorem 2.1.1 and Theorem 2.1.2 only give a partial solution towards Bers' question, it has an interesting consequence stated as follows:

Theorem 2.1.3 *Let Γ be a finitely generated Fuchsian group of the first kind of type (g, n). Then the Bers fiber space $F(\Gamma)$ is isomorphic to the Teichmüller space $T(g, n+1)$ if and only if one of the following conditions is satisfied:*

(1) *Γ is of type $(0,3)$;*

(2) *The signature of Γ is either $(0,4;2,2,2,\infty)$, or $(0,4;2,2,\infty,\infty)$;*

(3) *Γ is torsion free.*

Outline of proof. First we prove the "if" part. If Γ is of type $(0,3)$, then the assertion follows from (0.1) in the introduction. If Γ is torsion free, it is the Bers isomorphism theorem. If condition (2) holds, then from (0.4) of the introduction, one sees that $F(0,4;\infty,\infty,\infty,\infty) \cong F(0,4;2,2,2,\infty) \cong F(0,4;2,2,\infty,\infty) \cong T(0,5)$.

Next, we prove the "only if" part. Suppose that no conditions (1) (2) (3) hold, and that $F(\Gamma)$ is isomorphic to a Teichmüller space $T(\Gamma')$ for some group Γ' of type $(g, n+1)$. This means that Γ contains elliptic elements, and hence the condition of Theorem 0.1 is satisfied. By using that theorem, one finds that all elliptic elements of Γ must be of order 2, and moreover, by checking Table (B) in the introduction, one sees at once that only several possible situations can occur, which are:

(1) $(g,n;\nu_1,\ldots,\nu_n) = (0,5;\underbrace{2,\ldots,2}_{k},\underbrace{\infty,\ldots,\infty}_{5-k})$, $0 < k \leq 5$ and $(g',n') = (0,6)$;

(2) $(g,n;\nu_1,\ldots,\nu_n) = (1,2;2,2)$ or $(1,2,2,\infty)$ and $(g',n') = (1,3)$;

(3) $(g,n;\nu_1,\ldots,\nu_n) = (1,1;2)$, and $(g',n') = (1,2)$;

(4) $(g,n;\nu_1,\ldots,\nu_n) = (0,4;2,\infty,\infty,\infty)$ and $(g',n') = (0,5)$.

But all these cases are excluded by Theorem 2.1.1 (4) (5) and (6). This completes the proof. □

2.2 Topological aspects

The purpose of this section is to construct, by means of purely topological methods, some interesting periodic automorphisms of surfaces. We will also study the structure of the components of hyperelliptic loci in the Teichmüller space $T(2,1)$, and will prove that any two components of hyperelliptic loci in $T(2,1)$ are modular equivalent (Proposition 2.2.7). The result we obtain is interesting in its own right and will play a crucial role in proving Theorem 2.1.2 as well.

Throughout this paper, we use the phrase "self-map" to denote a quasiconformal self-map of a Riemann surface; and use the phrase "self-map in the sense of orbifolds" to denote a quasiconformal self-map of an orbifold which carries a branch point of order ν to a branch point with the same order.

Let $\Delta = \{z;\ |z| < 2\}$ be parametrized by the polar coordinates (r, α). Define $\hat{\sigma} : \Delta \to \Delta$ as $\hat{\sigma}(r, \alpha) = (r, \alpha - r\pi)$. Let x_i, $i = 1, 2, \ldots, n$, $n \geq 3$, be n distinct points on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and let τ_i, $i = 1, \ldots, n-1$, be any embeddings of Δ into $\hat{\mathbb{C}}$ with the properties that $\tau_i(\Delta)$ contains x_i and x_{i+1}, but is disjoint from all x_j for $j \neq i, i+1$. Suppose that $\tau_i(1,0) = x_i$, and

$\tau_i(1, \pi) = x_{i+1}$. We obtain a self-maps σ_i on $\hat{\mathbb{C}}$ defined by

$$\sigma_i(x) = \begin{cases} x & \text{if } x \notin \tau_i(\Delta) \\ \\ \tau_i \circ \hat{\sigma} \circ \tau_i^{-1} & \text{if } x \in \tau_i(\Delta). \end{cases} \tag{2.1}$$

Intuitively, σ_i is a self-map interchanging x_i and x_{i+1}, and is the identity outside a neighborhood of x_i and x_{i+1}. Observe that for $i = 1, 2, \ldots, n$, σ_i is a self-map on the punctured sphere $\hat{\mathbb{C}} - \{x_1 \ldots, x_n\}$. Note also that σ_i is a self-map (in the sense of orbifolds) of the orbifold X, where X is the Riemann sphere $\hat{\mathbb{C}}$ with n branch points $x_1, \ldots, x_n$ of order ν.

Remark. σ_i is usually called an *elementary twist.* For more details about the map σ_i, the reader is referred to Birman-Hilden [14]. Elementary twists can also be defined on an arbitrary orbifold provided that the two distinguished points involved have the same ramification number. Some interesting results on Teichmüller modular group may be obtained by more careful investigations of elementary twists. The study of this object is closely linked to the theory of knots and braids, see Birman [13] for a discussion.

Consider the following self-map of $\hat{\mathbb{C}}$:

$$\sigma = \sigma_{n-2} \circ \cdots \circ \sigma_1. \tag{2.2}$$

It is easily seen that σ fixes x_n and realizes a permutation of the set $\{x_1, \ldots, x_n\}$. A theorem of Magnus [31] shows that, as a self-map of $\hat{\mathbb{C}} - \{x_1, \ldots, x_n\}$, σ is periodic and its order is $n-1$ (up to isotopy). The isotopy $I\colon \hat{\mathbb{C}}\times[0,1]\to\hat{\mathbb{C}}$ between σ^{n-1} and the identity may be chosen so that $I(x_i,t)=x_i$, for $0\le t\le 1$ and $i=1,\ldots,n$.

In general, let σ_0 be a self-map of $\hat{\mathbb{C}}-\{x_1,\ldots,x_n\}$. Then σ_0 can be extended to a self-map of $\hat{\mathbb{C}}$ and therefore, σ_0 determines a permutation of $\{x_1,\ldots,x_n\}$. We need the following simple lemma.

Lemma 2.2.1 *If we assume that σ_0 is a periodic self-map (up to isotopy) of the punctured sphere $\hat{\mathbb{C}}-\{x_1,\ldots,x_n\}$, and also assume that σ_0 keeps $k\ge 3$ punctures $\in\{x_1,\ldots,x_n\}$ fixed. Then σ_0 is isotopic to the identity on $\hat{\mathbb{C}}-\{x_1,\ldots,x_n\}$.*

Proof. The extension of σ_0 to $\hat{\mathbb{C}}$ is denoted by $\tilde{\sigma}_0$. As we see above, $\tilde{\sigma}_0$ is a self-map with the property that it permutes the set $\{x_1,\ldots,x_n\}$. By hypothesis, σ_0 induces a elliptic modular transformation (in the sense of Bers [9]) on the Teichmüller space $T(\hat{\mathbb{C}}-\{x_1,\ldots,x_n\})$. (The definition of Teichmüller space of a Riemann surface is given in §3.2.) Thus Nielsen's theorem [41] implies that there is a n-punctured sphere, say $\hat{\mathbb{C}}-\{x_1,\ldots,x_n\}$, on which σ_0 is isotopic to a conformal self-map. The isotopy can be chosen so as to fix $x_1,\ldots,x_n$. So $\tilde{\sigma}_0$ must be isotopic to a Möbius transformation. Since $\tilde{\sigma}_0$ leaves more than two points fixed, this Möbius transformation is the identity. It follows that σ_0 is isotopic to the identity, as claimed. □

Let X be a Riemann surface of type (g,n) with $2g-2+n>0$, and let $\overline{X}$ be the compactification of X which is obtained by filling in all punctures of X. Suppose that X is hyperelliptic; that is, X admits a hyperelliptic involution J.

Here by a hyperelliptic involution on a Riemann surface X of type (g, n) with $2g-2+n>0$ and $n>0$, we mean a conformal involution on X (hence on $\overline{X}$) which has $2g+2$ fixed points on $\overline{X}$, interchanges pairwise the n punctures if n is even, and fixes one puncture and interchanges the other $n-1$ punctures pairwise if n is odd.

Now let X be a hyperelliptic Riemann surface, and let J be the corresponding hyperelliptic involution of X. Let $q: X \rightarrow X/<J>$ denote the natural projection. We know that J has $2g+2$ fixed points on $\overline{X}$, these fixed points are called *Weierstrass points* of $\overline{X}$. Their images under q are on $\overline{X}/<J>$, all of which are branch points of order 2. We see that $\overline{X}/<J>$ is an orbifold with signature $(0, 2g+2; 2, \ldots, 2)$. In what follows, $\overline{X}$ is always taken as a symmetrically embedded surface (about x-axis) in $\mathbb{R}^3$. In this setting, J is considered as a $180°$ rotation about x-axis.

Let f be a self-map of $X/<J>$ in the sense of orbifolds; that is, f carries a point over which the covering $q: X \rightarrow X/<J>$ is ramified of order 2 or ∞ into another such point with the same order. f always lifts to a self-map $\tilde{f}: X \rightarrow X$ (see Birman-Hilden [14] for a construction) so that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ X/<J> & \xrightarrow[f]{} & X/<J> \end{array}$$

It is easy to see that if $\tilde{f}$ is a lift of f, then $J \circ \tilde{f}$ is also a lift of f. And these are all possible lifts of f. Now choose a simple closed curve c on X with an arbitrary orientation so that it is symmetric about the x-axis (recall that X is embedded symmetrically about x-axis in $\mathbf{R}^3$). These curves are called *canonical curves* in the sequel. The following properties are immediate:

(1) $J(c) = c$;

(2) J reverse the orientation of c, and hence J is not isotopic to the identity.

It follows that $\tilde{f}$ and $J \circ \tilde{f}$ cannot be isotopic to each other. Observe also that $\tilde{f}$ and $J \circ \tilde{f}$ are all of the lifts of f and they lie in the normalizer of $< J >$; that is, $\tilde{f} \circ J \circ (\tilde{f})^{-1} = J$ and $(J \circ \tilde{f}) \circ J \circ (J \circ \tilde{f})^{-1} = J$. The following lemma is an easy consequence of a theorem of Birman-Hilden [14].

Lemma 2.2.2 *Under the same notation as above, f is isotopic to the identity on $X/< J >$ if and only if either $\tilde{f}$ or $J \circ \tilde{f}$ (but not both) is isotopic to the identity on X.*

Proof. Suppose there is an isotopy $I\colon X/{<J>} \times [0,1] \to X/{<J>}$, connecting f and the identity. By utilizing the result of [14], as we mentioned earlier, for a fixed t_0, $0 \le t_0 \le 1$, $I(x, t_0)$ can be lifted to a self-map of X. It follows from continuity that $I(x,t)$, $0 \le t \le 1$, $x \in X/< J >$ can be lifted to an isotopy $\tilde{I}\colon X \times [0,1] \to X$ such that for all $x \in X$, $\tilde{I}(x,0) = x$ and $\tilde{I}(\cdot, 1)$ is a lift of f, which is either $\tilde{f}$ or $J \circ \tilde{f}$. Since $\tilde{f}$ is not isotopic to $J \circ \tilde{f}$, only one case can occur. Conversely, we assume first that $\tilde{f}$ is isotopic to the identity. Since $\tilde{f}$ is J-symmetric, by a theorem of Birman-Hilden [14], $\tilde{f}$ is isotopic to the identity by an isotopy $\tilde{I}\colon X \times [0,1] \to X$ which is J-symmetric as well. This means that $\tilde{I}(\cdot, t)$ is J-symmetric for all $0 \le t \le 1$, which in turn implies that $\tilde{I}(x, t)$ can be projected to an isotopy connecting f and the identity. Similarly, if $J \circ \tilde{f}$ is isotopic to the identity, then the projection of $J \circ \tilde{f}$, which is also f, is isotopic to the identity as well. The lemma is proved. □

We consider now a general situation. Let X be a hyperelliptic Riemann surface of genus g with one puncture. The associated hyperelliptic involution is denoted by J. According to the definition given earlier, the puncture is one of the $2g+2$ Weierstrass points. Let $x_1, \dots, x_{2g+2} \in \overline{X}$ denote these Weierstrass points. Assume that x_{2g+2} is the puncture of X.

The geometric intersection number of two unoriented non-separating, simple closed curves α and β of X, denoted by $i(\alpha, \beta)$, is the minimal number of intersections of $\tilde{\alpha}$ and $\tilde{\beta}$ as $\tilde{\alpha}$ and $\tilde{\beta}$ run over free homotopy classes of α and β, respectively. By definition, a canonical curve α satisfies the condition that $J(\alpha) = \alpha$. Since $J\colon X \to X$ is considered as a $180°$ rotation about the x-axis, α must contain only two fixed points. Moreover, α can be parametrized as $\alpha(\theta)$, $0 \le \theta \le 2\pi$, so that $\alpha(\theta)$ contains only two Weierstrass points $\alpha(0)$ and $\alpha(\pi)$.

A *chain for* x_{2g+2} is a $2g$-tuple $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ of canonical curves on X with the following properties:

(1) $i(\alpha_j, \alpha_k) = 0$, $i(\beta_j, \beta_k) = 0$, $i(\alpha_j, \beta_j) = 1$, for $1 \le j, k \le g$;

(2) $i(\beta_j, \alpha_{j+1}) = 1$, for $1 \le j \le g-1$;

(3) $i(\alpha_j, \beta_k) = 0$, for $1 \le j, k \le g$, $j \ne k, k+1$.

Example 1. Figure 1 below exhibits a chain for x_{12} in the case of $g = 5$.

Remark. In [36], J. D. McCarthy introduced another concept for chain (which is called *maximal chain* in his language) for compact Riemann surfaces. Our approach is similar to [36].

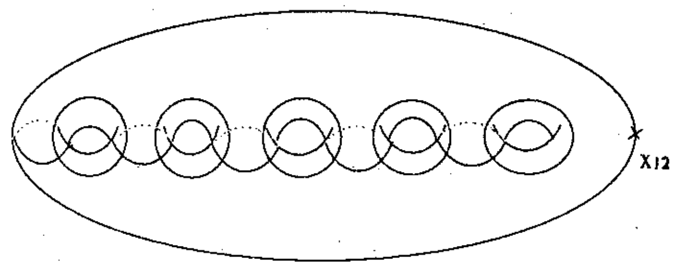

Figure 1.

A *Dehn twist* h_c about a simple closed curve c on a Riemann surface S is defined as follows. Let A be an annular neighborhood of c parametrized by the polar coordinates (r, θ), $-1 \leq r \leq 1$, and $0 \leq \theta \leq 2\pi$ such that c is defined by $r = 0$. Then, h_c is the identity outside A. Fix $x = (r, \theta) \in A$, we have

$$h_c(r, \theta) = \begin{cases} (r, \theta + 2\pi r), & 0 \leq r \leq 1, \\ (r, \theta), & -1 \leq r \leq 0. \end{cases}$$

A geometric interpretation of h_c is: cut S open along c, twist one end through 2π, and then glue back along c. Obviously, the homotopy class of h_c depends only on the free homotopy class of c. Thus, the modular transformation induced by h_c depends only on the free homotopy class of c. Another property is that if f is a self-map of S, then

$$f \circ h_c \circ f^{-1} = h_{f(c)}. \tag{2.3}$$

Our object now is to express the hyperelliptic involution J in terms of a composition of Dehn twists about some canonical curves. To each hyperelliptic Riemann surface X we can associate a set of canonical curves (that is, those curves on X which are symmetric about the x- axis). Now let $\mathcal{C} = (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ be a chain for x_{2g+2} ($\mathcal{C}$ is a set of canonical curves which satisfies the properties (1) (2) (3) in page 38). Define

$$h = h_{\beta_g} \circ h_{\alpha_g} \circ \cdots \circ h_{\beta_1} \circ h_{\alpha_1}. \tag{2.4}$$

The following lemma is based on a geometric argument of [14]. Recall the definition of σ is given by (2.2). For our purpose, the map σ can be viewed as a self-map of $X/<J>$ in the sense of orbifolds. Again, $q : X \to X/<J>$ is the natural covering map.

Lemma 2.2.3 *The self-map h of X defined in (2.4) is isotopic to a self-map h' which can be projected to a self-map of $X/<J>$ (in the sense of orbifolds) isotopic to σ.*

For completeness we sketch the proof. Consider a single Dehn twist h_{α_1} about the canonical curve α_1. Let x_1 and x_2 be the two Weierstrass points on α_1. The Dehn twist h_{α_1} is isotopic to the twist h'_{α_1} constructed as follows: cut X open along α_1, twist one end through π, twist the other end through π also, and then reattach X along c. We see that h'_{α_1} has the following properties:

(1) h'_{α_1} leaves invariant the canonical curve α_1 and interchanges x_1 and x_2;

(2) h'_{α_1} is the identity outside a small neighborhood $N(\alpha_1)$ of α_1;

(3) h'_{α_1} commutes with J.

Hence h'_{α_1} can be projected to a self-map σ'_1 on the orbifold $X/<J>$. By (1) (2) above, σ'_1 interchanges $q(x_1)$ and $q(x_2)$, and is the identity outside a disc enclosing $q(x_1)$ and $q(x_2)$. In particular, σ'_1 fixes $q(x_3), \ldots, q(x_{2g+2})$. Note that $q(x_{2g+2})$ is a puncture.

We claim that σ'_1 is isotopic to σ_1 (defined by (2.1)) on the $2g+2$-punctured sphere $X/<J> - \{q(x_1), \ldots, q(x_{2g+1})\}$. To see this, we need to investigate, in a small disc enclosing $q(x_1)$ and $q(x_2)$, the alteration of a foliation by the map $\sigma_1^{-1} \circ \sigma'_1$. A typical situation is illustrated in Figure 2:

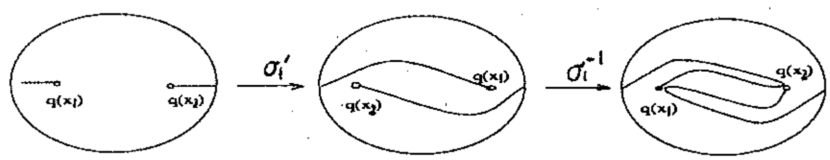

Figure 2.

See Thurston's lecture notes [48] for a discussion. Since this technique is not needed in this paper, we omit the details, and conclude that σ'_1 is isotopic to σ_1 on the $2g+2$-punctured sphere $X/<J> - \{q(x_1), \ldots, q(x_{2g+1})\}$.

Similarly, for each $i = 2, \ldots, g$, h_{α_i} (resp. h_{β_i}) is isotopic to a self-map h'_{α_i} (resp. h'_{β_i}) which can be projected to a self-map of $X/<J>$ (in the sense of orbifolds) isotopic to σ_{2i-1} (resp. σ_{2i}). From (2.4) we see that $h =$

$h_{\beta_g} \circ h_{\alpha_g} \circ \cdots \circ h_{\beta_1} \circ h_{\alpha_1}$ is isotopic to $h'_{\beta_g} \circ h'_{\alpha_g} \circ \cdots \circ h'_{\beta_1} \circ h'_{\alpha_1}$ which can be projected to the map σ' isotopic to $\sigma = \sigma_{2g} \circ \cdots \sigma_1$. This finishes the proof of Lemma 2.2.3. □

From Lemma 2.2.3, one sees immediately that h is isotopic to a lift of σ. Since σ^{2g+1} is isotopic to the identity on $X/<J> -\{q(x_1), \ldots, q(x_{2g+1})\}$, by Lemma 2.2.2, h^{2g+1} is either isotopic to J, or isotopic to the identity. But a simple computation tells us that h^{2g+1} reverses the orientation of the chain for x_{2g+2} of canonical curves (give arbitrarily an orientation to the curves before doing the Dehn twists). We see that h^{2g+2} is isotopic to J. We thus have

Lemma 2.2.4 *As a self-map of X, the hyperelliptic involution J is isotopic to $(h_{\beta_g} \circ h_{\alpha_g} \circ \cdots \circ h_{\beta_1} \circ h_{\alpha_1})^{2g+1}$.* □

Example 2. We consider a Riemann surface X of signature $(1,1;\infty)$. X is a punctured torus, which can be represented as $\mathbb{C}/G_\tau - \{0\}$, where $G_\tau =< A, B >$ is the group generated by translations $A : z \mapsto z+1$ and $B : z \mapsto z+\tau$, for some $\tau \in U$. A fundamental region for G_τ is shown in Figure 3(a), where the origin is the puncture. Consider the involution $j : z \mapsto -z$. A computation shows that

$$j \circ C(z) = C^{-1} \circ j(z), \quad \text{for any} \quad C \in G_\tau. \qquad (2.5)$$

This means that j can be projected to a conformal involution J of X. Note that $X/< J >$ is a Riemann surface of signature $(0,4;2,2,2,\infty)$.

Let $q(x_1), q(x_2), q(x_3) \in X/<J>$ denote the three branch points of order 2, and let $q(x_4)$ denote the puncture. By pasting the opposite sides in Figure 3(a), X can be drawn as Figure 4, where c_1', c_2' come from identifying the opposite sides c_i labeled in Figure 3(a). By construction, $J(c_i') = c_i'$ $(i = 1, 2)$, and $c_1' c_2' c_1'^{-1} c_2'^{-1}$ is isotopic to x_4. Moreover, $\{c_1', c_2'\}$ constitutes a chain for x_4. Define $h = h_{c_2'} \circ h_{c_1'}$. Obviously, h is a self-map of X fixing the puncture. By Lemma 2.2.4, h^3 is isotopic to J.

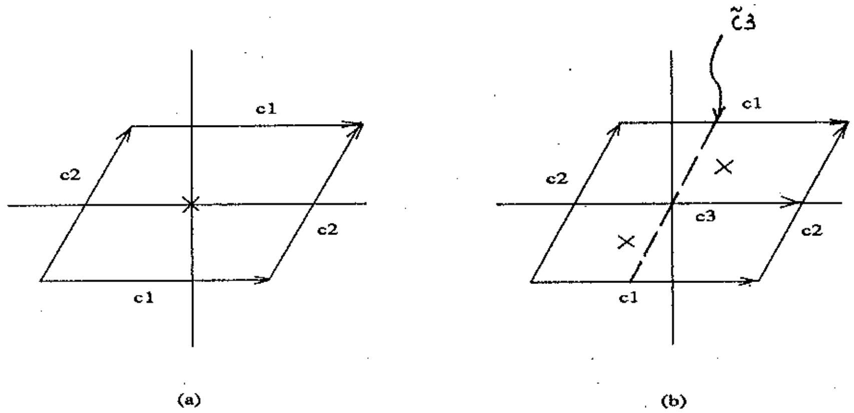

Figure 3.

Example 3. Now let X be a Riemann surface of signature $(1, 2; \infty, \infty)$, and let $\overline{X}$ be its compactification. Let G_τ be as in Example 2. Consider the universal covering $\mathbb{C}$ of the torus $\overline{X}$. The preimages of the pair of punctures (x_1, x_2) form two lattices. Let us denote $(\tilde{x}_1, \tilde{x}_2)$ one pair of preimages of (x_1, x_2). Any fundamental region for G_τ contains exactly one pair of preimages of (x_1, x_2). By composing with an Euclidean motion of $\mathbb{C}$ if necessary, one may assume that $(\tilde{x}_1, \tilde{x}_2)$ are symmetric about the origin. In this case a fundamental domain (parallelogram) D for G_τ can be chosen so that D is symmetric with respect to the origin and one pair of sides of D is parallel to the x-axis. Let $c_3 = D \cap \{x - \text{axis}\}$. Without loss of generality, we assume that $\tilde{x}_1$ and $\tilde{x}_2$ are not in c_3. Otherwise we choose $D \cap \{\tilde{c}_3\}$ as c_3, where $\tilde{c}_3$ is the segment in D which passes through the origin and parallel to the other pair of sides of D. See Figure 3(b). Again, the mapping $j : z \mapsto -z$ interchanges the two punctures $\tilde{x}_1$ and $\tilde{x}_2$ in D and satisfies (2.5). Hence, X admits a hyperelliptic involution J interchanging the two punctures (obtained by projecting j).

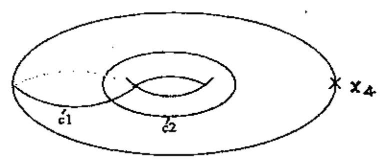

Figure 4.

Note also that $X/\langle J \rangle$ is a Riemann surface with signature $(0, 5; 2, \ldots, 2, \infty)$. By pasting the opposite sides in Figure 3(b), X can be drawn in Figure 5, where c_1', c_2', come from the boundary of D by "pasting procedure" and c_3' comes from $\tilde{c}_3$ by identifying the endpoints. One sees that $J(c_i') = c_i'$, for $i = 1, 2, 3$. Let $h = h_{c_3'} \circ h_{c_2'} \circ h_{c_1'}$. By the same argument as above, h is isotopic to a lift of σ (described in (2.2)). Since σ^4 is isotopic to the identity, by Lemma 2.2.2, h^4 is either isotopic to the identity, or isotopic to J. From the construction, h fixes the two punctures, while J interchanges the two punctures. We see that h^4 is not isotopic to J. So h^4 must be isotopic to the identity.

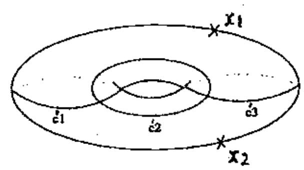

Figure 5.

The above two examples will play roles in proving Theorem 2.1.1 (4) (5) and (6). Now, we proceed to study the components of the hyperelliptic loci in $T(2,1)$. The discussion given below is crucial in proving Theorem 2.1.2. In our particular situation; that is, X is of type $(2,1)$, X is hyperelliptic if and only if its puncture is one of the six Weierstrass points of $\overline{X}$. Every component of the hyperelliptic loci of $T(2,1)$ is the fixed point set of an (elliptic with order 2) modular transformation induced by a hyperelliptic involution on a marked Riemann surface in $T(2,1)$. It is a connected, closed submanifold of $T(2,1)$. Conversely, every elliptic modular transformation induced by a hyperelliptic involution determines a fixed point set, which is a component of the hyperelliptic loci. We see that the set of components of the hyperelliptic loci in $T(2,1)$ is one-to-one correspondent with the set of isotopy classes of orientation-preserving self-maps represented by hyperelliptic involutions. Let H be a component of the hyperelliptic loci in $T(2,1)$. By using Lemma 2.2.4, we conclude that the corresponding hyperelliptic involution J is isotopic to $h^5 = (h_{\beta_2} \circ h_{\alpha_2} \circ h_{\beta_1} \circ h_{\alpha_1})^5$. Two chains for x_6 are called *equivalent* if they are invariant under the same hyperelliptic involution. Since h is uniquely determined by the chain $C = (\alpha_1, \beta_1, \alpha_2, \beta_2)$, we obtain

Lemma 2.2.5 *There is a bijection between the set of components of the hy-*

perelliptic loci in $T(2,1)$ and the set of chains for x_6 modulo the equivalence relation. □

Once again, let X be a hyperelliptic surface of type $(2,1)$ which is symmetrically embedded in $\mathbb{R}^3$ as shown in Figure 6. Our next purpose is to show that any chain for x_6 is homeomorphic (as a set of points) to a standard chain drawn in Figure 6:

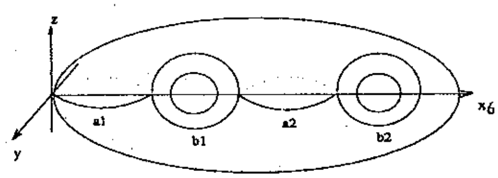

Figure 6.

where $a_1 = \{(x, y, 0) \in \mathbb{R}^3;\ (x-1)^2 + y^2 = 1\}$, $b_1 = \{(x, 0, z) \in \mathbb{R}^3;\ (x-3)^2 + z^2 = 1\}$, $a_2 = \{(x, y, 0) \in \mathbb{R}^3;\ (x-5)^2 + y^2 = 1\}$, and $b_2 = \{(x, 0, z) \in \mathbb{R}^3;\ (x-7)^2 + z^2 = 1\}$. Indeed, the desired homeomorphism f_b can be easily obtained by gluing 4 simple maps $f_1, \ldots, f_4$ together, where $f_1 : \alpha_1 \to a_1$, $f_2 : \beta_1 \to b_1$, $f_3 : \alpha_2 \to a_2$, and $f_4 : \beta_2 \to b_2$ are only defined on curves and can be constructed as follows. First, f_1 is defined as a homeomorphism of α_1 to a_1 with $f_1(\alpha_1 \cap \beta_1) = a_1 \cap b_1$; f_2 is a homeomorphism of β_1 to b_1 with the properties that $f_2(\alpha_1 \cap \beta_1) = a_1 \cap b_1$ and $f_2(\beta_1 \cap \alpha_2) = b_1 \cap a_2$. Since both $\beta_1 - \{\alpha_1 \cap \beta_1, \beta_1 \cap \alpha_2\}$ and $b_1 - \{a_1 \cap b_1, b_1 \cap a_2\}$ consist of two open intervals, f_2 can be easily constructed so as to satisfy the above properties. The constructions of f_3 and f_4 are similar to those of f_1 and f_2, respectively. From this construction, we have

Lemma 2.2.6 *Any two chains for x_6 are homeomorphic in the sense of one dimension.* $\square$

We are in the position to prove the following proposition which will play a crucial role in proving Theorem 2.1.2.

Proposition 2.2.7 *Any two components H and H' of the hyperelliptic loci in $T(2,1)$ are modular equivalent; that is, there exists $\Omega \in Mod(2,1)$ such that $\Omega(H) = H'$. Furthermore, let J and J' be the hyperelliptic involutions corresponding to H and H', respectively, then $\Omega \circ J \circ \Omega^{-1} = J'$.*

Remark. For any Teichmüller space of a compact Riemann surface, the result is well known; it is, however, not known whether or not the statement is also true for a Teichmüller space of a Riemann surface with more than one puncture. Our proof is similar to [36].

Proof. Lemma 2.2.5 says that we can choose two chains for x_6, $\mathcal{C}$ and $\mathcal{C}'$ on X corresponding to H and H', respectively. By Lemma 2.2.6, there is a homeomorphism $f_b\colon \mathcal{C} \to \mathcal{C}'$. f_b extends to a homeomorphism f_N of a tubular neighborhood $N(\mathcal{C})$ onto a tubular neighborhood $N(\mathcal{C}')$, where $N(\mathcal{C})$ is drawn in Figure 7. Hence, f_b determines a homeomorphism (call it f_b also) of $\partial N(\mathcal{C})$ onto $\partial N(\mathcal{C}')$. The Euler characteristic of $\overline{X}$ is $2-2g = -2$. On the other hand, by looking at the chain $\mathcal{C}$, we see that the number V of vertices of $\overline{X} - \mathcal{C}$ is 3, the number E of edges of $\overline{X} - \mathcal{C}$ is 6. Let F denote the number of faces of $\overline{X} - \mathcal{C}$. By computing the Euler characteristic by the formula $V + F - E$, we see that

$$-2 = F + 3 - 6,$$

and we get $F = 1$. In particular, we conclude that $X - N(\mathcal{C})$ is conformally equivalent to the punctured disk $\dot{\Delta} = \{z;\ 0 < |z| < 1\}$. We denote by ξ this conformal map. Similarly, there is a conformal map ζ of $X - N(\mathcal{C}')$ onto $\dot{\Delta}$.

Think of $\overline{X} - N(\mathcal{C})$ and $\overline{X} - N(\mathcal{C}')$ as polygons, we see that ξ and ζ can be extended to the closed polygons of $\overline{X} - N(\mathcal{C})$ and $\overline{X} - N(\mathcal{C}')$, respectively. Note that f_b establishes a boundary correspondence between the two polygons. It follows that

$$\zeta \circ f_b \circ \xi^{-1}$$

is a homeomorphism of $\mathbb{S}^1$ onto $\mathbb{S}^1$. Now $\zeta \circ f_b \circ \xi^{-1}$ can be extended to a self-map η of the closed unit disk by radial extension. It turns out that

$$f(x) = \begin{cases} \zeta^{-1} \circ \eta \circ \xi(x), & \text{if } x \in X - N(\mathcal{C}); \\ \\ f_N(x), & \text{if } x \in N(\mathcal{C}) \end{cases}$$

is a self-map of $\overline{X}$ which fixes x_6, and hence defines a self-map of X which carries $\mathcal{C}$ to $\mathcal{C}'$. Let $\mathcal{C} = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ and $\mathcal{C}' = (\alpha_1', \beta_1', \alpha_2', \beta_2')$. By (2.3), we obtain

$$f \circ (h_{\beta_2} \circ h_{\alpha_2} \circ h_{\beta_1} \circ h_{\alpha_1})^5 \circ f^{-1} = (h_{\beta_2'} \circ h_{\alpha_2'} \circ h_{\beta_1'} \circ h_{\alpha_1'})^5.$$

It follows from Lemma 2.2.4 that $f \circ J \circ f^{-1}$ is isotopic to J'. This implies that $\Omega \circ J \circ \Omega^{-1} = J'$, where $\Omega \in \mathrm{Mod}(2,1)$ is induced by f. This completes the proof. $\square$

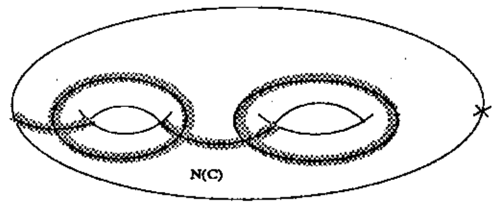

Figure 7.

2.3 Proof of Theorem 0.1

Our object in this section is to give another proof of Theorem 0.1. We first prove

Proposition 2.3.1 *Let G_0 be a torsion free finitely generated Fuchsian group of type (g_0, n_0) with $2g_0 - 2 + n_0 > 0$, and let $\chi_0 \in Mod\ G_0$ be an elliptic modular transformation of prime order, with the property that the restriction of χ_0 to a component l of the hyperelliptic loci is the identity. Then χ_0 is either the identity or a hyperelliptic involution.*

Proof. Suppose that $\chi_0 \in \text{Mod } G_0$ is non-trivial. Let

$$T(g_0, n_0)^{\chi_0} = \{x \in T(g_0, n_0); \text{ with } \chi_0(x) = x\}.$$

By hypothesis, l is included in the set of fixed points of χ_0. On the other hand, $T(g_0, n_0)^{\chi_0}$ is again a Teichmüller space of type (g^*, n^*) (see Kravetz [29], or Earle-Kra [20]), where g^* and n^* are defined as follows. Let χ_0 be induced by a conformal self-map h_0 on a Riemann surface X of type (g_0, n_0) (Nielsen's theorem [41] asserts that such an X exists), g^* is the genus of the surface $X/<h_0>$, and n^* is the number of the distinguished points (including all punctures) on $X/<h_0>$. Since we assume that l is a non-empty component of the hyperelliptic loci, Lemma 1 of Patterson [42] implies that

$$\dim l = 2g_0 - 1 + [\frac{n_0}{2}], \tag{2.6}$$

where $[x]$ denote the largest integer less than x. Let k denote the number of fixed points of h_0 on the compactification $\overline{X}$ of X, let m be the number of the punctures fixed by h_0. Since h_0 defines a branched covering $\overline{X} \to \overline{X}/<h_0>$, the Riemann-Hurwitz formula (see for example, Farkas-Kra [21]) shows that

$$2g_0 - 2 = (\mathrm{ord}(h_0)) \cdot (2g^* - 2) + B, \tag{2.7}$$

where B is the total branch number. The number of fixed points of a conformal automorphism on $\overline{X}$ is at most $2g_0 + 2$. By definition, there are $n_0 - m$ punctures on X which are not fixed by h_0. Since the order of h_0 is ≥ 2, the number of orbits of these $n_0 - m$ points under h_0 is at most $\frac{1}{2}(n_0 - m)$. Note that any one of these orbits projects to a distinguished point on $\overline{X}/<h>$. We thus obtain

$$n^* \leq k + \frac{1}{2}(n_0 - m). \tag{2.8}$$

First we assume that $g^* \geq 1$. Under this assumption, there are three possibilities.

Case 1. $g_0 = 0$. The left hand side of (2.7) is negative, while the right hand side is positive (since $g^* \geq 1$, by hypothesis). This is a contradiction.

Case 2. $g_0 \geq 2$. Since $\mathrm{ord}(h_0) \geq 2$, and $k \leq B$, from (2.7), we obtain

$$2g_0 - 2 = \mathrm{ord}(h_0) \cdot (2g^* - 2) + B \geq 2(2g^* - 2) + k,$$

or

$$g^* \leq \frac{g_0}{2} + \frac{1}{2} - \frac{k}{4}. \tag{2.9}$$

Since h_0 is not a hyperelliptic involution (otherwise $g^* = 0$), by Corollary 2 to V.1.5 and Proposition III. 7.11 of Farkas-Kra [21], we see that $k \leq 4$ if $\overline{X}$ is hyperelliptic, and $k \leq 2g_0 - 1$ otherwise. But our hypothesis says that $g_0 \geq 2$. We see that in any case,

$$k \leq 2g_0. \tag{2.10}$$

Now from (2.6), (2.8), (2.9), and (2.10), we obtain

$$2g_0 - 1 + [\frac{n_0}{2}] \; = \; \dim l \leq \; \dim T(g^*, n^*) = 3g^* - 3 + n^* \leq$$

$$\leq 3(\frac{g_0}{2} + \frac{1}{2} - \frac{k}{4}) - 3 + k + \frac{1}{2}(n_0 - m) \leq 2g_0 - \frac{3}{2} + \frac{n_0}{2} =$$

$$= \begin{cases} 2g_0 - \frac{3}{2} + [\frac{n_0}{2}], & \text{if } n_0 \text{ is even;} \\ 2g_0 - \frac{3}{2} + [\frac{n_0}{2}] + \frac{1}{2}, & \text{if } n_0 \text{ is odd.} \end{cases} \tag{2.11}$$

So (2.11) is a contradiction if n_0 is even. If n_0 is odd, we claim that either $\mathrm{ord}(h_0)$ is not 2, or $m > 0$. (Otherwise these n_0 points produces $n_0/2$ orbits, contradicting that n_0 is odd.) We see that (2.11) is still a contradiction when n_0 is odd. It follows that (2.11) cannot hold in any case.

Case 3. $g_0 = 1$. In this case, from (2.7) and the hypothesis that $g^* \geq 1$, we see that $g^* = 1$, $B = k = m = 0$, and $n^* \leq \frac{1}{2}n_0$. A similar computation as (2.11) shows that

$$1 + [\frac{n_0}{2}] \leq 3g^* - 3 + n^* \leq \frac{1}{2}n_0.$$

But this is impossible. In summary, we conclude that the case of $g^* \geq 1$ cannot occur.

Next, we consider the case of $g^* = 0$. From (2.8), we have

$$\begin{aligned} 2g_0 - 1 + [\frac{n_0}{2}] &= \dim l \leq -3 + k + \frac{1}{2}(n_0 - m) \\ &\leq -3 + 2g_0 + 2 + \frac{1}{2}(n_0 - m) \\ &= 2g_0 - 1 + \frac{1}{2}(n_0 - m). \end{aligned} \tag{2.12}$$

If n_0 is even, then we have equalities everywhere in (2.12), we must have $m = 0$, and $k = 2g_0 + 2$. This implies that h_0 is a hyperelliptic involution. If n_0 is odd, from (2.12) again, we have $m \leq 1$. We claim that m is not zero. Suppose for the contrary that $m = 0$. Then n_0 cannot be one, and we must have $n_0 \geq 3$. From (2.12) once again, we have $\mathrm{ord}(h_0) = 2$. Observe also that

$$n_0 = m_1 + 2m_2,$$

where m_i is the number of orbits of the punctures with period i. Note also that $m_1 = m$, we conclude that $m = 1$ if n_0 is odd, and zero if n_0 is even. It follows from the definition (given in §2.2) that h_0 is a hyperelliptic involution. This completes the proof. □

As an easy consequence, we obtain

Corollary 2.3.2 *A modular transformation χ_0 of $T(G_0)$ is the identity if its restriction to a subspace with dimension greater than the dimension of the hyperelliptic loci is the identity.*

Proof. χ_0 must be elliptic in the sense of Bers [9]. Let $a = bp$ be the order of χ_0, where p is a prime number. By the same computation as in Proposition 2.3.1, we conclude that $\chi_0^b = id$, which occurs only if $\chi_0 = id$, as claimed. □

Now we are able to prove Theorem 0.1. We need two simple lemmas.

Lemma 2.3.3 *Let Γ' be a torsion free finitely generated Fuchsian group of the first kind whose type is (g', n'). Assume that (g', n') is not $(0, 5)$, $(1, 3)$, $(0, 6)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, or $(3, 0)$. Then the codimension of the hyperelliptic loci of $T(\Gamma')$ is not one, and the codimension of the hyperelliptic loci of $T(\Gamma')$ is zero if and only if (g', n') is $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$, or $(2, 0)$.*

Proof. Suppose that the codimension of the hyperelliptic loci is one. The dimension of the hyperelliptic loci can be computed; it is $2g' - 1 + [\frac{n'}{2}]$. By assumption, we have

$$2g' - 1 + [\frac{n'}{2}] + 1 = 3g' - 3 + n',$$

which says that $(g', n') = (0, 5), (1, 3), (0, 6), (1, 4), (2, 1), (2, 2)$, or $(3, 0)$, contradicting our hypothesis. The second statement is true because there is a non-trivial modular transformation which acts trivially on the Teichmüller space $T(g', n')$ when $(g', n') = (0, 3), (0, 4), (1, 1), (1, 2)$ or $(2, 0)$. □

Lemma 2.3.4 *Let Γ be a finitely generated Fuchsian group of the first kind which contains elliptic elements and has type (g, n), let Γ' be another torsion free group with type (g', n'). Assume that $(g', n') = (0,5)$, $(1,3)$, $(0,6)$, $(1,4)$, $(2,1)$, $(2,2)$ or $(3,0)$, and that there is an isomorphism $\varphi : F(\Gamma) \to T(\Gamma')$. Then the pair of types $((g,n),(g',n'))$ is among the entries of the following table:*

((0,4),(0,5)), ((1,1),(0,5)), ((0,5),(1,3)), ((0,5),(0,6))
((1,2),(1,3)), ((1,2),(0,6)), ((0,6),(1,4)), ((0,6),(2,1))
((0,7),(2,2)), ((0,8),(3,0))

Table D

Proof. The Teichmüller space $T(\Gamma)$ is biholomorphically equivalent to the image of a canonical section which has codimension one in the fiber space $F(\Gamma)$. We thus have

$$\dim T(\Gamma) = \dim F(\Gamma) - 1 = \dim T(\Gamma') - 1.$$

That is,

$$3g - 3 + n = 3g' - 4 + n'.$$

The assertion then follows from the solution of the above equation. □

Remark. Table(D) constitutes a core part of Table (A) in the introduction. Table(A) can be easily obtained by adding relations (1.5) into Table(D).

Proof of Theorem 0.1. Suppose that there is an isomorphism $\varphi : F(\Gamma) \to T(\Gamma')$, and that the pair $((g,n),(g',n'))$ does not belong to the entries of Table (A) in the introduction. In particular, $((g,n),(g',n'))$ does not belong to the entries of Table (D). By Lemma 2.3.4, (g',n') is not of $(0,5),(1,3),(0,6),(1,4),(2,1),(2,2)$, or $(3,0)$. Since Γ has torsion, we can choose an elliptic element $\gamma \in \Gamma$ with prime order. The image of holomorphic section s of $\pi : F(\Gamma) \to T(\Gamma')$, which is determined by the fixed point of γ in U, is equivalent to $T(\Gamma)$. It is obvious that $\varphi \circ s(T(\Gamma))$ has codimension one. On the other hand, Lemma 2.3.3 says that the dimension of a component of the hyperelliptic loci of $T(\Gamma')$ is at least of codimension two. We conclude that

$$\dim \varphi \circ s(T(\Gamma)) > \dim \{\text{hyperelliptic loci of } T(\Gamma')\}.$$

Now $\gamma \in \text{mod}\,\Gamma$ fixes $s(T(\Gamma))$ pointwise, which implies that $\gamma' = \varphi \circ \gamma \circ \varphi^{-1} \in \text{Mod}\,\Gamma'$ fixes $\varphi \circ s(T(\Gamma))$ pointwise as well. It follows from Corollary 2.3.2 that $\gamma' = id$. This leads to a contradiction.

Also, from the proof of Proposition 2.3.1, we can deduce that every elliptic element of Γ must be of order 2^n for some positive integer n if an isomorphism of $F(\Gamma)$ onto $T(\Gamma')$ exists. Now the second part of Theorem 0.1 follows from the fact that any non-trivial modular transformation of $T(\Gamma')$ which acts trivially on another Teichmüller space $T(\Gamma'')$ ($\cong$ a component of the hyperelliptic loci in $T(\Gamma')$) must be of order 2. The details are omitted. □

Remark. If the pair of types $((g,n),(g',n'))$ lies in Table(A), then we shall see in §2.5 that $\varphi(s(T(\Gamma)))$ must be a component of the hyperelliptic loci of $T(\Gamma')$. The argument in this section leads to no contradiction. To extend

Theorem 0.1, more delicate methods must be introduced so as to construct new holomorphic automorphisms of $F(\Gamma)$. Details will appear in §2.5.

2.4 Proof of Theorem 2.1.2

The purpose of this section is to prove Theorem 2.1.2. The proof heavily relies on the properties of the space $T(2,1)$ discussed in §2.2.

Let Γ_1 be a finitely generated Fuchsian group of the first kind whose signature is $(2.0;\_)$. From a discussion in §1.2, Γ_1 is a normal subgroup of a Fuchsian group Γ_0 with Γ_0 of signature $(0,6,2,\ldots,2)$. Note that $\Gamma_1 \lhd \Gamma_0$ is of index 2 and $T(\Gamma_0) \cong T(\Gamma_1)$. Let Ξ denote this isomorphism. The considerations of §5 in Earle-Kra [19] leads to a holomorphic equivalence:

$$\lambda \colon F(\Gamma_0) \to F(\Gamma_1)$$

defined by sending $([\mu], z) \in F(\Gamma_0)$ to $(\Xi([\mu]), z) \in F(\Gamma_1)$.

To see that λ is well defined, we observe that for every $\mu \in M(\Gamma_0)$, there corresponds to a $\nu \in M(\Gamma_1)$ with $[\nu] = \Xi([\mu])$ and vise versa (since Ξ is an isomorphism). This implies that the set $w^{\mu'}(U)$ coincides with $w^{\nu'}(U)$ for $\mu' \sim \mu$ and $\nu' \sim \nu$.

To see that λ is holomorphic, note first that Ξ is a biholomorphic map. Next, when $[\mu'] \in T(\Gamma_0)$ lies in a sufficiently small neighborhood of $[\mu] \in T(\Gamma_0)$, z stays in $w^{\mu'}(U)$. It is trivial that λ is biholomorphic.

Let $\pi^0 \colon F(\Gamma_0) \to T(\Gamma_0) \cong T(0,6)$ be the natural projection. Since Γ_0 is of signature $(0,6;2,\ldots,2)$, all canonical sections s^0 of π^0 are determined by elliptic elements of Γ_0. Let $\mathcal{S}^0$ denote the set of all images $s^0(T(\Gamma_0))$. We first prove:

Lemma 2.4.1 *Let $\theta \in mod\,\Gamma_1$ and $\theta_0 = \lambda^{-1} \circ \theta \circ \lambda$. Then $\theta_0 \in mod\,\Gamma_0$. Furthermore, θ_0 keeps the set $\mathcal{S}^0$ invariant; that is, for any canonical section s^0 of π^0, $\theta_0(s^0(T(\Gamma_0)))$ is the image of a canonical section of π^0.*

Proof. Let $\theta \in \mathrm{mod}\,\Gamma_1$ be induced by a self-map f of U, and f_1 the projection of f to the surface U/Γ_1. Note that U/Γ_1 is a compact Riemann surface of genus 2 which is, of course, hyperelliptic.

A theorem of Lickorish [30] tells us that f_1 is isotopic to a self-map f_1' which is a product of Dehn twists about the curves belonging to the set of Lickorish's generators (see for example, Birman [13] for the definition and basic properties). But the set of Lickorish's generators are invariant under the hyperelliptic involution J. It follows that any Dehn twist about a Lickorish's generator is isotopic to a twist about that generator which commutes with J. (See the proof of Lemma 2.2.3.) This implies that f_1' is isotopic to a map (still called f_1') which commutes with J as well. It follows that f_1' projects to a self-map $f_0 \colon U/\Gamma_0 \to U/\Gamma_0$ in the sense of orbifolds.

Now lift the self-map f_1' of U/Γ_1 to the map $f' : U \to U$. Since f_1 is isotopic to f_1' on U/Γ_1, we can choose a lift so that f' is isotopic to f. On the other hand, f' is also a lift of f_0; that is, $f' \in N(\Gamma_0)$, the normalizer of Γ_0 in $Q(\Gamma_0)$. Hence, by definition of λ, the geometric isomorphism of Γ_0 induced by f' is exactly θ_0. It follows that $\theta_0 \in \mathrm{mod}\,\Gamma_0$. Since f_0 is a self-map in the sense of orbifolds (all branch points here are of order 2), it sends a branched point to a branched point. This implies that θ_0 sends an image of a canonical section to an image of a canonical section. The lemma is proved. □

Now let G be a finitely generated Fuchsian group of the first kind which is of type $(0,6)$ and contains an elliptic element ε of order 2. Let $s: T(G) \to F(G)$ be a canonical section determined by the fixed point of ε. Recall that the definition of hyperelliptic Riemann surfaces of finite analytic type is given in §2.2. We have

Lemma 2.4.2 *Suppose that there is an isomorphism $\varrho: F(G) \to T(G')$ for some torsion free Fuchsian group G'. Then $\varrho \circ s(T(G))$ is a component of the hyperelliptic loci in $T(G)$; that is, any marked Riemann surface $X' \in \varrho \circ s(T(G))$ admits a hyperelliptic involution determined by $\varepsilon' = \varrho \circ \varepsilon \circ \varrho^{-1}$.*

Proof. From Theorem 1.2.3, $\varepsilon' = \varrho \circ \varepsilon \circ \varrho^{-1}$ is an elliptic modular transformation of order 2. Next, Theorem 0.1 implies that G' is of type $(2,1)$ or $(1,4)$.

Case 1. G' is of type $(g_0', n_0') = (2,1)$. In this case, by Lemma 1 of Patterson [42], we conclude that

$$\dim \varrho \circ s(T(G)) \leq 2g_0' - 1 + [\frac{n_0'}{2}] = 3, \tag{2.13}$$

Since $s(T(G)) \subset F(G)$ is equivalent to $T(G)$, from (2.13), we obtain

$$3 = \dim T(G) = \dim \varrho \circ s(T(G)) \leq 3.$$

We thus obtain equality in the above inequality. In particular, we have

$$\dim \varrho \circ s(T(G)) = 2g_0' - 1 + [\frac{n_0'}{2}] = 3. \tag{2.14}$$

From (2.14) and the second part of Lemma 1 of Patterson [42], we conclude that $\varepsilon' = \varrho \circ \varepsilon \circ \varrho^{-1} \in \text{Mod}\,(2,1)$ is induced by a hyperelliptic involution on a hyperelliptic Riemann surface of type (2,1), which in turn implies that $\varrho \circ s(T(G))$ is a component of the hyperelliptic loci.

Case 2. G' is of type $(g_0', n_0') = (1,4)$. We use the same argument as above. Note that (2.13) and (2.14) also work in this case, we use Patterson's result once again. □

Remark. Lemma 2.4.2 is still valid if the pair of types $((g_0, n_0), (g_0', n_0'))$ of G and G' lies in Table (D) in §2.3. The proof is just a computation, and is not repeated here.

Let Γ_0 and Γ_1 be as above. We choose a torsion free group Γ' of type (2,1), and let $\psi\colon F(\Gamma_1) \to T(\Gamma')$ be the Bers isomorphism.

Now we proceed to prove Theorem 2.1.2. Let Γ be a finitely generated Fuchsian group of the first kind whose signature is $(0, 6; 2, \ldots, 2, \infty)$.

Suppose that there is an isomorphism $\varphi\colon F(\Gamma) \to T(\Gamma')$. It turns out that $\psi' = \psi \circ \lambda\colon F(\Gamma_0) \to T(\Gamma')$ is an isomorphism, we thus obtain an equivalence $\omega = \psi'^{-1} \circ \varphi\colon F(\Gamma) \to F(\Gamma_0)$. For convenience, we exhibit these isomorphisms in the following diagram:

$$\begin{array}{ccc} F(\Gamma_0) & \xrightarrow{\Lambda} & F(\Gamma_1) \\ \omega\uparrow & \searrow^{\psi'} & \downarrow\psi \\ F(\Gamma) & \xrightarrow[\varphi]{} & T(\Gamma') \end{array} \qquad (2.15)$$

Note that the diagram (2.15) is commutative. Let $\mathcal{S}$ denote the set of all images $s(T(\Gamma))$ under canonical sections s of $\pi : F(\Gamma) \to T(\Gamma) \cong T(0,6)$. We have

Lemma 2.4.3 *(1) ω carries $\mathcal{S}$ into $\mathcal{S}^0$.*

(2) Let e denote an elliptic element of Γ (e can be thought of as an element of mod Γ). Then $\omega \circ e \circ \omega^{-1} \in$ mod Γ_0. Furthermore, $\omega \circ e \circ \omega^{-1}$ is defined by an elliptic element of Γ_0.

Caution. The isomorphism ω need not be fiber-preserving.

Proof. Let $e_0 \in \Gamma_0$ be an elliptic elements of order 2, let s and s^0 be the canonical sections of $\pi : F(\Gamma) \to T(\Gamma) \cong T(0,6)$ and $\pi^0 : F(\Gamma_0) \to T(\Gamma_0) \cong T(0,6)$ corresponding to the fixed points e and e_0, respectively. We define $l' = \varphi(s(T(\Gamma)))$ and $l'_0 = \psi'(s^0(T(\Gamma_0)))$. Lemma 2.4.2 asserts that both l' and l'_0 are the components of the hyperelliptic loci in $T(\Gamma')$ and that both $e' = \varphi \circ e \circ \varphi^{-1}$ and $e'_0 = \psi' \circ e_0 \circ \psi'^{-1}$ are the corresponding hyperelliptic involutions. By Proposition 2.2.7, we see that there is a modular transformation $\chi' \in$ Mod Γ' such that $\chi'(l') = l'_0$, and $\chi' \circ e' \circ \chi'^{-1} = e'_0$. From a theorem of Bers (Theorem 10, Bers [8]), we know that $\theta = \psi^{-1} \circ \chi' \circ \psi$ is a modular transformation of

$F(\Gamma_1)$. Lemma 2.4.1 then says that $\theta_0 = \lambda^{-1} \circ \theta \circ \lambda$ keeps the set of the images of canonical sections invariant. We claim that $\omega(s(T(\Gamma))) = \theta_0^{-1}(s^0(T(\Gamma_0)))$.

Indeed, from the diagram (2.15) we can obtain

$$\begin{aligned}
\omega(s(T(\Gamma))) &= \psi'^{-1} \circ \varphi(s(T(\Gamma))) = \psi'^{-1}(l') \\
&= \psi'^{-1} \circ \chi'^{-1}(l'_0) = \psi'^{-1} \circ \chi'^{-1} \circ \psi'(s^0(T(\Gamma_0))) \\
&= \lambda^{-1} \circ \psi^{-1} \circ \psi \circ \theta^{-1} \circ \psi^{-1} \circ \psi \circ \lambda(s^0(T(\Gamma_0))) \\
&= \lambda^{-1} \circ \theta^{-1} \circ \lambda(s^0(T(\Gamma_0))) = \theta_0^{-1}(s^0(T(\Gamma_0))).
\end{aligned}$$

To prove the second statement of this lemma, we note that $e \in \operatorname{mod} \Gamma$ fixes $s(T(\Gamma))$ pointwise. Hence, $\omega \circ e \circ \omega^{-1}$ fixes $\theta_0^{-1}(s^0(T(\Gamma_0)))$ pointwise as well. By Lemma 2.4.2, we see that

$$\psi'(\theta_0^{-1}(s^0(T(\Gamma_0))) = l''_0$$

is a component of the hyperelliptic loci in $T(\Gamma')$. Now $\psi' \circ (\omega \circ e \circ \omega^{-1}) \circ \psi'^{-1} \in \operatorname{Mod} \Gamma'$ has the property that its restriction to l''_0 is the identity. By Proposition 2.3.1 and Corollary 2 to Proposition III.7.9 of Farkas-Kra [21], we conclude that $\psi' \circ (\omega \circ e \circ \omega^{-1}) \circ \psi'^{-1}$ is either the identity or equal to the hyperelliptic involution e''_0 corresponding to l''_0. But evidently, $\psi' \circ (\omega \circ e \circ \omega^{-1}) \circ \psi'^{-1}$ is not the identity, we see that,

$$\psi' \circ (\omega \circ e \circ \omega^{-1}) \circ \psi'^{-1} = e''_0. \tag{2.16}$$

On the other hand, if we denote by e^0 the elliptic element of Γ_0 corresponding to $\theta_0^{-1}(s^0(T(\Gamma_0)))$, by Proposition 2.3.1, we see that $\psi' \circ e^0 \circ \psi'^{-1} = e''_0$. It follows from (2.16) that $e^0 = \omega \circ e \circ \omega^{-1}$. This completes the proof of the lemma. □

To proceed, we need to recall a basic result of Kra [28]. Let $\tau \in T(\Gamma_1)$. By definition, $\pi_1^{-1}(\tau)$ is a quasidisc in $\hat{\mathbb{C}}$ bounded by the quasicircle $w^\tau(U)$ passing through 0, 1, and ∞. The domain $\pi_1^{-1}(\tau) = w^\tau(U)$ inherits two Γ_1-invariant metrics; those are, the Teichmüller metric $< \, , \, >$ and the Poincaré metric ρ which are defined in §1.1.

Lemma 2.4.4 (Proposition 4(a), Kra [28]) *There exists a differentiable, strictly increasing, real-valued function κ on $[0,\infty)$ with $\kappa(0) = 0$ so that for any x, $y \in \pi_1^{-1}(\tau)$, $x \neq y$, we have*

$$\kappa(\rho(x,y)) < \; < x, y > \; \leq \rho(x,y). \tag{2.17}$$

Remark. As a matter of fact, in his paper [28], Kra proves a much stronger result than Lemma 2.4.4. But we don't need that result here.

Proof of Theorem 2.1.2. Let $U = \{z; \operatorname{Im} z > 0\} \subset F(\Gamma_1)$ be the central fiber (by a central fiber we mean the fiber $\pi_1^{-1}([0])$). Thus, U is the central fiber of $F(\Gamma_0)$ as well. Let

$$\epsilon = \inf_{\gamma_1 \in \Gamma_1 - \{id\}} \inf_{x \in U} \;\; \rho(x, \gamma_1(x)). \tag{2.18}$$

Since Γ_1 is purely hyperbolic (type (2,0)), $\epsilon > 0$. Note also that the function κ (defined in Lemma 2.4.4) is strictly increasing, there is a small $\delta > 0$ such that

$$\rho(x, \gamma_1(x)) < \frac{\epsilon}{2}$$

as long as $\kappa(\rho(x, \gamma_1(x))) < \delta$ for $\gamma_1 \in \Gamma_1$. By looking at the diagram (2.15), we know that the restriction of φ to U is a holomorphic map into $T(\Gamma')$ (here U is also thought of as the central fiber of $F(\Gamma)$). Hence

$$d(\varphi(x), \varphi(y)) \;\leq\; \rho(x, y), \quad \text{for all } x, y \in U, \tag{2.19}$$

where d is the Kobayashi metric on $T(\Gamma')$. By Royden's theorem [43], the Kobayashi metric is the same as the Teichmüller metric. Therefore,

$$< \varphi(x), \varphi(y) > \;\leq\; \rho(x, y). \tag{2.20}$$

Unfortunately, there is no guarantee that $\omega(U)$ is a fiber in $F(\Gamma_0)$. To get rid of this difficulty, let $e_1, \ldots, e_5$, and e_∞ be a set of generators of Γ, where e_i, $i = 1, \ldots, 5$, are elliptic Möbius transformations of order 2, and e_∞ is a parabolic Möbius transformation. These generators may be chosen so as to satisfy the following relation:

$$e_5 \circ \cdots \circ e_1 = e_\infty. \tag{2.21}$$

Choose a point $x \in U$ so that

$$\rho(x, e_\infty(x)) < \delta. \tag{2.22}$$

This is possible because e_∞ is parabolic. Observe that $\omega(x) \in F(\Gamma_0)$. We denote $\pi_0^{-1}([\mu]) \in F(\Gamma_0)$ be the fiber to which the point $\omega(x)$ belongs. Then we construct a Bers' allowable mapping of $F(\Gamma_0)$ onto another isomorphic Bers fiber space, this Bers' allowable mapping can be defined by carrying the fiber $\pi_0^{-1}([\mu])$ to the central fiber of the new Bers fiber space. In this regard, we may assume, without loss of generality, that $\omega(x) \in \omega(U) \cap U \subset F(\Gamma_0)$, and hence also that $\lambda \circ \omega(x) \in U \subset F(\Gamma_1)$. Let $\tau' = \varphi(x) \in T(\Gamma')$, and let $\chi'_\infty = \varphi \circ e_\infty \circ \varphi^{-1}$. By Theorem 1.2.3, $\chi'_\infty \in \text{Mod}\,\Gamma'$. Moreover, (2.20) and (2.22) imply that

$$\begin{aligned} < \tau', \chi'_\infty(\tau') > &= < \varphi(x), \varphi \circ e_\infty \circ \varphi^{-1}(\tau') > \\ &= < \varphi(x), \varphi \circ (e_\infty(x)) > \\ &\leq \rho(x, e_\infty(x)) < \delta. \end{aligned} \tag{2.23}$$

As a holomorphic automorphism, $e_\infty \in \text{mod}\,\Gamma$ has no fixed point in $F(\Gamma)$, thus χ'_∞ has no fixed point in $T(\Gamma')$ either. It turns out that χ'_∞ is a parabolic modular transformation of $T(\Gamma')$.

From Theorem 6 of Bers [9], χ'_∞ is induced by a reducible self-map f' of a Riemann surface X' of type (2,1) (the puncture is denoted by x'). Let $X' = \varphi(x)$, and let $c' = \{c'_1, \ldots, c'_r\}$ be the corresponding admissible system of curves on X' which is reduced by f'.

Since χ'_∞ is parabolic, the restriction of f' to all parts of $X' - N(c')$ are either trivial or periodic (see Kra [28]), where $N(c')$ is an arbitrary small neighborhood of $c' = \{c'_1, \ldots, c'_r\}$. We may assume, by taking a power of f', that all restrictions of f' to $X' - N(c')$ are trivial. Then f' must be isotopic to some product of Dehn twists about $c'_1, \ldots, c'_n$ (see Abikoff [2]).

We need the following lemma.

Lemma 2.4.5 *Suppose that χ'_∞ is defined as above. Then $r = 2$, and χ'_∞ is actually induced by a power of the composition $h_{c'_2}^{-1} \circ h_{c'_1}$, where c'_1 and c'_2 bounds a cylinder which contains the puncture x'.*

Remark. A self-map of the form $h_{c'_2}^{-1} \circ h_{c'_1}$ is called a spin map in the literature. Spin maps have many interesting properties and are extremely useful in studying Bers fiber spaces. For details, see [13], [28] and the literature quoted there.

Proof of Lemma 2.4.5. Recall that χ'_∞ is induced by f', and f' fixes the puncture x'. Theorem 10 of Bers [8] asserts that $\psi^{-1} \circ \chi'_\infty \circ \psi \in \operatorname{mod} \Gamma_1$. From the proof of Lemma 2.4.1, $\lambda^{-1} \circ \psi^{-1} \circ \chi'_\infty \circ \psi \circ \lambda = \psi'^{-1} \circ \chi'_\infty \circ \psi'$ is an element of $\operatorname{mod} \Gamma_0$. In particular, $\psi'^{-1} \circ \chi'_\infty \circ \psi'$ is a fiber-preserving automorphism of $F(\Gamma_0)$. Consider the following commutative diagram

$$\begin{array}{ccc} T(\Gamma') & \xrightarrow{\chi'_\infty} & T(\Gamma') \\ {\scriptstyle \pi'_1}\downarrow & & \downarrow{\scriptstyle \pi'_1} \\ T(\Gamma') & \xrightarrow[\chi']{} & T(\Gamma') \end{array} \tag{2.24}$$

where $\pi'_1 = \pi_1 \circ \psi^{-1}$, and as before, $\psi \colon F(\Gamma_1) \to T(\Gamma')$ is the Bers isomorphism. The map χ' is defined by the formula $\pi'_1 \circ \chi'_\infty = \chi' \circ \pi'_1$. Since π'_1 is defined by forgetting the puncture x', χ' is defined by f' by filling in the puncture x'. We denote by $\bar{f}'$ the self-map on $\overline{X'}$ inducing χ'. Note that the map f' is a power of a spin if and only if $\bar{f}'$ is isotopic to the identity (for a proof of this fact, see Birman [13]). Suppose that χ' is not the identity. We see from (2.24) that

χ'_∞ sends a fiber to a different fiber. This implies that $\psi'^{-1} \circ \chi'_\infty \circ \psi'$ sends a fiber of $F(\Gamma_0)$ to another different fiber of $F(\Gamma_0)$. In particular, we see that $\psi'^{-1} \circ \chi'_\infty \circ \psi' \circ (\psi'^{-1}(\tau'))$ and $\psi'^{-1}(\tau')$ lie in different fibers.

On the other hand, the group Γ is generated by $e_1, \dots, e_5$. By Lemma 2.4.3, the ω-image of Γ is a subgroup of Γ_0. It follows that

$$\psi'^{-1}(\tau') = \psi'^{-1} \circ \varphi(x) = \omega(x). \tag{2.25}$$

We also have

$$\begin{aligned} \psi'^{-1} \circ \chi'_\infty \circ \psi' &= \psi'^{-1} \circ \varphi \circ e_\infty \circ \varphi^{-1} \circ \psi' = \omega \circ e_\infty \circ \omega^{-1} \\ &= \omega \circ (e_5 \circ \cdots \circ e_1) \circ \omega^{-1} = e_5^0 \circ \cdots \circ e_1^0, \end{aligned} \tag{2.26}$$

where e_i^0, $i = 1, \dots, 5$, are ω-images of e_i in Γ_0. We conclude from (2.25) and (2.26) that

$$\psi'^{-1} \circ \chi'_\infty \circ \psi' \circ (\psi'^{-1}(\tau')) = e_5^0 \circ \cdots \circ e_1^0 \circ (\omega(x)). \tag{2.27}$$

Since $\Gamma_0 \lhd \bmod \Gamma_0$ keeps all fibers of $F(\Gamma_0)$ invariant, and since $x \in U$, by (2.27), we see that $\psi'^{-1} \circ \chi'_\infty \circ \psi' \circ (\psi'^{-1}(\tau'))$ and $\psi'^{-1}(\tau')$ lie in the same fiber U. This is a contradiction. The lemma is proved. $\square$

Continuing with the Proof of Theorem 2.1.2. Observe that the spin map described in Lemma 2.4.5 determines a closed curve c' in $\overline{X'}$ passing through x' (c' is freely homotopic to both c'_1 and c'_2 in $\overline{X'}$, as shown in Figure 8). Thus the homotopy class of c' in $\overline{X'}$ determines an element of the fundamental group

$\pi_1(\overline{X'}, x')$, and hence corresponds to an element $\gamma_1 \in \Gamma_1$ which is hyperbolic because Γ_1 is purely hyperbolic.

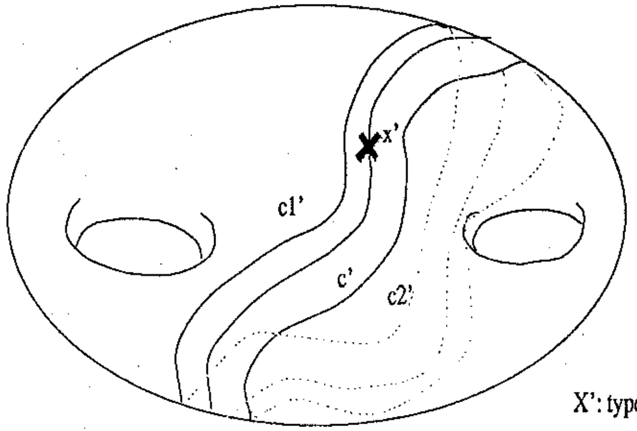

Figure 8.

It is now easy to see that $\psi^{-1} \circ \chi'_\infty \circ \psi = \gamma_1 \in \Gamma_1$, and the pair of points $(\tau', \chi'_\infty(\tau'))$ gets mapped under ψ^{-1} to the pair

$$(\psi^{-1}(\tau'), \psi^{-1} \circ \chi'_\infty(\tau')) = (\psi^{-1}(\tau'), \gamma_1 \circ \psi^{-1}(\tau')).$$

By (2.23), we have $< \tau', \chi'_\infty(\tau') >< \delta$. Note that $\psi : F(\Gamma_1) \to T(\Gamma')$ is the Bers isomorphism. From Lemma 2.4.4, we conclude that

$$\kappa(\rho(\psi^{-1}(\tau'), \psi^{-1} \circ \chi'_\infty(\tau'))) < \delta.$$

Hence,

$$\rho(\psi^{-1}(\tau'), \psi^{-1} \circ \chi'_\infty(\tau')) < \frac{\epsilon}{2};$$

that is,

$$\rho(\psi^{-1}(\tau'), \gamma_1 \circ (\psi^{-1}(\tau'))) < \frac{\epsilon}{2}.$$

But as we have seen, $\psi^{-1}(\tau')$ and $\gamma_1 \circ (\psi^{-1}(\tau'))$ lie in the central fiber U of $F(\Gamma)$. This contradicts to the definition of ϵ (see (2.18)), completing the proof of Theorem 2.1.2. $\square$

2.5 Holomorphic extensions of automorphisms of sections

In this section, we establish a proposition which asserts the existence of holomorphic extensions of some automorphisms defined on the images of certain canonical sections. The result will play an important role in proving Theorem 2.1.1. We also study in this section a uniqueness problem on extensions of automorphisms acting on certain canonical sections.

Let Γ be a finitely generated Fuchsian group of the first kind which acts on U and contains elliptic elements, and let f be a self-map of U/Γ in the sense of orbifolds. We emphasize that f maps regular points to regular points, punctures to punctures, and branch points to branch points with the same ramification number. By definition, f induces a modular transformation χ_f on $T(\Gamma)$. Let s denote a canonical section of $\pi\colon F(\Gamma) \to T(\Gamma)$. Then s induces a map

$$s_*\colon \text{Mod}\,\Gamma \to \text{Aut}\ s(T(\Gamma))$$

defined by the formula:

$$s_*(\chi_f)(x) = s \circ \chi_f \circ \pi(x) \quad \text{for } x \in s(T(\Gamma)).$$

It is easy to see that $s_*(\chi_f) : s(T(\Gamma)) \to s(T(\Gamma))$ is a holomorphic map. Suppose that $s_*(\chi_f)(x) = s_*(\chi_f)(y)$ with $x, y \in s(T(\Gamma))$. Then by the above formula, we have $s \circ \chi_f \circ \pi(x) = s \circ \chi_f \circ \pi(y)$. Since $s \circ \chi_f$ is injective, we must have $\pi(x) = \pi(y)$, which in turn implies that $x = y$. Hence $s_*(\chi_f)$ is injective. Similarly, one can show that $s_*(\chi_f)$ is surjective. We conclude that $s_*(\chi_f)$ is a holomorphic automorphism of $s(T(\Gamma))$.

Unfortunately, it is not true that for arbitrary s and arbitrary f, $s_*(\chi_f)$ can be extended holomorphically to the whole Bers fiber space. Our aim here is to choose a specific self-map f and a specific canonical section s so that the automorphism $s_*(\chi_f)$ of $s(T(\Gamma))$ is the restriction of a global holomorphic automorphism. More precisely, the self-maps we choose must satisfy the following properties:

(a) They are self-maps in the sense of orbifolds;

(b) Their fixed points must contain at least one branch point of U/Γ.

Remark. It is impossible to construct such maps in the case when Γ is torsion free. So our methods doesn't work in torsion free case. However, when Γ is torsion free, the Bers isomorphism theorem tells us that $F(\Gamma)$ may be identified with the Teichmüller space $T(g, n+1)$.

Let f be a self-map of U/Γ which fixes the branch point $\hat{z}_0$ determined by an elliptic element e of Γ, and let s be the canonical section of $\pi\colon F(\Gamma) \to T(\Gamma)$

which is determined by the fixed point of e in U. Under these circumstances, we have

Proposition 2.5.1 *(1) $s_*(\chi_f) \in Aut\, s(T(\Gamma))$ can be extended to an element χ of mod Γ;*

(2) χ commutes with e if e is also viewed as an element of mod Γ.

Remark. The motion $s_*(\chi_f)$ described in this proposition has at least two extensions which are elements of mod Γ. If χ is one of them, then $e \circ \chi (= \chi \circ e)$ is the other. The following proof provides these two extensions by means of a concrete elementary method.

Proof of Proposition 2.5.1. Let f be described as above. Lift f to a map $\hat{f}'$ on U so that the diagram below is commutative:

$$\begin{array}{ccc} U & \xrightarrow{\hat{f}'} & U \\ p\downarrow & & \downarrow p \\ U/\Gamma & \xrightarrow[f]{} & U/\Gamma \end{array}$$

By definition, we see that $\hat{f}' \in N(\Gamma)$ and $p \circ \hat{f}' = f \circ p$, where p is the natural projection of U onto U/Γ. Since for any $\gamma \in \Gamma$, we have

$$p \circ \gamma \circ \hat{f}' = p \circ \hat{f}' = f \circ p.$$

Then $\gamma \circ \hat{f}'$ is also a lift of f. We also see that $\gamma \circ \hat{f}' \circ e \circ (\gamma \circ \hat{f}')^{-1}$ is an order 2 elliptic element of Γ. By our hypothesis, we have

$$f \circ (p(z_0)) = p(z_0),$$

where z_0 is the fixed point of $e \in \Gamma$. This implies that there is $\gamma_0 \in \Gamma$, such that $\hat{f}'(z_0) = \gamma_0(z_0)$. Let $\hat{f} = \gamma_0^{-1} \circ \hat{f}'$. We have

$$\hat{f} \circ e \circ \hat{f}^{-1} = \gamma_0^{-1} \circ \hat{f}' \circ e \circ \hat{f}'^{-1} \circ \gamma_0 = e. \tag{2.28}$$

The equivalence class $[\hat{f}]$ of $\hat{f}$ (that is, all $\hat{f}' \in Q$ which lie in the normalizer of Γ with the property that $\hat{f}|_{\mathbb{R}} = \hat{f}'|_{\mathbb{R}}$) is an element of mod Γ. Let $\chi = [\hat{f}]$. We claim that χ is an extension of $s_*(\chi_f) : s(T(\Gamma)) \to s(T(\Gamma))$. To see this, first we note that

$$\chi_f([\mu]) = [\text{Beltrami coefficient of } (w_\mu \circ \hat{f}^{-1})],$$

for $[\mu] \in T(\Gamma)$. We see that the diagram

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\chi} & F(\Gamma) \\ \pi\downarrow & & \downarrow\pi \\ T(\Gamma) & \xrightarrow[\chi_f]{} & T(\Gamma) \end{array} \tag{2.29}$$

commutes (see Kra [28], or §1.3). Let $[\nu] = \chi_f([\mu])$. The diagram (2.29) shows that χ maps the fiber $w^\mu(U)$ over $[\mu]$ to the fiber $w^\nu(U)$ over $[\nu]$. In particular, χ maps the fiber that the point $s([\mu]) \in s(T(\Gamma))$ lies in to the fiber that the point $s([\nu]) \in s(T(\Gamma))$ lies in. But we have

$$\begin{aligned} s([\nu]) &= s \circ \chi_f([\mu]) = s \circ \chi_f \circ \pi([\mu], w^\mu(z_0)) \\ &= s_*(\chi_f)([\mu], w^\mu(z_0)) \in s(T(\Gamma)). \end{aligned} \tag{2.30}$$

On the other hand, to prove that $\chi \in$ mod Γ is an extension of $s_*(\chi_f)$ (that is, $\chi |_{s(T(\Gamma))} = s_*(\chi_f)$), we only need to show

$$\chi(s([\mu])) = s_*(\chi_f)([\mu], w^\mu(z_0)) \in s(T(\Gamma)).$$

An interesting question arises at this point as to whether or not there are any other holomorphic extensions of $s_*(\chi_f)$ (not necessarily fiber-preserving). The following proposition answers this question in some special situations. In general case, the problem remains open.

As before, let s denote the canonical section of $\pi\colon F(\Gamma) \to T(\Gamma)$, let $s_*(\chi_f)$ be the automorphism of $s(T(\Gamma))$ determined by s and f. Then Proposition 2.5.1 implies that $s_*(\chi_f)$ can be extended holomorphically to an automorphism χ of $F(\Gamma)$.

Proposition 2.5.2 *Let Γ be of signature $(0,4;2,2,\infty,\infty)$, $(0,4;2,2,2,\infty)$, $(0,5;2,2,2,2,\infty)$, or $(0,6;2,2,2,2,2,2)$. Then the only holomorphic extensions of $s_*(\chi_f)$ in Aut $F(\Gamma)$ are χ and $\chi \circ e$ $(= e \circ \chi)$.*

To prove this result, first we need to refine Proposition 2.3.1 in some special cases. Let Γ' be a finitely generated Fuchsian group of the first kind. Assume that l' $(\neq \emptyset)$ is a component of the hyperelliptic loci of $T(\Gamma')$.

Lemma 2.5.3 *Let Γ' be of type $(2,1)$, $(1,3)$ or $(0,n')$, for $n' \geq 5$. Assume that $\chi' \in$ Mod Γ' is a modular transformation whose restriction to l' is the identity. Then χ' must be of prime order, and hence χ' is either the identity, or equal to a hyperelliptic involution.*

Proof. Suppose for the contrary that $n = mp$ is the order of χ', where $m \geq 2$, and p is a prime. Then $\chi'' = \chi'^m$ is of order p. By using Proposition 2.3.1, we deduce that $p = 2$. Therefore, n must be of form 2^r for $r \geq 1$ an integer.

Observe that

$$l' \subset T(\Gamma')^{\chi'} \subset \cdots \subset T(\Gamma')^{\chi'^{2^{r-1}}}, \tag{2.33}$$

where $\chi'^{2^{r-1}}$ is induced by a hyperelliptic involution. We obtain

$$\dim T(\Gamma')^{\chi'^{2^{r-1}}} = \dim l'.$$

It follows from (2.33) that

$$\dim l' = \dim T(\Gamma')^{\chi'} = \cdots = \dim T(\Gamma')^{\chi'^{2^{r-1}}}. \tag{2.34}$$

In the proof given below, we denote by X' a Riemann surface of type $(2,1)$, $(1,3)$, or $(0,n')$ for $n' \geq 5$, by h' the conformal automorphism of $\overline{X'}$ which induces χ' (its order is 2^r by the above argument). Let $\mathrm{Fix}(h')$ be the set of the fixed points of h' on $\overline{X'}$, k' the number of fixed points of h' on $\overline{X'}$, and g'' the genus of $\overline{X'}/<h'>$.

Case 1. Γ' is of type $(2,1)$. It is obvious that

$$\mathrm{Fix}(h') \subset \mathrm{Fix}(h'^2) \subset \mathrm{Fix}(h'^4) \subset \cdots.$$

Since $h'^{2^{r-1}}$ is a hyperelliptic involution, it fixes all Weierstrass points $x'_1, \ldots, x'_6$ of $\overline{X'}$. Let x'_6 be the puncture. Observe that h' fixes x'_6 and at least one another Weierstrass point, say x'_1. Thus, h' determines a permutation of the remaining 4 Weierstrass points $x'_2, \ldots, x'_5$. If $\{x'_2, \ldots, x'_5\}$ is divided into two orbits under the iteration of h', then h'^2 is hyperelliptic and

$$\mathrm{Fix}(h'^3) \subset \mathrm{Fix}(h'^6) = \mathrm{Fix}(h'^2).$$

It follows that all fixed points of h'^j are the Weierstrass points, which implies that the surface $X'/ < h' >$ has 4 distinguished points. As an immediate consequence, $\dim T(\Gamma')^{\chi'} = 3g''+1$, contradicting that $\dim T(\Gamma')^{\chi'} = \dim l' = 2g'' - 1 + [n'/2] = 3$.

If $\{x'_2, \ldots, x'_5\}$ is a cycle under the iteration of h', then by the same argument as above, the surface $X'/<h'^2>$ has 4 distinguished points. This implies that $\dim T(\Gamma')^{\chi'^2} = 3g''' + 1$, contradicting that $\dim T(\Gamma')^{\chi'^2} = \dim T(\Gamma')^{\chi'} = 3$ (where g''' is the genus of $\overline{X'}/<h'^2>$). See (2.34).

If h' fixes all 6 Weierstrass points, then h' is a hyperelliptic involution.

Case 2. Γ' is of type $(1, 3)$. In this case, $h'^{2^{r-1}}$ is a hyperelliptic involution. By definition, it fixes only one puncture and interchanges other two punctures. Since the fixed points of h' is contained in the set of fixed points of $h'^{2^{r-1}}$, h' must fix one puncture and interchanges the other two punctures as well. But $h'^{2^{r-1}}$ is of even order (unless $r = 1$), it must fixes all three punctures. This is a contradiction. We conclude that $r = 1$ and h' is a hyperelliptic involution.

Case 3. Γ' is of type $(0, n')$, $n' \geq 5$. In this case, the number of the fixed points of all conformal automorphisms (Möbius transformations) h'^j, $j = 1, \ldots, 2m-1$, is two. Moreover, the fixed points of h'^j coincide with those of h'^i for all $i, j = 1, \ldots, 2m-1$. Note that $n' \geq 5$. By a simple calculation, we obtain

$$\dim T(0, n')^{\chi''} = \dim T(X'/<h'^m>) =$$

$$= \begin{cases} -3+(n'-2)/2+2 & \text{if two fixed points of } h' \text{ are punctures;} \\ -3+(n'-1)/2+2 & \text{if one fixed point of } h' \text{ is a puncture;} \\ -3+n'/2+2 & \text{if no fixed points of } h' \text{ are punctures.} \end{cases}$$

We thus have

$$\dim T(0,n')^{\chi''} \geq -3 + \frac{n'-2}{2} + 2 = \frac{n'-2}{2} - 1. \tag{2.35}$$

On the other hand, by the same argument as above, we see that

$$\dim T(0,n')^{\chi'} \leq -3 + \frac{n'}{2m} + 2 = \frac{n'}{2m} - 1. \tag{2.36}$$

Since $m \geq 2$, and $n' \geq 5$, a simple computation shows that

$$\frac{n'}{2m} < \frac{n'-2}{2}.$$

Together with (2.35) and (2.36), we thus have

$$\dim T(0,n')^{\chi''} > \dim T(0,n')^{\chi'},$$

contradicting (2.34). □

Lemma 2.5.4 *Under the same condition of Lemma 2.5.3, there is a unique non-trivial* $\chi' \in Mod\,\Gamma'$ *whose restriction to* l' *is the identity.*

Proof. We first consider the case when Γ' is of type $(2,1)$. Let $X' \in l'$ be a hyperelliptic Riemann surface of type (2,1). By Lemma 2.5.3, χ' is induced by a hyperelliptic involution on X'. By Corollary 2 to Proposition III 7.9 of Farkas-Kra [21], there is only one hyperelliptic involution on $\overline{X'}$. It follows that there is only one hyperelliptic involution on X'. The assertion then follows.

Next, we consider the case when Γ' is of type $(1,3)$. Again, let $X' \in l'$ be a (marked) hyperelliptic Riemann surface of type (1,3) (see §2.2 for the definition), and let x_1', x_2' and x_3' denote the three punctures on X'. Since every hyperelliptic involution on X' must fix one and only one puncture, if there are two hyperelliptic involutions J' and J'' on X' which fix the same puncture, then $J' \circ J''^{-1}$ is either a hyperelliptic involution or the identity by Lemma 2.5.3. Since $J' \circ J''^{-1}$ fixes all three punctures, we see that $J' \circ J''^{-1}$ is the identity; that is $J' = J''$. Now we assume that there are two distinct hyperelliptic involutions J_1' and J_2', and that J_i' fixes x_i', $i = 1, 2$. By a simple calculation, $J_1' \circ J_2'$ permutes all three punctures. On the other hand, by the same argument as above, $J_1' \circ J_2'$ is either a hyperelliptic involution or the identity, it must fix at least one puncture. This is a contradiction.

Finally, if Γ' is of type $(0, n')$, $n' \geq 5$, then we choose $X' \in l'$ and assume that there are two distinct hyperelliptic involutions J_1' and J_2' on X'. Let $h' = J_1' \circ J_2'$. The modular transformation χ' induced by h' is elliptic, and its restriction to l' is the identity. By Lemma 2.5.3, χ' is either the identity or a hyperelliptic involution. If h' is the identity, there is nothing to prove. We thus assume that h' is hyperelliptic. Similarly, $h'' = J_2' \circ J_1'$ is also a hyperelliptic involution. But $h' \circ h'' = J_1' \circ J_2' \circ J_2' \circ J_1' = id$. It follows that $h'' = h'$. we conclude that $J_1' \circ J_2' = J_2' \circ J_1'$. Since h', J_1', and J_2' are half-turns on $\bar{X}' = \mathbf{S}^2$, by Proposition B.5 of Maskit [34], the axes of h', J_1' and J_2' constitute an orthogonal basis as shown in Figure 9.

If n' is odd, then since h' is a hyperelliptic involution, by definition, either C or A (but not both) is a puncture. Without loss of generality, we assume that C is a puncture, and A is a regular point. Since J_2' is also hyperelliptic, it sends C to A, which is impossible. We see that n' must be even, and all the points A, B, ..., F, are not punctures.

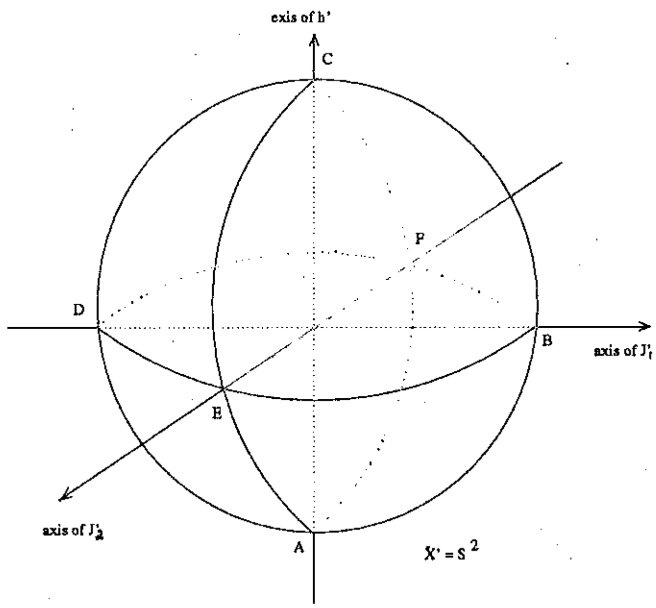

Figure 9.

Now let us denote by BCE the triangle in the sphere bounded by the geodesics BE, CF, and BC, and so forth. (In the spherical metric.) Observe that J_1', J_2', and h' send the triangle BCE to triangles ABF, ADE, and CDF, respectively. The punctures which are contained in BCE are mapped to punctures contained in ABF, ADE, and CDF, respectively. It is easy to see that there are no punctures lying in any boundary of a triangle. The same situation occurs for any of other triangles. This implies that $n' = 4k$ for some $k \in \mathbb{Z}^+$. Let χ_1' and χ_2' be the modular transformations induced by J_1' and J_2', respectively, and let Λ denote the subgroup of Mod Γ' generated by χ_1' and χ_2'. Since the quotient surface $X'/ < J_1', J_2' >$ has k punctures and three branched points of order 2, the dimension of $T(0, n')^\Lambda$ is k. On the other hand, by assumption dim $l' =$ dim $T(0, n')^\Lambda$. We thus obtain

$$k = \dim T(0,n')^\Lambda = \dim l' = \dim T(0,n')^{\chi_1'} = -1 + [\frac{n'}{2}] = 2k - 1.$$

But this occurs only if $k = 1$ and $n' = 4$. □

Remark. If $(g', n') = (0, 4)$, Earle-Kra [19] proved that any Riemann surface of type (0,4) has three (hyperelliptic) involutions, all of which induce the identity on $T(0, 4)$. Lemma 2.5.4 fails in this special case.

Proof of Proposition 2.5.2.

It is well-known that $F(0, 4; 2, 2, \infty, \infty) \cong F(0, 4; 2, 2, 2, \infty) \cong T(0, 5)$, $F(0, 5; 2, 2, 2, 2, \infty) \cong T(1, 3)$ and $F(0, 6; 2, 2, 2, 2, 2, 2) \cong T(2, 1)$ (see (0.2)–(0.4) in the introduction). Let Γ be of the signature which is one of those mentioned above, and let φ denote the corresponding isomorphism. Also, we denote by $s_*(\chi_f)$ a motion of the image $s(T(\Gamma))$ of a canonical section s which extends to a holomorphic automorphism χ of $F(\Gamma)$ (see Proposition 2.5.1). Suppose that there are another holomorphic extension χ_0 of $s_*(\chi_f)$. Then $\chi \circ \chi_0^{-1} \in$ Aut $F(\Gamma)$ is non-trivial but restricts to the identity map on $s(T(\Gamma))$.

From Lemma 2.4.2, we see that $l' = \varphi \circ s(T(\Gamma))$ is a component of hyperelliptic loci, and $\varphi \circ \chi \circ \chi_0^{-1} \circ \varphi^{-1} \in \text{Mod } \Gamma'$ is non-trivial but restricts to an identity map on l' as well. By Lemma 2.5.3, $\varphi \circ \chi \circ \chi_0^{-1} \circ \varphi^{-1} \in \text{Mod } \Gamma'$ is a hyperelliptic involution J' (since it is not the identity). On the other hand, we denote by $e \in \Gamma$ be the elliptic element corresponding to the canonical section s. Then $\varphi \circ e \circ \varphi^{-1} \in \text{Mod } \Gamma'$ is a hyperelliptic involution by Lemma 2.4.2 again. Hence, from Lemma 2.5.4, we conclude that $J' = \varphi \circ e \circ \varphi^{-1}$; that is, $e = \chi \circ \chi_0^{-1}$. This completes the proof of Proposition 2.5.2. □

The rest of this chapter is devoted to the proof of Theorem 2.1.1.

2.6 Periodic automorphisms of Bers fiber spaces

Let Γ be a finitely generated Fuchsian group of the first kind (operating on U) which contains elliptic elements and whose signature is $(g, n; \nu_1, \ldots, \nu_n)$. In the previous section, we proved that for certain self-maps f of U/Γ (in the sense of orbifolds) and certain canonical sections s of $\pi : F(\Gamma) \to T(\Gamma)$, $s_*(\chi_f) \in \text{Aut } s(T(\Gamma))$ is the restriction of a global holomorphic automorphism χ of the Bers fiber space $F(\Gamma)$. χ is, of course, fiber-preserving. In this section, we will choose more specific self-map f of U/Γ so that χ is a periodic holomorphic automorphism. Its order will be computable. The construction essentially depends on the signature of Γ. More importantly, the construction of χ allows us to compute the dimension of the fixed point set of χ in $F(\Gamma)$.

Let $e \in \Gamma$ be an elliptic element of order 2. e is also viewed as an element of mod Γ. In each case of the following proposition, s is the canonical section of $\pi : F(\Gamma) \to T(\Gamma)$ determined by e, f is a self-map of U/Γ in the sense of orbifolds which fixes the branch point on U/Γ determined by e, and χ always stands for the holomorphic extension od $s_*(\chi_f)$ to $F(\Gamma)$ which is obtained by Proposition 2.5.1. We have

Proposition 2.6.1 *(1) Assume that the signature $(g, n; \nu_1, \ldots, \nu_n)$ of Γ is one of the following:*

$$(0, 8; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{8-m}), \quad m = 1, 2, 4, 5, 7, 8.$$

Then there is a self-map f of U/Γ so that $\chi \in$ mod Γ is of order 3.

(2) Assume that $(g, n; \nu_1, \ldots, \nu_n)$ is one of the following:

$$(0, 7; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{7-m}), \quad m = 1, 3, 5, 7.$$

Then f can be chosen so that χ^2 is either the identity or equal to e.

(3) Assume that $(g, n; \nu_1, \ldots, \nu_n)$ is one of the following:

$$(0, 6; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{6-m}), \quad m = 1, 2, 5.$$

Then we may choose f so that χ^4 is either the identity or equal to e.

(4a) Assume that $(g, n; \nu_1, \ldots, \nu_n) = (0, 5; 2, 2, 2, \infty, \infty)$. Then f can be defined so that χ^2 is either the identity or equal to e.

(4b) If $(g, n; \nu_1, \ldots, \nu_n) = (0, 5; 2, 2, \infty, \infty, \infty)$*, then* $\chi \in$ *mod* Γ *can be constructed so that* χ^3 *is the identity.*

(4c) If $(g, n; \nu_1, \ldots, \nu_n)$ *is either* $(0, 5; 2, 2, 2, 2, 2)$ *or* $(0, 5; 2, \infty, \infty, \infty, \infty)$*, then* f *can be constructed so that* χ^4 *is either the identity or equal to* e*.*

(5) If $(g, n; \nu_1, \ldots, \nu_n)$ *is either* $(1, 2; 2, \infty)$ *or* $(1, 2; 2, 2)$*, then* f *can be chosen so that* χ^4 *is either the identity or equal to* e*.*

(6) If $(g, n; \nu_1, \ldots, \nu_n) = (0, 4, 2, \infty, \infty, \infty)$*, then* f *is defined so that* $\chi \in$ *mod* Γ *has order* 3.

(7) If $(g, n; \nu_1, \ldots, \nu_n) = (1, 1; 2)$*, then* f *may be chosen so that* $\chi \in$ *mod* Γ *has order* 3.

Proof. (1) Γ is of signature $(0, 8; 2, \ldots, 2)$; that is, U/Γ is a Riemann sphere with eight branch points $x_1, \ldots, x_8$ of order 2. We may choose Γ so that U/Γ is described as follows: take a standard unit sphere $\mathbb{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, let x_1 be the point $(0, 0, 1) \in \mathbb{S}^2$, x_2 the point $(0, 0, -1) \in \mathbb{S}^2$, and let A be the standard rotation about $z-$axis with rotation angle $2\pi/6$. Choose a point $x_3 \in \Sigma = \{(x, y, 0) \mid x^2 + y^2 = 1\}$. We then define $x_{i+1} = A(x_i)$ for $i = 3, \ldots, 7$. Then automatically, $A(x_8) = x_3$. See Figure 10. Let X be the Riemann sphere $\mathbb{S}^2$ with branch points $x_1, \ldots, x_8$ of order 2. The uniformization theorem asserts that there is a Γ so that $U/\Gamma = X$. A is an elliptic Möbius transformation fixing x_1 and x_2.

Let $e \in \Gamma$ be an elliptic element corresponding to the branch point $x_1 \in U/\Gamma$, and s the canonical section of $\pi : F(\Gamma) \to T(\Gamma)$ determined by $e \in \mathrm{mod}\, \Gamma$. Let $f = A^2$. Then f has order 3. By Proposition 2.5.1, there is $\chi \in \mathrm{mod}\, \Gamma$

with the following properties:

(i) χ leaves invariant the set $s(T(\Gamma))$;

(ii) χ commutes with e if both elements are viewed as elements of mod Γ;

(iii) χ^3 restricts to the identity map on $s(T(\Gamma))$.

From (iii) we see that $q_0(\chi^3) = id$, where q_0: mod $\Gamma \to$ Mod Γ is the quotient map. It follows that either $\chi^3 = id$ or $\chi^3 = e$. If $\chi^3 = id$, we are done; if $\chi^3 = e$, then we take $\chi_0 = e \circ \chi$. χ_0 has properties (i), (ii), (iii), and $\chi_0^3 = id$. We see that χ_0 does the job.

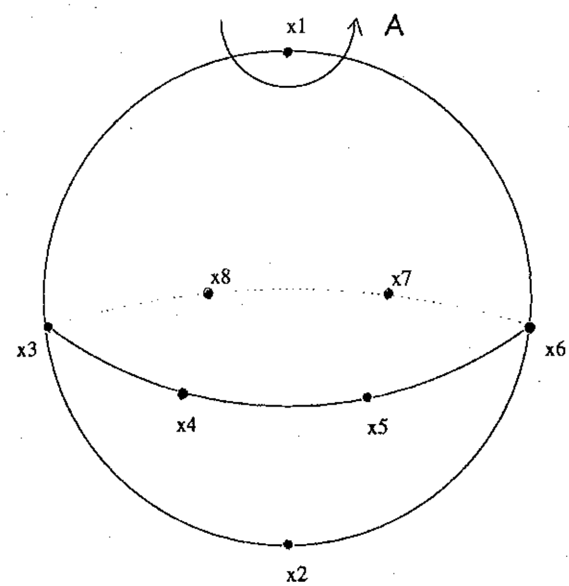

Figure 10.

If Γ is of signature $(0, 8; 2, \ldots, 2, \infty)$, let $x_1, x_3, \ldots, x_8$ be branch points of order 2, and x_2 a puncture.

If Γ is of signature $(0, 8; 2, \infty, \ldots, \infty)$, let x_1 be a branch point of order 2, and $x_2, \ldots, x_8$ punctures.

If Γ is of signature $(0, 8; 2, 2, 2, 2, \infty, \infty, \infty, \infty)$, let x_1, x_3, x_5, x_7 be branch points of order 2, and x_2, x_4, x_6, x_8 punctures.

It is easily seen that, in the above three cases, the map A^2 (A is the map constructed in the beginning of (1)) is a self-map of U/Γ in the sense of orbifolds. Note that A is not a self-map of U/Γ if Γ is of signature $(0, 8; 2, 2, 2, 2, \infty, \infty, \infty, \infty)$.

If Γ is of signature $(0, 8; 2, 2, \infty, \ldots, \infty)$, then $x_3, \ldots, x_8$ are set to be punctures, and x_1, x_2 are set to be branch points of order 2. By the same construction as above, A^2 is a self-map of U/Γ in the sense of orbifolds.

If Γ is of signature $(0, 8; 2, 2, 2, 2, 2, \infty, \infty, \infty)$, then x_1, x_2, x_3, x_5, x_7 are set to be branch points of order 2, and x_4, x_6, x_8 are set to be punctures. Again, A^2 is a self-map of U/Γ in the sense of orbifolds, and the previous argument is applied to finish the proof of case (1).

(2) Let Γ be of signature $(0, 7; 2, \ldots, 2)$. Take the sphere $\mathbb{S}^2$, and let $x_1 = (0, 0, 1) \in \mathbb{S}^2$ be a branch point of order 2. Let A be the same rotation as in (1). We choose an arbitrary point $x_2 \in \Sigma = \{(x, y, 0) \mid x^2 + y^2 = 1\}$. Define $x_{i+1} = A(x_i)$ for $i = 2, 3, 4, 5, 6$ (this implies that $A(x_7) = x_2$). See Figure 11. Let X be the Riemann sphere with 7 branch points $x_1, \ldots, x_7$ of order 2. By the uniformization theorem, there is a Γ such that $U/\Gamma = X$. (In Case (1) the rotation was made to fix two distinguished points. In current situation the rotation involved fixes only one branch point.) Let $f = A^3$. f is of order 2. By Proposition 2.5.1, there is $\chi \in \text{mod } \Gamma$ which satisfies properties (i) and (ii) in (1). By the same argument as in (1), we conclude that χ^2 is either the

identity or equal to e.

If Γ if of signature $(0,7;2,\infty,\ldots,\infty)$, then the same rotation A as in (1) is defined, but in this case X is the Riemann sphere with x_1 a branch point of order 2, and $x_2,\ldots,x_7$ punctures. See Figure 11. Note that A is a self-map of U/Γ in the sense of orbifolds. The same argument as in (1) also works in this case.

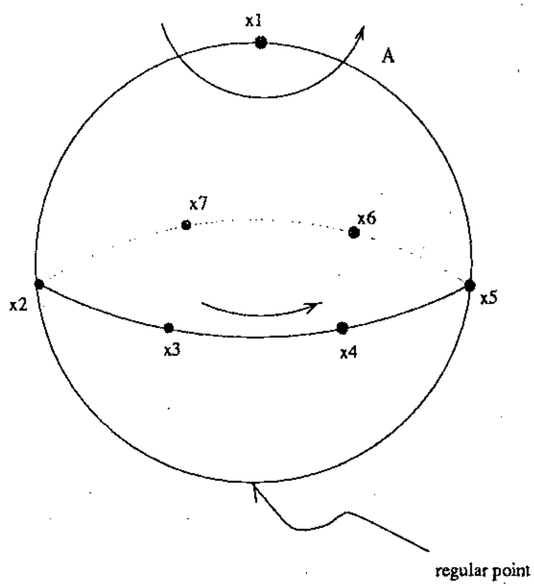

Figure 11.

If Γ is of signature $(0,7;2,2,2,\infty,\infty,\infty,\infty)$, then we still have the rotation A, but in this case X is the Riemann sphere with x_1, x_2, x_5 branch points of order 2, and x_3, x_4, x_6, x_7 punctures. Note that A^3 is a self-map of $X = U/\Gamma$, although A is not. Let $f = A^3$. Then the same argument as in (1) is used to obtain the required automorphism $\chi \in \bmod \Gamma$.

If Γ is of signature $(0,7;2,\ldots,2,\infty,\infty)$, then X here is the Riemann sphere with x_2 and x_5 punctures, and x_1, x_3, x_4, x_6, x_7 branch points of order 2. The above discussion also works.

(3) As in (1), let $x_1 = (0,0,1) \in \mathbb{S}^2$, $x_2 = (0,0,-1) \in \mathbb{S}^2$, and let A be the standard rotation about $z-$axis with rotation angle $2\pi/4$. Choose a point $x_3 \in \Sigma$ and let $x_{i+1} = A(x_i)$ for $i = 3,4,5$ (then $A(x_6) = x_3$). See Figure12.

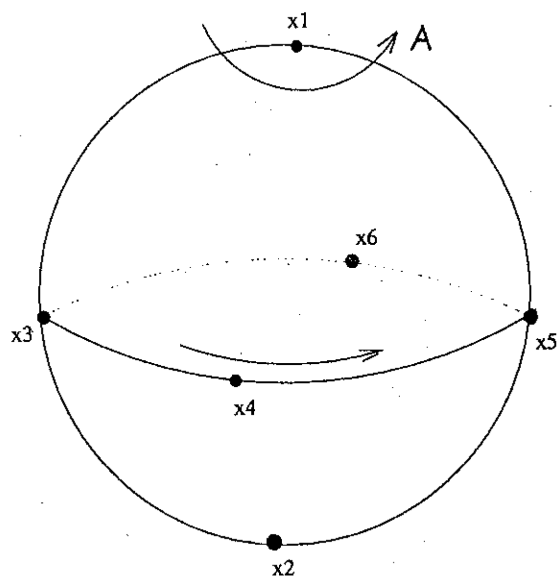

Figure 12.

Now X is defined as the Riemann sphere with 6 distinguished points $x_1,\ldots,x_6$, where:

(i) x_1 is a branch point of order 2 and $x_2,\ldots,x_6$ are punctures if Γ is of signature $(0,6;2,\infty,\ldots,\infty)$;

(ii) All x_i are branch points of order 2 if Γ is of signature $(0,6;2,\ldots,2)$;

(iii) x_1 and x_2 are branch points of order 2 and $x_3, \dots, x_6$ are punctures if Γ is of signature $(0,6;2,2,\infty,\dots,\infty)$;

(iv) x_2 is a puncture and all other distinguished points are branch points of order 2 if Γ is of signature $(0,6;2,\dots,2,\infty)$.

Observe that the rotation constructed above is a self-map of X in the sense of orbifolds, where X is in the cases of (i) (ii) (iii) (iv). Let $f = A$. By Proposition 2.5.1, we have $\chi \in \operatorname{mod}\Gamma$, which is the required automorphism. An important observation is that when the signature of Γ is in (i) (ii) (iii) (iv), χ is defined as an element of $\operatorname{mod}\Gamma$, but it is not the case when Γ is of signature $(0,6;2,2,2,\infty,\infty,\infty)$ or $(0,6;2,\dots,2,\infty,\infty)$. In latter two cases, only χ^2 is a well-defined element of $\operatorname{mod}\Gamma$.

(4a) As before, take the sphere $\mathbb{S}^2$. Let $x_1 = (0,0,1)$, and let A be the standard rotation about $z-$axis with rotation angle $2\pi/4$. Choose a point $x_2 \in \Sigma$ and set $x_{i+1} = A(x_i)$ for $i = 2,3,4$. See Figure 13.

Define X as the Riemann sphere with 5 distinguished points $x_1, \dots, x_5$, where x_1, x_3, x_5 are branch points of order 2, and x_2, x_4 are punctures.

Now $f = A^2$ is a self-map of U/Γ. By Proposition 2.5.1, there is an automorphism $\chi \in \operatorname{mod}\Gamma$ satisfying (i) and (ii) in (1) such that χ^2 is either the identity or equal to $e \in \operatorname{mod}\Gamma$.

(4b) Let $x_1 = (0,0,1)$, $x_2 = (0,0,-1)$, and A the standard rotation about $z-$axis with rotation angle $2\pi/3$. Choose an arbitrary point $x_3 \in \Sigma$ and set $x_{i+1} = A(x_i)$ for $i = 3,4$ (then $A(x_5) = x_3$). See Figure 14.

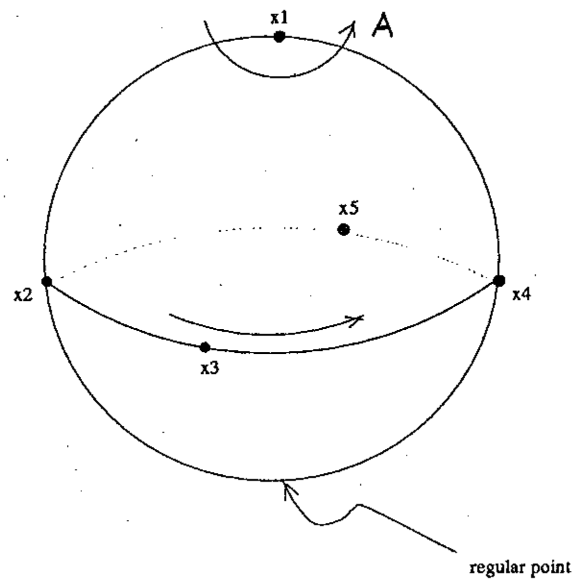

Figure 13.

In this case, X is the Riemann sphere with 5 distinguished points $x_1, \ldots, x_5$, where x_1, x_2 are branch points of order 2, and x_3, x_4, x_5 are punctures. Let $f = A$. By Proposition 2.5.1, we have $\chi \in \bmod \Gamma$ so that χ^3 is either the identity or equal to e. If $\chi^3 = e$, set $\chi_0 = e \circ \chi \in \bmod \Gamma$. Then χ_0 does the job.

(4c) Let $x_1 = (0, 0, 1)$ and x_2 an arbitrary point in Σ. Let A be the rotation with angle $2\pi/4$. Set $x_{i+1} = A(x_i)$ for $i = 2, 3, 4$. Now X is the Riemann sphere with 5 distinguished points $x_1, \ldots, x_5$, where x_1 is a branch point of order 2, and all other points are branch points of order 2 if Γ is of signature $(0, 5; 2, \ldots, 2)$; all other points are punctures if Γ is of signature $(0, 5; 2, \infty, \ldots, \infty)$. In these two cases we refer to Figure 13. The map $f = A$ is a self-map (in the sense of orbifolds) with order 4.

(5) In Example 3 of §2.2, a self-map h of a Riemann surface X of signa-

ture $(1,2;\infty,\infty)$ was constructed, h has the properties that it fixes the two punctures on X and h^4 is isotopic to the identity by an isotopy fixing the two punctures. The map h extends to a self-map (call it h also) of U/Γ if Γ is of signature $(1,2;2,\infty)$ or $(1,2;2,2)$. h fixes two distinguished points. Let $e \in \Gamma$ be the elliptic element corresponding to a branch point, and let s be the canonical section of $\pi : F(\Gamma) \to T(\Gamma)$ determined by the fixed point of e in U. By Proposition 2.5.1, an automorphism $\chi \in \text{mod}\,\Gamma$ is defined so that χ^4 restricts to the identity map of $s(T(\Gamma))$. It follows that χ^4 is either the identity or equal to e, as required.

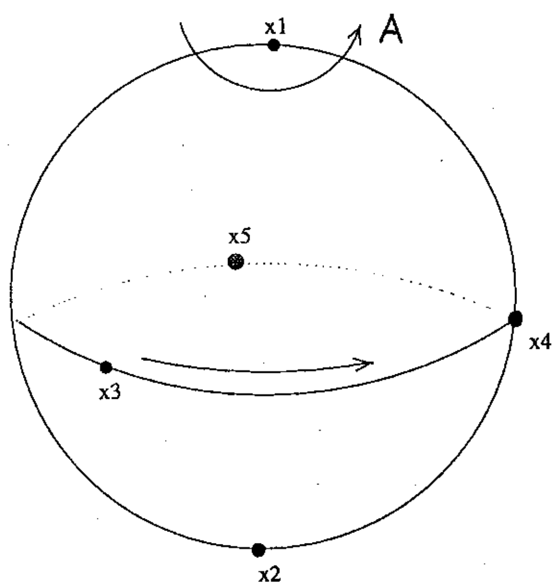

Figure 14.

(6) Let $x_1 = (0,0,1)$ and $x_2 \in \Sigma$ an arbitrary point, let A be the rotation with angle $2\pi/3$, and let $x_{i+1} = A(x_i)$ for $i = 2,3$ (then $A(x_4) = x_2$). See

Figure 15. In this case X is the Riemann sphere with 4 distinguished points $x_1, \ldots, x_4$, where x_1 is a branch point of order 2 and all other distinguished points are punctures. There is an $\chi \in \mathrm{mod}\ \Gamma$ satisfying (i) and (ii) in (1) such that χ^3 is either the identity or equal to e. If $\chi^3 = e$, then we set $\chi_0 = e \circ \chi$. χ_0 has order 3 and it also satisfies (i) and (ii) in (1), as required.

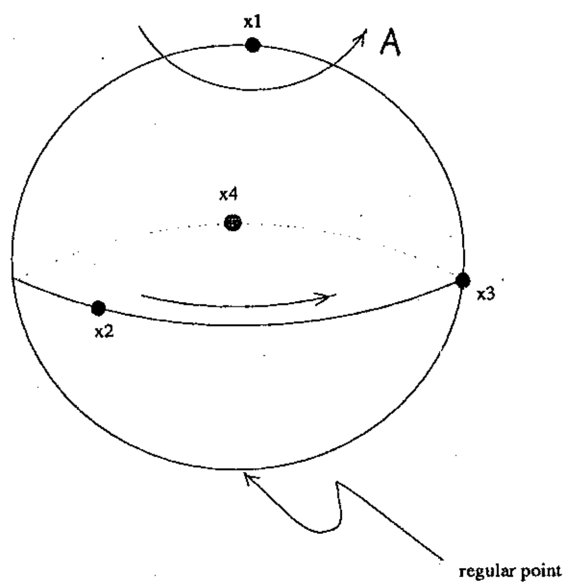

Figure 15.

Remark. If Γ is of signature $(0,4;2,2,2,\infty)$ or of signature $(0,4;2,2,\infty,\infty)$, then the map A (which satisfies the condition that x_1 is a branch point of order 2) is not a self map in the sense of orbifold. See Figure 15. In these two cases, we do know that there are isomorphisms $F(0,4;2,2,\infty,\infty) \cong T(0,5)$ and $F(0,4;2,2,2,\infty) \cong T(0,5)$. In this regard, the Bers question is solved in the case when Γ is of type $(0,4)$.

(7) Finally, if Γ is of signature $(1,1;2)$, then from Example 2 in §2.2, a self-map h of a surface X of signature $(1,1;\infty)$ was constructed with the property that h^3 is isotopic to the hyperelliptic involution (determined by the puncture). The map h extends to a self-map (call it h also) of U/Γ and h^3 is also isotopic to the hyperelliptic involution. Since the modular transformation χ_h^3 induced by h^3 acts trivially on $T(1,1)$, $s_*(\chi_h^3)$ is the identity map on $s(T(\Gamma))$. On the other hand, from Proposition 2.5.1, there is an element $\chi \in \bmod \Gamma$ such that $\chi^3|_{s(T(\Gamma))} = s_*(\chi_h^3) = id$. Therefore, χ^3 is either the identity or equal to e. If $\chi^3 = e$, we take $\chi_0 = e \circ \chi$ as the required element. This completes the proof of Proposition 2.6.1. □

Our next task is to compute the dimension of the set of fixed points of the periodic automorphism χ obtained from Proposition 2.6.1. (The proof of Proposition 2.6.1 also shows that the set of fixed points of χ is not empty.)

Let $F(\Gamma)^\chi$ denote the set of all points in $F(\Gamma)$ which are fixed by χ. We first prove:

Proposition 2.6.2 *Let s be the canonical section of $\pi: F(\Gamma) \to T(\Gamma)$ which is determined by an elliptic element $e \in \Gamma$. Let f be a self-map of U/Γ which fixes the branch point determined by e and let $\chi \in$ mod Γ be constructed via Proposition 2.5.1. Then we have*

$$F(\Gamma)^\chi = F(\Gamma)^{e\circ\chi} = s(T(\Gamma))^{s_*(\chi_f)}.$$

Proof. We only prove that $F(\Gamma)^{\chi} = s(T(\Gamma))^{s_*(\chi_f)}$; the proof of the other equality is the same. Since χ is a fiber-preserving extension of $s_*(\chi_f)$, it is trivial that

$$s(T(\Gamma))^{s_*(\chi_f)} \subset F(\Gamma)^{\chi}.$$

Suppose now that there is a point $x \in F(\Gamma)^{\chi}$ which is not in $s(T(\Gamma))^{s_*(\chi_f)}$, and that $x \in s(T(\Gamma))$. Since χ is an extension of $s_*(\chi_f)$, we have $x = \chi(x) = s_*(\chi_f)(x)$. This implies that $x \in s(T(\Gamma))^{s_*(\chi_f)}$, a contradiction. We conclude that $x \notin s(T(\Gamma))$. Therefore, in the fiber $\pi^{-1}(\pi(x))$, there are at least two points, x and the intersection $\pi^{-1}(\pi(x)) \cap s(T(\Gamma))$, which are fixed by χ. But the restriction of χ to the fiber $\pi^{-1}(\pi(x))$ is a conformal automorphism. It follows that χ is the identity map on $\pi^{-1}(\pi(x))$.

We need to investigate the action of χ on $\pi^{-1}(\pi(y))$ for any $y \in F(\Gamma)$. Following Bers [8], let h_μ be defined by

$$w_\mu = h_\mu \circ w^\mu | U$$

for any $\mu \in M(\Gamma)$. Then $h_\mu : w^\mu(U) \to U$ is a conformal map keeping 0, 1, ∞ fixed. It is easy to see that h_μ depends only on $[\mu]$. From (1.7), to each $x = ([\mu], z) \in F(\Gamma)$, we have $\chi([\mu], z) = ([\nu], \hat{z})$, where $\hat{z} = w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)$. Recall that $w_\nu = \alpha \circ w_\mu \circ \hat{f}^{-1}$, where $\alpha \in \mathrm{PSL}(2, \mathbb{R})$ is such that $\alpha \circ w_\mu \circ \hat{f}^{-1}$ is normalized. We thus have

$$\begin{aligned} \hat{z} &= w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z) = (h_\nu)^{-1} \circ w_\nu \circ \hat{f} \circ (w_\mu)^{-1} \circ h_\mu(z) \\ &= (h_\nu)^{-1} \circ \alpha \circ w_\mu \circ \hat{f}^{-1} \circ \hat{f} \circ (w_\mu)^{-1} \circ h_\mu(z) \\ &= (h_\nu)^{-1} \circ \alpha \circ h_\mu(z). \end{aligned} \tag{2.37}$$

Now we set $[\mu] = \pi(x) \in T(\Gamma)$. In this situation χ must keep the fiber $\pi^{-1}(\pi(x))$ invariant. This means that $[\mu] = [\nu]$ and $\chi([\mu], z) = ([\mu], (h_\mu)^{-1} \circ \alpha \circ h_\mu(z))$. By the above argument, the restriction of χ to $\pi^{-1}(\pi(x))$ is the identity. We see from (2.37) that α is the identity, and hence that $\hat{f}$ is normalized. Since $w_\mu = w_\mu \circ \hat{f}^{-1}$, $\hat{f}$ restricts to the identity on $\mathbb{R}$. It follows that $\hat{f}$ commutes with all elements of Γ, which in turn implies that f is isotopic to the identity on U/Γ, which leads to a contradiction. Therefore, $F(\Gamma)^\chi = s(T(\Gamma))^{s_*(\chi_f)}$, as we claimed. □

Now we are ready to prove the following result (recall that χ depends on the signature of Γ):

Proposition 2.6.3 *With the same notations as in proposition 2.6.1:*

(1) Assume that the signature $(g, n; \nu_1, \ldots, \nu_n)$ of Γ is one of the following:

$$(0, 8; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{8-m}), \quad m = 2, 4, 5, 7, 8.$$

Then dim $F(\Gamma)^\chi =$ dim $F(\Gamma)^{e \circ \chi} = 1$.

(2) Assume that $(g, n; \nu_1, \ldots, \nu_n)$ is one of the following:

$$(0, 7; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{7-m}), \quad m = 1, 3, 5, 7.$$

Then dim $F(\Gamma)^\chi =$ dim $F(\Gamma)^{e \circ \chi} = 2$.

(3) Assume that $(g, n; \nu_1, \ldots, \nu_n)$ is one of the following:

$$(0, 6; \underbrace{2, \ldots, 2}_{m}, \underbrace{\infty, \ldots, \infty}_{6-m}), \quad m = 1, 2, 5.$$

Then dim $F(\Gamma)^{\chi} =$ *dim* $F(\Gamma)^{e\circ\chi} = 0$, *and dim* $F(\Gamma)^{\chi^2} =$ *dim* $F(\Gamma)^{e\circ\chi^2} = 1$.

(4a) Assume that the signature $(g, n; \nu_1, \ldots, \nu_n) = (0, 5; 2, 2, \infty, \infty, \infty)$. *Then we have dim* $F(\Gamma)^{\chi} =$ *dim* $F(\Gamma)^{e\circ\chi} =$ *dim* $F(\Gamma)^{\chi^2} =$ *dim* $F(\Gamma)^{e\circ\chi^2} = 0$.

(4b) Assume that the signature $(g, n; \nu_1, \ldots, \nu_n) = (0, 5; 2, 2, 2, \infty, \infty)$. *Then we have dim* $F(\Gamma)^{\chi} =$ *dim* $F(\Gamma)^{e\circ\chi} = 1$.

Proof. By a theorem of Kravetz [29] (see also Earle-Kra [20] and Kra [28]), we know that for any elliptic modular transformation χ_0 of $T(g_0, n_0)$, the set $T(g_0, n_0)^{\chi_0}$ of fixed points of χ_0 is identified with another Teichmüller space $T(g^*, n^*)$ (where g^* and n^* are defined in the proof of Proposition 2.3.1). By definition, to each $x \in s(T(\Gamma))$, we have $s_*(\chi_f)(x) = s \circ \chi_f \circ \pi(x)$ (see §2.5). Hence, we obtain

$$T(\Gamma)^{\chi_f} = s(T(\Gamma))^{s\circ\chi_f\circ\pi} = s(T(\Gamma))^{s_*(\chi_f)}.$$

In particular, we have $\dim s(T(\Gamma))^{s_*(\chi_f)} = \dim T(\Gamma)^{\chi_f}$. From Proposition 2.6.2, we see that

$$\dim F(\Gamma)^{\chi} = \dim T(\Gamma)^{\chi_f}. \tag{2.38}$$

Now the rest of proof of Proposition 2.6.3 is just a simple computation. We consider the various cases.

(1) Notice that $f = A^2$ and that $X/\langle A^2 \rangle$ is a Riemann sphere with 4 distinguished points. Thus,

$$\dim F(\Gamma)^{\chi} = \dim T(X/\langle A^2 \rangle) = \dim T(0, 4) = 1.$$

(2) In this case $f = A^3$ and $X/ < A^3 >$ is a Riemann sphere with 5 distinguished points. This yields

$$\dim F(\Gamma)^{\chi} = \dim T(X/{<}A^3{>}) = \dim T(0,5) = 2.$$

(3) From Proposition 2.6.1 (3), we see that $f = A$ and $X/ < A >$ is a Riemann sphere with 3 distinguished points in this case. Hence,

$$\dim F(\Gamma)^{\chi} = \dim T(X/{<}A{>}) = \dim T(0,3) = 0.$$

Similarly, we have

$$\dim F(\Gamma)^{\chi^2} = \dim T(X/{<}A^2{>}) = \dim T(0,4) = 1.$$

(4a) Observe that $f = A$ and $X/ < A >$ is a Riemann sphere with 3 distinguished points in this case, we have

$$\dim F(\Gamma)^{\chi} = \dim T(X/{<}A{>}) = \dim T(0,3) = 0.$$

Similarly, we obtain

$$\dim F(\Gamma)^{\chi^2} = \dim T(X/{<}A^2{>}) = \dim T(0,3) = 0.$$

(4b) In this case, $f = A^2$ and $X/ < A^2 >$ is a Riemann sphere with signature $(0,4;2,2,4,\infty)$. Thus we obtain

$$\dim F(\Gamma)^{\chi} = \dim F(\Gamma)^{e\circ\chi} = \dim T(X/{<}A^2{>}) = 1. \qquad \square$$

2.7 Elliptic transformations of Teichmüller spaces

In this section, we compute the dimensions of the fixed point sets of some elliptic modular transformations of Teichmüller spaces in some low dimensional cases. With the help of the periodic automorphisms constructed in §2.6 and Theorem 1.2.3, some elliptic modular transformations of $T(\Gamma')$ are defined (via an isomorphism of $F(\Gamma)$ onto $T(\Gamma')$) so that their orders are known. Our purpose is to show that the dimensions of the fixed points of these elliptic modular transformations of $T(\Gamma')$ are actually different from those we obtained from Proposition 2.6.3 if the conditions of Theorem 2.1.1 are satisfied. This will finish the proof of our main theorem.

We assume that Γ' is a torsion free finitely generated Fuchsian group of the first kind. Let $\chi' \in \text{Mod}\,\Gamma'$ be an elliptic element in the sense of Bers [9]. As before, let $T(\Gamma')^{\chi'}$ denote the set of the fixed points of χ' in $T(\Gamma')$. (The set is always non-empty by Nielsen's theorem [41].)

Proposition 2.7.1 *In each of the following cases we assume that the element χ' commutes with a hyperelliptic involution $e' \in Mod\,\Gamma'$ but is not equal to e'.*

(1) Assume that Γ' is of type $(3,0)$ and that χ' is of order 3. Then $dim\ T(\Gamma')^{\chi'} = 2$;

(2) Assume that Γ' is of type $(2,2)$ and that ${\chi'}^2$ is either the identity or equal to e'. Then either $dim\ T(\Gamma')^{\chi'}$ or $dim\ T(\Gamma')^{e'\circ\chi'}$ equals 3;

(3) Assume that Γ' is of type $(1,4)$ and that ${\chi'}^4$ is either the identity or equal to e'. Then either $dim\ T(\Gamma')^{\chi'} \geq 1$, or $dim\ T(\Gamma')^{e'\circ\chi'} \geq 1$, or

dim $T(\Gamma')^{e'\circ\chi'^2} \geq 2$;

(4) Assume that Γ' *is of type* $(0,6)$ *and that* χ' *is of order* 3. *Then* *dim* $T(\Gamma')^{\chi'} = 1$.

Proof. First of all, as we mentioned above, there exists a fixed point $x' \in T(\Gamma')$ of χ' in each of the cases discussed below. Let X' denote the marked Riemann surface of type (g', n') which is represented by x', let h' be the conformal automorphism of X' which induces χ', and k' the number of the fixed points of h' on the compactification $\overline{X'}$ of X'. The symbol B' stands for the total branch number of the corresponding branched covering: $\overline{X'} \to \overline{X'}/\langle h' \rangle$, and g'' stands for the genus of the surface $X'' = X'/\langle h' \rangle$.

(1) From the Riemann-Hurwitz formula, we have

$$4 = 3(2g'' - 2) + B' = 3(2g'' - 2) + 2k'. \qquad (2.39)$$

If $g'' = 1$, then from (2.39), one sees that $k'' = 2$. By Kravetz's theorem [29], one obtains

$$\dim T(\Gamma')^{\chi'} = \dim T(X'') = 3g'' - 3 + k' = 2,$$

where X'' is the orbifold $X'/\langle h' \rangle$. The second equality holds because X' is compact and the number of the fixed points of h' on X' is the number of the branch points of X''.

If $g'' = 0$, then $k'' = 5$ and again, it is easy to see that

$$\dim T(\Gamma')^{\chi'} = 2.$$

This proves (1).

(2) By hypothesis, χ'^2 is either the identity or equal to e'. But since χ'^2 is induced by a square of a self-map h', and h'^2 must fix the two punctures on X', we see that χ'^2 must be the identity. Now the Riemann-Hurwitz formula tells us that

$$2 = 2(2g'' - 2) + k'. \tag{2.40}$$

If $g'' = 0$, then $k' = 6 = 2g' + 2$ and χ' is another hyperelliptic involution on X'. From Lemma 2.5.4, we see that the hyperelliptic involution on X' is unique. Hence, $h' = e'$, contradicting our hypothesis. Therefore, by (2.40), the only possibility is that $g'' = 1$, and $k' = 2$.

Case 1. h' fixes the two punctures. Then h' has no other fixed points. Since χ' commutes with e', h' can be projected to a conformal automorphism h'' of the surface $X''_0 = X'/<e'>$ in the sense of orbifolds. h'' is an elliptic Möbius transformation. This means that h'' has two fixed points a'' and b'', one of which, say a'', comes from the projection of the punctures. The set $\{q^{-1}(b'')\}$ (where $q : X' \to X'/<e'>$ is the branched covering) must contains exactly 2 points, otherwise $\{q^{-1}(b'')\}$ is a fixed point of h'. This is impossible. It follows that h' must interchange the two points $\{q^{-1}(b'')\}$.

Consider the modular transformation $e' \circ \chi'$ which is induced by $e' \circ h'$. Since h' commutes with e', the self-map $e' \circ h'$ is of order 2. Moreover, $e' \circ h'$ has the property that it fixes $\{q^{-1}(b'')\}$ pointwise, and interchanges the two punctures. Then, we apply the formula (2.40) for the map $e' \circ h'$ to conclude that $X'/\langle e' \circ h' \rangle$ is of signature $(1,3;2,2,\infty)$ and that

$$\dim T(\Gamma')^{e' \circ \chi'} = \dim T(X'/\langle e' \circ h' \rangle) = 3.$$

Case 2. h' interchanges the two punctures. Then h' has two fixed points elsewhere which are symmetric with respect to e' (since h' commutes with e'). It is rather easy to see that these two points can not be Weierstrass points of $\overline{X'}$ (otherwise, h'' would have three fixed points). Thus, $X'/\langle h' \rangle$ is of signature $(1,3;2,2,\infty)$ and we obtain

$$\dim T(\Gamma')^{\chi'} = \dim T(X'/\langle h' \rangle) = 3.$$

This proves (2).

(3) Let h' be a conformal automorphism of some Riemann surface X' which induces χ'. Since h'^4 fixes all of the punctures (which are denoted by x_1', x_2', x_3', and x_4'), we see that χ'^4 is not a hyperelliptic involution, and hence χ'^4 is the identity. Assume that X' is a hyperelliptic Riemann surface and that x_1', x_2', x_3', and x_4' are arranged so that $e'(x_1') = x_2'$ and $e'(x_3') = x_4'$. Since h' commutes with e', h' can be projected to a conformal automorphism h'' of the Riemann surface $X_0'' = X'/\langle e' \rangle$ in the sense of orbifolds. Hence, h'' is a Möbius transformation. Observe that X_0'' is of signature $(0,6;2,2,2,2,\infty,\infty)$. Let x_1'', x_2'' denote the two punctures coming from the four punctures x_1', x_2', x_3', and x_4', and x_3'', x_4'', x_5'', x_6'' the four branch points of order 2 on X_0''. Then either h'' fixes both x_1'' and x_2'', or interchanges these two punctures.

Case 1. h'' interchanges x_1'' and x_2''. In this case, since h'' is an elliptic Möbius transformation, it has two fixed points a'' and b''. If $\{a'', b''\} \subset$

$\{x''_3, x''_4, x''_5, x''_6\}$, then h''^2 fixes the set $\{x''_3, x''_4, x''_5, x''_6\}$ pointwise, which implies that h''^2 is the identity, a contradiction. If $\{a'', b''\} \cap \{x''_3, x''_4, x''_5, x''_6\}$ is a'' or b'', then h'' induces a permutation of the three points in $\{x''_3, x''_4, x''_5, x''_6\} - \{a'', b''\}$, contradicting the fact that h''^4 is the identity. Finally, if $\{a'', b''\}$ and $\{x''_3, x''_4, x''_5, x''_6\}$ are disjoint, then h''^2 fixes $\{a'', b'', x''_1, x''_2\}$, which says that h''^2 is the identity. So the case that h'' interchanges x''_1 and x''_2 cannot occur.

Case 2. h'' fixes both x''_1 and x''_2. In this case, there are three possibilities:

(i) h' fixes all x'_1, x'_2, x'_3, and x'_4,

(ii) $h'(x'_i) = e'(x'_i)$, for $i = 1, 2, 3, 4$,

(iii) $h'(x'_i) = e'(x'_i)$, for $i = 1, 2$ and $h'(x'_i) = x'_i$, for $i = 3, 4$.

If h' fixes all x'_1, x'_2, x'_3, and x'_4, then $X'/<h'>$ has at least 4 distinguished points, which means that

$$\dim T(\Gamma')^{\chi'} = \dim T(X'/<h'>) \geq 1.$$

If $h'(x'_i) = e'(x'_i)$, for $i = 1, 2, 3, 4$, then $e' \circ h'$ has order 4 and fixes all the punctures. It follows that $X'/<e' \circ h'>$ has at least 4 distinguished points coming from the fixed points (punctures) of $e' \circ h'$. This implies that

$$\dim T(\Gamma')^{e' \circ \chi'} = \dim T(X'/<e' \circ h'>) \geq 1.$$

If $h'(x'_i) = e'(x'_i)$ for $i = 1, 2$ and $h'(x'_i) = x'_i$ for $i = 3, 4$, then again, h'^2 fixes all punctures x'_1, x'_2, x'_3, and x'_4. In this case, $e' \circ h'^2$ has order 2 and fixes no punctures, and therefore by the Riemann-Hurwitz formula, we need to discuss two cases. If the genus g'' of $X'/<e' \circ h'^2>$ is one and k'' (k'' is the number of the fixed points of $e' \circ h'^2$ on $\overline{X'}$) is zero, then $X'/<e' \circ h'^2>$ is of signature $(1,2;\infty,\infty)$, which implies that dim $T(\Gamma')^{e'\circ\chi'^2} = 2$. If $g'' = 0$ and $k'' = 4$, then $X'/< e' \circ h'^2 >$ is a Riemann sphere with 6 distinguished points including the two punctures x_1'' and x_2''. It follows that dim $T(\Gamma')^{e'\circ\chi'^2} = 3$, as asserted.

(4) By hypothesis, χ' is of order 3, and h' is of order 3 too. If h' fixes two punctures, then h' permutes the remaining 4 punctures, which is impossible since h' is of order 3. If h' fixes one puncture, then again, h' permutes the remaining 5 punctures, but this case cannot happen either. It remains to consider the case that h' fixes no punctures. In this case, all 6 punctures of X' must be divided into two orbits under the iteration of h'. It follows that the surface $X'/< h' >$ has 4 distinguished points; more precisely, $X'/< h' >$ has signature $(0,4;2,2,\infty,\infty)$. Therefore, we obtain

$$T(\Gamma')^{\chi'} = T(X'/< h' >) = T(0,4).$$

This proves (4) and hence the proof of Proposition 2.7.1 is complete. □

2.8 Proof of Theorem 2.1.1

Let Γ and Γ' be finitely generated Fuchsian groups of the first kind. Assume that Γ contains elliptic elements and is of signature $(g,n;\nu_1,\ldots,\nu_n)$, and that Γ' is torsion free with type (g',n'). We need

Lemma 2.8.1 *Assume that Γ contains elliptic elements with type $(g,n) \neq (0,3)$ and that there is an isomorphism $\varphi\colon F(\Gamma) \to T(\Gamma')$. Then there is an* *elliptic modular transformation $\chi' \in$ Mod Γ' with the property that it commutes with a hyperelliptic involution.*

Proof. Since $F(\Gamma)$ and $T(\Gamma')$ are isomorphic, by Theorem 0.1, we see that the pair of types $((g,n),(g',n'))$ lies in table (A) in the introduction. Furthermore, every elliptic element of Γ is of order 2 (note that $(g,n) \neq (0,3)$). Let $e \in \Gamma$ be an elliptic element, e is also viewed as an element of mod Γ. Let $s \colon T(\Gamma) \to F(\Gamma)$ be the canonical section determined by the fixed point of e in U, and let $e' = \varphi \circ e \circ \varphi^{-1}$ (by assumption there is an isomorphism $\varphi \colon F(\Gamma) \to T(\Gamma')$). Theorem 1.2.3 says that e' belongs to Mod Γ'. e' is an involution since e is of order 2. It follows from Lemma 2.4.2 (see also its remark) that the fixed point set of e', which is $\varphi \circ s(T(\Gamma))$, is a component of the hyperelliptic locus determined by e'.

On the other hand, by Proposition 2.6.1, one may construct a periodic automorphism $\chi \in$ mod Γ of $F(\Gamma)$ with the property that it leaves invariant the set $s(T(\Gamma))$ which is biholomorphically equivalent to $T(\Gamma)$. Now by using Theorem 1.2.3 once again, one sees at once that $\chi' = \varphi \circ \chi \circ \varphi^{-1}$ is an elliptic modular transformation in the sense of Bers [9], and the order of χ' is the same as the order of χ. Since χ commutes with e, we have

$$\begin{aligned} \chi' \circ e' &= \varphi \circ \chi \circ \varphi^{-1} \circ \varphi \circ e \circ \varphi^{-1} = \varphi \circ \chi \circ e \circ \varphi^{-1} \\ &= \varphi \circ e \circ \chi \circ \varphi^{-1} = \varphi \circ e \circ \varphi^{-1} \circ \varphi \circ \chi \circ \varphi^{-1} \\ &= e' \circ \chi'. \end{aligned}$$

The lemma then follows. □

Now we prove our main results.

Proof of Theorem 2.1.1. (1) $((g,n),(g',n')) = ((0,8),(3,0))$. Let χ be constructed by Proposition 2.6.1 (1), and let χ' be defined in the proof of Lemma 2.8.1. Then both χ and χ' have order 3. By Proposition 2.7.1 (1), we see that

$$\dim T(\Gamma')^{\chi'} = 2. \tag{2.41}$$

But since φ is an isomorphism, the following equality holds:

$$\dim T(\Gamma')^{\chi'} = \dim F(\Gamma)^{\varphi^{-1}\circ\chi'\circ\varphi} = \dim F(\Gamma)^{\chi}.$$

From (2.41), we obtain $\dim F(\Gamma)^{\chi} = 2$, which contradicts to Proposition 2.6.3 (1).

Remark. If Γ is of signature $(0,8;2,2,2,\infty,\infty,\infty,\infty,\infty)$ or $(0,8;2,2,2,2,2,2,\infty,\infty)$, then the only periodic automorphism $\chi \in \operatorname{mod}\Gamma$ we can produce has the property that $\chi^2 = e$, and therefore, $\dim F(\Gamma)^{\chi} = 2$. On the other hand, $\chi' = \varphi \circ \chi \circ \varphi^{-1} \in \operatorname{Mod}\Gamma'$ commutes with the hyperelliptic involution $e' = \varphi \circ e \circ \varphi^{-1}$ (where $e \in \Gamma$ is an elliptic element corresponding to a branch point of order 2). And the corresponding map on X' projects to a map on $X'/<J'>$ which is again a rotation with two branch points as its fixed points. We see that the behavior of the action of χ' on $T(\Gamma')$ is quite similar to that of the action of χ on $F(\Gamma)$; in other words, we could not tell the difference between these two spaces if we only use the methods developed here. For similar reasons, if the signature of Γ lies in the 2nd row of Table (C'), we cannot find any contradiction either.

(2) $((g,n),(g',n')) = ((0,7),(2,2))$. The assertion follows from Proposition 2.6.1 (2), Proposition 2.6.3 (2) and Proposition 2.7.1 (2).

(3) $((g,n),(g',n')) = ((0,6),(1,4))$ and the signature of Γ is neither $(0,6;2,2,2,\infty,\infty,\infty)$ nor $(0,6;2,2,2,2,\infty,\infty)$. Again the assertion in this case follows directly from Proposition 2.6.1 (3), Proposition 2.6.3 (3) and Proposition 2.7.1 (3).

Remark. If Γ is of signature $(0,6;2,2,2,\infty,\infty,\infty)$ or $(0,6;2,2,2,2,\infty,\infty)$, we do not have such a χ, while χ^2 is still defined. So the method used above cannot be applied in these two cases; the treatment of these two cases needs a further understanding of conformal automorphisms of punctured Riemann surfaces. At this point we do not know whether χ^2 acts on $T(1,4)$.

(4a) $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,5;2,2,2,2,2),(1,3))$. In this case, the argument is different from that in (1), (2), and (3) above. From Proposition 2.6.1 (4c), a periodic automorphism $\chi \in \bmod \Gamma$ is constructed so that χ^4 is either the identity or equal to e.

By Lemma 2.8.1, there is an elliptic modular transformation χ' of $T(\Gamma')$ which commutes with $e' = \varphi \circ e \circ \varphi^{-1}$ such that χ'^4 is either the identity or equal to e'. Thus, we obtain a self-map h' of a marked Riemann surface $X' \in \varphi \circ s(T(\Gamma))$ so that $h' \circ e' \circ h'^{-1} = e'$ on X'. This means that h' can be projected to a self-map h'' of $X''_0 = X'/\langle e' \rangle$ in the sense of orbifolds. Since h'^4 is either isotopic to the identity or isotopic to e'. By Lemma 2.2.2, h''^4 is isotopic to the identity by an isotopy which fixes all distinguished points.

Note also that X_0'' is a Riemann surface of signature $(0,5;2,2,2,\infty,\infty)$, where one puncture comes from the puncture of X' fixed by h' (since X' is of type $(1,3)$ and h' commutes with e'). It follows that h'' fixes the two punctures pointwise. Since h'' is periodic, by Lemma 2.2.1, either h'' is isotopic to the identity or h''^3 is isotopic to the identity. In the first case, by Lemma 2.2.2, h' is either isotopic to the identity or isotopic to e', both of which cannot occur. If h''^3 is isotopic to the identity, then either h'^3 is isotopic to the identity, or h'^6 is isotopic to the identity. This means that χ' has order 3 or 6. This is a contradiction.

(4b) The case $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,5;2,\infty,\infty,\infty,\infty),(1,3))$ can be handled in a similar way.

(4c) $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,5;2,2,2,\infty,\infty),(1,3))$. The proof of this case is similar to (4a).

(4d) Assume now that the pair $((g,n;\nu_1\ldots,\nu_n),(g',n'))$ is either $((1,2;2,\infty),(1,3))$ or $((1,2;2,2),(1,3))$. From Proposition 2.6.1 (5) and Lemma 2.8.1, an element $\chi' \in \text{Mod}\,\Gamma'$ is defined so that χ'^4 is either the identity or equal to e'. By using the same argument as in (4a) above, we finish the argument of Theorem 2.1.1 (4).

Remark. We cannot settle the case when the pair $((g,n;\nu_1\ldots,\nu_n),(g',n'))$ is equal to $((0,5;2,2,\infty,\infty,\infty),(1,3))$. This seems to be a very difficult situation. The reason is the following: the automorphism $\chi \in \text{mod}\,\Gamma$ constructed in Proposition 2.6.1 (4b) can be "transplanted" as an automorphism of the space $F(0,5;2,2,2,2,\infty)$ which is, by our knowledge, isomorphic to $T(1,3)$. In other words, we "see" that χ really acts on $T(1,3)$. So the methods which are used in this paper do not work here.

(5a) Assume that $((g,n;\nu_1\dots,\nu_n),(g',n')) = ((0,5;2,2,2,\infty,\infty),(0,6))$. The proof is also similar to the proof of (4a). By Proposition 2.6.1 (4a) and Lemma 2.8.1, an element $\chi' \in \text{Mod}\,\Gamma'$ is defined so that its order is 2 or 4.

We may assume that h' (which induces χ') is a conformal automorphism on X' (this can be done by Nielsen's theorem [41]). Obviously, h' cannot fix one puncture and one regular point; otherwise the number of remaining punctures would be 5, contradicting that h' has order 2 or 4.

We assume that h' fixes two punctures, say x'_1 and x'_2. In this case h' has order 2. Since h' commutes with e', x'_1 and x'_2 must be e'-symmetric; that is, we have $e'(x'_1) = x'_2$. Also, it is easily seen that h' interchanges the two fixed points (not punctures) of e'. Observe that the remaining 4 punctures cannot be a single orbit under the iteration of h' since h'^2 is the identity.

The map h' can be projected to a self-map h'' of $X''_0 = X'/<e'>$ which is of signature $(0,5;2,2,\infty,\infty,\infty)$. Moreover, h'' fixes one puncture, interchanges the other two punctures and the two branch points. There is one more fixed point y'' of h''. This implies that h' interchanges the two points $\{q^{-1}(y'')\}$.

Consider the conformal automorphism $e' \circ h'$ of X'. Since e' commutes with h', $e' \circ h'$ is of order 2 and thus divides the 6 punctures into 3 orbits. There are also 2 branch points on $X'/<e' \circ h'>$, which come from the fixed points $\{q^{-1}(y'')\}$. We see that $X'/<e' \circ h'>$ is of signature $(0,5;2,2,\infty,\infty,\infty)$.

Therefore, we obtain

$$\dim T(\Gamma')^{e'\circ\chi'} = 2,$$

which implies that $\dim F(\Gamma)^{e\circ\chi} = \dim T(\Gamma')^{e'\circ\chi'} = 2$. This contradicts to Proposition 2.6.3 (4b).

Next, if h' fixes two regular points, then h' cannot be of order 4 unless h' is the identity (since h' defines a permutation of the 6 punctures). It follows that h' divides the 6 punctures into three orbits, which means that $X'/\langle h'\rangle$ is of signature $(0,5;2,2,\infty,\infty,\infty)$. In particular, $\dim T(\Gamma')^{\chi'} = 2$, which contradicts to Proposition 2.6.3.

(5b) The above argument also leads to a proof of the the following two cases: the pair $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,5;2,2,2,2,2),(0,6))$ or $((0,5;2,\infty,\infty,\infty,\infty),(0,6))$. Details are not repeated here.

(5c) Assume that $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,5;2,2,\infty,\infty,\infty),(0,6))$. Again, the assertion follows from Proposition 2.6.1 (4b), Proposition 2.7.1 (4), and Proposition 2.6.3 (4a).

(5d) Now we assume that $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((1,2;2,\infty),(0,6))$. In this case the proof is the same as (5a) since we have $\chi' \in \operatorname{Mod}\Gamma'$ so that ${\chi'}^4$ is the identity or equal to e'. By the same argument, we can settle the pair $((1,2;2,2),(0,6))$. Since $T(0,6) \cong T(2,0)$, the pairs $((1,2;2,\infty),(2,0))$ and $((1,2;2,2),(2,0))$ are handled similarly. This proves Theorem 2.1.1 (5).

(6a) Assume that $((g,n;\nu_1\ldots,\nu_n),(g',n')) = ((0,4;2,\infty,\infty,\infty),(0,5))$. Once again, by Proposition 2.6.1 (6) and Lemma 2.8.1, there is a $\chi' \in \operatorname{Mod}\Gamma'$

with order 3. Thus, there is a self-map h' of $X' \in \varphi \circ s(T(\Gamma))$ which induces χ' such that $h' \circ e' \circ h'^{-1} = e'$. Therefore, h' can be projected to a self-map h'' of $X_0'' = X'/ < e' >$ in the sense of orbifolds. By Lemma 2.2.2, we see that h'' is periodic up to isotopy. Observe that X_0'' is a surface of signature $(0, 4; 2, \infty, \infty, \infty)$, where one puncture comes from the fixed point (puncture) of e'. It follows that h'' fixes both the branch point and the puncture coming from the fixed point of e'. By Lemma 2.2.1, h''^2 must be isotopic to the identity on X_0''. Then Lemma 2.2.2 asserts that either h'^2 or h'^4 is isotopic to the identity. But this leads to a contradiction.

(6b) Assume that $((g, n; \nu_1 \ldots, \nu_n), (g', n')) = ((1, 1; 2), (0, 5))$. The assertion follows from Proposition 2.6.1 (7), Lemma 2.8.1, and the argument in (6a). Since $T(0, 5) \cong T(1, 2)$, the case of $((1, 1; 2), (1, 2))$ is handled similarly. This completes the proof of Theorem 2.1.1.

Chapter 3

Non extendibility of isomorphisms

In this chapter, we first assume that Γ is a torsion free finitely generated Fuchsian group of the first kind acting on U. Assume that U/Γ is of type (g, n). Choose an arbitrary point $a \in U$, let $A = \Gamma(a) = \{\gamma(a); \gamma \in \Gamma\}$, and let

$$v: U \to U - A$$

be a holomorphic universal covering. The Fuchsian model for the action of Γ on $U - A$ is the group

$$\dot{\Gamma} = \{\dot{\gamma} \in \mathrm{PSL}(2, \mathbb{R});\ \text{there is a } \gamma \in \Gamma \text{ with } v \circ \dot{\gamma} = \gamma \circ v\}.$$

One sees at once that $U/\dot{\Gamma}$ is conformally equivalent to $U/\Gamma - \{\hat{a}\}$, where $\hat{a} \in U/\Gamma$ is the image of a under the projection $p : U \to U/\Gamma$. A point in $F(\Gamma)$ is represented as a pair $([\mu], w^{\mu}(a))$ for a $\mu \in M(\Gamma)$ by Lemma 6.3 of Bers [8]. We also know that there is a surjective map $v^* : M(\dot{\Gamma}) \to M(\Gamma)$ defined as

$$(v^*(\nu)) \circ v = \nu \cdot (\overline{v'}/v'), \text{ for all } \nu \in M(\dot{\Gamma}).$$

Now we fix a point $x \in F(\Gamma)$, write $x = ([\mu], w^\mu(a))$ for some $\mu \in M(\Gamma)$. Since v^* is surjective, there is a $\nu \in M(\dot{\Gamma})$ such that $v^*(\nu) = \mu$. We may thus define a map $\psi : F(\Gamma) \to T(\dot{\Gamma})$ by sending x to $[\nu]$. Theorem 9 of Bers [8] asserts that ψ is a well defined biholomorphic map (which is called Bers' isomorphism in the literature).

As we discussed in §1.1, $T(\Gamma)$ can be identified with its Bers' embedding into the space $B_2(L, \Gamma)$. Thus the set of boundary points of $T(\Gamma)$ (called *Bers' boundary of Teichmüller space*) is naturally defined. We denote by $\partial T(\Gamma)$ the Bers boundary of $T(\Gamma)$.

On the other hand, the Bers fiber space $F(\Gamma)$ is represented as an open connected and simply connected subset of $B_2(L, \Gamma) \times \mathbb{C}$. The boundary of $F(\Gamma)$ is naturally defined as well. Let $\overline{F(\Gamma)}$ denote the closure of $F(\Gamma)$. Kra has asked if the Bers isomorphism of $F(\Gamma)$ onto $T(\dot{\Gamma}) \subset B_2(L, \dot{\Gamma})$ has a continuous extension to $\overline{F(\Gamma)}$. In §3.3 we settle this problem in the negative if U/Γ is not of type $(0,3)$; that is, we will prove Theorem 0.3 stated in the introduction.

The proof of Theorem 0.3 involves the Thurston-Bers classification for modular transformations as well as the iterates of modular transformations on a Teichmüller space.

Remark. that when U/Γ is of type $(0,3)$, then $F(\Gamma)$ is the unit disc Δ. It is well known that in this case $T(\dot{\Gamma}) = \Delta'$ is a connected and simply connected bounded subset of $B_2(L, \dot{\Gamma}) \approx \mathbb{C}$ (see §1.1 for details). It is also well known that any conformal map of Δ onto Δ' can be extended continuously if and only if Δ' has a locally connected boundary (see, for example, [24]). The problem under the consideration for type $(0,3)$ is closely related to a famous Bers' conjecture which states that Δ' is a Jordan domain.

In §3.4 we assume Γ is an arbitrary finitely generated Fuchsian group of the first kind, and continue to study the relationships between Bers fiber spaces and Teichmüller spaces.

The Bers conjecture, as we mentioned in the above remark, really says that the isomorphism (0.1) in the introduction admits a homeomorphic extension to the boundary. In §3.4 we prove Theorem 0.3′ which asserts that for any other Fuchsian group (whose type is not $(0,3)$), the answer to the *generalized Bers conjecture* is *no.*

We also study the iteration of hyperbolic modular transformation in Teichmüller space, and prove a stronger version of a theorem of Bers [10] which plays an important role in proving Theorem 0.3.

3.1 The Thurston-Bers classification of modular transformations

In [9] Bers introduced a classification for elements of the Teichmüller modular group Mod Γ for a torsion free finitely generated Fuchsian group Γ of the first kind. Let $\chi \in$ Mod Γ, and let

$$a(\chi) = \inf_{\tau \in T(\Gamma)} < \tau, \chi(\tau) >,$$

where $<,>$ is, of course, the Teichmüller distance defined in §1.1. χ is called *elliptic* if it has a fixed point in $T(\Gamma)$; *parabolic* if there is no fixed point and $a(\chi) = 0$; *hyperbolic* if $a(\chi) > 0$ and $a(\chi)$ is assumed; and *pseudo-hyperbolic* if $a(\chi) > 0$ and $a(\chi)$ is not assumed.

On the other hand, χ is induced by a self-map f of a surface S of type (g, n), $2g - 2 + n > 0$. The isotopy class of self-map f of S can be topologically classified as follows (see Thurston [47]). A (non-empty) finite set of simple curves $c = \{c_1, \ldots, c_r\}$ is called *admissible* if c_i is not homotopic to a point, a puncture, or some c_j, for $j \neq i$. f is called a *reduced map* if it keeps c invariant. f is *reducible* if it is isotopic to a reduced map. f is *irreducible* if it is not reducible. Thurston's classification theorem [47] asserts that any self-map of S is either isotopic to a periodic map, or to a reducible map, or to a irreducible map; any non-periodic irreducible map is isotopic to a pseudo-Anosov diffeomorphism (a pseudo-Anosov diffeomorphism f is a diffeomorphism satisfying $f\mathcal{F}^u = \lambda\mathcal{F}^u$ and $f\mathcal{F}^s = (1/\lambda)\mathcal{F}^s$, where $\mathcal{F}^u$ and $\mathcal{F}^s$ are a pair of transversal measured foliations, and $\lambda > 1$ is a real number. See Thurston [47] for more details). A relationship between Bers' classification for modular transformations and Thurston's classification for isotopy classes of self-maps is established by Bers [9]. The results are the following:

Theorem 3.1.1 *Let $\chi \in Mod\,\Gamma$ be induced by a self-map f of U/Γ.*

(1) An element $\chi \in Mod\,\Gamma$ is elliptic if and only if f is isotopic to a periodic map;

(2) Assume that f is not isotopic to a periodic map. Then $\chi \in Mod\,\Gamma$ is

hyperbolic if and only if f is an irreducible map.

Remark. Theorem 3.1.1 (2) implies that a reducible non-periodic self-map f corresponds to either parabolic or pseudo-hyperbolic element χ. More precisely, let f be a reducible non-periodic self-map. The corresponding system of admissible curves is denoted by $c = \{c_1, \ldots, c_r\}$. Then there exists an $n \in \mathbb{Z}^+$ such that f^n restricts to self-maps of all parts $S - \{c\}$. If f has the property that all restrictions of f^n to the parts are isotopic to a periodic map, then f must correspond to a parabolic element χ, otherwise, f corresponds to a pseudo-hyperbolic element χ.

For our application, we are mainly concerned with those modular transformations χ of $T(\dot{\Gamma})$ (or the corresponding self-maps of $U/\dot{\Gamma}$) for which there exist elements $\gamma \in \Gamma \subset \text{Aut } F(\Gamma)$ such that $\psi \circ \gamma \circ \psi^{-1} = \chi$, where $\psi \colon F(\Gamma) \to T(\dot{\Gamma})$ is the Bers isomorphism described in the beginning of this chapter.

As discussed in §1.3, the group Γ can be viewed as a Fuchsian group acting on U as well as a subgroup of holomorphic automorphisms of $F(\Gamma)$ which leaves invariant each fiber. The restriction of Γ to a fiber $\pi^{-1}([\mu])$, $[\mu] \in T(\Gamma)$, is the quasifuchsian group $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$.

An element $\gamma \in \Gamma \subset \text{PSL}(2, \mathbb{R})$ is a Möbius transformation; it is either elliptic, or parabolic, or hyperbolic. We call a hyperbolic element $\gamma \in \Gamma$ *simple* if γ is a power of an element whose axis projects to a Jordan curve on U/Γ. If γ is not simple, it is called *essential* if the axis of γ projects to a curve that intersects every admissible curve on U/Γ. γ is essential if and only if the complement in U/Γ of the projection of the axis of γ consists of a union of discs and punctured discs. (A curve satisfying this property is called a *filling curve.*) Since there are closed curves on U/Γ which are neither simple curves nor filling curves, we see that there are countably many non-simple non-essential hyperbolic elements of Γ.

Notice that $\gamma \in \Gamma$ is elliptic (resp. parabolic or hyperbolic) if and only if for all $[\mu] \in T(\Gamma)$, $\gamma^\mu \in \Gamma^\mu$ is elliptic (resp. parabolic or hyperbolic). Let Aut $F(\Gamma)$ denote the group of holomorphic automorphisms of $F(\Gamma)$. If we think of Γ as a subgroup of Aut $F(\Gamma)$ (via (1.8)), all elements of Γ can be classified as elliptic, parabolic, simple hyperbolic, essential hyperbolic, and non-simple non-essential hyperbolic transformations.

Let $\gamma \in \Gamma$. From the above discussion, Γ also defines a holomorphic automorphism of $F(\Gamma)$, and hence $\psi \circ \gamma \circ \psi^{-1}$ is a holomorphic automorphism of $T(\dot{\Gamma})$ as well. A relationship between the classification of Γ and the Bers classification of elements of $\psi \circ \Gamma \circ \psi^{-1} \subset \text{Mod } \dot{\Gamma}$ is established by a theorem of Kra [28], which is stated as follows.

Theorem 3.1.2 *(1) Assume that U/Γ is of type $(0,3)$. Then $\gamma \in \Gamma$ is parabolic (resp. hyperbolic) if and only if $\psi \circ \gamma \circ \psi^{-1} \in$ Mod $\dot{\Gamma}$ is parabolic (resp. hyperbolic). In particular, γ is parabolic if and only if $\psi \circ \gamma \circ \psi^{-1}$ is induced by a reducible map;*

(2)Assume that U/Γ is not of type $(0,3)$. Then

(a) An element $\gamma \in \Gamma$ is either parabolic or simple hyperbolic if and only if $\psi \circ \gamma \circ \psi^{-1} \in$ Mod $\dot{\Gamma}$ is parabolic;

(b) γ is essential hyperbolic if and only if $\psi \circ \gamma \circ \psi^{-1} \in$ Mod $\dot{\Gamma}$ is

hyperbolic;

(c) γ *is non-simple non-essential hyperbolic elements of* Γ *if and only if* $\psi \circ \gamma \circ \psi^{-1} \in Mod\, \dot{\Gamma}$ *is pseudo-hyperbolic.*

In particular, $\psi \circ \gamma \circ \psi^{-1}$ *is induced by a reducible map if and only if* γ *is not an essential hyperbolic element of* Γ.

For reader's convenience, we sum up all well known relations put forth so far in the following Table (E):

<table>
<tr><th>γ</th><th>χ</th><th>f</th></tr>
<tr><td>elliptic</td><td>elliptic</td><td>periodic</td></tr>
<tr><td>parabolic</td><td rowspan="2">parabolic</td><td rowspan="3">reducible</td></tr>
<tr><td>simple hyperbolic</td></tr>
<tr><td>non-simple
non-essential</td><td>pseudo-hyperbolic</td></tr>
<tr><td>essential hyperbolic</td><td>hyperbolic</td><td>irreducible (PA map)</td></tr>
</table>

In this table, γ stands for an element of $\Gamma \subset \text{Aut}\, F(\Gamma)$, $\chi = \psi \circ \gamma \circ \psi^{-1}$ is the corresponding modular transformation, and f is a self-map of $U/\dot{\Gamma}$ which induces χ.

3.2 On iterates of hyperbolic modular transformations of Teichmüller space

Let Γ be a torsion free finitely generated Fuchsian group of the first kind. Let $\Phi : T(\Gamma) \to B_2(L,\Gamma)$ denote the Bers embedding described in §1.1. To each point $\phi \in \Phi(T(\Gamma))$, we can associate a normalized univalent function W_ϕ on L which admits a quasiconformal extension to $\hat{\mathbb{C}}$ and whose Schwarzian derivative is ϕ. By Nehari-Kraus' theorem, $T(\Gamma)$ is bounded in $B_2(L,\Gamma)$. W_ϕ is always univalent not only for $\phi \in \Phi(T(\Gamma))$ but also for $\phi \in \overline{\Phi(T(\Gamma))} = \Phi(T(\Gamma)) \cup \partial T(\Gamma)$. More precisely, it was shown by Bers [6] that $\Gamma^\phi = W_\phi \Gamma W_\phi^{-1}$, $\phi \in \partial T(\Gamma)$, is a b-group (that is, a Kleinian group which has only one simply connected invariant component). The function W_ϕ induces a group homomorphism $\theta_\phi \colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ defined by the formula:

$$\theta_\phi(\gamma) \circ W_\phi = W_\phi \circ \gamma, \quad \gamma \in \Gamma.$$

θ_ϕ is an isomorphism of Γ onto its image Γ^ϕ if $\Gamma^\phi \in \overline{\Phi(T(\Gamma))}$. More precisely, Γ^ϕ is a quasifuchsian group if ϕ is in $\Phi(T(\Gamma))$. When $\phi \in \partial T(\Gamma)$, we call the group Γ^ϕ a *boundary group.* A boundary group Γ^ϕ is called *totally degenerate* if $W_\phi(L)$ is dense in $\hat{\mathbb{C}}$. A parabolic element of Γ^ϕ is *accidental* if it is of the form $\theta_\phi(\gamma)$ with $\gamma \in \Gamma$ hyperbolic. A boundary point ϕ is called a *cusp* if Γ^ϕ contains an accidental parabolic element.

Since the Teichmüller modular group Mod Γ acts as a discontinuous group of holomorphic self-maps of $T(\Gamma)$, to each $\chi \in \Gamma$ and each $n \in \mathbf{Z}^+$, the maps $\Phi \circ \chi^n \circ \Phi^{-1}|_{\Phi(T(\Gamma))}$ are bounded analytic functions. Therefore, the sequence

$\{\Phi \circ \chi^n \circ \Phi^{-1}\}_{n\in\mathbb{Z}^+}$ has a convergent subsequence $\{\Phi \circ \chi^{n_j} \circ \Phi^{-1}\}_{j\in\mathbb{Z}^+}$. In [10], Bers proved the following result:

Theorem 3.2.1 *Let $\chi \in Mod\,\Gamma$ be a hyperbolic element. Then for every convergent subsequence $\{\Phi \circ \chi^{n_j} \circ \Phi^{-1}\}_{j\in\mathbb{Z}^+}$ and every point $\tau \in \Phi(T(\Gamma))$, $\lim_{j\to\infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(\tau) \in \partial T(\Gamma)$ represents a totally degenerate b-group. Moreover, the limit point does not depend on the choice of τ.*

In his paper [10], Bers wrote the following remark: "Thurston informed me that he can prove that for a hyperbolic element χ the sequence $\{\chi^n\}$ converges. The proof is not yet published". The study of this problem requires an understanding of the image of a Teichmüller geodesic under the Bers embedding. By using an unpublished result of Gallo [22], the above problem can be settled affirmatively; the proof is rather straightforward. For completeness, however, we include a proof in this paper. First we state the result as follows.

Theorem 3.2.2 *Let $\chi \in Mod\,\Gamma$ be a hyperbolic element. Then the sequence $\{\Phi \circ \chi^n \circ \Phi^{-1}\}_{n\in\mathbb{Z}^+}$ converges to a single totally degenerate b-group for all $\tau \in \Phi(T(\Gamma))$.*

Remark. This result tells us that the behavior of iterates of hyperbolic modular transformations on a Teichmüller space is similar to those of hyperbolic Möbius transformations acting on the unit disc. It is not known, however, whether the two boundary points $\lim_{n\to\infty} \Phi \circ \chi^n \circ \Phi^{-1}(\tau)$ and $\lim_{n\to\infty} \Phi \circ \chi^{-n} \circ \Phi^{-1}(\tau)$ are distinct in the boundary of Teichmüller space.

To prove Theorem 3.2.2, We first need to review some basic facts about *quadratic differentials.* For more details about this subject, see Gardiner [23].

Let X_0 denote a Riemann surface of type (g, n) with $2g-2+n>0$. We define the Teichmüller space $T(X_0)$ of X_0 as a space of all equivalence classes of pairs (f, X), where $f\colon X_0 \to X$ is a (quasiconformal) homeomorphism and two pairs (f_1, X_1) and (f_2, X_2) are called *equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of X_1 onto X_2. The equivalence class of (f, X) is denoted by $[f, X]$. Let Γ be a torsion free finitely generated Fuchsian group of the first kind which acts on U. If $U/\Gamma = X_0$, then $T(X_0)$ is biholomorphically equivalent to $T(\Gamma)$. See Nag [39]. Let $Q(X_0)$ be the $3g-3+n$ complex dimensional vector space of holomorphic integrable quadratic differentials defined on X_0. An element of $Q(X_0)$ may have a simple pole at a puncture of X_0.

Let $x_0 \in \overline{X_0}$, and z a local parameter with $z(x_0) = z_0$. Let $\phi(z)$ be the local expression for ϕ in terms of the local coordinate chart $z\colon U_{x_0} \to \mathbb{C}$, where $U_{x_0} \subset \overline{X_0}$ is a small neighborhood of x_0. Assume that x_0 is a regular point for ϕ; that is $\phi(z_0) \neq 0$. We obtain a local parameter w by setting

$$w = \int_{z_0}^{z} \sqrt{\phi(z)}\, dz.$$

w is well-defined and is called a *natural parameter* for X_0 with respect to ϕ. A parametric curve c in X_0 is called a *horizontal* (or *vertical*) trajectory of ϕ if $\phi(z)dz^2 > 0$ (or $\phi(z)dz^2 < 0$) along c. All horizontal and vertical trajectories of ϕ produce two transverse foliations on $\overline{X_0}$ which are singular at zeros of ϕ and at punctures where ϕ has simple poles.

Let X_0, X_1 be Riemann surfaces with the same type (g,n). A quasiconformal homeomorphism $h\colon X_0 \to X_1$ is called a *Teichmüller map* if its complex dilatation μ is of the form

$$\mu = k|\phi|/\phi$$

where $0 < k < 1$ and $\phi \in Q(X_0)$. The Teichmüller theorem (cf. [45] [46]) asserts that every quasiconformal map of X_0 onto X_1 is homotopic to a unique extremal map and that a map $X_0 \to X_1$ is extremal if and only if it is either conformal or a Teichmüller map.

Given X_0, a Teichmüller deformation $h_{k,\phi} : X_0 \to X_{k,\phi}$ with respect to $\phi \in Q(X_0)$ and k, $0 \le k < 1$, is defined, see Bers [9] for a description. Here $h_{k,\phi}$ is a Teichmüller map. More precisely, if w is the natural parameter for X_0 with respect to ϕ, then $w' = w \circ h_{k,\phi}^{-1}$ is a natural parameter for $X_{k,\phi}$, and the map $h_{k,\phi}$ induces a quadratic differential $\psi_{k,\phi}$ on $X_{k,\phi}$ whose zeros and poles correspond to the zeros and poles of ϕ under the map $h_{k,\phi}$.

The *Teichmüller ray*, denoted by $r(\phi)$, is determined by a non-zero element $\phi \in Q(X_0)$. $r(\phi)$ is an arc consisting of all points $[h_{k,\phi}, X_{k,\phi}]$, $0 \le k < 1$.

Let $B^{6g-6+2n} \subset Q(X_0) \approx \mathbb{R}^{6g-6+2n}$ be the unit ball, and let $PQ(X_0)$ be the space of projective equivalence classes of non-zero elements of $Q(X_0)$. We define a map $T' : B^{6g-6+2n} \to T(X_0)$ by sending (k,ϕ), $0 \le k < 1$, $\phi \in PQ(X_0)$, to the equivalence class of $(h_{k,\phi}, X_{k,\phi})$. The Teichmüller theorem, as mentioned above, implies that T' is a homeomorphism of $B^{6g-6+2n}$ onto $T(X_0)$. The inverse map of T', which is denoted by T, is called *a Teichmüller embedding* of $T(X_0)$ into $\mathbb{R}^{6g-6+2n}$.

The proof of Theorem 3.2.2 is based on the following theorem.

Theorem 3.2.3 (Bers [9], Thurston [47]) *Let $\chi \in Mod\,\Gamma$ be a hyperbolic element. Then there exists a non-zero element $\phi \in Q(X_0)$ so that χ leaves the Teichmüller ray $r(\phi)$ invariant.*

The following result leads immediately to a proof of Theorem 3.2.2.

Theorem 3.2.4 (Gallo [22]) *Let $\phi \in PQ(X_0)$, $\phi \neq 0$. Then $\Phi(r(\phi))$ has a unique endpoint in $\partial T(\Gamma)$.*

For convenience of the reader, we sketch the proof here. Before proceeding, let us recall that a differential in $Q(X_0)$ ia called a *Jenkins-Strebel differential* if its horizontal trajectories are closed. A remarkable theorem (see [15]) asserts that the set of Jenkins-Strebel differentials in $Q(X_0)$ is a dense subset of $Q(X_0)$. We choose a torsion free Fuchsian group Γ acting on U such that $U/\Gamma = X_0$.

Proof of Theorem 3.2.4. First we choose a sequence $\{\phi_m\}$ of Jenkins-Strebel differentials on X_0 so that $\phi_m \to \phi$. Then we choose arbitrarily an endpoint $\alpha \in \partial T(\Gamma)$ of $\Phi(r(\phi))$. By definition, there is a sequence $\{k_n\}$, $k_n \to 1$ such that $[h_{k_n,\phi}, X_{k_n,\phi}] \in r(\phi)$ and $\{w^{k_n,\phi}|_L, \cdot\} \to \phi$, where $\mu_{k_n,\phi}$ is the complex dilatation of the Teichmüller deformation $h_{k_n,\phi}$ and $w^{k_n,\phi}$ is the normalized quasiconformal map with complex dilatation $\mu_{k_n,\phi}$ and is conformal in the lower half plane L.

Let $\Gamma^{k_n,\phi} = w^{k_n,\phi}\Gamma(w^{k_n,\phi})^{-1}$, and let $\alpha_{m,n} = \{w^{k_n,\phi_m}|_L, \cdot\}$. By a theorem of Masur [35], we see that for fixed m, $\lim_{n\to\infty} \alpha_{m,n}$ exists. We denote this limit by α_m. We claim the following inequality holds:

$$\| \alpha_m - \alpha_{m,n} \|_2 \leq \frac{3}{2}(1-k_n), \tag{3.1}$$

where $\| \cdot \|_2$ is the usual sup norm defined in §1.1. To see this, let $\mu_{m,n}$ be a Γ^{k_n,ϕ_m}-compatible Beltrami coefficient which is supported on $w^{k_n,\phi_m}(U)$ with $\| \mu_{m,n} \|_\infty = 1$. Define

$$\Phi_0([t\mu_{m,n}]) = \{w^{t\mu_{m,n}} \circ w^{k_n,\phi_m}|_L, \cdot\}$$

where $0 \leq t < 1$. By Corollary 2.3 of [37],

$$\| \dot{\Phi}_0(\mu_{m,n}) \|_2 = \| (d\{w^{t\mu_{m,n}} \circ w^{k_n,\phi_m}|_L, \cdot\}/dt)|_{t=0} \|_2 \leq \frac{3}{2}.$$

Let $f_{\phi_m}(k) = \{w^{k,\phi_m}|_L, \cdot\}$ for $0 < k < 1$. Then

$$\| f'_{\phi_m}(k_0) \|_2 = \| (df_{\phi_m}(k)/dk)|_{k=k_0} \|_2 =$$
$$= \| (d\{w^{t\mu_0} \circ w^{k_0,\phi_m}|_L, \cdot\}/dt)|_{t=0} \|_2 = \| \dot{\Phi}_0(\mu_0) \|_2 \leq \frac{3}{2},$$

where μ_0 denotes a Γ^{k_0,ϕ_m}-compatible Beltrami coefficient which is supported on $w^{k_0,\phi_m}(U)$ with $\| \mu_0 \|_\infty = 1$, and $k_0 \in (0,1)$ is arbitrary. Hence for a sequence $t_i \to 1$ with $0 < k_n < t_i < 1$, we have

$$\| f_{\phi_m}(t_i) - f_{\phi_m}(k_n) \|_2 \leq \frac{3}{2}(t_i - k_n).$$

Since $f_{\phi_m}(k_n) = \alpha_{m,n}$, $\lim_{i\to\infty} f_{\phi_m}(t_i) = \alpha_m$ (cf. Masur [35]), and $\lim_{n\to\infty} t_i = 1$, inequality (3.1) then follows by taking $i \to \infty$.

The next assertion is that for the sequence $\{k_n\}$ determined above, there is a subsequence $\{\phi_{m_n}\}$ of $\{\phi_m\}$ such that

$$\lim_{n\to\infty} \alpha_{m_n,n} \to \alpha. \tag{3.2}$$

To verify (3.2), let $\epsilon_n > 0$ be a sequence with $\epsilon_n \to 0$. For each fixed k_n, one may choose an element ϕ_{m_n} of $\{\phi_m\}$ such that

$$< [h_{k_n,\phi_{m_n}}, X_{k_n,\phi_{m_n}}], [h_{k_n,\phi}, X_{k_n,\phi}] > < \epsilon_n,$$

where $< \cdot, \cdot >$ is the Teichmüller metric. This implies that there is a normalized quasiconformal map w_n with bounded dilatation (whose norm is less than ϵ_n) and satisfies

$$w_n \Gamma^{k_n,\phi_{m,n}} (w_n)^{-1} = \Gamma^{k_n,\phi}.$$

Since $\epsilon_n \to 0$, there is a further subsequence (call it $\{w_n\}$ also) of $\{w_n\}$, so that $w_n \to \mathrm{id}$, as $n \to \infty$. The assertion then follows.

Now from (3.1), we obtain

$$\| \alpha_{m_n} - \alpha_{m_n,n} \|_2 \leq \frac{3}{2}(1 - k_n).$$

It follows from (3.2) that

$$\| \alpha_{m_n} - \alpha \|_2 \leq \| \alpha_{m_n} - \alpha_{m_n,n} \|_2 + \| \alpha_{m_n,n} - \alpha \|_2 \to 0, \quad \text{as} \quad n \to \infty.$$

Now suppose that α' is another endpoint of $\Phi(r(\phi))$. We may repeat the above argument to choose a subsequence $\{\alpha_{m_n}\} \in \partial T(\Gamma)$ which converges both α and α'. Thus we must have $\alpha' = \alpha$. This completes the proof of Theorem 3.2.4.

Now we are able to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. By Theorem 3.2.3, there is a $\phi \in Q(X_0)$ such that χ keeps the ray $r(\phi)$ invariant; this says $\Phi \circ \chi^n \circ \Phi^{-1}(\Phi(r(\phi))) = \Phi(r(\phi))$ for all $n \in \mathbb{Z}^+$. Suppose that there are two subsequences $\{\Phi \circ \chi^{n_j} \circ \Phi^{-1}\}_{j \in \mathbb{Z}^+}$, $\{\Phi \circ \chi^{m_j} \circ \Phi^{-1}\}_{j \in \mathbb{Z}^+}$ which are convergent. Let

$$\lim_{j \to \infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(x) = x' \in \partial T(\Gamma)$$

and

$$\lim_{j \to \infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(y) = y' \in \partial T(\Gamma),$$

where $x, y \in T(\Gamma)$. Choose a point $\tau \in \Phi(r(\phi))$. By Lemma 2 of Bers [10], we see that $\lim_{j\to\infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(x) = \lim_{j\to\infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(\tau) = x'$ and $\lim_{j\to\infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(y) = \lim_{j\to\infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(\tau) = y'$.

On the other hand, since $\tau \in \Phi(r(\phi))$, both $\lim_{j\to\infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(\tau)$ and $\lim_{j\to\infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(\tau)$ are endpoints of the ray $\Phi(r(\phi))$. By using Theorem 3.2.4, we conclude that

$$\lim_{j \to \infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(\tau) = \lim_{j \to \infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(\tau) \in \partial T(\Gamma).$$

It follows that $x' = y'$. By using Lemma 2 of Bers [10] once again, we conclude that

$$\lim_{j \to \infty} \Phi \circ \chi^{n_j} \circ \Phi^{-1}(T(\Gamma)) = \lim_{j \to \infty} \Phi \circ \chi^{m_j} \circ \Phi^{-1}(T(\Gamma))$$

is the endpoint of $\Phi(r(\phi))$. Since $\{\Phi \circ \chi^{n_j} \circ \Phi^{-1}\}_{j \in \mathbb{Z}^+}$ and $\{\Phi \circ \chi^{m_j} \circ \Phi^{-1}\}_{j \in \mathbb{Z}^+}$ are arbitrary two subsequences of $\{\Phi \circ \chi^{n} \circ \Phi^{-1}\}_{j \in \mathbb{Z}^+}$, Theorem 3.2.2 then follows. □

3.3 Proof of Theorem 0.3

Let Γ be a torsion free finitely generated Fuchsian group of the first kind. We need two lemmas.

Lemma 3.3.1 *The set of fixed points of essential hyperbolic elements of* Γ *are dense in* $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Proof. First observe that there are filling curves on U/Γ. For example, we can choose a set of generators in the fundamental group $\pi_1(U/\Gamma, x_0)$, a curve containing all these generators is a filling curve. Figure 16 below shows a filling curve c on a surface of type $(2,1)$. (See also Figure 3 in Kra [28].)

Since there is a closed geodesic in the homotopy class of a filling curve on U/Γ, this geodesic corresponds to an essential hyperbolic element. So there always exists a essential hyperbolic element γ in Γ.

It is well known that for any point $x \in \hat{\mathbb{R}}$, the orbit $\Gamma(x) = \{\gamma(x); \gamma \in \Gamma\}$ is dense in $\hat{\mathbb{R}}$. In particular, the orbit of the fixed point of an essential hyperbolic element γ is dense in $\hat{\mathbb{R}}$; these are fixed points of conjugates of γ, which are again essential hyperbolic elements. $\square$

Let $\psi\colon F(\Gamma) \to T(\dot{\Gamma})$ be the Bers isomorphism constructed in the beginning of this chapter. Since $T(\dot{\Gamma})$ can be identified with its image under the Bers embedding $\Phi : T(\dot{\Gamma}) \mapsto B_2(L, \dot{\Gamma})$, the composition $\Phi \circ \psi$, which we still call the Bers isomorphism and still denote by ψ, is a biholomorphic map of $F(\Gamma)$ onto its image in $B_2(L, \dot{\Gamma})$.

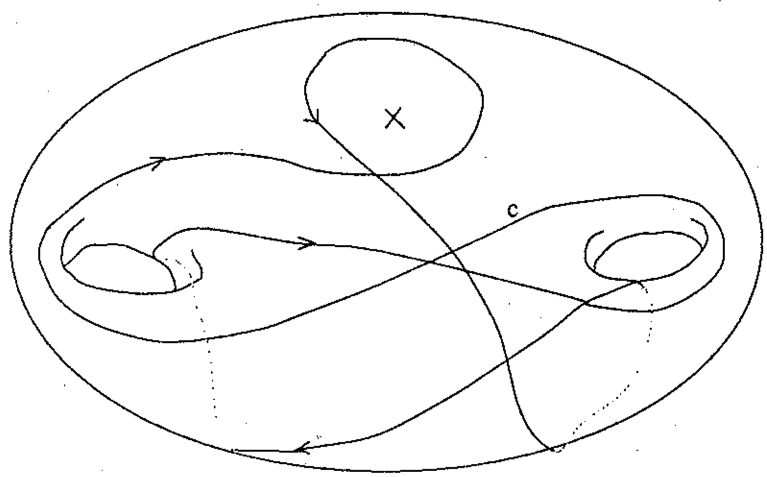

Figure 16.

By Theorem 10 of Bers [8], mod Γ is isomorphic to a subgroup (via the map ψ) of the Teichmüller modular group Mod $\dot{\Gamma}$ = mod $\dot{\Gamma}$ /$\dot{\Gamma}$ with finite index $n+1$, where n is the number of the punctures on U/Γ. More precisely, the elements of the image of mod Γ are induced by those quasiconformal homeomorphisms which fix one special puncture of $U/\dot{\Gamma}$. As we discussed in §1.3 and §3.1, the group Γ is also regarded as a subgroup of the Teichmüller modular group Mod $\dot{\Gamma}$.

Lemma 3.3.2 *Suppose that dim* $T(\Gamma) \geq 1$. *There is no continuous injective map* $\tilde{\psi}$ *of* $\overline{F(\Gamma)}$ *into* $T(\dot{\Gamma}) \cup \partial T(\dot{\Gamma})$ *extending* ψ.

Proof. Let $\gamma \in \Gamma$ be an essential hyperbolic element. Under the inclusion $\Gamma \subset$ mod $\Gamma \hookrightarrow$ Mod $\dot{\Gamma}$ described above, $\psi \circ \gamma \circ \psi^{-1} = \chi_\gamma$ is an element of Mod $\dot{\Gamma}$. By Theorem 3.1.2, χ_γ is a hyperbolic modular transformation in the sense of Bers (Theorem 3.1.1).

Choose two points $([\mu_1], z_1)$, $([\mu_2], z_2)$ in $F(\Gamma)$ lying in different fibers; that is, $w^{\mu_1} \neq w^{\mu_2}$ on $\hat{\mathbb{R}}$. Let us consider the two sequences $\{\gamma^n([\mu_i], z_i)\}$, $i = 1, 2$. Observe that the action of γ^n, $n = 1, 2, \ldots$, on $F(\Gamma)$ is defined in a quite natural way; that is,

$$\gamma^n([\mu_i], z_i) = ([\mu_i], (\gamma^{\mu_i})^n(z_i)) =$$

$$= ([\mu_i], w^{\mu_i} \circ \gamma^n \circ (w^{\mu_i})^{-1}(z_i)),$$

for $i = 1, 2$. It follows that the action of γ^n keeps both fibers $([\mu_1], w^{\mu_1}(U))$ and $([\mu_2], w^{\mu_2}(U))$ invariant. Observe also that on the fiber over $[\mu]$, γ^n acts as a hyperbolic Möbius transformation γ^μ in the quasifuchsian group $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$. Therefore, the sequence $\{(\gamma^{\mu_1})^n(z_1)\}$ must converge to the attractive fixed point of γ^{μ_1}, say z_1', lying in the quasicircle $w^{\mu_1}(\hat{\mathbb{R}})$. Similarly, the sequence $\{(\gamma^{\mu_2})^n(z_2)\}$ converges to the attractive fixed point z_2' of γ^{μ_2} lying in $w^{\mu_2}(\hat{\mathbb{R}})$. Since $\{\gamma^n([\mu_1], z_1)\}$ and $\{\gamma^n([\mu_2], z_2)\}$ lie in two different fibers, these two sequences converge to two different limit points $([\mu_1], z_1')$ and $([\mu_2], z_2')$. (Note that z_1' and z_2' may coincide.)

For $i = 1, 2$, let $[\nu_i]$ denote the ψ-image of $([\mu_i], z_i)$ in $T(\dot{\Gamma})$. We consider the sequence $\{\chi_\gamma^n([\nu_i])\}$. For any $n \geq 1$, we have

$$\chi_\gamma^n([\nu_i]) = \psi \circ \gamma^n \circ \psi^{-1}([\nu_i]) = \psi(\gamma^n([\mu_i], z_i)).$$

This implies that $\{\chi_\gamma^n([\nu_i])\}$ is the ψ-image of $\{\gamma^n([\mu_i], z_i)\}$. Selecting if need be a subsequence, we may assume that both sequences, $\{\chi_\gamma^n([\nu_i])\} = \psi(\gamma^n([\mu_i], z_i))$, $i = 1, 2$, are convergent. By Theorem 3.2.1, both sequences converge to the same boundary point which represents a totally degenerate b-group Γ' isomorphic to Γ. If ψ can be extended continuously and injectively to $\overline{F(\Gamma)}$, then

$$\tilde{\psi}([\mu_1], z_1') \neq \tilde{\psi}([\mu_2], z_2').$$

This is a contradiction. Hence, the lemma is proved. $\square$

Remark. By using Lemma 3.3.2, we can easily solve the problem related to the inverse of ψ; namely, we claim that there is no continuous extension of ψ^{-1} to the closure $T(\dot{\Gamma}) \cup \partial T(\dot{\Gamma})$. Indeed, as we saw before, for $i = 1, 2$, the sequence $\{\chi_\gamma^n([\nu_i])\}$ is the ψ-image of $\{\gamma^n([\mu_i], z_i)\}$. Suppose that $\tilde{\psi}^{-1}$ is continuous, since (choose a subsequence if necessary)

$$\lim_{n\to\infty} \chi_\gamma^n([\nu_1]) = \lim_{n\to\infty} \chi_\gamma^n([\nu_2]) = \phi',$$

where ϕ' corresponds to a totally degenerate b-group $\Gamma' = W_{\phi'}\Gamma W_{\phi'}^{-1}$, we must have

$$\lim_{n\to\infty} \gamma^n([\mu_1], z_1) = \lim_{n\to\infty} \gamma^n([\mu_2], z_2).$$

Thus,

$$([\mu_1], z_1') = ([\mu_2], z_2'),$$

but this is a contradiction; proving our assertion. To obtain the same conclusion for the isomorphism ψ, we must do some further work.

Proof of Theorem 0.3. First, we prove the theorem under the assumption that $\dim T(\Gamma) \geq 2$ and that $\psi\colon F(\Gamma) \to T(\dot{\Gamma})$ is the Bers isomorphism. Suppose that there is a continuous extension $\tilde{\psi}$ of ψ to the closure $\overline{F(\Gamma)}$. Let $\alpha \in \Gamma$ be a simple hyperbolic Möbius transformation; that is, the projection of $Axis(\alpha)$

under the projection $p: U \to U/\Gamma$ is a simple closed geodesic on $S = U/\Gamma$. By Theorem 3.1.2, as an element of Mod $\dot{\Gamma}$, $\psi \circ \alpha \circ \psi^{-1} = \chi_\alpha \in$ Mod $\dot{\Gamma}$ is a parabolic modular transformation in the sense of Bers.

To proceed, we need to investigate more carefully the action of the parabolic modular transformations χ_α which are determined by simple hyperbolic transformations α of Γ. We invoke Theorem 2 of Nag [38], which says that the self-map f_α which induces χ_α is isotopic to a spin map about $\hat{a}$, where $\hat{a}$ is the projection of a (defined in the beginning of Chapter 3) under $p: U \to U/\Gamma$. (For the definition of a spin map, see §2.4, [13], [28] and [38].) This means that the system of admissible curves defined by f_α is $c = \{c_1, c_2\}$, where c_1 and c_2 bound a cylinder A containing the puncture $\hat{a}$ and no other punctures. Further, since α is hyperbolic, neither c_1 nor c_2 bounds a punctured disk.

On the other hand, we know that the number of the curves in a maximal system for $U/\dot{\Gamma}$ is $3g - 2 + n$ and that χ_α is reduced by a system with 2 simple closed curves $c = \{c_1, c_2\}$. Thus, c is not of maximal system unless dim $T(G) = 0$ or 1; that is, unless $(g, n) = (0,3)$, $(0,4)$, or $(1,1)$.

For any $x \in T(\dot{\Gamma})$, let us consider the set $Ac(\chi_\alpha, x)$ of accumulation points of $\{\chi_\alpha^n(x)\}$. By Theorem 3 of Abikoff [2], $Ac(\chi_\alpha, x)$ consists of those quadratic differentials ϕ in $B_2(L, \dot{\Gamma})$ for which $W_\phi \dot{\Gamma} W_\phi^{-1}$ are regular b-groups. (Recall that W_ϕ is a normalized univalent function on L which admits a quasiconformal extension to $\mathbb{C}$ and whose Schwarzian derivative is ϕ.)

Fix $x \in T(\dot{\Gamma})$, by passing a subsequence if necessary, we assume that $\{\chi_\alpha^n(x)\}$ converges. This implies that $Ac(\chi_\alpha, x)$ consists of only one point, which corresponds to a regular b-group $B \in \partial T(\dot{\Gamma})$. Topologically, the upper part $(\Omega(B) - \Delta(B))/B$ of B (where $\Omega(B)$ is the set of points at which B acts discontinuously, and $\Delta(B)$ is the simply connected invariant domain of B) is obtained by squeezing the curves c on $U/\dot{\Gamma}$. See Theorem 5 of Maskit [33].

By the previous argument, we see that $U/\dot{\Gamma} - \{c_1, c_2\}$ consists of two components if c_1 (and c_2) is a non-dividing curve, and three components if c_1 (and c_2) is a dividing curve. Since $\{c_1, c_2\}$ is not maximal, we know that at least one component is not a pair of pants. Therefore, we can change the conformal structure on $\hat{S}_1 + \cdots + \hat{S}_m$, $m = 2$ or 3, where $\hat{S}_i$ are obtained from S_i by capping the punctured discs on the boundary curves. Fix a conformal structure on $\hat{S}_1 + \cdots + \hat{S}_m$. From Theorem 6 of Maskit [33], we conclude that there is a regular b-group B such that $(\Omega(B) - \Delta(B))/B = \hat{S}_1 + \cdots + \hat{S}_m$, and the corresponding quadratic differential lies in the boundary of $T(\dot{\Gamma})$. Different conformal structures on $\hat{S}_1 + \cdots + \hat{S}_m$ will produce different regular b-groups. Let B, B_0 be two distinct regular b-groups defined in this way, and let ϕ, $\phi_0 \in B_2(L, \dot{\Gamma})$ be the quadratic differentials corresponding to B and B_0, respectively; that is, $B = W_\phi \dot{\Gamma} (W_\phi)^{-1}$ and $B_0 = W_{\phi_0} \dot{\Gamma} (W_{\phi_0})^{-1}$.

By selecting a further subsequence, we assume that the sequence $\{\chi_\alpha^n\}$ of bounded analytic maps converges. Theorem 3 of Abikoff [2] then asserts that $\{\chi_\alpha^n\}$ converges to a limiting holomorphic map of $T(\dot{\Gamma})$ to $\partial T(\dot{\Gamma})$ which is a surjection of $T(\dot{\Gamma})$ onto the boundary Teichmüller space representing the corresponding congruence class (for the definition, see Abikoff [2]). This implies that there are points x, $y \in T(\dot{\Gamma})$ such that $\{\chi_\alpha^n(x)\}$ converges to ϕ, and

$\{\chi_\alpha^n(y)\}$ converges to ϕ_0.

Let $([\mu_1], z_1)$ and $([\mu_2], z_2) \in F(\Gamma)$ denote the preimages of x and y under $\psi : F(\Gamma) \to T(\dot{\Gamma})$, respectively. There are two cases.

Case 1. μ_1 is not equivalent to μ_2; that is, $([\mu_i], z_i)$, $i = 1, 2$, lie in different fibers. Consider the sequences $\{\alpha^n([\mu_i], z_i)\}$. These sequences are the preimages of $\{(\chi_\alpha)^n(x)\}$ and $\{(\chi_\alpha)^n(y)\}$. By using the same proof as in Lemma 3.3.2, we conclude that the limit points z_i' of $\{\alpha^n([\mu_i], z_i)\}$, $i = 1, 2$, lie in the boundaries of different fibers, $([\mu_i], w^{\mu_i}(\hat{\mathbb{R}}))$, and the images of $([\mu_1], z_1')$ and $([\mu_2], z_2')$ under $\tilde{\psi}$ is exactly ϕ and ϕ_0 described above, where $([\mu_i], z_i') = \lim_{n\to\infty} \gamma^n([\mu_i], z_i)$.

By Lemma 3.3.1, we can choose a sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points of essential hyperbolic Möbius transformations in Γ such that $\{u_n\}_{n=1}^{\infty}$ converges to a fixed point z' of $\alpha \in \Gamma$. But $w^{\mu_i}(z')$ is a fixed point of $\alpha^{\mu_i} \in \Gamma^{\mu_i}$, which is equal to z_i'. It follows that $(w^{\mu_1})^{-1}(z_1') = (w^{\mu_2})^{-1}(z_2')$. Let θ_n, $n = 1, 2, \ldots$, denote the ~~corresponding~~ essential hyperbolic elements of Γ corresponding to the fixed point u_n. Since w^{μ_i}, $i = 1, 2$, are global homeomorphisms, the sequences $\{u_{i,n}\}$ of the fixed points of $\{\theta_n^{\mu_i}\}$ also converge to z_i'. For $i = 1, 2$, choose $y_i \in F(\Gamma)$ so that y_i lie in the fibers $([\mu_i], w^{\mu_i}(U))$, respectively. Since $\Gamma \subset \operatorname{mod} \Gamma$ is a normal subgroup which leaves each fiber invariant, and since $u_{i,n}$ is a fixed point of $\theta_n^{\mu_i}$, fix n, the sequence $\{\theta_n^m(y_i)\}_{m=1}^{\infty}$ converges to $u_{i,n}$. (If $u_{i,n}$ is the repulsive fixed point, then we replace m by $-m$, the above argument still works.) We denote by x_i the ψ-image of y_i in $T(\dot{\Gamma})$ for $i = 1, 2$. The sequences $\{\theta_n^m(y_i)\}_{m=1}^{\infty}$ are mapped via ψ to the sequences $\{\chi_{\theta_n}^m(x_i)\}_{m=1}^{\infty}$.

By selecting a subsequence if necessary, we may assume that the two sequences $\{\chi_{\theta_n}^m(x_i)\}_{m=1}^{\infty}$, $i = 1, 2$, converge for every $n \in \mathbf{Z}^+$. By using the same proof as in Lemma 3.3.2, we conclude that for $i = 1, 2$ and a fixed n, the two sequences $\{\chi_{\theta_n}^m(x_i)\}$ converge to a single point ϕ_n. Let $\Gamma_n = W_{\phi_n}\dot{\Gamma}(W_{\phi_n})^{-1}$. Then, by the Theorem of Bers [10], all Γ_n are totally degenerate b-groups in $\partial T(\dot{\Gamma})$ isomorphic to $\dot{\Gamma}$. If the continuous extension $\tilde{\psi}$ of ψ exists, then we must have

$$\tilde{\psi}(u_{1,n}) = \tilde{\psi}(u_{2,n}) = \phi_n.$$

Since $\{u_{i,n}\}$, $i = 1, 2$, converge to z_i', and since $\tilde{\psi}(z_1') = \phi, \tilde{\psi}(z_2') = \phi_0$, the sequence $\{\phi_n\}$ must converge to both ϕ and ϕ_0. This is clearly impossible.

Case 2. μ_1 is equivalent to μ_2. In this case y_i, $i = 1, 2$, lie in the same fiber. This means that the sequence $\{\theta_n^m(y_i)\}$ converges to $u_n \in \partial w^{\mu_1}(\hat{\mathbf{R}})$ (n is fixed). It follows that $\tilde{\psi}(u_n)$ is the limit ϕ_n of the sequence $\{\chi_{\theta_n}^m(\psi(y_i))\}$. Since $\{u_n\}$ converges to $z_1' = z_2'$, $\tilde{\varphi}(u_n) = \phi_n$ converges to ϕ. Similarly, $\tilde{\psi}(u_n) = \phi_n$ also converges to ϕ_0. This is impossible.

Next, we deal with the case of dim $T(\Gamma) = 1$; that is, U/Γ is of type (0,4) or (1,1). This means that $\dot{\Gamma}$ is of type (0,5) or (1,2). Choose a spin map $s_p = h_{c_2} \circ h_{c_1}^{-1}$ about the puncture $\hat{a}$ (recall that $\hat{a}$ is the projection of a under $p\colon U \to U/\Gamma$), where h_{c_i}, $i = 1, 2$, is the Dehn twist about a simple closed curve c_i, and c_1 bounds a punctured disk, see Figure 17. (More details about Dehn twists are discussed in §2.2.) In this case, the spin map s_p defined on $U/\dot{\Gamma}$ is isotopic to the Dehn twist about c_2. (The Dehn twist about c_1 is isotopic to the identity.) Let $\chi \in$ Mod $\dot{\Gamma}$ be the (parabolic) modular transformation induced by s_p. By selecting a subsequence if necessary, we see that $\{\chi^n(x)\}$, $x \in T(\dot{\Gamma})$, converges to a quadratic differential ϕ' corresponding to a regular b-group B'. By Theorem 5 of Maskit [33], B' can be obtained topologically by squeezing the curve c_2 to a point. This gives $(\Omega(B') - \Delta(B'))/B' = \hat{S}_1' + \hat{S}_2'$, where $\hat{S}_1'$ is a thrice punctured sphere, $\hat{S}_2'$ is a 4-times punctured sphere if $U/\dot{\Gamma}$ is of type (0,5) and is a punctured torus if $U/\dot{\Gamma}$ is of type (1,2). In both cases, $\hat{S}_2'$ has moduli. Thus, we can change the conformal structure on $\hat{S}_1' + \hat{S}_2'$.

On the other hand, since χ is induced by s_p, and s_p fixes $\hat{a}$, by Theorem 10 of Bers [8], $\psi^{-1} \circ \chi \circ \psi \in \text{mod}\,\Gamma$. Note that the following diagram is commutative:

$$\begin{array}{ccc} T(\dot{\Gamma}) & \xrightarrow{\chi} & T(\dot{\Gamma}) \\ {\scriptstyle\dot{\pi}}\downarrow & & \downarrow{\scriptstyle\dot{\pi}} \\ T(\Gamma) & \xrightarrow[id]{} & T(\Gamma) \end{array}$$

where $\dot{\pi} = \pi \circ \psi^{-1}$, and $\pi : F(\Gamma) \to T(\Gamma)$ is the natural projection. We conclude that $\psi^{-1} \circ \chi \circ \psi = \alpha \in \Gamma$. It is easy to see that α is a parabolic element of Γ. The previous argument works equally well in this case. The details are omitted.

For general situation, suppose that $\psi' \colon F(\Gamma) \to T(\dot{\Gamma})$ is a biholomorphic map which can be extended continuously to the boundary. Then $\psi' \circ \psi^{-1}$ is a holomorphic automorphism of $T(\dot{\Gamma})$. From Theorem 1.2.3, $\psi' \circ \psi^{-1} \in \text{Mod}\,\dot{\Gamma}$. Let $\psi' \circ \psi^{-1}$ be induced by a self-map f of $U/\dot{\Gamma}$. By Theorem 3.1.2, an essential hyperbolic element γ of Γ determines a hyperbolic modular transformation

$\psi \circ \gamma \circ \psi^{-1}$ which is, of course, induced by a irreducible self-map f_0 on $U/\dot{\Gamma}$ (Theorem 3.1.1). f_0 is irreducible if and only if $f \circ f_0 \circ f^{-1}$ is irreducible. It follows that $\psi' \circ \gamma \circ \psi'^{-1} \in \text{Mod}\, \dot{\Gamma}$ is hyperbolic. Similarly, the self-map s_p of $U/\dot{\Gamma}$ is a spin map about $\hat{a}$ (that is, $s_p = h_{c_2} \circ h_{c_1}^{-1}$, where h_{c_i} is the Dehn twist about c_i, c_1 and c_2 bound a cylinder which contains the only puncture $\hat{a}$) if and only if $f \circ s_p \circ f^{-1}$ is a spin map about $f(\hat{a})$. More precisely, we see that $f \circ s_p \circ f^{-1}$ is isotopic to $h_{f(c_2)} \circ h_{f(c_1)}^{-1}$. Furthermore, c_1 bounds a punctured disk if and only if $f(c_1)$ bounds a punctured disk. Hence, the argument above carries over word by word for this general case. This completes the proof of Theorem 0.3. $\square$

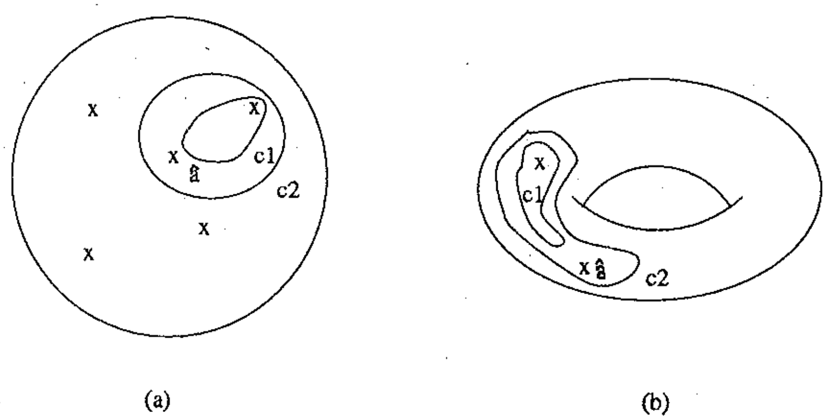

Figure 17.

As an application of Theorem 0.3, we observe that for a torsion free group Γ, the Teichmüller space $T(\Gamma)$ can be embedded into $\mathbf{R}^{6g-6+2n}$ via the Teichmüller embedding, as we discussed in §3.2. Therefore, we obtain an natural enbedding $\mathcal{T}: F(\Gamma) \hookrightarrow \mathbb{R}^{6g-6+2n} \times \mathbb{C} \approx \mathbb{R}^{6g-4+2n}$. Let us denote by $\mathcal{B}^{6g-4+2n}$ the image of $F(\Gamma)$ under $\mathcal{T}$. On the other hand, by means of the Bers isomorphism $\psi: F(\Gamma) \to T(\dot{\Gamma})$, we can define the Teichmüller embedding of $F(\Gamma)$ into $\mathbb{R}^{6g-4+2n}$ by the composition $T \circ \psi$, where $T : T(\dot{\Gamma}) \hookrightarrow \mathbb{R}^{6g-4+2n}$ is the Teichmüller embedding.

Theorem 3.3.3 *The homeomorphism $\psi_0 = T \circ \psi \circ \mathcal{T}^{-1}$ admits no homeomorphic extension to the boundary of $\mathcal{B}^{6g-4+2n}$.*

Proof. Consider the following commutative diagram of homeomorphisms:

$$\begin{array}{ccc} \mathcal{B}^{6g-4+2n} \subset \mathbb{R}^{6g-6+2n} \times \mathbb{C} & \xrightarrow{\psi_0} & \mathbb{B}^{6g-4+2n} \subset \mathbb{R}^{6g-4+2n} \\ \mathcal{T}\uparrow & & \uparrow T \\ F(\Gamma) & \xrightarrow[\psi]{} & T(\dot{\Gamma}) \end{array}$$

Again, let $\Phi : T(\dot{\Gamma}) \hookrightarrow B_2(L, \dot{\Gamma})$ denote the Bers embedding. By a theorem of Gallo [22], the homeomorphism $\Phi \circ T^{-1} : \mathbb{B}^{6g-4+2n} \hookrightarrow B_2(L, \dot{\Gamma})$ extends continuously to the boundary of $\mathbb{B}^{6g-4+2n}$. Suppose that ψ_0 admits a homeomorphic extension, then the above statement implies that $\Phi \circ T^{-1} \circ \psi_0$ admits a continuous extension. But $\Phi \circ T^{-1} \circ \psi_0 = \Phi \circ \psi \circ \mathcal{T}^{-1}$. We conclude that $\Phi \circ \psi \circ \mathcal{T}^{-1}$ admits a continuous extension to the boundary of $\mathcal{B}^{6g-4+2n}$. On the other hand, by Theorem 0.3, we assert that $\Phi \circ \psi \circ \mathcal{T}^{-1}$ admits no continuous extension to the boundary. This is a contradiction. The proof of Theorem 3.3.3 is complete. □

3.4 Proof of Theorem 0.3'

In this section, we prove another non extendibility theorem in more general cases. The methods used to prove Theorem 0.3 essentially rely on Theorem 3.1.2. By exploring Kra's proof, we find that his argument needs information from the Bers isomorphisms; that is, the assumption that Γ is torsion free is crucial. In order to prove Theorem 0.3', we must develop new methods since our objects here are arbitrary isomorphisms rather than the Bers isomorphisms.

We remark that the methods introduced in this section do not work in general torsion free cases. The reason is the following:

(1) U/Γ may be compact, or

(2) even if U/Γ is not compact, it is difficult (or even impossible in some cases) to construct periodic self-maps which fix a puncture on U/Γ.

We see that the main parts of the proofs of these two non-extendibility theorems are independent, although Theorem 0.3' includes Theorem 0.3 as a special case.

Also, Theorem 0.2 is a significant step in proving Theorem 0.3' since after the work of Earle and Kra (Theorem 0.1), some cases cannot be handled by only using the methods introduced below. For example, when Γ is of signature $(0,8;2,\ldots,2)$, $(0,7;2,\ldots,2)$ or $(0,5;2,\ldots,2)$), our methods do not work since there are no punctures on U/Γ. However, it is shown by Theorem 0.2 that the Bers fiber spaces of the groups with these signatures are not isomorphic to any Teichmüller spaces.

Let Γ, Γ' be finitely generated Fuchsian groups of the first kind. Assume that Γ is not of type $(0,3)$. Note that the group Γ' is always assumed to be torsion free. We need several lemmas.

Lemma 3.4.1 *Let $\chi' \in Mod\,\Gamma'$, and let α be a positive integer. Then that χ'^{α} is parabolic implies that χ' is parabolic.*

Proof. This lemma is a special case of Proposition 2 of Kra [28]. The proof is included for completeness. According to Bers [9], every element in Mod Γ' is either elliptic, or parabolic, or pseudo-hyperbolic, or hyperbolic. If χ' is elliptic, then so is χ'^{α} because a fixed point of χ' is a fixed point of χ'^{α}. If χ' is hyperbolic, then by Theorem 5 of Bers [9], χ' has an invariant line which is also an invariant line of χ'^{α}. By using Theorem 5 of [9] again, we see that χ'^{α} is hyperbolic. Finally, if χ' is pseudo-hyperbolic, then it is reducible and at least one of the restrictions is hyperbolic. This implies that χ'^{α} is pseudo-hyperbolic. The lemma is then proved. □

Now we assume that χ' and χ'^{α} are elements of Mod Γ'. Assume also that χ'^{α} is parabolic. By Theorem 4 and Theorem 7 of Bers [9], χ'^{α} is induced by a reducible self-map f'_{α} of U/Γ'. Let $C' = \{c'_1, \ldots, c'_r\}$ be the corresponding (non empty) system of admissible curves which is reduced by f'_{α}. By Lemma 5 of [9], we may also assume that f'_{α} is completely reduced by $C' = \{c'_1, \ldots, c'_r\}$. Now Lemma 3.4.1 tells us that $\chi' \in$ Mod Γ' is also parabolic. By using Theorem 4 and Theorem 7 of [9] once again, χ' is induced by another reducible self-map f' of U/Γ'. f' is completely reduced by a system $D' = \{d'_1, \ldots, d'_s\}$.

Lemma 3.4.2 *$r{=}s$ and for every $i = 1, 2, \ldots, r$, d'_i is freely homotopic to a c'_j for some $j = 1, 2, \ldots, s$.*

Proof. Without loss of generality, we may assume that all d'_i and all c'_j are closed simple geodesics on U/Γ'. Also, by taking a suitable power of both f' and f'_α, we assume that f'_α is of the form $h^{\alpha_1}_{c'_1} \circ \cdots \circ h^{\alpha_r}_{c'_r}$ and f' is of the form $h^{\beta_1}_{d'_1} \circ \cdots \circ h^{\beta_s}_{d'_s}$, where α_i, β_j are integers, and $h_{c'}$ is the Dehn twist about c' which is the identity outside an arbitrarily small tubular neighborhood $N(c') \subset U/\Gamma'$ of c'.

Suppose that for some i, d'_i intersects with c'_j for a j. Let τ' denote the point in $T(\Gamma')$ which represents the Riemann surface U/Γ'. We may assume, if need be a subsequence, that $\{\chi'^{\alpha n}(\tau')\}_{n=1}^{\infty}$ converges. The limit point (in $\partial T(\Gamma')$) represents a regular b-group. By Theorem 5 of Maskit [33], this regular b-group can be obtained by squeezing the geodesics $c'_1, \ldots, c'_r$. This implies that $l[f'^{\alpha n}(c'_j)] \to 0$ for $j = 1, \ldots, r$ (where $l[c]$ is the length of the geodesic freely homotopic to c). By hypothesis, d'_i intersects with c'_j. Keen's collar lemma (see also Lemma 2 of Bers [9]) shows that $l[f'^{\alpha n}(d'_i)] \to \infty$.

But on the other hand, $\{\chi'^{\alpha n}(\tau')\}_{n=1}^{\infty}$ is also a subsequence of $\{\chi'^{n}\}_{n=1}^{\infty}$. By the same argument as above, we have $l[f'^{\alpha n}(d'_i)] \to 0$. This is a contradiction.

We see that d'_i either coincides with c'_j for some j, or is disjoint from all c'_j. If $d'_i \neq c'_j$ for all $j = 1, \ldots, r$, then χ'^{α} is reduced by $\{c'_1, \ldots, c'_r, d'_i\}$ this contradicts to the fact that χ'^{α} is completely reduced by $C' = \{c'_1, \ldots, c'_r\}$. It follows that the two systems $C' = \{c'_1, \ldots, c'_r\}$ and $D' = \{d'_1, \ldots, d'_s\}$ are coincident (up to free homotopy). This proves the lemma. □

Lemma 3.4.3 *Let the signature $(g, n; \nu_1, \ldots, \nu_n)$ be among the entires of Table (C'). Then there is a non-trivial automorphism χ of $F(\Gamma)$ so that χ^2 is a parabolic element of Γ (again, Γ is viewed as a normal subgroup of mod Γ).*

Proof. We need to examine all cases exhibited in Table (C').

(1) $(g, n; \nu_1, \ldots, \nu_n) = (0, 8; 2, 2, 2, \infty, \ldots, \infty)$ or $(0, 8; 2, \ldots, 2, \infty, \infty)$. In these two cases, we refer to Figure 10, where x_2, x_3, x_6 are set to be branch points of order 2, x_1, x_4, x_5, x_7 are set to be punctures if $(g, n; \nu_1, \ldots, \nu_n) = (0, 8; 2, 2, 2, \infty, \ldots, \infty)$; and x_1, x_2 are set to be punctures, $x_3, \ldots, x_8$ are set to be branch points of order 2 if $(g, n; \nu_1, \ldots, \nu_n) = (0, 8; 2, \ldots, 2, \infty, \infty)$.

Let $\gamma \in \Gamma$ be a parabolic transformation corresponding to the puncture x_1, and let A be the rotation around z-axis with rotation angle $2\pi/6$. Then A fixes x_1 and x_2. Set $f = A^3$. It is easy to see that f is a self-map of U/Γ in the sense of orbifolds (in both cases). Since f fixes the puncture x_1, we may choose a lift $\hat{f}$ of f so that $\hat{f}$ fixes the fixed point of γ. We obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\hat{f}} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ U/\Gamma & \xrightarrow[f]{} & U/\Gamma \end{array}$$

Thus $\hat{f}$ induces an element $\chi \in \text{mod } \Gamma$. It is easy to see that $q_0(\chi) \in \text{Mod } \Gamma$ is induced by the map f (where $q_0 : \text{mod } \Gamma \to \text{Mod } \Gamma$ is the usual quotient homomorphism). Since $f^2 = A^6$ is the identity, $q_0(\chi^2) \in \text{Mod } \Gamma$ is the identity as well. This in turn implies that the non-trivial element χ^2 leaves invariant each fiber. As a matter of fact, χ^2 lies in the kernel of the homomorphism $q_0\colon \mathrm{mod}\,\Gamma \to \mathrm{Mod}\,\Gamma$. Hence, $\chi^2 \in \Gamma$. Since $\hat{f}$ fixes the fixed point of γ, we see that $\chi^2 = \gamma \in \Gamma$ is a parabolic Möbius transformation. Note that χ is not an element of Γ.

(2) If $(g, n; \nu_1, \ldots, \nu_n) = (0, 7; 2, 2, \infty, \ldots, \infty)$, then Figure 11 is referred, where x_1, x_3, x_4, x_6, x_7 are punctures, and x_2, x_5 are branch points of order 2. The rotation A is made by fixing x_1 and a regular point. Set $f = A^3$. More importantly, f is a self-map of U/Γ in the sense of orbifolds.

If $(g, n; \nu_1, \ldots, \nu_n) = (0, 7; 2, 2, 2, 2, \infty, \infty, \infty)$, we still look at Figure 11, but this time x_1, x_2, x_5 are punctures, and x_3, x_4, x_6, x_7 are branch points of order 2. A and f are the same as above. We see that f is a self-map of U/Γ in the sense of orbifolds.

If $(g, n; \nu_1, \ldots, \nu_n) = (0, 7; 2, \ldots, 2, \infty)$, Figure 11 is referred once again, where x_1 is a puncture, and $x_2, \ldots, x_7$ are branch points of order 2. A and f are the same as above. Obviously, f is a self-map of U/Γ in the sense of orbifolds.

The rest of the argument of Case 2 remains the same as in Case 1, details are omitted.

(3) If $(g, n; \nu_1, \ldots, \nu_n) = (0, 6; 2, \infty, \ldots, \infty)$, then Figure 12 is a reference picture, where $x_1, x_3, \ldots, x_6$ are punctures and x_2 is a branch point of order 2. A is a rotation with angle $2\pi/4$ and is made by fixing x_1 and x_2. Set $f = A^2$. We see that f is a self-map of U/Γ in the sense of orbifolds.

If $(g, n; \nu_1, \ldots, \nu_n) = (0, 6; 2, 2, \infty, \ldots, \infty)$, then we also refer to Figure

12, where x_1, x_2, x_4, x_6 are punctures, and x_3, x_5 are branch points of order 2. A and f are the same rotations as above. f is clearly a self-map of U/Γ in the sense of orbifolds.

If $(g, n; \nu_1, \ldots, \nu_n) = (0, 6; 2, 2, 2, \infty, \infty, \infty)$, then Figure 12 is referred, where x_1, x_3, x_5 are punctures, and x_2, x_4, x_6 are branch points of order 2. A and f are the same rotations as above. Then f is a self-map of U/Γ in the sense of orbifolds.

If $(g, n; \nu_1, \ldots, \nu_n) = (0, 6; 2, 2, 2, 2, \infty, \infty)$, then once again, Figure 12 is referred, where x_1, x_2 are punctures, and x_3, x_4, x_5, x_6 are branch points of order 2. A and f are the same rotations as above. Then f is a self-map of U/Γ in the sense of orbifolds.

The rest of the argument of Case 3 is similar to Case 1.

(4) If $(g, n; \nu_1, \ldots, \nu_n) = (0, 5; 2, 2, \infty, \infty, \infty)$, then we refer to Figure 13, where x_1, x_3, x_5 are punctures, and x_2, x_4 are branch points of order 2. A is the rotation with angle $2\pi/4$ and is made by fixing x_1 and a regular point. Set $f = A^2$. Once again, f is a self-map of U/Γ in the sense of orbifolds.

This case by case discussion finishes the proof of Lemma 3.4.3. □

Proof of Theorem 0.3'. When Γ is torsion free, the result is a weak version of Theorem 0.3. So we only need to prove the theorem in the case when Γ contains elliptic elements. Now Theorem 0.1 and Theorem 0.2 tell us that the Bers fiber space $F(\Gamma)$ cannot be isomorphic to any Teichmüller space $T(g', n')$ except when the signature $(g, n; \nu_1, \ldots, \nu_n)$ lies in Table (C). There are three special cases to consider.

When Γ is of signature $(0,6;2,\ldots,2)$, we have the equivalences:

$$F(0,6;2,\ldots,2) \overset{\lambda}{\cong} F(2,0;\_) \overset{\psi}{\cong} T(2,1), \tag{3.3}$$

where the equivalence $\lambda\colon F(0,6;2,\ldots,2) \cong F(2,0;\_)$ is defined by sending a point $([\mu],z) \in F(0,6;2,\ldots,2)$ to the point $(\Xi([\mu]),z) \in F(2,0;\_)$ (Ξ denotes an isomorphism in (1.5)). The other isomorphism ψ in (3.3) is a Bers isomorphism which cannot be continuously extended to the boundary of $F(2,0;\_)$ by Theorem 0.3. We conclude that $\psi \circ \lambda$ admit no continuous extension. It turns out that any isomorphism of $F(0,6;2,\ldots,2)$ onto $T(2,1)$ admits no continuous extension.

By the same argument, we can prove that the isomorphisms (0.3) and (0.4) admit no continuous extensions.

It remains to prove the result in the case when $(g,n;\nu_1,\ldots,\nu_n)$ lies in Table (C'). In this case, by using Lemma 3.4.3, we can find a non-trivial element $\chi \in \text{mod}\ \Gamma$, $\chi \notin \Gamma$, so that $q_0(\chi^2) \in \text{Mod}\ \Gamma$ acts trivially on $T(\Gamma)$.

Let $\pi : F(\Gamma) \to T(\Gamma)$ denote the natural projection. Note that Mod Γ acts faithfully on $T(\Gamma)$ if the signature of Γ lies in Table (C'). Choose a point $[\mu] \in T(\Gamma)$ so that $[\nu] = q_0(\chi)([\mu]) \neq [\mu]$. Without loss of generality, we may assume that $[\mu] = [0]$ (this means that $\pi^{-1}([\mu]) = U$ is the central fiber of $F(\Gamma)$). Since $q_0(\chi^2)$ acts trivially on $T(\Gamma)$, $\chi^2 \in \Gamma$. More precisely, we have $\chi^2 = \gamma$, a parabolic element of Γ.

Since $\chi^2 \in \Gamma$ is parabolic, for an arbitrarily small $\varepsilon > 0$, there is a point

$([0], x) \in U \subset F(\Gamma)$ so that

$$\rho_U(([0], x), ([0], \chi^2(x))) \ < \ \varepsilon,$$

where ρ_U is the Poincaré metric on U.

Suppose that there is an isomorphism $\varphi : F(\Gamma) \to T(\Gamma')$. Then the restriction $\varphi|_U$ is a holomorphic map of U into $T(\Gamma')$. By a theorem of Royden [43], the Kobayashi metric on $T(\Gamma')$ coincides with the Teichmüller metric $< , >$ on $T(\Gamma')$. We thus obtain

$$< \varphi([0], x), \varphi([0], \chi^2(x)) > < \ \varepsilon. \tag{3.4}$$

On the other hand, by Theorem 1.2.3, we see that both $\chi' = \varphi \circ \chi \circ \varphi^{-1}$ and $\chi'^2 = \varphi \circ \chi^2 \circ \varphi^{-1}$ are well defined modular transformations of $T(\Gamma')$. From (3.4) and (1.8) we obtain

$$< \varphi([0], x), \chi'^2(\varphi([0], x)) > = < \varphi([0], x), \varphi \circ \chi^2 \circ \varphi^{-1}(\varphi([0], x)) > =$$
$$= < \varphi([0], x), \varphi \circ \chi^2([0], x) > =< \varphi([0], x), \varphi([0], \chi^2(x)) > < \ \varepsilon.$$

It follows that

$$a(\chi'^2) \ = \ \inf_{\tau' \in T(\Gamma')} < \tau', \chi'^2(\tau') > < \ \varepsilon.$$

Since ε is arbitrary, and since χ'^2 is of infinite order, we see that χ'^2 is a parabolic modular transformation. By Lemma 3.4.1, χ' is parabolic. Then Lemma 3.4.2 asserts that we can find a common non-empty system $C' = \{c'_1, \ldots, c'_r\}$ of admissible curves which is completely reduced by both f'_2 and f', where $f' : U/\Gamma' \to U/\Gamma'$ induces χ' and $f'_2 : U/\Gamma \to U/\Gamma'$ induces χ'^2. By taking a suitable power if necessary, we may assume that both f'_2 and

f' restrict to the identity on each part of $U/\Gamma' - N(C')$, where $N(C')$ is an arbitrarily small tubular neighborhood of $C' = \{c'_1, \ldots, c'_r\}$. Since both f'^2 and f'_2 induce the same modular transformation, we conclude that f'_2 is isotopic to $f'^2 = f' \circ f'$.

By (1.7), we obtain $\chi([0], x) = ([\nu], w^\nu \circ f(x))$, where ν is the Beltrami coefficient of f^{-1}. Define

$$\tau' = \varphi([0], x) \in T(\Gamma') \quad \text{and} \quad \sigma' = \varphi([\nu], w^\nu \circ f(x)) \in T(\Gamma').$$

By the above argument, χ' is induced by a reduced map f', and χ'^2 is induced by f'^2. By selecting a subsequence if necessary, we may assume that the sequence $\{\chi'^{2n}(\tau')\}_{n=1}^{\infty}$ converges. By Theorem 5 of Maskit [33], $\{\chi'^{2n}(\tau')\}_{n=1}^{\infty}$ converges to a limit point ϕ' in $\partial T(\Gamma)$, and ϕ represents a regular b-group which is obtained by squeezing all curves $c'_1, \ldots, c'_r$. It is immediate that $\chi'(\phi') = \phi'$ (see Abikoff [2]). By (3.5), we obtain

$$\sigma' = \varphi([\nu], w^\nu \circ f(x)) = \varphi \circ \chi([0], x) = \chi'(\tau').$$

Hence, by a theorem of Abikoff [2], $\chi'^{2n}(\sigma') = \chi'^{2n} \circ \chi'(\tau') = \chi' \circ \chi'^{2n}(\tau') \to \chi'(\phi') = \phi'$, as $n \to \infty$. We also see from (3.5) that the φ-image of $\chi^{2n}([0], x)$ is

$$\varphi(\chi^{2n}([0], x)) = \chi'^{2n} \circ \varphi([0], x) = \chi'^{2n}(\tau')$$

and the φ-image of $\chi^{2n}([\nu], w^\nu \circ f(x))$ is

$$\varphi(\chi^{2n}([\nu], w^\nu \circ f(x))) = \chi'^{2n}(\sigma') = \chi'^{2n+1}(\tau').$$

Since $\chi^2 = \gamma \in \Gamma$, we have $\chi^{2n}([0], x) = \gamma^n([0], x) = ([0], \gamma^n(x))$. So all points $\chi^{2n}([0], x)$, $n = 1, 2, \cdots$, stay in the fiber $U \subset F(\Gamma)$ and converge to $([0], x')$, as $n \to \infty$, where $x' \in \mathbb{R}$ is the fixed point of γ. However,

$$\chi^{2n}([\nu], w^\nu \circ f(x)) = \gamma^n([\nu], w^\nu \circ f(x)) =$$
$$= ([\nu], (\gamma^\nu)^n \circ w^\nu \circ f(x)) = ([\nu], w^\nu \circ \gamma^n \circ f(x))$$

which stays in the fiber $\pi^{-1}([\nu])$ over $[\nu] \in T(\Gamma)$. The sequence $\{\chi^{2n}([\nu], w^\nu \circ f(x))\}_{n=1}^{\infty}$ converges to $([\nu], x'_\nu)$, where $x'_\nu \in \partial w^\nu(U)$ is the fixed point of $\gamma^\nu = w^\nu \circ \gamma \circ (w^\nu)^{-1}$.

Since $[0] \neq [\nu]$, we have $([0], x') \neq ([\nu], x'_\nu)$. Now it is obvious that if $\tilde{\varphi}$ is an extension of $\varphi \colon F(\Gamma) \to T(\Gamma')$, then we must have

$$\tilde{\varphi}([\nu], x'_\nu) = \tilde{\varphi}([0], x') = \phi'.$$

So $\tilde{\varphi}$ cannot be a homeomorphism. This completes the proof of Theorem 0.3'.

□

Remark. Our methods apparently fail in dealing with the case when the signature of Γ is $(0, 3; \nu_1, \nu_2, \nu_3)$ for $\nu_1, \nu_2, \nu_3 \in \{2, 3, \cdots\} \cup \{\infty\}$ with $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1$. In this case, the Teichmüller space $T(\Gamma)$ is a single point, there is only one fiber over $T(\Gamma)$ which is the unit disk.

We expect that all isomorphisms (if exist) between Bers fiber spaces and Teichmüller spaces admit no continuous extensions. The issue is to construct certain automorphisms of $F(\Gamma)$ with the property that they act as hyperbolic modular transformations on $T(\Gamma')$. Further discussion will appear elsewhere.

Chapter 4

On fiber-preserving isomorphisms

The aim of this chapter is to study fiber-preserving isomorphisms among Bers fiber spaces and isomorphisms among Teichmüller curves. Some results stated and proved in this chapter (in particular under the assumption that the Fuchsian groups involved are torsion free) are known to experts. Since these results do not appear in the literature and they are important in studying various fiber spaces over Teichmüller spaces as well, we fill in all details in this paper for completeness.

Let Γ and Γ' be finitely generated Fuchsian groups of the first kind. Then except for certain exceptional situations, all fiber-preserving isomorphisms between $F(\Gamma)$ and $F(\Gamma')$ are determined. (See Theorem 0.4 for a precise statement.) Since every isomorphism of Teichmüller curves is automatically fiber-preserving, some well-known results stated in Chapter 1 can be utilized to investigate isomorphisms among Teichmüller curves. First, we classify all Teichmüller curves under isomorphisms, then we prove that all isomorphisms among Teichmüller curves are "geometric".

4.1 Basic properties of Teichmüller curves

Let Γ be a finitely generated Fuchsian group of the first kind which acts on U and has type (g,n). As we discussed in §1.3, the Fuchsian group Γ can be thought of as the group of inner automorphisms of Γ. With this point of view, Γ is a normal subgroup of mod Γ. In particular, Γ acts on the Bers fiber space $F(\Gamma)$ by the formula (1.8). The quotient space

$$V(\Gamma) = F(\Gamma)/\Gamma \tag{4.1}$$

is a complex manifold with $\dim_{\mathbb{C}} V(\Gamma) = 3g - 2 + n$. (See Proposition 3.6 of Earle-Kra [20].) Due to the fact that the action of Γ keeps all fibers of $\pi : F(\Gamma) \to T(\Gamma)$ invariant, the natural projection π induces a holomorphic projection $\pi_0 \colon V(\Gamma) \to T(\Gamma)$ with $\pi_0^{-1}(x)$, $x \in T(\Gamma)$, an orbifold conformally equivalent to $w^\mu(U)/\Gamma^\mu$, where the quasifuchsian group Γ^μ is defined, as usual, by the formula $\Gamma^\mu = w^\mu \Gamma (w^\mu)^{-1}$. The object $V(\Gamma)$ is called the *Teichmüller curve*.

For a moment, we assume that Γ contains no parabolic elements. (We will see later on that $V(\Gamma)$ is isomorphic to $V(\Gamma')$ if Γ' is of type (g,n) with no parabolic elements, too.) In this case, $V(\Gamma)$ is called *n-pointed Teichmüller curve* and is denoted by $V(g,n)$. In particular, we see that $V(g,n)$ is a complex manifold with a holomorphic projection $\pi_n : V(g,n) \to T(g,n)$ onto $T(g,n)$

such that for each point $x \in T(g,n)$, $\pi_n^{-1}(x)$ is the closed orbifold of genus g determined by the surface of type (g,n) represented by x.

In what follows, we assume that Γ is of type (g,n) and may or may not contain parabolic elements. Let

$$U_\Gamma = \{z \in U; z \text{ is not a fixed point of any elliptic element of } \Gamma\}.$$

We define the *punctured Bers fiber space* $F_0(\Gamma)$ as the space

$$\{([\mu], z) \in T(\Gamma) \times \mathbb{C};\ \mu \in M(\Gamma) \text{ and } z \in w^\mu(U_\Gamma)\}.$$

Clearly, the group Γ acts on $F_0(\Gamma)$ freely and discontinuously as a group of holomorphic automorphisms which keeps each fiber invariant. The quotient space $V(g,n)' = F_0(\Gamma)/\Gamma$ is called a *punctured Teichmüller curve.* Let $\pi_n' : V(g,n)' \to T(g,n)$ denote the natural projection.

For every elliptic element of Γ, we have a canonical section of $\pi : F(\Gamma) \to T(\Gamma)$ defined in §1, 3; this section projects (via (4.1)) to a global holomorphic section, which is called a *canonical section* of $\pi_0 : V(\Gamma) \to T(\Gamma)$. It is easy to see that the image of a canonical section of π_0 is exactly a locus consisting of the branch points corresponding to an elliptic element of Γ. Conjugate elliptic elements of Γ determine a single holomorphic section of π_0. The above discussion leads to the following relation:

$$V(g,n)' = V(g,n) - \{\text{the images of all canonical sections of } \pi_n\}.$$

If Γ contains k conjugacy classes of parabolic elements, then we have

$$V(\Gamma) = V(g,n) - \cup_{j=1}^{k} s_j(T(g,n)),$$

where s_j, $j = 1, \ldots, k$, are the sections of $\pi_n : V(g,n) \to T(g,n)$ determined by k punctures.

Several important results, due to Hubbard [26], Earle-Kra [19] [20], give us almost full information on global holomorphic sections of $\pi_0 : V(\Gamma) \to T(\Gamma)$. The following result, which is a weak version of their results (the main theorem of [26], Theorem 4.6 of [19] and Theorem 2.2, Theorem 10.3 of [20]), is sufficient for our use in this paper.

Theorem 4.1.1 *Let Γ be a finitely generated Fuchsian group of the first kind of type (g,n), and let k (may be zero) be the number of conjugacy classes of parabolic elements of Γ. Then the number of global holomorphic section of $\pi_0 : V(\Gamma) \to T(\Gamma)$ is finite (it is zero in most torsion free cases) provided that Γ satisfies one of the following conditions:*

(1) $g \geq 2$;

(2) $g = 1$, $n \geq 2$, and $k > 0$;

(3) $g = 0$, $n \geq 5$, and $k > 0$.

Remark. The condition that $k > 0$ guarantees that there are punctures on U/Γ. When $g \leq 1$ and there are no punctures on U/Γ, we shall have uncountably many conformal involution of $V(\Gamma)$. The above theorem is not true.

The number of the holomorphic sections in the above theorem can be counted. We omit the calculations since they are not needed in this paper. Nevertheless, a particularly interesting case is still worth mentioning. When $g = 2$ and $k = 0$, there are 6 Weierstrass sections (see [20]) which are defined by the fixed point locus of the holomorphic involution $J\colon V(2,n) \to V(2,n)$ (the restriction of J to each fiber is the usual hyperelliptic involution on the corresponding compact Riemann surface of genus 2). There are also n canonical sections $s_1, \ldots, s_n$. So in this case, there are altogether $2n+6$ holomorphic sections:

$$\{s_1, \ldots, s_n;\ J \circ s_1, \ldots, J \circ s_n;\ \text{six Weierstrass sections}\}.$$

However, when $g = 2$ and $k > 0$, the number of sections is just $n - k$.

Finally, we point out that certain situations are still unclear; the question is: what can we say about holomorphic sections of $\pi_0\colon V(\Gamma) \to T(\Gamma)$ when Γ is "exceptional"? Here by exceptional cases we always mean the type (g, n) (or signature) of Γ belongs to the following Table (F):

$(g,n) = (0,3)$, $(0,4)$, or $(1,1)$
$g = 1$, $n \geq 2$, Γ contains no parabolic elements
$g = 0$, $n \geq 5$, Γ contains no parabolic elements

Table (F)

In considering the above table, there is nothing to say when Γ is of type $(0,3)$. Note that Theorem 4.6 (b) of [19] tells us that if $(g,n) = (0,4)$ or $(1,1)$, then for each point $x \in V(g,n)'$, there is a unique global holomorphic section s with $s(\pi_0(x)) = x$. Assume that $g = 0$ and $n \geq 5$. For each $x \in V(0,n)$, how many global holomorphic sections of $\pi_n : V(0,n) \to T(0,n)$ are there which pass through x? The picture remains unclear only when (g, n) belongs to the 2nd or 3rd row of Table (F).

4.2 A classification of Bers fiber spaces

Let Γ and Γ' be finitely generated Fuchsian groups of the first kind. Suppose that their signatures are $\sigma = (g, n; \nu_1, \ldots, \nu_n)$ and $\sigma' = (g', n'; \nu'_1, \ldots, \nu'_{n'})$, respectively. Before proceeding, we take some examples to see what kind of isomorphisms among Bers fiber spaces are known to us. Recall that $Q(\Gamma)$ consists of all quasiconformal self-maps w of U with $w\Gamma(w)^{-1}$ a Fuchsian group. First, we consider the case when $\sigma = \sigma'$. Then there is $w \in Q(\Gamma)$ such that $w\Gamma w^{-1} = \Gamma'$. By a theorem of Bers [8], w induces an isomorphism $[w]_*$ of $F(\Gamma)$ onto $F(\Gamma')$. More precisely, the isomorphism $[w]_*$ can be described by sending every point $([\mu], z) \in F(\Gamma)$ to the point $([\nu], w^{\nu} \circ w \circ (w^{\mu})^{-1}(z)) \in F(\Gamma')$, where $\nu \in M(\Gamma')$ is the Beltrami coefficient of $w^{\mu} \circ w^{-1}$. It is easy to check that $[w]_*$ is a fiber-preserving isomorphism. An isomorphism defined in this way is called a *Bers allowable mapping.*

Assume now that the (unordered) pair of signatures (σ, σ') is either

$$((1, 2; \nu, \nu), (0, 5; 2, \ldots, 2, \nu)), \quad \forall \nu \in \{2, 3, \cdots\} \cup \{\infty\}$$

or $((2, 0; \_\_), (0, 6; 2, \ldots, 2))$. In these two exceptional cases, an isomorphism can be constructed by carrying $([\mu], z) \in F(\Gamma)$ to $\gamma'(\Xi([\mu]), z) \in F(\Gamma')$, where $\gamma' \in \Gamma'$, and Ξ stands for the canonical isomorphism

$$T(2, 0) \cong T(0, 6) \quad (\text{or } T(1, 2) \cong T(0, 5))$$

as we described in §1.2. Observe that in general situation, all well-known isomorphisms among Bers fiber spaces are fiber-preserving (there are only a few examples showing that non fiber-preserving isomorphisms exist).

The main result of this section is a weak version of Theorem 0.4 which classifies (in a fiber-preserving way) all Bers fiber spaces $F(\Gamma)$ in general cases.

Theorem 4.2.1 *Let Γ be a finitely generated Fuchsian group of the first kind whose type (or signature) is not in Table (F). Assume that $(g, n; \nu_1, \ldots, \nu_n)$ is not $(2,0;\text{--})$, $(0,6;2,\ldots,2)$, $(1,2;\infty,\infty)$, or $(0,5;2,\ldots,2,\infty)$. Let Γ' be a group with signature $\sigma' = (g', n'; \nu'_1, \ldots, \nu'_{n'})$. Then the following conditions are equivalent:*

(i) $F(\Gamma)$ is fiber-preservingly isomorphic to $F(\Gamma')$;

(ii) $\sigma = \sigma'$.

We begin with a result whose proof is a direct consequence of Theorem 4.1.1. Let $\mathcal{S}$ denote the set of all images of holomorphic sections of $\pi_0 \colon V(\Gamma) \to T(\Gamma)$. By Theorem 4.1.1, if the type (or signature) of Γ is not in Table (F), then the cardinality of $\mathcal{S}$ is always finite. Let

$$p_0 \colon F(\Gamma) \to V(\Gamma)$$

be the natural projection determined by (4.1); that is, the image $p_0(x)$ of $x \in ([\mu], w^\mu(U)) \subset F(\Gamma)$ is its image under the natural projection $w^\mu(U) \to w^\mu(U)/\Gamma^\mu \subset V(\Gamma)$.

Proposition 4.2.2 *Suppose that $\mathcal{S}$ is not empty. Then for each $[\mu] \in T(\Gamma)$, $\mathcal{S}^{\mu} = \pi^{-1}([\mu]) \cap p_0^{-1}(\mathcal{S})$ is discrete and invariant under the action of $\Gamma^{\mu} = w^{\mu}\Gamma(w^{\mu})^{-1}$.* $\square$

To prove Theorem 4.2.1, we first need to study those holomorphic automorphisms which keep each fiber invariant. In what follows, we use the same symbol Γ to denote the Fuchsian group as well as the automorphism group of $F(\Gamma)$ it induces. The symbol Γ^{μ}, $\mu \in M(\Gamma)$, stands for the quasifuchsian group $w^{\mu}\Gamma(w^{\mu})^{-1}$, which can be identified with $\Gamma|_{\pi^{-1}([\mu])}$ in the action of Γ on $F(\Gamma)$. We need

Lemma 4.2.3 *Assume that the type (g, n) (or signature) of Γ is not in Table (F), and assume that η is a holomorphic automorphism of $F(\Gamma)$ which keeps each fiber invariant. Then $\Gamma_0 =< \Gamma, \eta|_{\pi^{-1}([0])} >$ is again a finitely generated Fuchsian group of the first kind.*

Proof. If Γ is torsion free, then Bers' isomorphism theorem [8] asserts that $F(\Gamma) \cong T(g, n+1)$. Thus, the group Aut $(F(\Gamma))$ of holomorphic automorphisms of $F(\Gamma)$ is isomorphic to the group of holomorphic automorphisms of $T(g, n+1)$ which is, by Theorem 1.2.3, the Teichmüller modular group Mod $(g, n+1)$. Since Mod $(g, n+1)$ acts discontinuously on $T(g, n+1)$, Aut $(F(\Gamma))$ acts discontinuously on $F(\Gamma)$ as well. It follows that Γ_0 is discrete.

Now we assume that Γ contains elliptic elements. Let U be the central fiber of $F(\Gamma)$. (By the central fiber we mean $U = \pi^{-1}([0])$, where $[0] \in T(\Gamma)$ is the origin.) Observe that $\eta|_{\pi^{-1}([0])}$ is a real Möbius transformation.

The set $\mathcal{S}^0 = p_0^{-1}(\mathcal{S}) \cap U$ $(\neq \emptyset)$ constructed above is not only Γ-invariant, but also Γ_0-invariant. For otherwise there is a point $x_0 \in \mathcal{S}^0$ with the property that $\eta|_{\pi^{-1}([0])}(x_0) \notin \mathcal{S}^0$. If we denote by $s_0 \colon T(\Gamma) \to F(\Gamma)$ the global canonical section defined by sending $[\mu] \in T(\Gamma)$ to $([\mu], w^\mu(x_0))$, the above argument shows that $\eta \circ s_0$ is a global holomorphic section of π whose image is not in $p_0^{-1}(\mathcal{S})$, contradicting Theorem 4.1.1. To prove that Γ_0 is discontinuous, it is equivalent to showing that Γ_0 is discrete (see, for example, Farkas-Kra [21]). Suppose for the contrary that there is a sequence $\{\gamma_n\} \in \Gamma_0$ such that $\gamma_n \to id$. This implies that for any point, in particular, for $x \in \mathcal{S}^0$, $\gamma_n(x) \to x$. Since $\mathcal{S}^0$ is Γ_0-invariant and discrete in U by Proposition 4.2.2, we find a contradiction. Lemma 4.2.3 is proved. □

Lemma 4.2.4 *Let Γ be a finitely generated Fuchsian group of the first kind. Then any fiber-preserving holomorphic automorphism $\varphi \colon F(\Gamma) \to F(\Gamma)$ projects to a holomorphic automorphism χ of $T(\Gamma)$ in the sense that*

$$\pi \circ \varphi(x) = \chi \circ \pi(x), \quad \textit{for all} \;\; x \in F(\Gamma). \tag{4.2}$$

Proof. Given the map φ, we define χ by (4.2). Clearly, χ is well defined. The only issue is to show that χ is holomorphic. Choose an arbitrary point $x \in ([\mu], w^\mu(U)) \subset F(\Gamma)$. There is a local holomorphic section $s \colon T(\Gamma) \to F(\Gamma)$ with $s([\mu]) = x$. For $[\nu]$ close to $[\mu]$, we can write

$$\chi([\nu]) = \chi \circ (\pi(x')) = \pi \circ \varphi(x') = \pi \circ \varphi \circ s([\nu]),$$

where $s([\nu]) = x'$. Since s, φ, and π are holomorphic, χ is holomorphic as well. The lemma is proved. $\square$

Lemma 4.2.5 *Let D be a simply connected domain in $\hat{\mathbb{C}}$ which misses at least three points of $\hat{\mathbb{C}}$, let f is a conformal self-map of D. Then f has at most one fixed point in D.*

Proof. Let $\alpha\colon U \to D$ be a Riemann mapping, where U is the upper half plane. Then $\alpha^{-1} \circ f \circ \alpha$ is conformal, and hence belongs to $\mathrm{PSL}(2, \mathbb{R})$. Therefore, $\alpha^{-1} \circ f \circ \alpha$ has at most one fixed point in U. $\square$

It is well-known that for any $\mu \in M(\Gamma)$, there is a unique quasiconformal self-map w_μ of U which fixes 0, 1, ∞, and satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$. (See §1.1.) Hence, to each $\mu \in M(\Gamma)$, there corresponds a Fuchsian group $\alpha_\mu(\Gamma) = \Gamma_\mu$ which depends only on the equivalence class $[\mu]$ of μ, where $\alpha_\mu : \Gamma \to \Gamma_\mu$ is an isomorphism defined by taking $\gamma \in \Gamma$ to $w_\mu \circ \gamma \circ (w_\mu)^{-1} \in \Gamma_\mu$. We see that $T(\Gamma)$ is identified with the set $\{\alpha_\mu : \Gamma \to \Gamma_\mu \subset \mathrm{PSL}(2, \mathbf{R});\ [\mu] \in T(\Gamma)\}$. Let us denote by $Max(\Gamma)$ the set of points $[\mu]$ in $T(\Gamma)$ which corresponds to a finite maximal Fuchsian group; that is, the group Γ_μ for which there does not exist any other Fuchsian group G such that $\Gamma_\mu \subset G$ and the index $[G : \Gamma_\mu]$ is finite.

Lemma 4.2.6 *Under the condition of Lemma 4.2.3, let η be a holomorphic automorphism of $F(\Gamma)$ which keeps each fiber invariant. Suppose that for all $[\mu] \in T(\Gamma)$, $\eta|_{\pi^{-1}([\mu])}$ is not in Γ^μ. Then the set $Max(\Gamma)$ is empty.*

Proof. Let $h_\mu\colon w^\mu(U) \to U$ be the Riemann mapping with $h_\mu(0) = 0$, $h_\mu(1) = 1$, and $h_\mu(\infty) = \infty$. It is easy to see that $h_\mu = w_\mu \circ (w^\mu)^{-1}$ and that $h_\mu \Gamma^\mu (h_\mu)^{-1}$ is properly contained in $h_\mu \Gamma_0^\mu (h_\mu)^{-1}$ for all $[\mu]$, where $\Gamma_0^\mu = < \Gamma^\mu, \eta|_{\pi^{-1}([\mu])} >$. Also, a simple computation shows that $h_\mu \Gamma^\mu (h_\mu)^{-1} = w_\mu \Gamma (w_\mu)^{-1}$. Since $h_\mu \circ \eta|_{\pi^{-1}([\mu])} \circ (h_\mu)^{-1}$ is a real Möbius transformation which is not in Γ_μ, Γ_μ is properly contained in $h_\mu \Gamma_0^\mu (h_\mu)^{-1}$. The discreteness of $h_\mu \Gamma_0^\mu (h_\mu)^{-1}$ for any μ follows from the proof of Lemma 4.2.3. Since $\alpha_\mu\colon \Gamma \to \Gamma_\mu$ runs over all points in $T(\Gamma)$, the lemma is established. □

Lemma 4.2.7 *Let Γ be the group which contain elliptic elements. Under the condition of Lemma 4.2.3, suppose that for some $[\mu] \in T(\Gamma)$, Γ^μ is properly contained in $\Gamma_0^\mu = < \Gamma^\mu, \eta|_{\pi^{-1}([\mu])} >$. Then the set $Max(\Gamma)$ is empty.*

Proof. Let $\mathcal{A}$ denote the set

$$\{[\mu] \in T(\Gamma);\ \Gamma^\mu \text{ is properly contained in } < \Gamma^\mu, \eta|_{\pi^{-1}([\mu])} >\}.$$

We claim that $\mathcal{A}$ is open. Indeed, we may assume, without loss of generality, that the origin [0] of $T(\Gamma)$ belongs to $\mathcal{A}$. (Otherwise a Bers allowable mapping will be constructed to carry a fiber over a point in $\mathcal{A}$ to a fiber over [0] of another Teichmüller space.) Choose a point x in U which is not fixed by any non-trivial element of Γ. We see that

$$\delta = \rho_U(\eta(x), \Gamma(x)) = \inf\{\rho_U(\eta(x), \gamma(x));\ \gamma \in \Gamma, \text{ and } \gamma \neq id\}$$

is positive, where ρ_E is the Poincaré metric on a domain E. For any sequence $\{\mu_n\} \in M(\Gamma)$ with $\mu_n \to 0$ almost everywhere, the sequence $\{w^{\mu_n}\}$ converges to the identity uniformly on compact sets (see [4], [23], or [39]). This implies that $w^{\mu_n}(U) \to U$ in the sense that for any compact set $E \subset U$, there exists a large n_0 such that $E \subset w^{\mu_n}(U)$ whenever $n \geq n_0$. Therefore, $\rho_{w^{\mu_n}(U)}(x,y)$ must converge to $\rho_U(x,y)$ for any pair $x, y \in U$. We conclude that if $[\mu] \in T(\Gamma)$ is in a sufficiently small neighborhood of $[0]$, the point x stays in $w^{\mu}(U)$ and satisfies the condition that

$$\rho_{w^{\mu}(U)}(\eta(x), \Gamma^{\mu}(x)) > \delta/2.$$

This implies that Γ^{μ} is properly contained in Γ_0^{μ} as well. Hence, $\mathcal{A}$ is open.

To show that $\mathcal{A} = T(\Gamma)$, it remains to verify that $\mathcal{A}$ is also closed. By assumption, Γ contains elliptic elements and its type (or signature) does not belong to Table (F). Choose a fixed point z_0 of some elliptic element, and choose an arbitrary $[\mu] \in \mathcal{A}^c$. By definition of $\mathcal{A}^c$, there is a $\gamma \in \Gamma$ so that $\eta|_{\pi^{-1}([\mu])} = \gamma^{\mu} = w^{\mu} \circ \gamma \circ (w^{\mu})^{-1}$. (Note that γ^{μ} depends only on the equivalence class $[\mu]$ of μ.) Let $\mathcal{B}$ be a finite subset of $p_0^{-1}(\mathcal{S})$ whose cardinality is ≥ 2. Then in particular, we have

$$\eta|_{\pi^{-1}([\mu])}(\mathcal{B} \cap \pi^{-1}([\mu])) = \gamma^{\mu}(\mathcal{B} \cap \pi^{-1}([\mu])).$$

Consider now an arbitrary sequence $\{\mu_i\} \in M(\Gamma)$ with $[\mu_i] \to [\mu]$ as $i \to \infty$. By Proposition 4.2.2, $p_0^{-1}(\mathcal{S}) \cap \pi^{-1}([\mu'])$ is discrete for any $[\mu'] \in T(\Gamma)$. For sufficiently large i, we must have

$$\eta|_{\pi^{-1}([\mu_i])}(\mathcal{B} \cap \pi^{-1}([\mu_i])) = \gamma^{\mu}(\mathcal{B} \cap \pi^{-1}([\mu_i]).$$

(Otherwise there would be a new holomorphic section whose image is not an element of $p_0^{-1}(\mathcal{S})$, contradicting Theorem 4.1.1). This implies that there is a neighborhood N_μ of $[\mu]$ such that for any $[\nu] \in N_\mu$, the restriction of the conformal self-map $\eta|_{\pi^{-1}([\nu])}$ to $\mathcal{B} \cap \pi^{-1}([\nu])$ coincides with $\gamma^\nu|_{\mathcal{B}\cap\pi^{-1}([\nu])}$. Now both $\eta|_{\pi^{-1}([\nu])}$ and γ^ν are conformal self-maps of $w^\nu(U)$, $(\eta|_{\pi^{-1}([\nu])})^{-1} \circ \gamma^\nu$ is thus a conformal self-map of $w^\nu(U)$ which fixes all points in $\mathcal{B} \cap \pi^{-1}([\nu])$. Hence, by Lemma 4.2.5, we see that $\eta|_{\pi^{-1}([\nu])} = \gamma^\nu$. (Since they are self-maps in the quasidisk and have the same values in at least two points.) It follows that $N_\mu \subset \mathcal{A}^c$, and $\mathcal{A}^c$ is open. This implies that for each $[\mu] \in T(\Gamma)$, Γ^μ is properly contained in $< \Gamma^\mu, \eta|_{\pi^{-1}([\mu])} >$. Lemma 4.2.6 then implies that $Max(\Gamma)$ is empty. $\square$

For torsion free Fuchsian group Γ, we have

Lemma 4.2.8 *Let* Γ *be a torsion free group whose type is not* $(0,3)$, $(0,4)$, *or* $(1,1)$. *Let* $\eta \in Aut\ F(\Gamma)$ *keeps each fiber invariant. Further assume that for some* $[\mu] \in T(\Gamma)$, Γ^μ *is properly contained in* $\Gamma_0^\mu =< \Gamma^\mu, \eta|_{\pi^{-1}([\mu])} >$. *Then the signature of* Γ *must be either* $(2,0;\_)$ *or* $(1,2;\infty,\infty)$.

Proof. Suppose that the signature of Γ is neither $(2,0;\_)$ nor $(1,2;\infty,\infty)$. Since there are no holomorphic sections of π_0, the argument applied in Lemma 4.2.7 does not work at this time, we must use another method.

By Bers' isomorphism theorem [8], there is an isomorphism $\psi\colon F(\Gamma) \to T(g, n+1)$. By Lemma 6.3 of Bers [8], given arbitrarily $a \in U$, any point of $F(\Gamma)$ can be represented as a pair $([\mu], w^\mu(a))$ for some $\mu \in M(\Gamma)$. Our first claim is that $\eta \in \text{mod}\ \Gamma$.

Suppose $\eta \notin \text{mod}\ \Gamma$. By Theorem 1.2.3, $\psi \circ \eta \circ \psi^{-1}$ is induced by a self-map f_0 of a surface X of type $(g, n+1)$, by Theorem 10 of Bers [8], f_0 must send the special puncture $\hat{a}$, where $\hat{a}$ is the image of a under the projection $U \to U/\Gamma$, to another puncture. This means that f_0 does not define a self-map of $\hat{X}$, where $\hat{X} = X \cup \{\hat{a}\}$. On the other hand, by Lemma 4.2.4, $\eta \in \text{Aut}\,(F(\Gamma))$ projects to a trivial action on $T(\Gamma)$, which says that $\pi \circ \eta = \pi$. Since the Bers isomorphism identifies the projection π onto the first factor with the forgetful map ϑ, we have the following commutative diagram:

$$\begin{array}{ccc} F(g,n;\infty,\ldots,\infty) & \xrightarrow{\psi} & T(g,n+1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\vartheta} \\ T(g,n) & \xrightarrow[id]{} & T(g,n) \end{array}$$

which gives

$$\vartheta \circ \psi \circ \eta \circ \psi^{-1} = \pi \circ \eta \circ \psi^{-1} = \pi \circ \psi^{-1} = \vartheta.$$

This implies that $\psi \circ \eta \circ \psi^{-1}$ projects to the identity via the forgetful map ϑ. But $\psi \circ \eta \circ \psi^{-1} \in \text{Mod}\,(g, n+1)$ is induced by a self-map f_0 of a Riemann surface of type $(g, n+1)$. We see that f_0 fixes $\hat{a}$, which leads to a contradiction. Therefore, $\eta \in \text{mod}\,\Gamma$, and we conclude that η is induced by a self-map f of U with $f \circ \gamma \circ f^{-1} \in \Gamma$ for all $\gamma \in \Gamma$. Since Γ is torsion free and the signature of Γ is neither $(2,0;\_)$ nor $(1,2;\infty,\infty)$, the Teichmüller modular group Mod Γ acts effectively on $T(\Gamma)$. Observe also that the kernel of the quotient homomorphism q_0:mod $\Gamma \to$ Mod Γ is Γ. We must have $\eta \in \Gamma$. This implies that for all $[\mu] \in T(\Gamma)$, $\eta|_{\pi^{-1}([\mu])}$ is an element of Γ^{μ}, contradicting our hypothesis. $\square$

Lemma 4.2.9 *Under the condition of Lemma 4.2.3, assume that the signature of* Γ *is neither* $(2,0;\_)$ *nor* $(1,2;\infty,\infty)$*. Then any holomorphic automorphism on* $F(\Gamma)$ *which leaves each fiber invariant is an element of* Γ*.*

Proof. By Lemma 4.2.8, we know that the lemma holds when Γ is torsion free. Now we assume that Γ has torsion. Let η be a holomorphic automorphism which satisfies the condition of the lemma. By Lemma 4.2.3, Γ and $\Gamma_0 =< \Gamma, \eta|_{\pi^{-1}([0])} >$ are two finitely generated Fuchsian groups of the first kind with $\Gamma \subset \Gamma_0$. Suppose that Γ is properly contained in Γ_0. The index $[\Gamma_0 : \Gamma] = \text{Area}(U/\Gamma)/\text{Area}(U/\Gamma_0) < \infty$. It follows from Lemma 4.2.7 that the set $Max(\Gamma)$ is empty. Hence, by Theorem 3A of Greenberg [25] or Theorem 1 of Singerman [44], we see that there is a unique group G such that Γ is a subgroup of finite index in G (the index can be proved to be equal to 2) and $T(G) \cong T(\Gamma)$. Furthermore, Γ must be of the signature $(1,2;\nu,\nu)$, where $\nu \geq 2$ is an integer or ∞. (Γ cannot be of signature $(2,0;\_)$ since our assumption says that Γ contains elliptic elements.) This is a contradiction. We conclude that $\Gamma = \Gamma_0$, and thus $\eta|_{\pi^{-1}([0])}$ is an element of Γ. By the argument of Lemma 4.2.7, we see that $\eta \in \Gamma$, as a group of automorphisms of $F(\Gamma)$. □

Proof of Theorem 4.2.1. Suppose that there is a fiber-preserving isomorphism $\varphi: F(\Gamma') \to F(\Gamma)$. The upper half plane U can be viewed as the central fiber of both $F(\Gamma)$ and $F(\Gamma')$. By composing with a Bers allowable mapping (which is fiber-preserving, see Bers [8]), we may assume, without loss of generality, that $\varphi(U) = U$. Consider the homomorphism α_φ of Γ' to Aut $(F(\Gamma))$ defined by $\alpha_\varphi(\gamma') = \varphi \circ \gamma' \circ \varphi^{-1}$ for all $\gamma' \in \Gamma'$. Since $\gamma' \in$ mod Γ' leaves each fiber invariant, $\alpha_\varphi(\gamma')$ is an automorphism of $F(\Gamma)$ which keeps each fiber invariant. Since σ is neither $(2,0;\_\_)$ nor $(1,2;\infty,\infty)$, by Lemma 4.2.9, $\alpha_\varphi(\gamma') \in \Gamma$ for all $\gamma' \in \Gamma'$. It follows that α_φ is a monomorphism of Γ' to Γ. Since σ is neither $(0,6;2,\ldots,2)$ nor $(0,5;2,\ldots,2,\infty)$, by Theorem 3A of Greenberg [25] or Theorem 1 of Singerman [44], we conclude that α_φ is an isomorphism of Γ' onto Γ. Since $\varphi|_U : U \to U$ is a real Möbius transformation, $\alpha_\varphi : \Gamma \to \Gamma'$ is type-preserving. We conclude that Γ and Γ' have the same signature. The reverse direction is trivial. This completes the proof of Theorem 4.2.1. □

4.3 Proof of Theorem 0.4

In this section, we prove Theorem 0.4 stated in the introduction. To proceed, let us recall a lemma which is proved by Royden [43] (see also Earle-Kra [20], Kra [28]) in the case when Γ is torsion free. However, it remains true even if Γ has torsion. See Gardiner ([23] section 9.6, pp 184-185) for a proof. We formulate it as

Lemma 4.3.1 *Assume that Γ is a finitely generated Fuchsian group of the first kind, and let $\chi: T(\Gamma) \to T(\Gamma)$ is a biholomorphic map. If for each $[\mu] \in T(\Gamma)$, there exists a $\chi_{[\mu]} \in Mod\,\Gamma$ such that*

$$\chi([\mu]) = \chi_{[\mu]}([\mu]). \tag{4.3}$$

Then $\chi \in Mod\,\Gamma$. □

Now we are ready to prove

Proposition 4.3.2 *Under the condition of Lemma 4.2.3, assume that the signature of* Γ *is neither* $(2,0;\_)$ *nor* $(1,2;\infty,\infty)$. *Then the following conditions are equivalent:*

(i) $\theta \in Aut\,(F(\Gamma))$ *is fiber-preserving;*

(ii) θ *projects to an element* $\chi \in Mod\,\Gamma$ *(which is induced by a self-map of* U/Γ*);*

(iii) θ *is an element of* $mod\,\Gamma$.

Proof. The proof that (iii) $\Rightarrow$ (i) is trivial. The implication (i) $\Rightarrow$ (ii) is not so obvious since we do not know θ can be projected to a modular transformation of $T(\Gamma)$. To prove that (i) $\Rightarrow$ (ii), we use Lemma 4.2.4, and see that θ projects to a holomorphic automorphism χ of $T(\Gamma)$ under the projection $\pi : F(\Gamma) \to T(\Gamma)$. Now Theorem 1.2.3 asserts that $\chi \in \text{Mod}\,(g,n)$; that is, χ is induced by a self-map f of the punctured Riemann surface U_Γ/Γ. We claim that f defines a self-map of U/Γ in the sense of orbifolds. For this purpose, let $\chi([0]) = [\mu]$. If we think of Γ as a group of automorphisms of $F(\Gamma)$, then for each $\gamma \in \Gamma$, $\theta \circ \gamma \circ \theta^{-1}$ is again an automorphism which leaves each fiber of $F(\Gamma)$ invariant. It follows from Lemma 4.2.9 that $\theta \circ \gamma \circ \theta^{-1} = \gamma_1$ for some $\gamma_1 \in \Gamma$. This implies that θ conjugates Γ to itself; in other words, θ can be projected to a biholomorphic self-map ζ on the Teichmüller curve $V(\Gamma)$. We thus obtain the following commutative diagram:

$$\begin{array}{ccc} V(\Gamma) & \xrightarrow{\zeta} & V(\Gamma) \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ T(\Gamma) & \xrightarrow[\chi]{} & T(\Gamma) \end{array} \tag{4.4}$$

Note that the Riemann surface $\pi_0^{-1}([0]) = U/\Gamma$ is represented by $[0] \in T(\Gamma)$, and $\pi_0^{-1}([\mu]) = w^\mu(U)/\Gamma^\mu$ is represented by $[\mu] \in T(\Gamma)$. Since ζ is biholomorphic, the restriction of ζ to each fiber of $V(\Gamma)$ is clearly conformal. By construction, ζ carries a branch point with ramification number ν to a branch point with the same ramification number. In particular, ζ realizes a conformal equivalence between $\pi_0^{-1}([0]) = U/\Gamma$ and $\pi_0^{-1}([\mu]) = w^\mu(U)/\Gamma^\mu$. This implies that the two points $[0]$ and $[\mu] \in T(\Gamma)$ are modular equivalent. Let us denote by χ_0 the corresponding modular transformation of $T(\Gamma)$ induced by $\zeta|_{\pi_0^{-1}([0])}$ and by f_ζ is the self-map of U/Γ which induces χ_0. Note also that $\chi_0 \in \text{Mod } \Gamma$.

Since the diagram (4.4) commutes, we have

$$\chi_0([0]) = [\mu] = \chi([0]).$$

Now choose an arbitrary point $[\nu] \in T(\Gamma)$. By using the same argument as above, we see that there exists a modular transformation $\chi_\nu \in \text{Mod } \Gamma$ such that $\chi_\nu([\nu]) = \chi([\nu])$. We arrive at the situation of Lemma 4.3.1, by which we conclude that $\chi \in \text{Mod } \Gamma$; that is, χ is induced by a self-map f of U/Γ which is isotopic to f_ζ keeping all distinguished points fixed. This finishes the argument of (i) $\Rightarrow$ (ii).

To verify (ii) $\Rightarrow$ (iii), again, we let $\chi \in \text{Mod } \Gamma$ to denote the projection of θ. By assumption of (ii), we see that χ is induced by a self-map f of U/Γ. f can be lifted to a self-map $\tilde{f}$ of U. Then the geometric isomorphism $\tilde{\chi}$ induced by $\tilde{f}$ is an element of mod Γ. We thus have the following commutative diagram:

$$\begin{array}{ccc} F(\Gamma) & \xrightarrow{\tilde{\chi}} & F(\Gamma) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ T(\Gamma) & \xrightarrow[\chi]{} & T(\Gamma) \end{array}$$

Then $\theta \circ \tilde{\chi}^{-1} \in \mathrm{Aut}\ (F(\Gamma))$ and $\pi \circ \theta \circ \tilde{\chi}^{-1} = \chi \circ \pi \circ \tilde{\chi}^{-1} = \pi$. Hence, $\theta \circ \tilde{\chi}^{-1}$ leaves each fiber invariant. Lemma 4.2.9 then asserts that $\theta \circ \tilde{\chi}^{-1} = \gamma \in \Gamma$. It follows that $\theta = \gamma \circ \tilde{\chi} \in \mathrm{mod}\ \Gamma$. This completes the proof of Theorem 4.3.2. □

Proof of Theorem 0.4. Suppose that $\varphi\colon F(\Gamma') \to F(\Gamma)$ is a fiber-preserving isomorphism. By Lemma 4.2.4, we know that φ can be projected to a biholomorphic map $\chi\colon T(\Gamma') \to T(\Gamma)$. By Theorem 4.2.1, we see that the signatures of Γ and Γ' are the same. Note that Γ and Γ' have the same signature if and only if there is $w \in Q(\Gamma')$ so that

$$w\Gamma' w^{-1} = \Gamma. \tag{4.5}$$

Hence, by Theorem 2 of Bers [8], w induces a Bers allowable mapping $[w]_*$ of $F(\Gamma')$ onto $F(\Gamma)$.

Consider the automorphism $\varphi_0 = [w]_*^{-1} \circ \varphi\colon F(\Gamma') \to F(\Gamma')$. It is easy to see that φ_0 is fiber-preserving and holomorphic. By Proposition 4.3.2, we assert that φ_0 is an element of mod Γ'. This implies that there is a quasiconformal self-map $\hat{f}$ of U with

$$\hat{f}\Gamma'\hat{f}^{-1} = \Gamma' \tag{4.6}$$

such that $\varphi_0 = [\hat{f}]_*$. It follows that $\varphi = [w]_* \circ [\hat{f}]_* = [w \circ \hat{f}]_*$, which says that φ is the Bers allowable mapping induced by the quasiconformal self-map $w \circ \hat{f}$ of U.

To be more precise, we see from (4.5) and (4.6) that $(w \circ \hat{f})\Gamma'(w \circ \hat{f})^{-1} = \Gamma$. Thus $w \circ \hat{f} \in Q(\Gamma')$. Hence, by a construction of Bers [8], the mapping φ is given by

$$\varphi([\nu], z) = [w \circ \hat{f}]_*([\nu], z) = (\chi([\nu]), w^{\mu} \circ w \circ \hat{f} \circ (w^{\nu})^{-1}(z)), \ \forall([\nu], z) \in F(\Gamma'),$$

where μ is the Beltrami coefficient of $w^{\nu} \circ (w \circ \hat{f})^{-1}$. By (4.5) and (4.6) again, we know that $\mu \in M(\Gamma)$. It is also not hard to see that χ is defined by sending the conformal structure $[\nu] \in T(\Gamma')$ to the conformal structure $[\mu] \in T(\Gamma)$. The reverse direction is completely trivial. This completes the proof of our main theorem. □

Remark. If Γ and Γ' are of type $(0,3)$, then both $F(\Gamma)$ and $F(\Gamma')$ are conformally equivalent to the unit disk. Thus, there are uncountably many conformal mappings (all of which are real Möbius transformations) of $F(\Gamma)$ onto $F(\Gamma')$. If Γ and Γ' are of type $(0,4)$ or $(1,1)$, then we know that there are at least three pairs (or triples) of Bers fiber spaces which are isomorphic to each other in a fiber-preserving way: $F(1,1,\nu) \cong F(0,4;2,2,2,2\nu)$, $F(0,4;\nu,\nu,\nu,\nu) \cong F(0,4;2,2,\nu,\nu) \cong F(0,4;2,2,2,\nu)$, and $F(0,4;\nu_1,\nu_1,\nu_2,\nu_2) \cong F(0,4;2,2,\nu_1,\nu_2)$, where $\nu, \nu_1, \nu_2 \geq 2$ are any integers or ∞. (See Earle-Kra [19].) Now the question arises as to whether there are some other pairs of Bers fiber spaces $(F(\Gamma), F(\Gamma'))$ with $F(\Gamma)$ isomorphic in a fiber-preserving way to $F(\Gamma')$ for (g, n) and (g', n') lying in Table (F). This is a difficult question. Results of further investigations will appear elsewhere.

As a digression, we reconsider the question in §2.5 which asks whether or not there is any other holomorphic extension of $s_*(\chi_f)$ except for the obvious ones constructed in Proposition 2.5.1. In §2.5 we studied some special cases; that is, when Γ is of signature $(0,4;2,2,\infty,\infty)$, $(0,4;2,2,2,\infty)$, $(0,5;2,2,2,2,\infty)$, or $(0,6;2,2,2,2,2,2)$. With the help of the fact that these fiber spaces are identified with some Teichmüller spaces, we showed that the only holomorphic extensions of $s_*(\chi_f)$ in Aut $F(\Gamma)$ are χ and $\chi \circ e (= e \circ \chi)$. We intend to consider a similar problem which asks whether or not there is any other fiber preserving holomorphic extensions of $s_*(\chi_f)$ when Γ has a general signature. Comparing with Proposition 2.5.2, we have the following:

Proposition 4.3.3 *With the same notation as in §2.5, assume that the signature of* Γ *is neither* $(2,0;\_)$, *nor* $(1,2;\infty,\infty)$, *nor in Table (F). Then* χ *and* $\chi \circ e (= e \circ \chi)$ *are the only two fiber-preserving holomorphic extensions of* $s_*(\chi_f)$.

Proof. Proposition 4.3.2 asserts that every fiber-preserving automorphism of $F(\Gamma)$ is an element of mod Γ. Suppose that χ_0 is a fiber-preserving extension of $s_*(\chi_f)$ distinct from χ and $\chi \circ e$ $(= e \circ \chi)$, then $\chi_0 \in$ mod Γ. Now $\chi_0 \circ \chi^{-1}$ is an element of mod Γ whose restriction to $s(T(\Gamma))$ is the identity. This implies that $\chi_0 \circ \chi^{-1}$ lies in the kernel of the quotient map q_0 : mod $\Gamma \to$ Mod Γ. Therefore, $\chi_0 \circ \chi^{-1} \in \Gamma$. On the other hand, since the restriction of $\chi_0 \circ \chi^{-1}$ to $s(T(\Gamma))$ is the identity, in particular, $\chi_0 \circ \chi^{-1}$ restricts to the central fiber U of $F(\Gamma)$ fixes the fixed point of $e \in \Gamma$. It follows that $\chi_0 \circ \chi^{-1} = id$ or e; that is, either $\chi_0 = \chi$ or $\chi_0 = e \circ \chi = \chi \circ e$. This completes the proof. □

We are in the position to discuss some special situations.

(A) When Γ is of type $(0,3)$, $T(\Gamma)$ is a point, so $F(\Gamma)$ is U. For each point $a \in U$, there are uncountably many Möbius transformations in PSL $(2, \mathbb{R})$ fixing a. Proposition 4.3.3 fails in this case.

(B) When Γ is of signature $(0,4;2,2,\infty,\infty)$ or $(0,4;2,2,2,\infty)$, by Proposition 2.5.2, we see that if a fiber-preserving extension χ of $s_*(\chi_f)$ exists, then χ and $\chi \circ e$ are the only two possible fiber-preserving extensions. So Proposition 4.3.3 remains true in these cases.

(C) If Γ is of signature $(2,0;\_)$, the conclusion of the above Proposition is also valid. To be more precise, let $s\colon T(\Gamma) \to F(\Gamma)$ be a Weierstrass section. Suppose that $s_*(\chi_f)$ can be extended holomorphically to a fiber-preserving $\chi \in$ mod Γ. We can conclude that χ and $\chi \circ e$ $(= e \circ \chi)$ are the only two fiber-preserving extensions of $s_*(\chi_f)$, where $e|_U$ is a lift of the hyperelliptic involution, it is an elliptic Möbius transformation with order 2. Clearly, $e|_U$ is not in the group Γ but in the group $< \Gamma, \tilde{J} >$ (which is generated by 5 elliptic elements of order 2), where $\tilde{J}$ is one of lifts of the hyperelliptic involution. Our assertion can be verified by applying Theorem 4.1.1, Lemma 4.2.3, and Theorem 3A of Greenberg [25] (or Theorem 1 of Singerman [44]). As a matter of fact, those results imply that any fiber preserving automorphism of $F(2,0;\_)$ which acts trivially on the image $s(T(2,0))$ of a canonical section s is actually a lift of the holomorphic involution on $V(2,0;\_)$.

(D) If Γ is of signature $(1,2;\infty,\infty)$, the conclusion is also true; the proof is basically the same as above. However, the assertion is no longer true if Γ is of signature $(1,2;\nu,\nu)$ for $2 \leq \nu < \infty$ since there are uncountably many (global) holomorphic sections of $\pi_0\colon V(1,2;\nu,\nu) \to T(1,2)$.

Remark. There are of course many cases in Table (F) for which we do not know if the motion $s_*(\chi_f)$ can be extended in a fiber preserving way. Further investigations of this question will be pursued in the near future.

4.4 Holomorphic maps of Teichmüller curves

In this section we continue to study the properties of holomorphic maps (or isomorphisms in particular) among Teichmüller curves.

Let Γ and Γ' be finitely generated Fuchsian groups of the first kind which have types (g,n) and (g',n') respectively. Theorem 1.2.1 tells us that $T(\Gamma)$ is isomorphic to $T(\Gamma')$ if Γ and Γ' have the same type. On the other hand, Theorem 1.2.2 asserts that an isomorphism of $T(\Gamma)$ onto $T(\Gamma')$ exists only if Γ and Γ' have the same type except for three special cases (see (1.5)). More precise information on these isomorphisms is given by Theorem 1.2.3, namely, all isomorphisms involved in the context must be "geometric".

The purpose of this section is to obtain similar results in the category of Teichmüller curves. We shall first classify all Teichmüller curves under isomorphisms, and then verify that in general all possible isomorphisms must be "geometric".

A natural question arises as to when two Teichmüller curves are isomorphic. Theorem 1.2.1 asserts that the Teichmüller space $T(\Gamma)$ depends only on the type of Γ while the Bers fiber space $F(\Gamma)$ depends on the signature of Γ. Some examples are given in §1.3 which illustrate that there exist groups with the same type and distinct signatures such that their Bers fiber spaces are essentially distinct. In contrast, the Teichmüller curve $V(\Gamma)$ is semi-independent of the type of Γ. More precisely, we have

Theorem 4.4.1 *Let Γ, Γ' be finitely generated Fuchsian groups of the first kind. Then $V(\Gamma)$ and $V(\Gamma')$ are isomorphic if and only if Γ and Γ' have the same type and contain the same number of conjugacy classes of parabolic elements.*

Remark. The proof of this result is well known. In fact, Earle-Kra [20] proved the result in the case of $g \geq 2$. Their methods are also valid in the cases of $g = 1$ and $g = 0$. The argument is suggested by Kra (oral communication).

Our discussion is based on the fact that every holomorphic automorphism of $V(\Gamma)$ is automatically fiber-preserving.

To verify this result, we refer to Kra [27]. First we assume that Γ contains no parabolic elements. In this case, for each point $x \in T(\Gamma)$, the fiber $\pi_0^{-1}(x)$

is either a compact Riemann surface or an orbifold with no punctures (where $\pi_0 \colon V(\Gamma) \to T(\Gamma)$ is the natural projection defined in §4.1). Let $\zeta \in \mathrm{Aut}\ V(\Gamma)$. Consider the holomorphic map $\pi_0 \circ \zeta \colon \pi_0^{-1}(x) \to T(\Gamma)$. By means of the Bers embedding, $T(\Gamma)$ can be viewed as a bounded domain in $B_2(L, \Gamma) \approx \mathbf{C}^{3g-3+n}$ (see §1.1 for details).

Let $z_j \colon \mathbf{C}^{3g-3+n} \to \mathbf{C}$, $j = 1, \cdots, 3g-3+n$, be coordinate functions. If ζ is not fiber-preserving, then $z_j \circ (\pi_0 \circ \zeta)|_{\pi_0^{-1}(x)} \colon \pi_0^{-1}(x) \to \mathbf{C}$ is a non constant analytic function. Since $\pi_0^{-1}(x)$ is compact, $z_j \circ (\pi_0 \circ \zeta)(\pi_0^{-1}(x))$ is both open and closed. This is impossible unless $z_j \circ (\pi_0 \circ \zeta)|_{\pi_0^{-1}(x)}$ is a constant map on $\pi_0^{-1}(x)$, $j = 1, \cdots, 3g-3+n$. Let $\pi_0 \circ \zeta(\pi_0^{-1}(x)) = y \in T(\Gamma)$. We see that ζ maps the fiber $\pi_0^{-1}(x)$ to the fiber $\pi_0^{-1}(y)$ over y. The assertion then follows in this special case.

The above argument also works in general cases by observing that any conformal map between two orbifolds can be extended to a conformal map between their compactifications.

Theorem 4.4.1 gives us a complete classification for Teichmüller curves. Further, we will see that all possible isomorphisms involved in Theorem 4.4.1 are actually "geometric" except for some special cases. By "geometric isomorphisms of $V(\Gamma)$ onto $V(\Gamma')$" we mean those isomorphisms induced by quasiconformal maps f of U/Γ onto U/Γ' with the property that f sends branch points to branch points, regular points to regular points, and punctures to punctures, but we do not require that f sends a branch point of order ν to a branch point of the same order. We see that the geometric isomorphisms among Teichmüller curves are slightly different from those among Teichmüller spaces.

Theorem 4.4.2 *Let* Γ *and* Γ' *be Fuchsian groups whose types (or signatures) do not lie in Table (F). Then every isomorphism (if exists) of* $V(\Gamma)$ *onto* $V(\Gamma')$ *is geometric.*

To prove this result, we must invoke a theorem in [16] which states that the group Aut $V(\Gamma)$ of holomorphic automorphisms of the Teichmüller curve $V(\Gamma)$ with U/Γ a compact Riemann surface of genus $g \geq 3$ is isomorphic to the Teichmüller modular group Mod Γ. We need to generalize this result to the case when U/Γ is an orbifold.

Let $\mathrm{Aut}_0 V(\Gamma)$ denote the subgroup of Aut $V(\Gamma)$ consisting of those (fiber-preserving) automorphisms ζ whose restriction to each fiber is a conformal equivalence; that is, ζ restricts to a conformal map between orbifolds which sends a branch point of order ν to a branch point of the same order.

It is useful to investigate relationships between the groups $\mathrm{Aut}_0 V(\Gamma)$ and Aut $V(\Gamma)$. First, if U/Γ is a Riemann surface (with no branch points), then we have $\mathrm{Aut}_0 V(\Gamma) = \mathrm{Aut}\ V(\Gamma)$.

Next, we assume that Γ contains elliptic elements, and that (g, n) is the type of Γ which does not lie in Table (F) in §4.1. Let $\zeta \in \mathrm{Aut}\ V(\Gamma)$. Suppose that for some $[\mu_0] \in T(\Gamma)$, $\zeta|_{\pi_0^{-1}([\mu_0])}$ is a conformal map sending each branch point of order ν to a branch point of the same order ν. We claim that for all $[\mu] \in T(\Gamma)$, $\zeta|_{\pi_0^{-1}([\mu])}$ has the same property.

To see this, we chose a canonical section $s: T(\Gamma) \to V(\Gamma)$. As usual, ζ can be projected to a holomorphic automorphism χ of $T(\Gamma)$. It is easy to verify that $\zeta \circ s \circ \chi^{-1}$ is a holomorphic section of $\pi_0: V(\Gamma) \to T(\Gamma)$ with the property that

$$\zeta \circ s \circ \chi^{-1}(\chi([\mu_0])) = s'(T(\Gamma)) \cap \pi_0^{-1}(\chi([\mu_0])),$$

where s' is the canonical section of π_0 determined by the branch point $\zeta(s(T(\Gamma)) \cap \pi_0^{-1}([\mu_0]))$. By using the same argument as in the proof of Lemma 4.2.7, we conclude that $\zeta \circ s \circ \chi^{-1}$ is a canonical section, as asserted.

Another fact is that $\mathrm{Aut}_0 V(\Gamma)$ is a subgroup (need not be normal) of Aut $V(\Gamma)$ of finite index.

Indeed, we know that the set $\mathcal{S}$ of images of all holomorphic sections of $\pi_0 : V(\Gamma) \to T(\Gamma)$ is finite (Theorem 4.1.1). Choose an arbitrary section s of π_0, and let $\zeta \in$ Aut $V(\Gamma)$. Then ζ can be projected to a biholomorphic self-map χ of $T(\Gamma)$. We see that $\zeta \circ s \circ \chi^{-1}$ is another holomorphic section of π_0. Since $\mathcal{S}$ contains all images of holomorphic sections, as a set, we have $\zeta \circ s \circ \chi^{-1}(T(\Gamma)) = \zeta \circ s(T(\Gamma)) \in \mathcal{S}$. It turns out that $\mathcal{S}$ is invariant under Aut $V(\Gamma)$. This in turn implies that there is a large $N \in \mathbb{Z}^+$ so that $\zeta^N \in \mathrm{Aut}_0 V(\Gamma)$ for each $\zeta \in$ Aut $V(\Gamma)$. The assertion then follows.

The proof of Theorem 4.4.2 follows immediately from the following two propositions.

Proposition 4.4.3 (First generalization of Duma's theorem [16]) *Let* Γ *be a* *finitely generated Fuchsian group of the first kind. Then $Aut_0 V(\Gamma)$ is isomorphic to the Teichmüller modular group Mod Γ.*

Let Mod $(g, n; \nu_1, \ldots, \nu_n)$ denote the Teichmüller modular group Mod Γ for Γ of signature $(g, n; \nu_1, \ldots, \nu_n)$, where $2 \leq \nu_1 \leq \ldots \leq \nu_n$ and $\nu_1, \ldots, \nu_n$ are integers or ∞. We also define the modular group mod $(g, n; \nu_1, \ldots, \nu_n)$ in the same way.

Proposition 4.4.4 (Second generalization of Duma's theorem [16]) *Let Γ be of type (g, n). assume that (g, n) does not lie in Table (F). Then Aut $V(\Gamma)$ is isomorphic to Mod $(g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$, where $2 \leq \nu < \infty$ and k is the number of conjugacy classes of parabolic elements of Γ.*

Remark. Proposition 4.4.4 is apparently false if the type of Γ lies in Table (F). For example, when Γ is of signature $(1, 2; \nu, \nu)$ with $\nu < \infty$, there are uncountably many automorphisms of $V(\Gamma)$ keeping each fiber invariant. However, Mod $(1, 2; \nu, \nu)$ is always countable.

The tools which are used to prove Proposition 4.4.3 and Proposition 4.4.4 are applications of the theorems stated in §1.1, §1.2 as well as the finiteness theorem of holomorphic sections of general Teichmüller curves (Theorem 4.1.1).

We also need two basic facts on Teichmüller curves, one of which is merely a restatement of Royden's theorem (see Theorem 1.2.3) stated as follows:

Lemma 4.4.5 *Let Γ be of type (g, n) which is not $(0, 4)$ or $(1, 1)$. Then every* *holomorphic automorphism of $T(g,n)$ can be lifted to an automorphism of the punctured Teichmüller curve $V(g,n)'$.*

The other fact is that, under certain conditions for Γ, every holomorphic map between two punctured Teichmüller curves $V(\Gamma)'$ and $V(\Gamma')'$ is the restriction of a holomorphic map of $V(\Gamma)$ onto $V(\Gamma')$. The precise statement is the following:

Lemma 4.4.6 *Let Γ and Γ' be of type $(g,n) \neq (0,4)$ or $(1,1)$. Assume that both Γ contains no parabolic elements. Then every isomorphism of $T(\Gamma)$ onto $T(\Gamma')$ can be lifted to an isomorphism of $V(\Gamma)$ onto $V(\Gamma')$.*

Finally, we consider two finitely generated Fuchsian groups Γ and Γ_1 of the first kind (acting on U) whose types are $(g,0)$ and $(g,1)$ with $g \geq 2$, respectively. The important Bers isomorphism theorem [8] asserts that the Teichmüller space $T(\Gamma_1) \cong T(g,1)$ is biholomorphically equivalent to a Bers fiber space; or we can say $T(g,1)$ admits a natural fibration structure. Associated to this natural fibration structure there is a holomorphic submersion $t\colon T(g,1) \to T(g,0)$ whose Fréchet derivative is complex linear at each point. The submersion t is usually called the *forgetful map* in the literature since it is defined by forgetting the distinguished point. A natural question arises as to whether or not a fibration $\tau : T(g,1) \to T(g,0)$ with fiber conformally equivalent to the unit disk is a natural (Bers) fibration structure of $T(g,1)$. As a final result of this paper, we will prove:

Theorem 4.4.7 *Let $\tau\colon T(g,1) \to T(g,0)$ be a fibration with fiber conformally* *equivalent to the unit disk. Assume that $g \geq 2$. Then, up to a modular transformation on $T(g,0)$, τ defines a natural (Bers) fibration structure of $T(g,1)$ if and only if there is a holomorphic map $\zeta: V(g,1) \to V(g,0)$ so that the following diagram is commutative:*

$$\begin{array}{ccc} V(g,1) & \xrightarrow{\zeta} & V(g,0) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ T(g,1) & \xrightarrow[\tau]{} & T(g,0) \end{array} \tag{4.7}$$

The next section is devoted to the proof of these results.

4.5 Proofs of Theorems 4.4.1, 4.4.2 and 4.4.7

Proof of Theorem 4.4.1. Let Γ amd Γ' be of type (g,n) and (g',n'), respectively. Suppose that we have an isomorphism $\xi : V(\Gamma) \to V(\Gamma')$. Then ξ is fiber-preserving. Hence ξ can be projected to an isomorphism of $T(\Gamma)$ onto $T(\Gamma')$. From Theorem 1.2.2, we conclude that $(g,n) = (g',n')$ unless the unordered pair $((g,n),(g',n'))$ lies in the following short list:

$$((2,0),(0,6)), \quad ((1,2),(0,5)), \quad ((1,1),(0,4)).$$

Since the restriction $\xi_{[\mu]}$ of ξ to a fiber $\pi_0^{-1}([\mu])$ is holomorphic; it is a conformal map between two orbifolds. It turns out that the above exceptional cases cannot happen. We see that $g = g'$ and the two groups Γ, Γ' have the same number of conjugacy classes of parabolic elements.

The reverse direction of this result is given by Earle-Kra [20] in the case of g, $g' \geq 2$. In the case of g, $g' = 0$ or 1, Kra suggest a similar proof (oral communication). We outline a proof here for convenience of the reader.

Again, let Γ be of type (g, n), and let k, $0 \leq k \leq n$, be the number of punctures on U/Γ.

Case 1. $g = g' \geq 2$ and $k \geq 0$. Let Γ_k be a torsion free group of type (g, k) so that $U/\Gamma = U/\Gamma_k = S$ as Riemann surfaces (with k punctures). Let $p : U \to S$ and $p_k : U \to S$ be the projections with covering groups Γ and Γ_k respectively. Then there is a surjective holomorphic holomorphic map $h_k : U \to U$ with $p = p_k \circ h_k$ and an epimorphism $\chi_k : \Gamma \to \Gamma_k$ such that $h_k \circ \gamma = \chi_k(\gamma) \circ h_k$ for all $\gamma \in \Gamma$ (see Lemma 4.1 of [20]). For every $\mu \in M(\Gamma)$ (see §1.1 for the definition). We define $h_{k,*}(\mu)$ by the equation

$$h_{k,*}(\mu) \circ h_k = \mu h_k' / \overline{h_k'}. \tag{4.8}$$

It is easy to see that $h_{k,*}$ is a well-defined bijective isometry of $M(\Gamma)$ onto $M(\Gamma_k)$. Consider now the map

$$h_k^\mu = w^\nu \circ h_k \circ (w^\mu)^{-1}, \quad \nu = h_{k,*}(\mu). \tag{4.9}$$

From Corollary 4.3 and Lemma 4.2 of [20], h_k^μ is independent of choice of element in the equivalence class of μ and depends holomorphically on μ. Define $t_k \colon T(\Gamma) \to T(\Gamma_k)$ and a map $f_k \colon F(\Gamma) \to F(\Gamma_k)$ by

$$t_k([\mu]) = [h_{k,*}(\mu)] \tag{4.10}$$

$$f_k([\mu], z) = (t([\mu]), h_k^\mu(z)). \tag{4.11}$$

The map t_k and f_k defined above are the forgetful maps. We thus obtain a holomorphic map ξ_k so that the following diagram commutes:

$$\begin{array}{ccc} V(\Gamma) & \xrightarrow{\xi_k} & V(g,k)' \\ {\scriptstyle \pi_n}\big\downarrow & & \big\downarrow{\scriptstyle \pi_k} \\ T(\Gamma) & \xrightarrow[t_k]{} & T(g,k) \end{array} \tag{4.12}$$

where $V(g,k)' \cong V(\Gamma_k)$ is the punctured Teichmüller curve of type (g,k) (see §4.1 for the definition). From Theorem 4.6 of [20], the map $\pi_n \times \zeta_k : V(\Gamma) \to T(\Gamma) \times V(g,k)'$ defines a biholomorphic map of $V(\Gamma)$ onto the closed submanifold

$$W = \{(x,z);\ t_k(x) = \pi_k'(z)\}.$$

Similarly, there exists a holomorphic map $\xi_k' \colon V(\Gamma') \to V(g,k)'$ (with respect to the forgetful map $t_k' : T(\Gamma') \to T(g,k)$) so that the diagram (4.12) also commutes (where the Teichmüller curve $\pi_n : V(\Gamma) \to T(\Gamma)$ is replaced by $\pi_n' \colon V(\Gamma') \to T(\Gamma')$). We may assume (without loss of generality) that $U/\Gamma = U/\Gamma' = S$ as Riemann surfaces. There is an isomorphism $\lambda : T(\Gamma) \to T(\Gamma')$ which is called the Bers-Greenberg isomorphism (see [11]). We thus obtain

$$t_k = t_k' \circ \lambda.$$

This implies in particular that

$$t_k(T(\Gamma)) = t_k'(T(\Gamma')) \qquad \text{and} \qquad \zeta_k'(V(\Gamma')) = \zeta_k(V(\Gamma)).$$

Again, from Theorem 4.6 of [20], $V(\Gamma')$ is biholomorphically equivalent to the closed submanifold $W' = \{(x',z);\ t_k'(x') = \pi_k'(z)\}$. Let $(x,z) \in W$. Since

$t'_k(\lambda(x)) = t_k(x) = \pi'_k(z)$, we have $(\lambda(x), z) \in W'$. Thus the map defined by sending $(x, z) \in W$ to $(\lambda(x), z) \in W'$ is a biholomorphic map. It follows that $V(\Gamma)$ is biholomorphically equivalent to $V(\Gamma')$.

Case 2. $g = g' = 0$, $k > 3$. The same argument as in Case 1 applies.

Case 3. $g = g' = 1$, $k \geq 1$. Again, use a similar argument as in Case 1.

The following two cases are treated in a slightly different way; the reason is: when all branch points on U/Γ are neglected, we will obtain a Riemann surface which is not hyperbolic. So in general we do not have equations (4.8)–(4.11).

Case 4. $g = g' = 0$ and $k \leq 3$. Again, by forgetting all branch points of finite order, we obtain the commutative diagram:

$$\begin{array}{ccc} V(\Gamma) & \xrightarrow{\xi_0} & \tilde{\mathbb{C}} \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ T(\Gamma) & \xrightarrow[t_0]{} & \{\text{point}\} \end{array}$$

where $\tilde{\mathbb{C}}$ is the Riemann sphere with $k(\leq 3)$ points removed. t_0 is holomorphic, and it is also easy to see that ζ_0 is holomorphic. So $V(\Gamma)$ is isomorphic to the trivial bundle $T(\Gamma) \times \tilde{\mathbb{C}}$. Similarly, $V(\Gamma')$ is isomorphic to the product $T(\Gamma') \times \tilde{\mathbb{C}}$. Since $T(\Gamma) \cong T(\Gamma')$, the result follows.

Case 5. $g = g' = 1$ and $k = 0$. Let $(1, n; \nu_1, \ldots, \nu_n)$ and $(1, n'; \nu'_1, \ldots, \nu'_{n'})$ denote the signatures of Γ and Γ', respectively. If there is a branch point x_i

on U/Γ whose order ν_i is the same as the order of a branch point x'_j on U/Γ', then the method of Case 1 is applicable; the forgetful map ξ_0 (resp. ξ'_0) are obtained by forgetting all branch points except x_i (resp. x'_j).

If $n > 1$ and $\nu_i \neq \nu'_j$ for all i and j, then we choose a branch point x_i on U/Γ and a branch point x'_j on U/Γ', the forgetful map ξ_0 (resp. ξ'_0) is obtained by forgetting all branch points except x_i (resp. x'_j). So it remains to show that $V(1,1;\nu)$ is isomorphic to $V(1,1;\nu')$ with $\nu \neq \nu'$. But this fact is clear, since the map $\tilde{\lambda}\colon V(1,1;\nu) \to V(1,1;\nu')$ defined by $\tilde{\lambda}([\mu],z) = (\lambda([\mu]),z)$ does the job, where, as before, λ is the Bers-Greenberg isomorphism. □

Proof of Proposition 4.4.3. Let $\zeta \in \mathrm{Aut}_0 V(\Gamma)$. We know that ζ is fiber-preserving. Hence ζ can be projected to an automorphism χ of $T(\Gamma)$. From the proof of Proposition 4.3.2, we see that $\chi \in \mathrm{Mod}\,\Gamma$. Define

$$\Theta\colon \mathrm{Aut}_0 V(\Gamma) \to \mathrm{Mod}\,\Gamma$$

by sending ζ to χ.

On the other hand, for every $\chi \in \mathrm{Mod}\,\Gamma$, there is $\tilde{\chi} \in \mathrm{mod}\,\Gamma$ in the preimage of χ under the quotient map $q_0\colon \mathrm{mod}\,\Gamma \to \mathrm{Mod}\,\Gamma$ (defined in §1.2). Then there is $\zeta' \in \mathrm{Aut}_0 V(\Gamma)$ so that $p_0 \circ \tilde{\chi} = \zeta' \circ p_0$, where $p_0\colon F(\Gamma) \to V(\Gamma)$ is defined in §4.2. Define

$$\Theta'\colon \mathrm{Mod}\,\Gamma \to \mathrm{Aut}_0 V(\Gamma)$$

by sending χ to ζ'. It is easy to check that ζ' is independent of choice in the preimage of χ, so it is well defined. To see that $\Theta' \circ \Theta$ is the identity on $\mathrm{Aut}_0 V(\Gamma)$, we must show that $\zeta' \circ \zeta^{-1} \in \mathrm{Aut}_0 V(\Gamma)$ is the identity. By definition,

$\zeta' \circ \zeta^{-1}$ is an automorphism which leaves invariant each fiber. There are several cases to consider.

Case 1. Γ is not of type $(0,3)$, $(0,4)$, $(1,1)$, $(1,2)$, or $(2,0)$. In this case, the action of Mod Γ on $T(\Gamma)$ is effective. Observe that the conformal map $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([0])}$ of U/Γ to itself determines a modular transformation which acts trivially on $T(\Gamma)$ (Lemma 4.3.1). We conclude that $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([0])}$ is isotopic to the identity. Hence, $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([0])}$ is the identity. Similarly, we can prove that $\zeta' \circ \zeta^{-1}$ is trivial on each fiber.

Case 2. Γ is of type $(0,3)$ or $(0,4)$. By definition of ζ', we know that for every $[\mu] \in T(\Gamma)$, $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])}$ is a conformal self-map keeping all distinguished points fixed. Since $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])} \in \mathrm{PSL}(2,\mathbb{C})$, one sees at once that $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])}$ is the identity.

Case 3. Γ is of signature $(1,2,\nu,\mu)$ with $\mu \neq \nu$. In this case, $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])}$ fixes the two branch points for all $[\mu] \in T(\Gamma)$. Let X_μ denote $\pi_0^{-1}([\mu])$ with the two branch points removed. Then $\zeta' \circ \zeta^{-1}$ is a conformal automorphism of X_μ for all $[\mu] \in T(\Gamma)$. It induces a trivail action on $T(1,2) \cong T(0,5)$. Since Mod $(1,2)/\mathbb{Z} \cong$ Mod $(0,5)$, we see that $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])}$ is either the identity or J_μ (where J_μ is the hyperelliptic involution of X_μ. Since $\zeta' \circ \zeta^{-1}|_{\pi_0^{-1}([\mu])}$ fixes the two punctures, we are done.

Case 4. Γ is of signature $(1,1;\nu)$, $(1,2;\nu,\nu)$, or $(2,0;\_)$, where $\nu \geq 2$ is an integer or ∞. In this case, let f be a self-map of a surface $X = U/\Gamma$ of signature $(2,0;\text{--})$ which induces $\chi \in \text{Mod}\,(2,0)$. Choose $\hat{f}: U \to U$ so that the diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{\hat{f}} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ U/\Gamma & \xrightarrow[f]{} & U/\Gamma \end{array}$$

Let $[\hat{f}]$ be the set consisting of all self-maps $\hat{f}'$ of U with the properties:

(1) $\hat{f}'\Gamma\hat{f}'^{-1} = \Gamma$, and

(2) $\hat{f}'$ is isotopic to $\hat{f}$ in U.

Then $[\hat{f}] = \tilde{\chi} \in \text{mod}\,\Gamma$ is a preimage of χ under q_0. It follows that $\zeta' \circ \zeta^{-1}$ is not the hyperelliptic involution. The discussion is the same when Γ is of signature $(1,1;\nu)$ or $(1,2;\nu,\nu)$. The argument in Case 3 can then be applied in this case.

We have shown that $\Theta' \circ \Theta$ is the identity on $\text{Aut}_0V(\Gamma)$. The proof that $\Theta \circ \Theta'$ is the identity on Mod Γ is obvious by construction. In particular, we see that Θ is a bijection. The proof that Θ is a group homomorphism is trivial. This proves Proposition 4.4.3. $\square$

Remark. The group $\text{Aut}_0V(\Gamma)$ acts on $V(\Gamma)$ in the following way: we take $\zeta \in \text{Aut}_0V(\Gamma)$. For any point $x \in V(\Gamma)$. Choose a point $\tilde{x}$ in the set $p_0^{-1}(x)$, and choose a lift $\tilde{\zeta}: F(\Gamma) \to F(\Gamma)$ of ζ. We have the following diagram:

$$
\begin{array}{ccc}
F(\Gamma) & \xrightarrow{\tilde{\zeta}} & F(\Gamma) \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_0} \\
V(\Gamma) & \xrightarrow[\zeta]{} & V(\Gamma)
\end{array}
\tag{4.13}
$$

We see that $\zeta(x)$ is defined by $p_0 \circ \tilde{\zeta}(\tilde{x})$. Now it is easy to check from the diagram (4.13) that ζ is a well defined holomorphic automorphism.

Proof of Lemma 4.4.5. Let χ be a holomorphic automorphism of $T(g,n)$. Theorem 1.2.1 implies that χ defines a holomorphic automorphism (call it χ also) of $T(\Gamma)$ for a torsion free group Γ of type (g,n). Assume that Γ is not of type $(0,3)$, $(0,4)$, $(1,1)$, $(1,2)$ or $(2,0)$. By using Theorem 1.2.3, we know that χ is an element of Mod Γ. Let $\tilde{\chi} \in$ mod Γ be a preimage of χ under $q_0\colon$ mod $\Gamma \to$ Mod Γ. $\tilde{\chi}$ can be projected to a holomorphic automorphism ζ of $V(\Gamma) \cong V(g,n)'$. Of course, we should check that ζ is well defined. But this is done in the proof of Proposition 4.4.3. There are several exceptional cases to consider.

If Γ is of type $(0,3)$, there is nothing to prove. We assume that Γ is of signature $(2,0;\_)$. Since $T(2,0) \cong T(0,6)$, χ actually defines a holomorphic automorphism (call it χ also) of $T(0,6)$. By Theorem 1.2.3 and the above argument, χ can be lifted to a holomorphic automorphism of $V(0,6;\infty,\dots,\infty)$. But since every (quasiconformal) self-map of a surface of signature $(0,6;\infty,\dots,\infty)$ can be lifted to its double cover whose compactification is of signature $(2,0;\_)$, the assertion then follows. The case that Γ is of signature $(1,2;\infty,\infty)$ can be handled similarly; the only issue is that every holomorphic automorphism of $T(1,2;\infty,\infty)$ determines a holomorphic automorphism of $T(0,5;2,2,2,2,\infty)$ which is induced by a self-map of a surface of signature $(0,5;2,2,2,2,\infty)$. Now the above argument works equally well in this case. □

Remark. When $(g,n) = (0,4)$ or $(1,1)$, we do not know whether or not the lemma is still true. It is obvious that any modular transformation of $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ (or $T(1,1;\nu)$, where ν_1,ν_2,ν_3,ν_4 and $\nu \geq 2$ are integers) can be lifted to an automorphism of $V(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$. On the other hand, we have Aut $T(0,4) \cong \mathrm{PSL}(2,\mathbb{R}) \cong$ Aut $T(1,1)$. The question is: whether or not any other automorphism ($\in \mathrm{PSL}(2,\mathbb{R})$) can also be lifted to an automorphism of $V(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$. We expect but cannot prove that an $\eta \in$ Aut $T(0,4)$ can be lifted to an automorphism of $V(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ if and only if η belongs to the group generated by Mod $(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ and the three conformal involutions.

Proof of Lemma 4.4.6 By Theorem 4.4.1, we see that Γ and Γ' have the same type (g,n). We assume $(g,n) \neq (0,3)$ (if $(g,n) = (0,3)$, the lemma is trivial). By hypothesis, dim $T(\Gamma) =$ dim $T(\Gamma') \geq 2$. By Theorem 4.4.1, we can find isomorphisms $\xi : V(\Gamma) \to V(\Gamma_0)$ and $\xi' : V(\Gamma') \to V(\Gamma_0)$, where Γ_0 is a Fuchsian group of the signature $(g,n;2,\ldots,2)$. Observe that both of the isomorphisms carry the images of canonical sections to the images of canonical sections.

Let θ be an isomorphism of $T(\Gamma)$ onto $T(\Gamma')$. Theorem 1.2.1 implies that θ actually defines an isomorphism (call it θ also) of $T(\Gamma_\infty)$ onto $T(\Gamma'_\infty)$, where Γ_∞ (resp. Γ'_∞) is the Fuchsian group so that U/Γ_∞ is conformally equivalent to U/Γ − {all branch points} (resp. U/Γ' − {all branch points}). By Lemma 4.4.5, θ can be lifted to an isomorphism ζ of $V(\Gamma_\infty)$ ($\cong V(\Gamma)'$) onto $V(\Gamma'_\infty)$ ($\cong V(\Gamma')'$).

Consider the map $\eta = \xi' \circ \zeta \circ \xi^{-1}|_{V(\Gamma_0)'} \colon V(\Gamma_0)' \to V(\Gamma_0)'$. We claim that η can be extended holomorphically to an automorphism $\tilde{\eta} \colon V(\Gamma_0) \to V(\Gamma_0)$.

To see this, first observe that η is fiber-preserving. Hence η can be projected to a biholomorphic self-map χ of $T(\Gamma_0)$ which is induced by a self-map f of U_{Γ_0}/Γ_0 (see §4.1 for the definition of U_{Γ_0}). By assumption, all branch points of U/Γ_0 have the same order and f sends a puncture to a puncture, we see that f can be extended to a self-map (denoted by f again) of U/Γ_0 in the sense of orbifolds. Clearly, f can be lifted to a self-map $\tilde{f}$ of U with the property that $\tilde{f}\Gamma_0\tilde{f}^{-1} = \Gamma_0$. Furthermore, $\tilde{f}$ induces a Bers allowable mapping $[\tilde{f}]_*$ of $F(\Gamma_0)$ onto itself. This means that $[\tilde{f}]_* \in \operatorname{mod}\Gamma$ and $[\tilde{f}]_*$ can be projected to a biholomorphic map $\tilde{\eta}$ of $V(\Gamma_0)$ onto itself.

Clearly, $\tilde{\eta}$ can be projected to the automorphism χ defined above. It follows that $\tilde{\eta}|_{V(\Gamma_0)'} \circ \eta^{-1}$ can be projected to the identity, and hence $\tilde{\eta}|_{V(\Gamma_0)'} \circ \eta^{-1}$ is an automorphism which leaves invariant each fiber. By using the same proof of Proposition 4.4.3, we conclude that $\tilde{\eta}|_{V(\Gamma_0)'} \circ \eta^{-1}$ is the identity. It follows that $\eta = \tilde{\eta}|_{V(\Gamma_0)'}$. Therefore, $\tilde{\zeta} = \xi'^{-1} \circ \tilde{\eta} \circ \xi \colon V(\Gamma) \to V(\Gamma')$ is the desired lifting of θ. This completes the proof. □

Proof of Proposition 4.4.4 (Sketch). First we assume that the type of Γ is neither $(2,0)$ nor $(1,2)$. Let $\zeta \in \operatorname{Aut} V(\Gamma)$. By Theorem 4.1.1, the set $\mathcal{S}$

of images of (canonical) sections of $\pi_0: V(\Gamma) \to T(\Gamma)$ is finite. By the same argument as in Lemma 4.2.7, we obtain $\zeta(\mathcal{S}) = \mathcal{S}$. This implies that ζ restricts to an automorphism (call it ζ also) in Aut $V(\Gamma)'$. By Lemma 4.4.6 (with a slight modification), we see that ζ extends to an automorphism (call it ζ also) of $V(\Gamma_0)$, where Γ_0 is a group of signature $(g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$, k is the number of conjugacy classes of parabolic elements of Γ, and $\nu \geq 2$ is an integer. By using the same argument of Proposition 4.3.2, we conclude that ζ can be projected to $\chi \in \text{Mod}\ (g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$. Define

$$\Theta \colon \text{Aut}\ V(\Gamma) \to \text{Mod}\ (g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$$

by sending ζ to χ.

On the other hand, let $\chi \in \text{Mod}\ (g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$, and let $\tilde{\chi} \in \text{mod}\ (g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k})$ be a preimage of χ under the quotient homomorphism q_0. As usual, $\tilde{\chi}$ can be projected to an automorphism $\zeta' \in \text{Aut}\ V(\Gamma_0)$ in the sense that $p_0 \circ \tilde{\chi} = \zeta' \circ p_0$, where $p_0 \colon F(\Gamma_0) \to V(\Gamma_0)$ is defined in §4.2.

Clearly, ζ' restricts to an element (call it ζ' also) in Aut $V(\Gamma_0)'$. By Lemma 4.4.6, ζ' defines an automorphism (call it ζ' again) of Aut $V(\Gamma)$. Define

$$\Theta' \colon \text{Mod}\ (g, n; \nu, \ldots, \nu, \underbrace{\infty, \ldots, \infty}_{k}) \to \text{Aut}\ V(\Gamma)$$

by sending χ to ζ'. It is easy to check that ζ' is well defined. The only issue is to show that $\Theta' \circ \Theta$ is the identity; that is, we have to show that $\zeta' \circ \zeta^{-1}$ restricts to the identity map on each fiber. Now the proof remains the same as Proposition 4.4.3.

If Γ is of type $(2,0)$, then by a discussion in §4.4, we have $\mathrm{Aut}\ V(\Gamma) \cong \mathrm{Aut}_0 V(\Gamma)$. The assertion follows from Proposition 4.4.3.

If Γ is of signature $(1,2;\nu,\infty)$ with $2 \leq \nu < \infty$, we use the same argument as in Case 3 of Proposition 4.4.3. Note that each $\zeta \in \mathrm{Aut}\ V(\Gamma)$ cannot send a holomorphic section to the section determined by the puncture.

If Γ is of signature $(1,2;\infty,\infty)$, then the argument in Case 4 of Proposition 4.4.3 applies. □

Proof of Theorem 4.4.2. Let $\zeta : V(\Gamma) \to V(\Gamma')$ be an isomorphism. By Theorem 4.4.1, Γ and Γ' have the same type (g,n) and contain the same number k of conjugacy classes of parabolic elements. Let G, G' be Fuchsian groups of signature $(g,n;\nu,\ldots,\nu,\underbrace{\infty,\ldots,\infty}_{k})$ ($\nu \geq 2$ is an integer) such that

$$U_\Gamma/\Gamma \cong U_G/G \quad \text{and} \quad U_{\Gamma'}/\Gamma' \cong U_{G'}/G'.$$

By definition, we know that there is a quasiconformal map (in the sense of orbifolds) $f : U/G \to U/G'$ which induces a geometric isomorphism $f_* : V(G) \to V(G')$. There are also conformal maps $\alpha : U_\Gamma/\Gamma \to U_G/G$ and $\alpha' : U_{\Gamma'}/\Gamma' \to U_{G'}/G'$.

By Lemma 4.4.6 and our definition of geometric isomorphisms (see §4.4), the map α (resp. α') induces a geometric isomorphism α_* of $V(\Gamma)$ onto $V(G)$ (resp. α_* of $V(\Gamma')$ onto $V(G')$). Consider the automorphism

$$\zeta_0 = \alpha_*'^{-1} \circ f_* \circ \alpha_* \circ \zeta^{-1} \in \mathrm{Aut}\ V(\Gamma').$$

By Proposition 4.4.4, ζ_0 is a geometric isomorphism. Hence, ζ is a geometric isomorphism. This completes the proof of Theorem 4.4.2.

Summary. Theorem 1.2.1, Theorem 1.2.2 and Theorem 1.2.3 tell us that, except three special cases (see (1.5)), all isomorphisms between Teichmüller spaces are "weakly geometric", by which we mean all isomorphisms are induced by quasiconformal maps between punctured Riemann surfaces. Theorem 4.2.1 and Theorem 0.4 say that except Table (F) all fiber-preserving isomorphisms among Bers fiber spaces must be geometric, by which we mean all fiber-preserving isomorphisms are induced by quasiconformal maps in the sense of orbifolds. Theorem 4.4.1 and Theorem 4.4.2 assert that except Table (F) all isomorphisms among Teichmüller curves are "semi geometric", by which we mean all isomorphisms are induced by those quasiconformal maps between two punctured Riemann surfaces which send punctures to punctures (on original orbifolds).

Proof of Theorem 4.4.7. The "only if" part is a special case of Theorem 4.5 of [20] (see also Case 1 in the proof of Theorem 4.4.1). We only need to prove the reverse direction.

Suppose that we have commutative diagram (4.7). It is immediate that ζ is fiber-preserving. Hence the restriction of ζ to each fiber $\pi_1^{-1}(x)$, $x \in T(g,1)$, is a holomorphic map from the orbifold (represented by x) to the Riemann surface X_0 of type $(g,0)$ (represented by $\tau(x) \in T(g,0)$). Notice that $\pi_1^{-1}(x)$ is a compact orbifold of genus g with one branch point. The map ζ actually defines a holomorphic surjective map $\zeta_x : X_x \to X_0$ between two Riemann surfaces of type $(g,0)$. Thus ζ_x is a covering map. (To see this, note that for every point $a \in X_0$, $\zeta_x^{-1}(a)$ is a finite set and the cardinality of this set is independent of choice of $a \in X_0$.) Since $g \geq 2$, Riemann-Hurwitz formula (see, for example, Farkas-Kra [21]) implies that ζ_x is a covering map with degree 1; that is, ζ_x is a conformal map for each $x \in T(g,1)$.

Choose an arbitrary fiber $D = \tau^{-1}(x)$, $x \in T(g,0)$. By hypothesis, D is conformally equivalent to the unit disk. We claim that the map $t|_D$ must be a constant map (recall that t is the forgetful map).

Suppose not, then the image of D under t is a connected and path-connected subset of $T(g,0)$. Define a map

$$\lambda \colon V(g,1) \to V(g,0)$$

by forgetting the branch point. Thus the following diagram is clearly commutative:

$$\begin{array}{ccc} V(g,1) & \xrightarrow{\lambda} & V(g,0) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ T(g,1) & \xrightarrow[t]{} & T(g,0) \end{array}$$

For each point $x \in D$, $\zeta|_{\pi_1^{-1}(x)}$ is holomorphic as a map between two orbifolds. By definition of λ the two compact Riemann surfaces $\lambda(\pi_1^{-1}(x))$ and $\zeta(\pi_1^{-1}(x))$ are conformally equivalent. Hence, we can find a modular transformation $\chi_x \in$ Mod $(g,0)$ which sends the point $\pi_0 \circ \zeta(\pi_1^{-1}(x)) \in T(g,0)$ to the point $\pi_0 \circ \lambda(\pi_1^{-1}(x)) \in T(g,0)$, namely, χ_x sends $\tau(x)$ to $t(x)$.

On the other hand, by definition, the set $\{\tau(x);\ x \in D\}$ is a single point, whereas $t(D)$ contains a continuous arc $c(r)$, $r \in [0,1]$, in $T(g,0)$. We thus obtain a continuous family $\{\chi_x(r)\}_{r \in [0,1]} \in \mathrm{Mod}(g,0)$ sending the point $\tau(x) \in$

$T(g,0)$ to the continuous arc $c(r)$, contradicting the fact that the Teichmüller modular group $\mathrm{Mod}(g,0)$ acts discontinuously on $T(g,0)$. We conclude that for each fiber D, $t|_D$ is a constant map, as asserted. Now we define a self-map $\chi : T(g,0) \to T(g,0)$ by carrying each point x to $\chi(x) = t(y)$, where y is a preimage of $\tau^{-1}(x)$. The above argument shows that $\chi(x)$ is independent of choice of y in $\tau^{-1}(x)$.

It is easy to show that the map χ defined above is one to one and onto. Since τ and t are holomorphic and since t has a local holomorphic section passing through $y \in t^{-1}(x)$, we conclude that χ is a biholomorphic automorphism of $T(g,0)$. It follows from Theorem 1.2.3 that $\chi \in \mathrm{Mod}(g,0)$. Therefore, $\chi \circ \tau = t$, and the fibration structure of $T(g,1)$ determined by t coincides with that determined by $\chi \circ \tau$. This completes the proof of Theorem 4.4.7.

□

Theorem 4.4.7 has a direct consequence, which is stated as follows.

Corollary 4.5.1 *Let Γ_1 and Γ be as in Theorem 4.4.7, Assume that there are holomorphic maps $t' : T(\Gamma_1) \to T(\Gamma)$, $f' : F(\Gamma_1) \to F(\Gamma)$, and $\zeta' : V(\Gamma_1) \to V(\Gamma)$ so that the diagram*

$$\begin{array}{ccc} F(\Gamma_1) & \xrightarrow{f'} & F(\Gamma) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ V(\Gamma_1) & \xrightarrow[\zeta']{} & V(\Gamma) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ T(\Gamma_1) & \xrightarrow[t']{} & T(\Gamma) \end{array} \tag{4.14}$$

commutes, where p_0, p_1, π_0, *and* π_1 *are the natural projections. Assume further that for each* $x \in T(\Gamma)$, $t'^{-1}(x)$ *is conformally equivalent to the unit disk. Then there is a* $\theta \in mod\,\Gamma$ *so that* $f' = \theta \circ f$, *and* $t' = q_0(\theta) \circ t$, *where* f *and* t *are defined by (4.11), (4.10), and* $q_0 : mod\,\Gamma \to Mod\,\Gamma$ *is the natural quotient homomorphism.*

To prove this corollary, we need a simple lemma.

Lemma 4.5.2 *Let* G *be a finitely generated Fuchsian group of the first kind, let* $\{\gamma_{[\mu]}\}$, $[\mu] \in T(G)$, *be a continuous family of Möbius transformations so that for each* $[\mu] \in T(G)$, $\gamma_{[\mu]} \in G^\mu$. *Suppose that* $\gamma_{[0]} = \gamma$ *for a* $\gamma \in G$. *Then* $\{\gamma_{[\mu]}\}$ *is a holomorphic family and for each* $[\mu] \in T(G)$, $\gamma_{[\mu]} = \gamma^\mu$.

Proof. Since G acts discontinuously on U, we can choose a point $z \in U$, which is not a fixed point of elliptic elements of G, so that $\epsilon = \inf\,\{\rho_U(\gamma(z), \sigma(z));\ \sigma \in G,\ \sigma \neq \gamma\}$ is a positive number, where ρ_E is the Poincaré metric on the domain E. Since $\{\gamma_{[\mu]}\}$ is a continuous family, for $[\mu]$ sufficiently close to $[0]$, z stays in $w^\mu(U)$ and satisfies

$$\inf\,\{\rho_{w^\mu(U)}(\gamma_{[\mu]}(z), \sigma^\mu(z));\ \sigma^\mu \in G^\mu,\ \sigma^\mu \neq \gamma^\mu\} > \epsilon/2.$$

It turns out that the set $\mathcal{A}_1 = \{[\mu] \in T(G);\ \gamma^\mu = \gamma_{[\mu]}\}$ is an open neighborhood of $[0]$. But it is easy to see that the complement $\mathcal{A}_1^c$ of $\mathcal{A}_1$ is also open. Thus, $\mathcal{A}_1 = T(G)$, as we claimed. □

Proof of Corollary 4.5.1. By Theorem 4.4.7, we see at once that $t' = \chi \circ t$ and $\zeta' = \hat{\chi} \circ \zeta$, where t, ζ are the forgetful maps defined by (4.10) and (4.11), respectively. $\chi \in \text{Mod}\,\Gamma$ and $\hat{\chi} \in \text{Aut}(V(\Gamma))$ are determined by Proposition 4.4.3. Clearly, $\hat{\chi}$ can be lifted to a geometric automorphism $\tilde{\chi} \in \text{mod}\,\Gamma$. Hence, $q_0(\tilde{\chi}) = \chi$.

It remains to prove the corollary under the assumption that t' and ζ' are the forgetful maps defined by (4.10) and (4.11). We select from the upper half of diagram (4.13) an arbitrary slice, say that determined by the fiber $([\mu], w^\mu(U))$ for a $\mu \in M(\Gamma_1)$, and obtain the following commutative diagram:

$$\begin{array}{ccc} w^\mu(U) & \xrightarrow{f'} & w^\nu(U) \\ {\scriptstyle p_\mu}\big\downarrow & & \big\downarrow{\scriptstyle p_\nu} \\ w^\mu(U)/\Gamma^\mu & \xrightarrow[\zeta']{} & w^\nu(U)/\Gamma^\nu \end{array} \qquad (4.15)$$

where p_μ and p_ν are the natural projections. Note that ζ' is the identity map on the base Riemann surface. From Lemma 4.1 of [20], f' is a surjective map and it induces an epimorphism $\chi_*\colon \Gamma^\mu \to \Gamma^\nu$ defined by

$$f' \circ \gamma^\mu = \chi_*(\gamma^\mu) \circ f' \quad \text{for any } \gamma^\mu \in \Gamma^\mu.$$

On the other hand, the map $h^\mu\colon w^\mu(U) \to w^\nu(U)$ defined by (4.9) satisfies the condition that $p_\nu \circ h^\mu = \zeta' \circ p_\mu$ (that is, we also have the above diagram where f' is replaced by the map h^μ). It follows from (4.14) that for each $z \in w^\mu(U)$, there is a $\gamma_{z,[\nu]} \in \Gamma^\nu$ so that

$$h^\mu(z) = \gamma_{z,[\nu]} \circ f'(z). \qquad (4.16)$$

Since both f' and h^μ are continuous in terms of z and Γ^ν is discrete, $\gamma_{z,[\nu]}$ is independent of choice of $z \in w^\mu(U)$. Let $\gamma_{z,[\nu]} = \gamma_{[\nu]}$. Equation (4.15) yields

$$h^\mu(z) = \gamma_{[\nu]} \circ f'(z), \quad \text{for a } \gamma_{[\nu]} \in \Gamma^\nu. \tag{4.17}$$

Now let $\mu' \in M(\Gamma_1)$ be so close to μ that z stays in $w^{\mu'}(U)$. Since h^μ depends holomorphically on $\mu \in M(\Gamma)$ (Lemma 4.2 of [20]) and f' is holomorphic on $F(\Gamma_1)$, we conclude that the family $\{\gamma_{[\nu]}\}$, $[\nu] \in T(\Gamma)$, defined by (4.16) is a continuous family. From Lemma 4.5.2, we see that $\gamma_{[\nu]} = \gamma^\nu$, $[\nu] \in T(\Gamma)$. Therefore, we have

$$h = \gamma \circ f',$$

where γ is viewed as a holomorphic automorphism of $F(\Gamma)$, and h is defined by sending the point $([\mu], z)$ to the point $(t([\mu]), h^\mu(z))$. Observe that γ is an element of mod Γ and that γ lies in the kernel of the quotient map $q_0\colon \text{mod } \Gamma \to \text{Mod } \Gamma$. Taking $\tilde{\chi}_0 = \gamma \circ \tilde{\chi}$, we have $q_0(\tilde{\chi}_0) = q_0(\gamma \circ \tilde{\chi}) = q_0(\gamma) \circ q_0(\tilde{\chi}) = \chi$. This completes the proof of the corollary. $\square$

Appendix: Some unsolved problems

We now discuss several unsolved problems suggested by the work of this paper.

(1) *Bers' question.* An equivalent version of this question is the conjecture of Earle and Kra which is stated in the introduction of this paper. A proof of this conjecture will give us full information on isomorphisms between Bers fiber spaces and Teichmüller spaces. The above conjecture states that if Γ contains elliptic elements, then (0.1)–(0.4) in the introduction exhaust all possible isomorphisms between Bers fiber spaces and Teichmüller spaces. Earle and Kra [19] proved that in almost all cases, the answer to Bers' question is negative; that is, they proved that the Bers fiber space $F(\Gamma)$ is in general not isomorphic to any Teichmüller space. There are, however, 39 unclear cases (for the signature of Γ) which remain to be settled. The work of this thesis settled 27 cases out of the 39 cases mentioned above. What happens if the signature of Γ lies in the remaining 12 situations? To give a complete solution of this problem, some new ideas and techniques must be introduced. For more detailed discussion on the issue of this conjecture, see Chapter 1 and Chapter 2 of this paper.

(2) *Structure of the group of automorphisms of Bers fiber space.* In order to study isomorphisms among Bers fiber spaces, as we mentioned in (1), we

need to find new automorphisms of Bers fiber spaces. In general, all fiber-preserving automorphisms of Bers fiber spaces are modular transformations (see Proposition 4.3.2). In this setting, we must find non-fiber-preserving automorphisms of the Bers fiber space $F(\Gamma)$. What does a non fiber-preserving automorphism look like? The answer is not so clear even if Γ is torsion free, in which case, $F(\Gamma)$ can be identified with a Teichmüller space $T(g, n+1)$ by Bers' isomorphism theorem [8] (here we assume that Γ is of type (g, n)). We see that all fiber-preserving automorphisms of $F(\Gamma)$ can be obtained from self-maps of a surface of type $(g, n+1)$ with a "special puncture" fixed. This in turn implies that we can have a lot of non fiber-preserving automorphisms (all of them are obtained by doing elementary twist (see §2.2 for the definition) involving that special puncture) and, if U/Γ is compact, there will be no non fiber-preserving automorphisms of $F(\Gamma)$. In general situation; that is, if Γ contains elliptic elements, the question remains unanswered.

(3) *Fiber-preserving isomorphisms among Bers fiber spaces in lower genus cases.* An interesting question is to find all fiber-preserving automorphisms of $F(\Gamma)$, where Γ is of type (g, n) with $g = 0$ or 1. It seems that, even if in the case when both Γ and Γ' are of type $(0, 4)$, the general picture is very hard to predict. For instance, we have $F(0,4;2,2,\infty,\infty) \cong F(0,4;2,2,2,\infty)$ in a fiber-preserving way, but $F(0,4;2,\infty,\infty,\infty)$ cannot be isomorphic to $F(0,4;2,2,\infty,\infty)$ (or $F(0,4;2,2,2,\infty)$). See §4.3 for a discussion.

(4) Continuing with Problem 3, we are also interested in investigating holomorphic sections of $\pi_0 : V(\Gamma) \to T(\Gamma)$ if the type of Γ lies in Table (F)

in §4.1. The earlier work of Hubbard [26] and Earle-Kra [19] [20] suggested that all (global) holomorphic sections be obtained from canonical ones if genus is ≥ 3; when $g = 2$ and Γ has no parabolics, then all (global) holomorphic sections be obtained from canonical ones, the images of canonical ones under a hyperelliptic involution, and Weierstrass sections. There are similar results addressing some specific cases when $g = 0, 1$ (see Theorem 4.1.1). But in general, the picture is still unclear. There are uncountably many holomorphic sections of π_0 if (g, n) lies in Table (F) and, of course, hyperelliptic involution operations contribute a lot of sections which are distinct from canonical ones. Is there any other (global) holomorphic section of π_0 which is basically distinct from the ones obtained by hyperelliptic involutions?

(5) Theorem 4.4.2 asserts that any holomorphic automorphism of $T(\Gamma)$ always lifts to a holomorphic automorphism of $V(\Gamma)$ if Γ is torsion free and is not of type $(0,3)$, $(0,4)$, $(1,1)$, $(1,2)$ and $(2,0)$. This result is nothing but a restatement of the theorem of Royden and Earle-Kra [43], [19]. In the light of this formulation, instead of considering automorphisms of a Teichmüller space, we are interested in studying holomorphic maps of $T(g,1)$ onto $T(g,0)$ for $g \geq 2$. Theorem 4.4.5 says that if a holomorphic map f is a submersion and determines a fibration with fibers conformally equivalent to the unit disk, then f can be lifted to a holomorphic map of $V(g,1)$ onto $V(g,0)$ if and only if f is the so called forgetful map. A natural question arises at this point as to whether or not every holomorphic map of $T(g,1)$ onto $T(g,0)$ is forgetful; that is, whether or not every holomorphic map of $T(g,1)$ onto $T(g,0)$ can be

lifted to a holomorphic map of $V(g,1)$ onto $V(g,0)$.

(6) Continuing with problem (5), we assume that Γ contains no parabolic elements, and consider a lifting problem: whether or not every holomorphic automorphism of $T(\Gamma)$ can be lifted to an automorphism of $V(\Gamma)$? By Royden's theorem (see also Earle-Kra [19]), the answer to this question is "yes" except when $(g,n) = (0,4)$ or $(1,1)$. It is obvious that any modular transformation of $T(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ (or $T(1,1;\nu)$, where ν_1,ν_2,ν_3,ν_4 and $\nu \geq 2$ are integers) can be lifted to an automorphism of $V(0,4;\nu_1,\ldots,\nu_4)$. However, Aut $T(0,4) \cong \mathrm{PSL}(2,\mathbb{R}) \cong$ Aut $T(1,1)$. A natural question is: whether or not any other automorphism ($\in \mathrm{PSL}(2,\mathbb{R})$) can also be lifted to an automorphism of $V(0,4;\nu_1,\ldots,\nu_4)$. We expect but cannot prove that an $\eta \in$ Aut $T(0,4)$ can be lifted if and only if η belongs to the group generated by Mod $(0,4;\nu_1,\nu_2,\nu_3,\nu_4)$ and the three conformal involutions described in the introduction.

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