Singularities of Maps Between 4-Manifolds

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Abstract of the Dissertation

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In this work we study rank two singularities of smooth maps between compact 4-manifolds without boundary. In the stable case we explicitly determine when the points can be pairwise cancelled by deformation. In particular we give the exact obstruction to removing such points.

To Opa

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Section 1

INTRODUCTION

In the study of smooth maps between manifolds it is often useful to associate local invariants to singularities which, when collected in some fashion, give global invariants. In general one cannot do this for all smooth mappings but only for some select collection. One example is the theorem of Hopf for sections of a tangent bundle and their relation to the Euler characteristic. Another example of a different nature is the study of Morse theory, with its restricted class of Morse functions, where relations are given between the homology of the manifold and the index of the singularities of the maps.

One of the early examples is given by the Riemann-Hurwitz formula. In this case we take holomorphic maps between compact Riemann surfaces otherwise known as ramified covers. If $f : \Sigma_1 \to \Sigma_2$ is a holomorphic map between two compact Riemann surfaces we get the Riemann-Hurwitz formula:

$$(deg f)\chi(\Sigma_2) - \chi(\Sigma_1) = \sum_{p \in \Sigma_1} brind_p(f)$$

where $deg f$ is the degree of the map f, $\chi(X)$ is the Euler characteristic of X, and $brind_p(f)$ is the branching degree associated to each singularity p of f.

To each point $p \in \Sigma_1$ and $f(p) \in \Sigma_2$ we can find complex coordinates so that $f(z) = z^n$ in those coordinates. The integer $n - 1$ is the branching index and is independent of the chosen coordinates.

The aim of the first two sections of this thesis is to give a formula of Riemann-Hurwitz type for oriented 4-manifolds. Assume that $f : M \to N$ is a smooth map between two compact oriented 4-manifolds. The further restriction that we take is to require the points where f is of rank two to be isolated and thus finite in number. It should be pointed out that not all maps satisfy this but that it is an open and dense condition. In the first section we will give an integer index, $ind_p^2(f)$, associated to these isolated rank two singularities. Then in section 2 we prove the equation

$$(deg f)p_{_N} - p_{_M} = \sum_{p \in M} ind_p^2(f)$$

where $deg f$ is still the degree of f but $p_{_X}$ is now the first Pontryagin number of X. For transverse maps, where $ind_p^2(f) = \pm 1$, this goes back to the work of MacPherson, [Mac71] and Ronga, [Ron71].

It is a peculiarity of mathematics that what is generally true is not often true of the examples that we initially understand. A good example is given by polynomials versus smooth maps. In the setting of maps between 4-manifolds it is generically true that the maps have isolated rank two degeneracy but it is not the case for most known examples. A good collection of examples is given by ramified covers along 2-dimensional surfaces. In this case the rank two points are exactly the ramification surface and thus certainly not isolated. In section 3 we show how to canonically perturb such a map so that it has isolated rank two singularities. We further calculate the index of the singularities when they do occur. Note that this can only give transverse singularities when the ramification is degree 2.

Although we prove the 4-dimensional Riemann-Hurwitz formula for maps with isolated rank two degeneracies, a more restricted class of functions is already residual. It is an amazing result of Mather that stable maps between two n-manifolds are generic when $n < 9$. In section 4 we give the necessary information about stable maps between 4-manifolds and their corresponding singularities. The stable singularities are of two main types. The first type, the Morin singularities, is given as the rank 3 singularities and occurs along a 3-dimensional submanifold. The second type, the umbilics, is given as the rank two singularities and occurs at isolated points. The umbilics further divide into two types called the hyperbolic and elliptic umbilics. It should be pointed out that both types of umbilics can take the indices ± 1 as defined in section 1.

From this we see that the new Riemann-Hurwitz formula gives a lower bound on the amount of rank two degeneracy that a map $f : M \to N$ must have given a fixed degree. It is one of the main results here that this lower bound can be obtained by isotopy. In all that follows isotopic means homotopic through smooth maps. In section 5 we state the main theorem.

Theorem III: *$f : M \to N$ is isotopic to a map g with exactly $|(deg f)p_N - p_M|$ umbilics of index ± 1 and no other rank two degeneracies.*

We further note that this implies the

Corollary: The number $(deg f)p_N - p_M$ is the exact obstruction to removing the rank two degeneracy by isotopy.

In this section we also give the outline of the proof which is dependent on the results of sections 6, 7, and 8 and the language of section 4.

In section 6 we show that each map isotopes to a map that has only hyperbolic umbilics. This is done by giving a local isotopy which converts elliptic umbilics to hyperbolic umbilics. In section 7 we show that we can isotope a map so that the complement of the singularities has two components, one where the map is orientation preserving and one where it is orientation reversing. This gives us a path between any two singularities that does not have any other singularities. Finally in section 8 we show that we can cancel two hyperbolic umbilics of opposite signed index.

Finally we point out that these results shed an interesting light on the understanding of the smooth structures on 4-manifolds. If we take two smooth homeomorphic 4-manifolds, M and N, and a smooth map, f, near an homeomorphism we now see that it is isotopic to a map which has only rank 3 singularities occurring exactly along a smooth *closed* 3-dimensional submanifold. Although the results of sections 6, 7, and 8 are given to prove the main theorem of section 5 they can also be used to gain insight into the nature of this 3-dimensional submanifold. With all the new results about smooth 4-manifolds it would be interesting to know how they are determined by these associated submanifolds. What then remains to be understood is when they can be removed to give diffeomorphisms.

Section 2

INDICES

In this section we introduce the notion of the index of an isolated zero of a section of an oriented vector bundle. From this we can then define the indices that will be used throughout the rest of this work.

Take $f : \mathbb{R}^n \to \mathbb{R}^n$ where $f^{-1}(0) = 0$ and $n \geq 2$. Let

$$S^{n-1}(\epsilon) = \{x \in \mathbb{R}^n | \, |x| = \epsilon\}$$

have the induced orientation given from $\mathbb{R}^n$. This is given by taking $[v_1, ..., v_{n-1}]$ to be the orientation of S^{n-1} at q if $[v, v_1, ..., v_{n-1}]$ is the standard orientation of $\mathbb{R}^n$ and v is the outward pointing normal vector at q. Now we can define $F_\epsilon(x) = f(x)/|f(x)|$ as a map from $S^{n-1}(\epsilon)$ to $S^{n-1}(1)$ and take the index of f at 0, written $ind_0(f)$, to be the degree of $F_\epsilon(x)$. This is well-defined independent of ϵ because the degree is a homotopy invariant and the $F_\epsilon(x)$'s are homotopic.

Take X to be a smooth, oriented n-dimensional manifold. Further take E to be a real oriented n-plane bundle over X with a section s and projection π.

The expected dimension of the zeroes of a generic s is 0 and we will assume that the zeroes are isolated. We assume nothing about the transversality of s at the isolated zeroes. Take $p \in X$ such that $s(p) = 0 \in E$. If we take a small neighborhood about p the bundle E can be trivialized. Take $U \subset X$ to be such a neighborhood which is also a coordinate chart for X about p and let $\phi : (U, p) \to (\mathbb{R}^n, 0)$, and $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$. Assume that ϕ takes the orientation on U to the standard orientation of $\mathbb{R}^n$ and that ψ does the same thing for the fibers. Take $f : \phi(U) \to \mathbb{R}^n$ as $f = \pi_2 \circ \psi \circ s \circ \phi^{-1}$ where $\pi_2 : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the second factor. Define $ind_p(s) = ind_0(f)$. This is well defined independent of U, ϕ, and ψ. In the case that E is the tangent bundle of X this is just the Hopf index of a vector field with isolated zeroes.

Now assume that we have a smooth map $g : M \to N$ where M and N are smooth 4-manifolds. It is further necessary to have M oriented. We say that g has rank k, written $rk_p(g) = k$, at $p \in M$ if $(dg)_p : T_pM \to T_{g(p)}N$ has rank k as a linear map, where dg is the derivative of g. We also assume that $rk_p(g) \geq 2$ for all $p \in M$ and $rk_p(g) = 2$ at isolated points, Σ. Associated to each of the isolated rank 2 points we will give a corresponding integer. In this case $G = Gr_2(TM)$ is defined as a smooth orientable manifold of dimension 8, where this corresponds to the Grassmann bundle of 2-planes in the tangent space over each point of M. This comes equipped with a projection $\pi : G \to M$. We can also define $H = Hom(U_2(TM), \pi^* \circ g^*TN)$ as an orientable 8-plane bundle over G where $U_2(TM)$ is the tautological bundle over G. From g we also get a section s of H over G with isolated zeroes exactly at those points (p, P) where $(dg_p)|P$ vanishes. This is given by restricting dg to the subplane P. To apply the previous index we must fix an orientation for G and H. We know that $TG \simeq TM \oplus Hom(U_2(TM), U_2(TM)^{\perp})$ and thus we can take the orientation on TM but we still need to fix the orientation for $Hom(U_2(TM), U_2(TM)^{\perp})$. We can take this case and H at the same time. In both cases we have $Hom(E, F)$ where E and F are even dimensional. Taking any bases $e_1, ..., e_{2j}$ and $f_1, ..., f_{2k}$ of E^* and F we fix the orientation of $Hom(E, F)$ as

$$e_1 \otimes f_1, e_2 \otimes f_1, ..., e_1 \otimes f_{2k}, ..., e_{2j} \otimes f_{2k}.$$

Because $2j$ and $2k$ are even this is well-defined independent of the bases taken. Now this fixes the necessary orientations to define the index,

$$ind_p^2(g) = ind_{(p,K)}(s)$$

where p is the point where the rank drops to 2 and $K = ker(dg)_p$ is the kernel plane.

At this point we have defined the index that will be used in the next section to prove the 4-dimensional Riemann-Hurwitz equation. On the other hand it is often useful to have an easier and more direct method of computing this index. In particular we wish to calculate the index associated to certain mapping germs in various ways.

First note that if $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is a smooth proper map and $f^{-1}(0) = 0$ then $deg(f) = ind_0(f)$, where $degf$ is the degree associated to a proper map. Often times the maps of interest are of the form

$$f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

(assumed proper) where $f(x,y) = (x, s(x,y))$, i.e. an $\mathbb{R}^m$-parameterized collection of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. In this case we immediately get $s(0,y) : \mathbb{R}^n \to \mathbb{R}^n$ is proper and $deg(f) = deg(s)$. This can be seen by taking a regular value v of s. Then $(0,v)$ is a regular value for f and if $s^{-1}(v) = \{w_1, ..., w_j\}$ then $f^{-1}(0,v) = \{(0,w_1), ..., (0,w_j)\}$ and f is orientation preserving (reversing) at $(0,w_i)$ when s is at w_i.

When calculating the index of an isolated rank two degenerate point one can show that the map locally is given by

$$g(x,y,z,w) = (x, y, A(x,y,z,w), B(x,y,z,w))$$

where $(x,y,z,w) = 0$ is the only rank two point.

Lemma 2.1 *If the section s associated to g is proper then $ind_0^2(g) = -deg(\tilde{g})$ where $\tilde{g}(x,y,z,w) = (A_z, B_z, A_w, B_w)$.*

Note that we could take $-deg(\tilde{g})$ as the definition of the index of an isolated rank two point instead of the one given here.

Proof: The kernel, K, of $(dg)_0$ is given by $< \frac{\partial}{\partial z}, \frac{\partial}{\partial w} >$. Thus we can take a coordinate chart about the point $(0,K)$ in G given by

$$\phi(x,y,z,w,< \alpha\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \beta\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial y} + \frac{\partial}{\partial w} >) = (x,y,z,w,\alpha,\beta,\gamma,\delta)$$

and take the previously defined orientation as given by $[\frac{\partial}{\partial x}, ..., \frac{\partial}{\partial \delta}]$. Then H is trivial over this chart with oriented coordinates given by

$$\Upsilon(a_1 e_1^* \otimes \frac{\partial}{\partial x} + a_2 e_2^* \otimes \frac{\partial}{\partial x} + ... + a_8 e_2^* \otimes \frac{\partial}{\partial z}) = (a_1, a_2, ..., a_8).$$

Then the section s is given by

$$s(x, y, z, w, \alpha, \beta, \gamma, \delta) =$$

$$(\alpha, \gamma, A_z + \alpha A_x + \gamma A_y, B_z + \alpha B_x + \gamma B_y, \beta, \delta, A_w + \beta A_x + \delta A_y, B_w + \beta B_x + \delta B_y).$$

After an orientation reversing permutation of the coordinates of H we get

$$\tilde{s}(x, y, z, w, \alpha, \beta, \gamma, \delta) =$$

$$(A_z + \alpha A_x + \delta A_y. B_z + \alpha B_x + \delta B_y; A_w + \beta A_x + \delta A_y, B_w + \beta B_x + \delta B_y, \alpha, \beta, \gamma, \delta).$$

If this is proper we then have $ind_0(\tilde{s}) = deg(\tilde{s}) = deg(\tilde{g})$.) On the other hand we have $ind_0^2(g) = -ind_0(\tilde{s})$ so $ind_0^2(g) = -deg(\tilde{g})$.

It should be pointed out that the conditions on s are not really necessary. The cases where we use the lemma satisfy the extra condition and thus we include it to get the easier proof.

In the next section we will take the index, $ind_p^2(g)$, and give a Riemann-Hurwitz type formula for the first Pontryagin number.

Section 3

4-DIMENSIONAL RIEMANN-HURWITZ

If we have a nonconstant holomorphic map $f : \Sigma^1 \to \Sigma^2$ between two compact Riemann surfaces then the Riemann-Hurwitz formula is:

$$(degf)\chi(\Sigma^2) - \chi(\Sigma^1) = \sum_{p \in \Sigma 1} brind_p(f),$$

where $degf$ is the degree of f, $\chi(X)$ is the Euler characteristic of X, and $brind_p(f)$ is the branching index of f at p. Because f is holomorphic the points where f is ramified are isolated. Because Σ^1 is compact this means that there is only a finite collection of points where the index is not one. Thus the sum on the right makes sense.

In this section we will prove a corresponding formula on 4-dimensional manifolds. The Euler characteristic will be replaced by the first Pontryagin numbers and the ramification index will be replaced by the index associated to isolated rank two degenerate points from the last section. Thus we will get the formula:

$$(degf)p_{_N} - p_{_M} = \sum_{p \in \Sigma^1} ind_p^2(f),$$

where $degf$ is still the degree of f, p_X is the first Pontryagin number of X, and $ind_p^2(f)$ is the index previously defined for the isolated rank two points. We also extend $ind_p^2(f)$ to be 0 if p is a point where the rank is greater than 2.

Now assume that we have two smooth, compact 4-dimensional manifolds, M and N. Further assume that M is oriented and that we have a smooth map $f : M \to N$. In this case define $(degf)p_N - p_M$ to be the relative Pontryagin number of f.

Now we can state the result of this section:

Theorem I: *Let f be a smooth map between smooth, compact 4-dimensional manifolds M and N with isolated points where the rank drops to 2. Then*

$$(degf)p_N - p_M = \sum_{p \in M} ind_p^2(f).$$

Proof: From $f : M \to N$ we get the bundle $H = Hom(U_2(TM), \pi^*TN)$ over $G = Gr_2(TM)$ with projection π and section s given as in section one. In [H-L] they construct a smooth 8-form χ_G^H on G and prove $\pi_*\chi_G^H = f^*p_1(N) - p_1(M)$ where $p_1(X) = \frac{1}{4\pi^2}tr((R^{TX})^2)$ is the Pontryagin form on X associated to R^{TX}, the curvature given from a connection on X. Thus $(\pi_*\chi_G^H, [M]) = ((f^*p_1(N) - p_1(M)), [M]) = degfp_N - p_M$ where $(\omega, [X])$ is the pairing given by integrating ω over the space X. On the other hand $(\pi_*\chi_G^H, [M]) = (\chi_G^H, \pi^*[M]) = (\chi_G^H, [G]) = \chi_G(H)$, the Euler number of the bundle H over G. It thus remains to show that $\chi_G(H) = \sum_{p \in M} ind_p^2(f)$. But the definition of $ind_p^2(f)$ was given at the level of G and H. Thus it is enough to show $\chi_G(H) = \sum_{(p,K)} ind_{(p,K)}(s)$ where $ind_{(p,K)}(s)$ is again 0 unless p is a point where the rank drops to 2 and K is the 2-plane kernel of $(df)_p$. The result then follows from the

Lemma 3.1 *Given a smooth section s of a smooth oriented $2n$-plane bundle V over the smooth oriented $2n$-manifold X we get $\chi_X(V) = \sum_p ind_p(s)$, where we have assumed that s has isolated zeros.*

Proof: The proof is by the same method as given in the fundamental paper by Chern, [Che44]. We use

$$\chi_X(V) = (\chi_X^V, [X]) = \int_X \chi_X^V$$

where χ_X^V is the smooth n-form on X defined as $\chi_X^V = Pf(\frac{-1}{2\pi}R^V)$ given by a connection, D, on V. In this case we get $\chi_X^V = s^*\chi(R^{\pi^*V})$ where we have taken the pull-back bundle π^*V over V with the pull-back connection. Thus

$$\chi_X(V) = \int_X \chi_X^V = \int_X s^*\chi(R^{\pi^*V}) = \int_{s(X)} \chi(R^{\pi^*V}).$$

Now take $D_\epsilon(X) = \cup_{p\in Z} s(B_\epsilon(p))$, $Z = Zeros(s) \subset X$, $B_\epsilon(p)$ as closed ϵ-balls about the zeros of s, $S_\epsilon(X)$ and $S_\epsilon(p)$ as their boundaries. Away from X Harvey and Lawson give $\chi(R^{\pi^*V}) = d\sigma$ where σ is a smooth $(2n-1)$-form which vertically is just the volume form of a sphere normalized to 1 and pulled back by radial projection to V minus the zero section, [HL]. Thus

$$\int_{s(X)} \chi(R^{\pi^*V}) = \lim_{\epsilon\to 0}\int_{s(X)-D_\epsilon(X)} \chi(R^{\pi^*V}) = \lim_{\epsilon\to 0}\int_{s(X)-D_\epsilon(X)} d\sigma$$
$$= \lim_{\epsilon\to 0}\int_{S_\epsilon(X)} \sigma = \sum_{p\in Z} ind_p(s).$$

Thus the lemma and the theorem follow.

Section 4

RAMIFIED COVERS

Although rank two degeneracy generically occurs at isolated points this is not true for most explicit standard examples of maps between 4-manifolds. One such class of maps is given by ramified covers. In this case the rank two singularities occur along the ramification surface which is 2-dimensional.

To fix the setting take $f : X \to Y$, a ramified cover of degree n along Σ where X and Y are complex 2-dimensional surfaces. Here we take Σ to be the smooth, connected complex curve embedded in both X and Y and denote by η_X and η_Y the normal bundles of Σ in X and Y respectively. Take $AC(E,F)$ to be the bundle of anticonformal homomorphisms from E to F. In this case we get the following

Theorem II: *To each section $A \in \Gamma_\Sigma(AC(\eta_X, f^*\eta_Y))$ there corresponds a perturbation of f to a map $\tilde{f}$ with rank two degeneracies precisely at points of Σ where A vanishes. If A vanishes at isolated points the index of the umbilic is $(n-1)$ times the index of the zero of A.*

Then from a calculation we get the

Corollary: If we have a degree n ramified cover, $f : X \to \mathbb{CP}^2$, over a smooth, connected curve of degree n we get $p_X = -n(n-2)(n+2)$.

Proof(of corollary):

$$(degf)p_{\mathbb{CP}^2} - p_X = (n-1)\chi(AC(\eta_X, f^*\eta_Y)) = (n-1)[\chi(\eta_X \otimes f^*\eta_Y)]$$

$$= (n-1)[\chi(\eta_X) + \chi(\eta_Y)] = (n-1)[n+n^2] = (n-1)n(n+1)$$

and thus

$$p_X = 3n - (n-1)n(n+1) = -n(n-2)(n+2).$$

Proof(of theorem): Fix C^∞ tubular neighborhoods of Σ in X and Y and identify them with η_X and η_Y. Also take local trivializations of the bundles

$$\eta_X|_U = U \times \mathbb{C}, \eta_Y|_U = U \times \mathbb{C}$$

so that $w = f(z) = z^n$ where z and w are the normal coordinates for U in X and Y respectively. Now define $M = max_{p\in\Sigma}(|A(p)|)$ where $|A|$ is the metric induced on $(AC(\eta_X, f^*\eta_Y))$ from unitary metrics on η_X and η_Y. Take $f(x,y,z) = (x,y,z^n)$ in some coordinate chart. Further fix a smooth function $p : \mathbb{R}^+ \to [0,1]$ with the following properties:

1. $p(r) = 1$ for all $r < 1$,
2. there exists R so that $p(r) = 0$ for all $r > R$,
3. $0 \leq |p'(r)| < \frac{n}{2}$ for $n \geq 3$ and $0 \leq |p'(r)| < \frac{2}{R}$ for $n = 2$,

Now define $\check{f}(x,y,z) = (x,y,z^n + p\epsilon\bar{z})$ where $\epsilon = \epsilon(x,y)$ is A in the local coordinates and $p = p(\frac{|z|^2}{M})$ or $p(\frac{z\bar{z}}{M})$. Clearly $\check{f}$ is rank two if and only if $\check{f}_z = \check{f}_{\bar{z}} = 0$. This occurs when $nz^{n-1}M + p'\epsilon z\bar{z} = 0$ and $p\epsilon M + p'\epsilon\bar{z}^2 = 0$. If $z = 0$ we get $p\epsilon = 0$. But at $z = 0$ we will have $p = 1$ so we get rank 2 along Σ only at points where $\epsilon = 0$, i.e. where the section A vanishes. The claim then is that this is the only case of rank 2 degeneracy. Assume $z \neq 0$. If $\epsilon = 0$ then we get $nz^{n-1} = 0$ and thus $z = 0$. Thus we can also assume $\epsilon \neq 0$. Then $p' = \frac{-nz^{n-1}M}{\epsilon\bar{z}^2}$ implies $|p'| = \frac{nM}{|\epsilon|}|z|^{n-3}$. This means that $p' \neq 0$ and thus $R > |z| > 1$. Now $|p'| = \frac{n|z|^{n-1}M}{\epsilon|z|^2} \geq \frac{nM}{|\epsilon|} \geq n$ when $n \geq 3$ and $|p'| = \frac{2|z|M}{\epsilon|z|^2} \geq \frac{2}{R}$ when $n = 2$ but these contradict the third condition on p.

Thus we have exhibited a local perturbation associated to A which has rank two degeneracies precisely when A vanishes. It remains to determine what the index is at an isolated point where A vanishes. First assume that the zero of A is transverse. Also write $R^{(n)} = Re[(u+iv)^n]$ and $I^{(n)} = Im[(u+iv)^n]$ where we have $z = u + iv$.

Lemma 4.1 *Assume* $F(x,y) = (A(x,y), B(x,y))$ *has a transverse* 0 *at* 0 *and* $f(x,y,u,v) = (x,y,R^{(n)} + Au + Bv, I^{(n)} - Av + Bu)$. *Then*

$$ind_0^2(f) = (n-1)deg(f) = \pm(n-1)$$

with the sign given by the sign of $A_xB_y - A_yB_x$.

Note that along Σ $\check{f}$ now looks like this where $A = Re(\epsilon)$ and $B = Im(\epsilon)$. Also note that F thought of as a map on all of $\mathbb{R}^4$ is proper.

Proof: From the lemma in section 2 we have that $ind_0^2(f) = deg(\tilde{f})$ where $\tilde{f}(x, y, u, v) = (nR^{(n-1)} + A, nI^{(n-1)} + B, -nI^{(n-1)} + B, nR^{(n-1)} - A)$. But

$$|d\tilde{f}| = 4n^2(n-1)^2(A_xB_y - A_yB_x)[(I^{(n-2)})^2 + (R^{(n-2)})^2]$$

$$= 4n^2(n-1)^2(A_xB_y - A_yB_x)(u^2 + v^2)^{n-2}.$$

Now because F is transverse $A_xB_y - A_yB_x \neq 0$. If it is positive(negative) then $|d\tilde{f}| > 0$ (< 0) provided $(u, v) \neq (0, 0)$. Thus all points (x, y, u, v) with $(u, v) \neq (0, 0)$ are regular points and count positively (negatively) towards the degree. Thus it suffices to count the number of preimages of some point with $(u, v) \neq (0, 0)$. Take $\tilde{f}(x, y, u, v) = (a_1, a_2, a_3, a_4)$. Then we get $R^{(n-1)} = \frac{a_1+a_4}{2n}$, $I^{(n-1)} = \frac{a_2-a_3}{2n}$, $A = \frac{a_1-a_4}{2}$, and $B = \frac{a_2+a_3}{2}$. Provided either $a_1 + a_4 \neq 0$ or $a_2 - a_3 \neq 0$ the solutions for $R^{(n-1)}$ and $I^{(n-1)}$ are the $n-1$-roots of a nonzero complex number. By the transversality of F, $F^{-1}(\frac{a_1-a_4}{2}, \frac{a_2+a_3}{2})$ is one point and the number of preimages of $\tilde{f}^{-1}(a_1, a_2, a_3, a_4)$ is $(n-1) \cdot 1 = n-1$ and the result follows.

To finish the computation when A is not transverse we recognize that a local perturbation of the section of A near a point (x_0, y_0) will give transverse points. On the other hand the total degree is perturbation invariant as is ind^2 and thus the general result follows.

It should further be pointed out that a result similar to the theorem can be shown for double fold maps. These would be maps $f : X \to Y$ with Σ, a surface within both, where the normal bundle to Σ in both X and Y is trivial and the map along Σ looks like $f(x, y, z, w) = (x, y, z^2, w^2)$. Natural examples of where this would occur are given if we take $f_1 : \Sigma_1^1 \to \Sigma_1^2$ and

$f_1 : \Sigma_2^1 \to \Sigma_2^2$ where Σ_i^j are smooth surfaces and the f_i only have folds. In this case $f_1 \times f_2 : \Sigma_1^1 \times \Sigma_2^1 \to \Sigma_1^2 \times \Sigma_2^2$ would be as described.

Section 5

SINGULARITIES

In this section we will give enough information about singularities of stable mappings between 4-manifolds to proceed with the results in the rest of this work. For a more complete picture of stable maps and their singularities see [GG73]. With only the condition that the map is smooth the singularities can be quite complicated. To simplify the structure of the singularities we will restrict to the space of stable mappings. Let M and N be two fixed smooth 4 dimensional manifolds, and $C^{\infty}(M, N)$ be the space of infinitely smooth mappings between them with the Whitney topology. Then we say that $f \in C^{\infty}(M, N)$ is stable if there is some neighborhood N_f of f so that for all $g \in N_f$ there are diffeomorphisms, h and j, of M and N, respectively, so that $j \circ f = g \circ h$. In general this restriction is too much to ask but it is one of the amazing results of Mather that in dimensions below nine, the stable mappings are dense, [Mat71]. In particular, in four dimensions the space of stable mappings is dense.

With this global result the next natural question might be to ask what the singularities can look like. We say that $p \in M$ is a singularity for f if $r = rk(df_p) < 4$. In the stable case there are actually only two possibilities. Either $r = 2$, which we call an umbilic point, or $r = 3$, which we call a Morin singularity. In the simplest case of a Morin singularity, there are local coordinates around p and $f(p)$ so that f looks like $f(x, y, z, w) = (x, y, z, w^2)$, called folds. There are really only two things which we need to know about Morin singularities. In the stable case these are that the space of Morin singularities forms a smooth 3-dimensional submanifold Σ_1 of M and that all points but a smooth 2-dimensional submanifold $\Sigma_{1,1}$ of Σ_1 have local forms which are the same as folds. If it were not for the umbilics Σ_1 and $\Sigma_{1,1}$ would be closed submanifolds.

Whereas we needed to know little about the Morin singularities, we need to know a little more about the umbilics. The first thing to note is that they are isolated. This means that they are isolated from each other but not from other singularities. In fact, the reason that Σ_1 and $\Sigma_{1,1}$ are not closed is that the umbilics would locally form the vertex to a cone over a torus for Σ_1 and the vertex to a cone over S^1's for $\Sigma_{1,1}$. Thus closing Σ_1 in M only adds the umbilics.

The second thing of interest about umbilics is that they come in two types. The first type of umbilic is called elliptic. It appears as the generic perturbation of a ramified cover of degree two over a surface. Similar to folds we can give a local normal form by

$$f(x, y, z, w) = (x, y, z^2 - w^2 + xz + yw, zw - xw + yz).$$

The second type of umbilic is a hyperbolic umbilic and its local normal form is given by

$$f(x,y,z,w) = (x,y,z^2+yw,w^2+xz)$$

and appears as the generic perturbation of a 'double fold' along surface. Because there are two types of umbilics it is desirable to have some way to separate them other than from their local forms. One method is to understand their local mapping rings. Since this will only be necessary for polynomial maps between $\mathbb{R}^4$'s we will only give the definition in this case. If $f(x,y,z,w) = (f_1,f_2,f_3,f_4)$ then the local mapping ring M_f is given by

$$M_f = \mathbb{R}[x,y,z,w]/(f_1,f_2,f_3,f_4),$$

where $\mathbb{R}[x,y,z,w]$ is the polynomial algebra in variables x,y,z and w and (f_1,f_2,f_3,f_4) is the ideal generated by the f_i's. In the elliptic case

$$M_f \simeq \mathbb{R}[z,w]/(z^2-w^2,zw)$$

and in the hyperbolic case

$$M_f \simeq \mathbb{R}[z,w]/(z^2,w^2).$$

If $f(x,y,z,w) = (x,y,f_3(x,y,z,w),f_4(x,y,z,w))$ and the mapping ring is 4-dimensional over $\mathbb{R}$ then with only the further condition that $|dF| \neq 0$ at 0, where $F(x,y,z,w) = (\frac{\partial f_3}{\partial z},\frac{\partial f_3}{\partial w},\frac{\partial f_4}{\partial z},\frac{\partial f_4}{\partial w})$, we get that there are local coordinates so that f is locally given by either the normal form of the elliptic umbilic or the hyperbolic umbilic. It should be pointed out that the rank 3-singularities near an umbilic do not distinguish between the two types because both sets are cones over a torus with the umbilic point as the vertex.

With just these basics about singularities between 4-manifolds we will be able to show exactly when umbilics are really necessary and when they can be removed. Remembering that umbilics are stable we can see that it does not suffice to perturb the mappings to remove them. It is natural to ask if we can isotopy the map to eliminate them and certainly the results of the first two sections tells us that this is not the case in general. In fact it will turn out that the only obstruction to removing rank two degeneracy is the relative Pontryagin number of a map. This is essentially the main theorem of the next section.

Section 6

MAIN THEOREM

In this chapter we will state the main result and its corollaries and give their proofs dependent on the results in the following three sections. Again take $f : M \to N$ to be a smooth map between two smooth closed 4-manifolds. Then we get the

Theorem IV: *f is isotopic to a map g with exactly $|(deg f)p_N - p_M|$ umbilics all of index 1 or all of index -1 and no other rank two degeneracies.*

Remark: By the results of section 2 we know that all the indices must be of the same sign. Also from the proof it is clear that we can insist that all the umbilics be hyperbolic. From this we get the immediate

Corollary: The number $(deg f)p_N - p_M$ is the exact obstruction to removing rank 2 degeneracy by isotopy.

If we start with two smooth manifolds M and N that are homeomorphic then we also get the

Corollary: There is a smooth map $f : M \to N$ that is everywhere rank 3 or higher.

Remark: Because the set of maps that are rank 3 or greater is an open subset of the set of maps we know that f is then isotopic to a map with Morin singularities precisely along an embedded 3-dimensional submanifold and no other singularities.

Proof (of the second corollary): The space of smooth maps is dense in the space of continuous maps and thus we can apply the preceding corollary to a smooth map close to a homeomorphism. But the first Pontryagin number is a topological invariant and the degree will still be one. Thus the obstruction vanishes and the result follows.

Proof (of theorem): We know that f is isotopic to a stable map. Thus assume it was stable in the beginning. Now by the work in section 6 we can take all umbilics to be hyperbolic. Then in section 7 we show that we can isotope f so that there is a curve between any two umbilics that does not cross any other singularities. Lastly in section 8 we show that under such conditions we can isotope further to pairwise cancel the hyperbolic umbilics with indices of opposite sign. Note that it may be necessary after cancelling a pair of umbilics to repeat the process of section 7. As the work of section 7 only changes the fold locus this does not create any new umbilics. Thus we cancel until there are only umbilics of one sign. As they are all hyperbolic umbilics their index is either ± 1.

In the next three sections we will prove the results needed in this proof.

Although the results are only used to prove the theorem they also shed further light on the fold locus and more generally, all the Morin singularities.

Section 7

CONVERSION OF UMBILICS

In this chapter we show that up to local isotopy elliptic and hyperbolic umbilics are interchangeable. Heuristically this can be done by showing the existence of a map $f : \mathbb{R}^4 \to \mathbb{R}^4$ which outside a ball in the domain looks like the elliptic umbilic and near the origin looks like the hyperbolic umbilic. Then in local coordinates we can take an elliptic umbilic and take the straight line isotopy to this map. It is clear from the proof that going from hyperbolic to elliptic umbilics is similar. From this we will get the following result.

Theorem V: *Given a stable map $g : M \to N$ there is an isotopic map h that has only umbilics of one type.*

In the proof of the main theorem we convert all elliptic umbilics to hyperbolic umbilics.

Proof: The real problem is to show that this can be done in a neighborhood of each umbilic. Thus the problem is to show the existence of f above. The idea is to study the straight line homotopy of a positive elliptic to a positive hyperbolic. To this end define

$$F(x, y, z, w, a) = (x, y, g(x, y, z, w, a), h(x, y, z, w, a)),$$

$$g(x, y, z, w, a) = a(z^2 - w^2 + xz + yw) + (1 - a)(z^2 + yw),$$

$$h(x, y, z, w, a) = a(2zw - xw + yz) + (1 - a)(w^2 + xz),$$

as maps from $\mathbb{R}^4 \times [0, +\infty)$ to $\mathbb{R}^4$ and $f_a(x, y, z, w) = F(x, y, z, w, a)$ as a map from $\mathbb{R}^4$ to $\mathbb{R}^4$ for fixed a. For fixed a df_a has rank 2 if and only if

$$a(2z + x) + (1 - a)(2z) = 0$$

$$a(y - 2w) + (1 - a)(y) = 0$$

$$a(2w + y) + (1 - a)(x) = 0$$

$$a(2z - x) + (1 - a)(2w) = 0.$$

By simple linear algebra this yields $x = y = z = w = 0$. Thus we further get that $F : \mathbb{R}^5 \to \mathbb{R}^4$ is rank two exactly when $x = y = z = w = 0$ and a is arbitrary. Because these terms are linear in x, y, z and w we also get that $|dF| \neq 0$ for $F(x, y, z, w) = (\frac{\partial (f_a)_3}{\partial z}, \frac{\partial (f_a)_3}{\partial w}, \frac{\partial (f_a)_4}{\partial z}, \frac{\partial (f_a)_4}{\partial w})$. This gives one of the two conditions from section 5 to guarantee that the rank two points are umbilics.

If we know that the local mapping ring is 4-dimensional over $\mathbb{R}$ we get the other condition. Then we know that each point is either an elliptic or hyperbolic umbilic. But the local mapping ring of f_a is given as

$$M_{f_a} = \mathbb{R}[z,w]/(z^2 - aw^2, 2azw + (1-a)w^2).$$

Call $I = (z^2 - aw^2, 2azw + (1-a)w^2)$. For example, $a = 0$ gives $\mathbb{R}[z,w]/(z^2, w^2)$, the expected local mapping ring of a hyperbolic umbilic.

By the homogeneity of g and h in x, y, z and w, I is generated, at most, by two homogeneous quadratic terms. Thus the dimension of $M_{f_a} \geq 4$ and M_{f_a} is, at least, generated by $\{1, z, w, other\ quadratic\ term\}$. Now the claim is that M_{f_a} has multiplicity 4 provided $(1-a)^2 \neq 4a^3$ which is true except at one point $a_0 \in [0,1]$. The result is immediate in the two cases $a = 0$ or 1 so assume $a \neq 0$ or 1. We show that $\{1, z, w, zw\}$ is a basis for M_{f_a}. Take $w(z^2 - aw^2)$, $w[2azw + (1-a)w^2]$, and $z[2azw + (1-a)w^2] \in I$. Now these three elements are in the vector space spanned by $\{z^2w, zw^2, w^3\}$ and generate it if and only if they are linearly independent. This occurs when $(1-a)^2 \neq 4a^3$. Further note that $z(z^2 - aw^2) \in I$ then gives that $z^3 \in I$. But $w^2 = \frac{2a}{a-1}zw$ and $z^2 = \frac{2a^2}{a-1}zw$ shows that $dim_{\mathbb{R}} M_{f_a} \leq 4$ provided $(1-a)^2 \neq 4a^3$. Thus $dim_{\mathbb{R}} M_{f_a} = 4$ unless $(1-a)^2 = 4a^3$. Now we know also that the umbilics are stable and thus all umbilic points $a < a_0$ are hyperbolic umbilics and all umbilic points $a > a_0$ are elliptic umbilics. Figure 7.1 sums up the acquired information and gives the picture of the embedded $\mathbb{R}^4$.

The proof is completed at this point by embedding an $\mathbb{R}^4$ in $\mathbb{R}^4 \times \mathbb{R}$ that is sufficiently vertical so that F restricted to it is rank 3 or higher except when it crosses the line $x = y = z = w = 0$. Near this line it should be entirely contained in a fixed $\mathbb{R}^4 \times \{a^*\}$, with $a^* < a_0$. Outside some compact set of

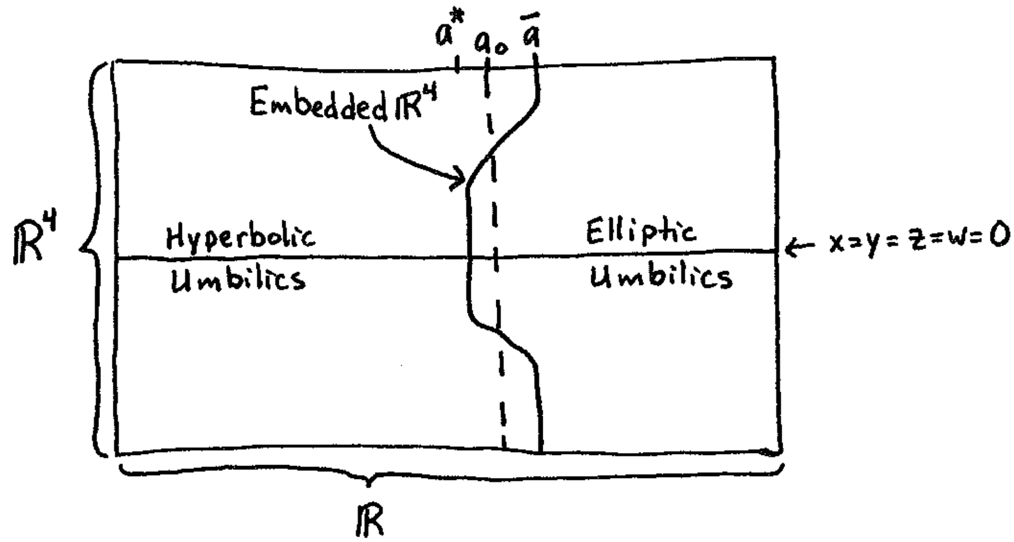

Figure 7.1:

$\mathbb{R}^4 \times \mathbb{R}$ it should be entirely contained in a fixed $\mathbb{R}^4 \times \{\bar{a}\}$ with $\bar{a} > a_0$. Then restricting F to this embedded $\mathbb{R}^4$ gives a map $f : \mathbb{R}^4 \to \mathbb{R}^4$ with the desired properties.

A couple of comments are in order. First it is clear that converting to elliptic umbilics is no different. More important is the statement that an umbilic has coordinates where it looks like its normal form but only perhaps for a small portion of the domain $\mathbb{R}^4$. Thus it may only be possible to give f on some small subset of $\mathbb{R}^4$ which does not contain B. The point here is that this is not the case because the normal forms for both umbilics are in a sense self-similar. If $f : B(\epsilon) \to \mathbb{R}^4$ is given by $f(x, y, z, w) = (h_1, h_2, h_3, h_4)$ where each h_i is homogeneous in the variables x, y, z, and w and $B(\epsilon)$ is an ϵ-ball in $\mathbb{R}^4$ then through a linear change of coordinates in both the image and the domain f can be defined by the same equations but on any size ball desired.

But clearly all the f_a's are of this form.

Now the straight line homotopy between $f_{\bar{a}}$ and f gives the local isotopy from a map with an elliptic umbilic to one with a hyperbolic umbilic and the theorem follows.

As mentioned in the introduction it would be interesting to know how this affects the Morin singularities in a neighborhood of the umbilics. The total rank 3 degeneracy is a cone over a torus but the higher order Morin singularities are different and it is unclear how this difference comes about. In the next two sections we will be taking similar transformations of f and we will be able to describe the effects on the Morin singularities.

Section 8

CONNECTIVITY OF THE REGULAR LOCUS

The essence of this chapter is that by an isotopy of a stable map between two oriented 4-manifolds one can fix it so that there is only one component of regular points where the map is orientation preserving and one where it is orientation reversing. This procedure will give us the necessary curve between points in the proof of the main theorem but is also of independent interest as mentioned in the introduction. In the case that there is no rank two degeneracy but the map is still stable we get that the complement of the submanifold representing the Morin singularities, the rank 3 degeneracies, only divides the manifold into, at most, two pieces. It should also be pointed out that this connection can be done regardless of the dimension of the manifolds. The more general proof would follow in exactly the same manner.

Thus assume that we have a stable map $f : M \to N$ between two 4-manifolds. Then we get the

Theorem V: *There is an isotopy of f to a map g fixing all lower order Morin singularities and the rank two points and where the complement of all singularities has at most two components. If there is only one component then the map g is an immersion.*

Although we keep fixed the lower order Morin singularities of f the map g will have new rank 3 singularities.

First we shall take the local case where we have a map

$$f(x_1, ..., x_4) = (x_1^3 - 3x_1, x_2, ..., x_4).$$

In this case we show that we can give a bounded isotopy of f to a map F where the components of the fold locus of F are connected. Essentially the fold locus of F is a connected sum of $\{x_1 = -1\}$ and $\{x_1 = 1\}$, the fold components of f.

Lemma 8.1 *Fix $\epsilon > 0$. There is an F isotopic to f and equal to f outside a ball. This F has the further property that the complement of the space of singularities has two components.*

Proof(of lemma):

First fix a smooth $\mathbb{R}$-parameterized family f_a of functions from $\mathbb{R}$ to $\mathbb{R}$ with the following properties:

1. $f_a(x) = x^3 - 3ax$ for $|x| < \theta$ ($\theta > max(0, \sqrt{3a})$,

2. $f_a(x) = x^3 - 3x$ for $|x| > \phi$ for $(\phi > \sqrt{3})$,

3. $f_a'(x) \neq 0$, $\theta \leq x \leq \phi$.

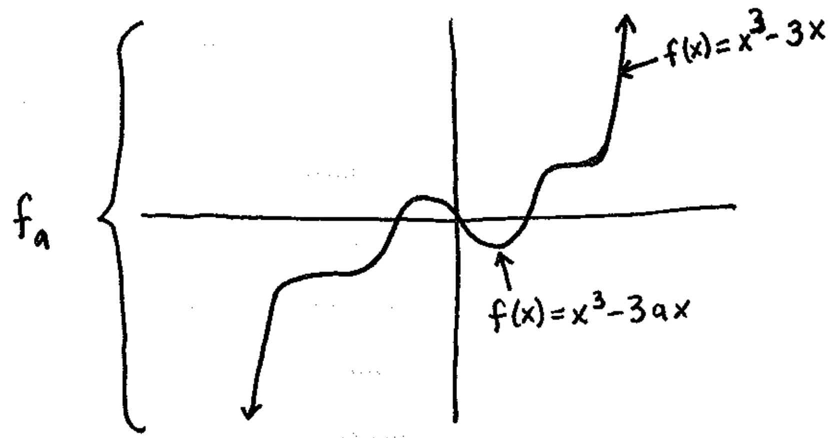

Figure 8.1:

Note that it is necessary to have $\phi^3 - 3\phi \geq \theta^3 - 3a\theta$ to satisfy condition 3. Also note that ϕ can be made as close to $\sqrt{3}$ as desired. Now define

$$F(x_1, ..., x_4) = (f_{p(r^2)}(x_1), x_2, x_3, x_4)$$

where $r^2 = x_2^2 + x_3^2 + x_4^2$ and $p : \mathbb{R}^+ \to \mathbb{R}$ is any smooth function given by the following properties:

1. there is an $r_0 > 0$ with $p(r) = -\epsilon$ for $r < r_0$;

2. there is an $R_0 > 1 + \epsilon$ with $p(r) = 1$ for $r > R_0$;

3. $p'(r) \geq 0$ for all r.

Essentially we take $p(r) = max(min(-\epsilon, r - \epsilon), 1)$ smoothed out near $r = 0$ and $r = 1 + \epsilon$.

Under these conditions F has rank ≥ 3 with equality precisely when $f'_{p(r^2)}(x_1) = 0$. From the definition of f_a this is only possible when $a = p(r^2)$ is nonnegative and then $x_1 = \pm\sqrt{3a}$. If we take the line $x_2 = x_3 = x_4 = 0$ we have a is negative and thus there are now only regular points and we have a function which we can isotopy to which has joined the original two components. Also by the definition of p we see that, outside of some compact set, F is the same as f. In effect we have taken an isotopy which has given a connected sum of the two planes $\{x_1 = 1\}$ and $\{x_1 = -1\}$ and given a tunnel between the two components that had previously been seperated. Note that we have not created any other components of the complement of the rank 3 points. We see this because there are only zero, one, or two fold points over each point of the (x_2, x_3, x_4)-plane. The points with two are the extensions of the old components and the points with only one are where they are connected. With this normal form connection we can now proceed to the proof of the theorem.

Proof(of theorem):

Assume that we have a map with the fewest possible number of components of the complement of the rank 3 points but that the number is greater than 2. Then without loss of generality we can assume that there must be at least two components where the map f is orientation preserving. Take a point p in one component and q in the other.

Now take a curve, $c : I \to M$, from p to q whose singularities are a minimum number of fold points. This is possible because the other singularities

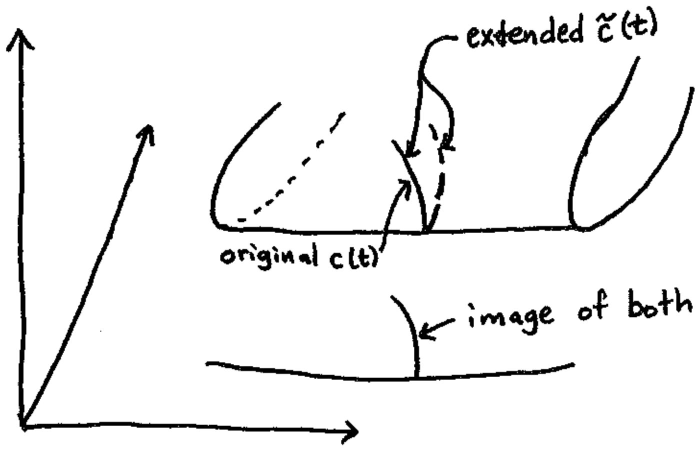

Figure 8.2:

are codimension two or higher. Further assume that there are only two fold points, f_1 and f_2. If this is not the case we can take two points that are closer on $c(I)$ where it is $c : [a, b] = J \subset I \to M, c(a) = f_1, c(b) = f_2$, as a curve from f_1 to f_2 through regular points of M. Clearly we can take c where $c(J)$ and $f \circ c(J)$ are embedded by perturbing c if necessary. Finally assume that the reflection through the fold extends c smoothly at the fold. This can be done by a C^0 perturbation. The point of this is that we want c to exactly come into the fold locus in the normal direction to the fold.

With these conditions we define a new curve $\tilde{c}$ in M extending c to the preimage of $f \circ c(J)$. Geometrically we extend c by lifting the image of c in N to the other leaf of the fold. Take any nonsingular parameterization $\tilde{c} : K \to M$. The picture is given by Figure 8.2.

This can be thought of merely as maps of curves and then by suppressing

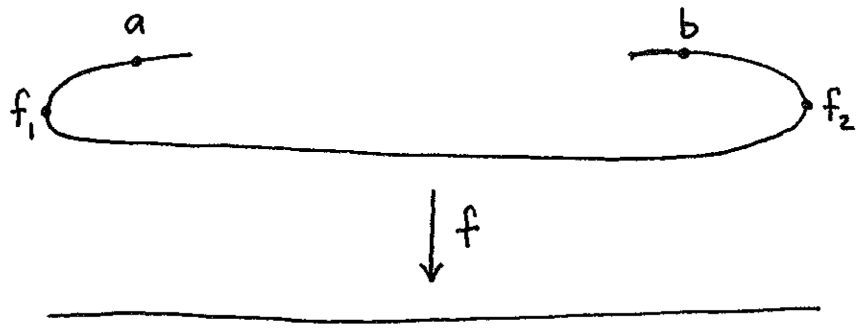

Figure 8.3:

the other three directions we get Figure 8.3.

Note that a and b will be points in the same component as p and q respectively. Thus it suffices to connect them.

Now the point is that we continue this extension until one of four things happens. Either this closes into a loop, or both extensions fold again, or one extension folds and the other exits, or both exit. By 'exit' we mean that to extend the curve we would have to leave the preimage of $f \circ \tilde{c}(J)$.

See Figures 8.4 to 8.7.

In each case we have chosen an a and a b. In the first case note that we already have a curve from a to b without folds. The point in the last three cases is that we can apply the local normal form connection to connect them.

The idea is that each of them has a region which looks like Figure 8.9. Although initially this may not be true for Figure 8.5, Figure 8.8 describes an

Figure 8.4:

isotopy where this is the case in a neighborhood of the curve.

Now it remains to show that near $\tilde{c}(K') = \tilde{L}$ and $f \circ \tilde{c}(K') = L$ there are local coordinates so that f looks like the local normal form where K' is the subinterval of K which parameterizes $\tilde{c}$ in the region described in Figure 8.9. It is enough to show the x_1-coordinate part and then extend to the normal bundle. The fact that we can coordinatize so that $f_1(x) = x^3 - 3x$ is immediate. The extension to the normal bundle is slightly more difficult.

We already know that we have coordinates near both f_1 and f_2 so that

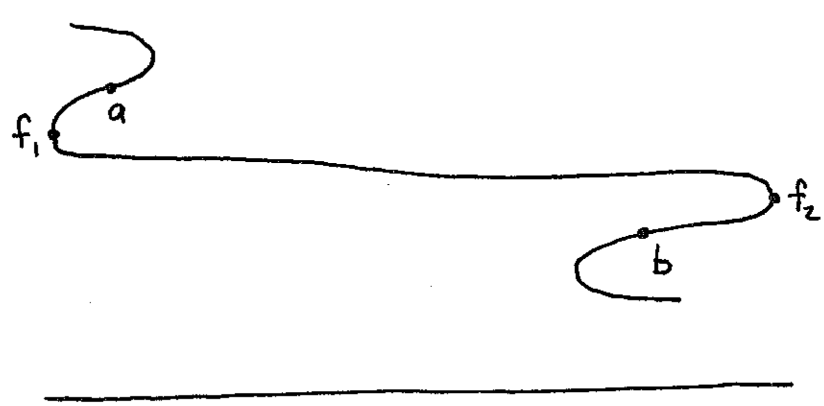

Figure 8.5:

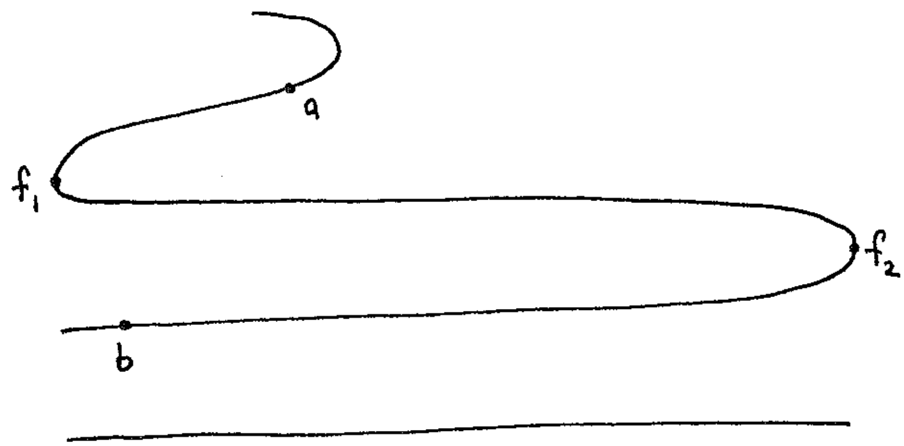

Figure 8.6:

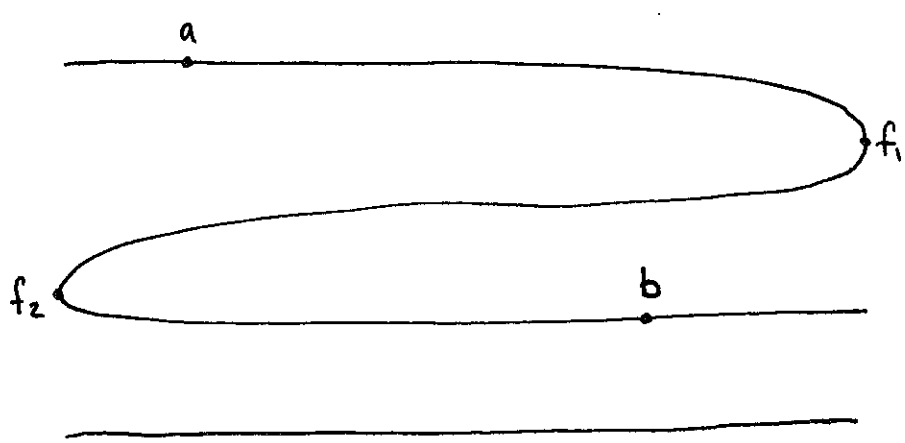

Figure 8.7:

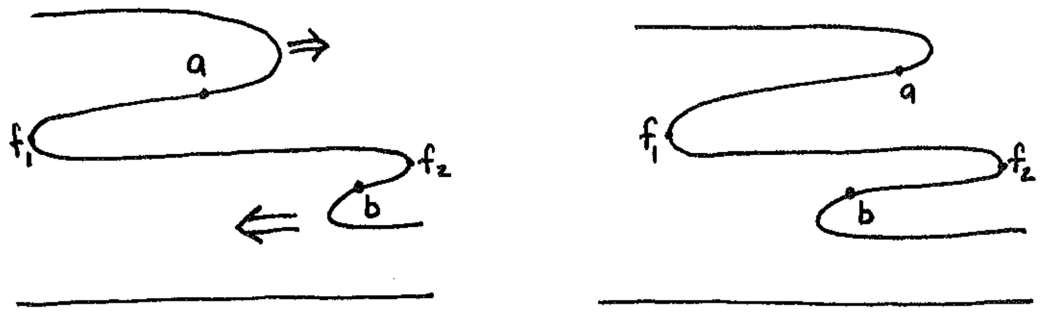

Figure 8.8:

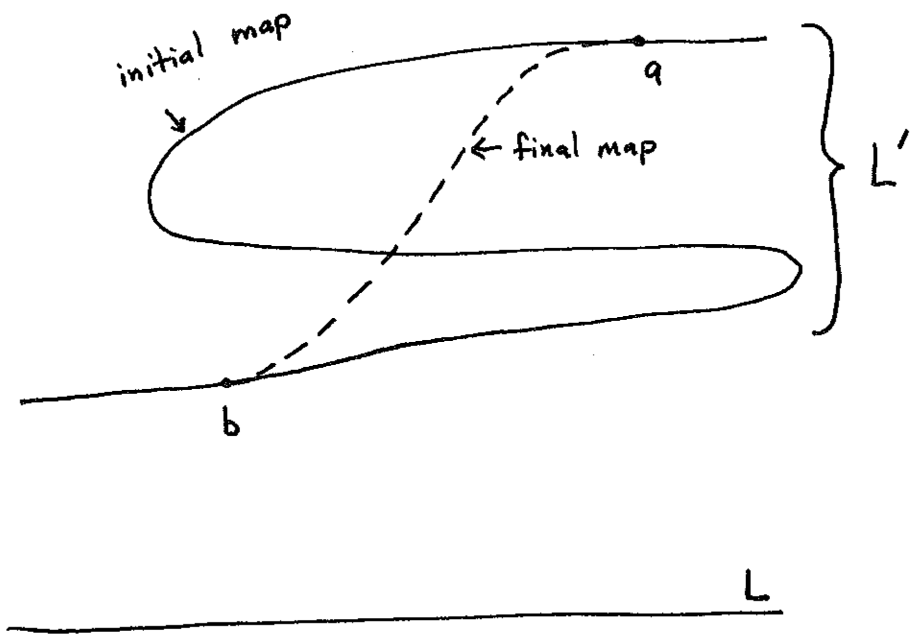

Figure 8.9:

$f(x,y,z,w) = (x^2,y,z,w)$ from section 3 where both f_i and $f(f_i)$ would correspond to $0 \in \mathbb{R}^4$. Ignoring the last three coordinates we can give an explicit change of coordinates so that $f(x,y,z,w) = (x^3 - 3x, y, z, w)$ with f_i corresponding to $((-1)^i, 0, 0, 0)$ and $f(f_i)$ corresponding to $((-1)^{i+1}2, 0, 0, 0)$. Take the original coordinate that gave $x = \tilde{x}^2$ where x is the coordinate for N and $\tilde{x}$ for M. If we are near f_i we take the change of coordinates given by $\tilde{x} = \tilde{u} + (-1)^i[2 + (-1)^{i+1}\tilde{u}]^{1/2}$ and $x = 2 + (-1)^{i+1}u$. This clearly gives $u = \tilde{u}^3 - 3\tilde{u}$ and can be inverted. Thus we have coordinates near the f_i and $f(f_i)$ of the desired form.

The trick now is to extend the coordinates in N and then lift them to M by f. This works away from the folds points because f is nonsingular there. Of course we will have a three valued function but the choices are clear by extension from the ends. We do have to take care to make sure that $x = 0$ ends up seperating the images of a and b so that we can apply the earlier result to the local normal form. But we can now extend to L the x-coordinate by choosing any point between $f(a)$ and $f(b)$ along L and assigning to it the value 0. Now take any smooth extension. Note that R_0 in the lemma can be taken arbitrarily near 1 from above and thus we will be able to apply the lemma.

It remains to show that we can extend the coordinates in the normal directions to L. We show that we can give normal bundle coordinates and then since the normal bundle is trivial we are done. Thus take any trivialization of the normal bundle along L. Then altering the trivialization so that it is compatible with the ends gives the desired result. Take $U \subset N, \phi : U \rightarrow \mathbb{R}^4$, to be the coordinate chart that gave the local normal form $f(x,y,z,w) =$

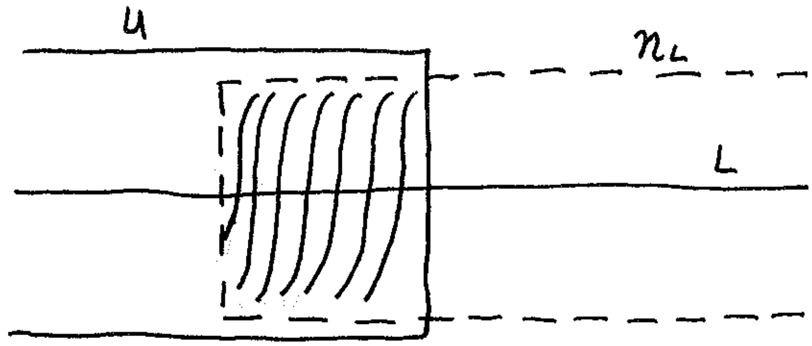

Figure 8.10:

(x^2, y, z, w). Further take $\upsilon : \eta_L \to \mathbb{R}^4$ where η_L is any normal bundle to L. Assume both ϕ and υ take the orientation on N to the standard orientation of $\mathbb{R}^4$ and further assume that $\upsilon \circ \phi^{-1}(x, 0, 0, 0) = (x, 0, 0, 0)$ where we assume that this curve corresponds to L.

In Figure 8.10 we see the fixed $\bar{x}$ planes for η_L compared to the vertical fixed x-planes for U. The point is to flatten these out as in Figure 8.11.

Take

$$A(\bar{x}, \bar{y}, \bar{z}, \bar{w}) = \phi \circ \upsilon^{-1}(\bar{x}, \bar{y}, \bar{z}, \bar{w}).$$

Now we would like to take

$$T = [1 - p(\bar{x})](\bar{x}, \bar{y}, \bar{z}, \bar{w}) + p(\bar{x})A(\bar{x}, \bar{y}, \bar{z}, \bar{w})$$

as an intermediate chart for the normal bundle between U and η_L where we discard the trivialization of ϕ and υ where they do not match with T. As

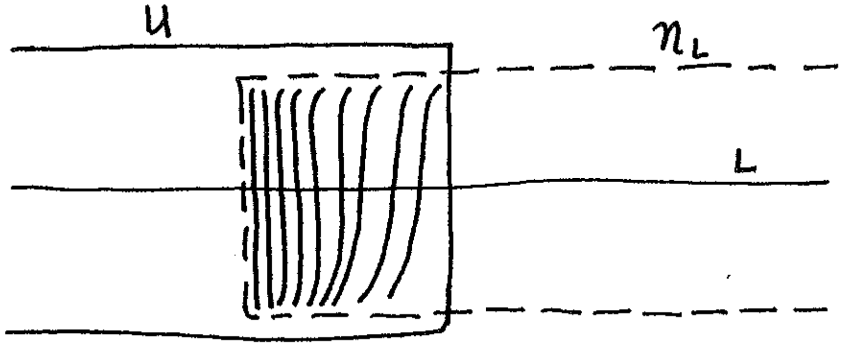

Figure 8.11:

always we will take a function p which in a neighborhood about $f(f_i)$ is 0 and similarly for some point on L that is not $f(f_i)$ we will take p to be 1. This all works provided T is a trivialization. Thus we need $dT \neq 0$. But

$$dT = [1 - p(\bar{x})]I_4 + p(\bar{x})dA + p'(\bar{x})[A - (\bar{x}, \bar{y}, \bar{z}, \bar{w})].$$

At $(\bar{x}, 0, 0, 0)$ we have $A = (\bar{x}, 0, 0, 0)$ and thus $dT = [1 - p(\bar{x})]I_4 + p(\bar{x})dA$. But this is nonsingular provided dA does not have negative eigenvalues which can be guaranteed since both ϕ and ψ were orientation preserving charts. Although it is not the case that there will be no negative eigenvalues we can always multiply pairs of coordinates by -1 if necessary so there will be no negative eigenvalues. Once this is the case we see that $dTv = 0$ implies $dAv = [1 - \frac{1}{p(\bar{x})}]v$ giving the desired result provided $p(\bar{x}) \neq 0$. On the other hand $p(\bar{x}) = 0$ gives I_4, the identity on $\mathbb{R}^4$. With this we have a coordinate chart which gives the transition from U to η_L and the rest follows. Note that the calculation of dT

was only along L and thus we may have to take a smaller neighborhood of L than either U or η_L.

Once in this normal form we apply the results on the lemma to finish the proof. Again note that there is nothing really special in this argument about being in dimension 4.

Section 9

CANCELLATION

In this section we prove the final step in the main theorem. Here we cancel oppositely signed hyperbolic umbilics that have a path between them which does not intersect any other singularities. The ideas were inspired by the work in section 4 on perturbing ramified covers to maps with isolated rank two points. We do similar things here except that we are working with perturbations of a double fold $f(x, y, z, w) = (x, y, z^2, w^2)$.

Theorem VI: *Let $f : M \to N$ be a map with hyperbolic umbilics at $p_i \in M$ with index $(-1)^i$. $(i = 1, 2)$. Assume that there is a curve between p_1 and p_2 which does otherwise meet the locus of singularities of f. Then taking an appropriate curve $c : [a, b] = I \to M$ with $c(a) = p_1$ and $c(b) = p_2$ there is an isotopy of f, fixing f outside of a neighborhood of I, to a map that has no rank two degeneracy in this neighborhood . In other words, the umbilics have been cancelled.*

Before we cancel umbilics we give a lemma necessary for the proof.

Lemma 9.1 *Assume that $(A_i(x,y), B_i(x,y))$, $(i = 1,2)$, are two smooth functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ which are the same outside of some compact set J. Further assume that $|A_i|, |B_i| < M < 1$ for some positive constant M. Take $f_i(x,y,z,w) = (x, y, z^2 + A_i(x,y)w, w^2 + B_i(x,y)z)$. Then there is an isotopy from f_1 defined on $\mathbb{R}^2 \times B(K)$, with $K > 3M^2$, to a function F with the following properties:*

1. *$F(x,y,z,w) = f_1(x,y,z,w) = f_2(x,y,z,w)$ for $(x,y,z,w) \in J \times B(K)$,*
2. *there is an $\epsilon > 0$ with $F(x,y,z,w) = f_2(x,y,z,w)$ for*
$$(x,y,z,w) \in \mathbb{R}^2 \times B(\epsilon),$$
3. *there is a positive $\theta < K$ so that $F(x,y,z,w) = f_1(x,y,z,w)$ for*
$$(x,y,z,w) \in \mathbb{R}^2 \times \{B(K) \setminus B(\theta)\},$$
4. *F has rank two degeneracy only when $A_2 = B_2 = 0$ and $z = w = 0$.*

In particular, if there are no points (x,y) where $A_2 = B_2 = 0$ then F has no rank two degeneracy.

Note that we have taken $B(R)$ to be the 2-ball of radius R.

Proof(of lemma):

Take $\rho(r)$ to be any smooth function from $\mathbb{R}$ to $\mathbb{R}$ that has the following properties:

1. there is an $\epsilon > 0$ so that $\rho(r) = 0$ for $r \leq \epsilon$,
2. there is a $\theta < K$ so that $\rho(r) = 1$ for $r \geq \theta$, and

3. $0 \leq \rho'(r) \leq \frac{1}{3M^2}$.

Such a function exists provided $K > 3M^2$.

Now define

$$F(x, y, z, w) = (x, y, z^2 + [\rho A_1 + (1-\rho)A_2]w, w^2 + [\rho B_1 + (1-\rho)B_2]z)$$

where $\rho = \rho(z^2 + w^2)$. F is rank two if and only if

$$2z + (A_1 - A_2)(2zw\rho') = 0$$

$$[\rho A_1 + (1-\rho)A_2] + (A_1 - A_2)(2w^2\rho') = 0$$

$$[\rho B_1 + (1-\rho)B_2] + (B_1 - B_2)(2z^2\rho') = 0$$

$$2w + (B_1 - B_2)(2zw\rho') = 0.$$

The claim then follows by showing that this is the case only when $w = z = 0$ and $A_2(x,y) = B_2(x,y) = 0$. If $w = z = 0$ and $A_2(x,y) = B_2(x,y) = 0$ we get rank two because $\rho = \rho' = 0$. Thus assume that the rank is two. If $z^2 + w^2 < \epsilon$ then we get $\rho = 0$. $w = z = 0$ and $A_2(x,y) = B_2(x,y) = 0$. If $w = 0$ we get $z = 0$ from the first equation and thus $z^2 + w^2 < \epsilon$. The same is true for $z = 0$. Thus assume that $z, w \neq 0$. Also note that $A_1 = A_2$ gives $z = 0$ and similarly for the B_i's. Thus assume that $A_1 \neq A_2$ and $B_1 \neq B_2$. Then we get that

$$1 + (A_1 - A_2)w\rho' = 0.$$

By eliminating w from this and

$$[\rho A_1 + (1-\rho)A_2] + (A_1 - A_2)(2w\rho') = 0$$

we get

$$\rho' = \frac{-4}{(A_1 - A_2)[A_2 + (A_1 - A_2)\rho]}.$$

But this gives

$$|\rho'| = \frac{4}{|A_1 - A_2||A_2 + (A_1 - A_2)\rho|} \geq \frac{2}{3M^2}$$

which contradicts

$$0 \leq |\rho'(r)| \leq \frac{1}{3M^2}.$$

Thus the lemma is shown.

With this we are now in a position to cancel hyperbolic umbilics.

Proof(of theorem):

Because the p_i's are umbilics we know that there are coordinates about p_i and $f(p_i)$ so that $f(x, y, z, w) = (x, y, z^2 + xw, w^2 + yz)$. Now we wish to choose a curve connecting p_1 and p_2 which does not meet the locus of singularities of f except at the endpoints. Further require that the curve in the coordinates about each p_i is entirely contained within the (x, y)-plane and its image by f is embedded in N. Note that this is possible if $f(p_1) \neq f(p_2)$ and that this is true generically.

The problem of taking the curve within the (x, y)-plane is the reason that we work with hyperbolic umbilics as opposed to elliptic umbilics. In each case the rank 3 singularities locally are given by a cone over a torus. This cuts the complement into two connected components. In the case of hyperbolic umbilics both components contain portions of the (x, y)-plane. This is not the case for the elliptic umbilics. The equations for the fold locus are given by

$xy = zw$ for the hyperbolics and by $x^2 + y^2 = z^2 + w^2$ for the elliptics. If $z = w = 0$ we have $xy = 0$ and $x^2 + y^2 = 0$ and thus the fold locus restricted to the (x, y)-plane is given by the coordinate axes in the first case and only by a point in the second. But clearly a point does not seperate the components. Thus we may have to take smaller coordinate charts to get the curve where we want it. By a change of coordinates in the (x, y)-plane both in M and N we can further insist that our curve is the x-axis and also that the p_i go to $((\pm 1)^i, 0, 0, 0)$. The effects of such a change is that our normal forms become $f(x, y, z, w) = (x, y, z^2 + A(x, y)w, w^2 + B(x, y)z)$ for some functions A and B which still only have simultaneous zeros when $(x, y) = (\pm 1, 0)$.

Now we are in the same position as in section 7 where we extend the coordinates about the image of the curve in N. On the other hand, in section 7 we then used the normal form, $f(x, y, z, w) = (x^3 - 3x, y, z, w)$, to lift the coordinates back upstairs. This worked because it was nonsingular outside of where it was already defined. We can do the same thing here once we extend the function f. At present, we only have f defined in small neighborhoods about $((\pm 1)^i, 0, 0, 0)$. As we already have the general form of f it is only necessary to extend the A and B's. The only condition of the extension that is needed is that they do not vanish. We need this to have a nonsingular map. This can be extended provided the (A, B)'s on each end are in the same quadrant. Because the (z, w) coordinates on the ends are still independent in M we can adjust by $(\tilde{z}, \tilde{w}) = (\pm z, \pm w)$ with an appropriate choice of signs to make A, B in the same quadrant. Now we can extend. With our function $f(x, y, z, w) = (x, y, z^2 + A(x, y)w, w^2 + B(x, y)z)$ we can now lift coordinates to

M as in section 7. As always this can only be done in the small neighborhood about the x-axis where f is nonsingular. Now all that is left to do is to see that we can apply the lemma. First note that there is a section $(A'(x,y), B'(x,y))$ which is the same as $(A(x,y), B(x,y))$ except where we isotope to eliminate the two zeros. This can be viewed as the cancellation of singularities of vector fields and thus exists provided their Hopf indices add to 0. But this is exactly the same as the umbilics being oppositely signed by lemma 2.1. Finally if it is necessary we can scale in the (z,w)-plane to satisfy the conditions of the lemma pertaining to M and K.

Thus we can cancel the oppositely signed hyperbolic umbilics and the theorem follows.

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