Hofer's Geometry on Compact Surfaces

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Abstract of the Dissertation
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Sufficient conditions for an Hamiltonian isotopy to minimize its length in Hofer's geometry are obtained. Given a Morse function H that separates critical points on a compact Riemann surface Σ equipped with a symplectic form ω, we can associate a 1-dimensional CW-complex, its *graph*, that takes into account the values of the function.

The conditions will depend entirely on this graph, and a function τ that gives the length of the periods at the level h. This extends Hofer's results on autonomous Hamiltonians on $\mathbb{R}^{2n}$ to surfaces, see [Ho] but our results are stronger in the sense that our conditions are weaker. In fact a proof in the same spirit produces length-minimizing paths in $\mathbb{R}^2$ with orbits of arbitrary small period which can therefore wind around as much as we wish. In the case of higher genus surfaces, coverings are used to get further results. The extent to which the techniques can be applied to non-autonomous systems is also studied. In particular, when the Hamiltonian is of the form $H(t, x) = f(t)H(x)$, the method applies.

A Guette, sa Maîtresse, et à la Mémoire de Martin Boucher.

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MOMO

Chapter 1

Introduction

1.1 Hamiltonian Diffeomorphism

Let $(\mathbf{M}, \omega)$ be a compact symplectic manifold. To an isotopy ϕ_t joining the identity to $\phi = \phi_1$, there corresponds a vector field X_t, defined as $X_t(\phi_t(p)) = \frac{d}{dt}\phi_t(p)$. Such an isotopy is a path in the group of symplectomorphisms, $\text{Symp}(\text{M}, \omega)$ if and only if $\mathcal{L}_{X_t}\omega = 0$. But for $\phi_t \in \text{Symp}(\mathbf{M}, \omega)$, we get

$$\begin{aligned} \mathcal{L}_{X_t}\omega &= d\iota_{X_t}\omega + \iota_{X_t}d\omega \\ &= d\iota_{X_t}\omega + 0 \\ &= 0 \end{aligned}$$

So $\iota_{X_t}\omega$ is closed. If it is exact, i.e. $\iota_{X_t}\omega = dH_t$, then by definition, we will say that $\phi_t \in \text{Ham}(\mathbf{M}, \omega)$, the group of Hamiltonian diffeomorphisms. H_t is only defined up to a constant, but we say that ϕ_t is generated by H_t. To fix notation, we will denote the Hamiltonian isotopy generated by H :

$\mathbf{M} \times [0,1] \to \mathbb{R}$ by ϕ_H^t.

A more geometric definition can be made using the *Flux Homomorphism*, [Mc-Sa]. This translates into the following: for every loop $\gamma : S^1 \to M$, we construct its image under the flow $\beta_\gamma : [0,1] \times S^1 \to M$ defined by

$$\beta_\gamma(t,s) = \phi_t(\gamma(s))$$

The condition for a symplectic isotopy ϕ_t to be Hamiltonian is

$$\int_{[0,1]\times S^1} \beta_\gamma^* \omega = 0, \quad (\forall \gamma)$$

A natural question arising in symplectic topology is the following: If $H_n \to 0$ in the C^0 norm, does $\phi_{H_n} \to \mathbb{1}$? Notice that we only have a C^0 control but the conclusion is of C^1 nature. It is mainly to answer this question in $\mathbb{R}^{2n}$, that Hofer introduced his metric.

1.2 Hofer's metric

In [Ho], Hofer introduced a bi-invariant metric on the group of compactly supported Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$, which was later generalized for any symplectic manifold by Lalonde and McDuff, [La-Mc1].

This metric arises from a bi-invariant norm, which gives $\mathrm{Ham}(\mathbf{M},\omega)$ the structure of an infinite dimensional Finsler manifold. As in the case of bi-invariant metrics on finite dimensional Lie groups, it suffices to first define such a norm on its Lie algebra, and then use integration and the bi-invariance. Since the Lie algebra of $\mathrm{Ham}(\mathbf{M},\omega)$ can be identified with:

$$C^\infty(\mathbf{M};\mathbb{R})/\{\text{constants}\}$$

one can use the norm

$$\|H\| = \sup_{x\in \mathrm{M}} H(x) - \inf_{x\in \mathrm{M}} H(x).$$

Then the length of the isotopy ϕ_t generated by H_t, where $0 \leq t \leq 1$ is :

$$\mathcal{L}(\phi_t) = \int_0^1 \left(\sup_{x\in \mathrm{M}} H_t(x) - \inf_{x\in \mathrm{M}} H_t(x) \right) dt = \int_0^1 \|H_t\| dt$$

The distance between ϕ to the identity, called the energy, and denoted by $\mathbf{E}(\phi)$, is defined as the infimum of the length of all paths joining the identity to ϕ. Using the bi-invariance, one can also define the distance between any two elements ϕ, ψ of $\mathrm{Ham}(\mathbf{M}, \omega)$ by;

$$d(\phi, \psi) = \mathbf{E}(\phi\psi^{-1})$$

To show that d indeed is a metric i.e. that $\mathbf{E}(\phi) = 0$ implies $\phi = \mathbb{1}$, is an highly non-trivial result. Hofer's proof, which only works in $\mathbb{R}^{2n}$ uses heavy machinery from nonlinear analysis, mainly the study of the action functional defined on the loop space of $\mathbb{R}^{2n}$ with some Sobolev norm, see[Ho-Ze].

In [La-Mc1], Lalonde and McDuff use pseudo-holomorphic techniques to produce a proof that works for any symplectic manifold. Essentially, it follows from the generalised unsqueezing theorem, which was discovered by Gromov [Gr] for $\mathbb{R}^{2n}$, and which they generalised for any $(\mathbf{M}, \omega)$. This states:

Theorem 1.2.1 Generalized Unsqueezing Theorem[Gr],[La-Mc1]

If there is a symplectic embbedding of $\mathrm{B}(\mathrm{r})^{2n+2} \hookrightarrow \mathbf{M} \times \mathrm{B}(\mathrm{R})$, *then* $\mathrm{r} \leq \mathrm{R}$.

1.3 Length Minimizing Geodesics

Motivated by Riemannian geometry, geodesics are defined in a variational way. Given $\psi \in \mathrm{Ham}(\mathrm{M},\omega)$, let $\mathcal{P}(\psi)$ be the space of all C^∞ paths $\gamma = \{\phi_t\}_{t\in[0,1]}$ from $\mathbb{1}$ to ψ with the C^∞-topology. (Thus two paths γ and γ' are close if the associated maps $M \times [0,1] \to M$ are C^∞-close.) For each $\gamma \in \mathcal{P}(\psi)$ let $\mathcal{P}_\gamma$ be the path-connected component of $\mathcal{P}(\psi)$ containing γ. A path $\gamma = \{\phi_t\}_{t\in[a,b]}$ is said to be *regular* if its tangent vector $\dot{\phi}_t$ is non-zero for all $t \in [a,b]$. Further, γ is said to be a *local minimum* of $\mathcal{L}$ if it has a neighbourhood $\mathcal{N}(\gamma)$ in $\mathcal{P}_\gamma$ such that

$$\mathcal{L}(\gamma) \leq \mathcal{L}(\gamma'), \quad \text{for all } \gamma' \in \mathcal{N}(\gamma).$$

Given an interval $I \subset \mathbb{R}$, we will say that a path $\{\phi_t\}_{t\in I}$ is a *geodesic* if it is regular and if every $s \in I$ has a closed neighbourhood $\mathcal{N}(s) = [a_s, b_s]$ in I such that the path $\{\phi_{\beta(t)}\}_{t\in\mathcal{N}(s)}$ is a local minimum of $\mathcal{L}$, where $\beta : \mathcal{N}(s) \to [0,1]$ is the linear reparametrization $\beta(t) = (t - a_s)/(b_s - a_s)$.

Following the work of Bialy-Polterovich on $\mathbb{R}^{2n}$, [Bi-Po] as well as Lalonde-McDuff on general manifolds,[La-Mc2],[La-Mc3] geodesics are now well understood in Hofer's metric. This is not the case for length minimizing geodesics on general manifolds.

Necessary conditions which holds for any symplectic manifold, were established by Lalonde-Mcduff [La-Mc2], and say that in order to minimize length, H_t has to be *quasi-autonomous*, i.e., there exist points x_+ and x_- , such that for all t:

$$x_+ \in \{x \in \mathbf{M} : H_t(x) = \max H_t\}.$$
$$x_- \in \{x \in \mathbf{M} : H_t(x) = \min H_t\}.$$

Notice that the values at these points may vary. In fact, this condition is sufficient for the isotopy generated by such Hamiltonian to be geodesics.

On $\mathbb{R}^{2n}$ sufficient conditions for a geodesic to minimize length are well understood. Hofer in [Ho] showed that autonomous Hamiltonians minimize length as long as there are no non-constant periodic orbits. This was later generalized by Bialy-Polterovich, [Bi-Po] and again by Siburg, [Si] to the quasi-autonomous case, under certain additionnal hypothesis, mainly Ustilovsky's conditions, [Us], on the nondegeneracy at the fixed minimum and maximum of H_t. Siburg's result states that under the non-degeneracy condition, a quasi-autonomous Hamiltonian is length minimizing provided that ϕ_t does not admit any non-constant closed trajectories for any $t \in [0,1]$.

In [La-Mc3] , Lalonde-McDuff considered the space

$$\Gamma_H = \{(m,t,z) \in \Sigma \times \mathbb{R} \times [0,1] \mid z = H_t(m)\},$$

which separates $\Sigma \times \mathbb{R} \times [0,1]$ into two components, $\Gamma_H^{\pm}$. They introduced

$$\rho(H) = Min\{c_G(\Gamma_H^-), c_G(\Gamma_H^+)\},$$

where $c_G(X)$, the *Gromov radius* of X, see [Gr] is defined as

$$c_G(X) = sup_\varphi\{\pi \mathbf{r}^2 \mid \varphi : \mathrm{B(r)}^{2n} \hookrightarrow X\}$$

where φ ranges over all symplectic embedding.

In [La-Mc3], the following theorem is proved:

Theorem 1.3.1 *Let* $(\mathbf{M}, \omega)$ *be a compact Riemann surface with area form* ω. *If* $\rho(H) = \mathcal{L}(\phi_{H_t})$ *then* ϕ_{H_t} *is length-minimizing in its homotopy class with fixed endpoint.*

As Ham$(\mathbf{M}, \omega)$ is contractible for any surface of genus bigger than zero, this implies that ϕ_{H_t} is length-minimizing, thus of energy $\mathcal{L}(\phi_{H_t})$. For the sphere an extra assumption is needed, as Ham$(\mathbf{M}, \omega)$ retracts into $SO(3)$. Another way to produce length minimizing paths is to use the *Energy-Capacity Inequality* :

Theorem 1.3.2 Energy-Capacity Inequality,[Ho],[La-Mc1]

$$\mathbf{E}(\phi) \geq \tfrac{1}{2} \sup\{c_G(A) : \phi(A) \cap A = \emptyset\}$$

In the case of $\mathbb{R}^{2n}$, *it remains true without multiplying by* $\frac{1}{2}$.

As the right side of the inequality is clearly bigger than zero when $\phi \neq \mathbb{1}$, this shows that the pseudo-metric is indeed a metric.

1.4 Outline of the Work

Theorem 1.3.1 and 1.3.2 constitute the foundation of this work. In order to find neccesary conditions for a geodesic in Ham$(\mathbf{M}, \omega)$ to minimize the distance between its endpoint, the latter results suggest two methods:

Embedding of Balls Method Find conditions on the Hamiltonian H, under which the equality $\rho(H) = \mathcal{L}(\phi_{H_t})$ holds. So one must embed balls in $\Gamma_H^{\pm}$, in order to apply 1.3.1.

Covering Space Method Use the Energy-Capacity Inequality, 1.3.2 on some covering of a Riemann surface of higher genus to show that a path is length-minimizing.

Chapter 2

Statement of the Main Results

2.1 Definition of the Graph

Let (Σ, ω) be a compact surface with symplectic form ω. For a given Morse function $H : \Sigma \to \mathbb{R}$ that separates critical points, we can construct a graph (see Figure 2.1) as follows: At every critical point, we put a vertex, the height of the vertex being its H-value. We join two vertices together if there is exactly an open cylinder between them.

The graph satisfies the following:

(i) Each vertex is a critical point

(ii) Each edge represents an open cylinder, disjoint from each others

(iii) A critical point of index 1 is where either one open cylinder splits in two, or two open cylinders merge as one.

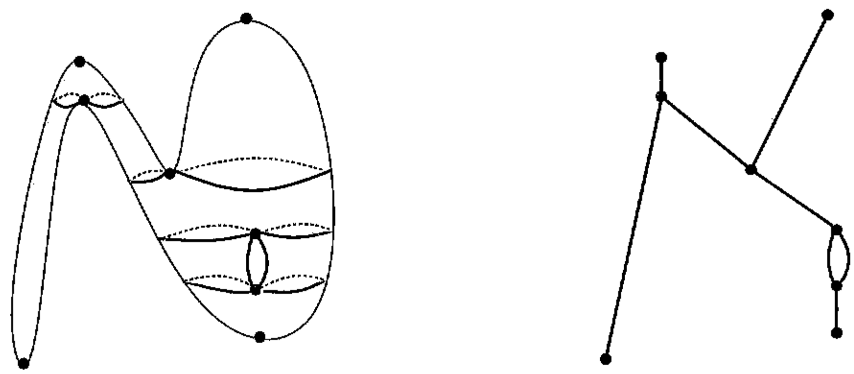

Figure 2.1: A slanted height function on the torus.

Definition 2.1.1 On the edge σ_i, representing the cylinder $A_i \subseteq H^{-1}(b_{j-1}, b_j)$ between the critical values (b_{j-1}, b_j), each $h \in (b_{i-1}, b_j)$ corresponds to a unique point $\bar{h}$ in σ_i. We can thus define $\tau_i : \sigma_i \to \mathbb{R}$, as:

$$\tau_i(\bar{h}) = \text{the period of the orbit at level } h \text{ , for } \quad h \in (b_{i-1}, b_i).$$

There is a nice relation which will turn out to be crucial in this work, between τ_i and the area of the cylinder A_i, namely the "Area-Equality" :

Proposition 2.1.2

$$Area(A_i) = \int_{A_i} \omega = |\int_{b_{i-1}}^{b_i} \tau_i(h) dh|$$

This tells us that a "thin" tube will have on average small periods. We will denote by $Area_{\sigma_i}$ this common value.

Definition 2.1.3 By a *admissible path* σ on the graph, we mean a sequence of edges σ_i that start at the maximum, which satisfies the two following conditions:

(i) Two consecutive edges must be connected by a critical point

(ii) Neither edges nor vertices can be used twice

An admissible path $\sigma = (\sigma_1, \dots, \sigma_n)$ thus determines a sequence of critical points $(x_0, \dots, x_n)$ and of critical values $a_0 = H(x_0), a_1 = H(x_1), \dots, a_n = H(x_n)$ so the edge σ_i is determined by x_{i-1}, x_i. To such a path σ on the graph, we can assign a (non-unique!) path γ on Σ, which goes through the critical points $x_0 = max(H), x_1, ..., x_n$, and is transverse to H outside of this set.

Definition 2.1.4

$$\sigma_i \text{ is called } \begin{cases} \textit{downwards} & \text{if } a_{i-1} > a_i; \\ \\ \textit{upwards} & \text{if } a_{i-1} < a_i; \end{cases}$$

Thus for an admissible path σ, σ_1 is always downwards.

2.2 Definition of τ

Definition 2.2.1 The variation of the values of H along σ allows us to define a natural parameterization for an admissible path σ. Thus, we can identify elements of σ with an $h \in \mathbb{R}$; if $var_{\sigma_k} = |a_k - a_{k-1}|$, then denote $|\sigma_i|$ by :

$$|\sigma_i| = \begin{cases} 0 & \text{if } i = 0 \\ \\ \sum_{k=1}^{i} var_{\sigma_k} & 0 < i \leq n \end{cases}$$

The parameterization on $(|\sigma_{i-1}|, |\sigma_i|)$ will be given by:

$$t \mapsto \frac{a_i - a_{i-1}}{|\sigma_i| - |\sigma_{i-1}|}(t - |\sigma_{i-1}|) + a_{i-1} = (\pm 1)(t - |\sigma_{i-1}|) + a_{i-1}$$

From now on we shall make no difference between the path and its parametrization, i.e., between an element of σ and a real number $h \in (0, |\sigma_n|)$.

Notice that $h \in (|\sigma_{i-1}|, |\sigma_i|) \quad \Leftrightarrow \quad h$ parameterizes a element of σ_i, and that $H(|\sigma_i|) = a_i$. As γ is transverse to H on A_i, we can also reparameterize the curve γ, so that on A_i, from x_{i-1} to x_i, we have $\gamma_i : (|\sigma_{i-1}|, |\sigma_i|) \to A_i$, so that:

$$H(\gamma_i(h)) = \begin{cases} H(|\sigma_{i-1}|) - h + |\sigma_{i-1}| & \text{if } \sigma_i \text{ is downwards,} \\ \\ H(|\sigma_{i-1}|) + h - |\sigma_{i-1}| & \text{if } \sigma_i \text{ is upwards} \end{cases}$$

We can now define τ on the whole path using this identification:

Definition 2.2.2 Definition of τ

$$\tau : (0, |\sigma_n|) \setminus \{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_{n-1}|\} \to \mathbb{R},$$

where

$\tau(h) =$ the period of the orbit at the point parameterized by h.

Note that $\lim_{h \to |\sigma_i|} \tau(h) = \infty, \quad 0 < i < n.$

2.3 Statement of the Main Theorem

For the next definition, we will assume that H is *normalised*, i.e. it has maximum M and minimum 0. To do so one has to consider $H - min(H)$, a process which leaves the Hamiltonian flow invariant.

Definition 2.3.1 Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an *admissible path* on the graph determined by H. The *reached area*, $\mathcal{R}_\sigma$ and $\mathcal{H}_\sigma$, the H-value of the point on the graph parameterized by h, are given by

$$\begin{aligned} \mathcal{R}_\sigma(h) &= \int_0^h \tau(\eta)d\eta \\ \mathcal{H}(h) &= \begin{cases} a_{i-1} + |\sigma_{i-1}| - h & \text{if } h \in \sigma_i,\ \sigma_i \text{ downwards} \\ a_{i-1} - |\sigma_{i-1}| + h & \text{if } h \in \sigma_i,\ \sigma_i \text{ upwards} \end{cases} \end{aligned}$$

Remark 2.3.2 We see that $\mathcal{H}(h) = H(\gamma(h))$, where γ is the curve on Σ constructed from σ with the appropriate parameterization. Also $\mathcal{H}_{\sigma_i}(h) = \ell_i(h)$, where ℓ_i is the linear function in the (h,k)-plane going through the points $(|\sigma_{i-1}|, a_{i-1})$ and $(|\sigma_i|, a_i)$.

Definition 2.3.3 Let H be a Morse function with minimum 0 and maximum K on a compact Riemann surface (Σ, ω). An admissible path $\sigma = (\sigma_1, \ldots, \sigma_n)$ from the maximum is said to "*cover enough area*" if the following holds:

$$\sup_{h \in (0, |\sigma_n|)} \mathcal{R}_\sigma(h) \geq K$$

Remark 2.3.4 The previous definitions only made sense for a *admissible path* σ starting at the maximum. An *admissible path from the minimum* will be by definition an admissible path for $H' = K - H$.

Theorem 2.3.5 [Main Theorem]

Let H be a Morse function with minimum 0 and maximum K on a compact Riemann surface (Σ, ω) of genus > 1 that separates critical points. Suppose also that the graph admits admissible paths σ^+ and σ^- from the maximum and minimum, which cover enough area and satisfy:

$$\mathcal{R}_{\sigma^\pm}(h) \geq K - \mathcal{H}_{\sigma^\pm}(h)$$

Then the time one map of the Hamiltonian system generated by H, ϕ_H is length minimizing, and

$$\mathbf{E}(\phi_H) = K$$

In the case of the sphere, we also need $Max(H) \leq \frac{1}{2} Area(\Sigma)$, as $\mathrm{Ham}(\mathrm{S}^2, \omega)$ *is not contractible.*

Remark 2.3.6 Suppose that $h \in \sigma_j$, for σ_j upwards: as $h > \sigma_{j-1}$, then $\mathcal{H}_\sigma(h) = a_{j-1} + h - |\sigma_{j-1}| > a_{j-1}$. So $K - \mathcal{H}_\sigma(h) < K - a_{j-1}$. Let σ_k, for $k < j$, be the last downward path before σ_j; then $\mathcal{R}_\sigma(|\sigma_k|) \geq K - \mathcal{H}_\sigma(|\sigma_k|) = K - a_k$; as $a_k \leq a_{j-1}$, then $K - a_k \geq K - a_{j-1}$. Summing up, we get:

$$\mathcal{R}_\sigma(h) \geq \mathcal{R}_\sigma(|\sigma_k|) \geq K - a_k \geq K - a_{j-1} \geq K - \mathcal{H}_\sigma(h)$$

That means that the conditions $\mathcal{R}_\sigma(h) \geq K - \mathcal{H}_\sigma(h)$ are automatically satisfied on upwards paths, provided they are satisfied on previous downwards one. So our conditions is only on downwards paths.

As a special case, we generalize Hofer's result for autonomous flows on $\mathbb{R}^{2n}$, see [Ho], Theorem 3, to the case of arbitrary surfaces.

Corollary 2.3.7 *If ϕ_H does not admit periodic orbits of period T, $0 < T \leq 1$, then*

$$\mathbf{E}(\phi_H) = Max(H) - Min(H).$$

Proof: Normalize H such that it has maximum K and minimum *0*. In that case, we have $\tau \geq 1$, so $\mathcal{R}_\sigma(h) \geq \int_0^h d\eta \geq h$. Take a path σ from the maximum to the minimum . As we only have to verify our hypothesis on downwards paths, using

$$K - a_{j-1} \leq |\sigma_{j-1}| \Rightarrow K - a_{j-1} + h - |\sigma_{j-1}| \leq h,$$

on the downwards path σ_j we get

$$\begin{aligned} \mathcal{R}_\sigma(h) &\geq h \\ &\geq K - a_{j-1} + h - |\sigma_{j-1}| \\ &= K - \mathcal{H}_\sigma(h) \end{aligned}$$

□

2.4 The non-autonomous case

The aim is to generalize the results of Bialy-Polterovich, see [Bi-Po], as well as Siburg, see [Si] in $\mathbb{R}^{2n}$, on the conditions for a given non-autonomous Hamiltonian H, to minimize the distance between its endpoints. For a general Hamiltonian, our technique does not apply. However for H satisfying the *foliated property*, under an additional hypothesis over the average of H, $a_H(x) = \int_0^1 H_t(x)dt$, the technique works:

Definition 2.4.1 $H : [0,1] \times M \to \mathbb{R}$ satisfies the *foliated property* if the following holds for all p, q in M:

$$\text{If} \quad H_{t_o}(p) = H_{t_o}(q) \quad \text{for some} \quad t_o \in [0,1]$$

then

$$H_t(p) = H_t(q) \quad \text{for all} \quad t \in [0,1].$$

Remark 2.4.2 The foliated property implies that in fact

$$H_t = \rho_t \circ H_0, \quad \rho_t : \mathbb{R} \to \mathbb{R}, \quad t \in [0,1]$$

For example, any autonomous Hamiltonian satisfies this property, as well for Hamiltonian of the form $H(t,x) = f(t)H(x)$, where $f : [0,1] \to \mathbb{R}$.

In fact, we won't need the foliated property everywhere, but only along admissible paths $\gamma^{\pm}$ on Σ constructed from admissible paths $\sigma^{\pm}$ which cover enough area from the maximum and minimum of $a_H(x)$. In [Bi-Po], the following is proved:

Lemma 2.4.3 *If $H : \Sigma \times [0,1] \to \mathbb{R}$ is quasi-autonomous, then*

$$\mathcal{L}(a_H) = \mathcal{L}(H)$$

For the following corollary, we suppose that a_H is a Morse function. We now state our result for the non-autonomous case:

Corollary 2.4.4 *Let $H : [0,1] \times \Sigma \to \mathbb{R}$ satisfying the foliated property, along admissibles paths σ_+ and σ_i which covers enough area from the maximum and the minimum of $a_H(x) = \int_0^1 H_t(x)dt$, and which satisfy:*

$$\mathcal{R}_{\sigma_\pm}(h) \geq K - \mathcal{H}_{\sigma_\pm}(h)$$

Then H produce a length-minimizing isotopy, so that

$$\mathbf{E}(\phi_H) = \mathcal{L}(\phi_{H_t})$$

If $H(t,x) = f(t)H(x)$, we get

$$\mathbf{E}(\phi_H) = (max(H) - min(H)) \int_0^1 f(t)dt$$

Proof:

Remark 2.4.5 The ingredient needed to get rid of the *foliated property* condition can be phrased as the following question:

Let $\mathcal{U}$ be a simply connected open set of $\mathbb{R}^2$ equipped with the standard symplectic form ω_o. Let $H : \mathcal{U} \times [0,1] \to \mathbb{R}$ be a smooth function. Denoting by $B^2(K)$, the disk of area K, let:

$$\mathcal{U}_B = \{(x,y) \in \mathcal{U} \times \mathbb{R}^2 : y \in B^2(a_H(x))\}$$
$$\mathcal{U}_H = \{(x,t,h) \in \mathcal{U} \times [0,1] \times \mathbb{R} : 0 \leq h \leq H_t(x)\}$$

If the following were affirmative, 2.4.4 would be true without the foliated property assumption.

Question 2.4.6 [Dropping the Foliated Property Assumption]
Is there a fiber preserving symplectomorphism of $(\mathcal{U}_B, \omega_o \oplus \omega_o)$ into $(\mathcal{U}_H, \omega_o \oplus dt \wedge dh)$ which fixes the base $\mathcal{U}$?

This is essentially a parameterized version of Moser's theorem, [Mo]. It boils down to finding a smooth family

$$\{\Phi_x\}_{x \in \mathcal{U}} : B^2(a_H(x)) \to \{(t,h) \in [0,1] \times \mathbb{R} : 0 \leq h \leq H_t(x)\}$$

satisfying

$$\Phi^*(dt \wedge dh) = \omega_o$$

In general, $\Phi^*(dt \wedge dh)$ would depend on x, but if H satisfies the foliated property, we can find $\Phi : B^2(K) \to \mathbb{R}^2$ which works for all x.

Even if we could drop the foliated property assumption, we are still far from Siburg criterion, which purely is a statement about trajectories of ϕ_{H_t}. To avoid making any statement about $a_H(x)$, it seems that one has to use the

Hofer-Zehnder capacities [Ho-Ze], instead of the Gromov radius. In [La-Mc3], this capacity is used to prove corollary 2.3.7 in higher dimensions. However, there are two major drawbacks as the Hofer-Zehnder capacity is very difficult to calculate. Worse it can be infinite for compact sets, as it is imply by Herman's non-closing lemma[Ho-Ze].

Chapter 3

Generalized Trapezoids

In order to use the *Embedding of Balls Method*, according to 1.3.1 one has to show that given an Hamiltonian such that $K = max(H) - min(H)$, we can symplectically embed balls of capacity K into $\Gamma_H^{\pm}$. Instead, we will symplectically embbed a *Generalized Trapezoid*, which are objects of capacity K, provided τ satisfies the hypothesis of Theorem 2.3.5. Here τ is the function defined in 2.1.1 which depends on the Hamiltonian H and on a given path of its associated graph.

3.1 Trapezoid

Before defining generalised trapezoid, we recall the definition of *trapezoid* as in [La-Mc3], and then compute their capacity. From now on, let $B^{2n}(c)$ denotes the ball of capacity c (thus of radius $\sqrt{\frac{c}{\pi}}$).

Definition 3.1.1 Let $K > 0$; then $T(K)$, the trapezoid of capacity K is defined as

$$T(K) = \{(u, v, x) \in (0, K) \times (0, 1) \times B^2(K) \mid x \in B^2(K - u)\}$$

By the unsqueezing theorem 1.2.1, we have $c_G(T(K)) \leq K$. In fact, the following stronger assertion holds:

Lemma 3.1.2

$$c_G(T(K)) = K$$

Proof: To prove $c_G(T(K)) \geq K$, for any $\epsilon > 0$ we will construct a fibered embedding

$$F : B^4(M - \epsilon) \to T(K).$$

As $B^4(M - \epsilon)$ fibers over $B^2(M - \epsilon)$ [1] and $T(K)$ fibers over $t_K = \{((u, v) \in (0, K) \times (0, 1)\}$ with balls of varying sizes, it suffices to construct an embedding which preserves the fibers. More precisely, let $h_B(x, y) : B^2(M - \epsilon) \to \mathbb{R}$ be the function which at (x, y) gives the capacity of the fiber at this point. Then $h_B(x, y) = M - \epsilon - \pi(x^2 + y^2)$. Let h_t be the similar function on t_K. Then $h_t(u, v) = M - u$. it Thus to define F, it suffices to find an area preserving map

$$f : B^2(M - \epsilon) \to t_K, \quad \text{which satisfies} \quad h_t(f(x, y)) \geq h_B(x, y).$$

Then we can let $F = f \times \mathbb{1}$.

The construction of f will be given in two steps:

[1] If we denote by (z_1, z_2) the element of $B^4(M - \epsilon)$, then the fibering is given by $(z_1, z_2) \to z_1$.

Step 1 We will construct a C^∞ family of loops $\Phi(r,\theta) : B^2(M-\epsilon) \to t_K$ which will preserve the area radially and satisfy $h_t(\Phi(r,\theta)) \geq h_B(r,\theta)$.

Step 2 We will reparameterize our family $\Phi(r,\theta)$, along θ, so that the map becomes area preserving. This will be our f.

To construct such a map, consider a family of loops with disjoint images,

$$\Phi(r,\theta) : S^1 \to t_K, \quad 0 \leq r < \sqrt{\frac{K-\epsilon}{\pi}}$$

such that the following are satisfied:

(i) $\Phi(0,\theta)$ goes to the constant loop $(\frac{\epsilon}{2}, \frac{1}{2})$

(ii) $Area(\bigcup_{r<\rho} \Phi(\{r\} \times S^1)) = Area(\{(r,\theta) \mid r < \rho\}) = \pi\rho^2$, for all ρ

(iii) $p_u(\Phi(\{r\} \times S^1)) \leq \pi r^2 + \epsilon$, for all r, where p_u is the projection on the u-axis.

Condition **(iii)** implies $h_t(\Phi(r,\theta)) \geq h_B(r,\theta)$, and its verification is implied by

$$\begin{aligned} Area\{(\theta,r) \mid h_B(\theta,r) \geq \lambda\} &= K - \lambda - \epsilon \\ Area\{(u,v) \mid h_t(u,v) \geq \lambda\} &= K - \lambda \end{aligned}$$

We now can pass to step 2.

Consider

$$\omega = k(r,\theta) dr \wedge d\theta = \Phi^*(dx \wedge dy).$$

We claim that there exist a coordinate change of $B^2(K-\epsilon)$ of the form

$$G(r,\theta) = (r, g(r,\theta)) \quad \text{so that} \quad G^*(\omega) = r dr \wedge d\theta.$$

As r is preserved, conditions **(i)**, **(ii)** , **(iii)** will still be true, so we can let

$$f = \Phi \circ G.$$

As $G^*(\omega) = k(r, g(r,\theta)) g_\theta dr \wedge d\theta$, it suffices to solve:

$$k(r, g(r,\theta)) g_\theta(r,\theta) = r, \quad \text{for } g \text{ periodic in } \quad \theta. \qquad (*)$$

The coordinate change will essentially be a reparametrization by arc-length.

Consider the following family of functions of θ parameterized by r:

$$\psi^r(\theta) = \frac{1}{r}\int_0^t k(r,\eta)d\eta, \qquad \text{for } \theta \in (0, 2\pi),$$

where $0 < r < \sqrt{\frac{K-\epsilon}{\pi}}$.

Here are some properties of $\psi^r(\theta)$:

(a) $\psi^r(0) = \frac{1}{r}\int_0^0 k(r,\eta)d\eta = 0.$

(b) As Φ preserves the area of concentric disks, we get:

$$\begin{aligned} \int_0^\rho \int_0^{2\pi} k(r,\eta) dr d\eta &= \pi\rho^2; \quad \text{taking derivatives, we get} \\ \int_0^{2\pi} k(\rho,\eta)d\eta &= 2\pi\rho. \end{aligned}$$

This gives $\psi^r(2\pi) = 2\pi$

Let $g(r,\theta)$ be the inverse of $\psi^r(\theta)$.

We claim that $g(r,\theta)$ is a solution of $(*)$. Periodicity follows from properties **(a)** and **(b)**. To simplify notation, we will denote $\psi^r(\theta)$ by $\psi(r,\theta)$. Taking the θ derivative on both sides, we get:

$$\begin{aligned} g(r,\theta) &= \psi^{-1}(r,\theta), \quad \text{so} \\ g_\theta(r,\theta) &= \frac{1}{\psi'(\psi^{-1}(r,\theta))} \\ &= \frac{1}{\psi'(g(r,\theta))} \\ &= \frac{r}{k(r,g(r,\theta))} \end{aligned}$$

This implies $k(r,g(r,\theta))g_\theta(r,\theta) = r$.

□

3.2 Main Property of Generalised Trapezoids

We will construct a generalized version of a trapezoid, abstracting the hypothesis of Theorem 2.3.5.

Let $|\sigma_0| = 0, |\sigma_1|, \ldots, |\sigma_n|$ and $a_0 = K, a_1 \ldots, a_n$ be sequences of numbers. The first sequence is required to be increasing. Denoting by σ the interval $(|\sigma_{i-1}|, |\sigma_i|)$, we say that σ is downwards if $|\sigma_i| - |\sigma_{i-1}| < 0$, upwards if $|\sigma_i| - |\sigma_{i-1}| > 0$. We do not admit a pair of sequences such that $|\sigma_i| - |\sigma_{i-1}| = 0$, for any i.

Let $\mathcal{H} : (0, |\sigma_n|) \to \mathbb{R}^+$ be a continuous piecewise linear function and also $\tau : (0, |\sigma_n|) \to (0, \infty$ be a positive integrable C^∞ function such that

(i) $\mathcal{H}(0) = 0 = a_0, \quad \mathcal{H}(|\sigma_i|) = a_i$, for all i.

(ii) By continuity of $\mathcal{H}$, we must have

$$\mathcal{H}(h) = \begin{cases} a_{i-1} + |\sigma_{i-1}| - h & \text{if } h \in \sigma_i,\ \sigma_i \text{ downwards} \\ \\ a_{i-1} - |\sigma_{i-1}| + h & \text{if } h \in \sigma_i,\ \sigma_i \text{ upwards} \end{cases}$$

Definition 3.2.1 Let $\{|\sigma_i|\}_{0 \le i \le n}$, $\{a_i\}_{0 \le i \le n}$, $\mathcal{H}$ and τ be defined as above; then the *generalized trapezoid* $T_{\sigma,\mathcal{H}}(\tau)$ is defined as

$$T_{\sigma,\mathcal{H}}(\tau) = \{(h, k, x) \in (0, |\sigma_n|) \times \mathbb{R} \times B^2(K) \mid 0 < k < \tau(h), x \in B^2(\mathcal{H}(h))\}$$

Theorem 3.2.2 *Suppose that* $\int_0^{|\sigma_n|} \tau(\eta) d\eta \ge K$, *and that* $\int_0^h \tau(\eta) d\eta \ge K - \mathcal{H}(h)$ *for all* h. *Then*

$$c_G(T_{\sigma,\mathcal{H}}(\tau)) = K$$

Proof: As in lemma 3.1.2, we will construct a fibered embedding. But we will use trapezoid instead of balls, taking full advantage of lemma 3.1.2. Let $t_{K-\epsilon} = \{(u, v) \in (0, K - \epsilon) \times (0, 1)\}$, and $t_\tau = \{(h, k) \in (0, |\sigma_n|) \times \mathbb{R} \mid 0 < k < \tau(h)\}$. $T(K - \epsilon)$ fibers over $t_{K-\epsilon}$ and $T_{\sigma,\mathcal{H}}(\tau)$ fibers over t_τ with balls of variable sizes. Denoting by $h_t : t_{K-\epsilon} \to \mathbb{R}$, the function which at (u, v) gives the capacity of the fibers, and by h_σ the similar function on t_σ, we get that

$$h_t(u, v) = k - \epsilon - u \quad and \quad h_\sigma(h, k) = \mathcal{H}(h).$$

Thus to prove our theorem, it suffices to find, for every ϵ, an area preserving map:

$$f : t_{K-\epsilon} \to t_\tau \quad \text{such that} \quad h_\sigma(f(u,v)) \geq K - \epsilon - u$$

Our embedding will then be $F = f \times \mathbb{1}$.

bo We now construct our f:

Let $g(u) = \int_0^u \tau(\eta)d\eta$ and $h_\epsilon = g^{-1}(K-\epsilon)$ We can now define

$$\phi : (0, h_\epsilon) \to (0, |\sigma_n|)$$

by

$$\phi(u) \quad \text{goes to} \quad g^{-1}(u).$$

Remark 3.2.3 *$\phi(u)$ is where on $(0, |\sigma_n|)$, we have accumulated an amount u of area.*

As

$$\phi'(u) = \frac{1}{g'(\phi(u))} = \frac{1}{\tau(\phi(u))},$$

The map f given by

$$f(u,v) = (\phi(u), \tau(\phi(u))v)$$

will be area preserving, as its Jacobian at (u, v) is 1. It remains to show that $h_\sigma(f(u,v)) = \mathcal{H}(\phi(u)) \geq K - \epsilon - u$. But by hypothesis

$$u \;=\; \int_0^{\phi(u)} \tau(\eta)d\eta$$

$$\geq \; K - \mathcal{H}(\phi(u))$$

We get $\mathcal{H}(\phi(u)) \geq K - u > K - \epsilon - u$.

□

Chapter 4

Proof of the Main Theorem

4.1 Symplectic Embedding of T_τ into Σ

Let σ be an admissible path, and γ the curve on Σ constructed from it, as in definition 2.2.1. Recall that we parameterized γ such that:

$$H(\gamma(h)) = \begin{cases} a_{i-1} - h + |\sigma_{i-1}| & \text{If } h \in \sigma_i \text{ downwards} \\ \\ a_{i-1} + h - |\sigma_{i-1}| & \text{If } h \in \sigma_i \text{ upwards} \end{cases}$$

We use the Hamiltonian flow ϕ_H^t to embed $t_{\tau_i} = \{(h,k) \in \mathbb{R}^2 \mid |\sigma_{i-1}| < h < |\sigma_i| \quad 0 < k < \tau_i(h)\}$ into A_i, the cylinder determined by the edge σ_i.

Let $(\mathbf{M}, \omega)$ be a symplectic mainifold. We say that an almost complex structure J is *compatible* if $g_J(v,w) = \omega(v, Jw)$ forms a Riemann metric for which J acts as an isometry. Such J always exists, see [Gr]. Moreover, using g_J, the Hamiltonian vector field, X_H becomes:

$$X_H = -Jgrad(H)$$

Here grad denotes the gradient in the metric g_J.

The following is essentially a construction of Action-Angle coordinates, see [Me-Ha].

Proposition 4.1.1 *Consider t_{τ_i} with the symplectic form $dh \wedge dk$. Then the following is a symplectic embedding:*

$$\begin{aligned} \Psi_i : T_{\tau_i} &\to A_i \\ (h,k) &\mapsto \begin{cases} \phi_H^k(\gamma_i(h)), & \textit{if } \sigma_i \textit{ is downwards} \\ \\ \phi_H^{-k}(\gamma_i(h)), & \textit{if } \sigma_i \textit{ is upwards} \end{cases} \end{aligned}$$

Proof: Let J be a compatible almost complex structure. Since on A_i, dH is non zero, we can define the one form α by:

$$\alpha = \frac{1}{\|X_H\|_{g_J}^2} J^* dH, \quad i.e. \alpha(k) = \frac{1}{\|X_H\|_{g_J}^2} dH(Jk)$$

As $dH(v) = \omega(X_H, v)$, we get

$$\begin{aligned} \alpha(X_H) &= \frac{1}{\|X_H\|_{g_J}^2} dH(JX_H) \\ &= \frac{1}{\|X_H\|_{g_J}^2} \omega(X_H, JX_H) \\ &= \frac{1}{\|X_H\|_{g_J}^2} g_J(X_H, X_H) \\ &= 1 \end{aligned}$$

Moreover,

$$\begin{aligned}\alpha \wedge dH(X_H, JX_H) &= \alpha(X_H)dH(JX_H) - \alpha(JX_H)dH(X_H) \\ &= 1 \times dH(JX_H) - 0 \quad \text{by conservation of energy} \\ &= dH(JX_H) \\ &= \omega(X_H, JX_H)\end{aligned}$$

As $\{X_H, JX_H\}$ forms a basis of the tangent spaces to A_i, we get

$$\omega = \alpha \wedge dH$$

So in order to prove the proposition, it suffices to show that

$$\Psi_i^*(\alpha \wedge dH) = dh \wedge dk$$

First, we look at some properties of Ψ_i:

$$\frac{\partial}{\partial h} H(\Psi(h,k)) = \begin{cases} -1 & \text{if } \sigma_i \text{ is downwards} \\ 1 & \text{if } \sigma_i \text{ is upwards} \end{cases} \tag{4.1}$$

$$\frac{\partial}{\partial k} H(\Psi(h,k)) = 0 \tag{4.2}$$

$$\Psi_*(\frac{\partial}{\partial_v}) = \begin{cases} X_H & \text{if } \sigma_i \text{ is downwards} \\ -X_H & \text{if } \sigma_i \text{ is upwards} \end{cases} \tag{4.3}$$

From (3.1) and (3.2), it follows that

$$\Psi_i^*(dH) = \begin{cases} -dh & \text{if } \sigma_i \text{ is downwards} \\ \\ dh & \text{if } \sigma_i \text{ is upwards} \end{cases}$$

From (3.3), we get

$$\Psi_i^*\alpha(\frac{\partial}{\partial_k}) = \alpha(\Psi_*(\partial_k)) = \begin{cases} -1 & \text{if } \sigma_i \text{ is downwards} \\ \\ 1 & \text{if } \sigma_i \text{ is upwards} \end{cases}$$

This implies that

$$\Psi_i^*(\alpha) = \begin{cases} dk + f(h,k)dh & \text{if } \sigma_i \text{ is downwards} \\ \\ -dk + g(h,k)dh & \text{if } \sigma_i \text{ is upwards} \end{cases}$$

Taking the pullback of the product, we get:

$$\Psi_i^*(\alpha \wedge dH) = \begin{cases} (dk + f(h,k)dh) \wedge -dh = du \wedge dk & \text{if } \sigma_i \text{ is downwards} \\ \\ (-dk + g(h,k)dh) \wedge dh = du \wedge dk & \text{if } \sigma_i \text{ is upwards} \end{cases}$$

□

Corollary 4.1.2 [Proposition 2.1.2]

Proof:

$$\int_{A_i} \omega = \int_{t_{\tau_i}} dh \wedge dk == \int_{|\sigma_{i-1}|}^{|\sigma_i|} \tau_i(h)dh$$

□

4.2 Embedding under the graph

Outline of the Construction

Using an admissible path σ from the maximum, we consider γ, its associated path with its given parameterization, as in definition 2.2.1. One can also define $\Psi : t_\tau \to \Sigma$ as the following:

$$\Psi(h,k) = \begin{cases} \Psi_i(h,k) & \text{if } h \in (|\sigma_{i-1}|, |\sigma_i|), \\ \gamma(|\sigma_i|) & \text{if } h \in \{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|\}. \end{cases}$$

To apply 1.3.1, we need to show that

$$\rho(H) = Min\{c_G(\Gamma_H^-), c_G(\Gamma_H^+)\} = K$$

Thus we want to embed a generalized trapezoid in $\Gamma_H^\pm$. Here:

$$\begin{aligned} \Gamma_H^- &= \{(m,t,z) \in \Sigma \times \mathbb{R} \times [0,1] \mid z \le H_t(m)\}, \\ \Gamma_H^+ &= \{(m,t,z) \in \Sigma \times \mathbb{R} \times [0,1] \mid z \ge H_t(m)\}. \end{aligned}$$

We call Γ_H^-, "the space under the graph" and Γ_H^+ "the space over the graph". Except for $h \in \{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_n|\}$, $T_{\sigma,\mathcal{H}}(\tau)$ fibers over t_τ with fibers of capacity $\mathcal{H}(h)$. As the fiber over $\Psi(h,k)$ in Γ_H^- is the rectangle $[0,\mathcal{H}(h)] \times [0,1]$, see remark 2.3.2, there exist a map $\beta : B^2(\mathcal{H}(h)) \to [0, M-h] \times [0,1]$ which preserves the area for all $h's$. Here , as in the discussion following the foliated property, it is essential that this map does not depend on h or k.

Equipped with this map β, one is even more tempted to define the following fibered "embedding":

$$\begin{aligned} \Phi^- \quad : \quad & T_{\sigma,\mathcal{H}}(\tau) \to \Gamma_H^- \\ & (h,k,x) \mapsto (\Psi(h,k), \beta(x)) \end{aligned}$$

For Γ_H^+, the area under the graph, one does the same thing for an admissible path σ of $H' = K - H$. As $c_G(T_{\sigma,\mathcal{H}}(\tau) = K$, we would get, if $\Phi^\pm$ were symplectic embeddings, $\rho(H) = K$, thus the proof of the Main Theorem, 2.3.5.

However, there are two major flaws in the latter construction:

(i) The map Ψ is not injective on the set $h \in \{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_{n-1}|\}$.

(ii) As $\lim_{h\to|\sigma_i|} \tau_i = \infty$, serious problems occur, as there is no way to extend the map to $h \in \{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_{n-1}|\}$

Remark 4.2.1 If there are admissible paths from the maximum and minimum whose level sets avoid critical values, the latter construction works, and ϕ_H^t is length minimizing.

To circumvent the latter difficulties, we will perturb τ a little.

Let $\epsilon > 0$. Let $\tilde{\tau} : (0, |\sigma_n|) \to (0, \infty)$ satisfy the following:

(i) $\displaystyle\sup_{h \in, (0,|\sigma_n|)} \int_0^h \tilde{\tau}(\eta) d\eta \geq K - \epsilon$

(ii) $\int_0^h \tilde{\tau}(\eta) d\eta \geq K - \epsilon - \mathcal{H}(h)$

(iii) $\tilde{\tau}(\eta) \leq \tau(\eta)$

By 3.2.2, we have

$$c_G(T_{\sigma,\mathcal{H}}(\tilde{\tau})) = K - \epsilon$$

Thus it suffices to show that for every $\epsilon > 0$, we can find a symplectic embedding of $T_{\sigma,\mathcal{H}}(\tilde{\tau})$ into $\Gamma^{\pm}_{H}$.

Remark 4.2.2 Suppose there exists $h_\epsilon \in \sigma_k$ with $\int_0^{h_\epsilon} \tau(\eta) d\eta = K - \epsilon$; Let $I_{|\sigma_i|}$ be an open interval containing the point $|\sigma_i|$, for $0 < i < k$, such that $\int_{I_{|\sigma_i|}} \tau < \frac{\epsilon}{k+1}$. Define $\tilde{\tau}(h)$ by

$$\tilde{\tau}(h) = \begin{cases} 0 & \text{if } h \in I_{|\sigma_i|} \\ \tau(h) & \text{if not} \end{cases}$$

Then $\tilde{\tau}$ satisfies properties **(i)**, **(ii)** and **(iii)**. It not continuous but the point is that a pertubation can be as small as we want on the intervals $I_{|\sigma_i|}$. This will turn out to be useful.

Perturbation of τ

Let $\mathcal{U}_{x_i}, \mathcal{V}_{x_i}$ be open neighbourhoods of x_i and $J_{|\sigma_i|}$ an open neighbourhoods of $|\sigma_i|$ such that

(a) $|H(x) - H(x_i)| < \epsilon, \quad \text{on } \mathcal{U}_{x_i}$

(b) $\overline{\mathcal{V}_{x_i}} \subset \mathcal{U}_{x_i}$

(c) $\gamma(J_{|\sigma_i|}) \subset \mathcal{V}_{x_i}$

On $I_{|\sigma_i|} \setminus J_{|\sigma_i|}$ we let $\tilde{\tau} = \tau$. It remains to define $\tilde{\tau}$ on $J_{|\sigma_i|}$ (see Figure 4.1) Suppose $J_{|\sigma_i|} = (|\sigma_i| - \delta_i, |\sigma_i| + \delta_i)$; Consider a one-sided thickening of $\gamma((|\sigma_i| - \frac{\delta_i}{2}, |\sigma_i| + \frac{\delta_i}{2}))$, such that the thickening is included in $\mathcal{V}_{x_i}$ and has area δT_i. On the intervals $(|\sigma_i| - \delta_i, |\sigma_i| - \frac{\delta_i}{2})$ and $(|\sigma_i| + \frac{\delta_i}{2}, |\sigma_i| + \delta_i)$, we "slow down" the flow, (see Figure 4.2) in the sense that we make τ smaller, until it reaches the constant map $\tilde{\tau} = T_i$. Denote by f_i and g_i these slowing down maps.Then on $(|\sigma_i| - \frac{\delta_i}{2}, |\sigma_i| + \frac{\delta_i}{2})$, we make $\tilde{\tau} \equiv T_i$. The same construction as in theorem 3.2.2 gives a symplectomorphism ϕ_i from $\{(h,k) \mid h \in (|\sigma_i| - \frac{\delta_i}{2}, |\sigma_i| + \frac{\delta_i}{2}), k = T_i\}$ to the thickening. We can now extend Ψ on the intervals $I'_{|\sigma_i|}s$. To recapitulate,

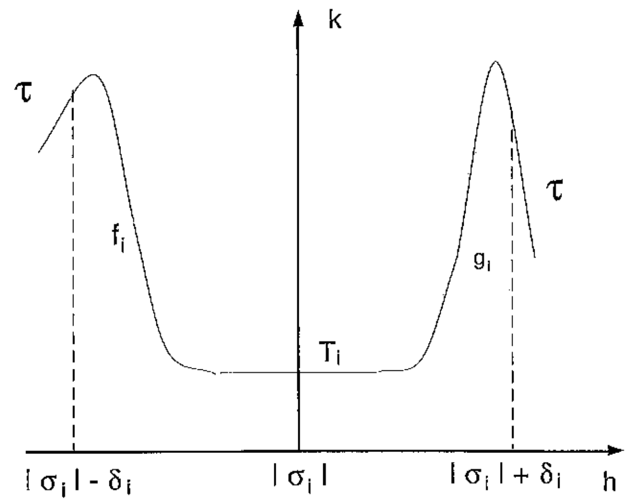

Figure 4.1: Graph of $\tilde{\tau}$ near $|\sigma_i|$

we define

$$\tilde{\tau}(h) = \begin{cases} T_i & \text{if } h \in (|\sigma_i| - \frac{\delta_i}{2}, |\sigma_i| + \frac{\delta_i}{2}), \text{ for some } i \\ f_i(h) & \text{if } h \in (|\sigma_i| - \delta_i, |\sigma_i| - \frac{\delta_i}{2}), \text{for some } i \\ g_i(h) & \text{if } h \in (|\sigma_i| + \frac{\delta_i}{2}, |\sigma_i| + \delta_i) \text{for some } i \\ \tau(h) & \text{if not} \end{cases}$$

We now define the desired embedding $\widetilde{\Psi}_{\tilde{\tau}} : t_{\tilde{\tau}} \to \Sigma$:

$$\widetilde{\Psi_{\tilde{\tau}}}(h,k) = \begin{cases} \phi_i(h,k) & \text{if } h \in (|\sigma_i| - \frac{\delta_i}{2}, |\sigma_i| + \frac{\delta_i}{2}), \text{ for some } i \\ \Psi(h,k) & \text{if not} \end{cases}$$

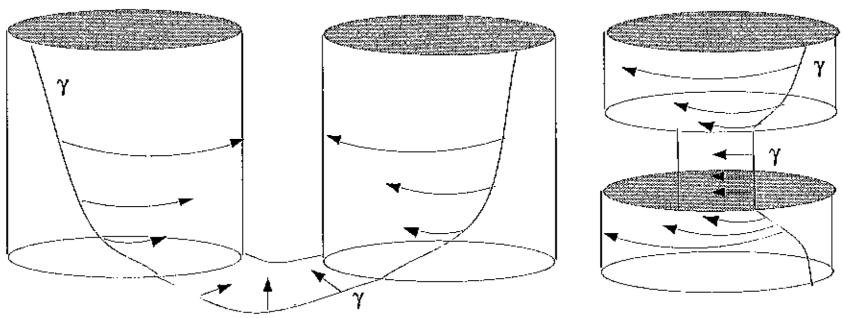

Figure 4.2: The Glueing Construction

Proof of the Main Theorem, 2.3.5

For any $\epsilon > 0$, we use the pertubation $\tilde{\tau}$ of τ to constructed an embedding $\widetilde{\Psi_{\tilde{\tau}}}$ of $t_{\tilde{\tau}}$ into Σ. To embed $T_{\sigma,\mathcal{H}}(\tilde{\tau})$ into Γ_H^-, we use a fibered embedding, namely

$$\widetilde{\Psi_{\tilde{\tau}}} \times \mathbb{1} : T_{\sigma,\mathcal{H}}(\tilde{\tau}) \to \Gamma_H^-.$$

This gives $c_G(\Gamma_H^-) \geq K-\epsilon$. For Γ_H^+, we do the same for $H' = K-H$. We obtain $min\{c_G(\Gamma_H^+), c_G(\Gamma_H^-)\} = K$, so by Theorem 1.3.1, ϕ_H^t is length-minimizing. □

4.3 Proof of the Foliated Property Case

If a_H is a Morse function which generates an isotopy which satisfies the hypothesis of Theorem 2.3.5, then we can embed balls of capacity $\max(a_H) - \min(a_H)$ in $\Gamma_{a_H}^{\pm}$. But if H satisfies the foliated property, we can find symplectomorphisms between $\Gamma_H^{\pm}$ and $\Gamma_{a_H}^{\pm}$, thus we can embed balls in $\Gamma_H^{\pm}$. □

Chapter 5

Examples and Computations

5.1 A length minimizing isotopy which admits a set of positive measure consisting of periodic orbits

Using 1.3.1, one concludes the following:

Corollary 5.1.1 *Suppose H attains its maximum and minimum on sets of capacity $M = Max(H) - Min(H)$, then* $\mathrm{E}(\phi_H) = M$

Using 5.1.1, we will construct a length minimizing path in $\mathrm{Ham}(\mathrm{S}^2, \omega_o)$ such that there is a set of positive area where all the orbits have small periods. Here small means smaller than one, but in fact we can prescribe the orbit to be as small as we wish. Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with its usual volume form, ω_o. We will use the symplectic coordinates (sometimes called chordal) $(\theta, z), 0 < \theta < 2\pi, \quad -1 < z < 1$, so that $\omega_o = d\theta \wedge dz$.

Consider H of the form $f(z)$, a function on Σ that only depends on the height. In that case, the isotopy is given by $\phi_t(\theta, z) = (\theta + tf'(z), z)$, so orbits close whenever $tf'(z) \in 2\pi\mathbb{Z}$.

Let us choose f such that $f'(z)$ satisfies:

(i) $f'(z)$ is even

(ii) $f'(0) \geq 2\pi K$, for K a big integer

(iii) $f'(z) = 0, z \leq -L$, for $L > 0$ very small

(iv) $\int_{-1}^{1} f'(z)dz = M < 2\pi$

By choosing K very large, then ϕ_t admits orbits of arbitrary small period which can therefore wind around as much as we wish. But as $f(z) = \int_{-1}^{z} f'(x)dx$, (see Figure 5.1), we still have in virtue of (iv), control over the maximum M of f.

To use 5.1.1, it suffices to choose K and L so that $M < (1 - L)2\pi$. We then obtain that ϕ_t is length-minimizing.

The set $\{z \mid |z| < L\}$ consists of periodic orbits. □

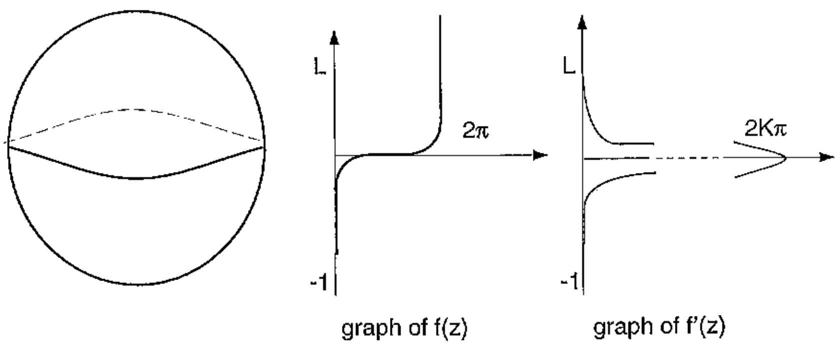

Figure 5.1: A length minimizing Hamiltonian with "a lot" of closed orbits

Remark 5.1.2 A similar construction using a function depending $r^2 = x^2 + y^2$ on $\mathbb{R}^2$ would give a length-minimizing Hamiltonian isotopy with compact support on $\mathbb{R}^2$ which has periodic orbits on a set of positive measure.

5.2 Some calculations on the torus

In [La-Mc3] it was proven that $\mathbb{T}^2$ is unbounded in Hofer's metric. A construction was given, of an Hamiltonian isotopy which minimizes length for all time, by passing to the universal cover $\mathbb{R}^2$. We will give an explicit construction, using a finite cover. But first we need a lemma, getting rid of the $\frac{1}{2}$ in the Energy-Capacity Inequality,1.3.2.

Lemma 5.2.1 *Let* $(\mathbb{T}^2, \omega)$ *be the torus with area* $\omega(\mathbb{T}^2)$.
Then the following holds:

$$\mathbf{E}(\phi) \geq \sup\{c_G(A) : \phi(A) \cap A = \emptyset\}$$

Proof: Let $\mathcal{U}$ the symplectic image of a ball be such that

(i) $\phi(\mathcal{U}) \cap \mathcal{U} = \emptyset$

(ii) $c_g(\mathcal{U}) > \mathbf{E}(\phi)$

Then there exists $H \in C^\infty(\mathbb{T}^2 \times \mathbb{R})$ such that $\|H\| < c_g(\mathcal{U})$, and $\phi_H^1 = \phi$. Let $\widetilde{\phi_H^t}$ be the unique lift of ϕ_H^t to $\mathbb{R}^2$ such that $\widetilde{\phi_H^o} = 1\!\!1$: Then:

$$\mathcal{L}(\widetilde{\phi_H^t}) \leq \mathcal{L}(\phi_H^t) < c_g(\mathcal{U})$$

As $\widetilde{\phi^1}(\mathcal{U}) \cap \mathcal{U} = \emptyset$, this contradicts 1.3.2. □

5.3 An isotopy of the torus which minimizes length for all time

In the case of $\mathbb{R}^{2n}$, the Hofer norm is continuous in the C^o topology as it was discovered by Hofer, see [Ho] , as the following inequality holds

Theorem 5.3.1 *For any $\phi \in \mathbf{Ham}(\mathbb{R}^{2n}, \omega_o)$,*

$$\mathbf{E}(\phi) \leq 256 \cdot \mathrm{diam}(supp(\phi)) \cdot \|\phi\|_{C^0}$$

where the C^0-norm is $\sup |\phi(x) - x|$.

In particular, no function of compact support can generate an isotopy which minimize length for alltime. But in the case of surfaces, the continuity of the Hofer norm is still unknown, see [La].

On the torus, we can minimize length for all time, as the following holds:

Lemma 5.3.2 *The isotopy generated by the Hamiltonian $H(x, y) = \frac{1}{2}\cos(2\pi x)$ on $\mathbb{T}^2$ minimizes length for all time.*

Proof: By Moser's classification [Mo] we can suppose $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$ with the induced standard symplectic form $dx \wedge dy$.

Let $H(x, y) = \frac{1}{2}\cos(2\pi x)$. We get the Hamiltonian flow

$$\phi^t_H(x, y) = (x, y + t\pi \sin(2\pi x)).$$

As H is autonomous, the following holds:

$$\phi^t_H = \phi^1_{tH}$$

Both notations will be used interchangeably. Thus it suffices to show that ϕ^1_{tH} minimizes length. If we think of ϕ^t_H as the unique lift on the universal cover, $\mathbb{R}^2$ which at $t = 0$ is the identity, it displaces the set

$$\Delta_t = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < \tfrac{1}{2}, 0 < y < t\pi \sin(2\pi x)\}$$

As

$$c_G(\Delta_t) = \int_0^{\frac{1}{2}} t\pi \sin(2\pi x) dx = t = \mathcal{L}(tH),$$

we see that ϕ^t_H displaces a set of capacity equal to its length, but viewed as a map from $\mathbb{R}^2$. Let

$$n > \sup_{x \in (0, \frac{1}{2})} t\pi \sin(2\pi x) = t\pi;$$

Consider the n-sheeted cover $\widetilde{\mathbb{T}^2}$ of $\mathbb{T}^2$ given by $\mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/n\mathbb{Z}$. Let $\widetilde{\phi^t_H}$ denote the unique lifting of ϕ^t_H such that $\widetilde{\phi^0_H} = \mathbb{1}$.

Then $\widetilde{\phi^t_H}$ displace Δ_t, a set of capacity t.

For fixed t, let $\{\psi^t_{K, s \in [0,1]}\}$ be another isotopy such that $\psi^t_{K,1} = \phi^1_{tH}$; Then

$$\widetilde{\psi^t_{K,1}} = \widetilde{\phi^1_{tH}} + c, \quad c \in \mathbb{Z}_4$$

As $\widetilde{\psi^t_{K,1}}$ and $\widetilde{\phi^1_{tH}}$ fix some points, this implies $c = 0$. Thus $\widetilde{\psi^t_{K,1}}$ displaces Δ_t. From lemma 5.2.1, we get:

$$\mathcal{L}(\psi^t_{K,s}) \geq \mathcal{L}(\widetilde{\psi^t_{K,s}}) \geq t = \mathcal{L}(\phi^t_H)$$

Thus ϕ^t_H is length-minimizing for all t. □

5.4 Morse functions which generate high energy Hamiltonian Diffeomorphisms

As the group $\mathrm{Ham}(\mathbb{T}^2, \omega_o)$ is unbounded, and the Morse functions are dense in $C^\infty(\mathbb{T}^2, \mathbb{R})$, one would expect that there exist Morse functions which generate isotopies of arbitrary length. As 2.3.5 only produces length-minimizing paths for which $\mathbf{E}(\phi) \leq \omega(\Sigma)$, to show that there exist Morse functions which generate length minimizing paths with energy bigger than the area of the underlying surface, a construction is required.

Another problem arises: $(\mathrm{Ham}(\mathbf{M}, \omega), d_H)$ is not complete, as was shown by [La-Mc2]. In fact, they constructed a $\phi \in \mathbf{Ham}(S^2, \omega)$ for which there is no shortest path from $\mathbb{1}$ to ϕ. Using the results in Bialy-Polterovich, [Bi-Po], one can show that this holds for any symplectic manifold. This was pointed out to the author by Polterovich. In other words, for any symplectic manifold (M, ω) there exist $\phi \in \mathrm{Ham}(\mathbf{M}, \omega)$ such that there is no shortest path from the identity to ϕ. In [La-Mc2], necessary conditions for a geodesic to minimize length are obtained. We will give the version for Hamiltonians generated from a time independant Morse function, thus simplifying the statement. First we define the *linearized flow*:

Definition 5.4.1 Consider a path ϕ_t with $\phi_0 = \mathbb{1}$. Suppose that x is a fixed extremum of the Hamiltonian $\{H_t\}_{t\in[0,1]}$ and consider the linearizations

$$L_t = d\phi_t(x) : T_x(M) \to T_x(M)$$

of the ϕ_t at x. This is the symplectic isotopy generated by the Hessian of H_t at q.

In the case of a Morse function H, in a Darboux chart [Mc-Sa], the Hessian at any local maximum or minimum does not depend on t. Moreover, it is a positively or negatively definite symmetric bilinear form. The linearized flow at those point becomes:

$$L_t = \exp(J_o Hess(x)t), \quad where \quad J_o = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

From classical Hamiltonian mechanics, as in [Me-Ha], there exists a linear symplectic change of variable Ξ such that the system becomes:

$$\Xi^{-1} J_o Hess(x) \Xi = \begin{cases} \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \\ or \\ \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \end{cases}$$

This will be called the normal form of L_t. So in these charts the system has solution $\mathcal{R}_{\pm t\theta}$, the rotation with angle $t\theta$ or $-t\theta$. This system admits closed orbits as soon as $t\theta \in 2\pi\mathbb{Z}$. If, for every $v \in T_x(M)$ and every $t' \in (0,T)$, the only trajectories $\alpha(t) = L_t(v), 0 \leq t \leq t'$, with $v = L_0(v) = L_{t'}(v)$ are single points, we will say that the linearized flow at x has no non-trivial closed trajectories in the time interval $(0, T)$.

Theorem 5.4.2 *[La-Mc2]*

Suppose ϕ_t is a length-minimizing path generated by the Morse function H on the compact surface (Σ, ω). Then there is at least one fixed maximum and one fixed minimum at which the linearized flow has no non-trivial closed trajectory in the open interval $(0, 1)$.

From the normal form of L_t, it follows:

Corollary 5.4.3 *Let (Σ, ω) be any compact surface. Then there is no isotopy generated by a Morse function which minimizes length for all time.*

Proof: As noted, $\mathcal{R}_{\pm t\theta}$ admits closed trajectories for all points of index zero or two, as soon as t is big enough. □

Construction a Morse function G on $\mathbb{T}^2$ such that $\{\phi_G^t\}_{t\in[0,1]}$ is length minimizing and $\max(G) - min(G) > \omega(\mathbb{T}^2)$.

From the proof in Section 5.3, if we can construct a function G such that for its flow ϕ_G^t, its unique lift to the universal cover starting at the identity $\widetilde{\phi_G^t}$, displaces a set of capacity equal to $\max(G) - \min(G)$, then ϕ_G^t minimizes

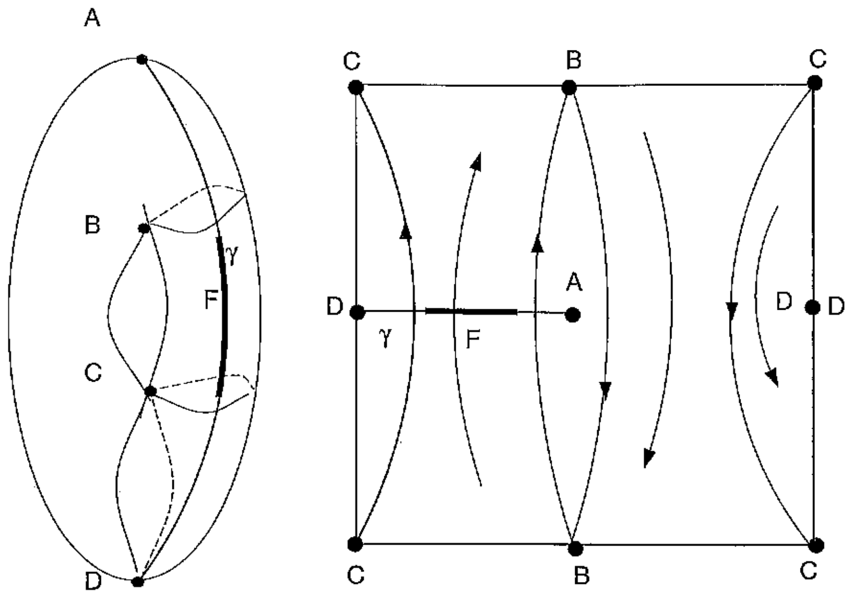

Figure 5.2: The flow of the height function on the torus

length. So it suffices to find a Morse function G with $\max(G) - min(G) > \omega(\mathbb{T}^2)$ satisfying the desired property.

Suppose we have an embedded torus in $\mathbb{R}^3$, such that it stands on the plane $z = 0$. Let H denotes its heigth function with minimum 0 . Let X_H be its Hamiltonian vector field. Consider a curve γ on the torus, from A to D , which is transverse to H outside critical levels, see Figure 5.2 and moreover does not pass through any other critical points.

Let $\mathbf{F}$ be a compact set of the image of γ between the the two critical points of index 1 B and C. We can suppose, if the torus is big enough that the cylinder between z_1 and z_2 along $\mathbf{F}$ is the flat round cylinder of radius one, which we will denote by A_{z_1,z_2}. Thus $area(A_{z_1,z_2}) = 2\pi(z_2 - z_1)$ On A_{z_1,z_2}, if we put the coordinates (z, θ) with $z_1 < z < z_2$ and $0 < \theta < 2\pi$, , then for the heigth function H, we get $\tau(z) \equiv 2\pi$. If we compose H with a $f : \mathbb{R} \to \mathbb{R}$ such that f' never vanishes, letting $G = f \circ H$, we obtain $X_G = f'(z)X_H$. Choose f such that $f(z) = \lambda z$ on (z_1, z_2). Then for the function G, we have $\tau(z) \equiv \frac{2\pi}{\lambda}$ Choose λ so that $\frac{2\pi}{\lambda} = \frac{1}{N}$ Then if $\tilde{\phi}_G^t$ is the unique lift to the universal cover that starts at the identity, $\tilde{\phi}_G^1$ displaces a set of capacity $N2\pi(z_2 - z_1) = \lambda(z_2 - z_1)$

But as the variation of f between z_1 and z_2 is equal to $\lambda(z_2 - z_1) = 2\pi N(z_2 - z_1)$, we need room to get to the maximum and the minimum. Let δ_{max}, δ_{min} and δ_F denote the following:

$$\delta_{max} = G(A) - G(B) \quad \delta_{min} = G(C) - G(D) \quad \delta_F = G(B) - G(C) - 2\pi N(z_2 - z_1)$$

Then $K = 2\pi N(z_2 - z_1) + \delta_{max} + \delta_{min} + \delta_F$ (see figure 5.3.) We want f so that $\tilde{\phi}_G^t$ displaces a set of capacity $K - \epsilon$, for any $\epsilon > 0$.

Lemma 5.4.4 *We can choose f such that $\mathbb{T}_{max} = \{x \in \mathbb{T}^2 \mid G(B) < G(x) < G(A)\}$ and $\mathbb{T}_{min} = \{x \in \mathbb{T}^2 \mid G(D) < G(x) < G(C)\}$ both admit sets $\mathcal{U}_{max}$ and $\mathcal{U}_{min}$ of capacity δ_{max} and δ_{min} which are displaced by $\tilde{\phi}_G^1$.*

Proof: Choose f such that $\tau > 2$ on both these sets. Consider the symplectic coordinates $\{(h, k) \mid 0 < k < \tau(h)\}$ introduced earlier. We consider the $\mathbb{T}_{max}$ case, $\mathbb{T}_{min}$ being similar.

$$\tau > 2 \Rightarrow \{(h, k) \mid G(B) < h < G(B), 0 < k < 1\} \subset \mathbb{T}_{max}$$

Letting $\mathcal{U}_{max} = \{(h, k) \mid G(B) < h < G(B), 0 < k < 1\}$, this set is displaced and has capacity $G(B) - G(A)$. To make $\tau < 2$, it suffices to choose f such that f' is small enough for $z > z_2$. $\square$

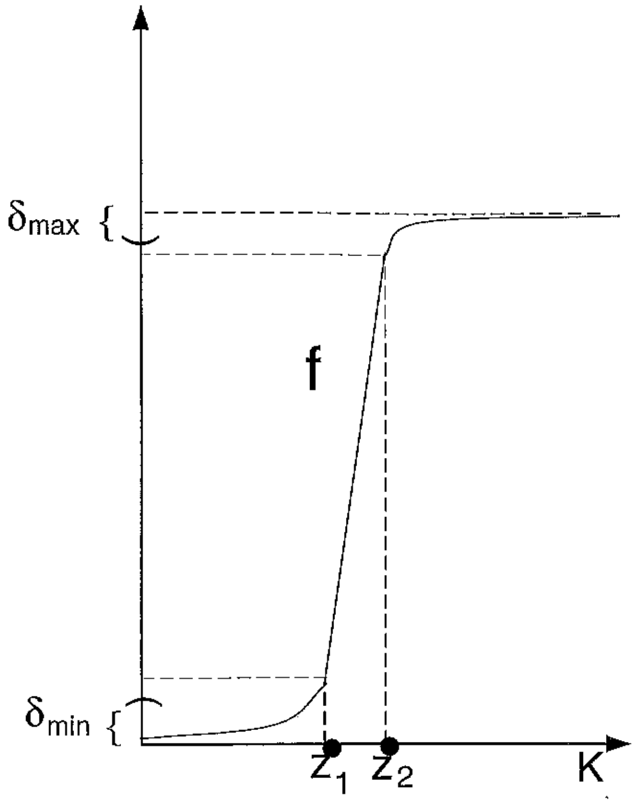

Figure 5.3: Composing the height function

Lemma 5.4.5 *The cylinder* $\mathbb{A}$ *between the critical points* B *and* C *admits a simply connected set* $\mathcal{U}_F$ *of capacity* $2\pi N(z_2 - z_1) + \delta_F$ *which is displaced by* $\tilde{\phi}^1_G$.

Proof: We only consider the case $\{z \in \mathbb{A} \mid G(z_2) < G(z) < G(B)\}$, the case $\{z \in \mathbb{A} \mid G(C) < G(z) < G(z_1)\}$ being similar. As $\tau \to \infty$ as $z \to B$, and on $A_{z_1,z_2}, \quad \tau < 1$, we can choose f such that τ decreases in a monotone way from B to z_2. We have two cases:

$\tau(z) > 1:$ Let w such that $\tau(w) = 1$. Then in the universal cover, ϕ^1_G displace the set $\{(h,k) \mid G(B) < h < w, 0 < k < 1\}$ of capacity $G(B) - G(w)$.

$\tau(z) < 1$: Then in the universal cover, ϕ_G^1 displaces the set $\{(h,k) \mid G(w) < h < G(z_2), 0 < k < \frac{1}{\tau(h)}\}$ of capacity bigger than $G(z_2) - G(w)$, as $\tau > 1$.

Taking the union of A_{z_1,z_2} the two former set, joined along the curve γ, then $\tilde{\phi}_G^1$ displaces a set of capacity $G(B) - G(z_2) + 2\pi N(z_2 - z_1)$. To get the remaining part of δ_F, we do the same for $\{z \in \mathbb{A} \mid G(C) < z < G(z_2)\}$ and let $\mathcal{U}_F$ be the unions of theses five sets. □

Proposition 5.4.6 *There exist a Morse function G on $\mathbb{T}^2$ which generates an isotopy of minimal length such that* $\max(G) - \min(G) > \omega(\mathbb{T}^2)$.

Proof: We can choose N such that $2\pi \cdot N(z_2 - z_1) > \omega(\mathbb{T}^2)$;
Choose f as in the two preeceding lemmas. Let $\epsilon > 0$: Let $\tilde{\gamma}$ be a lift of γ in the universal cover. Attach $\mathcal{U}_{max}, \mathcal{U}_{min}$ and $\mathcal{U}_F$ along $\tilde{\gamma}$.

We can excise small neighborhood of $\mathcal{U}_{max}, \mathcal{U}_{min}$ and $\mathcal{U}_F$ such that we do not delete more than an amount of ϵ area. Thicken the path $\tilde{\gamma}$ along the deleted area such that $\tilde{\phi}_H^1$ displaces the thickened area and that the union of the thickening and $\mathcal{U}_{max}, \mathcal{U}_{min}$ and $\mathcal{U}_F$ is homeomorphic to a ball. Then $\tilde{\phi}_G^1$ displaces a ball of capacity more than $K - \epsilon$. □

5.5 Generalizations

We now discuss some ways to generalized theorem 2.3.5:

Extending the class of admissible paths We could admit paths that pass through critical points of order zero or two, and then come back. Thus an edge could be used twice. But one has to be careful to "share the area" as the path travels through the critical point. More precisely, suppose that σ_i is an edge from the point x_{i-1} of order one going to the point x_i of order two. Let γ_i be a loop based at x_{i-1} transerve to H on the open cylinder A_i determined by the two points. Then γ_i decomposes as γ_i^+ and γ_i^-, the upwards and downwards parts. On γ_i^+, we flow along X_{-H}, using τ_i, but until we reach γ_i^-. On γ_i^-, we flow along X_H, using τ_i, but until we reach γ_i^+. The point is that we dont use all of τ_i on any of the paths. But by using both paths, we can recover all of the area of A_i.

Allowing functions which are not of the Morse type If a function f has a finite number of critical points, we can define its graph, thus τ and the theorem generalized to this case. For a general $f : \Sigma \to \mathbb{R}$ with minimum 0 and maximum K, let $Reg(f) \subset (0, K)$ be the set of regular values of f. Let $\epsilon > 0$;

Then by Sard's Theorem, we can find a finit number of mutually disjoint intervals $I_j = (a_i, b_j)$ of $Reg(f)$ such that

$$\sum_j (b_j - a_j) > K - \epsilon.$$

One could then define τ on these intervals to get some results.

Approximation by Morse Functions One could use the C^∞-density of Morse function to approximate a general function f.

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