# The Deformation Spaces of Certain Subgroups of Kleinian Groups

A Dissertation Presented

by

Eduardo Mendoza Reyes III

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

May 1995

State University of New York
at Stony Brook

The Graduate School

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Graduate School

# Abstract of the Dissertation

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Ahlfors' Finiteness Theorem says that any non-elementary and finitely generated Kleinian group G acting (properly) discontinuously on $\hat{\mathbb{C}}$ represents a union $S_1 \cup S_2 \cup \ldots \cup S_N$ of Riemann surfaces such that for every i = 1, 2, ... N, $S_i$ is of finite type. We investigate whether there exists another Kleinian group F such that F is properly contained in G with finite index such that both have the same deformation spaces. In the case of Fuchsian groups, Greenberg, Singerman, and others had shown that such pairs do not occur very often. These pairs of non-elementary and finitely generated Fuchsian groups satisfy the following properties:

(i) one is a (normal) subgroup of the other and the index is finite

(ii) the Teichmuller spaces of the two groups are equal

and the above authors specifically listed down the pairs of signatures associted with these pairs of Fuchsian groups.

In the general case of Kleinian groups, we found it to be true as well that pairs of Kleinian groups, satisfying similar properties as the ones above, do not occur frequently. The main results are obtained by focusing on Q(F) and Q(G). In the proof of one of our propositions (Proposition 1), we make use of the natural decompositions of this pair of (Banach) spaces of quadratic differentials and the 1-1 correspondence that results thereafter on the assumption that they are equal, Q(F) = Q(G). In Proposition 3, we found necessary conditions that F and G must satisfy. They are clearly generalizations of those in the Fuchsian case. If F and G are Kleinian groups with an invariant component, the so-called function groups, then the signatures of their Fuchsian eqiuvalents are crucial in determining whether they are included among these pairs of Kleinian groups. For the general case, the component subgroups that do not correspond to thrice-punctured spheres play a key role. Using standard theories, we establish the equality of their deformation spaces in Chapter 4.

Taos puso kong inihahandog nang buong dalisay sa aking mag-anak: Luis, Vicente, Jose Rizalino, at higit sa lahat sa aking ina, Marta.

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## Acknowledgements

First and foremost, I would like to express my thanks to Prof. Bernie Maskit who gave me this rare and unique opportunity. Secondly, I would like to recognize Irwin Kra for suggesting this thesis problem and for initializing the notion of improper inclusion pairs which was fundamental here. My gratitude again to Bernie Maskit for his help and guidance in working with Kleinian groups and in pointing out the right paths to take in the course of my work. He has been an invaluable adviser and mentor. I would also like to send my sincerest appreciation to L. Takhtajan and A. Phillips for their support and to all my grad student colleagues who lent kind words of encouragement, thanks. Last but not the least, best wishes to Bill & Tracey Brunet & family for the home away from home during my stay in Stony Brook.

As a personal note, I would like to mention some people whose company were inspirational: my friends and relatives in California especially Mr & Mrs Ireneo and Gloria Mendoza & family, Mr & Mrs Romy and Susan Dial & family and Mr & Mrs Zenon and Cynthia Reyes & family; my near and dear friends from Connecticut, Mr & Mrs Benito Joel and Ma. Guia Ditching-Cuevo & family, for the memorable holidays. Sa inyong lahat, maraming salamat.

# Chapter 1

# Background

## 1.1 Kleinian Groups

The Lie group PSL(2,$\mathbb{C}$) = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid$ a,b,c,d $\in \mathbb{C}$ & ad-bc=1}/{I,-I}

acts on $\hat{\mathbb{C}}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$; it is well-known that PSL(2,$\mathbb{C}$) is (isomorphic) to the full group, Aut($\hat{\mathbb{C}}$), of orientation-preserving conformal automorphisms of $\hat{\mathbb{C}}$. Elements of PSL(2,$\mathbb{C}$) are categorized into three types: elliptic, parabolic and loxodromic (which includes hyperbolic). An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of PSL(2, $\mathbb{C}$) is called elliptic if the square of its trace, $(a+d)^2 \in [0,4)$; it is parabolic if $(a+d)^2 = 4$; loxodromic otherwise.

Suppose that G is a discrete subgroup of PSL(2,$\mathbb{C}$). G is said to act (properly) discontinuously at $z \in \hat{\mathbb{C}}$ if there is a neighborhood U of z such that $g(U) \cap U = \emptyset$ for all but a finite number of $g \in G$. The action of G on $\hat{\mathbb{C}}$ produces two sets, $\Omega(G)$ and $\Lambda(G)$, with the characteristics that

(i) $\Omega(G) \cap \Lambda(G) = \emptyset$

(ii) $\Omega(G) \cup \Lambda(G) = \hat{\mathbb{C}}$

(iii) $\Omega(G)$ is the largest open, G-invariant set in $\hat{\mathbb{C}}$ where G acts discontinuously.

(iv) $\Lambda(G)$ is the closure of the set whose members are the fixed points of all parabolic and loxodromic elements of G.

$\Omega(G)$ and $\Lambda(G)$ are called the ordinary set and the limit set of G respectively. If $\Omega(G) \neq \emptyset$, then G is called a Kleinian group of the second kind (or simply a Kleinian group) and the limit set of such a group is nowhere dense in $\hat{\mathbb{C}}$. In case G is a Kleinian group of the second kind which is non-elementary, that is card$\Lambda(G) \neq 0, 1, 2$, then $\Lambda(G)$ is the minimal closed, nowhere dense and G-invariant subset of $\hat{\mathbb{C}}$. If, on the other hand, $\Omega(G) = \emptyset$, then G is called a Kleinian group of the first kind. These groups are primarily linked with 3-manifolds and will not be our main object of study here. Hence, we will agree from now on that whenever $\Omega(G)$ is used, it will mean the non-empty ordinary set of G in $\hat{\mathbb{C}}$. We will also, from this point on, refer to Kleinian groups of the second kind simply as Kleinian groups.

A component of a Kleinian group G is any connected component of $\Omega(G)$. If $\Omega(G)$ is itself connected, then G has only one component, $\Omega(G)$. According to Ahlfors' Finiteness Theorem [**B4**], if G is finitely generated, then the 2-

orbifold $\Omega(G)/G$ is a finite union of compact Riemann surfaces with a finite number of special points. Let S be one of these Riemann surfaces and let $\{x_1, x_2, ..., x_n\}$ be its finite set of special points. The order of $x_i$ is defined to be the order of the element g $\in$ G that corresponds to the homotopy class, $[\lambda]$, of a simple loop $\lambda$ about $x_i$. If a Kleinian group G has a component, $\Delta$, which is invariant under G, then G is called a function group. If G is a function group and, in addition, its invariant component is simply-connected, then G is called a b-group.

**Definition 1** *If $(G,\Delta)$ is a b-group and $j \in G$ is parabolic such that $\beta_*(j)$ is hyperbolic for $\beta:\Delta \longmapsto U$ a Riemann map, then j is called an accidental parabolic.*

*The axis of j is defined to be the inverse image under $\beta$ of the axis of $\beta_*(j)$ which is the geodesic in U connecting its two fixed points.*

The following lemmas are known. We show proofs for completeness.

**Lemma 1** *Let F and G be non-elementary Kleinian groups with $F \preceq G$ and $[G:F] = N < \infty$. Then, $\Lambda(F) = \Lambda(G)$*

**Proof:** Clearly, $\Lambda(F) \subseteq \Lambda(G)$. Let $G = Fg_0 \cup Fg_1 \cup Fg_2 \cup ... \cup Fg_{N-1}$ be a partition of G into (left) cosets with respect to F, where $Fg_0 = F$ and $g_0 = \mathbf{1}$.

Let $z_0 \in \Lambda(G)$. Then, there exists a sequence of distinct elements $\{\gamma_i\}$ in G such that $\lim_i \gamma_i(z) = z_0$ for some z in $\Omega(G)$. This sequence has a subsequence $\{\tilde{\gamma}_i\}$ such that $\tilde{\gamma}_i(z) \longrightarrow z_0$ for any point z in $\Omega(G)$. This subsequence is chosen as follows. After an appropriate normalization of G that sends $z \mapsto \infty$, each element $\tilde{\gamma}_i$ of this subsequence satisfies the property that the center of the isometric circle of its inverse, $\tilde{\gamma}_i(\infty)$, tend to $z_0$. Without loss of generality, we take the subsequence to be the original sequence. From the finite partition above, it follows that for infinitely many indices j, $\gamma_j \in Fg_k$ for some k = 0,1,...,N-1. If k=0, then we are done. Otherwise, we have $\gamma_j \circ g_k^{-1} = f_j \in$ F. Now, we observe that $g_k^{-1}$(z) $\in \Omega$(G). Hence, it follows that $\lim_j \gamma_j(g_k^{-1}(z)) = z_0$ which implies that $\lim_j f_j(z) = z_0$. But $\Omega$(G)$\subseteq\Omega$(F) so that z$\in\Omega(F)$. Thus, $z_0 \in \Lambda$(F) which completes the proof. □

**Lemma 2** *Suppose G is a non-elementary Kleinian group. If G has a component $\Delta$ which is invariant under G, then $\Lambda(G) = \partial\Delta$, the boundary of $\Delta$.*

**Proof:** Clearly, $\partial\Delta \subseteq \Lambda$(G) since $\partial\Delta$ is in the complement of $\Omega$(G), and $\hat{\mathbb{C}}$ - $\Omega$(G) = $\Lambda$(G). Now, let $z_0 \in \Delta$ and let g be a loxodromic element of G. By iterating g and g$^{-1}$ and applying it to $z_0$, we see that the resulting sequences of points converge to the two distinct fixed points of g, one is attracting and the other is repelling. Hence, the fixed points of g lie on $\partial\Delta$. Since the set of fixed points of loxodromic elements in G is dense in $\Lambda$(G), we are done. □

The two lemmas above result in the corollary below which is easy to prove.

**Corollary 1** *If F and G are non-elementary function groups with F$\preceq$G of finite index, then there exist invariant components, $\Delta(F)$ and $\Delta(G)$, for F and G respectively such that $\Delta(F) = \Delta(G)$. Furthermore (i) $\Omega(G) = \Omega(F)$ and consequently (ii) $\Omega(G)$ - $\Delta(G) = \Omega(F)$ - $\Delta(F)$.*

**Proof:** Given $\Delta$(G) for G. Since F$\preceq$G, we can choose $\Delta$(F) for F such that $\Delta$(G)$\subseteq\Delta$(F). Suppose x $\in \Delta$(F) - $\Delta$(G). Either x $\in \partial\Delta$(G) in which case, by directly applying lemmas 1 and 2, we obtain x $\in \Lambda$(F) contradicting the assumption that x $\in \Delta$(F); or x $\in \Delta(F) - \overline{\Delta(G)}$. Let f be a loxodromic element of F and consider the sequence $\{f^n(x)\}$, n = 1,2,... Then $f^n(x) \longrightarrow x_0 \in \Lambda(F)$. By assumption, $\Delta(F) - \overline{\Delta(G)}$ is a nonempty open set and we can thus choose an open set O containing $x_0$ such that $O \cap \overline{\Delta(G)} = \emptyset$. But $\Lambda(F) = \Lambda(G)$ and $\Lambda(G) = \partial\Delta(G)$ by lemmas 1 and 2. Thus, there exists a sequence $\{z_j\}$ of distinct points in $\Delta$(G) such that $z_j \longrightarrow x_0$ and hence, $z_j \in O$ for infinitely many j's. This contradicts the choice of O. Hence, we get $\Delta(F) - \Delta(G) = \emptyset$ and $\Delta(F) = \Delta(G)$. The rest follows trivially. □

## 1.2 Quadratic Differentials

Throughout the following list of well-known facts and definitions, G will denote a non-elementary and finitely generated Kleinian group of the second kind and $\Omega$(G) its (nonempty) ordinary set in $\hat{\mathbb{C}}$. We will use the notation $\mathbb{T}$(G) for the deformation space of G, the formal definition of which is given in succeeding chapters. If G is Fuchsian, we will denote the Teichmuller space of G as T(G) or T(G,U) where U is the upper half-plane.

(1) A (holomorphic) quadratic differential for G on $\Omega$(G) (in general, on any open set $D \subseteq \Omega(G)$) is a holomorphic function $\phi$ on $\Omega$(G) with the property that $\phi(g(z))g'(z)^2 = \phi(z)$ for all g $\in$ G and for all z $\in \Omega$(G). This last equality says that $\phi$ is an automorphic (2,0)-form or simply a 2-form on $\Omega$(G) with respect to G.

(2) The function $\phi$ is called integrable if

$$\int\int_{\Omega(G)/G} |\phi(z)\frac{dz\overline{dz}}{2}| < +\infty$$

(3) The Banach space of integrable, holomorphic quadratic differentials on $\Omega$(G) is denoted by Q(G,$\Omega$(G)) with norm equal to the given integral above.

**Remark:** There is naturally a corresponding space Q($\Omega$(G)/G) of quadratic differentials on $\Omega$(G)/G and the two (Banach) spaces Q(G,$\Omega$(G)) and Q($\Omega$(G)/G) are, in fact, isometric. Moreover, Q(G,$\Omega$(G)) is (canonically isomorphic to) the cotangent space of $\mathbb{T}$(G) at the identity.

## 1.3 The Planarity Theorem

We state without proof the Planarity Theorem **[M]**.

**Theorem 1 (Planarity Theorem)** *Let $p{:}\tilde{S}\longmapsto S$ be a regular covering of the topologically finite Riemann surface $S$, where $\tilde{S}$ is planar. Then, there is a finite set $\omega = \{\omega'_m\}$ of disjoint loops on $S$, where each $\omega'_m$ is a power of a simple loop so that $p{:}\tilde{S}\longmapsto S$ is the highest regular covering of $S$ for which the loops $\{\omega'_m\}$ all lift to loops.*

A short discussion of this theorem may be helpful. Consider the Riemann surface $\Delta$/G = $\Sigma_G$ which is necessarily of finite type. Since $\pi{:}\Delta\longmapsto\Sigma_G$ is a planar and regular covering, the Planarity Theorem applies. We get as a result, a finite set $\{w'_m\}$ of disjoint loops on $\Sigma_G$ satisfying the properties mentioned in the theorem. Furthermore, there exists a finite set of homotopically independent loops $\{\alpha'_n\}$, all of which lift to axes of accidental parabolic transformations $\tau_G(\alpha'_n)$ which are elements of some b-groups that are subgroups of G. The set $\{w'_m\} \cup \{\alpha'_n\}$ is a (complete) set of $\nu$-dividers, $\nu<\infty$, and $\infty$-dividers on $\Sigma_G$. We note here that this set is composed of both dividing and non-dividing loops.

The lifts, $w_m$, of the loops $w'_m$ traversed $\nu$ times are simple disjoint loops in $\Delta$. These loops divide $\Delta$ into regions $R'_1$,$R'_2$,... which are called preliminary structure regions. The lifts, $\alpha_n$, of the loops $\alpha'_n$ are, apart from the parabolic fixed points of $\tau_G(\alpha'_n)$ in $\Delta$, simple disjoint loops as well and each one lies entirely in some $R'_i$. The loops $w_m$ and $\alpha_n$ are referred to as the structure loops of G and they divide $\Delta$ into regions $R_1$,$R_2$,... called G-structure regions. Let $H_i = Stab_G R_i = \{ g \in G \mid g(R_i) = R_i \}$. These groups in turn are called G-structure subgroups. These groups are b-groups without accidental parabolics.

The complement of the set of dividers in $\Sigma_G$ is a finite number of building blocks $Y_1$,$Y_2$,...,$Y_q$ which are subsurfaces of $\Sigma_G$ and whose pre-image under the covering has some $R_i$ as a connected component. Since $q<\infty$, there exist up to G-equivalence only a finite number of structure regions in $\Delta$. Let $R_1$,$R_2$,...,$R_q$ be a complete list of G-inequivalent structure regions. From the building blocks, we obtain the associated 2-complex, K, together with a number of 1-complexes called connectors. K gives rise to what is referred to as the graph of $\Sigma_G$ with the building blocks as vertices and the 1-connectors as edges. K also gives rise to a marking, $\Sigma_1{}^+$,$\Sigma_2{}^+$,...,$\Sigma_S{}^+$, where $\Sigma_1{}^+ \cup \Sigma_2{}^+ \cup ... \cup \Sigma_S{}^+$ is topologically equivalent, in fact conformally equivalent, to the disjoint union of component surfaces of G. A component surface of G is $\Omega_i/\hat{G}_i$ where $\Omega_i$ is a component of $\Omega(G)$ and $\hat{G}_i = Stab_G\Omega_i$. This subgroup is called a component subgroup of G.

Apart from the 1-complexes called connectors, K is a disjoint union of finite Riemann surfaces, $K_i$. Hence, one can affix an (orbifold) Euler characteristic to each $K_i$. All the $K_i$'s are of hyperbolic characteristic. If G is regular, all these $K_i$'s are elements of the marking. Otherwise a number of them may have disappeared or thrown away together with those with euclidean and spherical characteristics. G is usually called partially degenerate in this case. Hence, we see that the number of component surfaces of a function group (G,$\Delta$) is not necessarily equal to the number of building blocks of $\Sigma_G$ = $\Delta$/G. Note that it is also possible to consider the signatures of the elements of the marking instead of their (orbifold) characteristics as the two are clearly related.

# Chapter 2

# Preliminary Results

## 2.1 Improper Inclusion Pairs

Using the facts and definitions established in the previous chapter, we now present the following definition. It is assumed that F $\neq$ G.

**Definition 2** *Let F and G be non-elementary, finitely generated Kleinian groups such that* $F \prec G$ *with index* $[G:F] < \infty$. *Let* $Q(F,\Omega)$ *and* $Q(G,\Omega)$ *be the space of holomorphic, integrable quadratic differentials compatible with F and G respectively and supported on* $\Omega = \Omega(F) = \Omega(G)$*, the common ordinary set of F and G. F and G is called an* **improper inclusion pair** *if* $Q(F,\Omega) = Q(G,\Omega)$.

Below is the first of our preliminary results. Its proof uses the natural decompositions of Q(F,$\Omega$) and Q(G,$\Omega$) into subspaces associated with the component subgroups of F and G. This will be used repeatedly in our succeeding discussions.

Suppose that F and G is an improper inclusion pair. Let $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \ldots$ be a disjoint union of components of $\Omega$ where $\Omega$ is the common ordinary set of F and G. Notice that if $\Omega_i$ is G-inequivalent to $\Omega_j$, then there is no g $\in$ G and in particular no g $\in$ F for which $g(\Omega_i) = \Omega_j$. Hence, $\Omega_i$ is also F-inequivalent to $\Omega_j$.

Therefore, if $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_S$ is a maximal disjoint union of G-inequivalent components of $\Omega$, then $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_S \cup \ldots \cup \Omega_{S'}$ is the corresponding disjoint union for F where S'$\geq$S. Ahlfors' Finiteness Theorem guarantees that S,S' $< \infty$. The associated component subgroups of F and G are denoted respectively by $\hat{F}_i$ and $\hat{G}_i$.

**Proposition 1** *If F and G is an improper inclusion pair, then* $Q(\hat{F}_i,\Omega_i) = Q(\hat{G}_i,\Omega_i)$ *for all i. In particular, if F and G are function groups, then* $Q(F,\Delta) = Q(G,\Delta)$ *where* $\Delta = \Delta(F) = \Delta(G)$ *is the invariant component common to F and G.*

**Proof:** Observe that $Q(F,\Omega)$ and $Q(G,\Omega)$ can be decomposed as follows:

$$Q(G,\Omega) = Q(\hat{G}_0,\Omega_0) \oplus Q(\hat{G}_1,\Omega_1) \oplus \ldots \oplus Q(\hat{G}_S,\Omega_S)$$

$$Q(F,\Omega) = Q(\hat{F}_0,\Omega_0) \oplus Q(\hat{F}_1,\Omega_1) \oplus \ldots \oplus Q(\hat{F}_S,\Omega_S) \oplus Q(\hat{F}_{S+1},\Omega_{S+1}) \oplus \ldots \oplus Q(\hat{F}_{S'},\Omega_{S'})$$

where $\hat{G}_i = Stab_G \Omega_i$ and $\hat{F}_j = Stab_F \Omega_j$ are the associated component subgroups for all i = 0,1,2,...,S , j = 0,1,2,...,S,S+1,...,S'.

As $\hat{F}_i \preceq \hat{G}_i$, we note that $Q(\hat{G}_i,\Omega_i) \hookrightarrow Q(\hat{F}_i,\Omega_i)$. Since F and G is an improper inclusion pair by assumption, we have $Q(F,\Omega) = Q(G,\Omega)$. Thus, $Q(\hat{F}_j,\Omega_j)$ is trivial for j = S+1,S+2,...,S'. Without loss of generality, we will assume that S = S'. Suppose $\exists$ an index k, k = 0,1,2,...,S such that the inclusion Q($\hat{G}_k$,$\Omega_k$)$\hookrightarrow$Q($\hat{F}_k$,$\Omega_k$) between these two Banach spaces is not surjective. Then, dimQ($\hat{G}_k$,$\Omega_k$) < dimQ($\hat{F}_k$,$\Omega_k$) and thus, dimQ(G,$\Omega$) < dimQ(F,$\Omega$). This contradicts the fact that Q(G,$\Omega$) = Q(F,$\Omega$). Hence, for every i = 0,1,2,...,S, we must have Q($\hat{G}_i$,$\Omega_i$) = Q($\hat{F}_i$,$\Omega_i$). The remaining conclusion follows immediately. □

**Remark:** The definition of improper inclusion pairs can be extended to elementary groups. This is, in fact, included among Greenberg's results **[Gr]**. He showed that for each possible elementary group, $F_i$, i = 1, 2, ..., 12, each of whose signatures is indicated below, there exists a group $G_i$, i = 1, 2, ..., 12, with $F_i \prec G_i$ of finite index such that their Teichmuller spaces, and thus their spaces of quadratic differentials, are equal. Here is the list of all such possible elementary groups, $F_i$, i = 1, 2, ..., 12, accompanied by their respective sigantures.

List of Elementary Groups

| **Name of Group** | **Signature** |
|---|---|
| 1. Finite Cyclic Groups | (0,2;n,n) |
| 2. Dihedral n-groups | (0,3;n,2,2) |
| 3. Solid Rotation Groups | (0,3;2,3,e), e = 3, 4, or 5 |
| 4. Parabolic Cyclic Groups | (0,2;$\infty$,$\infty$) |
| 5. Finite $\mathbb{Z}_2$-extension of parabolic cyclic groups | (0,3;$\infty$,2,2) |
| 6. Rank 2 parabolic groups | (1,0) |
| 7. Euclidean Triangle Groups | (0,3;3,3,3), (0,3;4,4,2) |

or (0,3;6,3,2)

8. Eucildean Four Punctured Sphere Groups (0,4;2,2,2,2)

9. Loxodromic Cyclic Groups (1,0)

10. Abelian index n≥2 extension of loxodromic cyclic groups (1,0)

11. Non-abelian index 2 extension of loxodromic cyclic groups (0,4;2,2,2,2)

12. Double Dihedral n-groups (0,4;2,2,2,2)

## 2.2 Fuchsian Case and Greenberg's Theorem

Let Γ be a Fuchsian group. A well-known corollary of the Riemann-Roch formula is the following result on the complex dimension of the space of automorphic q-forms, q ≥ 2, for Γ. We state it as a theorem.

**Theorem 2** *If Γ is a Fuchsian group and has signature $(p,k;\nu_1,\nu_2,\ldots,\nu_k)$, then the dimension of the space of automorphic q-forms for Γ is given by*

$$(2q - 1)(p - 1) + \sum_{j=1}^{k} \left[q - \frac{q}{\nu_j}\right]$$

*where [x] is the integral part of x and it is agreed that $[q - q/\infty] = 1$.*

One can easily see that dim Q(Γ,U) = 3p - 3 + k by setting q = 2 and noting that, since $\nu_j \geq 2$, the integral part $[2 - 2/\nu_j] = 1$ for every j. The signature of Γ is the same as the signature of the associated orbifold U/Γ. Its orbifold characteristic, $\chi$, and its area can then be calculated from this signature. It is given by the formulae below.

$$\chi(\mathrm{U}/\Gamma) = -(2\mathrm{p} - 2 + (\textstyle\sum_{j=1}^{k} \left[1 - \frac{1}{\nu_j}\right]));$$
$$\mathrm{Area}(\mathrm{U}/\Gamma) = -2\pi\chi(\mathrm{U}/\Gamma)$$

**Definition 3** *A signature $(p,k;\nu_1,\nu_2,\ldots,\nu_k)$ is called hyperbolic if and only if $\chi < 0$; euclidean if $\chi = 0$; and spherical if $\chi > 0$.*

The concept of Improper Inclusion Pairs in the Fuchsian case is certainly known. This is the content of the theorem of Greenberg **[Gr]** or of Singerman **[S]**. The table of hyperbolic signatures below summarizes these results for finitely generated Fuchsian groups $F_0$ and $G_0$ of the first kind with $F_0 \triangleleft G_0$ and index $[G_0{:}F_0] = 2 < \infty$.

**Table 1:**

| $F_0$ | $G_0$ |
|---|---|
| (2,0) | (0,6;2,2,2,2,2,2) |
| (1,2;m,m) | (0,5;2,2,2,2,m) |
| (1,1;m) | (0,4;2,2,2,2m) |
| (0,4;m,m,n,n) | (0,4;2,2,m,n), m $\neq$ n |

The variables m and n are allowed to take on the value $\infty$. If m = n in the last pair of signatures, then there exists a $G_0'$ of signature (0,4;2,2,2,m) such that $F_0 \triangleleft G_0 \triangleleft G_0'$ with $[G_0'{:}G_0] = [G_0{:}F_0] = 2$ and $[G_0'{:}F_0] = 4$. More importantly, $F_0$ and $G_0'$ is an improper inclusion pair. This will be treated separately in some of our succeeding results in the next chapter.

The rest of the pairs of signatures correspond to pairs $F_0$ and $G_0$ which are both triangle groups, where the index $[G_0{:}F_0] \geq 2$. In general, these are not normal inclusions. Those pairs with normal inclusion have indices and signatures which are as follows: $[G_0{:}F_0] = 2$ where $F_0$ has signature (0,3;m,m,n) and $G_0$ has signature (0,3;2,m,2n); $[G_0{:}F_0] = 3$ where $F_0$ has signature (0,3;m,m,m) and $G_0$ has signature (0,3;3,3,m); $[G_0{:}F_0] = 6$ where $F_0$ has signature (0,3;m,m,m) and $G_0$ has signature (0,3;2,3,2m).

**Definition 4** *A pair of signatures, $\sigma$ and $\sigma'$, will be called a maximal pair of signatures if it is among the pairs of signatures listed in Table 1.*

**Remark:** All non-hyperbolic signatures are signatures of elementary groups, and conversely. The signatures $(1,1;\nu)$ and $(0,2;\nu,\mu)$ with $\nu \neq \mu$ never occur and are called non-geometric signatures.

The following lemma relates the notion of improper inclusion pairs with maximal pairs of signatures for the Fuchsian case.

**Lemma 3** *$(F_0, U)$ and $(G_0, U)$ is an improper inclusion pair of Fuchsian groups (of the first kind) if and only if their associated pair of signatures is a maximal pair.*

**Proof:** ($\Rightarrow$) This direction is Greenberg's Theorem **[Gr]**.

($\Leftarrow$) Conversely, if the pair of signatures from $F_0$ and $G_0$ is maximal, then by using the formula dim $Q(\Gamma,U) = 3p - 3 + n$ we have above for $\Gamma$, a Fuchsian group, we get that dim $Q(F_0,U)$ = dim $Q(G_0,U)$. Since the inclusion, $Q(G_0,U) \hookrightarrow Q(F_0,U)$, between these two Banach spaces already holds, it must be true that $Q(F_0,U) = Q(G_0,U)$. Hence, $F_0$ and $G_0$ is an improper inclusion pair. □

We now extend this lemma to function groups.

**Lemma 4** *Suppose that (F,$\Delta$) and (G,$\Delta$) are non-elementary and finitely generated function groups such that F$\prec$G with [G:F] $< \infty$. Then, Q(F,$\Delta$) = Q(G,$\Delta$) if and only if the pair of signatures from $\Delta/F$ and $\Delta/G$ is a maximal pair.*

**Proof:** Denoting the upper half-plane by U, let ($F_0$,U) and ($G_0$,U) be the Fuchsian equivalents of F and G respectively. Then clearly, $\Delta$/F and U/$F_0$ have the same signatures. This is also true for $\Delta$/G and U/$G_0$. There are induced isomorphisms Q($F_0$,U)$\cong$Q(F,$\Delta$) and Q($G_0$,U)$\cong$Q(G,$\Delta$) given by

$$Q(F_0,U) \ni q \longmapsto (q \circ f^{-1}) \cdot ((f^{-1})')^2 \in Q(F,\Delta)$$

where f is the covering map from U onto $\Delta$. The induced isomorphism between Q($G_0$,U) and Q(G,$\Delta$) is defined in exactly the same manner. Obviously, we have Q($G_0$,U)$\hookrightarrow$Q($F_0$,U). Since Q(F,$\Delta$) = Q(G,$\Delta$) by assumption, the isomorphisms above yield dimQ($F_0$,U) = dimQ($G_0$,U) and thus Q($F_0$,U) = Q($G_0$,U). Hence, the signatures of U/$F_0$ and U/$G_0$ and therefore of $\Delta$/F and $\Delta$/G must be among the maximal pairs of signatures for Fuchsian groups.

The converse is clear. □

**Remark:** Since Q($\hat{F}_i$,$\Omega_i$) = Q($\hat{G}_i$,$\Omega_i$) for all other components $\Omega_i$ as a consequence of Proposition 1, we get that the pairs of signatures from the Riemann surfaces of finite type, $\Omega_i/\hat{F}_i$ and $\Omega_i/\hat{G}_i$, i = 1, 2,....,S, must also be maximal or else, $\hat{F}_i = \hat{G}_i$. Moreover, one can conclude from a result of Accola **[M1]** that $\Omega_1$, $\Omega_2$, ..., $\Omega_S$ are all simply-connected. It then follows that for all i = 1, 2, ..., S, the Teichmuller spaces of $\Omega_i/\hat{F}_i$ and $\Omega_i/\hat{G}_i$ are biholomorphic to the Teichmuller spaces of the corresponding surfaces uniformized by the Fuchsian equivalents of $\hat{F}_i$ and $\hat{G}_i$ respectively. This is shown easily via the conformal similarity that arises from the Riemann map between $\Omega_i$ and U. Also, from elementary results on Teichmuller spaces of Fuchsian groups of finite type, we get dim T($F_0$) = dim Q($F_0$,U) = dim Q($G_0$,U) = dim T($G_0$). We will refer to these observations again in succeeding chapters.

## 2.3 Hyperellipticity and Some Consequences

The definition of a hyperelliptic Riemann surface in the compact case can be extended to the case of Riemann surfaces of finite type:

**Definition 5** *A Riemann surface M of finite type (g,n) is called hyperelliptic if and only if there exists a meromorphic function f on M, f:M$\longmapsto\hat{\mathbb{C}}$, such that deg(f) = 2 and f either*

*(i) maps two distinct distinguished points $x_i$ and $y_i$ of the same order, i = 1,2,...,n (these include punctures) on M onto the same image in $\hat{\mathbb{C}}$ or*

*(ii) f is 2-1 there.*

**Remark:** Riemann surfaces of signature $\sigma_1 = (1,2;m,m)$, $\sigma_2 = (0,4;m,m,n,n)$, or $\sigma_3 = (1,1;m)$ are all hyperelliptic. Indeed, the underlying meromorphic function on each of these surfaces is the covering map induced by its hyperelliptic involution, say $\hat{\gamma}_i$ for i = 1, 2, 3. Let us denote a Riemann surface with signature $\sigma_i$ by $S_i$. The action of $\hat{\gamma}_i$ gives rise precisely to a Riemann surface $S_i'$ of signature $\sigma_1' = (0,5;2,2,2,2,m)$, $\sigma_2' = (0,4;2,2,m,n)$, or $\sigma_3' = (0,4;2,2,2,2m)$ respectively. We notice that $S_i$ and $S_i$' satisfy the property that $Q(S_i) = Q(S_i')$, i = 1, 2, 3.

**Lemma 5** *For any uniformization of a hyperelliptic $S_i$ by a non-elementary, finitely generated function group $(F,\Delta)$, there exists a half-turn $\gamma$ such that $\gamma(\Delta) = \Delta$ and $\gamma$ is induced by the hyperelliptic involution.*

**Remark:** This is a strengthened statement **[RST]** of a theorem by Kra-Maskit. We give a sketch of the arguments.

**Proof:** First, we pass to the Fuchsian equivalent, $(F_0,U)$, of $(F,\Delta)$, thereby defining a deformation $(\pi_0,\theta_0)$ of $F_0$ onto F such that for every $f_0 \in F_0$, we have $\pi_0 f_0 = f\pi_0$ where $\theta_0(f_0) = f \in F$. Let $\gamma_0$ be the lift of the hyperelliptic involution on $\Delta/F$. Clearly, $\gamma_0$ is a real, elliptic Moebius transformation of order two. Let $G_0 = <F_0,\gamma_0>$ be the $\mathbb{Z}_2$-extension of $F_0$ by $\gamma_0$. We observe that $Q(F_0,U) = Q(G_0,U)$. This last equality between the spaces of quadratic differentials allows us to extend $\theta_0$ to $G_0$. By using the holomorphicity of an induced map from the Banach space $Q(G_0,U)$ to $\mathbb{C}$, we get that $\mathrm{tr}^2\theta_0(\gamma_0) = 0$ and the desired half-turn $\gamma = \theta_0(\gamma_0)$. □

**Lemma 6** *The $\mathbb{Z}_2$-extension of F by $\gamma$ is a function group whose Fuchsian equivalent is $G_0$.*

**Proof:** This follows from the fact that the image of $G_0 = <F_0,\gamma_0>$ under the same deformation $(\pi_0,\theta_0)$ as above yields the $\mathbb{Z}_2$-extension of F by $\gamma$,

$<F,\gamma>$. This is clearly a function group, $(<F,\gamma>,\Delta)$, whose Fuchsian equivalent is precisely $G_0$. □

In the following corollary, we are assuming that the pair of function groups are not triangle groups. The notations used will be the same as those in the preceding lemma.

**Corollary 2** *Suppose that (F,Δ) and (G,Δ) is a pair of non-elementary and finitely generated function groups such that* $F \prec G$ *with* $[G:F] < \infty$. *If* $Q(F,\Delta) = Q(G,\Delta)$ *and these have positive dimension, then* $G = <F,\gamma>$ *for some half-turn* $\gamma$ *and* $F \lhd G$ *with* $[G:F] = 2$.

**Proof:** This corollary is certainly known for Fuchsian groups of the first kind where $\gamma = \gamma_0$, the (real)elliptic Moebius transformation of order two which is induced by the hyperelliptic involution on $U/F_0$.

For the general case of function groups, we first observe, by Lemma 5, that $\Delta/F$ and $\Delta/G$ must have a maximal pair of signatures. Pass to the Fuchsian equivalents of F and G, say $F_0$ and $G_0$. Our assumption that $Q(F,\Delta) = Q(G,\Delta)$ yields $Q(F_0,U) = Q(G_0,U)$ and thus, $G_0 = <F_0,\gamma_0>$. From our remarks above on the previous lemma, the Fuchsian equivalent of $<F,\gamma>$, where $\gamma$ is the half-turn such that $\gamma = \theta_0(\gamma_0)$, is precisely $<F_0,\gamma_0> = G_0$. Hence, $G = <F,\gamma>$. Furthermore, since $Q(F_0,U) = Q(G_0,U)$, we get that $F_0$ and $G_0$ is an improper inclusion pair of Fuchsian groups (of the first kind) with $[G_0:F_0] = 2$. Thus, it clearly follows that $F \lhd G$ and that $[G:F] = 2$. □

## 2.4 Dimension Zero Case for Function Groups

We gave the pairs of signatures for certain pairs of triangle groups, $F_0$ and $G_0$ with $[G_0{:}F_0] < \infty$ such that $F_0 \cdot\!\!\not\,\triangleleft G_0$, in a previous section. Table 2 below gives the rest of the pairs of triangle groups and the corresponding group indices, included among the results of Greenberg's Theorem where normality is absent.

**Table 2:**

| $F_0$ | $G_0$ | $[G_0{:}F_0]$ |
|---|---|---|
| (0,3;2,m,2m) | (0,3;2,3,2m) | 3 |
| (0,3;3,m,3m) | (0,3;2,3,3m) | 4 |
| (0,3;m,2m,2m) | (0,3;2,4,2m) | 4 |
| (0,3;m,4m,4m) | (0,3;2,3,4m) | 6 |
| (0,3;4,4,5) | (0,3;2,4,5) | 6 |
| (0,3;7,7,7) | (0,3;2,3,7) | 24 |
| (0,3;2,7,7) | (0,3;2,3,7) | 9 |
| (0,3;3,3,7) | (0,3;2,3,7) | 8 |
| (0,3;4,8,8) | (0,3;2,3,8) | 12 |
| (0,3;3,8,8) | (0,3;2,3,8) | 10 |
| (0,3;9,9,9) | (0,3;2,3,9) | 12 |

**Proposition 2** *If (F,Δ) and (G,Δ) is an improper inclusion pair of function groups such that the inclusion is not normal and [G:F] > 2, then F and G is a pair of triangle groups whose signatures appear as a maximal pair in Greenberg's theorem.*

**Proof:** Our hypothesis implies that Q(F,$\Delta$) = Q(G,$\Delta$). If we look at their Fuchsian equivalents $G_0$ and $F_0$, we observe that $[G_0{:}F_0]$ = [G:F] > 2. Moreover, by using the isomorphisms described before, Q(G,$\Delta$) = Q(F,$\Delta$) will imply that Q($G_0$,U) = Q($F_0$,U). From this and the indicated hypotheses, we get that $G_0$ and $F_0$ must be a pair of Fuchsian triangle groups whose signatures appear above. This will imply that dim Q(G,$\Delta$) = dim Q(F,$\Delta$) = 0. Clearly, the dimension zero case for function groups occurs if and only if the groups are triangle groups. Thus, F = $F_0$ and G = $G_0$. □

# Chapter 3

## THE MAIN PROPOSITION

We present, at this point, a proposition that summarizes our results regarding improper inclusion pairs in the more general cases of Kleinian groups. Before we state it, however, we consider the following lemma. Let $(\Omega(F)/F)^*$ and $(\Omega(G)/G)^*$ denote the component surfaces in $\Omega(F)/F$ and $\Omega(G)/G$ respectively which are not thrice-punctured spheres.

**Lemma 7** *Suppose that F and G is an improper inclusion pair. Then, ∃ a 1-1 correspondence between $(\Omega(F)/F)^*$ and $(\Omega(G)/G)^*$.*

**Proof:** By our assumptions, we have $Q(F,\Omega) = Q(G,\Omega)$. These spaces have natural decompositions as observed in our preliminary results. Furthermore, we have seen that G has at most as many components, $\Omega_k$, as F and any excess components of F must correspond to thrice-punctured spheres. This results in a pairing of component subgroups of F and G which arises via the obvious 1-1 correspondence induced by inclusion.

By denoting the paired component subgroups as $\hat{F}_k$ and $\hat{G}_k$ respectively, we can thus establish an induced 1-1 correspondence between $(\Omega(F)/F)^*$ and

$(\Omega(G)/G)^*$ given by $(\Omega(F)/F)^* \ni \Omega_k/\hat{F}_k = S_k \longmapsto S_k' = \Omega_k/\hat{G}_k \in (\Omega(G)/G)^*$.

□

## 3.1 Statement of the Proposition

**Proposition 3** *Let F and G be an improper inclusion pair. Then, either*

*(a) dim Q(F,Ω) = dim Q(G,Ω) = 0*

*or*

*(b) G - F has at least one half-turn that stabilizes a component of F.*

**Proof:** If dim Q(F,Ω) = 0, then dim Q(G,Ω) = 0 and we get (a). Thus, assume that dim Q(F,Ω) > 0. This clearly implies that $(\Omega(F)/F)^* \neq \emptyset$. Let $S_i \in (\Omega(F)/F)^*$. Let $\gamma$ be an arbitrary element of G - F and $\Omega_i$ be a component of Ω such that $S_i = \Omega_i/\hat{F}_i$ with $\hat{F}_i = Stab_F\Omega_i$. Denote by $S_i'$ the corresponding surface in $(\Omega(G)/G)^*$ given by the 1-1 correspondence above so that $S_i' = \Omega_i/\hat{G}_i$ with $\hat{G}_i = Stab_G\Omega_i$.

Now, note that $\Omega_i$ and $\gamma(\Omega_i)$ are automatically G-equivalent. Hence, by choice of $S_i$, $\Omega_i$ and $\gamma(\Omega_i)$ cannot be F-inequivalent for otherwise, the dimension of Q(G,Ω) will be strictly less than the dimension of Q(F,Ω) which is impossible since F and G is an improper inclusion pair. Hence, $\gamma(\Omega_i) = f(\Omega_i)$ for some f ∈ F. From this, observe that $f^{-1}\gamma$ stabilizes $\Omega_i$ with $f^{-1}\gamma \in \hat{G}_i - \hat{F}_i$. By our hypothesis and Proposition 1, we must have $Q(\hat{F}_i,\Omega_i) = Q(\hat{G}_i,\Omega_i)$. Thus, the signatures of $S_i$ and $S_i'$ must be a maximal pair. As $\hat{F}_i$ is clearly a function group with invariant component $\Omega_i$, Corollary 2 implies that $\hat{F}_i \lhd \hat{G}_i$

with $[\hat{G}_i:\hat{F}_i] = 2$ such that $\hat{G}_i = <\hat{F}_i,\gamma_i>$ where $\gamma_i$ is the resulting half-turn found in Lemma 6. Hence, conclusion (b) follows immediately. □

**Remark:** We note that $f^{-1}\gamma$ must necessarily be an element of the unique nontrivial coset of $\hat{G}_i/\hat{F}_i$ represented by $\gamma_i$.

Suppose that $S_1$, $S_2$, ..., $S_N$ is a list of all the component surfaces in $(\Omega(F)/F)^*$ and $(S_i,S_i')$, $S_i' \in (\Omega(G)/G)^*$, is the pair obtained from the 1-1 correspondence in the previous lemma. We have the following corollary. The proof directly follows from the proof of the previous proposition.

**Corollary 3** *Assume that F and G are as above. Then, there are only two possibilities. Either all the pairs $(S_i,S_i')$ have maximal pairs of signatures or both F and G represent only thrice-punctured spheres.*

*Furthermore, for every component, $\Omega_i$, of F where $\Omega_i/\hat{F}_i$ is not a thrice-punctured sphere, there is a half-turn $\gamma_i \in \hat{G}_i$ - $\hat{F}_i$.*

**Proof:** By using our main Proposition, we get two possibilities. If case (a) holds, then it is clear that both F and G represent only thrice-punctured spheres. If case (b) holds, then by our proof above, we can assume that $\exists$ k such that $S_k = \Omega/\hat{F}_k$ and $S_k' = \Omega/\hat{G}_k$ have a maximal pair of signatures. This implies that $\hat{G}_k = <\hat{F}_k,\gamma_k>$ for some half-turn $\gamma_k$, as a result of Lemma 6. Suppose that $S_j = \Omega_j/\hat{F}_j$ is another element of $(\Omega(F)/F)^*$ with $S_j \neq S_k$. By choice of $S_j$, we get that $\gamma_k(\Omega_j) = f(\Omega_j)$ for some $f \in F - \{1\}$. Hence, $f^{-1}\gamma_k$ stabilizes $\Omega_j$ and we can conclude by using similar arguments as in our previous proof that $\hat{G}_j = <\hat{F}_j,\gamma_j>$ for some half-turn $\gamma_j$ and $(S_j,S_j')$ have a maximal pair of signatures. Since $S_k$ and $S_j$ were chosen arbitrarily, we get the conclusion of our corollary. The rest of the conclusion follows directly. □

It is worthwhile to ask what the realizable indices, [G:F], are between F and G. We will answer this in some special cases. First, we shall say that F is **nontriangular** if $(\Omega(F)/F)$ - $(\Omega(F)/F)^* = \emptyset$, that is, F does not represent any thrice-punctured spheres. Clearly, if F is nontriangular, then G is also nontriangular.

**Lemma 8** *If F and G is an improper inclusion pair where both are nontriangular, then all the pairs of component surfaces of F and G have maximal pairs of signatures.*

**Proof:** The hypothesis implies that case (b) of the main proposition must hold. Now, the conclusion follows immediately by applying the preceding corollary. □

**Proposition 4** *Suppose that F and G is an improper inclusion pair. If both F and G are nontriangular, then the index [G:F] is either 2 or 4.*

**Proof:** By the lemma above, all the pairs of component surfaces, $(S_i,S_i')$, of F and G have signatures that are listed in Table 1. For reference in our proof, we denote these maximal pairs of signatures shown below by $(\sigma_j,\sigma_j')$, j = 1, 2, 3, 4, 5, 6, 7. The pair of signatures $(\sigma_6,\sigma_6')$ is clearly a special case of $(\sigma_4,\sigma_4')$ where n = 2.

| $j$ | $\sigma_j$ | $\sigma'_j$ | *index* |
|---|---|---|---|
| 1 | (2,0) | (0,6;2,2,2,2,2,2) | 2 |
| 2 | (1,2;m,m) | (0,5;2,2,2,2,m) | 2 |
| 3 | (1,1;m) | (0,4;2,2,2,2m) | 2 |
| 4 | (0,4;m,m,n,n) | (0,4;2,2,m,n), m $\neq$ n | 2 |
| 5 | (0,4;m,m,m,m) | (0,4;2,2,m,m) | 2 |
| 6 | (0,4;2,2,m,m) | (0,4;2,2,2,m) | 2 |
| 7 | (0,4;m,m,m,m) | (0,4;2,2,2,m) | 4 |

Let $S'_i = \Omega_i/\hat{G}_i = \Omega_i/<\hat{F}_i,\gamma_i>$ be a component surface of G, where $\gamma_i$ is the half-turn from Lemma 6. We will break down our proof into two cases.

**Case 1: Some component surface $S'_k = \Omega_k/\hat{G}_k$ of G is of signature $\sigma'_1$, $\sigma'_2$, $\sigma'_3$, or $\sigma'_4$.**

Let $\gamma_k$ be the half-turn associated with $\hat{G}_k$. Suppose that $S'_i = \Omega_i/\hat{G}_i$ is another component surface of G, i $\neq$ k. Then, as we noted in the previous results, the (unique) nontrivial coset in $\hat{G}_i$ represented by its associated half-turn, $\gamma_i$, must contain $f^{-1}\gamma_k$ for some f $\in$ F. This implies that $\gamma_i \in <F,\gamma_k>$ for all i. Hence, $G = <F,\gamma_1,\gamma_2,...,\gamma_N> = <F,\gamma_k>$ where N is taken to be the number of component surfaces of G. This yields [G:F] = 2. Here in case 1, observe that [G:F] cannot be larger than 2, for otherwise we will have a pair of signatures $\tau' \subseteq \tau''$[1], where $\tau'$ is one of the signatures we specified above. Clearly, such a pair will not be a maximal pair, simply by checking Table 1.

[1]This notation is used, e.g. by Singerman, to signify the inclusion relation between the two groups associated with the given signatures.

**Case 2: No component surface of G has a signature equal to $\sigma'_j$, j = 1, 2, 3, 4. Hence, all component surfaces, $S'_i$, of G have only two kinds of signatures: $\sigma'_5$ or $\sigma'_6$ since $\sigma'_6 = \sigma'_7$**

**Subcase 1: All component surfaces, $S'_i$, of G are of signature $\sigma'_5$.**

We now apply our previous corollary and get that all component surfaces, $S_i$, of F are of signature $\sigma_5$ and [G:F] = 2.

**Subcase 2: All component surfaces, $S'_i$, of G are of signature $\sigma'_6$.**

Applying again our previous corollary, we find that there are two possibilities since $\sigma'_6 = \sigma'_7$. Either all component surfaces, $S_i$, of F are of signature $\sigma_6$, in which case [G:F] = 2; or all are of signature $\sigma_7$, in which case [G:F] = 4. This is clearly the only situation under our hypothesis that yields [G:F] = 4.

**Subcase 3: Some component surfaces, $S'_k$, of G are of signature $\sigma'_5$ and some are of signature $\sigma'_6$.**

We find that all component surfaces, $S_i$, of F have only two kinds of signatures: $\sigma_5$ or $\sigma_6$ and that [G:F] = 2. The index here cannot be larger than 2 for the same reason as in case 1 above.

This completes the proof of our proposition. □

## 3.2 The Function Group Case

For function groups, the existence of the half-turn in case (b) of the main Proposition (Proposition 3) is a direct consequence of our earlier results. To see this, suppose that (F,$\Delta$) and (G,$\Delta$) is an improper inclusion pair of function groups. Then, by Proposition 1, Q(F,$\Delta$) = Q(G,$\Delta$). By Lemma 4, the pair of signatures from $\Delta/F$ and $\Delta/G$ must be found in Table 1. Furthermore, G is a $\mathbb{Z}_2$-extension of F by some half-turn $\gamma$ induced by the hyperelliptic involution, $\hat{\gamma}$, on $\Delta/F$ such that $(\Delta/F)/\{\ \mathbf{1}, \hat{\gamma}\ \} \cong \Delta/G$. This follows from Lemma 6 and Corollary 2. Hence, conclusion (b) in our main proposition is illustrated by this half-turn $\gamma$ for the case of function groups.

Recall the definitions of regular and partially degenerate given in the background from Chapter 1. Suppose that (F,$\Delta$) and (G,$\Delta$) are non-elementary and finitely generated function groups such that F$\prec$G with [G:F] $< \infty$. Consider the actions of F and G on $\mathbb{H}^3$ which are properly discontinuous actions. To each group, there is an associated polyhedron in $\mathbb{H}^3$, called a fundamental polyhedron, having the property that no two of its interior points are equivalent under the group. If this fundamental polyhedron has finitely many sides, then we say that the group is a **geometrically finite** Kleinian group. It is known that F is regular if and only if F is geometrically finite and, thus partially degenerate if and only if F is geometrically infinite. However, any finite extension of a geometrically finite Kleinian group is, in fact, geometrically finite. Hence, if F is regular, then G is regular.

**Remark:** Bers' Area Inequalities **[B4]** state that for a non-elementary, finitely generated function group $\Gamma$, Area $(\Omega(\Gamma)/\Gamma) \leq 2\cdot\text{Area}(\Delta(\Gamma)/\Gamma)$; $\Gamma$ is regular if equality is attained and partially degenerate otherwise.

Now, we prove a Proposition that can be viewed as a partial converse of Proposition 1 in the function group case.

**Proposition 5** *Let $(F,\Delta)$ and $(G,\Delta)$ be non-elementary, geometrically finite function groups with $F \prec G$ of finite index. If $Q(F,\Delta) = Q(G,\Delta)$, where $\Delta = \Delta(F) = \Delta(G)$ is the common invariant component of F and G, then F and G is an improper inclusion pair. (i.e. $Q(F,\Omega) = Q(G,\Omega)$, $\Omega = \Omega(F) = \Omega(G)$ is the common ordinary set of F and G.)*

Note that since the proposition is trivial for triangle groups, we will assume that neither F nor G is a triangle group. We will consider some special cases before tackling the general one.

### 3.2.1 Special Cases

There are special cases where we can replace the geometrically finite assumption above with the weaker hypothesis of finitely generated. The special cases are listed below. We note that the groups in cases (i) and (iii) are necessarily geometrically finite.

*(i) F and G are both quasi-Fuchsian groups*

*(ii) F and G are both totally degenerate b-groups*

*(iii) F and G are both terminal(regular) b-groups.*

In (i), we have

$$\dim Q(G,\Omega) = 2\dim Q(G,\Delta)$$

$$\dim Q(F,\Omega) = 2\dim Q(F,\Delta)$$

while in (ii) and (iii), we observe that

$$\dim Q(G,\Omega) = \dim Q(G,\Delta)$$

$$\dim Q(F,\Omega) = \dim Q(F,\Delta)$$

By assumption, $Q(F,\Delta) = Q(G,\Delta)$. This yields $\dim Q(G,\Omega) = \dim Q(F,\Omega)$ in all three cases (i), (ii), and (iii). From this, it follows easily that $Q(G,\Omega) = Q(F,\Omega)$ and thus, we have an improper inclusion pair.

We can say, in addition, that establishing the above Proposition for these special function groups can easily be reduced to the Fuchsian case. To show why, let f be a covering map from the upper half-plane U to one of the invariant components, $\Delta$, of F ( If F is totally degenerate, $\Delta = \Omega$ ). Since $\Delta$ is simply-connected in all three cases, we can assume that f is a Riemann map. Then f is a conformal similarity between $(F,\Delta)$ and $(G,\Delta)$ to their respective Fuchsian equivalents $(F_0,U)$ and $(G_0,U)$. Observe, therefore, that the signatures of $\Delta/F$ and $\Delta/G$ are the same as the signatures of $U/F_0$ and $U/G_0$ respectively. Since the component surfaces of a terminal(regular) b-group are all thrice-punctured spheres and hence have rigid conformal structures, it suffices then to just consider the signatures of $\Delta/F$ and $\Delta/G$ in all of the cases (i), (ii), (iii). With the observations above, we get $\dim Q(F_0,U) = \dim Q(F,\Delta) = \dim Q(G,\Delta) = \dim Q(G_0,U)$ and thus, we are reduced to the Fuchsian case as we claimed. This means that if $F_0$ and $G_0$ have a maximal pair of signatures, then $(F,\Delta)$ and $(G,\Delta)$ is immediately an improper inclusion pair. Moreover, by the conformal similarity above, the Teichmuller spaces of $U/F_0$ and $U/G_0$ are biholomorphic to the Teichmuller spaces of $\Delta/F$ and $\Delta/G$ respectively and they are, in fact, equal.

**Remarks:** 1. A quasi-Fuchsian group which is not a Fuchsian group is a b-group with exactly two (simply-connected) invariant components $\Delta_1$ and $\Delta_2$ such that $\Delta_1 \cup \Delta_2 = \Omega$, its ordinary set. Clearly, $\partial\Delta_1 = \partial\Delta_2$ and this common boundary is the quasi-circle image of $\mathbf{R}$ with respect to the underlying quasi-conformal deformation. By Lemma 2, this quasi-circle and the limit set coincide. In this sense, we see that a quasi-Fuchsian group that is not a Fuchsian group is always of the first kind. This is a difference in terminology between quasi-Fuchsian groups and the usual Fuchsian groups. Therefore, even if not specifically mentioned, any quasi-Fuchsian group referred to above is understood to be of the first kind.

2. We note that a Fuchsian group of the second kind when viewed as a Kleinian group acting on $\hat{\mathbb{C}}$ is not a b-group since its invariant component will not be simply-connected. Moreover, there exist Kleinian groups which are called extended Fuchsian groups characterized by having a Fuchsian subgroup of index two. These are also not b-groups.

3. Maskit **[M1]**proved that a Kleinian group F is a Schottky group if and only if F is finitely generated, free and purely loxodromic. From this, it follows that any $\mathbf{Z}_2$-extension of a Schottky group by a half-turn $\gamma$ cannot be a Schottky group, the reason being that the extension will not be purely loxodromic. Hence, it is not possible for an improper inclusion pair (F,$\Omega$) and (G,$\Omega$) to have both groups be Schottky groups.

### 3.2.2 General Case for Function Groups

This subsection is entirely devoted to the proof of Proposition 5 for the general cases. Before beginning with the proof, however, we give the following theorems. Theorem 3 was stated as a corollary in **[M1]**. Its proof is a straightforward application of the characterization of an accidental parabolic in **[M1]** and the fact that $[\tilde{G}\text{:G}] < \infty$ implies that their limit sets coincide.

**Theorem 3** [**M1**] *If $(G,\Delta)$ and $(\tilde{G},\tilde{\Delta})$ are b-groups where $G \prec \tilde{G}$ of finite index and j is an accidental parabolic transformation of G, then j is an accidental parabolic transformation in $\tilde{G}$.*

The following theorem due to Haas and Susskind will be extremely useful to our succeeding analyses and discussions. With minor modifications, it has a generalization, as was pointed out in **[HS]**, to certain Riemann surfaces with punctures or ramification points. For our purposes, we only need the generalization to Riemann surfaces of signatures (1,2;m,m), (1,1;m), and (0,4;m,m,n,n).

**Theorem 4** [**HS**] *Let J be the hyperelliptic involution of a genus two Riemann surface M. Then, every simple closed geodesic on M is mapped onto itself by J.*

By using these theorems, we can prove the following lemma.

**Lemma 9** *Assume that $(F,\Delta)$ and $(G,\Delta)$ satisfy the hypotheses of Proposition 5 where $\Delta/F$ is of signature (2,0) and $\Delta/G$ is of signature (0,6;2,2,2,2,2,2).*

*Then, F and G have the same number of inequivalent maximal cyclic subgroups generated by accidental parabolics.*

**Proof:** Assume that F has no accidental parabolics but that G has an accidental parabolic, say g ∈ G - F. As [G:F] = 2, we must have $g^2 \in F$. By using Theorem 3, we can conclude that $g^2$ is an accidental parabolic transformation in F. (Contra!)

Now, suppose that G has only one accidental parabolic with primitive, say g. Then, $g^2$ must be an accidental parabolic for F as in the proof above. Obviously, $g^2$ cannot be a primitive in F since g was chosen to be primitive in G. Hence, g must also be primitive in F and we have at least one accidental parabolic transformation in F. Suppose that F has another accidental parabolic with primitive equal to, say h, such that <g> and <h> are inequivalent in F. We will now show that <g> and <h> must necessarily be inequivalent in G and thus, arrive at a contradiction. Denote by $L_g$ and $L_h$ the two disjoint simple loops on $\Delta/F$ that correspond to g and h respectively. Clearly, $L_g$ and $L_h$ must be homotopically independent. There are only two possibilities for $L_g$ or $L_h$:

(i) $L_g$ or $L_h$ is non-dividing, or

(ii) $L_g$ or $L_h$ is dividing

Suppose that both $L_g$ and $L_h$ are non-dividing. We can assume that $L_g$ and $L_h$ are the projections onto $\Delta/F$ of the axes (see background for definition) of g and h respectively which do not contain the fixed points of the half-turn induced by the hyperelliptic involution acting on $\Delta/F$. Thus, $L_g$ and $L_h$ are

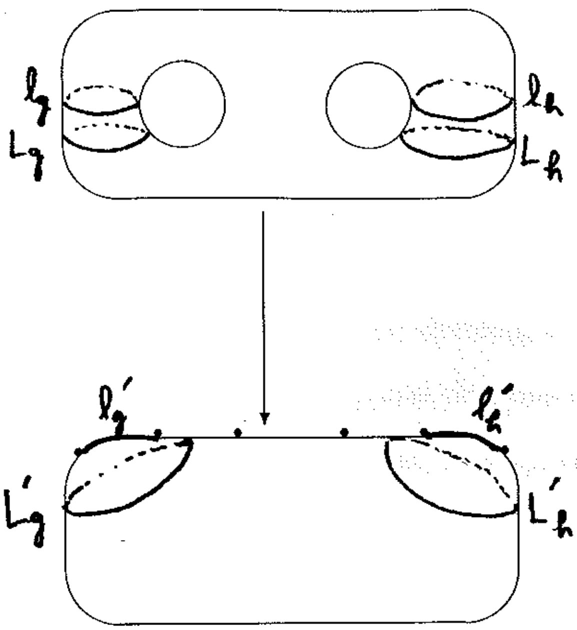


Figure 3.1: Induced Covering

both neighboring simple loops to the non-dividing geodesics, say $l_g$ and $l_h$, in their respective homotopy classes. (see Figure 3.1)

Let us look at the degree 2 covering, say p, of $\Delta/F$ onto $\Delta/G$ induced by the hyperelliptic involution acting on $\Delta/F$. By using the theorem above, each of $l_g$ and $l_h$ is mapped onto itself by the hyperelliptic involution. We ,therefore, observe that the corresponding images of $l_g$ and $l_h$, say $l'_g$ and $l'_h$, are paths on $\Delta/G$ running from one distinguished point ($P_{1,g}$ and $P_{1,h}$ respectively) of order two to another distinguished point of order two ($P_{2,g}$ and $P_{2,h}$ respectively) and back again. If we denote the images of $L_g$ and $L_h$ on $\Delta/G$ under the covering map, p by $L'_g$ and $L'_h$ respectively, then it is clear that $L'_g$ must bound a disc which contains $P_{1,g}$, $P_{2,g}$, and $l'_g$ while $L'_h$ must bound a disc which contains

$P_{1,h}$, $P_{2,h}$, and $l'_h$ (see Figure 3.1). Clearly, $L'_g$ and $L'_h$ are disjoint, simple loops on $\Delta$/G since $L_g$ and $L_h$ are and moreover, they are homotopically independent. This implies that <g> and <h> are inequivalent in G and we get a contradiction.

Now, note that $L_g$ and $L_h$ cannot be both dividing since <g> and <h> are inequivalent in F. Hence, let us now suppose that one of them, say $L_g$, is dividing. We can allow here the possibility that $L_g = l_g$, the simple geodesic loop in its homotopy class. By the above theorem, $l_g$ must be mapped onto itself by the hyperelliptic involution. Consider the image, $L'_g$, of $L_g$ on $\Delta$/G. We note that the signature (0,2;2,$\infty$) is non-geometric and thus, never occurs. Hence, there are also only two cases for $L'_g$:

(i)′ $L'_g$ separates two distinguished points on $\Delta$/G from the other four, or

(ii)′ $L'_g$ separates three distinguished points on $\Delta$/G from the other three.

By our choice of $L_g$, it is clear that $L'_g$ must not satisfy (i)′. Hence, $L'_g$ satisfies (ii)′ which yields that the two disjoint simple loops, $L'_g$ and $L'_h$, are homotopically independent (see Figure 3.1). We again get that <g> and <h> are inequivalent in G and thus, a contradiction. Hence, F must only have one primitive accidental parabolic, g.

For the remaining case of G having only two accidental parabolics, we can conclude that F must also have exactly two accidental parabolics by arguing similarly. An appropriate remark to be made, however, is that if F has three accidental parabolics with primitives, say g, h and j, such that <g>, <h>, <j> are pairwise inequivalent, then no two of their corresponding simple loops $L_g$, $L_h$ and $L_j$ on $\Delta$/F can both be dividing. □

Now, we begin with the proof of Proposition 5. Using Lemma 4, the pair of signatures from $\Delta/F$ and $\Delta/G$ must be found in Table 1 since $Q(F,\Delta) = Q(G,\Delta)$ by hypothesis. When $[G{:}F] = 2$, there are four cases to be considered. First, we will assume that both F and G are b-groups.

**Case I.:** Consider the case of the pair (2,0) and (0,6;2,2,2,2,2,2). Observe that the maximum number of homotopically independent loops on $\Delta/F$ and $\Delta/G$ that correspond to primitive accidental parabolic transformations is 3. This implies that if F(or G) has the maximum number of primitive accidental parabolics, then $(F,\Delta)$ and $(G,\Delta)$ are both terminal(regular) b-groups and the conclusion $Q(G,\Omega) = Q(F,\Omega)$ follows from the results in the special cases. On the other hand, if G has no primitive accidental parabolic transformations, then clearly F has no accidental parabolics and either F and G are both quasi-Fuchsian or both totally degenerate. The conclusion $Q(G,\Omega) = Q(F,\Omega)$ now follows again from the results in the special cases. It can also be deduced from Lemma 9 that if F has no accidental parabolics, then neither does G. Thus, either F and G are both quasi-Fuchsian or both totally degenerate and the conclusion $Q(G,\Omega) = Q(F,\Omega)$ follows again from the special cases discussed earlier.

Now, suppose that F has only one (primitive) accidental parabolic, say f. Then, G has only one (primitive) accidental parabolic which is necessarily equal to f by using the lemma above. Let $L_f$ and $L'_f$ be the simple loops on $\Delta/F$ and $\Delta/G$ respectively that correspond to f. As we observed in the proof of the lemma above, $L_f$ either satisfies property (i) or (ii) and $L'_f$ either satisfies property (i)$'$ or (ii)$''$ and that if $L_f$ satisfies (i) (or (ii)), then $L'_f$ must satisfy (i)′ (or (ii)′).

If $L_f$ and $L'_f$ satisfy (i) and (i)′ respectively, then F represents two surfaces, one of signature (2,0) and the other of signature (1,2;∞,∞) while G represents two surfaces, one of signature (0,6;2,2,2,2,2,2) and the other of signature (0,5;2,2,2,2,∞). (see Figure 3.2) We note that (0,3;2,2,∞) is a euclidean signature. By simply summing up the dimensions of the resulting spaces(subspaces) of quadratic differentials associated with these component surfaces, we get that Q(F,Ω) = Q(G,Ω).

If $L_f$ and $L'_f$ satisfy (ii) and (ii)′ respectively, then F represents three surfaces, one of signature (2,0) and the other two are both of signature (1,1;∞) while G represents three surfaces, one of signature (0,6;2,2,2,2,2,2) and the other two are both of signature (0,4;2,2,2,∞). (see Figure 3.2) A dimension computation again yields that Q(F,Ω) = Q(G,Ω).

We, now, suppose that F ( and thus, also G by Lemma 9) has only two (primitive) accidental parabolics, say $f_1$ and $f_2$. Let $L_{f_1}$ and $L_{f_2}$ be the two disjoint homotopically independent simple loops on Δ/F that correspond to $f_1$ and $f_2$ respectively. Denote the disjoint simple loops on Δ/G by $L'_{f_1}$ and $L'_{f_2}$ that correspond to $f_1$ and $f_2$ respectively. We find that there are two possibilities for $L_{f_1}$ and $L_{f_2}$. Either $L_{f_1}$ and $L_{f_2}$ are both non-dividing or one of them, say $L_{f_2}$ is dividing. Note that they cannot be both dividing as they are non-homotopic.

If $L_{f_1}$ and $L_{f_2}$ are both non-dividing and thus satisfy (i) above, then we find by using our previous arguments that $L'_{f_1}$ and $L'_{f_2}$ must both satisfy (i)′. Thus, F represents two surfaces, one of signature (2,0) and the other of

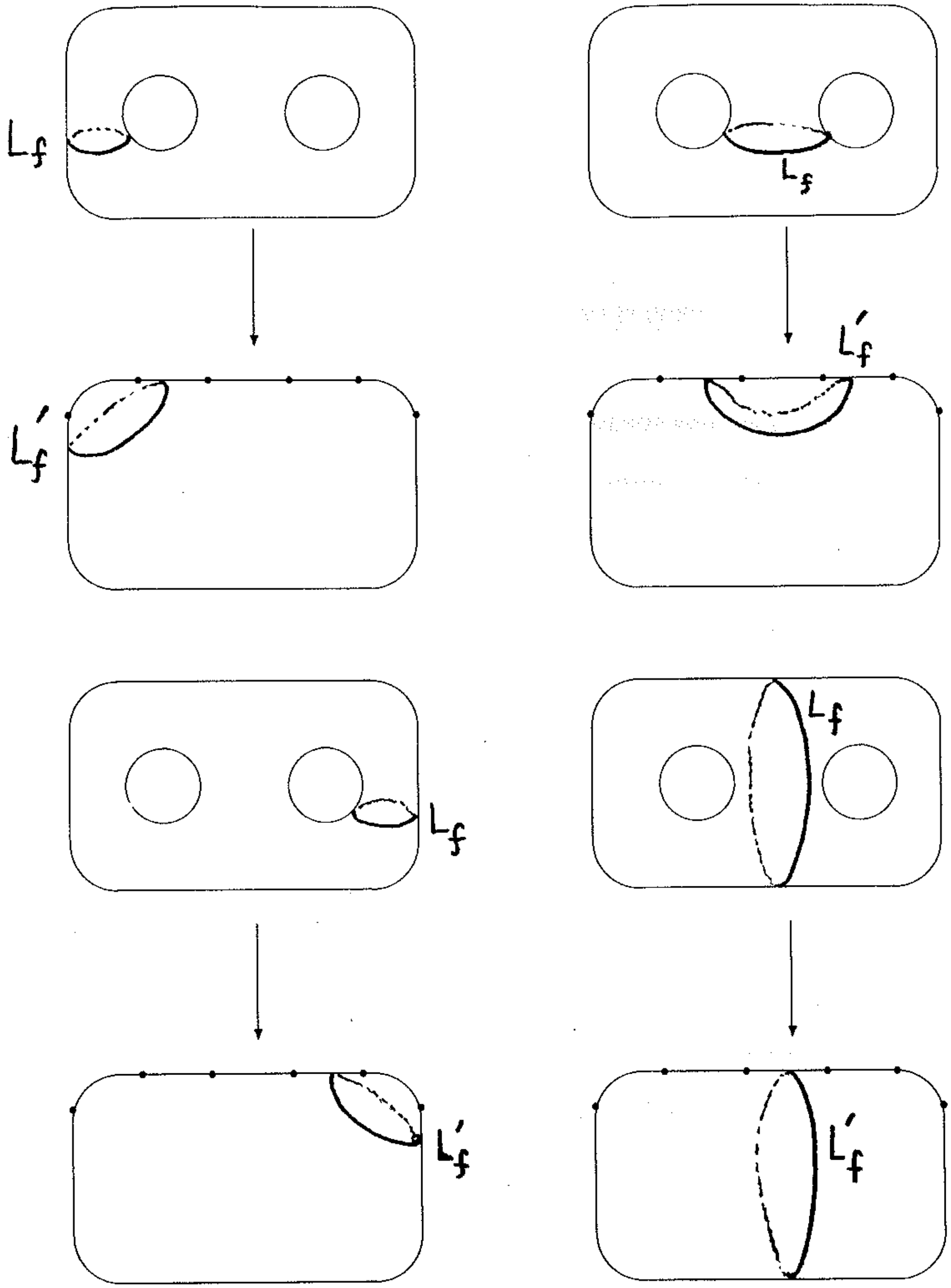


Figure 3.2: Decomposition I

signature (0,4;$\infty$,$\infty$,$\infty$,$\infty$) while G represents two surfaces, one of signature (0,6;2,2,2,2,2,2) and the other of signature (0,4;2,2,$\infty$,$\infty$). (see Figure 3.3) We note again that (0,3;2,2,$\infty$) is euclidean. Hence, by considering the sum of the dimensions of the resulting spaces(subspaces) of quadratic differentials, we get that Q(F,$\Omega$) = Q(G,$\Omega$).

The other case is when $L_{f_1}$ is non-dividing and satisfies (i) and $L_{f_2}$ is dividing and satisfies (ii). We find that $L'_{f_1}$ must satisfy (i)$'$ while $L'_{f_2}$ must satisfy (ii)$'$. Since the space of quadratic differentials for a thrice-punctured sphere is 0-dimensional, we will only give those component surfaces of F which are not thrice-punctured spheres. Hence, F represents two surfaces, one of signature (2,0) and the other of signature (1,1;$\infty$) while G represents two surfaces, one of signature (0,6;2,2,2,2,2,2) and the other of signature (0,4;2,2,2,$\infty$). (see Figure 3.3) By using a dimension argument once again, we get Q(F,$\Omega$) = Q(G,$\Omega$). This completes our analysis and proof of case I for the pair of signatures (2,0) and (0,6;2,2,2,2,2,2).

**Case II.**: Consider the case of the pair (1,2;m,m) and (0,5;2,2,2,2,m). Observe that the maximum number of homotopically independent loops on $\Delta$/F and $\Delta$/G that correspond to primitive accidental parabolic transformations is 2. This implies that if F(or G) has the maximum number of primitive accidental parabolics, then (F,$\Delta$) and (G,$\Delta$) are both terminal(regular) b-groups and the conclusion Q(G,$\Omega$) = Q(F,$\Omega$) follows from the results in the special cases.

If Theorem 4 is generalized to the case of M having a signature (1,2;m,m), then we observe that we can state and prove a modified but very similar version

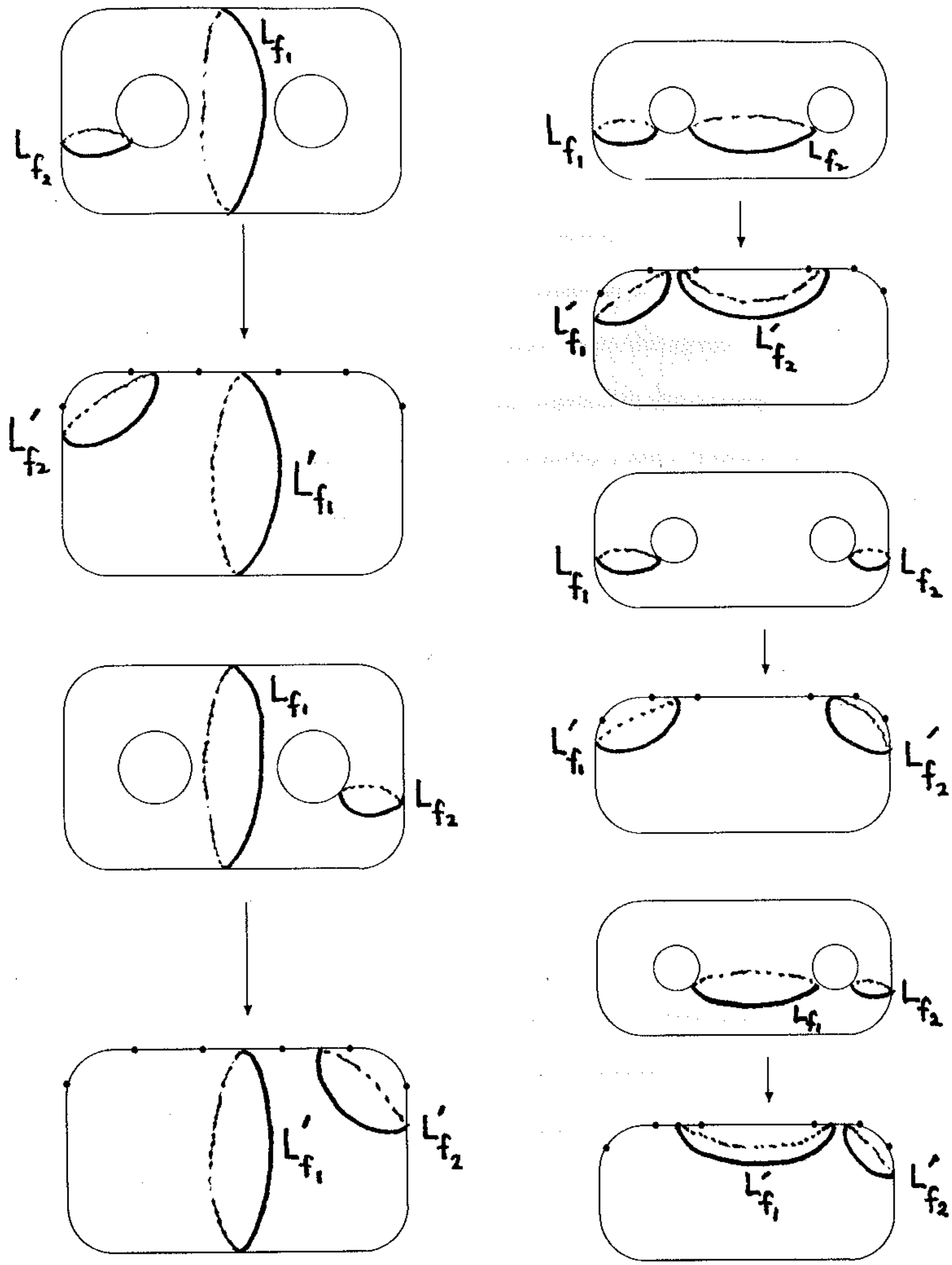


Figure 3.3: Decomposition II

of Lemma 9 for case II to show that F and G must have the same number of inequivalent maximal cyclic subgroups of accidental parabolics. Thus, if F (or G) has no primitive accidental parabolic transformations, then either F and G are both quasi-Fuchsian or both totally degenerate. The conclusion $Q(G,\Omega) = Q(F,\Omega)$ now follows again from the results in the special cases.

To proceed, we observe that we only have this one situation to consider: F has exactly one (primitive) accidental parabolic, say f which is necessarily also the only (primitive) accidental parabolic for G. Let $L_f$ and $L'_f$ be the simple loops on $\Delta/F$ and $\Delta/G$ respectively that correspond to f. On $\Delta/F$, there are only two possibilities for $L_f$:

(i) $L_f$ is non-dividing, or

(ii) $L_f$ is dividing

If $L_f$ is non-dividing, then we assume, as before, that $L_f$ is the projection of the axis of f that do not contain any of the fixed points of the half-turn induced by the hyperelliptic involution. Now, consider $L'_f$ on $\Delta/G$. We again note that $(0,2;2,\infty)$ never occurs. Moreover, $(0,2;m,\infty)$ is also non-geometric, unless $m = \infty$. Let $P'$ be the distinguished point with order m on $\Delta/G$. We have three possibilities but only two are non-trivial:

(i)′ $L'_f$ separates two distinguished points on $\Delta/G$, none of which is $P'$, from the other three, or

(ii)′ $L'_f$ separates two distinguished points on $\Delta/G$, one of which is $P'$, from the other three, or

(iii)′ If $m = \infty$ so that $\Delta/G$ has signature $(0,5;2,2,2,2,\infty)$, $L'_f$ separates $P'$ from the other four distinguished points.

For (iii)$'$, we get that $L'_f$ must correspond to a puncture, namely $P'$, on $\Delta/G$. This is a contradiction since f is an accidental parabolic in G. Hence, we only have (i)$'$ and (ii)$'$.

We now refer to the degree 2 covering, say p, of $\Delta/F$ onto $\Delta/G$ induced by the hyperelliptic involution acting on $\Delta/F$. By using the generalized Theorem 4 for case II, we can conclude that if $L_f$ satisfies (i) (or (ii)), then $L'_f$ must satisfy (i)$'$ (or (ii)$'$).

If $L_f$ and $L'_f$ satisfy (i) and (i)$'$ respectively, then (see Figure 3.4) F represents two surfaces, one of signature (1,2;m,m) and the other of signature (0,4;m,m,$\infty$,$\infty$) while G represents two surfaces, one of signature (0,5;2,2,2,2,m) and the other of signature (0,4;2,2,m,$\infty$). (The signature (0,3;2,2,$\infty$) is euclidean.) By simply summing up the dimensions of the resulting spaces(subspaces) of quadratic differentials associated with these component surfaces, we get that Q(F,$\Omega$) = Q(G,$\Omega$).

If $L_f$ and $L'_f$ satisfy (ii) and (ii)$'$ respectively, then we again make the remark that the space of quadratic differentials for thrice-punctured spheres is 0-dimensional and thus, can be ignored. Hence, F represents two surfaces, one of signature (1,2;m,m) and the other of signature (1,1;$\infty$) while G represents two surfaces, one of signature (0,5;2,2,2,2,m) and the other of signature (0,4;2,2,2,$\infty$). (see Figure 3.4) Clearly, we also get Q(F,$\Omega$) = Q(G,$\Omega$). This completes case II.

**Case III.:** Consider the case of the pair (1,1;m) and (0,4;2,2,2,2m). Observe that the maximum number of homotopically independent loops on $\Delta/F$ and $\Delta/G$ that correspond to primitive accidental parabolic transformations is

1. This implies that if F(or G) has the maximum number of primitive accidental parabolics, then (F,$\Delta$) and (G,$\Delta$) are both terminal(regular) b-groups and the conclusion Q(G,$\Omega$) = Q(F,$\Omega$) follows from the results in the special cases.

**Remark:** Let f be a (primitive) accidental parabolic in F and thus, in G. Let $L_f$ and $L'_f$ be the simple loops on $\Delta$/F and $\Delta$/G respectively that correspond to f. We notice that $L_f$ cannot be dividing for otherwise, we will get a non-geometric signature (0,2;m,$\infty$) if m $\neq \infty$ or we will have $L_f$ corresponding to a puncture on $\Delta$/F if m = $\infty$. Thus, $L_f$ must be non-dividing and we get that $L'_f$ must separate two distinguished points from the other two. (see Figure 3.5) In particular, $L'_f$ must bound a disc that has exactly two distinguished points of order 2. We can see this from a version of Theorem 4, modified for case III.

We also get a modified but very similar version of Lemma 9 which shows that F and G must have the same number of inequivalent maximal cyclic subgroups of accidental parabolics. Hence, if F(or G) has no primitive accidental parabolic transformations, then either F and G are both quasi-Fuchsian or both totally degenerate. The conclusion Q(G,$\Omega$) = Q(F,$\Omega$) now follows again from the results in the special cases. This proves case III.

**Case IV.:** Consider the case of the pair (0,4;m,m,n,n) and (0,4;2,2,m,n) where the index is two. Observe that the maximum number of homotopically independent loops on $\Delta$/F and $\Delta$/G that correspond to primitive accidental parabolic transformations is 1. This implies that if F(or G) has the maximum number of primitive accidental parabolics, then (F,$\Delta$) and (G,$\Delta$) are both terminal(regular) b-groups and the conclusion $Q(G,\Omega) = Q(F,\Omega)$ follows from the results in the special cases. On the other hand, if F (or G by a corresponding modified version of Lemma 9) has no primitive accidental parabolic transformations, then either F and G are both quasi-Fuchsian or both totally degenerate. The conclusion $Q(G,\Omega) = Q(F,\Omega)$ now follows again from the results in the special cases. Hence, case IV is done and this completes our analysis of the cases when both F and G are b-groups.

The general case of function groups where $\Delta$ is connected but not simply-connected can be treated in a very similar fashion. We first assume that $\Delta/F$ has signature (2,0) and has n dividers which are composed of say, a $(\nu_1)$-divider, ..., $(\nu_n)$-divider, n = 3, 2, 1, 0. We again notice that these simple loops must be the same (up to homotopy equivalence) as the loops we considered in Figures 3.2 - 3.3. The projections of these loops on the surface of signature (0,6;2,2,2,2,2,2) are then easily seen to be exactly similar to those found in the b-group case. We, thus, observe that the only difference in the decomposition of $\Delta/F$ and the corresponding one for $\Delta/G$ between the general case of function groups and the special case of b-groups is that the simple disjoint loops may correspond to elliptic cyclic subgroups and/or parabolic cyclic subgroups. The other pairs of signatures for $\Delta/F$ and $\Delta/G$ with $[G:F] = 2$ can be treated similarly (see Figures 3.4 - 3.6).

To describe the resulting decompositions more efficiently, the notion of graph type, which was defined in the background, will be useful to us. Specifically, the graph types of the Riemann surfaces $\Delta/F$ and $\Delta/G$ will be the focus of the next paragraphs. We observe, though, that since the dimension of the

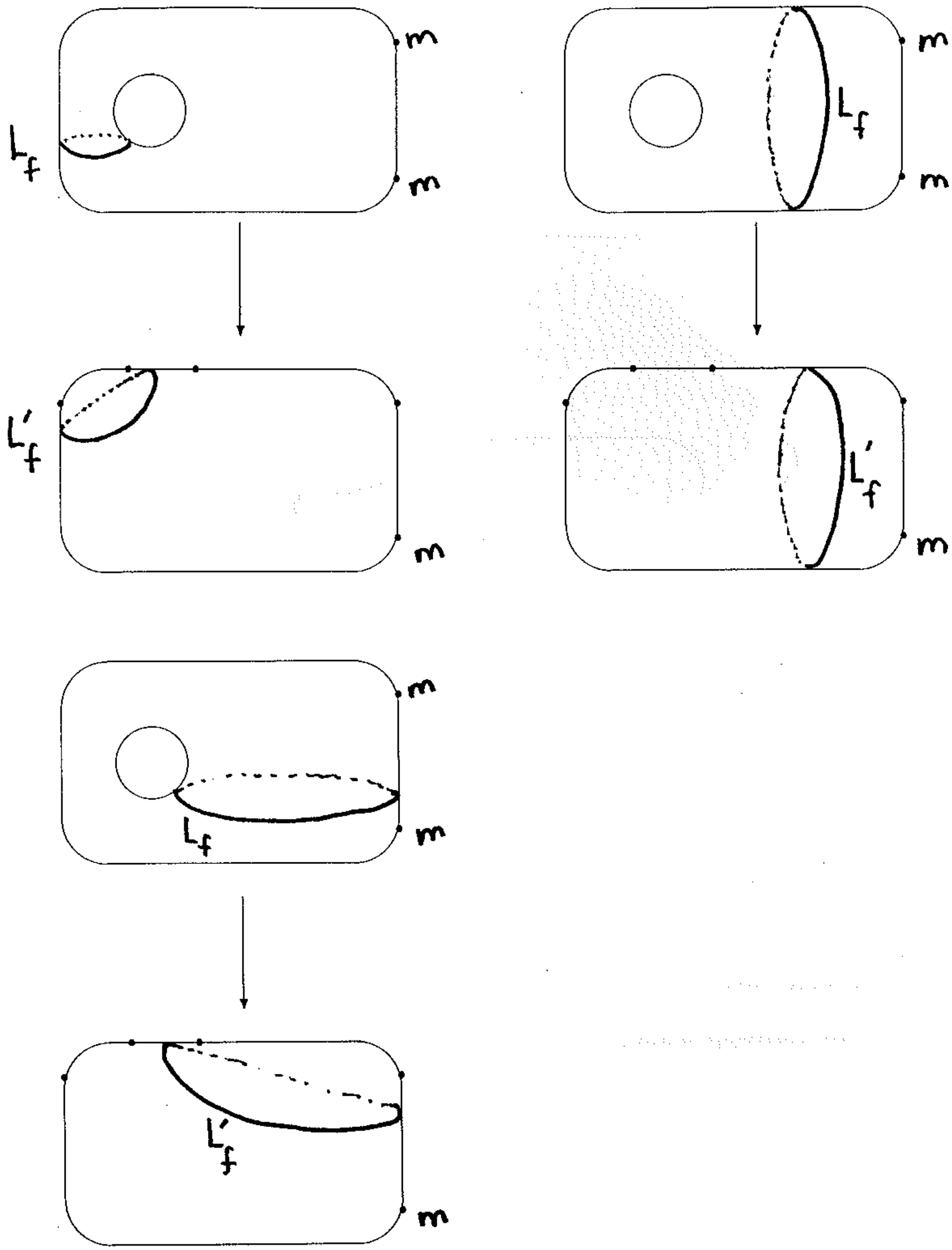


Figure 3.4: Type (1,2)

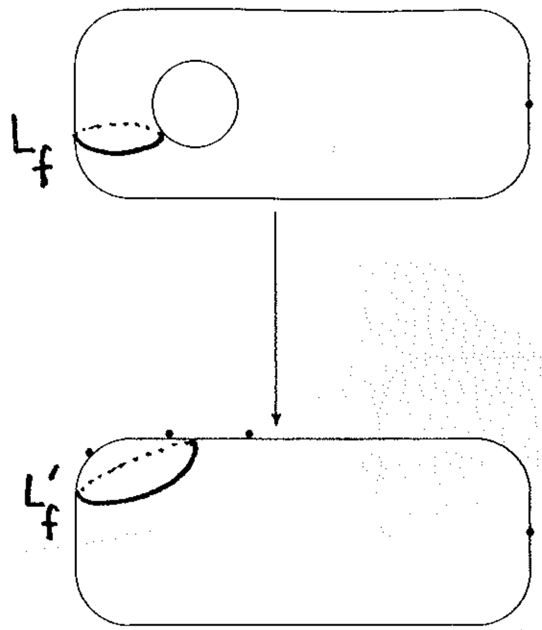


Figure 3.5: Type (1,1)

space of quadratic differentials for a surface S does not depend on its signature but merely on its type, we need only specify the types of the surfaces arising from the possible decompositions of S. Furthermore, we will only be interested in the non-trivial decompositions of S, by which we mean those decompositions that have at least one component surface whose space of quadratic differentials has positive dimension.

**Type** (2,0): The graph of this type may consist of two surfaces (vertices), one of type (1,1) and the other is a thrice-punctured sphere; or two surfaces both of type (1,1); or a single surface, the type of which may be (0,4) or (1,2). Apart from the thrice-punctured spheres, F therefore represents at most two component surfaces. Corresponding to these, the graph of type (0,6) either

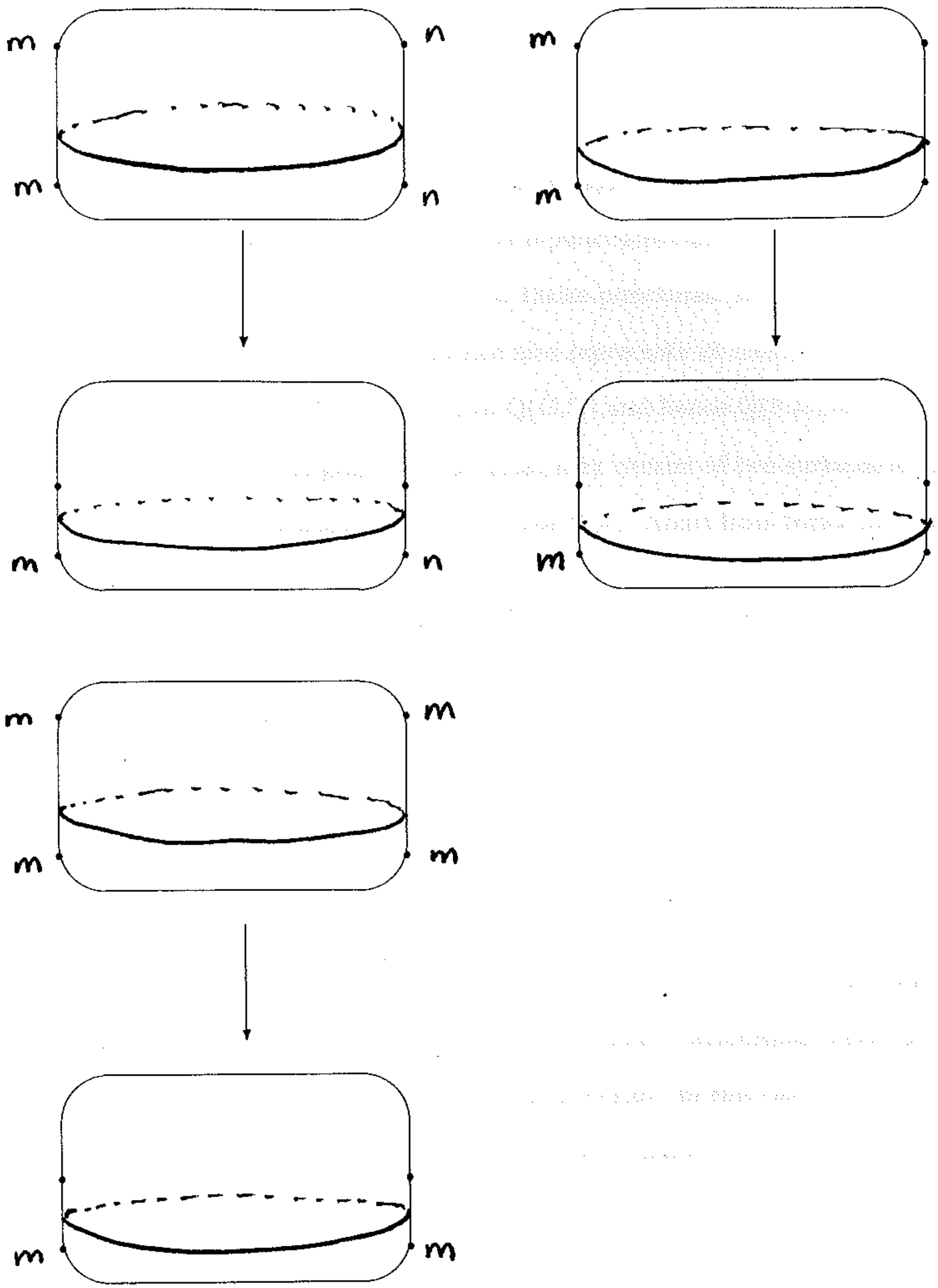


Figure 3.6: Type (0,4)

consists of three surfaces, one of type (0,4) and the other two are thrice-punctured spheres or two surfaces, both of type (0,4); three surfaces, one of type (0,4) and the other two are thrice-punctured spheres; or two surfaces, one of type (0,5) and the other is a thrice-punctured sphere. Apart from thrice-punctured spheres, G therefore also represents at most two component surfaces. Clearly, dim $Q(F,\Omega)$ = dim $Q(G,\Omega)$ and hence $Q(F,\Omega) = Q(G,\Omega)$.

**Type** (1,2): The graph of type (1,2) may consist of two surfaces of types (1,1) and (0,3); or of a single surface of type (0,4). Apart from thrice-punctured spheres, F therefore represents at most one component surface. Corresponding to these, the graphs of type (0,5) either consists of two surfaces of types (0,4) and (0,3); or two surfaces of types (0,4) and (0,3). Apart from thrice-punctured spheres, G therefore represents at most one component surface. We also see that $Q(F,\Omega) = Q(G,\Omega)$ from these pairs of resulting component surfaces.

**Types** (0,4) and (1,1): There are no non-trivial decompositions for these two types. It therefore follows directly that $Q(F,\Omega) = Q(G,\Omega)$.

**Remark:** Observe however in Type (2,0) that F may also be an amalgamated (free) product of elementary groups, namely, loxodromic cyclic groups or rank 2 parabolic groups, both of signature (1,0). In this case, G is an amalgamated product of elementary groups as well, namely, the $\mathbb{Z}_2$-extensions of the loxodromic cyclic groups or the rank 2 parabolic groups which are both of signature (0,4;2,2,2,2). Since all these elementary groups have euclidean signatures, $\Omega/F$ represents only one surface of signature (2,0) and $\Omega/G$ also represents only one surface of signature (0,6;2,2,2,2,2,2). Therefore, these particular decompositions still imply that $Q(F,\Omega) = Q(G,\Omega)$. The pair of groups

F and G where $\Delta/F$ having signature (1,2;m,m) and $\Delta/G$ having signature (0,5;2,2,2,2,n) can also be built up from elementary groups. In particular, F can be an amalgamated (free) product of a loxodromic or a rank 2 parabolic group and a finite cyclic group whose signature is (0,2;m,m). Correspondigly, G is an amalgamated product of the $\mathbb{Z}_2$-extensions of these groups where we note that the $\mathbb{Z}_2$-extension of a finite cyclic group of signature (0,2;m,m) is a Dihedral m-group of signature (0,3;m,2,2). Obviously, we will still have $Q(F,\Omega) = Q(G,\Omega)$.

### 3.2.3 The Index 4 Case

Consider now the case of $[G{:}F] = 4$. Then, G/F is either cyclic or of rank 2, that is, $G/F \cong \mathbb{Z}_4$ or $G/F \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the Klein-4 group). In either case however, G - F has an element $\gamma$ such that $\gamma^2 \in F$. We denote by K, the $\mathbb{Z}_2$-extension $<F,\gamma>$ of F by $\gamma$. Clearly, $[K{:}F] = 2 = [G{:}K]$. Obviously, K is a function group in this case with invariant component equal to $\Delta$, the common invariant component of F and G. This is true since the group indices we have here are all finite (=2). Moreover $\Omega(K) = \Omega$, the common ordinary set of F and G.

Pass to the Fuchsian equivalents, say $F_0$, $K_0$ and $G_0$, of F, K and G respectively. We find that $F_0$ must have signature (0,4;m,m,m,m) and $G_0$ must have signature (0,4;2,2,2,m). Hence, by the results in the Fuchsian case, $F_0$ and $G_0$ is an improper inclusion pair. Observe that $[G_0{:}K_0] = [K_0{:}F_0] = 2$. We now use this easy corollary.

**Corollary 4** *If F and G is an improper inclusion pair, then for any intermediate group K where $F \prec K \prec G$, the pairs F and K as well as K and G are also improper inclusion pairs.*

**Proof:** Clearly, we get $\dim Q(G,\Omega) \leq \dim Q(K,\Omega) \leq \dim Q(F,\Omega)$. Since $Q(F,\Omega) = Q(G,\Omega)$, we get $\dim Q(F,\Omega) = \dim Q(K,\Omega) = \dim Q(G,\Omega)$ and $Q(F,\Omega) = Q(K,\Omega) = Q(G,\Omega)$. □

By the corollary above, the pairs $F_0$, $K_0$ and $K_0$,$G_0$ are therefore improper inclusion pairs. Since we know the signatures of $F_0$ and $G_0$, we simply refer to Table 1 in order to identify the signature of $K_0$. Clearly, it must be (0,4;2,2,m,m). Hence, $\Delta/F$, $\Delta/K$ and $\Delta/G$ are of signatures (0,4;m,m,m,m), (0,4;2,2,m,m) and (0,4;2,2,2,m) respectively. We notice that these reduce to the index two cases of type (0,4) for the two pairs F, K and K, G. Hence $Q(F,\Omega) = Q(K,\Omega)$ and $Q(K,\Omega) = Q(G,\Omega)$ which yields $Q(F,\Omega) = Q(G,\Omega)$. This completes the proof of Proposition 5. □

**Remark:** Notice that $G/F \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the Klein-4 group) since this is the case for their Fuchsian equivalents. $G_0/F_0$ acts on $U/F_0$ of signature (0,4;m,m,m,m) to give rise to $U/G_0$ which is of signature (0,4;2,2,2,m) (see Figure 3.4)

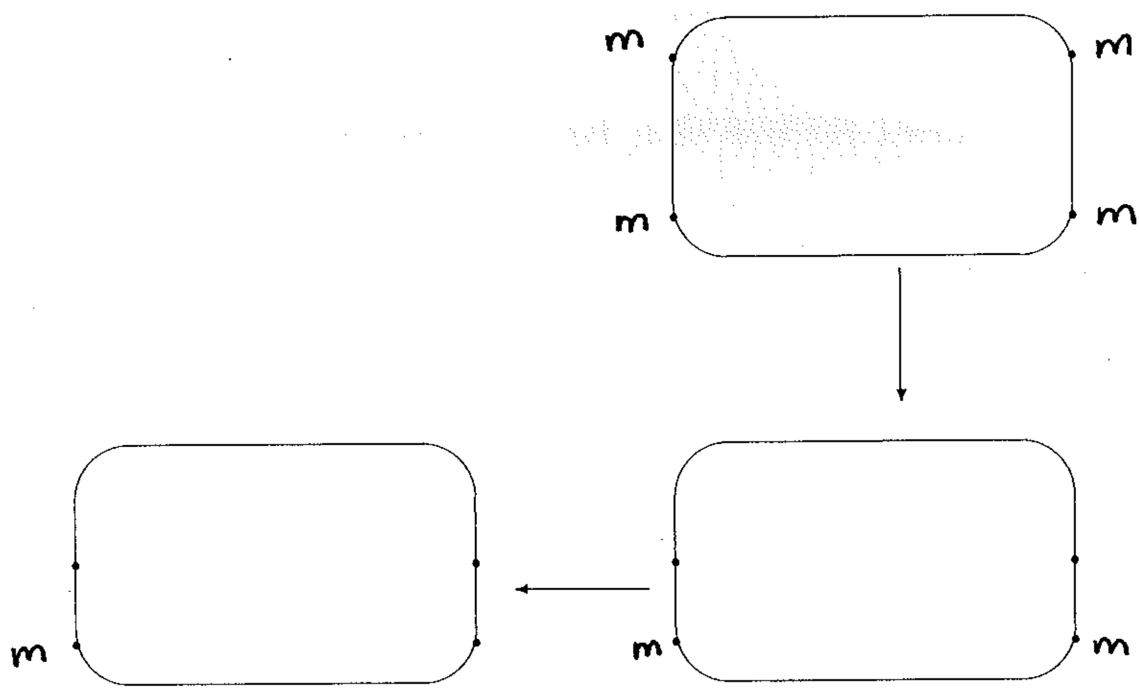


Figure 3.7: Index 4 action

# Chapter 4

# Deformation Spaces of Improper Inclusion Pairs

## 4.1 Ahlfors-Bers Theory and Teichmuller Theory

If G is a Kleinian group, a Beltrami coefficient $\mu$ on $\mathbb{C}$ is called a Beltrami coefficient on G provided that $\mu(g(z))\cdot\frac{\overline{g'(z)}}{g'(z)} = \mu(z)$, for every $g \in G$ and $\mu|_{\Lambda(G)}$ $= 0$. Denote by $w^\mu$ the unique automorphism $z \mapsto w^\mu(z)$ of $\hat{\mathbb{C}}$ from the theory of Ahlfors and Bers [**AB**] which leaves 0, 1, and $\infty$ pointwise fixed and is $\mu$-conformal, that is, $w^\mu$ satisfies the Beltrami equation : $w^\mu{}_{\bar{z}} = \mu w^\mu{}_z$. Through a direct calculation, for every $g \in G$ the function $w^\mu \circ g$ is again a $\mu$-conformal automorphism of $\hat{\mathbb{C}}$. It follows from the same theory that $w^\mu$ and $w^\mu \circ g$ are related via the equation $w^\mu \circ g = g_1 \circ w^\mu$ for some Moebius transformation $g_1$. It can easily be shown that the collection of all such $g_1$'s, which is denoted by $G^\mu = w^\mu G(w^\mu)^{-1}$, form a group. In fact, it is a Kleinian group with ordinary set $\Omega(G^\mu) = w^\mu(\Omega(G))$. The mapping that sends $g \mapsto g_1$ is called a quasiconformal isomorphism defined by $\mu$ or a $\mu$-conformal deformation. It is clear that $w^\mu:\Omega(G)\longmapsto\Omega(G^\mu)$ induces quasiconformal mappings of the components of $\Omega(G)$ onto the corresponding components of $\Omega(G^\mu)$. A general principle in the classical uniformization theorem implies that we also get quasi-conformal mappings between the associated pairs of component surfaces from G and $G^\mu$.

We will assume from now on that G is non-elementary and finitely generated. If $\mu$ and $\nu$ are Beltrami coefficients on G, then $\mu$ is said to be equivalent to $\nu$, $\mu \simeq \nu$, if and only if $w^\mu \circ (w^\nu)^{-1}|_{\Lambda(G)}$ is Moebius. The set of Beltrami coefficients on G modulo the above equivalence relation is, by definition, $\mathbb{T}(G)$, the deformation space of G. Alternatively, if we implicitly assume without loss of generality that 0, 1, and $\infty \in \Lambda(G)$, $\mathbb{T}(G)$ may also be defined as the set of normalized[1] quasi-conformal self-maps w of $\hat{\mathbb{C}}$ that is G-compatible (i.e. $wGw^{-1}$ is Kleinian) modulo the equivalence relation $w_1 \simeq w_2$ (we sometimes say that $w_1$ is G-equivalent to $w_2$) if and only if $w_1 \circ (w_2)^{-1}|_{\Lambda(G)} = \mathbf{1}$. Kra-Maskit have shown by using stratifications that $\mathbb{T}(G)$ is biholomorphically equivalent to a domain in $\mathbb{C}^d$ for some d. In addition, $\mathbb{T}(G)$ is, in fact, a domain of holomorphy **[KMa]**.

Moreover earlier works of Bers, Maskit and Kra **[B3]**, **[KMa]** or **[M4]** established that the holomorphic universal covering of $\mathbb{T}(G)$ is the product of Teichmuller spaces of the Fuchsian groups belonging to the Fuchsian model for G. We can deduce from this that if G is, in particular, a b-group, then $\mathbb{T}(G)$ is (isomorphic to) its own holomorphic universal covering. We capitalize on this and the results in Proposition 1 in proving the proposition that immediately

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[1]by normalized, we mean that $w(0) = 0$, $w(1) = 1$ and $w(\infty) = \infty$

follows. Recall that in Proposition 1, the spaces Q(F,$\Omega$) and Q(G,$\Omega$) of an improper inclusion pair F and G naturally decomposes into spaces of quadratic differentials for their component subgroups and these spaces are identical in pairs, $Q(\hat{F}_i,\Omega_i) = Q(\hat{G}_i,\Omega_i)$.

**Proposition 6** *If F and G are non-elementary, finitely generated b-groups which is an improper inclusion pair, then* $\mathbb{T}(F) = \mathbb{T}(G)$, *where* $\mathbb{T}(F)$ *and* $\mathbb{T}(G)$ *are the deformation spaces of F and G respectively.*

**Proof:** Without loss of generality, assume once again as we have done before, that F and G have the same number of component subgroups. When applied to b-groups, the result by Bers, Maskit and Kra mentioned above will imply that

$\mathbb{T}(F) \cong T(F_0,U) \times T(F_1,U) \times \ldots \times T(F_S,U)$ where $\{F_0,F_1,\ldots,F_S\}$ is the Fuchsian model for F and $\cong$ denotes biholomorphic equivalence, and similarly for $\mathbb{T}(G)$.

Recall that the signatures of $U/F_i$ and $\Omega_i/\hat{F}_i$ are the same and both are Riemann surfaces of finite type. Hence, $\dim Q(\hat{F}_i,\Omega_i) = \dim Q(F_i,U) = \dim T(F_i,U)$. The first equality is from the isomorphism between the two spaces of quadratic differentials. The second equality is from the assumption that F is finitely generated as we noted in a remark in Chapter 2. This is a basic result from the theory of finite-dimensional Teichmuller spaces. From Proposition 1 and the results of Greenberg and others, we can therefore conclude that $T(F_i,U) = T(G_i,U)$ for i = 0, 1, 2,..., S. This implies that both $\mathbb{T}(F)$ and $\mathbb{T}(G)$ are biholomorphically equivalent to the same product of Teichmuller spaces and are therefore biholomorphically equivalent, $\mathbb{T}(F) \cong \mathbb{T}(G)$. Since $\mathbb{T}(F)$ contains $\mathbb{T}(G)$, the only possibility is that $\mathbb{T}(F) = \mathbb{T}(G)$. □

As an immediate consequence to the above Proposition, we have by virtue of Proposition 5 the following easy corollary

**Corollary 5** *If (F,Δ) and (G,Δ) are non-elementary, finitely generated b-groups with* $F \lhd G$ *and* $Q(F,\Delta) = Q(G,\Delta)$*, then* $\mathbb{T}(F) = \mathbb{T}(G)$*.*

For the general case of Kleinian groups in Proposition 6, we only need to show the inclusion $\mathbb{T}(F) \hookrightarrow \mathbb{T}(G)$ since the other inclusion holds already. This is clearly immediate from our hypothesis.

First, we need to recall some important concepts in Teichmuller theory. Let F be a Fuchsian group. A quasi-conformal map $W: S \longmapsto S'$ between Riemann surfaces of finite type, necessarily of the same signature, is called a Teichmuller mapping if its Beltrami coefficient is of the form $W_{\bar{z}}/W_z = \mu = k.\frac{|\phi|}{\phi}$. This is sometimes called the (initial) Teichmuller differential on S with $k = ||\mu||_\infty \in (0,1)$ and $\phi \in Q(S)$, the space of quadratic differentials on S. Now, let $\tilde{f}$ be an F-compatible quasi-conformal automorphism of U and $k(\tilde{f}) = ||\frac{\tilde{f}_{\bar{z}}}{\tilde{f}_z}||_\infty$. The number $K = \frac{1+k(\tilde{f})}{1-k(\tilde{f})}$ is called the dilatation of $\tilde{f}$ and $\tilde{f}$ is called F-extremal if and only if this number satisfies: $K \leq K(f)$ for all quasi-conformal F-compatible automorphisms f of U that is F-equivalent to $\tilde{f}$. The function $\tilde{f}$ induces a quasi-conformal mapping between orbifolds $W_{\tilde{f}}: U/F \longrightarrow U/\tilde{f}F\tilde{f}^{-1}$. In fact, every mapping such as W above is induced this way. $\tilde{f}$ will be called a Teichmuller mapping if $W_{\tilde{f}}$ is. In such a case, the Beltrami coefficient of $\tilde{f}$ also satisfies the analogous Teichmuller condition: $\mu(\tilde{f}) = k(\tilde{f}).\frac{|\tilde{\phi}|}{\tilde{\phi}}$, $\tilde{\phi} \in Q(F,U)$.

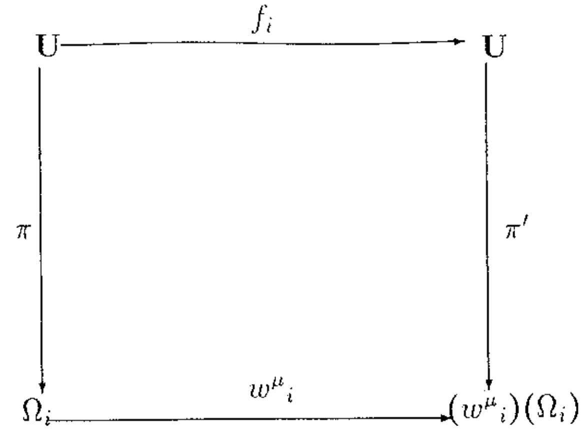


Figure 4.1: <diagram 1>

We can now prove the generalized version of Proposition 6.

**Proposition 7** *If F and G is an improper inclusion pair, then* $\mathbb{T}(F) = \mathbb{T}(G)$.

**Proof:** Let $\{F_1, ..., F_S\}$ and $\{G_1, ..., G_S\}$ be the Fuchsian models of F and G respectively where the extra elements of the Fuchsian model of F have been discarded. Recall that these uniformize thrice-punctured spheres. Clearly, $F_i$ and $G_i$ must necessarily be finitely generated Fuchsian groups of the first kind for every i = 1, ..., S. We also use the same notations as in Proposition 1 and consider $\Omega_i$, i = 1, 2, ..., S. This is a maximal collection of disjoint components of $\Omega$, inequivalent with respect to G and to F. Let $[\mu] \in$ $\mathbb{T}$(F).

By the remarks above in the beginning of this chapter, $w^{\mu}$ induces $\hat{F}_i$-compatible quasi-conformal mappings, $w^{\mu}{}_i$, of $\Omega_i$ onto $w^{\mu}{}_i(\Omega_i)$, i = 1, 2, ..., S, where the $\hat{F}_i$'s are the component subgroups of F relative to the $\Omega_i$'s. We observe that U is the universal covering space of both $\Omega_i$ and $w^{\mu}{}_i(\Omega_i)$ for i = 1, 2, ... ,S. We can thus lift $w^{\mu}{}_i$ to an automorphism of U such that we have a commutative diagram (see diagram 1). We observe that this lift must be

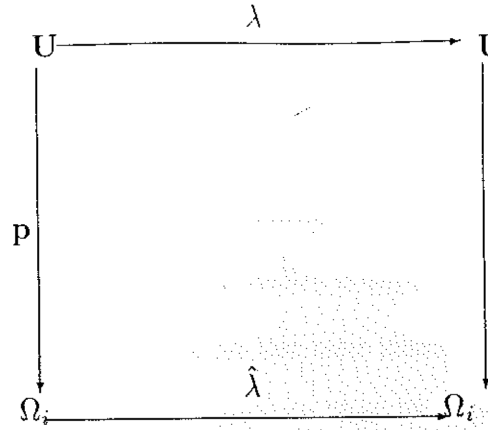


Figure 4.2: <diagram 2>

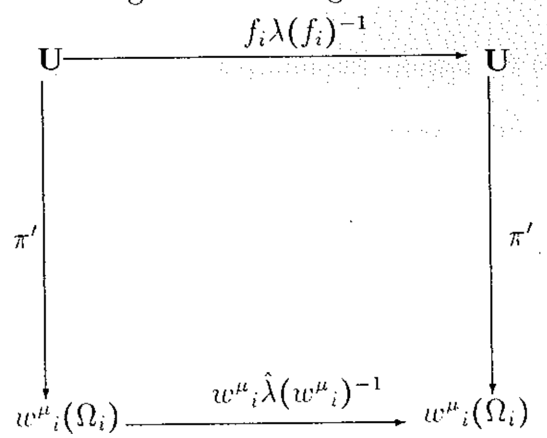


Figure 4.3: <diagram 3>

quasi-conformal. This follows from the earlier remarks by noting that $\Omega_i$ and ${w^\mu}_i(\Omega_i)$, which are open subsets of $\hat{\mathbb{C}}$, are Riemann surfaces. Let us call this lift $f_i$. We now show that $f_i$ is $F_i$-compatible. Let $\lambda \in F_i$. Now, $\lambda$ corresponds to some element $\hat{\lambda} \in \hat{F}_i$ such that diagram 2 commutes. Now, observe that ${w_i}^\mu\hat{\lambda}({w_i}^\mu)^{-1}$ lifts precisely to the map $f_i\lambda {f_i}^{-1}$. Furthermore, diagram 3 is a commutative diagram since diagram 2 is. Recall that ${w_i}^\mu$ is $\hat{F}_i$-compatible. Therefore, we have that ${w_i}^\mu\hat{\lambda}({w_i}^\mu)^{-1}$ is a Moebius transformation. Hence, $f_i\lambda {f_i}^{-1} \in$ PSL(2,$\mathbb{R}$) which shows that $f_i$ is $F_i$-compatible.

Teichmuller's Theorem implies that $f_i$ is $F_i$-equivalent to a unique $F_i$-

compatible and $F_i$-extremal quasi-conformal automorphism of U. Let us denote it by $\tilde{f}_i$. Teichmuller's Theorem further implies that $\tilde{f}_i$ is either a (real) Moebius transformation or is a Teichmuller mapping. However, one can easily check that the former corresponds to the case where $\mu = 0$ (trivial case). So, without loss of generality, we will assume that the latter holds. Hence if we denote the Beltrami coefficient of $\tilde{f}_i$ by $\tilde{\mu}_i$, then $\tilde{\mu}_i = \frac{K(\tilde{f}_i)-1}{K(\tilde{f}_i)+1} \cdot \frac{|\tilde{\phi}_i|}{\tilde{\phi}_i}$ for some $\tilde{\phi}_i$ $\in$ Q($F_i$,U).

Now, let $\{\hat{G}_1, ..., \hat{G}_S\}$ be the component subgroups of G relative to the $G_i$'s. Since Q($\hat{F}_i$,$\Omega_i$) = Q($\hat{G}_i$,$\Omega_i$), we can use a proof similar to that of Lemma 4 to get Q($F_i$,U) = Q($G_i$,U) for every i = 1, ..., S. Thus, $\tilde{\phi}_i \in$ Q($G_i$,U) and $\tilde{f}_i$ as well as $f_i$ are $G_i$-compatible. By chasing diagram 3 in the direction which is opposite to that of the previous one above, we conclude that $w^{\mu}{}_i$ is $\hat{G}_i$-compatible for all i = 1, ... S. Since the $w^{\mu}{}_i$'s are induced by $w^{\mu}$, we now claim that $w^{\mu}$ is G-compatible. To show this, we observe that:

(i) $w^{\mu}\hat{G}_i(w^{\mu})^{-1} = w^{\mu}{}_i\hat{G}_i(w^{\mu}{}_i)^{-1}$ for each i = 1, ..., S

(ii) $\bigcup_{i=1}^{S} w^{\mu}(\Omega_i) = \bigcup_{i=1}^{S} w^{\mu}{}_i(\Omega_i)$ is a maximal disjoint $w^{\mu}G(w^{\mu})^{-1}$ - invariant union of components of $w^{\mu}(\Omega)$

(iii) $w^{\mu}{}_i\hat{G}_i(w^{\mu}{}_i)^{-1}$ is a Kleinian group which acts invariantly and (properly) discontinuously on $w^{\mu}{}_i(\Omega_i)$ for each i = 1, ..., S

(iv) $<w^{\mu}{}_1\hat{G}_1(w^{\mu}{}_1)^{-1}, ..., w^{\mu}{}_S\hat{G}_S(w^{\mu}{}_S)^{-1}> = w^{\mu}G(w^{\mu})^{-1}$

(v) $w^{\mu}G(w^{\mu})^{-1}$ acts invariantly and (properly) discontinuously on $w^{\mu}(\Omega)$ which is clearly nonempty since $\Omega \neq \emptyset$

Therefore, $w^{\mu}G(w^{\mu})^{-1}$ is a Kleinian group and the claim is shown. Hence, $\mu$ is a Beltrami coefficient on G and $[\mu] \in$ T(G), from which we get that

$\mathbb{T}(F) \hookrightarrow \mathbb{T}(G)$. By our first few remarks in this chapter, this is sufficient to conclude that $\mathbb{T}(F) = \mathbb{T}(G)$. □

## 4.2 Kleinian Modular Groups

We motivate here the definition of the modular group of a Kleinian group by recalling some facts from the Fuchsian case. Given a finitely generated Fuchsian group F. We construct a set. $\Gamma(F)$. for F defined as follows: Let $\gamma: F \longmapsto PSL(2,\mathbb{R})$ be an isomorphism-into. Assume that $\gamma(F)$ is again a Fuchsian group and $\gamma$ is type-preserving and orientation-preserving. The collection of all such $\gamma$'s is what constitute the set $\Gamma(F)$.

On $\Gamma(F)$, we define a relation which is clearly an equivalence relation. Two elements $\gamma_1$, $\gamma_2$ of $\Gamma(F)$ will be called equivalent, $\gamma_1 \simeq \gamma_2$, if and only if there exists $\lambda \in PSL(2,\mathbb{R})$ such that $\gamma_1(f) = \lambda\gamma_2(f)\lambda^{-1}$ for all $f \in F$. The quotient space, $\Gamma(F)/\simeq$, is the Teichmuller space of F which we denote by T(F). It is well-known that there are other formulations for the definition of T(F). We choose to adopt this particular one for our motivations.

Now, if we denote by A(F) the group of all elements $\gamma \in \Gamma(F)$ which are automorphisms of F (i.e. $\gamma(F) = F$) and by I(F) the group of inner automorphisms in A(F), then clearly I(F) is a (normal) subgroup of A(F). The modular group, M(F), of F is defined as the quotient group $M(F) = A(F)/I(F)$. It naturally acts on T(F) by right translation and whenever F is finitely generated, this action is properly discontinuous.

To describe the modular group for a non-elementary, finitely generated

Kleinian group F, we first start with its Fuchsian model $\{F_1, F_2, ..., F_N\}$. We can associate to F a set which is the product $\Gamma(F_1)$ x $\Gamma(F_2)$ x ... x $\Gamma(F_N)$ of the $\Gamma(F_i)$'s whose definitions follow the Fuchsian case definition in a direct fashion. There is a natural equivalence relation on $\Gamma(F_1)$ x $\Gamma(F_2)$ x ... x $\Gamma(F_N)$ via application component-wise of the equivalence relation $\simeq$ defined above. This gives rise to a product of Teichmuller spaces $T(F_1)$ x $T(F_2)$ x ... x $T(F_N)$. Each of the Teichmuller spaces, $T(F_i)$, is homeomorphic to the open unit ball in $\mathbb{C}^{d_i}$ where $d_i = \dim T(F_i)$, and thus is simply-connected. Consequently, so is the above product. As we mentioned in the previous chapters, this is in fact the holomorphic universal cover of $\mathbb{T}$(F).

**Definition 6** *The modular group of F is defined abstractly to be a group, $\mathbb{M}(F)$, of automorphisms of $T(F_1)$ x $T(F_2)$ x ... x $T(F_N)$ such that $\mathbb{T}(F) = (T(F_1)$ x $T(F_2)$ x ... x $T(F_N))/\mathbb{M}(F)$ and $\mathbb{M}(F) \cong \pi_1(\mathbb{T}(F))$.*

**Remark:** $\mathbb{M}$(F) is known to be isomorphic to a product which we will denote by $m(F_1)$ x $m(F_2)$ x ... x $m(F_N)$ where each $m(F_i)$ satisfies $m(F_i) \preceq M(F_i)$, i = 1, 2, ..., N. Hence, the modular group $\mathbb{M}$(F) is a discrete subgroup of $M(F_1)$ x $M(F_2)$ x ... x $M(F_N)$.

Our current objective is to give another criterion in order that F admit a finite extension G such that $\mathbb{T}$(F) = $\mathbb{T}$(G). First, let us go back to the Fuchsian case and define a set $T(F)_{max}$. For a finitely generated and non-elementary Fuchsian group F, the set $T(F)_{max}$ is defined to be the collection of all $[\gamma] \in$ T(F) such that there does not exist another Fuchsian group containing $\gamma$(F) as a subgroup of finite index. Greenberg has shown **[Gr]** that $T(F)_{max}$ is everywhere dense in T(F) or else, it is empty. $T(F)_{max} = \emptyset$ implies that there exists a group G containing F as a subgroup of finite index such that T(G) = T(F).

To extend this to the Kleinian case, we define $\mathbb{T}(F)_{max} = \{ \{[x]\} = \{([x_1],[x_2],...,[x_N])\} \in \mathbb{T}(F) \mid [x_i] \in T(F_i)_{max}$ for all i $\}$ where $\{[x]\}$ denotes the class of $[x] \doteq ([x_1],[x_2],...,[x_N]) \in T(F_1) \times T(F_2) \times ... \times T(F_N)$ in $\mathbb{T}(F)$. Here, it is understood that F is a non-elementary and finitely generated Kleinian group and $\{F_1,F_2,...,F_N\}$ is its Fuchsian model.

**Lemma 10** *$T(F_1)_{max}$ x $T(F_2)_{max}$ x ... x $T(F_N)_{max}$ covers $\mathbb{T}(F)_{max}$.*

**Proof:** This follows from the definition of $\mathbb{T}(F)_{max}$ and the fact that the (holomorphic) universal cover of $\mathbb{T}(F)$ is $T(F_1) \times T(F_2) \times ... \times T(F_N)$. □

Now, we can prove the following proposition.

**Proposition 8** *Let F be a non-elementary, finitely generated Kleinian group. Then, F admits a finite extension with the same deformation space as F if and only if $\mathbb{T}(F)_{max}$ is not everywhere dense in $\mathbb{T}(F)$.*

**Proof:** First, we lift $\mathbb{T}(F)_{max}$ to its cover given by the lemma above. We have exactly two cases to consider which are described below.

__Case1 :__ There exists at least one i $\in \{1, 2, ..., N\}$ such that $T(F_i)_{max} = \emptyset$. By the result of Greenberg, there must exist a group $\overline{F_i}$ which contains $F_i$ with finite index such that $T(\overline{F_i}) = T(F_i)$. In this case, $\mathbb{T}(F)_{max}$ is not everywhere dense in $\mathbb{T}(F)$ following the results of Greenberg and our preceding lemma. We then claim that there is a finite extension, $\overline{F}$, of F such that $\mathbb{T}(F) = \mathbb{T}(\overline{F})$. To see why, first we observe that the Fuchsian model of this extension is clearly $\{F_1,...,\overline{F_i},...,F_N\}$ where each $F_i$ is replaced with $\overline{F_i}$ whenever $T(F_i)_{max} = \emptyset$. It is sufficient to show that the action on $T(F_1)$ x $T(F_2)$ x ... x $T(F_N)$ of $M(\overline{F})$ $\cong$ $m(F_1)$ x ...x $m(\overline{F_i})$ x ... x $m(F_N)$ is ineffective when restricted to $T(F_i)$.

The nontrivial element $\gamma_i$ in $\overline{F_i}/F_i$ corresponds to some nontrivial element $\overline{\gamma_i}$ in $m(\overline{F_i}) \preceq M(\overline{F_i})$. The equality $T(F_i) = T(\overline{F_i})$ between the two Teichmuller spaces implies that the action of $\gamma_i$ on the Riemann surface $U/F_i$ does not result in a new conformal structure on $U/F_i$. Hence, the corresponding action of $\overline{\gamma_i}$ on $T(F_i)$ is ineffective. This proves our claim and completes case 1.

**Case2 :** For every i, the set $T(F_i)_{max} \neq \emptyset$. In this case, we clearly have $T(F)_{max}$ is everywhere dense in T(F). Moreover, there is no finite extension of F whose deformation space is equal to T(F) for otherwise we will arrive at a contradiction to the Fuchsian case.

Since, these are the only possible cases, the proof of our proposition is complete. □

## 4.3 The Schwarzian Derivative

If (F,Δ) and (G,Δ) is a pair of function groups which is an improper inclusion pair, then another way of showing Proposition 7 is by proceeding as follows. We first observe that the decompositions of T(F) and T(G) in Proposition 6 is slightly modified such that except for the first pair of factors, the remaining ones are still the Teichmuller spaces of the other Fuchsian groups in the Fuchsian models of F and G. By a result of Accola mentioned in a previous chapter, they are biholomorphically equivalent in pairs. Now, let us consider the first pair of factors associated with the common invariant component $\Delta$, which in this case is not necessarily simply-connected. These factors are usually denoted by $T(F,\Delta)$ and $T(G,\Delta)$ respectively.

**Remark:** It is known that $T(F,\Delta) = T(F_0,U)/m(F_0)$ and $T(G,\Delta) = T(G_0,U)/m(G_0)$ where $m(F_0)$ and $m(G_0)$ are subgroups of the modular groups of $F_0$ and $G_0$ respectively. The actions of $m(F_0)$ and $m(G_0)$ are properly discontinuous and these groups are, in fact, the respective deck groups for the two coverings $p_1$:$T(F_0,U) \mapsto T(F,\Delta)$ and $p_2$:$T(G_0,U) \mapsto T(G,\Delta)$. This follows from the fact that $T(F_0,U)$ and $T(G_0,U)$ are the (holomorphic) universal covers of $T(F,\Delta)$ and $T(G,\Delta)$ respectively.

The elements of $T(F,\Delta)$ and $T(G,\Delta)$ consist respectively of classes of quasi-conformal self-maps of $\hat{\mathbb{C}}$ supported on $\Delta$ such that $wFw^{-1}$ and $wGw^{-1}$ are Kleinian groups. Let $[w]_F \in T(F,\Delta)$. Now, it is clear that $T(G,\Delta) \hookrightarrow T(F,\Delta)$. Hence, if we show that $[w]_F \in T(G,\Delta)$, we will get $T(F,\Delta) = T(G,\Delta)$ and our desired equality $T(F) = T(G)$ will be established.

To begin, recall that $(F,\Delta)$ and $(G,\Delta)$ is an improper inclusion pair by assumption. Hence, $G = <F,\gamma>$ by Corollary 2 (where $\gamma$ is a half-turn). From this, one can deduce that the quasi-conformal map w satisfies $wGw^{-1}$ is a Kleinian group if and only if $w\gamma w^{-1}$ is a Moebius transformation. We now aim to show that the latter is true. First we prove the following two lemmas. Note that we are following the same notations as before.

**Lemma 11** *If $w$ is a quasi-conformal map of $\hat{\mathbb{C}}$ supported on $\Delta$ that is com-*

*patible with F, then $S(w_0) \in Q(G_0,U)$ where S is the Schwarzian derivative operator and $w_0$ is the lift of w.*

**Proof:** Since w is compatible with F, we get that $wfw^{-1}$ is a Moebius transformation for any f ∈ F. Fix an f ∈ F. Consider a diagram similar to diagram 3 where $\pi$ and $\pi'$ are covering maps. Thus $wfw^{-1}$ lifts via $\pi'$ to a (real) Moebius transformation, say $\hat{f}_0$. Now clearly, w lifts to a quasi-conformal map $w_0$ of U. Let $f_0$ be the corresponding element to f in the Fuchsian equivalent, $F_0$, of F. Observe that $w_0 f_0 w_0^{-1} = \hat{f}_0$. Write this equality as $w_0 f_0 = \hat{f}_0 w_0$ and take the Schwarzian derivatives of both sides. Recall that S(f) = 0 if and only if f is a Moebius transformation. Furthermore, S satisfies the following identity known as Cayley's identity: $S(f \circ g) = (S(f) \circ g)\cdot(g')^2 + S(g)$.

By using these two facts, we get $(S(w_0) \circ f_0)\cdot(f_0')^2 = S(w_0)$. This implies that $S(w_0) \in Q(F_0,U)$. Since $Q(F_0,U) = Q(G_0,U)$, we get $S(w_0) \in Q(G_0,U)$.

□

**Lemma 12** *The mapping $w\gamma w^{-1}:\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ lifts to a (real) Moebius transformation.*

**Proof:** By the previous lemma, $S(w_0) \in Q(G_0,U)$. Since $G_0 = <F_0,\gamma_0>$. we have $(S(w_0) \circ \gamma_0)\cdot(\gamma_0')^2 = S(w_0)$. Hence, $S(w_0 \circ \gamma_0) = S(w_0)$. However if the Schwarzian derivatives of two mappings are the same, then they differ by a Moebius transformation. Therefore, $w_0 \circ \gamma_0 = A \circ w_0$ for some Moebius transformation A. Observe that A leaves U invariant which shows that $A \in PSL(2,\mathbb{R}) \cong Aut(U)$. Clearly $w_0 \gamma_0 w_0^{-1}$ is the lift of $w\gamma w^{-1}$ and the desired result now follows directly. □

For the rest of our proof, Lemma 12 is the key. Set $\gamma_w = \mathrm{w}\gamma\mathrm{w}^{-1}$. The map $\gamma_w$ is clearly a quasi-conformal automorphism of $\hat{\mathbb{C}}$ and maps w($\Delta$) onto itself. By lemma 12, its lift $w_0\gamma_0 w_0{}^{-1}$ is a (real) Moebius transformation A and thus is a conformal self-map of U. However, it is obvious that A can also be viewed as a conformal self-map of $\hat{\mathbb{C}}$ since it is a Moebius transformation. Therefore, $\gamma_w \in$ PSL(2,C) $\cong$ Aut($\hat{\mathbb{C}}$) and $\gamma_w$ is a Moebius transformation. If we set $G_w = <\mathrm{wFw}^{-1},\gamma_w>$, then we clearly get that $G_w = \mathrm{wGw}^{-1}$. Hence, the ordinary set of $G_w$ satisfies $\Omega(G_w) = \Omega(\mathrm{wGw}^{-1}) = \mathrm{w}(\Omega) \neq \emptyset$, where $\Omega = \Omega(\mathrm{F}) = \Omega(\mathrm{G})$ is the common ordinary set of F and G. This shows that $G_w$ is a Kleinian group and thus, w is G-compatible. Now, denote the deformation class of w in T(G,$\Delta$) by $[w]_G$ when viewing $\Delta$ as the invariant component of G. We have therefore shown that $[w]_F = [w]_G \in$ T(G,$\Delta$) and T(F,$\Delta$)$\hookrightarrow$T(G,$\Delta$). The desired conclusion now follows directly as we noted above. □

**Remark:** Notice that similar to what we just employed, we only needed to make use of the quasi-conformal mapping restricted to the invariant component of F in our proof of proposition 7.

As a byproduct, the previous lemmas imply the following corollary almost immediately. The map $\hat{w}$ referred to below is the induced quasi-conformal mapping between the two Riemann surfaces $\Delta$/F and w($\Delta$)/wFw$^{-1}$. $\hat{\gamma}$ is, as usual, the hyperelliptic involution on $\Delta$/F.

**Corollary 6** *The function $\hat{w}\hat{\gamma}\hat{w}^{-1}$:$w(\Delta)/wFw^{-1} \longmapsto w(\Delta)/wFw^{-1}$ is a conformal self-map of $w(\Delta)/wFw^{-1}$ i. e. $\hat{w}\hat{\gamma}\hat{w}^{-1} \in Aut(w(\Delta)/wFw^{-1})$.*

**Proof:** To prove this, we simply observe that $\hat{w}\hat{\gamma}\hat{w}^{-1}$ lifts precisely to the (real) Moebius transformation $A = w_0 \gamma_0 w_0{}^{-1} \in \mathrm{Aut}(U)$. Hence $\hat{w}\hat{\gamma}\hat{w}^{-1}$ is conformal and thus an element of $\mathrm{Aut}(w(\Delta)/wFw^{-1})$. □

**Remark:** Set $\hat{\gamma}_w = \hat{w}\hat{\gamma}\hat{w}^{-1}$ (the map in Corollary 6). Clearly, $\hat{\gamma}_w$ is of order two. Its action on $w(\Delta)/wFw^{-1}$ results to a Riemann surface which is quasi-conformally equivalent to $\Delta/G$ via $\hat{w}$. The map $\hat{\gamma}_w$ is the involution on $w(\Delta)/wFw^{-1}$ analogous to $\hat{\gamma} \in \mathrm{Aut}(\Delta/F)$ which does not give rise to any new conformal structure on $\Delta/F$. As we saw above, the action of $\hat{\gamma}_w$ on $w(\Delta)/wFw^{-1}$ also results only into an ineffective (or trivial) action on $T(F_0, U)$.

# Chapter 5

## Surfaces with Ideal Boundary Curves and Some Loxodromic Cyclic Constructions

Our objective in this chapter is to construct improper inclusion pairs by using some known techniques which are found in **[M1]**, **[M2]**. We first recall the following definition.

**Definition 7** *Let Γ be a finitely generated Fuchsian group. Γ is said to be of the first kind if* $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\} = \hat{\mathbb{R}}$, *and of the second kind if* $\Lambda(\Gamma) \subsetneq \hat{\mathbb{R}}$.

A Fuchsian group of the second kind therefore acts discontinuously on certain segments of the real axis. Each such segment L is kept invariant by a hyperbolic cyclic subgroup H of G, and every element of G not in H maps L onto some other segment. A generator h of H is called a boundary element of G; H is called a boundary subgroup of G. If C is the axis of h ( i.e. C is the geodesic line segment joining the fixed points of h ), then the non-Euclidean half-plane between C and L is precisely invariant under H, and is called a boundary half-plane of the Fuchsian group of the second kind.

The results of Greenberg **[Gr]** and Singerman **[S]** included such pairs where the Fuchsian groups are finitely generated but of the second kind. We will let $\Omega^{\#}(\Gamma) = \Omega(\Gamma)\cap\hat{\mathbb{R}}$ and $(g,n;\nu_1,\nu_2,...,\nu_n;b)$ denote the (conformal) signature of the associated surface $S = U^{\#}/\Gamma = U\cup\Omega^{\#}(\Gamma)/\Gamma$. The surface S has genus g, with n distinguished points (these include punctures), and b boundary curves. Notice though that $\Omega(\Gamma)/\Gamma$, which is the double of S, is necessarily of finite type by Ahlfors' Finiteness Theorem. The type of $\Omega(\Gamma)/\Gamma$ can be seen to be (2g + b - 1,2n). It is known in this case that $dim_{\mathbb{R}}T(\Gamma)$ = 6g - 6 + 2n + 3b. We summarize the results in **[Gr]** and **[S]** on Fuchsian groups of the second kind in the following theorem.

**Theorem 5 (Greenberg-Singerman)** *Suppose that $F_0$ and $G_0$ are finitely generated, non-elementary Fuchsian groups of the second kind with $F_0 \triangleleft G_0$, $[G_0:F_0]<\infty$. $T(F_0,U) = T(G_0,U)$ if and only if either $U^{\#}/F_0$ has signature (1,0,1) and $U^{\#}/G_0$ has signature (0,3;2,2,2;1) with $[G_0:F_0] = 2$ or $U^{\#}/F_0$ has signature (0,2;n,n;1) and $U^{\#}/G_0$ has signature (0,2;2,n;1), n = 2,3,... or $\infty$ with $[G_0:F_0] = 2$.*

**A.** Our aim at the present is to construct pairs of improper inclusion pairs with the preceding theorem as the springboard of our loxodromic cyclic constructions.

Start with a non-elementary and finitely generated Fuchsian group $F_0$ of the second kind of signature (1,0,1), normalized such that the second quadrant is a boundary half-plane and such that the sector $\{ z| \ \delta \leq \arg(z) \leq 2\pi - \delta \}$ is precisely invariant under $Stab_F(\{0,\infty\})$, where $\delta < 2\pi/n$ for some n $\in \mathbb{Z}^+$.

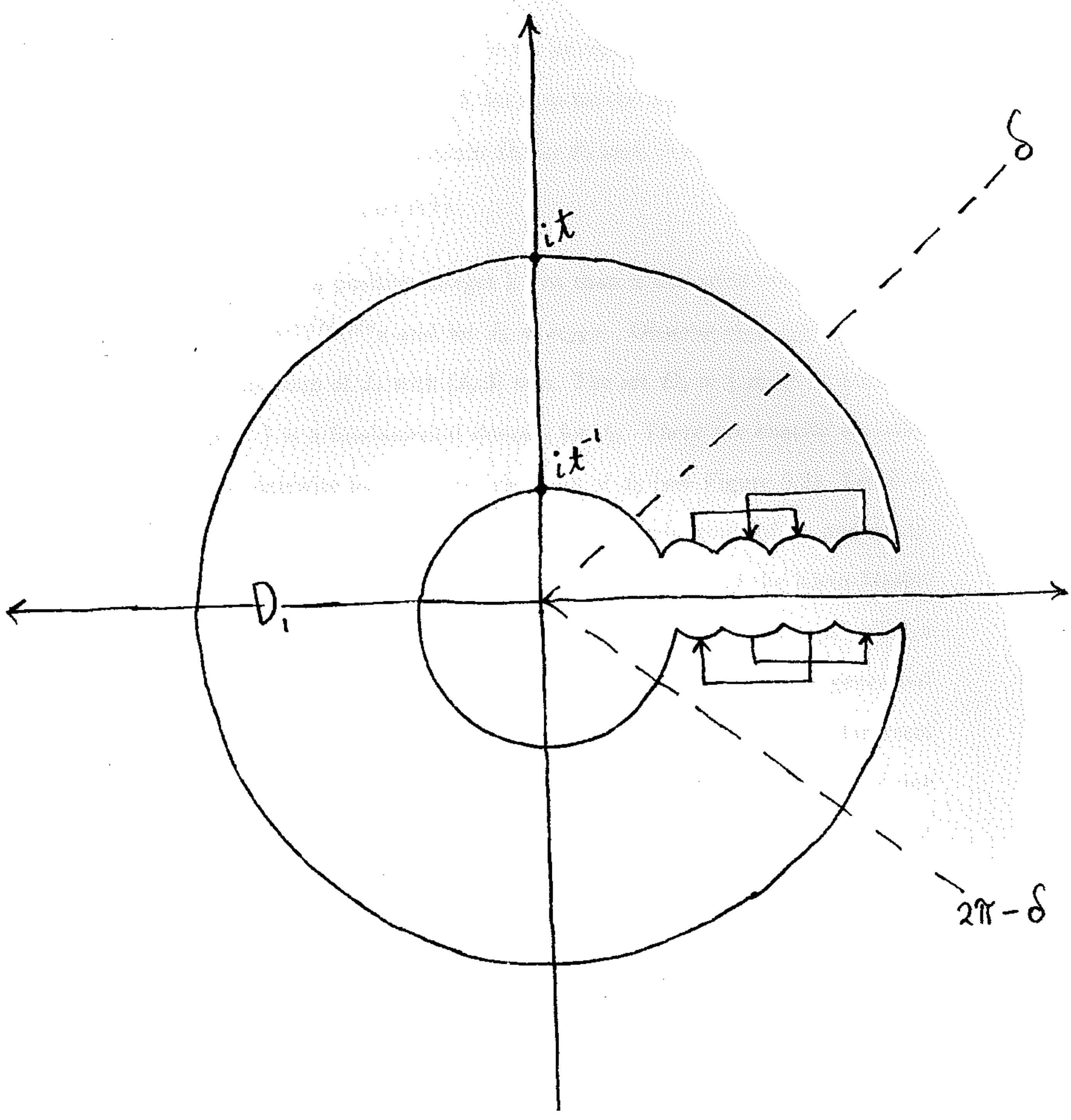


Figure 5.1: Construction I(Fundamental Domain)

Set J = $Stab_F(\{0,\infty\})$ and let $j(z) = t^2 z$ be the generator of J where $t > 1$. Choose the fundamental domain E for J to be the annulus $\{ z | \ t^{-1} < |z| < t \}$ and choose a fundamental domain $D_1$ for $F_0$ such that $\{ z | \ \delta \leq \arg(z) \leq 2\pi - \delta \} \subset D_1 \subset E$. (see Figure 5.1)

Now, pick out a positive integer k less than n and relatively prime to n. Define $j_0(z) = e^{2\pi ik/n} t^{2/n} z$ and let $J_0 = <j_0>$. Observe that $(j_0)^n = j$ and that J is a subgroup of $J_0$ with $[J_0{:}J] = n$. The set $D_2 = \{ z | \ \frac{-\pi}{n} < \arg(z) < \frac{\pi}{n}, \ t^{-1} < |z| \ t \}$ is a fundamental domain for $J_0$. There are exactly six sides of $D_2$ that is pairwise identified by elements of $J_0$ (see Figure 5.2). Let F = $<F_0,J_0>$ where J is a common loxodromic cyclic subgroup of $F_0$ and of $J_0$. Call the sets indicated in Figure 5.2 as $B_1$ and $B_2$. Applying the Klein-Maskit combination theorem along J to $(B_1{}^{\circ}, B_2{}^{\circ})$ (this pair of open sets is called a proper intercative pair), we get that F is a non-elementary, finitely generated Kleinian group with $D = D_1 \cap D_2 = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ as a fundamental domain (see Figure 5.2). D has two connected components, but a side of one component is identified by elements of $J_0$ with a side of the other component as we saw earlier. Hence, $\Omega(F)/F$ represents only one surface which is of signature (2,0).

**Remark:** We remark that this group F is an example of a group where every component is simply-connected and no component is invariant under F. However, there are exactly n special components of F that are invariant under J, but $J_0$ keeps no component invariant. The reader is referred to **[M1]** for more details.

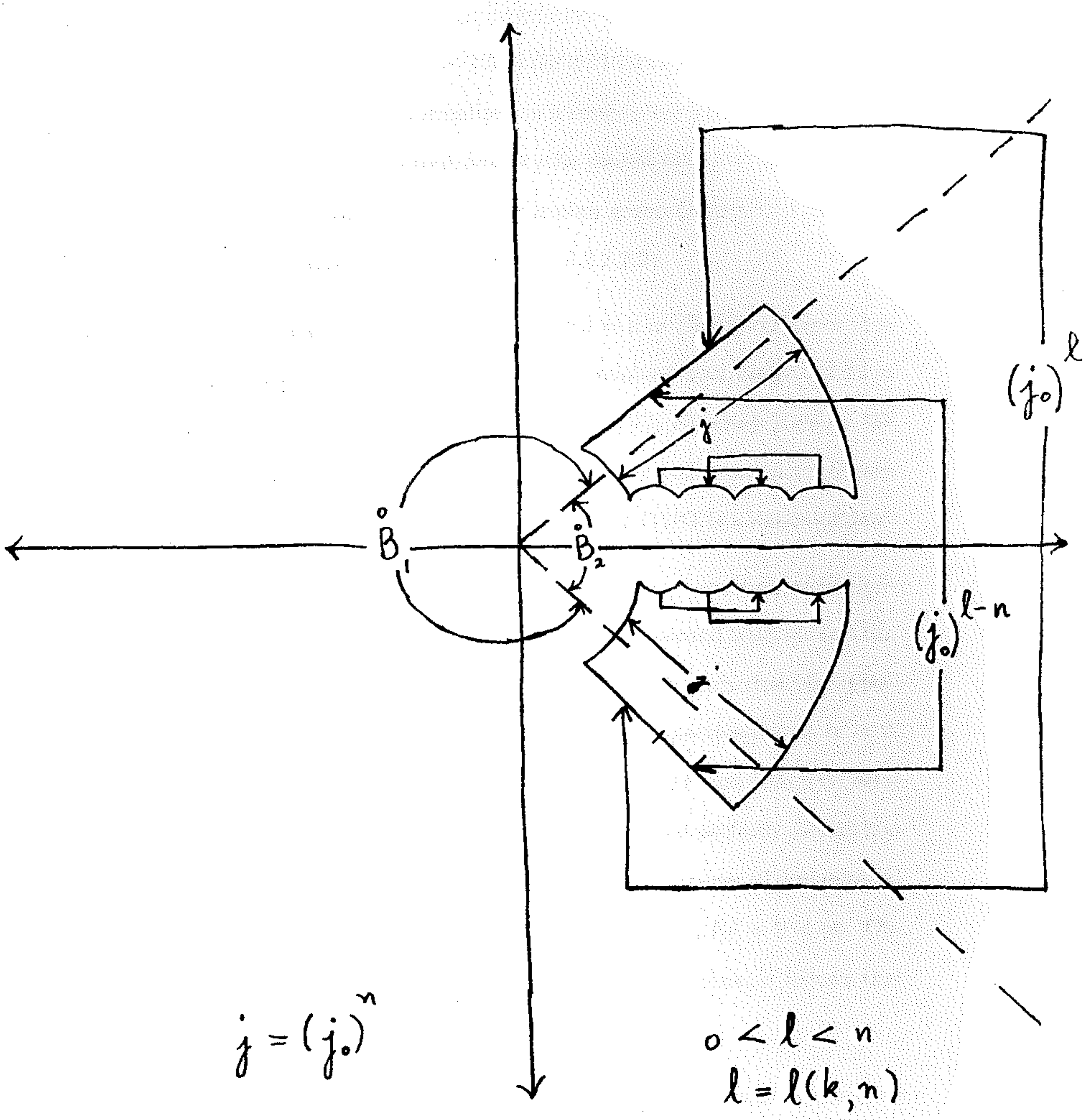


Figure 5.2: Construction I(Side-Pairing)

Now, let $G_0$ be a non-elementary, finitely generated Fuchsian group of the second kind with $F_0 \lhd G_0$, $[G_0:F_0] = 2$ and $G_0$ is of signature (0,3;2,2,2;1). We assume that $G_0$ is chosen and normalized in a similar way as $F_0$ was. Starting with $G_0$ and doing the same loxodromic cyclic constructions as in above, we get a non-elementary, finitely generated Kleinian group G such that $\Omega$(G)/G represents a single surface of signature (0,6;2,2,2,2,2,2).

Since $(j_0)^n = \mathrm{j} \in F_0$, we clearly have $[\mathrm{F}:F_0] = \mathrm{n} = [\mathrm{G}:G_0]$. Observe that the following equation relating various indices involved here holds: [G:F] = $[\mathrm{G}:G_0]\cdot[G_0:F_0]/[\mathrm{F}:F_0]$. Hence. [G:F] = n(2)/n = 2 = $[G_0:F_0]$ which is finite. We can see immediately that Q(F,$\Omega$) = Q(G,$\Omega$), where $\Omega$ is their common ordinary set, since the pair of signatures from $\Omega$(F)/F and $\Omega$(G)/G is a maximal pair.

We can repeat the above arguments this time by starting with non-elementary, finitely generated Fuchsian groups of the second kind $F_0$ and $G_0$, where $G_0$ contains $F_0$ as a normal subgroup of index 2, such that the signatures of $F_0$ and $G_0$ are (0.2:m.m:1) and (0.2:2.m:1) respectively with m = 2, 3, ..., or $\infty$. The Riemann surfaces of finite type, $\Omega$(F)/F and $\Omega$(G)/G, associated to the resulting Kleinian groups F and G after we perform similar loxodromic cyclic constructions as in above, will have signatures (0,4;m,m,m,m) and (0,4;2,2,m,m) respectively. Furthermore, [G:F] = $[G_0:F_0] = 2 < \infty$. Observe that their pair of signatures is a maximal pair and thus, it follows that Q(F,$\Omega$) = Q(G,$\Omega$) in this case as well.

**B.** Next, we do a construction of a pair of function groups which is an improper inclusion pair. The general idea is again found in **[M1]**.

Let $F_1$ be a finitely generated Fuchsian group of the first kind of signature $(1,1;\nu)$ with $\nu = 3$. Normalize $F_1$ so that the following holds: it acts on the unit disc, J = $Stab_{F_1}(0) = Stab_{F_1}(\infty)$ has order $3 = \nu$ and $B_1 = \{|z| \leq \delta\}$ is precisely invariant under J in $F_1$ for some $\delta > 0$. Choose a fundamental domain $D_1$ for $F_1$ which is contained in $\{z \mid 0 \leq \arg(z) \leq \pi\} = \overline{U}$ (the fundamental domain for J) and is maximal for $B_1$.

Now, let $F_2$ be another such Fuchsian group, but this time normalized (by a dilatation of the form $z \mapsto tz$, $0 < t < 1$) so that $F_2$ operates on a smaller disc than $B_1$ centered at the origin and $B_2 = \{z \mid |z| \geq \delta\} \cup \{\infty\}$ is precisely invariant under J in $F_2$. Choose a fundamental domain $D_2$ for $F_2$, similar to that of $F_1$. (Here, $F_2$ is taken to be of signature (1,1;3) as well but the only requirement is that $F_2$ have an elliptic element of order $\nu$ as does $F_1$.)

Set F = $<F_1, F_2>$ with J being a common subgroup to $F_1$ and $F_2$. Set D = $(D_1 \cap B_2) \cup (D_2 \cap B_1)$. It is clear that F represents three component surfaces, two of which have signature (1,1;3) and the remaining surface has signature (2,0). (See Figure 5.3) Let $\hat{F}_1$ and $\hat{F}_3$ be the component subgroups of F corresponding to the component surfaces with signature (1,1;3) and let $\hat{F}_2$ be the component subgroup corresponding to the component surface with signature (2,0).

$\hat{F}_3$ stabilizes the component equal to an unbounded disc, $\hat{F}_1$ stabilizes the component equal to a small disc centered at the origin and $\hat{F}_2$ stabilizes the annular region between them that contains $W_1 = \partial B_1$. Indeed, $\hat{F}_2$ = F and F is actually a function group with this annular region as its invariant component.

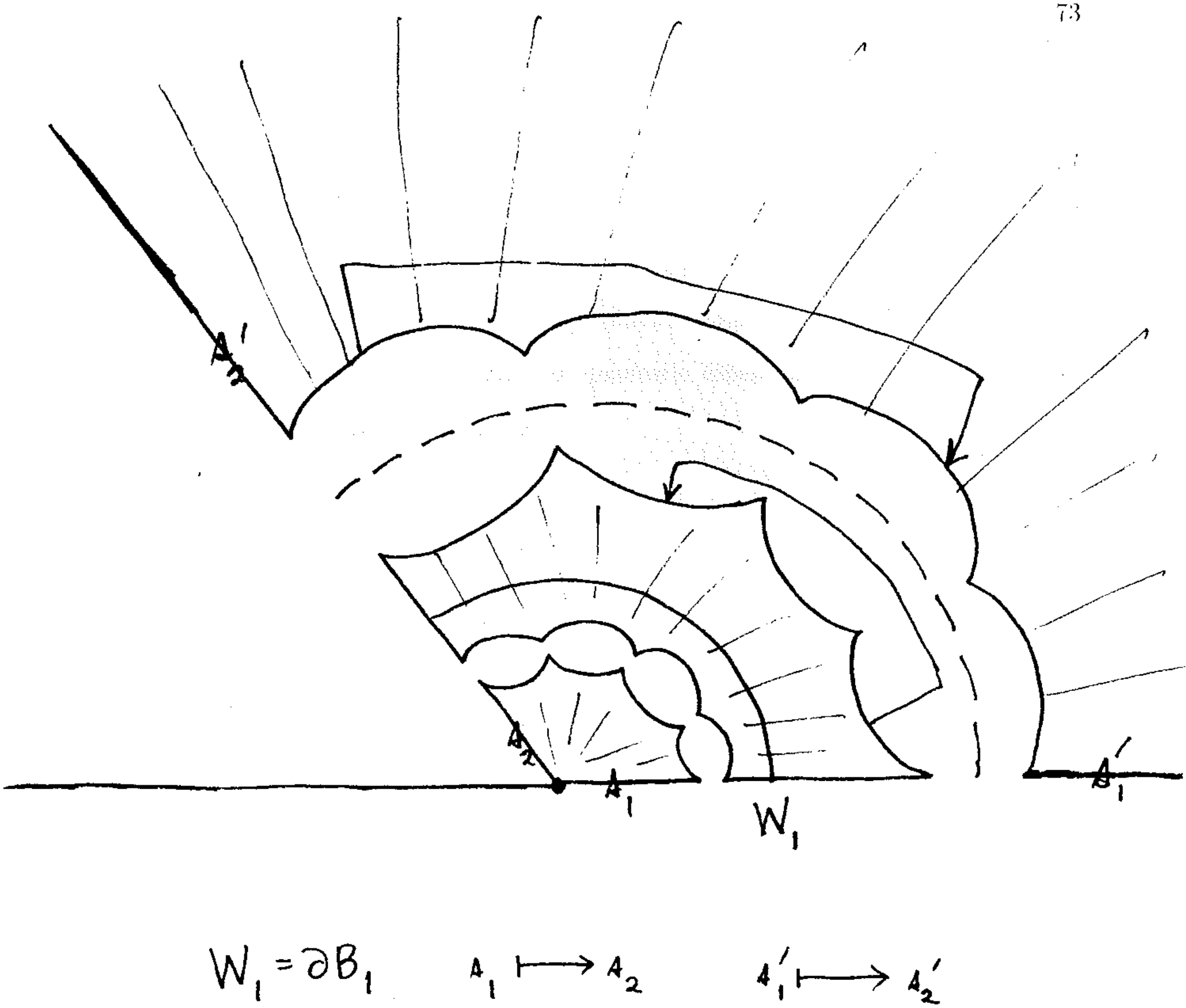


Figure 5.3: Construction II

Now, we adjoin the Moebius transformation, $\gamma$:z$\mapsto$ -z, to F and let G = <F,$\gamma$> so that [G:F] = 2 < $\infty$. This half-turn $\gamma$ will stabilize all three components of F. The result is that G is a function group such that $\Omega(G)/G$ represents three surfaces of signatures (0,4;2,2,2,6), (0,6;2,2,2,2,2,2) and (0,4;2,2,2,6). Clearly, the dimensions of the spaces of quadratic differentials for F and G are equal and hence, $Q(F,\Omega) = Q(F,\Omega)$.

# Chapter 6

## OTHER RELATED RESULTS

This chapter discusses a number of topics that may provide possible routes in further exploring our central ideas. In line with this, we raise a conjecture at the end of each section.

### 6.1 An Exact Sequence of Kleinian Groups

The result we have here is mainly an extension of a theorem due to Harvey [**H**]. This theorem says that if F and G are non-elementary, finitely generated Fuchsian groups and a certain short exact sequence involving F and G exists, then T(G) is the fixed point set of some finite subgroup, H, of the modular group of F, M(F). The converse to this theorem is in fact true as well and was shown also by Harvey [**H**]. First, we consider the following.

**Proposition 9** *Let $(F, \Delta(F))$ and $(G, \Delta)$ be non-elementary, finitely generated function groups and $\tau: F \hookrightarrow G$ be a 1-1 homomorphism (monomorphism) that is type-preserving such that $[G:\tau(F)] < \infty$. Then,* *(i) $(\tau(F),\Delta)$ is a function group.*

*(ii) If $Q(F,\Delta(F))\cong Q(G,\Delta)$, then $(\tau(F),\Delta)$ and $(G,\Delta)$ is an improper inclusion pair of function groups.*

**Proof:** Clearly, $\tau$(F) has the same ordinary set $\Omega$ as G which yields that $\Delta$ is necessarily a component of F. Furthermore, if f $\in \tau$(F), then it is in G and therefore f($\Delta$) = $\Delta$. Hence, f $\in Stab_{\tau(F)}\Delta$. This implies that $\Delta$ is also invariant under $\tau$(F). Thus, ($\tau$(F),$\Delta$) is a function group.

We notice that since $\tau$ is type-preserving, the surfaces $\Delta$(F)/F and $\Delta/\tau$(F) have the same signatures. Therefore, Q($\tau$(F),$\Delta$)$\cong$Q(F,$\Delta$(F)) and by the hypothesis we get Q($\tau$(F),$\Delta$)$\cong$Q(G,$\Delta$). By a dimension argument, this implies directly that Q($\tau$(F),$\Delta$) = Q(G,$\Delta$). By Proposition 5, we get that $\tau$(F) and G is an improper inclusion pair. □

Now, we have:

**Proposition 10** *Let $(F,\Delta(F))$ and $(G,\Delta(G))$ be non-elementary, finitely generated b-groups and $\tau:F\longmapsto G$ be a 1-1 homomorphism (monomorphism) that is type-preserving. Suppose that the sequence*

$$1\longrightarrow F\xrightarrow{\tau} G\xrightarrow{\xi} H\longrightarrow 1$$

*is exact for some finite group H.*

*Then, the set $\hat{\tau}(\mathbb{T}(G))$ is the fixed point set, $F(H')$, of some finite subgroup, $H'$, of the automorphism group of $\mathbb{T}(F)$ where $\hat{\tau}$ is the map from $\mathbb{T}(G)$ to $\mathbb{T}(F)$ induced by $\tau$.*

**Proof:** Since the sequence

$$1 \longrightarrow \mathrm{F} \xrightarrow{\tau} G \xrightarrow{\xi} H \longrightarrow 1$$

is exact, we get $\xi$ is surjective, im$\tau$ = ker$\xi$ and G/ker$\xi$ = G/im$\tau \cong$ H. In particular observe that we also get im$\tau = \tau(\mathrm{F}) \trianglelefteq$ G and by finiteness of H, the index [G:$\tau$(F)] is finite.

Using the first part of the above proposition, we get that $\tau$(F) is a function group with invariant component $\Delta = \Delta(\mathrm{G})$. Since (G,$\Delta$) is a b-group, $\Delta$ is simply-connected. Thus, ($\tau$(F).$\Delta$) is in fact also a b-group. It is known (see chapter 4) that for b-groups. we have a biholomorphism, $\psi$,

$$\mathrm{T}(F_0) \times \mathrm{T}(F_1) \times \mathrm{T}(F_2) \times \ldots \times \mathrm{T}(F_{N'}) \overset{\psi}{\cong} \mathrm{T}(\mathrm{F})$$

where $\{F_0, F_1, F_2, \ldots, F_{N'}\}$ is the Fuchsian model of F. The rest of our arguments will rely upon this biholomorphism. Let $\{G_0, G_1, G_2, \ldots, G_N\}$ be the Fuchsian model of G, where N' $\geq$ N.

Now, $\tau$ induces a finite family of monomorphisms $\{\tau_i |$ i = 0, 1, 2, ..., N $\}$ between $F_i$ and $G_i$ and a finite family of short exact sequences

$$1 \longrightarrow F_i \xrightarrow{\tau_i} G_i \xrightarrow{\xi_i} H_i \longrightarrow 1$$

where $H_i$ are the finite groups $G_i/\tau_i(F_i)$.

Then, by Harvey's results, $\hat{\tau}_i(\mathrm{T}(G_i))$ is the fixed point set of some finite subgroup, $H_i'$, of the modular group, M($F_i$), of $F_i$ for i = 0, 1, 2, ..., N. We now form the set $H_0' \times H_1' \times H_2' \times \ldots \times H_N'$ which is clearly a finite set. Now, we restrict our biholomorphism, $\psi$, on $\mathrm{T}(F_0) \times \mathrm{T}(F_1) \times \mathrm{T}(F_2) \times \ldots \times \mathrm{T}(F_N)$ and call this restriction $\psi$ as well, for simplicity of notation. Consider the push forward, $\psi_*(H_0' \times H_1' \times H_2' \times \ldots \times H_N')$, of $H_0' \times H_1' \times H_2' \times \ldots \times H_N'$ via the restriction $\psi$. Clearly, this is a finite subgroup of the automorphism group[1] of $\mathbb{T}$(F). Set

$$H' = \psi_*(H_0' \times H_1' \times H_2' \times \ldots \times H_N')$$

Notice that $\hat{\tau}_i(\mathrm{T}(G_i)) = \emptyset$ since $\mathrm{T}(G_i) = \emptyset$ for i = N+1, N+2, ..., N'. Then, by construction, we get that $\hat{\tau}(\mathbb{T}(\mathrm{G}))$ is the fixed point set of H'. This completes our proof. □

**Conjecture:** Let F and G be non-elementary, finitely generated Kleinian groups and $\tau$:F$\longmapsto$G be a 1-1 homomorphism (monomorphism) that is type-preserving. Assume that

$$1 \longrightarrow \mathrm{F} \xrightarrow{\tau} G \xrightarrow{\xi} H \longrightarrow 1$$

is exact for some finite group H.

Then, the set $\hat{\tau}(\mathbb{T}(\mathrm{G}))$ is the fixed point set, F(H'), of some (finite) subgroup, H', of the automorphism group of $\mathbb{T}$(F).

## 6.2 Kleinian groups of the First Kind

Here, we investigate the case when F is a non-elementary and finitely generated Kleinian group of the first kind. Recall that this implies $\Omega(\mathrm{F}) = \emptyset$. Let us assume that F$\prec$G, where G is a non-elementary and finitely generated Kleinian group and [G:F] $< \infty$. Then, as $\Omega(\mathrm{G}) \subseteq \Omega(\mathrm{F})$, we get $\Omega(\mathrm{G}) = \emptyset$ and G is also a Kleinian group of the first kind.

---

[1] Here, by automorphism group we mean the group of biholomorphic self-mappings.

All our previous results cannot be applied but if we assume that F is geometrically finite, then it is well-known that G is also geometrically finite. Their associated 3-manifolds, P(F)/F and P(G)/G where P(F) and P(G) are the fundamental polyhedra for F and G respectively, will both be compact (without boundary). *We can thus conclude from these that P(F)/F and P(G)/G will each have a unique hyperbolic structure as a consequence of the famous Mostow Rigidity Theorem* **[Mo]**.

## 6.3 Conditions for Adjoining Elements

Our present goal is to investigate this question: if F is a non-elementary and finitely generated Kleinian group and $<F,\gamma_a,...,\gamma_z>$ is also a Kleinian group, where $\gamma_a,...,\gamma_z$ are elliptic elements necessarily of finite order, such that $<F,\gamma_a,...,\gamma_z>$ and F is an improper inclusion pair, what properties must the elliptic elements satisfy? We will assume that $(\Omega(F)/F)^* \neq \emptyset$ in our succeeding discussions. The case where dim $Q(F,\Omega) = 0$ has more interesting implications and will be considered towards the end. The results that we have are mainly a series of conditions that must be satisfied by the adjoined elliptic elements. It is easy to verify that these conditions will guarantee the result of having an improper inclusion pair.

Let $\gamma$ be among the $\gamma_a$, ..., $\gamma_z$. We will proceed by looking at the action of $\gamma$ on $\Omega$. This will depend on the location of the fixed points of $\gamma$. Denote by p and q the two distinct fixed points of $\gamma$. There are four possible cases:

**<u>Case1 :</u>** p and q both lie in the same component, say $\Omega_{pq}$.

**Case2 :** p and q lie in two distinct components, say $\Omega_p$ and $\Omega_q$ respectively.

**Case3 :** p lies in some component, say $\Omega_p$, and q lies in the limit set of F.

**Case4 :** p and q both lie in the limit set of F.

I. Let us look at Cases 1-3 where p and q $\in \Omega$. Consider the two component surfaces of F, namely, $\Omega_p/\hat{F}_p$ and $\Omega_q/\hat{F}_q$, where $\hat{F}_p$ and $\hat{F}_q$ are the component subgroups associated with $\Omega_p$ and $\Omega_q$, respectively. Note that if $\Omega_p$ and $\Omega_q$ are F-equivalent, then $\Omega_p$ and $\Omega_q$ correspond to the same component surface in $\Omega$(F)/F. Our first condition is the following. The notations and assumptions are the same as in above.

*Condition I.: Unless they are thrice-punctured spheres, $\Omega_p/\hat{F}_p$ and $\Omega_q/\hat{F}_q$ must have signatures that are listed in Table 1 whenever p and $q \in \Omega$*

II. To proceed, assume first that F is nontriangular. Denote one of the component surfaces which has a signature listed in Table 1 by $S_1$. Such a component surface exists by Lemma 8. In particular, $S_1$ is a hyperelliptic Riemann surface of finite type. As usual, let $\Omega_1$ be a component of $\Omega$ and $\hat{F}_1$ be a component subgroup of F such that $\Omega_1/\hat{F}_1 = S_1$. Let $\gamma_1$ be the resulting half-turn in Lemma 5 which is induced by the hyperelliptic involution on $S_1$. Denote the two distinct fixed points of $\gamma_1$ by $x_1$ and $x_2$. Hence, we see that $\Omega_1$ must contain at least one of these two points. This eliminates Case 4 above. We now have this condition which was often used in the proofs of our earlier propositions.

*Condition II.: If $\Omega_i$ is any other component of $\Omega$ not containing $x_1$ or $x_2$, then $\gamma_1(\Omega_i) = f(\Omega_i)$ for some $f \in F$.*

III. Now, let us consider the case of $\Omega$(F)/F - $(\Omega(F)/F)^* \neq \emptyset$. This is the case where there are thrice-punctured spheres in the orbit space of F. With this assumption, it is clear that we must satisfy the following.

*Condition III.: Let $\Omega_i$ be an arbitrary component of $\Omega$ which does not contain any of the fixed points of $\gamma_1$ and $\hat{F}_i$ be the component subgroup of F associated with $\Omega_i$. Then, at least one of the following must be true:*

*(i) $\gamma_1(\Omega_i) = f(\Omega_i)$ for some $f \in F$.*

*(ii) $\Omega_i/\hat{F}_i$ is a thrice-punctured sphere.*

IV. Now, suppose that $\tilde{S}_1 \in \Omega(F)/F$ - $(\Omega(F)/F)^*$ with $\tilde{S}_1 = \tilde{\Omega}_1/\tilde{F}_1$. Recall that Greenberg's Theorem included some pairs of triangle groups. Hence, it is possible that there exists an elliptic element $\tilde{\gamma}_1$, which is not necessarily a half-turn, that gives rise to such a pair of triangle groups, $\tilde{F}_1$ and $<\tilde{F}_1, \tilde{\gamma}_1>$. Let $k_1$ be the order of $\tilde{\gamma}_1$. The following condition is clearly the generalization of Condition III when the order of the elliptic element is bigger than 2.

*Condition IV.: Let $\Omega_i$ be an arbitrary component of $\Omega$ which does not contain any of the fixed points of $\tilde{\gamma}_1$ and $\hat{F}_i$ be the component subgroup of F associated with $\Omega_i$. Then, at least one of the following must be true:*

*(i) for every $n \in \{ 1, 2, \ldots, k_1 - 1 \}$, $\tilde{\gamma}_1{}^n(\Omega_i) = f_n(\Omega_i)$ for some $f_n \in F$*

*(ii) $\Omega_i/\hat{F}_i$ is a thrice-punctured sphere.*

V. Now, let $\overline{\gamma}$ be elliptic such that $\overline{\gamma} \notin$ F and furthermore, assume that $\overline{\gamma}$ is not among the possible $\gamma_i$'s and $\tilde{\gamma}_j$'s described above which satisfy the foregoing conditions. Let $\overline{p}$ and $\overline{q}$ be the fixed points of $\overline{\gamma}$.

**Lemma 13** *$\overline{p}$ and $\overline{q}$ are both in the limit set of F.*

**Proof:** Suppose that one of the fixed points, say $\overline{p}$, is in some component, $\Omega_{\overline{p}}$, of $\Omega$. Then, $\overline{\gamma}$ acts invariantly on $\Omega_{\overline{p}}$ and induces a conformal self-map on $S_{\overline{p}} = \Omega_{\overline{p}}/\hat{F}_{\overline{p}}$ where $\hat{F}_{\overline{p}}$ is the component subgroup of F associated with $\Omega_{\overline{p}}$. Since $\overline{\gamma}$ is chosen such that it is not among the $\gamma_i$'s and $\tilde{\gamma}_j$'s, the resulting pair of signatures by the action of the induced conformal self-map on $S_{\overline{p}}$ must not be maximal. This will imply that dim $Q(<\hat{F}_{\overline{p}},\overline{\gamma}>,\Omega_{\overline{p}}) <$ dim $Q(\hat{F}_{\overline{p}},\Omega_{\overline{p}})$ which is not acceptable. Therefore, $\overline{\gamma}$ must not act with fixed points on $\Omega$. This shows that $\overline{p}$ and $\overline{q}$ must both be in the limit set of F. □

This is Case 4 above. Moreover, we see that $\overline{\gamma}$ leaves no component of $\Omega$ invariant. If we denote the order of $\overline{\gamma}$ by k, then observe that $\overline{\gamma}^s(\Omega_i) \cap \overline{\gamma}^t(\Omega_i) = \emptyset$ for any s $\neq$ t, s and t $\in$ { 1,2,...,k } since $\Omega_i$, $\overline{\gamma}(\Omega_i)$, $\overline{\gamma}^2(\Omega_i)$, ..., $\overline{\gamma}^{k-1}(\Omega_i)$ are all components of the extension of F that contains $\overline{\gamma}$.

*Condition V.: At least one of the following is true for any component $\Omega_i$ of $\Omega$.*

*(i) for every n $\in$ { 1,2,...,k-1 }, $\overline{\gamma}^n(\Omega_i) = f_n(\Omega_i)$ for some $f_n \in F$.*

*(ii) $\Omega_i/\hat{F}_i$ is a thrice-punctured sphere, where $\hat{F}_i = Stab_F\Omega_i$.*

VI. It is clear that if $\{\gamma_1,...,\tilde{\gamma}_1,...,\overline{\gamma},...\}$ is a finite set of elliptic elements having the properties described above, then F and $<F,\gamma_1,...,\tilde{\gamma}_1,...,\overline{\gamma},...>$ is an improper inclusion pair. However, suppose that $<F,\gamma_1,...,\tilde{\gamma}_1,...,\overline{\gamma},...,g_1>$ is a Kleinian group which is a finite extension of F, $g_1 \notin$ F, and that $g_1$ is not elliptic. Since the extension is finite, there must exist a smallest postive integer $m_1 \geq 2$ such that ${g_1}^{m_1} \in$ F.

*Condition VI.: At least one of the following is true for any component $\Omega_i$ of $\Omega$.*

*(i) for every* $n \in \{ 1,2,\ldots,m_1 - 1 \}$, $g_1{}^n(\Omega_i) = f_n(\Omega_i)$ *for some* $f_n \in F$

*(ii)* $\Omega_i/\hat{F}_i$ *is a thrice-punctured sphere.*

VII. Now, as a separate case, let us consider the situation where $(\Omega(F)/F)^* = \emptyset$. Obviously, dim $Q(F,\Omega) = 0$. This implies that any finite extension of F by any type of Moebius transformation must have the same 0-dimensional space of quadratic differentials as F. Recall that this will fall under case (a) in our main proposition in Chapter 3.

VIII. Assume once again that $(\Omega(F)/F)^* \neq \emptyset$. In this case, there is clearly a maximum number of appropriate extra elements, elliptics or otherwise, that can be adjoined to F in order that F and this finite extension be an improper inclusion pair. We get the following corollary. Its proof is straightforward.

**Corollary 7** *Let* $\{K_i | \ i = 0, 1, 2, \ldots, k\}$ *be the resulting ascending set of intermediate groups arising from the adjoinment of the extra elements to F, ordered by inclusion where we set* $K_0 = F$ *and* $K_k = G$. *Then, any pair* $K_i$ *and* $K_j$ *with* $i < j$ , $i = 0, 1, 2, \ldots, k-1$, $j = 1, 2, \ldots, k$ *is an improper inclusion pair.*

**Conjecture:** If $(\Omega(F)/F)^* = \emptyset$, then such an ascending set of intermediate groups can be infinite.

## 6.4 Exceptional Pairs

The following pairs of non-elementary and finitely generated Kleinian groups (of the second kind), F and G, with $F \prec G$ of finite index cannot be improper inclusion pairs. Again, we are assuming that neither F nor G is a triangle group.

*(i) F is quasi-Fuchsian and G is totally degenerate*

*(ii) F is terminal(regular) and G is totally degenerate*

*(iii) F is terminal(regular) and G is quasi-Fuchsian*

*(iv) F is quasi-Fuchsian and G is terminal(regular)*

*(v) F is totally degenerate and G is terminal(regular)*

*(vi) F is totally degenerate and G is quasi-Fuchsian*

*(vii) F and G are both Schottky groups*

We should note, however, that there may be other, more subtle, examples besides the ones listed above. Let us pursue the first three cases (i), (ii), and (iii):

F is quasi-Fuchsian and G is totally degenerate

F is terminal(regular) and G is totally degenerate

F is terminal(regular) and G is quasi-Fuchsian

Recall that $\Omega(G) \subseteq \Omega(F)$ (in fact, $\Omega(G) = \Omega(F)$) must hold. These three pairs clearly violate this inclusion relation.

Now consider (iv). If F is quasi-Fuchsian and G is terminal(regular), then

$$\dim Q(F,\Omega) = 2\dim Q(F,\Delta)$$

$$\dim Q(G,\Omega) = \dim Q(G,\Delta)$$

Obviously, $Q(F,\Omega) = Q(G,\Omega)$ if and only if their dimensions are equal and thus if and only if $2\dim Q(F,\Delta) = \dim Q(G,\Delta)$. This can only happen if and only if $\dim Q(F,\Delta) = \dim Q(G,\Delta) = 0$ since $0 \leq \dim Q(G,\Delta) \leq \dim Q(F,\Delta)$. F and G are both triangle groups in this case. (contra!)

For (v) and (vi), since F is totally degenerate and F has a common ordinary set $\Omega$ and invariant component $\Delta$ as G, we conclude that $\Omega - \Delta = \emptyset$

for G in both cases. This implies that $\Omega/G$ is a single surface. If G is quasi-Fuchsian, $\Omega/G$ is the disjoint union of exactly two surfaces, hence we have a contradiction. If G is terminal(regular), $\Omega/G$ consists of at least two surfaces which also leads to a contradiction. For (vii), we need only recall a remark made in Chapter 2 regarding Schottky groups.

**Conjecture:** If F is a Schottky group of rank 2, under what conditions does there exist a terminal(regular) b-group G containing F such that F and G have the same space of quadratic differentials and will the index still be finite in this case ? If F is a terminal(regular) b-group that represents a surface of genus 2 and two thrice-punctured spheres, does there exist a Kleinian group G properly containing F with finite index such that G only represents a surface of genus 2 and will G still be finitely generated in this case ?

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