Minimal Submanifolds with Various Curvature Bounds

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Abstract of the Dissertation

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This thesis consists of three main results, all of which are part of the long term project of understanding the global behavior of minimal submanifolds of Euclidean space. If $M^n \subset \mathbb{R}^N$ is a minimal submanifold, then the *total scalar curvature of* M, $\mathcal{A}(M)$, is defined to be

$$\mathcal{A}(M) := \int_{M^n} |A|^n \, dV$$

where A is the second fundamental form of M, $|A|$ is its Euclidean norm, and dV is the induced volume form on M.

The only known examples of complete minimal submanifolds $M^n \subset \mathbb{R}^N$, $n \geq 3$ with finite total scalar curvature are planes and catenoids. R. Schoen ([Sch83]) proved that any complete minimally immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ which has finite total scalar curvature and two ends, is either a catenoid or a pair of planes. I extend his result to show that if $n \geq 3$ and $n \geq \frac{N+1}{2}$, then the only such nonplanar minimal submanifolds are catenoids.

The second part of my thesis begins with the examination of another natural curvature expression,

$$\mathcal{B}_x(M) := \sup_r \frac{1}{r^{n-2}} \int_{D_x(r)} |A|^2 \, dV.$$

I prove a Gap Theorem, which says that if $M^k \subset \mathbb{C}^N$ is a smooth, complete, complex (minimal) submanifold, then either $\mathcal{B}_x(M)$ is uniformly bounded away from 0 for all $x \in M$, or M is planar. I show that as a consequence, the plane is the only smooth complete complex submanifold of complex dimension at least two with finite total scalar curvature. This advances the classification of all complete minimal submanifolds with finite total scalar curvature.

The third part of my thesis contains a result on the singular set of the tangent cones at infinity of complete stable minimal hypersurfaces in Euclidean space with bounded volume growth. I prove that the singular set has codimension 2.

For everyone who loves math

Contents

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Chapter 1

Introduction

This work was motivated by results concerning minimal surfaces M^2 in Euclidean space $\mathbb{R}^3$. R. Osserman ([Oss63]) proved that there are essentially two types of behavior, and that they are distinguished by the *total (Gauss) curvature of* M, $\mathcal{C}(M) := \int_M |K| \, dV$, where K is the Gauss curvature. For a complete minimal surface $M^2 \subset \mathbb{R}^3$, $\mathcal{C}(M)$ is finite if and only if M has a compactification via the Gauss map. (In this case, M is conformally equivalent to a compact Riemann surface $\overline{M}$ minus a finite number of points.) Later, Chern and Osserman ([CO67]) generalized this result to hold for complete minimal surfaces $M^2 \subset \mathbb{R}^N$. These theorems suggested the problem of classifying complete minimal surfaces with finite total curvature.

Until the last decade, there were very few results, even in the case of M^2 embedded in $\mathbb{R}^3$. For more than two hundred years, the only known examples of complete embedded minimal surfaces $M^2 \subset \mathbb{R}^3$ with finite total curvature, were planes and catenoids. It was widely conjectured and believed that these were the only such examples ([Hof87]). In fact, R. Schoen ([Sch83]) and Jorge and

Meeks ([JM83]) were able to rule out any other possibilities in certain cases. Then, in the early 1980's, C. Costa exhibited a minimal surface which was complete and had finite total curvature ([Cos82]). A few years later, Hoffman and Meeks proved that it was embedded ([HM85]). Eventually, Hoffman and Meeks constructed an infinite family of complete embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature ([HM90]).

In higher dimensions, there is a theorem which generalizes the Chern-Osserman theorem for minimal surfaces. Define the *total scalar curvature*, $\mathcal{A}(M)$, of a minimal submanifold $M^n \subset \mathbb{R}^N$ to be

$$\mathcal{A}(M) := \int_{M^n} |A|^n \, dV,$$

where A is the second fundamental form of M, $|A|$ is the Euclidean norm of A, and dV is the induced volume form on M. $\mathcal{A}$ is called total scalar curvature because, for minimal submanifolds, the scalar curvature is equal to $-|A|^2$. Note that $\mathcal{A}$ is scale-invariant and, if $M^2 \subset \mathbb{R}^3$, then $\mathcal{A}(M) < \infty$ if and only if $\mathcal{C}(M) < \infty$. The following was proved by M. Anderson ([And85], [And84]):

> **Theorem** *If $M^n \subset \mathbb{R}^N$, $n \geq 2$, is a complete minimal submanifold, then $\mathcal{A}(M) < \infty$ if and only if M has a compactification via the Gauss map, which gives a conformal diffeomorphism between M and a compact Riemannian manifold $\overline{M}$ minus a finite number of points.*

This result motivated the question of classification of higher-dimensional minimal submanifolds with finite total scalar curvature. Much less is known than in the case of minimal surfaces. The only known examples of complete minimal submanifolds $M^n \subset \mathbb{R}^N$, $n \geq 3$, with $\mathcal{A}(M) < \infty$, are (higher-dimensional) catenoids (see [Bla75]) and planes. An outstanding question in this area is whether or not any other examples exist. R. Schoen ([Sch83]) proved that any complete minimally immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ which is "regular at infinity" (a condition which is equivalent to $\mathcal{A}(M) < \infty$) and has two ends, is either a pair of planes or is a catenoid. In Chapter 2, I prove that if M is a complete minimal immersion of arbitrary codimension, $M^n \subset \mathbb{R}^N$, $n \geq 3$, with $\mathcal{A}(M) < \infty$, and if M has two ends, then M is either embedded or planar. Moreover, if M is embedded, and if $n \geq \frac{N+1}{2}$, I prove that M^n actually lies between two parallel n-planes in some subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^N$. When combined with Schoen's result, this yields the following:

> **Theorem** *Let $M^n \subset \mathbb{R}^N$, $n \geq 3$, $n \geq \frac{N+1}{2}$, be a complete minimal immersion having two ends and $\mathcal{A}(M) < \infty$. Then either M is planar or M is a catenoid.*

In Chapter 3, I examine the case of complex submanifolds of $\mathbb{C}^N \cong \mathbb{R}^{2N}$, and prove the following:

> **Theorem** *Let $M^k \subset \mathbb{C}^N$, be a smooth complete connected complex submanifold of complex dimension $k \geq 2$, with $\mathcal{A}(M) < \infty$. Then M is a plane.*

This contrasts the situation for complex dimension $k = 1$. In that case, there are many examples with $\mathcal{A}(M) < \infty$. The main step in the proof of the result above is the following Gap Theorem:

<u>**Theorem**</u> *Let M^k be a smooth complete connected complex submanifold in $\mathbb{C}^N$. There exists $\varepsilon = \varepsilon(k,N) > 0$ such that if*

$$\mathcal{B}(M) := \sup_r \frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 \, dV \leq \varepsilon$$

for some $x \in M$, then M is a plane.

Here k is the complex dimension of M, $D_x(r)$ is the geodesic ball about x of radius r in M, A is the second fundamental form of M, and dV is the induced volume form on M. The quantity $\mathcal{B}(M)$ comes from a term in the expansion of Weyl's formula for the volume of a tube ([Gri78], [And86], [Wey39]), which has been made to be invariant under rescaling. The Gap Theorem says that for the given class of submanifolds, there is an interval of values $(0, \varepsilon]$ which the curvature quantity $\mathcal{B}$ does not attain. (See [And84] and [Kas86] for related results.) In further work, I hope to use the Gap Theorem to obtain a theory for complex submanifolds M with $\mathcal{B}(M) < \infty$, which would give global geometric and topological information, similar to that in [And84] for minimal submanifolds M with $\mathcal{A}(M) < \infty$.

In Chapter 4, I prove a result on the singular set of the tangent cones at infinity of complete stable minimal hypersurfaces in Euclidean space with bounded volume growth. I have proved that the singular set of such tangent cones has codimension 2. It appears likely that the singular set has even higher codimension. It is known that for complete area-minimizing minimal hypersurfaces, the singular set of the tangent cones at infinity has codimension 7, and there are examples with singular set of codimension exactly 7. ([Fed70]) Thus, the best result to hope for in the case of stable minimal hypersurfaces is a singular set of codimension between 2 and 7. It would be interesting to know if there are any examples of stable minimal hypersurfaces with bounded volume growth which are not area-minimizing.

Chapter 2

A Result for Higher Codimensions

2.1 Collected Theorems (for reference)

I will use the following theorems, which I list here for convenience:

__Theorem A__ *Let $M^n \subset \mathbb{R}^N$ be a complete minimal immersion, $n \geq 2$, with finite total scalar curvature. Then each end of M has a unique n-plane as its tangent cone at infinity.* [And84, 3.1, p. 17]

__Theorem B__ *Let $M^n \subset \mathbb{R}^N$ be a complete minimal immersion, $n \geq 2$, with finite total scalar curvature. Then M has only finitely many ends, each of finite topological type.* [And84, 2.5, p. 13]

__Theorem C__ *Let $M^n \subset \mathbb{R}^N$, be a complete embedding of finite topological type, $n \geq 3$, $n - 1 = \frac{N}{2}$, with a well-defined tangent* *plane at infinity on each end. Then for each end* V_i,

$$\Sigma_i(r) := \frac{1}{r}(V_i \cap S^{N-1}(r))$$

converges to the same subsphere S^{n-1} *of* S^{N-1} *as* r *goes to infinity.* [JM83, Theorem 3, p. 209]

Theorem D *Assume* $n \geq 3$, *and* $M^n \subset \mathbb{R}^{n+1}$ *is a minimal immersion with the property that* $M - K$, *for some compact* K, *is a union of* $M_1, ..., M_r$, *where each* M_i *is a graph of bounded slope over the exterior of a bounded region in a hyperplane. Then* M *is regular at infinity. (Such an* M *is said to be "regular at infinity" if it satisfies the hypotheses of this theorem and, additionally,*

$$u_i(x) = b + a|x|^{2-n} + \sum_{j=1}^{n} c_j x_j |x|^{-n} + \mathcal{O}(|x|^{-n}),$$

is an expression of the graph M_i.*)* [Sch83, Prop. 3, p. 802]

Theorem E *Let* $M^n \subset \mathbb{R}^N$ *be a complete minimal immersion,* $n \geq 2$, *with finite total scalar curvature. Then* M *is properly immersed. (Recall that an immersion is said to be proper if the inverse image of any compact set is compact.)* [And84, 2.4, p. 13]

2.2 Statements of Results

The main results in this chapter are the following:

Theorem 1 *Let $M^n \subset \mathbb{R}^{n+1}$ be a complete connected minimal embedding, $n \geq 3$, with finite total scalar curvature. Then M^n lies between two parallel n-planes in $\mathbb{R}^{n+1}$.*

Theorem 2 *Let $M^n \subset \mathbb{R}^N$, $n \geq 3$, be a complete oriented minimal immersion, with finite total scalar curvature. Suppose M has two ends. Then either M is the union of two n-planes, or M is connected and embedded.*

Theorem 2 is related to a result obtained by R. Schoen ([Sch83]) for the case $N = n + 1$. The proof I give here is a simple argument using the monotonicity formula and a theorem of M. Anderson.

Theorem 3 *Let $M^n \subset \mathbb{R}^N$, $n \geq 3$, $n \geq \frac{N+1}{2}$, be a complete connected oriented minimal embedding with finite total scalar curvature. If M has two ends, then M lies between two parallel n-planes in some $\mathbb{R}^{n+1} \subset \mathbb{R}^N$.*

Due to Schoen's work, the following is a corollary of Theorem 3:

Theorem *Let $M^n \subset \mathbb{R}^N$, $n \geq 3$, $n \geq \frac{N+1}{2}$, be a complete connected oriented minimal immersion with finite total scalar curvature. Suppose M has two ends. Then M is a catenoid.*

A related result is the following theorem of Anderson: if $M^n \subset \mathbb{R}^N$, $n \geq 3$, is a complete oriented minimal immersion with $\mathcal{A}(M) < \infty$ and one end, then M is a plane. [And84, Theorem 5.2]

2.3 Proof of Theorem 1

Proof From Theorem A, M has tangent n-planes at infinity. Denote the collection of these n-planes by $\{P_i\}_{i \in I}$, where I is some index. Theorem B says that M has only finitely many ends, and thus finitely many tangent n-planes at infinity, say $P_1, ..., P_k$. All of the $\{P_i\}$ are parallel in $\mathbb{R}^{n+1}$ by Theorem C. These may be assumed to be of the form

$$P_i = \{(y_1, ..., y_{n+1}) \in \mathbb{R}^{n+1} | y_{n+1} = c_i\},$$

with $c_{i+1} \geq c_i$ for $i = 1, 2, ..., k-1$.

Define the set Δ by

$$\Delta := \{(y_1, ..., y_{n+1}) \in \mathbb{R}^{n+1} | c_1 \leq y_{n+1} \leq c_k\}.$$

For every $\varepsilon > 0$, let

$$Y_\varepsilon := \{q \in M \subset \mathbb{R}^{n+1} | d(q, \Delta) \geq \varepsilon\}.$$

Y_ε is the set of points in M which are bounded away from Δ by ε. From Theorem A, the ends of M are asymptotically flat. Also, each end is asymptotic to an n-plane, which must be one of the n-planes $P_i \subset \Delta$. Thus each set Y_ε is compact. If M is written as $\{u(x)\} = \{(u_1(x), ..., u_{n+1}(x)\}$, then the coordinate functions u_β, $\beta = 1, ..., n+1$, are harmonic. Theorem E says that u is a proper map, so, for each $\varepsilon > 0$, the preimage $X_\varepsilon := u^{-1}(Y_\varepsilon)$ of the compact set Y_ε, is compact.

Fix $\varepsilon_0 > 0$. Suppose there exists a point $\tilde{y} = (\tilde{y}_1, ..., \tilde{y}_{n+1}) \in Y_{\varepsilon_0}$ such that $\tilde{y}_{n+1} \geq \varepsilon_0$. By the maximum principle for harmonic functions, u_{n+1} achieves a maximum value m on the compact set $X_{\varepsilon_0} = u^{-1}(Y_{\varepsilon_0})$, and $m \geq \varepsilon_0$. Since u_{n+1} has values strictly less than ε_0 on $u^{-1}(M - \tilde{Y}_{\varepsilon_0})$, m must be a global maximum of u_{n+1} on $u^{-1}(M)$. But $\partial M = \emptyset$, so $\partial(u^{-1}(M)) = \emptyset$. So it must be that $u_{n+1} \equiv m \geq \varepsilon_0 > 0$. This contradicts the fact that the ends of M are asymptotic to the n-planes $P_i \subset \Delta$, $i = 1, ..., k$. Thus there can be no point $\tilde{y}$ in Y_{ε_0} with $\tilde{y}_{n+1} \geq \varepsilon_0$. This argument holds for all $\varepsilon_0 \geq 0$, so $u_{n+1} \leq c_k$. A similar argument yields $u_{n+1} \geq c_1$, so that M lies between the two n-planes P_1 and P_k. □

Remark: In this case, M is either a plane, or M has at least two ends which are asymptotic to catenoid ends. This follows from [Sch83, p. 800], where the definition of *regular at infinity* is extended to include minimal submanifolds of arbitrary codimension.

2.4 Proof of Theorem 2

Proof Since M is a complete minimal immersion, every connected component of M has at least one end (there cannot be any compact components). So M having two ends implies that M has either one or two components. If M has two components, then each component must have one end, and the following theorem can be applied to each component:

> **Theorem** *Let $\tilde{M}^n \subset \mathbb{R}^N$ be a complete connected minimal immersion, $n \geq 3$, with one end and finite total scalar curvature. Then $\tilde{M}$ is a plane.* [And84, p. 28]

So M is either a pair of planes, or M has only one component. That is, M is either a pair of planes, or M is connected.

Now define

$$v_p(r) := \frac{\text{vol}(M^n \cap B_p^N(r))}{r^n},$$

for $p \in M$, $r > 0$. I will use the following:

<u>Theorem</u> *If $M^n \subset \mathbb{R}^N$ is minimal and $p \in M$, then $v_p(r)$ is monotonically nondecreasing in r.* [Fed69, 5.4.3], [Sim83, Section 17]

<u>Theorem</u> *If $M^n \subset \mathbb{R}^N$ is analytic at p, then*

$$v_p(0) := \lim_{r \to 0^+} v_p(r) \geq \alpha(n),$$

where $\alpha(n)$ is the n-dimensional volume of a unit n-ball. [Fed69, 3.2.19]

<u>Theorem</u> *Let $M^n \subset \mathbb{R}^N$ be complete oriented minimal immersion, with finite total scalar curvature. Then M is diffeomorphic to a Riemannian manifold with finitely many points removed. Also, each end of M has multiplicity one and is embedded.* [And84, Theorem 3.2 and Theorem 5.1]

By Theorem A and the theorems above, M can be written as $M = V_1 \cup V_2 \cup S$, where the V_i are embedded ends, $i = 1, 2$, and S is compact. Since each end has an n-plane as its tangent cone at infinity,

$$\lim_{r \to \infty} \frac{\text{vol}(V_i \cap B_p^N(r))}{r^n} = \alpha(n),$$

for each $i = 1, 2$. Also,

$$\lim_{r \to \infty} v_p(r) = \lim_{r \to \infty} \frac{\mathrm{vol}((V_1 \cup V_2) \cap B_p^N(r))}{r^n} \leq 2\alpha(n),$$

since analyticity implies that $V_1 \cap V_2$ has zero n-dimensional measure, unless V_1 and V_2 coincide everywhere.

Let $\tilde{p}$ be a point where M self-intersects. Then $v_{\tilde{p}}(0) \geq 2\alpha(n)$, unless M coincides with itself in a neighborhood of $\tilde{p}$. In the case of co-incidence, analyticity forces M to have two components, and thus M is a pair of coincident planes.

Assume that M is not a pair of coincident planes, and let $\tilde{p}$ be a point where M self-intersects. Then $v_{\tilde{p}}(0) \geq 2\alpha(n)$. But

$$\lim_{r \to \infty} v_{\tilde{p}}(r) \leq 2\alpha(n),$$

and $v_{\tilde{p}}(r)$ is monotonically nondecreasing in r, so $v_{\tilde{p}}(r) \equiv 2\alpha(n)$. Thus M must be the union of two planes. (Otherwise, M would have $v_{\tilde{p}}(r) > 2\alpha(n)$ for some $r > 0$.)

If M is not the union of two planes, then there is no such $\tilde{p}$, and M is embedded. □

2.5 Proof of Theorem 3

First, I prove the following lemma, which generalizes a result proved by Jorge and Meeks for the case $n = \frac{N+1}{2}$ ([JM83, Theorem 3, p. 209]).

Lemma *Let $M^n \subset \mathbb{R}^N$ be a complete embedding of finite topological type, $n \geq 3$, $n \geq \frac{N+1}{2}$, with a well-defined tangent plane at infinity on each end.*

Then the components of

$$X_r := \frac{1}{r}(M^n \cap S^{N-1}(r))$$

converge C^1 *to the same subsphere* S^{n-1} *of* S^{N-1} *as* r *goes to infinity.*

Proof of Lemma Part (2) of Theorem 2 in [JM83] holds, to give that each component C_r^i of X_r converges C^1 to a totally geodesic sphere S_i^{n-1} in S^{N-1}. Suppose S_i^{n-1} and S_j^{n-1} are distinct. Let P_i^n, P_j^n be the n-planes containing S_i^{n-1}, S_j^{n-1}, respectively. Since S_i^{n-1}, S_j^{n-1} are totally geodesic, $\vec{0} \in P_i^n \cap P_j^n$. The condition $n \geq \frac{N+1}{2}$ implies $2n \geq N+1$, which implies that $P_i^n \cap P_j^n \subset \mathbb{R}^N$ is at least a one-dimensional plane. Thus the intersection of the great spheres S_i^{n-1} and S_j^{n-1} contains at least some great sphere S^0. Since it was supposed that the S_i^{n-1} and S_j^{n-1} are distinct, they must intersect in S^{N-1} without coinciding. The C^1-convergence in Theorem 2 in [JM83] implies that C_r^i and C_r^j intersect transversely for large r, which contradicts the hypothesis that M^n is embedded. Thus S_i^{n-1} and S_j^{n-1} must be the same great sphere in S^{N-1}. □

Proof of Theorem 3 Let V_1 and V_2 be the ends of M. Let Π_1 and Π_2 be the n-dimensional tangent planes at infinity of V_1 and V_2, respectively. The lemma implies that Π_1 and Π_2 are parallel, in the sense that they differ only by a translation normal to the Π_i's.

Let Δ be the convex hull of $\Pi_1 \cup \Pi_2$; that is, the intersection of all N-dimensional halfspaces in $\mathbb{R}^N$ containing $\Pi_1 \cup \Pi_2$. Let L be a line which intersects Π_1 and Π_2, and which is also normal to Π_1 and Π_2. Consider the set of all points on lines which intersect Π_1 and Π_2 and are parallel to L. This set is an $(n+1)$-dimensional subspace of $\mathbb{R}^N$, which I will denote by $\mathbb{R}^{n+1}$. Clearly $\Delta \subset \mathbb{R}^{n+1}$.

The n-planes Π_1 and Π_2 are contained in Δ by definition. Theorem A implies that the ends of M are asymptotically flat, and approach the planes Π_1 and Π_2. Thus M could only be bounded away from Δ on a compact set, but the maximum principle rules this out. Since no part of M can be bounded any distance away from Δ, it must be true that $M \subset \Delta \subset \mathbb{R}^{n+1}$. □

2.6 Consequences

Theorem *Let $M^n \subset \mathbb{R}^N$, $n \geq 3$, $n \geq \frac{N+1}{2}$, be a complete connected oriented minimal immersion with finite total scalar curvature. Suppose M has two ends and is non-planar. Then M is a catenoid.*

Proof By Theorem 1, M is embedded. Thus, by Theorem 2, M^n lies in some $\mathbb{R}^{n+1} \subset \mathbb{R}^N$. R. Schoen [Sch83] proved that in this case, M is a catenoid. □

Chapter 3

Results on Complex Submanifolds

3.1 Preliminary Remarks

For a complete minimal submanifold $M^n \subset \mathbb{R}^N$, consider the curvature quantity $\mathcal{B}_x(M)$ defined by

$$\mathcal{B}_x(M) := \sup_r \frac{1}{r^{n-2}} \int_{M \cap B_x(r)} |A|^2 \, dV,$$

where $B_x(r)$ is the Euclidean N-ball about the point x in M with radius r.

Proposition *If $\mathcal{A}(M)$ is finite, then so is $\mathcal{B}_x(M)$, for all $x \in M$. And if $\mathcal{B}_x(M)$ is infinite for any $x \in M$, then $\mathcal{A}(M)$ is also infinite.*

Proof Suppose $\mathcal{A}(M)$ is finite, and let x be any point in M. By Hölder's inequality, for any $r > 0$,

$$\begin{aligned} \int_{M \cap B_x(r)} |A|^2 \, dV &\leq \left(\int_{M \cap B_x(r)} |A|^n \, dV \right)^{\frac{2}{n}} \cdot \left(\int_{M \cap B_x(r)} 1 \, dV \right)^{\frac{n-2}{n}} \\ &= \left(\int_{M \cap B_x(r)} |A|^n \, dV \right)^{\frac{2}{n}} \cdot [\mathrm{Vol}(M \cap B_x(r))]^{\frac{n-2}{n}} . \end{aligned}$$

So

$$\frac{1}{r^{n-2}}\int_{M\cap B_x(r)}|A|^2\,dV \le \left(\int_{M\cap B_x(r)}|A|^n\,dV\right)^{\frac{2}{n}}\cdot\left[\frac{\mathrm{Vol}(M\cap B_x(r))}{r^n}\right]^{\frac{n-2}{n}}.$$

The condition $\mathcal{A}(M)<\infty$ implies that M has a finite number, say l, of tangent n-planes as its tangent cone at infinity, and therefore implies that

$$\lim_{r\to\infty}\frac{\mathrm{Vol}(M\cap B_x(r))}{r^n}=l.$$

By the monotonicity property of minimal submanifolds ([Fed69, 5.4.3], [Sim83, Section 17]),

$$\frac{\mathrm{Vol}(M\cap B_x(r))}{r^n}\le l$$

for all $r>0$. Since

$$\int_{M\cap B_x(r)}|A|^n\,dV\le\int_M|A|^n\,dV$$

for all $r>0$, then

$$\frac{1}{r^{n-2}}\int_{M\cap B_x(r)}|A|^2\,dV\le\left(\int_M|A|^n\,dV\right)^{\frac{2}{n}}\cdot(l)^{\frac{n-2}{n}}<\infty,$$

for all finite $r>0$. Also,

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r^{n-2}}\int_{M\cap B_x(r)}|A|^2\,dV\\
\le\ &\lim_{r\to\infty}\left\{\left(\int_{M\cap B_x(r)}|A|^n\,dV\right)^{\frac{2}{n}}\cdot\left[\frac{\mathrm{Vol}(M\cap B_x(r))}{r^n}\right]^{\frac{n-2}{n}}\right\}\\
=\ &\left(\int_M|A|^n\,dV\right)^{\frac{2}{n}}\cdot(l)^{\frac{n-2}{n}}<\infty.
\end{aligned}$$

So then

$$\sup_{r\to\infty}\frac{1}{r^{n-2}}\int_{M\cap B_x(r)}|A|^2\,dV<\infty.$$

In the case that $\mathcal{B}_x(M)$ is infinite for some $x \in M$, the same inequalities show that $\mathcal{A}(M)$ is also infinite. □

If $n = 2$, then $\mathcal{A}(M) \equiv \mathcal{B}_x(M)$ for all $x \in M$. For $n \geq 3$, there are examples of submanifolds with $\mathcal{B}_x(M) < \infty$ for all $x \in M$, which do not have $\mathcal{A}(M) < \infty$. Thus the class of minimal submanifolds with $\mathcal{B}_x(M)$ bounded for all $x \in M$ is larger than the class with $\mathcal{A}(M)$ bounded. It would be interesting to have a description of the global geometric and topological behavior of the class with $\mathcal{B}_x(M)$ bounded.

3.2 Proof of the Gap Theorem

Lemma 1 *Let X^k be a smooth complex submanifold in $\mathbb{C}^N$ with $dim_{\mathbb{C}} X = k$. Let $B_x(r)$ be the ball in $\mathbb{C}^N$ of radius r centered about x, and let $D_x(r) := X \cap B_x(r)$. Suppose $D_x(1) \cap \partial X = \emptyset$ for some $x \in X$. There exists $\varepsilon > 0$ such that if*

$$\sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 \, dV \leq \varepsilon,$$

then

$$\sup_{t\in[0,1]} \left\{ t^2 \cdot \sup_{p\in D_x(1-t)} |A|^2(p) \right\} \leq 4.$$

Proof Suppose the lemma is false. Then there exists a sequence of smooth complex minimal immersions $h_i : X_i^k \to \mathbb{C}^N$ with $D_{x_i}(1) \cap \partial X_i = \emptyset$ for some points $x_i \in X_i$, such that

$$\sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_{x_i}(r)} |A|_i^2 \, dV_i \to 0,$$

as $i \to \infty$, but

$$\sup_{t\in[0,1]} \left\{ t^2 \cdot \sup_{p\in D_{x_i}(1-t)} |A|_i^2(p) \right\} > 4$$

for all $i = 1, 2, 3, \ldots$. Choose $t_i \in [0,1]$ such that

$$t_i^2 \cdot \sup_{p\in D_{x_i}(1-t_i)} |A|_i^2(p) = \sup_{t\in[0,1]} \left\{ t^2 \cdot \sup_{p\in D_{x_i}(1-t)} |A|_i^2(p) \right\}$$

and choose $y_i \in \overline{D_{x_i}(1-t_i)}$ such that

$$|A|_i^2(y_i) = \sup_{p\in D_{x_i}(1-t_i)} |A|_i^2(p).$$

Then

$$t_i^2 |A|_i^2(y_i) = t_i^2 \cdot \sup_{p\in D_{x_i}(1-t_i)} |A|_i^2(p) = \sup_{t\in[0,1]} \left\{ t^2 \cdot \sup_{p\in D_x(1-t)} |A|^2(p) \right\}$$

$$\geq \left(\frac{t_i}{2}\right)^2 \cdot \sup_{p\in D_{x_i}\left(1-\frac{t_i}{2}\right)} |A|_i^2(p)$$

implies

$$4|A|_i^2(y_i) \geq \sup_{p\in D_{x_i}\left(1-\frac{t_i}{2}\right)} |A|_i^2(p) \geq \sup_{p\in D_{y_i}\left(\frac{t_i}{2}\right)} |A|_i^2(p),$$

since

$$D_{y_i}\left(\frac{t_i}{2}\right) \subset D_{x_i}\left(1-\frac{t_i}{2}\right).$$

Let $d\tilde{s}_i^2 = |A|_i^2(y_i) ds_i^2$ be the metric on X_i induced by the complex immersion $\widetilde{h}_i = \delta_i \cdot h_i$, where δ_i is dilation of $\mathbb{C}^N$ about $h_i(y_i)$ by the factor $|A|_i(y_i)$.

It may be assumed that $h_i(y_i) = 0$, by translating, if necessary. Note the following relationships, where $\widetilde{}$ signifies a quantity with respect to the induced metric:

$$\widetilde{D}_p(s) = D_p\left(\frac{1}{|A|_i(y_i)} \cdot s\right),$$

$$\widetilde{r} = |A|_i(y_i) \cdot r,$$

$$\widetilde{dV}_i = |A|_i^{2k}(y_i) dV_i,$$

and

$$|\widetilde{A}|_i(p) = \frac{1}{|A|_i(y_i)} \cdot |A|_i(p),$$

where $r = |x|$, $x \in \mathbb{C}^N$. Then

$$\sup_{p \in \widetilde{D}_{y_i}(|A|_i(y_i) \cdot \frac{t_i}{2})} |A|_i^2(y_i) |\widetilde{A}|_i^2(p) = \sup_{p \in D_{y_i}(\frac{t_i}{2})} |A|_i^2(p) \le 4|A|_i^2(y_i),$$

and so

$$\sup_{p \in \widetilde{D}_{y_i}(|A|_i(y_i) \cdot \frac{t_i}{2})} |\widetilde{A}|_i^2(p) \le 4.$$

By assumption, $t_i^2 |A|_i^2(y_i) > 4$, so $|A|_i(y_i) \cdot \frac{t_i}{2} > 1$, and $\widetilde{D}_{y_i}(1) \subset \widetilde{D}_{y_i}(|A|_i(y_i) \cdot \frac{t_i}{2})$. Thus

$$\sup_{p \in \widetilde{D}_{y_i}(1)} |\widetilde{A}|_i^2(p) \le \sup_{p \in \widetilde{D}_{y_i}(|A|_i(y_i) \cdot \frac{t_i}{2})} |\widetilde{A}|_i^2(p) \le 4.$$

The sequence $\widetilde{h}_i : \widetilde{D}_{y_i}(1) \to \mathbb{C}^N$ is then a sequence of smooth complex immersions of open geodesic balls of radius 1 with uniformly bounded curvature, translated so that $\widetilde{h}_i(y_i) = 0$. By the smooth compactness theorem, a subsequence converges in the C^∞- topology on compact subsets of $\mathbb{C}^N$, to a complex analytic immersion $\widetilde{h}_\infty : \widetilde{D}_{y_\infty}(1) \to \mathbb{C}^N$ with $\sup_{p \in \widetilde{D}_{y_\infty}(1)} |\widetilde{A}|_\infty^2 \le 4$.

This convergence implies

$$\int_{\widetilde{D}_{y_i}} |\widetilde{A}|_i^2 \widetilde{dV}_i \to \int_{\widetilde{D}_{y_\infty}} |\widetilde{A}|_\infty^2 \widetilde{dV}_\infty.$$

Note that $|\widetilde{A}|_i(y_i) = \frac{1}{|A|_i(y_i)} \cdot |A|_i(y_i) = 1$, for all $i = 1, 2, 3, \ldots$ implies $|\widetilde{A}|_\infty(y_\infty) = 1$.

By a result of Griffiths [Gri78, p. 471], the quantity

$$\frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 dV$$

is monotonically increasing as a function of r if X is a smooth complex analytic variety. Thus

$$\begin{aligned} \sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_{x_i}(r)} |A|_i^2 dV_i &= \sup_{0<\tilde{r}\leq 2|A|_i(y_i)} \frac{1}{\tilde{r}^{2k-2}} \int_{\widetilde{D}_{x_i}(\tilde{r})} |\widetilde{A}|_i^2 \widetilde{dV}_i \\ &= \frac{1}{(2|A|_i(y_i))^{2k-2}} \int_{\widetilde{D}_{x_i}(2|A|_i(y_i))} |\widetilde{A}|_i^2 \widetilde{dV}_i \to 0. \end{aligned}$$

Since $y_i \in D_{x_i}(1-t) \subset D_{x_i}(1)$, then $D_{y_i}(1) \subset D_{x_i}(2)$ and

$$\widetilde{D}_{y_i}(|A|_i(y_i)) \subset \widetilde{D}_{x_i}(2|A|_i(y_i)).$$

So

$$\begin{aligned} \int_{\widetilde{D}_{y_i}(1)} |\widetilde{A}|_i^2 \widetilde{dV}_i &\leq \frac{1}{|A|^{2k-2}(y_i)} \int_{\widetilde{D}_{y_i}(|A|_i(y_i))} |\widetilde{A}|_i^2 \widetilde{dV}_i \\ &\leq \frac{1}{|A|^{2k-2}(y_i)} \int_{\widetilde{D}_{x_i}(2|A|_i(y_i))} |\widetilde{A}|_i^2 \widetilde{dV}_i \to 0. \end{aligned}$$

But

$$\int_{\widetilde{D}_{y_\infty}(1)} |\widetilde{A}|_\infty^2 \widetilde{dV}_\infty = 0$$

contradicts $|\widetilde{A}|_\infty(y_\infty) = 1$. This contradiction means that the lemma must be true. □

Lemma 2 *Let X^k be a smooth complex analytic submanifold in $\mathbb{C}^N$, with $D_x(1) \cup \partial X = \emptyset$ for some $x \in X$. There exists $\varepsilon_0 > 0$ such that if*

$$\sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 dV = \varepsilon \leq \varepsilon_0,$$

then

$$\sup_{p\in D_x(\frac{1}{2})} |A|^2(p) < \delta,$$

where $\delta \to 0$ as $\varepsilon \to 0$.

Proof Suppose Lemma 2 is false. Then there exists a sequence of smooth complex analytic immersions $g_i : Z_i^k \to \mathbb{C}^N$ with $g_i(z_i) = 0$ and $D_{z_i}(1) \cup \partial Z_i = \emptyset$ for some $z_i \in Z_i$, such that

$$\sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_{z_i}(r)} |A|_i^2 dV_i \to 0,$$

but

$$\sup_{p\in D_{z_i}(\frac{1}{2})} |A|_i^2(z_i) \geq C,$$

for some fixed constant $C > 0$. From the previous lemma,

$$\sup_{t\in[0,1]} \left\{ t^2 \cdot \sup_{p\in D_{z_i}(1-t)} |A|_i^2(p) \right\} \leq 4,$$

for i sufficiently large. So

$$\left(\frac{1}{2}\right)^2 \cdot \sup_{p\in D_{z_i}(\frac{1}{2})} |A|_i^2(p) \leq 4, \text{ and } \sup_{p\in D_{z_i}(\frac{1}{2})} |A|_i^2(p) \leq 16.$$

By the smooth compactness theorem, a subsequence of $g_i : D_{z_i}(\frac{1}{2}) \to \mathbb{C}^N$ converges in the C^∞-topology on compact subsets of $\mathbb{C}^N$, to a smooth minimal immersion $g_\infty : D_{z_\infty}(\frac{1}{2}) \to \mathbb{C}^N$. Since

$$\sup_{0<r\leq\frac{1}{2}} \frac{1}{r^{2k-2}} \int_{D_{z_i}(r)} |A|_i^2 dV_i \leq \sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_{z_i}(r)} |A|_i^2 dV_i \to 0,$$

and

$$\sup_{0<r\leq\frac{1}{2}} \frac{1}{r^{2k-2}} \int_{D_{z_i}(r)} |A|_i^2 dV_i \to \sup_{0<r\leq\frac{1}{2}} \frac{1}{r^{2k-2}} \int_{D_{z_\infty}(r)} |A|_\infty^2 dV_\infty,$$

it follows that

$$\sup_{0<r\leq\frac{1}{2}} \frac{1}{r^{2k-2}} \int_{D_{z_\infty}(r)} |A|_\infty^2 dV_\infty = 0.$$

Further,

$$\frac{1}{(\frac{1}{2})^{2k-2}} \int_{D_{z_\infty}(\frac{1}{2})} |A|_\infty^2 dV_\infty \leq \sup_{0<r\leq\frac{1}{2}} \frac{1}{r^{2k-2}} \int_{D_{z_\infty}(r)} |A|_\infty^2 dV_\infty = 0$$

implies

$$\int_{D_{z_\infty}(\frac{1}{2})} |A|_\infty^2 dV_\infty = 0.$$

But it is also true that

$$\sup_{p\in D_{z_\infty}(\frac{1}{2})} |A|_\infty^2(p) \geq C > 0,$$

which gives a contradiction. Thus the lemma must be true. □

Gap Theorem *Let $h : M^k \to \mathbb{C}^N$ be a smooth complete connected complex analytic immersion. There exists $\varepsilon_0 = \varepsilon_0(k, N)$ such that if*

$$\sup_r \frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 dV \leq \varepsilon_0$$

for some $x \in M$, then M is a plane.

Proof By translation, assume that $h(x) = 0$, and let δ_R be dilation of $\mathbb{C}^N$ by $1/R$. Let the new metric quantities with respect to the immersions $\delta_R \circ h$

be denoted by a subscript of R. E.g., $r_R = \frac{1}{R} \cdot r$, $dV_R = \frac{1}{R^{2k}} dV$, $(D_R)_x(s) = D_x(R \cdot s)$, and $|A|_R(x) = R \cdot |A|(x)$. Let $r' \in (0, 2]$ be such that

$$\frac{1}{(r')^{2k-2}} \int_{D_x(r')} |A|^2 dV = \sup_{0<r\leq 2} \frac{1}{(r)^{2k-2}} \int_{D_x(r)} |A|^2 dV.$$

By monotonicity, such an r' exists, and $r' = 2$ suffices. Note that

$$\frac{1}{(r')^{2k-2}} \int_{D_x(r')} |A|^2 dV = \frac{1}{(r'_R)^{2k-2}} \int_{(D_R)_x(r'_R)} |A|_R^2 \, dV_R$$

for all $R > 0$. So the quantity

$$\sup_{0<r\leq 2} \frac{1}{r^{2k-2}} \int_{D_x(r)} |A|^2 dV = \frac{1}{(r')^{2k-2}} \int_{D_x(r')} |A|^2 dV$$

is invariant under rescaling. Let ε_0 be the constant given in Lemma 2. So

$$\frac{1}{(r')^{2k-2}} \int_{D_x(r')} |A|^2 dV = \frac{1}{(r'_R)^{2k-2}} \int_{(D_R)_x(r'_R)} |A|_R^2 \, dV_R = \widetilde{\varepsilon} \leq \varepsilon_0$$

for the immersions $\delta_R \circ h$, for all $R > 0$. Then

$$\sup_{p \in (D_R)_x(\frac{1}{2})} |A|_R^2(p) < \widetilde{\delta}$$

for all $R > 0$, where $\widetilde{\delta} \to 0$ as $\widetilde{\varepsilon} \to 0$. So

$$\sup_{p \in (D)_x(\frac{R}{2})} R^2 |A|^2(p) < \widetilde{\delta}$$

and

$$\sup_{p \in (D)_x(\frac{R}{2})} |A|^2(p) < \frac{\widetilde{\delta}}{R^2}$$

for all $R > 0$. Letting $R \to \infty$ yields $\sup_{p \in X} |A|^2(p) = 0$, which implies $|A| \equiv 0$ on X. Thus X is planar. $\square$

Remark: The preceeding proof runs along similar lines to the proof of Proposition 2.2 in [And84].

Corollary *Let $M^n \subset \mathbb{C}^N$ be a smooth complete connected complex submanifold with complex dimension $n \geq 2$ and finite total scalar curvature. Then M is a plane.*

Proof I will use the following pointwise estimate of Anderson [And84, p. 22]:

$$|A|(x) \leq \frac{c}{|x|^{2n}}$$

for large $|x|$, where $c > 0$ is a constant.

Note that

$$\lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{D_{x_0}(r)} |A|^2 dV \leq \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{B_{x_0}(r)\cap M} |A|^2 dV.$$

Also, for any constant $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{B_{x_0}(r)\cap M} |A|^2 dV < \infty \Leftrightarrow \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{(B_{x_0}(r)-B_{x_0}(\varepsilon))\cap M} |A|^2 dV < \infty,$$

since $|A|$ is smooth and bounded on M (by the theory of minimal submanifolds with finite total scalar curvature [And84]). So then

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{r^{2n-2}} & \int_{(B_{x_0}(r)-B_{x_0}(\varepsilon))\cap M} |A|^2 dV \\
&\leq \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{(B_{x_0}(r)-B_{x_0}(\varepsilon))\cap M} \frac{c}{|x|^{4n}}\, dV \\
&\leq \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{B_{x_0}(r)-B_{x_0}(\varepsilon)} \frac{c}{|x|^{4n}}\, dV_{\mathbb{R}^N} \\
&= \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{\varepsilon}^{r} \int_{S^{N-1}(1)} \frac{c}{\rho^{4n}} \cdot \rho^{n-1}\, d\sigma\, d\rho
\end{aligned}$$

(where $dV_{\mathbb{R}^N}$ is the volume form on $\mathbb{R}^N$, $\rho = |x|$, and $d\sigma$ is the volume form on the unit sphere $S^{N-1}(1)$)

$$
\begin{aligned}
&= \lim_{r\to\infty} \frac{c}{r^{2n-2}} \cdot \mathrm{Vol}(S^{N-1}(1)) \int_\varepsilon^r \rho^{-3n-1} d\rho \\
&= \lim_{r\to\infty} \frac{c}{r^{2n-2}} \cdot \mathrm{Vol}(S^{N-1}(1)) \cdot \left.\frac{\rho^{-3n}}{-3n}\right|_\varepsilon^r \\
&= \lim_{r\to\infty} \frac{c}{r^{2n-2}} \cdot \mathrm{Vol}(S^{N-1}(1)) \cdot \frac{-1}{3n}\left(\frac{1}{r^{3n}} - \frac{1}{\varepsilon^{3n}}\right) \\
&= \frac{-c}{3n} \cdot \mathrm{Vol}(S^{N-1}(1)) \cdot \lim_{r\to\infty} \frac{1}{r^{2n-2}} \cdot \left(\frac{1}{r^{3n}} - \frac{1}{\varepsilon^{3n}}\right)
\end{aligned}
$$

Since n was assumed to be greater than or equal to two, the last line is zero. This argument holds for all $\varepsilon > 0$, so choose ε small enough such that

$$\frac{1}{\varepsilon^{2n-2}} \int_{B_{x_0}(\varepsilon)\cap M} |A|^2 dV \leq \varepsilon_0,$$

where ε_0 is the constant given in the Gap Theorem. Then

$$
\begin{aligned}
\lim_{r\to\infty} & \frac{1}{r^{2n-2}} \int_{D_{x_0}(r)} |A|^2 dV \\
&\leq \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{B_{x_0}(r)\cap M} |A|^2 dV \\
&= \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{(B_{x_0}(r)-B_{x_0}(\varepsilon))\cap M} |A|^2 dV \\
&\qquad + \lim_{r\to\infty} \frac{1}{r^{2n-2}} \int_{B_{x_0}(\varepsilon)\cap M} |A|^2 dV \\
&\leq \varepsilon_0.
\end{aligned}
$$

Thus M is planar. □

As was noted in the introduction, this result does not hold if $n = 1$. For example, the graph of the complex function $f(z) = z^2$ has total scalar curvature $\frac{2\pi}{5}$.

Chapter 4

Results on Stable Minimal Submanifolds

Proposition Let $M^n \subset \mathbb{R}^{n+1}$ be a complete stable minimal immersion with

$$\lim_{r\to\infty} \frac{\mathrm{Vol}^n(M^n \cap B^N_{x_0}(r))}{r^n} \leq K$$

for some $x_0 \in M$, and some positive constant K. Then

$$\sup_r \frac{1}{r^{n-2}} \int_{B_x(r)\cap M} |A|^2 dV \leq \widetilde{K}$$

for all $x \in M$, where $\widetilde{K}$ is a positive constant which depends on K and n.

Proof Since M is stable,

$$\int_M |A|^2 f^2 \leq \int_M |\nabla f|^2$$

for all Lipschitz functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with compact support in M. (See [Sim83, p. 53].) Let $\vec{0} \in M$, let $R > 0$, and let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } |x| \le R, \\ 1 - \frac{1}{R}(|x| - R) & \text{if } R \le |x| \le 2R, \\ 0 & \text{if } 2R \le |x|. \end{cases}$$

Then

$$\int_M |A|^2 f^2 \le \int_M |\nabla f|^2$$

implies

$$\int_{B_0(R)\cap M} |A|^2 \; + \; \int_{(B_0(2R)-B_0(R))\cap M} |A|^2 \left(2 - \frac{|x|}{R}\right) \; \le \; \int_{(B_0(2R)-B_0(R))\cap M} \frac{1}{R^2} \; .$$

Thus

$$\int_{B_0(R)\cap M} |A|^2 \; \le \; \int_{(B_0(2R)-B_0(R))\cap M} \frac{1}{R^2} \; \le \; \int_{B_0(2R)\cap M} \frac{1}{R^2} \; ,$$

$$\frac{1}{R^{n-2}} \int_{B_0(R)\cap M} |A|^2 \; \le \; \frac{1}{R^n} \cdot \mathrm{Vol}^n(B(2R)\cap M) \; ,$$

and

$$\lim_{R\to\infty} \frac{1}{R^{n-2}} \int_{B_0(R)\cap M} |A|^2 \; \le \; 2^n \cdot \lim_{R\to\infty} \frac{\mathrm{Vol}(B(2R)\cap M)}{(2R)^n} \; \le 2^n \cdot K = \widetilde{K} \; . \; \square$$

<u>**Theorem**</u> Let $M^n \subset \mathbb{R}^{n+1}$, $n \ge 3$, be a complete stable minimal immersion. Suppose there exists a point $x_0 \in M$ and a constant $K < \infty$ such that

$$\frac{\mathrm{Vol}^n(M^n \cap B^N_{x_0}(r))}{r^n} \le K$$

for all $r > 0$. Then each tangent cone at infinity of M has a singular set of codimension 2.

Proof First, note that

$$\frac{\mathrm{Vol}^n(M^n \cap B^N_{x_0}(r))}{r^n} \leq K$$

for some $x_0 \in M$ implies

$$\lim_{r\to\infty} \frac{\mathrm{Vol}^n(M^n \cap B^N_x(r))}{r^n} \leq K$$

for all $x \in M$. This is true because

$$\frac{\mathrm{Vol}[D_y(r) - D_x(r)]}{\mathrm{Vol}(D(r))} < \frac{(2r)^{n-1}|x-y|}{\mathrm{const}\cdot r^n} = \mathrm{const}\cdot\frac{1}{r} \to 0$$

as $r \to \infty$.

Choose a sequence $r_i \to \infty$. Define

$$T_i := \frac{1}{r_i}\left(M \cap B_{x_o}(r_i)\right),$$

and

$$T_\infty := \lim_{i\to\infty} \frac{1}{r_i}\left(M \cap B_{x_o}(r_i)\right).$$

Since M is stable, T_i is stable for each i. By the Compactness Theorem of geometric measure theory ([Fed69, 4.2.17], [Mor88, 5.5]), the limit T_∞ exists in the flat norm, and is minimal. This gives existence of tangent cones at infinity, but not uniqueness, since various choices for the sequence r_i could lead to various limits.

Claim: there exists a constant $N = N(n)$ such that for each i, there is a covering of T_i by $\frac{1}{r_i}$-balls with no point of T_i contained in more than N of the balls in the cover. Fix i. Take a maximal disjoint covering of T_i by $\frac{1}{2r_i}$-balls. Then the $\frac{1}{r_i}$-balls with the same centers cover T_i. Take a cube C with sides of length a. C intersects a finite number of $\frac{1}{2r_i}$-balls. Shrinking C to $\tilde{C}$ with sides of length $a - \frac{1}{r_i}$, $\tilde{C}$ intersects the same number of $\frac{1}{r_i}$-balls. Since T_i can be covered with a finite number of $\tilde{C}$'s, there is a number N such that no point of T_i lies in more than N of the $\frac{1}{r_i}$-balls. This number N depends on n and is independent of i. [Sim83, p. 127]

Let this cover of T_i be denoted by $\mathcal{O}_i$. Suppose the balls in $\mathcal{O}_i$ are denoted by $B_{x_{\alpha_i}}\left(\frac{1}{r_i}\right)$, for $\alpha_i \in \Gamma_i$, where Γ_i is some index. Let

$$U_{\alpha_i} := B_{x_{\alpha_i}}\left(\frac{1}{r_i}\right) \cap T_i.$$

Fix $\varepsilon > 0$. For each i, let Λ_i be the subset of Γ_i consisting of those α_i such that

$$\int_{U_{\alpha_i}} |A|^2 dV \geq \varepsilon.$$

By the Proposition,

$$\sup_r \frac{1}{r^{n-2}} \int_{B_x(r)\cap M} |A|^2 dV \leq \widetilde{K}$$

for all $x \in M$. So

$$\frac{1}{r^{n-2}} \int_{B_x(r)\cap M} |A|^2 dV \leq \widetilde{K}$$

for all $r > 0$, and for all $x \in M$. Since $U_{\alpha_i} \subset \frac{1}{r_i}\left(M \cap B_{x_0}(r_i)\right)$,

$$\frac{1}{r_i^{n-2}} \int_{U_{\alpha_i}} |A|^2 dV \leq \frac{1}{r_i^{n-2}} \int_{\frac{1}{r_i}(M\cap B_{x_0}(r_i))} |A|^2 dV.$$

Rescale the metric on M by a factor of $\lambda = r_i$ about the point $x_0 \in M$, and denote quantities with respect to this new metric by a subscript of λ. Then

$$\frac{1}{r_i^{n-2}} \int_{\frac{1}{r_i}(M \cap B_{x_0}(r_i))} |A|^2 dV = \frac{1}{(r_i)_\lambda^{n-2}} \int_{\left(\frac{1}{r_i}(M \cap B_{x_0}(r_i))\right)_\lambda} |A|_\lambda^2 dV_\lambda$$

$$= \frac{1}{\lambda^{n-2} \cdot r_i^{n-2}} \int_{\lambda \cdot \frac{1}{r_i}(M \cap B_{x_0}(r_i))} \frac{1}{\lambda^2} |A|^2 \lambda^n dV = \frac{1}{r_i^{n-2}} \int_{(M \cap B_{x_0}(r_i))} |A|^2 dV \leq \widetilde{K},$$

and so

$$\frac{1}{r_i^{n-2}} \int_{U_{\alpha_i}} |A|^2 dV \leq \widetilde{K}.$$

Since $U_{\alpha_i} = B_{x_{\alpha_i}}\left(\frac{1}{r_i}\right) \cap T_i$, no point of M is in more than N of the sets U_{α_i}. Thus, for any given i,

$$|\Lambda_i| \cdot \varepsilon \leq \sum_{\alpha_i \in \Lambda_i} \varepsilon \leq \sum_{\alpha_i \in \Lambda_i} \int_{U_{\alpha_i}} |A|^2 dV = r_i^{n-2} \sum_{\alpha_i \in \Lambda_i} \frac{1}{r_i^{n-2}} \int_{U_{\alpha_i}} |A|^2 dV$$

$$\leq r_i^{n-2} \cdot N \cdot \frac{1}{r_i^{n-2}} \int_{U_{\alpha_i}} |A|^2 dV \leq r_i^{n-2} \cdot N \cdot \widetilde{K}.$$

Dividing by ε yields

$$|\Lambda_i| \leq \text{const} \cdot r_i^{n-2}.$$

Since T_i has Hausdorff dimension n, $|\Gamma_i|$, the number of $\frac{1}{r_i}$-balls in the cover $\mathcal{O}_i$ of T_i, is proportional to r_i^n. The singular set of T_i is contained in the U_{α_i} for $\alpha_i \in \Lambda_i$. By the definition of m-dimensional Hausdorff measure ([Fed69, 2.10.2]), the singular set of each T_i is of codimension 2, and so the singular set of any T_∞ is of codimension 2. □

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