Quaternionic Discrete Series of Semisimple Lie Groups

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Abstract of the Dissertation

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This thesis investigates the discrete series of linear connected semisimple noncompact groups G. These are irreducible unitary representations that occur as direct summands of $L^2(G)$.

Harish-Chandra produced discrete series representations, now called holomorphic discrete series representations, for groups G with the property that, if K is a maximal compact subgroup, then G/K has a complex structure such that G acts holomorphically. Holomorphic discrete series are extraordinarily explicit, it being possible to determine all the elements in the space and the action by Lie algebra of G.

Later Harish-Chandra parametrized the discrete series in general. His argument did not give an actual realization of the representations, but later authors found realizations in spaces defined by homology or cohomology.

These realizations have the property that it is not apparent what elements are in the space and what the action of the Lie algebra G is.

The point of this thesis is to find some intermediate ground between the holomorphic discrete series and the general discrete series, so that the intermediate cases may be used to get nontrivial insights into the internal structure of the discrete series in the general case.

The thesis examines the Vogan-Zuckerman realization of discrete series by means of cohomological induction. An explicit complex for computing the homology on the level of a K module was already known. Also, Duflo and Vergne had given information about how to compute the action of the Lie algebra of G.

The holomorphic discrete series are exactly those cases where the representations can be realized in homology of degree 0. The intermediate cases that are studied are those where the representation can be realized in homology of degree 1. Many of the intermediate cases correspond to the situation where G/K has a quaternionic structure. The thesis obtains full results in special cases of this situation and partial results in general.

To my family with love

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INTRODUCTION

This thesis investigates the discrete series of linear connected semisimple noncompact groups G. These are irreducible unitary representations that occur as direct summands of $L^2(G)$.

It was Bargmann [Bar] who discovered that such representations can actually exist. In classifying the irreducible unitary representations for $G = SL(2, \mathbb{R})$, he found two countable families of discrete series , one in spaces of analytic functions on the unit disk and one in the space of complex conjugates.

Later Harish-Chandra ([HC1], [HC2] and [HC3]) was able to abstract Bargmann's construction and generalize it somewhat. Let K be a maximal compact subgroup of G. The setting for Harish-Chandra's generalization is that G/K has a complex structure such that G acts holomorphically. For each irreducible finite-dimensional representation of K whose highest weight satisfies a certain negativity property, Harish-Chandra considered the space of scalar-valued square-integrable holomorphic functions on G that transform under a maximal torus of K by the given highest weight, with G acting by translation on the functions. Harish-Chandra found that this space gave a discrete series representation, and such representations have come to be known as holomorphic discrete series. Holomorphic discrete series are extraordinarily explicit, and one can read off with relative ease what elements are in the space and how the Lie algebra of G operates.

Harish-Chandra's early work on the Plancherel formula for semisimple groups suggested that other groups should have discrete series, not just those with G/K complex, and yet no such representations were discovered for a number of years. Then in 1960 Dixmier [Dix] was able to classify the irreducible unitary representations of $G = SO(4,1)$, as well as its double cover, and to prove that some of his representations were discrete series. Dixmier's student Takahashi [Tak] gave global realizations of these representations, ostensibly explicit, and for the first time one had nonholomorphic examples.

In 1966 Harish-Chandra [HC4] succeeded in parametrizing all the discrete series for all semisimple groups G for which the rank of G equals the rank of the maximal compact K, and he showed that there were no discrete series if the equal rank condition failed. His parametrization was in terms of features of the global characters of such representations and did not give a global realization of any kind, other than as an unspecified subspace of $L^2(G)$.

Global realizations were the subject of the Langlands conjecture [Lan], which ultimately was proved by Schmid ([S1], [S2] and [S3]). It was shown that the discrete series can be realized as spaces of L^2 cohomology sections over G/T, where T is a maximal torus of G. Although the result was a space that one could define, neither the methods of proof nor direct computations showed how to produce a single nonzero element in the representation space.

Thus, except for $SO(4,1)$ and some similar examples that were considered afterward, there was no intermediate ground between the very concrete holomorphic discrete series and the very abstract general discrete series. And, in fact, closer examination of Takahashi's construction shows that it is less explicit than one might at first suppose. It does allow for the computation of nonzero elements in discrete series, one representation at a time, but it does not really show what the whole space is like nor what the action of the Lie algebra of G is.

The point of this thesis is to find some intermediate ground between the holomorphic discrete series and the general discrete series, with the hope that one can use the intermediate cases to get nontrivial insights into the internal structure of the representations in the general case.

Thus we seek a kind of discrete series, more complicated than the holomorphic type, for which a global realization is relatively concrete. We insist on being able to identify the elements in the space, and we want to understand the action of the Lie algebra of G rather explicitly.

There are by now several "constructions" of general discrete series. In addition to Schmid's, we mention the Hotta-Parthasarathy realization [H-R] using a Dirac operator, the Enright-Varadarajan realization [E-V] using an iterative cohomological construction, the Flensted-Jensen realization ([FJ1] and [FJ2]) using a kind of spherical function for a dual group, and the Vogan-Zuckerman realization ([Vog] and [Zuc]) using cohomological induction.

One approach to the problem is to find one of these general constructions that is relatively manageable in some situations. Another approach is to look for a kind of realization that is valid only in special situations.

We shall in fact pursue both approaches. The general construction that we examine is the Vogan-Zuckerman realization using cohomological induction. This particular construction seems favorable since the homology or cohomology is to be computed on the level of a K module from an explicit complex and since Duflo and Vergne [D-V] have given information about how to compute the action by the Lie algebra of G. All the cases of holomorphic discrete series, and only those, turn out to be realizable in homology or cohomology of degree 0, and we propose cases where the realization is in degree 0 or 1 as the intermediate case.

The concrete realization that we pursue is suggested by Takahashi's extensive use of quaternions in his work with $SO(4,1)$. A key ingredient for Harish-Chandra's treatment of holomorphic discrete series is the use of a global decomposition of G that imbeds G/K explicitly as a bounded domain in some $\mathbb{C}^n$. We try to find a similar decomposition when G/K has some quaternion structure and to use it to construct representations.

Actually the two approaches have something in common. Wolf [Wol] has classified those groups for which G/K has a reasonable quaternion structure, and it turns out that most of the situations where Vogan-Zuckerman representations are realized in degree 1 are of the kind on Wolf's list. The study of discrete series that arise from Wolf's situation is not new but has been considered by Enright, Parthasarathy, Wallach, and Wolf ([EW1] and [EW2]), although these authors had completely different objectives.

At this point we state the main results of this thesis. In Chapter 3, rather than work with $SO(4,1)$, the author considers the group $Sp(1,1)$, which is locally $SO(4,1)$. Takahashi [Tak] exploits this fact as well.

In Chapter 1, the Vogan-Zuckerman construction of discrete series by means of cohomological induction is described. Crucial items are the K modules (1.12) and the maps ∂_n and $\partial_n^\natural$ acting on these modules.

In Chapter 3, Section 1, we consider the case where $G = Sp(1,1)$ and where the discrete series is of a special kind called $A_\mathfrak{q}(\lambda)$ (Definition 1.18). Using the modules and maps just mentioned, we are able to determine a basis for the multiplicity space (Definition 3.3a) of each K type (defined in Chapter 1, Section 1) appearing in $A_\mathfrak{q}(\lambda)$. This result is Theorem 3.17. As a result of Theorem 3.17, we can determine a homology basis for each K type appearing in the $A_\mathfrak{q}(\lambda)$ discrete series. This result is Definition 3.19. A theorem of Duflo and Vergne (mentioned above) is then used in order to compute the action of the (complexified) Lie algebra of G on a homology basis for each K type. This result is Theorem 3.29. Hereafter, we will mean the complexified Lie algebra of G when we write "the Lie algebra of G".

In Chapter 3, Section 2, we continue with $G = Sp(1,1)$ but now consider a discrete series $\mathcal{L}_1(V_{be_1+e_2})$ (see (1.3) and Chapter 1, Section 3) for which the corresponding Dixmier diagram (defined in Chapter 1, Section 3) has two rows of K types. (The special discrete series $A_\mathfrak{q}(\lambda)$ mentioned in the previous paragraph is $\mathcal{L}_1(\mathbb{C}_{ae_1})$ and has one row of K types.) An interesting phenomenon occurs in each row. For the row of K types containing the minimal K type, the space Im $\partial_2^\natural$, used in determining the multiplicity space for each K type, is 0. Theorem 3.32 exhibits a basis for the multiplicity space of each K type in this row. In the other row of K types, the space Im $\partial_2^\natural$ is one-dimensional for each K type. Theorem 3.34 produces, for each K type in this row, a basis vector of Ker $\partial^{\natural}$ that does not vanish in the quotient space Ker $\partial^{\natural}$ / Im $\partial_2^{\natural}$. Using these two theorems, we state Definition 3.36, which provides a set of basis vectors in Ker ∂ for each K type appearing in $\mathcal{L}_1(V_{be_1+e_2})$ with the property that none of these basis vectors vanishes in the quotient space Ker ∂ / Im ∂_2. Theorem 3.37 uses the Duflo and Vergne result in order to compute the action of the Lie algebra of G on any vector stated in Definition 3.36.

In Chapter 3, Section 3, we continue with G as in the previous two sections but now consider the general discrete series $\mathcal{L}_1(V_{de_1+(R-1)e_2})$, where R is an integer ≥ 1 and d is an integer satisfying $d \geq R-1$. The corresponding Dixmier diagram has R rows. Based on Theorems 3.17, 3.32, and 3.34, we state Conjecture 3.39, which gives for any fixed K type in any of the R rows, a basis vector of Ker $\partial^{\natural}$ that should not vanish in the quotient space Ker $\partial^{\natural}$ / Im $\partial_2^{\natural}$. Using that conjecture, we prove Theorem 3.40. This theorem provides a framework for computing explicit formulas of the action of the Lie algebra of G in each K type. In fact, using Theorem 3.40 in combination with Corollary 2.12, one can readily calcutate explicit formulas like those that appear in Theorems 3.29 and 3.37.

In Chapter 4, we address the more general question of $A_{\mathfrak{q}}(\lambda)$ discrete series for $G = Sp(1,n)$, $n \geq 1$. Proposition 4.2 gives a description of the K types appearing, and Theorem 4.14 proves that the question of determining explicit formulas for the action of the Lie algebra of G, as was done in Theorem 3.29, is a solvable problem, under the assumption that we have an explicit decomposition into irreducible components of the K representation

$\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$. The group K and the representation $\pi_{(P,Q)}$ of Chapter 4 are generalizations of those in Chapters 2 and 3.

In Chapter 5, we consider the discrete series $A_{\mathfrak{q}}(\lambda)$ for any group G satisfying the conditions of Theorem 1.6(a). The first main result is Proposition 5.3, which states explicitly a basis vector for the multiplicity space of the minimal K type. Other results include Proposition 5.5, which gives a general form for any K type appearing in $A_{\mathfrak{q}}(\lambda)$. A consequence of the proof of Proposition 5.6 is the relation (5.6g), which gives a general form for any K type η that has $C_2^*(\mathbb{C}_\lambda)|_\eta$ nonzero $\left(C_2^*(\mathbb{C}_\lambda)|_\eta\right.$ defined in Definition 3.3a$\left.\right)$. Proposition 5.7 shows that computation of the action of the Lie algebra of G on a homology basis for the minimal K type is a solvable problem. Finally, Proposition 5.8 proves that there are K types η for which $C_2^*(\mathbb{C}_\lambda)|_\eta$ is nonzero. This result is important when determining a homology basis.

Theorem 6.7 of Chapter 6 defines a unitary equivalence between two different realizations of quaternionic discrete series for $Sp(1,1)$. One realization uses the fact $Sp(1,1)/(SU(2) \times SU(2))$ has a reasonable quaternionic structure, in the sense of [Wol], and the discrete series is defined in a way that imitates Harish-Chandra's work with holomorphic discrete series. The other realization comes from Takahashi. It consists of a special subspace of vector-valued functions of $L^2(G)$ that are square integrable with respect to a certain inner product. Under the assumption that every function in Takahashi's space can be extended to a function on a larger set satisfying certain properties, we give in Theorem 6.7 an explicit unitary equivalence between the realizations.

CHAPTER 1

CONSTRUCTION OF DISCRETE SERIES BY COHOMOLOGICAL INDUCTION

1. Discrete Series

For a unimodular group G, an irreducible unitary representation π is in the **discrete series** if it is a direct summand of the right regular representation on $L^2(G)$, or equivalently if some (or equivalently every) nonzero matrix coefficient $(\pi(g)v_1, v_2)$ is in $L^2(G)$ [K-V, pg.20].

Let G be a linear connected semisimple noncompact Lie group, and let K be a maximal compact subgroup. The discrete series representations for G were parametrized by Harish-Chandra [HC4]. A key result is that the discrete series representations exist for G if and only if a maximal torus T of K is maximally abelian in G. This is equivalent to saying rank $G =$ rank K. For an exposition, see [K3, pg.454], especially Theorem 9.20.

The Vogan-Zuckerman construction known as cohomological induction is a way of constructing discrete series. At this point we shall give a brief sketch of cohomological induction, omitting details of a number of definitions. In the course of our description, we shall note under what conditions the construction yields discrete series. We shall give a more detailed description of cohomological induction in the next section. Motivation for cohomological induction may be found in [K-V, Introduction], particularly sections 3 and

5, or [K1, Section 1(Setting)].

We begin with G and K as stated above. For T a torus subgroup of K (not necessarily maximally abelian), we let $L = Z_G(T)$ be the centralizer of T in G. Eventually we shall assume that $L \subseteq K$, but we do not make this assumption yet. Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$, let $\mathfrak{k}$ and $\mathfrak{l}$ be the complexifications of the Lie algebras of K and L, respectively, and let $\mathfrak{q}$ be a θ stable parabolic subalgebra of $\mathfrak{g}$ of the form $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilpotent radical. Here and elsewhere, we shall use the notation $\mathfrak{m}_0$, with subscript 0, to refer to a real Lie algebra, and we shall use the notation $\mathfrak{m}$ without subscript to refer to a complexified Lie algebra ($\mathfrak{m} = (\mathfrak{m}_0)^{\mathbb{C}}$). Define $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$, where bar denotes conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \bar{\mathfrak{u}}$.

What follows mirrors closely the discussion in [K-V, pgs.26-27]. Suppose Z is an irreducible $(\mathfrak{l}, L \cap K)$ module. We define $Z^{\#}$, an $(\mathfrak{l}, L \cap K)$ module, by

$$Z^{\#} = Z \otimes \textstyle\bigwedge^{\mathrm{top}} \mathfrak{u}, \tag{1.1}$$

where top $= \dim(\mathfrak{u})$. Here and elsewhere we use the notation $\otimes$ with no subscript to mean $\otimes_{\mathbb{C}}$, the tensor product over $\mathbb{C}$.

We regard $Z^{\#}$ as a $(\bar{\mathfrak{q}}, L \cap K)$ module on which $\bar{\mathfrak{u}}$ acts as 0. The next step is to apply an 'algebraic induction' functor to form a $(\mathfrak{g}, L \cap K)$ module

$$\mathrm{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^{\#}) = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} Z^{\#} \cong U(\mathfrak{u}) \otimes Z^{\#}. \tag{1.2}$$

After this we apply the j^{th} derived functor of the "Bernstein functor" $\Pi_{\mathfrak{g},L\cap K}^{\mathfrak{g},K}$ to obtain a $(\mathfrak{g}, K)$ module:

$$\mathcal{L}_j(Z) = (\Pi_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})_j(\operatorname{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(Z^{\#})). \tag{1.3}$$

As mentioned above, under certain conditions, namely for the particular value $j = S = \dim(\mathfrak{u} \cap \mathfrak{k})$ and for certain irreducible modules Z, the $(\mathfrak{g}, K)$ module $\mathcal{L}_j(Z)$ is a discrete series representation. We will state these conditions on Z presently, but first we finish discussing how discrete series are constructed using $\mathcal{L}_j(Z)$. Under the assumption that the conditions for j and Z are satisfied, $\mathcal{L}_j(Z)$ can be computed directly by means of a complex. See formula (1.12) of the next section for details. Although $\mathcal{L}_j(Z)$ is a $(\mathfrak{g}, K)$ module, the complex whose members come from (1.12) will provide us with a $(\mathfrak{k}, K)$ module. This $(\mathfrak{k}, K)$ module is the S^{th} homology of the complex, and we can reimpose the full $\mathfrak{g}$ action on this $(\mathfrak{k}, K)$ module by using a theorem of Duflo and Vergne [K-V, Proposition 3.80] in combination with another result [K-V, Proposition 3.83]. Examples where this $\mathfrak{g}$ action is reimposed for $G = Sp(1,1)$ are provided in Chapter 3 (Theorems 3.29 and 3.37). Because these theorems will be cited extensively in this thesis, we present both of them now.

Proposition 3.80 (Duflo-Vergne). Let $(\mathfrak{g}, L) \hookrightarrow (\mathfrak{g}, K)$ be an inclusion of pairs, and let V be in $\mathcal{C}(\mathfrak{g}, L)$. Regard $\bigwedge^n \mathfrak{c}$ as a trivial $(\mathfrak{g}, L)$ module, and make $\mathcal{F}V$ into a $(\mathfrak{g}, L)$ module so that its action is the same as in V. Let Φ_0 be the Mackey isomorphism relative to $(\mathfrak{l}, L)$ and $(\mathfrak{k}, K)$. Then the $\mathcal{C}(\mathfrak{k}, K)$ map α_n that makes the diagram

$$
\begin{array}{ccc}
\mathfrak{g} \otimes_{\mathbb{C}} (R(K) \otimes_L (\bigwedge^n \mathfrak{c} \otimes_{\mathbb{C}} \mathcal{F}V)) & \xrightarrow{\Phi_0^{-1}} & R(K) \otimes_L (\mathfrak{g} \otimes_{\mathbb{C}} \bigwedge^n \mathfrak{c} \otimes_{\mathbb{C}} \mathcal{F}V) \\
\downarrow \alpha_n & \swarrow {\scriptstyle 1 \otimes \mu_{(\bigwedge^n \mathfrak{c} \otimes \mathcal{F}V)}} & \\
R(K) \otimes_L (\bigwedge^n \mathfrak{c} \otimes_{\mathbb{C}} \mathcal{F}V)) & &
\end{array}
\tag{3.81a}
$$

commute is given by

$$
\alpha_n(X \otimes (T \otimes w)) = T \otimes (\mathrm{Ad}(\,\cdot\,)^{-1} X) w \qquad \text{for } X \in \mathfrak{g},\ T \in R(K),
$$

$$
\text{and } w \in \textstyle\bigwedge^n \mathfrak{c} \otimes \mathcal{F}V.
$$

Proposition 3.83. Put

$$
V_n^{\Pi} = \textstyle\bigwedge^n \mathfrak{c} \otimes_{\mathbb{C}} \mathcal{F}V \otimes_{\mathbb{C}} (\bigwedge^m \mathfrak{c})^*.
$$

Then the $\mathcal{C}(\mathfrak{k}, K)$ diagram

$$
\begin{array}{ccc}
\left\{ \begin{matrix} \ker(1 \otimes \partial) \text{ in} \\ \mathfrak{g} \otimes_{\mathbb{C}} (R(K) \otimes_L V_n^{\Pi}) \end{matrix} \right\} & \xrightarrow{\alpha_n} & \left\{ \begin{matrix} \ker \partial \text{ in} \\ R(K) \otimes_L V_n^{\Pi} \end{matrix} \right\} \\
\downarrow & & \downarrow \\
\mathfrak{g} \otimes_{\mathbb{C}} \Pi_n^K(\mathcal{F}(V \otimes_{\mathbb{C}} (\bigwedge^m \mathfrak{c})^*)) & \xrightarrow{\mu} & \Pi_n^K(\mathcal{F}(V \otimes_{\mathbb{C}} (\bigwedge^m \mathfrak{c})^*))
\end{array}
$$

commutes.

For our purposes, the important portion of Proposition 3.80 is the map α_n. In Proposition 3.83, we may ignore the term $(\bigwedge^m \mathfrak{c})^*$, since it is not relevant for our situation. Also, the functor Π_n^K is the n^{th} derived functor of $P_{\mathfrak{k},L}^{\mathfrak{k},K}$ (see (1.9b)), $\mathcal{F} = \mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}$ (see (1.10)), and the map μ is "multiplication by $\mathfrak{g}$", i.e., the $\mathfrak{g}$ action.

At this point, we describe the the conditions on the irreducible $(\mathfrak{l}, L \cap K)$ module Z in order that $\mathcal{L}_S(Z)$ be a discrete series. The first condition is that L be compact, so that $L \subseteq K$. Under this condition, let us fix a maximal torus B in L. B will be a maximal torus also in K, and we let $\mathfrak{b}$ be the complexified Lie algebra. Form the roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{b})$, and choose a positive system Δ^+ compatible with $\mathfrak{q}$. If Z is an irreducible $(\mathfrak{l}, L)$ module with highest weight λ, then the infinitesimal character of $\mathcal{L}_S(Z)$ is $\lambda + \delta(\mathfrak{g})$, where $\delta(\mathfrak{g})$ is one half the sum of the positive roots of $\mathfrak{g}$. The necessary condition for $\mathcal{L}_S(Z)$ to be a discrete series is that $\lambda + \delta(\mathfrak{g})$ be dominant and nonsingular for $\mathfrak{g}$. In other words,

$$\text{(1.4a)} \qquad \langle \lambda + \delta(\mathfrak{g}), \alpha \rangle > 0 \qquad \text{for all } \alpha \in \Delta^+.$$

In this case $\mathcal{L}_S(Z)$ is the discrete series with Harish-Chandra parameter $\lambda + \delta(\mathfrak{g})$, in the sense of [HC4]. It turns out that $\delta(\mathfrak{g}) = \delta(\mathfrak{l}) + \delta(\mathfrak{u})$, and because L is compact for our cases of interest, then $\lambda + \delta(\mathfrak{l})$ is the infinitesimal character of Z. We have arranged that $\Delta(\mathfrak{u})$, the roots of $\mathfrak{u}$, are all positive, and we have seen that $\delta(\mathfrak{u}) \perp \Delta(\mathfrak{l})$. Because $\lambda + \delta(\mathfrak{l})$ is strictly dominant with respect to the roots of $\mathfrak{l}$, we see that $\lambda + \delta(\mathfrak{g})$ is automatically strictly dominant with respect to $\Delta^+(\mathfrak{l})$. Therefore, the inequality (1.4a) can be rewritten

$$\text{(1.4b)} \quad \langle (\text{infinitesimal character of } Z) + \delta(\mathfrak{u}), \alpha \rangle > 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

We mentioned in the introduction that the value $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ may be of interest in singling out special families of discrete series representations. We therefore state theorems that classify the groups G and parabolic subalgebras $\mathfrak{q}$ corresponding to the cases $S = 0$ and $S = 1$. The proofs may be found in Section 4 of this chapter.

Theorem 1.5 Let G be a noncompact simple group with rank $G =$ rank K, and suppose that the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ has $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ equal to 0. Then either $\mathfrak{q} = \mathfrak{g}$ or else G/K is Hermitian symmetric with $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$ in a suitable good ordering on the roots.

Theorem 1.6 Let G be a noncompact simple group with rank $G =$ rank K, and suppose that the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ has $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ equal to 1. Let β be the unique positive compact root in $\Delta(\mathfrak{u})$. Then $\{\beta, -\beta\}$ is a simple component in the root system of $\mathfrak{k}$, and the following is a classification of the possibilities for the Dynkin diagram of $\mathfrak{g}$ and the roles of $\mathfrak{l}$ and $\mathfrak{u}$:

(a) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is noncompact, all roots of $\mathfrak{l}$ are compact, β is the largest root and contains β_0 in its simple-root expansion with coefficient 2, and β_0 is characterized as the unique simple root nonorthogonal to β

(b) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is noncompact, exactly two other simple roots are noncompact and they are adjacent, and β is the sum of the simple roots from β_0 through the nearer noncompact simple root of $\mathfrak{l}$

(c) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is noncompact, exactly one other simple root is noncompact and it is the other node, and β is the sum of all the simple roots

(d) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is compact, the simple root adjacent to β_0 is the one and only noncompact simple root, and β equals β_0

(e) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is $\mathfrak{sp}(2,\mathbb{R})$ of type C_2, β_0 is compact and short, the other simple root is noncompact and long, and β equals β_0

(f) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is split G_2, β_0 is long and noncompact, the other simple root is short and compact, β contains β_0 in its simple-root expansion with coefficient 2, and β is the largest short root

(g) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, the Dynkin diagram of $\mathfrak{g}$ is of type A_n with $n \geq 2$, β_1 and β_2 are the nodes, β_1 and β_2 are noncompact, all other simple roots are compact, and β is the sum of all the simple roots

(h) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, β_1 is noncompact and β_2 is compact, the Dynkin diagram of $\mathfrak{g}$ is of type A_n, β_2 is a node, β_1 is adjacent to β_2, the simple roots of $\mathfrak{l}$ are all compact, and β is β_2

(i) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, $\mathfrak{g}_0$ is $\mathfrak{sp}(2,\mathbb{R})$

(c) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is noncompact, exactly one other simple root is noncompact and it is the other node, and β is the sum of all the simple roots

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(e) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is $\mathfrak{sp}(2, \mathbb{R})$ of type C_2, β_0 is compact and short, the other simple root is noncompact and long, and β equals β_0

(f) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is split G_2, β_0 is long and noncompact, the other simple root is short and compact, β contains β_0 in its simple-root expansion with coefficient 2, and β is the largest short root

(g) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, the Dynkin diagram of $\mathfrak{g}$ is of type A_n with $n \geq 2$, β_1 and β_2 are the nodes, β_1 and β_2 are noncompact, all other simple roots are compact, and β is the sum of all the simple roots

(h) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, β_1 is noncompact and β_2 is compact, the Dynkin diagram of $\mathfrak{g}$ is of type A_n, β_2 is a node, β_1 is adjacent to β_2, the simple roots of $\mathfrak{l}$ are all compact, and β is β_2

(i) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, $\mathfrak{g}_0$ is $\mathfrak{sp}(2, \mathbb{R})$

of type C_2, at least one of β_1 and β_2 is noncompact, and β is the unique positive compact root.

The situation described in Theorem 1.5 where $S = 0$ leads exactly to holomorphic discrete series, but in a realization as vector-valued holomorphic functions on G/K. Alternatively these representations are being presented on the level of $(\mathfrak{g}, K)$ modules as generalized Verma modules, which are reasonably well understood. Our approach toward finding intermediate cases of discrete series using cohomological induction will be to look at situations where $S = 1$. The main case of Theorem 1.6 is (a), the other cases corresponding to particular groups. Case (a) is closely related to work of Wolf [Wol] on quaternion structure for G/K. See also [Sud], [Bes], [F1], [F2] and [F3]. We do not have a complete theory of discrete series for case (a) of Theorem 1.6. Therefore, we are going to start with the example $Sp(1,1)$, see what features of the theory for this group generalize to $Sp(1,n)$, and see what features of the theory for $Sp(1,n)$ generalize fully to case (a). The term **quaternionic discrete series** will be used to refer to discrete series representations of a group G whose complexified Lie algebra $\mathfrak{g}$ satisfies the conditions of case (a) in Theorem 1.6. We consider $Sp(1,1)$ in Chapter 3, $Sp(1,n)$ in Chapter 4, and the general case in Chapter 5.

Section 2 of this chapter gives a more detailed description of cohomological induction, providing definitions and giving concrete formulas for the complexes used in determining discrete series representations. Section 3 gives a description of some diagrams introduced by Dixmier [Dix] and used in determining K types (defined below) for $Sp(1,1)$ discrete series. Section 4 restates Theorems 1.5 and 1.6 and provides proofs of these theorems; this section may be skipped on first reading.

In order to define the term K type, we need some theory. The Peter-Weyl Theorem [K3, Theorem 1.12] states that $\mathcal{L}_S(Z)$, when thought of as a K representation, will decompose as the direct sum of irreducible, finite-dimensional K representations. Each of these irreducible representations is parametrized, up to unitary equivalence, by its highest weight (Theorem of the Highest Weight, [K3, Theorem 4.28]). The highest weights that appear with nonzero multiplicity in such a direct sum decomposition are called the **K types of $\mathcal{L}_S(Z)$.**

2. Cohomological Induction

We begin this section by defining some terms used in our discussion of cohomological induction. After these definitions are stated, we present a more thorough analysis of the objects and maps used in constructing discrete series representations by means of cohomological induction. This will be the underlying structure for most of the work done in this thesis.

Definition 1.7 We shall define the **Hecke Algebra** $R(K)$ as the direct sum

$$R(K) = \bigoplus_{\mu \in \hat{K}} V_\mu \otimes (V_\mu)^*,$$

where $\hat{K}$ is the set of equivalence classes of irreducible representations of K, (π, V_μ) is an irreducible representation of K of highest weight μ and $(\pi^*, (V_\mu)^*)$ is the contragredient representation. A more detailed definition is given in [K-V, (1.33)]. From this reference we see that both l = left regular representation and r = right regular representation are defined on $R(K)$. A fact we shall use repeatedly throughout this thesis is the following: For any pure tensor $v \otimes v^*$ in a summand $V_\mu \otimes (V_\mu)^*$ of $R(K)$,

$$\begin{aligned} l(k)(v \otimes v^*) &= (\pi(k)v) \otimes v^* \\ r(k)(v \otimes v^*) &= v \otimes (\pi^*(k)v^*). \end{aligned} \tag{1.7a}$$

This extends to linear combinations of pure tensors, as well as finite sums of the summands $V_\mu \otimes (V_\mu)^*$. A thorough discussion of $R(K)$ can be found in [K-V, Chapter 1]. The summands $V_\mu \otimes (V_\mu)^*$ of $R(K)$ are called the K **isotypic components of** $R(K)$ **of type** μ**.** See [K-V, Proposition 1.18] for a further discussion.

Definition 1.8 We define the **Hecke Algebra** $R(\mathfrak{g}, K)$ to be the tensor product

$$R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}),$$

where $U(\mathfrak{k})$ is the universal enveloping algebra of $\mathfrak{k}$, contained in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Technically speaking, this tensor product is isomorphic to the actual Hecke Algebra $R(\mathfrak{g}, K)$ [K-V, Corollary 1.71], but for our purposes the tensor product is sufficient. Note that $R(K)$ is a right $U(\mathfrak{k})$ module via the action

$$Tv = r(v^t)T, \tag{1.8a}$$

where $T \in R(K)$ and $v \in U(\mathfrak{k})$. The transpose map $v \mapsto v^t$ is an antiautomorphism of $U(\mathfrak{k})$ characterized by

$$X^t = -X \qquad \text{and} \qquad (u\,v)^t = v^t u^t, \tag{1.8b}$$

for $X \in \mathfrak{k}$ and $u, v \in U(\mathfrak{k})$. In (1.8a), r is right regular representation extended to $U(\mathfrak{k})$. Note also that $U(\mathfrak{g})$ is a left $U(\mathfrak{k})$ module in a natural way, namely by left multiplication. A thorough discussion of $R(\mathfrak{g}, K)$ is given in [K-V, Chapter 1]. In the first section of that chapter is a discussion of the transpose map and the extension of r to the universal enveloping algebra.

Definition 1.9 Suppose that $i:(\mathfrak{h}, L) \to (\mathfrak{g}, K)$ is a map of pairs [K-V, (2.6)], i.e., a pair of maps

$$i_{\mathrm{alg}} : \mathfrak{h} \to \mathfrak{g}, \qquad \text{a Lie algebra homomorphism}$$

$$i_{\mathrm{gp}} : L \to K, \qquad \text{a Lie group homomorphism}$$

satisfying the compatibility conditions

(i) $i_{\mathrm{alg}} \circ \iota_L = \iota_K \circ di_{\mathrm{gp}}$, where di_{gp} is the differential of i_{gp};

(ii) $i_{\mathrm{alg}} \circ \mathrm{Ad}_L(l) = \mathrm{Ad}_K(i_{\mathrm{gp}}(l)) \circ i_{\mathrm{alg}}$ for $l \in L$.

For V an approximately unital $(\mathfrak{h}, L)$ module, the functor $P(\cdot)$ is defined by

$$P(V) = P_{\mathfrak{h},L}^{\mathfrak{g},K}(V) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{h},L)} V. \tag{1.9a}$$

This is a $(\mathfrak{g}, K)$ module. A special example of the P functor occurs when $\mathfrak{h} = \mathfrak{g}$. For V an approximately unital $(\mathfrak{g}, L)$ module, the **Bernstein functor** $P_{\mathfrak{g},L}^{\mathfrak{g},K}$, denoted by $\Pi_{\mathfrak{g},L}^{\mathfrak{g},K}$, satisfies the K isomorphism

$$P_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = \Pi_{\mathfrak{g},L}^{\mathfrak{g},K}(V) = R(K) \otimes_{R(\mathfrak{k},L)} V. \tag{1.9b}$$

This is proved in Proposition 2.69 of [K-V]. Another example of the P functor occurs when $L = K$. For V an approximately unital $(\mathfrak{h}, L)$ module, the functor $P_{\mathfrak{h},L}^{\mathfrak{g},L}$, called **Lie algebra induction,** satisfies the equality

$$P_{\mathfrak{h},L}^{\mathfrak{g},L}(V) \cong \mathrm{ind}_{\mathfrak{h},L}^{\mathfrak{g},L}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, \tag{1.9c}$$

with L acting by the tensor product of Ad and the action on V. This is proved in Proposition 2.57 of [K-V].

The previous definition was given for a general map of pairs. In what follows, we specialize to the case of quaternionic discrete series, so that $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, $L \subseteq K$ is compact, $\mathfrak{u}$ is the nilradical, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, and $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$.

Definition 1.10 The functor $\mathcal{F}(\cdot)$, the **forgetful functor,** will arise in two ways in this thesis. For the first case, suppose V an approximately unital $(\mathfrak{l}, L)$ module, $\mathcal{F}_{\mathfrak{l},L}^{\bar{\mathfrak{q}},L}(V)$ is the $(\mathfrak{l}, L)$ module V, extended to be a $(\bar{\mathfrak{q}}, L)$ module by defining $Xv = 0$ for $X \in \bar{\mathfrak{u}}$ and $v \in V$.

In the second case, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$; this is discussed more fully in Chapter 2, Section 1. The functor $\mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}(V)$ is the $(\mathfrak{g}, L)$ module V, reduced to a $(\mathfrak{k}, L)$ module by ignoring the action of $\mathfrak{p}$ on V. A precise definition of the functor $\mathcal{F}$ can be found in [K-V, pg.109].

We now state precisely what complexes are used in determining $\mathcal{L}_S(Z)$. For this discussion we will continue with the notation for groups and Lie algebras stated in Section 1 of this chapter. We will assume that appropriate choices have been made for S and Z so that $\mathcal{L}_S(Z)$ is a discrete series representation (see Section 1). If we abbreviate $\mathrm{ind}_{\mathfrak{q},L}^{\mathfrak{g},L}(Z^{\#})$ by V_Z, then the precise complex used consists of the $(\mathfrak{g}, K)$ modules

$$R(\mathfrak{g}, K) \otimes_L \left(\bigwedge^n(\mathfrak{u} \oplus \bar{\mathfrak{u}}) \otimes V_Z\right) \tag{1.11a}$$

and a map ∂_n, which maps from $\bigwedge^n$ to $\bigwedge^{n-1}$, defined on a module (1.11a) as

$$\begin{aligned} &\partial_n(R \otimes (Y_1 \wedge \cdots \wedge Y_n \otimes v)) \\ &= \sum_{l=1}^{n} (-1)^{l+1}(RY_l \otimes (Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes v)) \\ &\quad + \sum_{l=1}^{n} (-1)^{l}(R \otimes (Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes Y_l v)) \end{aligned} \tag{1.11b}$$

for $R \in R(\mathfrak{g}, K)$, $Y_1, \ldots, Y_n \in \mathfrak{u} \oplus \bar{\mathfrak{u}}$, and $v \in V_Z$. The value n in (1.11a) ranges between 0 and $N = \dim(\mathfrak{g}/\mathfrak{l})$. This complex is discussed thoroughly in [K-V, Chapter 2, Section 7]. Formulas (1.11a) and (1.11b) above are taken from formulas (2.128a) and (2.128b) of this reference and modified to fit our situation. Using this notation, $\mathcal{L}_S(Z)$ is the $(\mathfrak{g}, K)$ module $\mathrm{Ker}\, \partial_S / \mathrm{Im}\, \partial_{S+1}$.

Although we could calculate the discrete series in this way, we choose to modify this construction somewhat and produce the discrete series in steps. More specifically, the construction above gives us a $(\mathfrak{g}, K)$ module, which is what we desire ultimately. However, we proceed by first changing the complex (1.11a) to a $(\mathfrak{k}, K)$ module and then reconstructing the full $\mathfrak{g}$ action by means of the theorem of Duflo and Vergne (Proposition 3.80 of [K-V]) and one other result (Proposition 3.83 of [K-V]). These results are stated in the previous section. The reason for this change is that our new complex replaces $R(\mathfrak{g}, K)$ with $R(K)$, which is much easier to manipulate. To make the change, we use Corollary 3.26(c) of [K-V], which allows us to replace $(\Pi_{\mathfrak{g},L}^{\mathfrak{g},K})_S(\mathrm{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L} Z\#)$ by $(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_S(\mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}(\mathrm{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L} Z\#))$. If we abbreviate $\mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}(\mathrm{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L} Z\#)$ by $\mathcal{F}(V_Z)$, then (3.27) of the same reference tells us that the complex whose S^{th} homology is $(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_S(\mathcal{F}(V_Z))$ is given by a complex whose modules are

$$R(K) \otimes_L \left(\bigwedge\nolimits^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z\right) \tag{1.12}$$

and whose maps ∂_n are the same as that of (1.11b), except for notational changes. Note, however, that the domain of n decreases when considering the $(\mathfrak{k}, K)$ modules (1.12) instead of the $(\mathfrak{g}, K)$ modules (1.11a). The $(\mathfrak{k}, K)$ module $(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_S(\mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}(\mathrm{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L} Z\#))$ is given by Ker $\partial_S/$ Im ∂_{S+1}, with ∂_S and ∂_{S+1} being maps on the modules (1.12).

A special kind of discrete series we will consider occurs when the $(\mathfrak{l}, L)$ module Z mentioned before (1.1) is one dimensional.

Definition 1.13 Let B be a Cartan subgroup of G with $\mathfrak{h}$ the complexified Lie algebra of B. Let λ be an analytically integral linear functional on $\mathfrak{h}$ that is orthogonal to all the members of $\Delta(\mathfrak{l})$, and let $\mathbb{C}_\lambda$ be the corresponding $(\mathfrak{h}, B)$ module. By [K-V, Theorem 4.52], there is an irreducible $(\mathfrak{l}, L)$ module Z with highest weight λ. Since $\lambda \perp \Delta(\mathfrak{l})$, Z is one dimensional, i.e., $Z = \mathbb{C}_\lambda$. Thus Z becomes a one dimensional representation of L. We define a $(\mathfrak{g}, K)$ module by

$$\mathrm{A}_{\mathfrak{q}}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda),$$

where, as before, $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. For a fuller discussion see [K-V, Chapter 5, Example 2].

To make our calculations with ∂_n simpler, we will work with the space of L invariants of

$$R(K) \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z; \tag{1.14a}$$

denoted

$$\left(R(K) \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z\right)^L, \tag{1.14b}$$

rather than with the space of L coinvariants of (1.14a), namely (1.12). We can lift ∂_n from (1.12) to (1.14a) where the formula for ∂_n is still the same but we no longer expect $\partial_n^2 = 0$. If ξ denotes the representation of L on V_Z, then L acts on (1.14a) by $r \otimes \mathrm{Ad} \otimes \xi$ and we can recover (1.12), apart from a canonical isomorphism, as the subspace (1.14b) of (1.14a). The advantage of doing this is that we now are working with a subset of vectors rather than equivalence classes of vectors. For a fuller discussion, see [K-V, pg.193].

We can simplify the process of determining elements in Ker ∂_n for the $(\mathfrak{k}, K)$ modules

$$(V_\mu \otimes V_\mu{}^* \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z)^L, \tag{1.15a}$$

where μ is a K type in the discrete series $\mathcal{L}_S(Z)$ and $V_\mu \otimes V_\mu{}^*$ is the K isotypic component of $R(K)$ of type μ. We do this by first considering a pure tensor $v \otimes v^* \otimes Y_1 \wedge \cdots \wedge Y_n \otimes z$ in

$$V_\mu \otimes V_\mu{}^* \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z. \tag{1.15b}$$

Here $v \otimes v^* \in V_\mu \otimes {V_\mu}^*$ and z is a pure tensor in V_Z. From (1.11b) and (1.8) we have

$$\begin{aligned}
&\partial_n(v \otimes v^* \otimes Y_1 \wedge \cdots \wedge Y_n \otimes z) \\
(1.16) \qquad &= \sum_{l=1}^{n} (-1)^l (r(Y_l)(v \otimes v^*) \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes z) \\
&\quad + \sum_{l=1}^{n} (-1)^l (v \otimes v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes Y_l z)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1}^{n} (-1)^l (v \otimes \pi^*(Y_l) v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes z) \\
&\quad + \sum_{l=1}^{n} (-1)^l (v \otimes v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes Y_l z) \qquad \text{from (1.7a)} \\
&= \sum_{l=1}^{n} v \otimes \big[(-1)^l (\pi^*(Y_l) v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes z) \\
&\quad + (-1)^l (v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes Y_l z)\big] \\
&= v \otimes \sum_{l=1}^{n} \big[(-1)^l (\pi^*(Y_l) v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes z) \\
&\quad + (-1)^l (v^* \otimes Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_n \otimes Y_l z)\big].
\end{aligned}$$

Here, $r(Y_l)$ and $\pi^*(Y_l)$ refer to the differentials of the representations r and π^*, both of which are defined on the group K. Here and elsewhere, we omit writing π^* and simply write the element of the Lie algebra acting on a vector. With (1.16) as motivation, we define the map $\partial_n^\natural$ on the vector space

$$(1.17a) \qquad V_\mu^* \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes V_Z$$

by

$$(1.17b)\qquad \begin{aligned}&\partial_n^\natural(v^*\otimes Y_1\wedge\cdots\wedge Y_n\otimes z)\\&=\sum_{l=1}^{n}(-1)^l(\pi^*(Y_l)v^*\otimes Y_1\wedge\cdots\wedge\widehat{Y_l}\wedge\cdots\wedge Y_n\otimes z)\\&\quad+(-1)^l(v^*\otimes Y_1\wedge\cdots\wedge\widehat{Y_l}\wedge\cdots\wedge Y_n\otimes Y_l z).\end{aligned}$$

This descends to a map that we shall also denote by $\partial_n^\natural$ on the space of L invariants

$$(1.17c)\qquad (V_\mu^*\otimes{\textstyle\bigwedge}^n((\mathfrak{u}\oplus\bar{\mathfrak{u}})\cap\mathfrak{k})\otimes V_Z)^L.$$

Because ∂_n and $\partial_n^\natural$ both commute with the invariants functor, we see that producing an element of Ker ∂_n for (1.15a) amounts to first producing an element of Ker $\partial_n^\natural$ for the space (1.17c) and then tensoring it with **any** element of V_μ. This is the procedure we will follow when determining elements of Ker ∂_1 in Chapters 3–5. Here and elsewhere, when $n=1$ in ∂_n or $\partial_n^\natural$, we shall drop the subscript 1 and simply write ∂ or $\partial^\natural$.

As a final note about calculating $\mathcal{L}_S(Z)$, in particular the $(\mathfrak{k},K)$ module that comes from the complex with modules (1.14b), it is important to remember that, for $Y_l\in(\mathfrak{u}\oplus\bar{\mathfrak{u}})\cap\mathfrak{k}$ and $T\in R(K)$,

$$(1.18)\qquad TY_l=-r(Y_l)T.$$

We used this relation when computing the formula (1.16). It is also of particular importance when checking if a certain vector is in Ker ∂_S or if a certain vector is in Im ∂_{S+1}.

The author wishes to thank A. W. Knapp, who provided the proofs for these theorems.

Theorem 1.5 Let G be a noncompact simple group with rank $G =$ rank K, and suppose that the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ has $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ equal to 0. Then either $\mathfrak{q} = \mathfrak{g}$ or else G/K is Hermitian symmetric with $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$ in a suitable good ordering on the roots.

PROOF. Let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$, introduce a positive system $\Delta^+(\mathfrak{l}, \mathfrak{b})$ for $\mathfrak{l}$, and let $\Delta^+(\mathfrak{g}, \mathfrak{b}) = \Delta^+(\mathfrak{l}, \mathfrak{b}) \cup \Delta(\mathfrak{u})$ be the corresponding positive system for $\mathfrak{g}$. Since $S = 0$, the compact positive roots are all in $\Delta^+(\mathfrak{l}, \mathfrak{b})$. Therefore $\mathfrak{l}$ contains the semisimple part of $\mathfrak{k}$. Since center$(\mathfrak{k}) \subseteq \mathfrak{b} \subseteq \mathfrak{l}$, $\mathfrak{l}$ contains $\mathfrak{k}$. Consequently

$$\Delta(\mathfrak{p}) = \Delta(\mathfrak{l} \cap \mathfrak{p}, \mathfrak{b}) \cup \Delta(\mathfrak{u}) \cup \Delta(\bar{\mathfrak{u}}),$$

with each corresponding subspace of $\mathfrak{g}$ invariant under $\operatorname{ad}\mathfrak{k}$. If $\Delta(\mathfrak{u}) \neq \emptyset$, then $\Delta(\bar{\mathfrak{u}}) \neq \emptyset$ and Problem 16 on pg. 166 of [K3] shows that $\Delta(\mathfrak{l} \cap \mathfrak{p}, \mathfrak{b}) = \emptyset$ and center$(\mathfrak{k}) \neq 0$. Then $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u} = \mathfrak{p}^+$, and $\bar{\mathfrak{u}} = \mathfrak{p}^-$.

Theorem 1.6 Let G be a noncompact simple group with rank $G =$ rank K, and suppose that the θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ has $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ equal to 1. Let β be the unique positive compact root in $\Delta(\mathfrak{u})$. Then $\{\beta, -\beta\}$ is a simple component in the root system of $\mathfrak{k}$, and the following is a classification of the possibilities for the Dynkin diagram of $\mathfrak{g}$ and the roles of $\mathfrak{l}$ and $\mathfrak{u}$:

(a) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is noncompact, all roots of $\mathfrak{l}$ are compact, β is the largest root and contains β_0 in its simple-root expansion with coefficient 2, and β_0 is characterized as the unique simple root nonorthogonal to β

(b) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is noncompact, exactly two other simple roots are noncompact and they are adjacent, and β is the sum of the simple roots from β_0 through the nearer noncompact simple root of $\mathfrak{l}$

(c) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is noncompact, exactly one other simple root is noncompact and it is the other node, and β is the sum of all the simple roots

(d) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, β_0 is a node in a Dynkin diagram of type A_n with $n \geq 2$, β_0 is compact, the simple root adjacent to β_0 is the one and only noncompact simple root, and β equals β_0

(e) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is $\mathfrak{sp}(2, \mathbb{R})$ of type C_2, β_0 is compact and short, the other simple root is noncompact and long, and β equals β_0

(f) $\Delta(\mathfrak{u})$ contains exactly one simple root β_0 of $\mathfrak{g}$, $\mathfrak{g}_0$ is split G_2, β_0 is long and noncompact, the other simple root is short and compact, β contains β_0 in its simple-root expansion with coefficient 2, and β is the largest short root

(g) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, the Dynkin diagram of $\mathfrak{g}$ is of type A_n with $n \geq 2$, β_1 and β_2 are the nodes, β_1 and β_2 are noncompact, all other simple roots are compact, and β is the sum of all the simple roots

(h) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, β_1 is noncompact and β_2 is compact, the Dynkin diagram of $\mathfrak{g}$ is of type A_n, β_2 is a node, β_1 is adjacent to β_2, the simple roots of $\mathfrak{l}$ are all compact, and β is β_2

(i) $\Delta(\mathfrak{u})$ contains exactly two simple roots β_1 and β_2, $\mathfrak{g}_0$ is $\mathfrak{sp}(2,\mathbb{R})$ of type C_2, at least one of β_1 and β_2 is noncompact, and β is the unique positive compact root.

REMARKS.

1) Only case (a) is fairly general. It is immediately clear that at most one situation fits case (a) per complex simple Lie algebra. For type A_n, no coefficient 2 occurs in a root expansion, and case (a) is never applicable. But for the other types of irreducible Dynkin diagrams, case-by-case inspection shows that case (a) is applicable.

2) The cases other than (a) apply only to special groups. In cases (b), (c), (d), (g), and (h),the Lie algebra $\mathfrak{g}_0$ is $\mathfrak{su}(n-1,2)$. In cases (e) and (i), $\mathfrak{g}_0$ is $\mathfrak{sp}(2,\mathbb{R})$. In case (f), $\mathfrak{g}_0$ is split G_2.

3) The proof will make repeated use of the following fact. A sum of distinct simple roots is always a root if the set of simple roots in question is connected in the Dynkin diagram.

4) The proof will make use also of the following fact. In a Dynkin diagram other than A_n, any simple root other than a node has coefficient

≥ 2 when the largest root is expanded in terms of simple roots.

PROOF. Let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$, introduce a positive system $\Delta^+(\mathfrak{l}, \mathfrak{b})$ for $\mathfrak{l}$, and let $\Delta^+(\mathfrak{g}, \mathfrak{b}) = \Delta^+(\mathfrak{l}, \mathfrak{b}) \cup \Delta(\mathfrak{u})$ be the corresponding positive system for $\mathfrak{g}$. Since $S = 1$, there is a unique compact root β in $\Delta(\mathfrak{u})$. Let $\mathfrak{s}$ be the $\mathfrak{sl}(2, \mathbb{C})$ generated by β. We claim that $\mathfrak{s}$ is an ideal in $\mathfrak{k}$. In fact, let Δ' be the set of compact roots not corresponding to $\mathfrak{s}$, let $\mathfrak{b}'$ be the orthogonal complement to $\mathbb{C}H_\beta = [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ in $\mathfrak{b}$, and put $\mathfrak{r}' = \mathfrak{b}' \oplus \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha$. Then $\mathfrak{r}' \oplus \mathfrak{s} = \mathfrak{k}$ as vector spaces. Take $\alpha \in \Delta'$. Then α is in $\Delta(\mathfrak{l}, \mathfrak{b})$ since $S = 1$. If $\alpha + \beta$ is a root, then $\alpha + \beta$ is compact and is in $\Delta(\mathfrak{u})$, contradiction. Hence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ and similarly $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta] = 0$. From the latter we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}] = 0$, and thus $\mathfrak{g}_\alpha$ brackets into 0 the algebra generated by $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$, namely $\mathfrak{s}$. Since $[\mathfrak{b}', \mathfrak{s}] = 0$, it follows that $[\mathfrak{r}', \mathfrak{s}] = 0$. Thus $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$, and $\mathfrak{s}$ is an ideal in $\mathfrak{k}$.

Let us prove that

(1) If β_1 and β_2 are simple roots within $\Delta(\mathfrak{u})$ and if all simple roots lying between them in the Dynkin diagram of $\mathfrak{g}$ are in $\Delta(\mathfrak{l}, \mathfrak{b})$, then all simple roots lying between them are compact. If the set of such roots is nonempty, then β_1 and β_2 are noncompact.

In fact, the roots β_1 and β_2 are both in $\Delta(\mathfrak{u})$ and hence cannot both be compact. Without loss of generality, let β_1 be noncompact. If there is a noncompact simple root between β_1 and β_2, let α_1 be the closest such to β_1, and let α_2 be the closest such to β_2. The sum γ_1 of the simple roots from

β_1 through α_1 is compact in $\Delta(\mathfrak{u})$, and hence the root β_2 in $\Delta(\mathfrak{u})$ cannot be compact. On the other hand, β_2 cannot be noncompact, since then the sum of the simple roots from α_2 through β_2 would be compact in $\Delta(\mathfrak{u})$. Hence all simple roots between β_1 and β_2 are compact. If there is such a root, let α be the closest such root to β_2. We have normalized matters so that β_1 is noncompact. If β_2 were to be compact, then β_2 and $\beta_2 + \alpha$ would be distinct compact roots in $\Delta(\mathfrak{u})$, and we would have a contradiction. This proves (1).

Next let us prove that

(2) If β_1 and β_2 are noncompact simple roots within $\Delta(\mathfrak{u})$ and if all simple roots lying between them are in $\Delta(\mathfrak{l}, \mathfrak{b})$, then β_1 and β_2 are nodes in the Dynkin diagram of $\mathfrak{g}$, and the Dynkin diagram is of type A.

Assuming the contrary, suppose that β_1 is not a node. Let α be the sum of the simple roots from β_1 through β_2, and let γ be a simple root adjacent to β_1 but not contributing to α. From (1) we know that α is compact in $\Delta(\mathfrak{u})$. If γ is compact, then $\alpha + \gamma$ is a second compact root in $\Delta(\mathfrak{u})$, while if γ is noncompact, then $\beta_1 + \gamma$ is a second compact root in $\Delta(\mathfrak{u})$. In either case we have a contradiction, and we conclude that β_1 and β_2 are both nodes.

To complete the proof of (2), we still have to show that the Dynkin diagram is of type A. First suppose, continuing with arguments by contradiction, that the Dynkin diagram has a triple point α, necessarily between β_1 and β_2. Let ε_1 be the sum of the simple roots from β_1 to α, and let ε_2 be the sum of the simple roots from β_2 through α. Let γ be the first root on the third ray from α. The expression $\varepsilon = \varepsilon_1 + \varepsilon_2 - \alpha$ is the compact root in $\Delta(\mathfrak{u})$. If γ is compact, then $\varepsilon + \gamma$ is another compact root in $\Delta(\mathfrak{u})$, while if γ is noncompact, then $\varepsilon_1 + \gamma$ is another compact root in $\Delta(\mathfrak{u})$. In either case we have a contradiction, and thus there is no triple point.

Next suppose that the diagram is of type B_n with $\beta_1 = e_1 - e_2$, $\beta_n = e_n$, and $n \geq 3$. Then e_1 and $e_2 + e_n$ are distinct compact roots in $\Delta(\mathfrak{u})$, contradiction.

Next suppose that the diagram is of type C_n with $\beta_1 = e_1 - e_2$, $\beta_n = 2e_n$, and $n \geq 3$. Then $e_1 + e_2$ and $e_1 + e_n$ are distinct compact roots in $\Delta(\mathfrak{u})$, contradiction.

Next suppose that the diagram is of type F_4 with consecutive simple roots $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, e_4, $e_3 - e_4$, $e_2 - e_3$, the first two being short. The sum of the simple roots, namely $\gamma = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ is compact and is in $\Delta(\mathfrak{u})$. But $\gamma + e_4 = \frac{1}{2}(e_1 + e_2 - e_3 + e_4)$ is another compact root in $\Delta(\mathfrak{u})$, and we have a contradiction.

Finally suppose that the diagram is of type G_2. Since $\mathfrak{k}_0$ is equal to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, there are two positive compact roots. Since one of them is in $\mathfrak{u}$, the other must be in $\mathfrak{l}$. Thus $\mathfrak{l}$ has a nonempty set of roots, and not every simple root can be in $\Delta(\mathfrak{u})$. This completes the proof of (2).

We can now prove that $\Delta(\mathfrak{u})$ contains at most two simple roots of $\mathfrak{g}$. In fact, let β_1, β_2, and β_3 be simple roots in $\Delta(\mathfrak{u})$. Without loss of generality, we may assume that no simple root of $\Delta(\mathfrak{u})$ lies between β_1 and β_2 or between β_2 and β_3. Referring to (2), we see that β_1 and β_2 cannot both be noncompact, and nor can β_2 and β_3. On the other hand, at most one of these three simple roots can be compact, since they are all in $\Delta(\mathfrak{u})$. We conclude that β_1 and β_3 are noncompact and β_2 is compact. From (1) we see that β_1, β_2, and β_3 are consecutive. Then β_2 and $\beta_1 + \beta_2 + \beta_3$ are distinct compact roots in $\Delta(\mathfrak{u})$, and we have a contradiction. Thus $\Delta(\mathfrak{u})$ has at most two simple roots.

Suppose $\Delta(\mathfrak{u})$ has exactly two simple roots, β_1 and β_2. If they are both noncompact, then (2) shows that we are in case (g) of the theorem. Otherwise let us show that we are in case (h) or (i). The roots β_1 and β_2 cannot both be compact. Thus suppose that β_1 is noncompact and β_2 is compact. By (1) these roots are adjacent in the Dynkin diagram. Let us see that β_2 is a node. If, on the contrary, γ is another simple root adjacent to it, then γ cannot be compact since β_2 and $\beta_2 + \gamma$ would be distinct compact roots in $\Delta(\mathfrak{u})$. And γ cannot be noncompact since β_2 and $\beta_1 + \beta_2 + \gamma$ would be distinct compact roots in $\Delta(\mathfrak{u})$. We conclude that β_2 is a node.

In this situation the remaining simple roots must be compact. In fact, if α is a noncompact simple root other than β_1 and β_2, we may assume that α is as close as possible to β_1. Then the sum of the simple roots from α through β_1 is compact in $\Delta(\mathfrak{u})$, and so is β_2, contradiction.

Thus a root is compact or noncompact according as the coefficient of β_1 in its expansion in terms of simple roots is even or odd. To avoid a second compact root in $\Delta(\mathfrak{u})$, the coefficient must never be even and greater than 0. Hence it must always be 0 or 1, and it must in particular be 1 in the case of the largest root. Assuming that the rank is greater than 2, so that β_1 is not a node, we refer to Remark 4 and see that the coefficient of β_1 in the expansion of the largest root is at least 2 except when the Dynkin diagram is of type A_n. Type A_n gives case (h) of the theorem. If the group has rank exactly two, then $\mathfrak{l}$ has no roots. Hence $\mathfrak{k}$ has only one positive root, and G_2 and $\mathfrak{so}(4,1)$ are excluded. The only remaining rank 2 cases are A_2 and $\mathfrak{sp}(2,\mathbb{R})$, which are covered by cases (h) and (i). Thus if $\Delta(\mathfrak{u})$ has exactly two simple roots, we are in one of cases (g), (h), or (i).

Now suppose $\Delta(\mathfrak{u})$ has exactly one simple root β_0. Suppose first that β_0 is compact. To avoid having more than one compact root in $\Delta(\mathfrak{u})$, we note that any simple root adjacent to β_0 must be noncompact. If there are two such simple roots α_1 and α_2, then $\alpha_1 + \beta_0 + \alpha_2$ is a second compact root in $\Delta(\mathfrak{u})$, contradiction. Thus β_0 is a node. The simple root β_1 adjacent to β_0 must be noncompact, and all other simple roots must be compact. If the rank is ≥ 3 and if $\mathfrak{g}$ is not of type A_n, then Remark 4 shows that the coefficient of β_1 in the expansion of the largest root is ≥ 2, and it follows that there exists a root for which the coefficient of β_1 is 2. If such a root does not involve β_0, the sum with β_0 will be a root. Thus there exists a root in $\Delta(\mathfrak{u})$ for which the coefficient of β_1 is 2. Such a root is compact and distinct from β_0, contradiction. We conclude that $\mathfrak{g}$ is of type A_n as in case (d), or else $\mathfrak{g}$ has rank 2. In the latter case the possibilities for $\mathfrak{g}_0$ are $\mathfrak{so}(4,1)$, $\mathfrak{sp}(2,\mathbb{R})$, and G_2. In $\mathfrak{so}(4,1)$ the short root is noncompact, and the long root is compact. Hence $\mathfrak{l}$ corresponds to the short root, and both compact long roots are in $\Delta(\mathfrak{u})$, contradiction. In $\mathfrak{sp}(2,\mathbb{R})$ the situation is as in case (e). In G_2 we have seen that $\Delta^+(\mathfrak{l},\mathfrak{b})$ must contain one compact root, and thus the unique simple root of $\mathfrak{l}$ cannot be noncompact. We conclude that if β_0 is compact, the situation is as in case (d) or case (e).

Next suppose that β_0 is noncompact. First suppose that $\mathfrak{l}$ is compact. Then a root of $\mathfrak{g}$ is compact or noncompact according as its expansion in terms of simple roots contains β_0 with coefficient even or odd. Exactly one root has coefficient 2, and that root is β. Computing $\langle\beta,\beta\rangle$ by expanding one of the factors β in terms of simple roots and by using the orthogonality of β with the other members of $\Delta^+(\mathfrak{k},\mathfrak{b})$, we see that β has positive inner product with β_0. Again using the orthogonality of β with the other members of $\Delta^+(\mathfrak{k},\mathfrak{b})$, we see that β has positive inner product with each positive noncompact root. If $\mathfrak{g}$ is not of type G_2, it follows that $\beta+\alpha$ is not a root when α is positive noncompact. Consequently the expansion of β has coefficient 2 for β_0. Since $\beta+\alpha$ is not a root when α is positive compact, it follows that β is the largest root. This is case (a). If $\mathfrak{g} = G_2$, we still have case (a) when β is long. But the argument breaks down when β is short. The other positive compact root is long, being orthogonal to β, and is the positive root of $\mathfrak{l}$. This is case (f).

With β_0 still noncompact, we now suppose that $\mathfrak{l}$ is noncompact, i.e., that some simple root of $\mathfrak{l}$ is noncompact. Let α be a noncompact simple root of $\mathfrak{l}$ as close as possible to β_0 in the Dynkin diagram. The sum ε of the simple roots from α through β_0 is one compact root in $\Delta(\mathfrak{u})$. Let us show that β_0 is a node. Let γ be a simple root adjacent to β_0 but not contributing to ε. If γ is compact, then $\gamma+\varepsilon$ is a second compact root in $\Delta(\mathfrak{u})$, while if γ is noncompact, then $\gamma+\beta_0$ is a second compact root in $\Delta(\mathfrak{u})$, contradiction. We conclude that β_0 is a node.

Since $\mathfrak{l}$ is noncompact, $\Delta^+(\mathfrak{l},\mathfrak{b})$ has a noncompact simple root. Let α_0

be such a root that is as close as possible to β_0 in the Dynkin diagram. The sum ε of the simple roots from α_0 through β_0 is compact in $\Delta(\mathfrak{u})$. If α_0 is not a node, let α_1 be an adjacent simple root not contributing to ε. Then α_1 must be noncompact to avoid having $\alpha_1 + \varepsilon$ as a second compact root in $\Delta(\mathfrak{u})$, and all remaining simple roots must be compact. (In particular, α_0 cannot be a triple point.)

To complete the proof that we are in case (b) or case (c), we prove that the Dynkin diagram is of type A_n. Let Δ' be the Dynkin diagram obtained by including the simple roots α_0 through β_0, together with the full third ray extending from a triple point, if any, that lies between α_0 and β_0. Then the case-by-case argument for (2) that concludes type A is applicable here and shows that Δ' is of type A. In particular if α_0 is a node, we are in case (c).

Now consider the full Dynkin diagram. We claim it contains no triple point. Since Δ' contains no triple point, a triple point has to be at α_1 or beyond. Then we can form a Dynkin subdiagram of type D_n with β_0 as $e_1 - e_2$, α_0 as $e_{k-2} - e_{k-1}$ and α_1 as $e_{k-1} - e_k$, $k \le n-1$. The root $e_1 + e_{k-1}$ within this subdiagram, when expanded in terms of simple roots, is of the form

$$e_1 + e_{k-1} = \beta_0 + \cdots + \alpha_0 + 2\alpha_1 + \ldots$$

and is therefore compact. Since it is in $\Delta(\mathfrak{u})$, we have at least two compact roots in $\Delta(\mathfrak{u})$, contradiction. Thus there can be no triple point.

In a diagram B_n or C_n, we know that β_0 must be $e_1 - e_2$, since Δ' is of type A. Let α_0 be $e_{k-1} - e_k$, and let α_1 be the next simple root. In B_n, the roots $e_1 - e_k$ and $e_1 + e_k$ are compact in $\Delta(\mathfrak{u})$, while in C_n the roots $e_1 - e_k$

and $e_1 + e_{k-1}$ are compact in $\Delta(\mathfrak{u})$. In either case we have a contradiction.

Since the rank is ≥ 3, we are left with F_4. Since Δ' is of type A, the roots β_0, α_0, α_1 are consecutive with β_0 as a node. The diagram generated by these three simple roots is of type B_3 or C_3, and we have just seen how to produce a second compact root in $\Delta(\mathfrak{u})$ in either of these diagrams. Thus F_4 is excluded, and the proof that we are in case (b) is complete.

CHAPTER 2

BACKGROUND FOR $Sp(1,1)$

1. Notation for $Sp(1,1)$

This chapter consists of technical results that will be used in later chapters. The reader may skip this chapter on first reading and refer back to it as needed.

In the first section, we introduce notation for $Sp(1,1)$ that will be used in this chapter, as well as Chapters 3 and 4. To begin, on the Lie algebra level,

$$\mathfrak{g}_0 = \mathfrak{sp}(1,1),$$

and

$$\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}.$$

According to [K3, pg. 1],

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

is a Cartan decomposition for $\mathfrak{g}_0$, where $\mathfrak{k}_0$ is the set of skew-Hermitian members of $\mathfrak{g}_0$, and $\mathfrak{p}_0$ is the set of Hermitian members. We can represent these vector spaces as matrices by using [K3, pg. 27, Exercise 21]. The matrices listed below are obtained from this exercise by applying the map

$a_{ij} \mapsto a_{\sigma(i)\sigma(j)}$ to each entry of a matrix $A = (a_{ij})$, where $\sigma \in \mathfrak{S}_4$ is the transposition (23). Hence, if $u_1, \ldots, w_3 \in \mathbb{R}$, then a matrix in $\mathfrak{k}_0$ looks like

$$\begin{pmatrix} iu_1 & -u_2 - iu_3 & 0 & 0 \\ u_2 - iu_3 & -iu_1 & 0 & 0 \\ 0 & 0 & iw_1 & -w_2 - iw_3 \\ 0 & 0 & w_2 - iw_3 & -iw_1 \end{pmatrix}.$$

The complexification $\mathfrak{k}$ is given by

$$\mathfrak{k} = \mathfrak{k}_0 \oplus i\mathfrak{k}_0.$$

Also, for $x_1, \ldots, x_4 \in \mathbb{R}$, a matrix in $\mathfrak{p}_0$ looks like

$$\begin{pmatrix} 0 & 0 & x_1 + ix_2 & -x_3 - ix_4 \\ 0 & 0 & x_3 - ix_4 & x_1 - ix_2 \\ x_1 - ix_2 & x_3 + ix_4 & 0 & 0 \\ -x_3 + ix_4 & x_1 + ix_2 & 0 & 0 \end{pmatrix}.$$

The complexification $\mathfrak{p}$ is given by

$$\mathfrak{p} = \mathfrak{p}_0 \oplus i\mathfrak{p}_0.$$

This is one possible choice for $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{k}$, and $\mathfrak{p}$. It is the choice we will use in this thesis. From the matrix formulas of $\mathfrak{k}_0$ and $\mathfrak{k}$ we see that

$$\mathfrak{k}_0 = \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

and

$$\mathfrak{k} = (\mathfrak{k}_0)^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

As a representative matrix of the Cartan subalgebra $\mathfrak{h}_0$ we choose, for u_1 and $w_1 \in \mathbb{R}$, the matrix

$$\begin{pmatrix} iu_1 & 0 & 0 & 0 \\ 0 & -iu_1 & 0 & 0 \\ 0 & 0 & iw_1 & 0 \\ 0 & 0 & 0 & -iw_1 \end{pmatrix}.$$

Then the Cartan subalgebra of $\mathfrak{g}$ is given by

$$\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0.$$

The notation for linear functionals on $\mathfrak{h}$ is

$$e_j \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & -a_2 \end{pmatrix} = a_j.$$

Here $j = 1$ or 2, and $a_1, a_2 \in \mathbb{C}$.

Using matrices of $\mathfrak{k}$ and $\mathfrak{p}$, we can make assignments of root vectors in $\mathfrak{g}$. Let E_{ij} be the matrix of $\mathfrak{gl}(4,\mathbb{C})$ with 1 in the $(i,j)^{\text{th}}$ coordinate, 0 elsewhere. Then we define

Root Vectors for $\mathfrak{k}$:

$$X_{2e_1} = E_{12} \qquad X_{-2e_1} = E_{21}$$

(2.1a)

$$X_{2e_2} = E_{34} \qquad X_{-2e_2} = E_{43}.$$

Root Vectors for $\mathfrak{p}$:

$$X_{e_1+e_2} = -E_{14} + E_{32} \qquad X_{-e_1-e_2} = -E_{41} + E_{23}$$

(2.1b)

$$X_{e_1-e_2} = E_{13} + E_{42} \qquad X_{-e_1+e_2} = E_{24} + E_{31}.$$

Basis Vectors for $\mathfrak{h}$:

(2.1c)
$$H_1 = E_{11} - E_{22} \qquad H_2 = E_{33} - E_{44}.$$

In the decomposition of $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ (next section), it will be important to know the lengths of root vectors in $\mathfrak{p}$. For this purpose, we use an altered version of the Hermitian form

$$\langle U, V \rangle = -B(U, \theta\bar{V}),$$

where B is the Killing form on $\mathfrak{g}$, $U, V \in \mathfrak{g}$, and $V \mapsto \theta\bar{V}$ is conjugation of $\mathfrak{g}$ with respect to the compact form $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ [K-W, (2.4)]. The altered version replaces B by $\frac{1}{2}$ Trace form. It is also true that the Hermitian form

(2.2)
$$\langle U, V \rangle = -\frac{1}{2} Tr(U, \theta\bar{V})$$

is a positive definite inner product on $\mathfrak{g}$ for $U, V \in \mathfrak{g}$.

On the group level, when

$$G = Sp(1,1),$$

then

$$K = SU(2) \times SU(2).$$

According to the discussion in Chapter 1, Section 2, the discrete series representations come from modules (1.12) and (1.17c) and maps ∂_n and $\partial_n^{\natural}$. In order to describe more explicitly these modules and maps, we need to know the groups T and L and the Lie algebras $\mathfrak{l}$, $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ (see Chapter 1, Section 1). For $\mathfrak{g} = \mathfrak{sp}(1,1)^{\mathbb{C}}$, we choose $T = S^1 \times \{Id\} \subseteq SU(2) \times SU(2)$. It follows that

$$L = S^1 \times SU(2) \tag{2.3a}$$

$$\mathfrak{l} = \mathfrak{h} \oplus \mathbb{C}X_{2e_2} \oplus \mathbb{C}X_{-2e_2} \tag{2.3b}$$

$$\mathfrak{u} = \mathbb{C}X_{2e_1} \oplus \mathbb{C}X_{e_1+e_2} \oplus \mathbb{C}X_{e_1-e_2} \tag{2.3c}$$

$$\bar{\mathfrak{u}} = \mathbb{C}X_{-2e_1} \oplus \mathbb{C}X_{-e_1-e_2} \oplus \mathbb{C}X_{-e_1+e_2} \tag{2.3d}$$

and

$$\bar{\mathfrak{q}} = \bar{\mathfrak{u}} \oplus \mathfrak{l}. \tag{2.3e}$$

In Chapter 3, we will be computing L invariant vectors for $G = Sp(1,1)$. Important bracket relations that will be used in those sections are

$$\begin{gathered} [X_{2e_2}, X_{e_1-e_2}] = X_{e_1+e_2} \\ [X_{2e_2}, X_{e_1+e_2}] = [X_{2e_2}, X_{2e_1}] = 0, \end{gathered} \tag{2.4}$$

which follow from the definitions (2.1a) and (2.1b).

Throughout the remainder of this chapter, all symbols will refer back to notation defined in this first section.

2. Decomposition of $\mathbf{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$

Our goal for this section is a concrete realization of the decomposition of the K representation $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$, where $\pi_{(P,Q)}$ is an irreducible representation of K with highest weight $Pe_1 + Qe_2$. This will be accomplished by first presenting a concrete realization of $\pi_{(P,Q)}$.

To begin, we note that, with $\Delta_K^+ = \{2e_1, 2e_2\}$, all irreducible representations of K are parametrized by the ordered pair (P,Q), P and Q both nonnegative integers. This is a consequence of the Theorem of the Highest Weight [K3, Theorem 4.28]. Therefore, the notation $\pi_{(P,Q)}$ means P and Q are both nonnegative integers. In constructing $\pi_{(P,Q)}$, we construct first an irreducible representation of $SU(2)$ of highest weight Pe_1, call it π_1, and then an irreducible representation of $SU(2)$ of highest weight Qe_2, call it π_2, and then define $\pi_{(P,Q)}$ as

$$\pi_{(P,Q)} = \pi_1 \otimes \pi_2.$$

The representation π_1 is given as follows:

$$V_P = \{\text{Space of homogeneous polynomials in } z_1 \text{ and } z_2 \text{ of degree } P\},$$

and

$$\pi_1(k_1)\mathcal{P}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathcal{P}\left(k_1^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right),$$

for $k_1 \in SU(2)$, $\mathcal{P}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in V_P$. The weight vectors we shall be using for our calculations are denoted by $v_{(P-2i_1)e_1}$, $0 \leq i_1 \leq P$. Specifically,

(2.5a) $$v_{-Pe_1} = z_1^P,$$

and $v_{(P-2i_1)e_1}$ is defined so that

(2.5b) $$X_{2e_1} \cdot v_{(P-2i_1)e_1} = \begin{cases} v_{(P-2i_1+2)e_1}, & \text{for } 1 \leq i_1 \leq P \\ 0, & \text{for } i_1 = 0. \end{cases}$$

For the action (2.5b), we think of X_{2e_1} as the left upper block of the 4×4 matrix E_{12}, but eventually, X_{2e_1} will be E_{12}. With weight vectors defined in this way, it follows that

(2.5c)
$$X_{-2e_1} \cdot v_{(P-2i_1)e_1} = \begin{cases} (P-i_1)(i_1+1)v_{(P-2i_1-2)e_1}, & \text{for } 0 \leq i_1 \leq P-1 \\ 0, & \text{for } i_1 = P. \end{cases}$$

For the action (2.5c), we think of X_{-2e_1} as the left upper block of the 4×4 matrix E_{21}, but eventually, X_{-2e_1} will be E_{21}.

Using the inner product

$$\langle \mathcal{P}, \mathcal{Q} \rangle = \partial(\mathcal{P})\mathcal{Q}^*, \tag{2.6}$$

where the operator ∂ is defined on $\mathcal{P}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1^k z_2^l$ as

$$\partial(\mathcal{P}) = \frac{\partial^{k+l}}{\partial z_1^k \partial z_2^l},$$

and $\mathcal{Q} \mapsto \mathcal{Q}^*$ is conjugate linear conjugation (the coefficients of $\mathcal{Q}$ are conjugated), we see that

$$\|v_{(P-2i_1)e_1}\|^2 = \frac{(P!)^2(P-i_1)!}{i_1!}. \tag{2.7}$$

Notice that this inner product is unitary with respect to π_1 and is linear in the first coordinate, conjugate linear in the second.

Similarly, we can define an irreducible representation π_2 of $SU(2)$ with highest weight Qe_2, where the vector space is

$$W_Q = \{\text{Space of homogeneous polynomials in } z_1 \text{ and } z_2 \text{ of degree } Q\},$$

and the weight vectors $w_{(Q-2i_2)e_2}$, $0 \leq i_2 \leq Q$, satisfy

$$v_{-Qe_2} = z_1^Q, \tag{2.8a}$$

$$(2.8b)\qquad X_{2e_2}\cdot w_{(Q-2i_2)e_2}=\begin{cases} w_{(Q-2i_2+2)e_2}, & \text{for } 1\le i_2\le Q\\ 0, & \text{for } i_2=0\end{cases}$$

(2.8c)

$$X_{-2e_2}\cdot w_{(Q-2i_2)e_2}=\begin{cases} (Q-i_2)(i_2+1)w_{(Q-2i_2-2)e_2}, & \text{for } 0\le i_2\le Q-1\\ 0, & \text{for } i_2=Q\end{cases}$$

and

$$(2.9)\qquad \|w_{(Q-2i_2)e_2}\|^2=\frac{(Q!)^2(Q-i_2)!}{i_2!}.$$

In formulas (2.8b) and (2.8c), X_{2e_2} and X_{-2e_2} are, for the time being, the lower right blocks of the 4×4 matrices E_{34} and E_{43}, respectively.

As mentioned earlier, $\pi_{(P,Q)}=\pi_1\otimes\pi_2$. With X_{2e_1}, X_{2e_2}, X_{-2e_1}, and X_{-2e_2} now given by the formulas (2.1a), we have

$$(2.10a)\qquad X_{2e_1}(v_{(P-2i_1)e_1}\otimes w_{(Q-2i_2)e_2})=\begin{cases} v_{(P-2i_1+2)e_1}\otimes w_{(Q-2i_2)e_2}, & \text{for } 1\le i_1\le P\\ 0, & \text{for } i_1=0\end{cases}$$

(2.10b)

$$X_{-2e_1}(v_{(P-2i_1)e_1}\otimes w_{(Q-2i_2)e_2})=\begin{cases} (P-i_1)(i_1+1)v_{(P-2i_1-2)e_1}\otimes w_{(Q-2i_2)e_2}, & \text{for } 0\le i_1\le P-1\\ 0, & \text{for } i_1=P\end{cases}$$

$$(2.10c)\qquad X_{2e_2}(v_{(P-2i_1)e_1}\otimes w_{(Q-2i_2)e_2})=\begin{cases} v_{(P-2i_1)e_1}\otimes w_{(Q-2i_2+2)e_2}, & \text{for } 1\le i_2\le Q\\ 0, & \text{for } i_2=0\end{cases}$$

(2.10d)

$$X_{-2e_2}(v_{(P-2i_1)e_1} \otimes w_{(Q-2i_2)e_2}) = \begin{cases} (Q-i_2)(i_2+1)v_{(P-2i_1)e_1} \otimes w_{(Q-2i_2-2)e_2}, & \text{for } 0 \leq i_2 \leq Q-1 \\ 0, & \text{for } i_2 = Q \end{cases}$$

and

$$\|v_{(P-2i_1)e_1} \otimes w_{(Q-2i_2)e_2}\|^2 = \frac{(P!)^2(Q!)^2(P-i_1)!(Q-i_2)!}{i_1!i_2!}. \tag{2.10e}$$

REMARKS. In subsequent text, we will use the notation $(\cdot,\cdot)$, as opposed to $\langle\cdot,\cdot\rangle$, when refering to the inner product on $V_P \otimes W_Q$. However, when refering to the inner product on $\mathrm{Ad}|_{\mathfrak{p}} \otimes V_P \otimes W_Q$, we will use $\langle\cdot,\cdot\rangle$. Also, beginning with Proposition 2.11, and throughout the remainder of this chapter and also in Chapter 3, we drop e_1 from the vector $v_{(P-2i_1)e_1}$ and write simply $v_{(P-2i_1)}$. Also, we drop e_2 from the vector $w_{(Q-2i_2)e_2}$ and write simply $w_{(Q-2i_2)}$. No confusion will result, since, for irreducible representations of K, v will always correspond to e_1 and w will always correspond to e_2.

We are now ready to decompose $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ into its irreducible components. Notice that by using [BS-K, Corollary 1.4], we have the irreducible components $(P+1,Q+1)$, $(P+1,Q-1)$, $(P-1,Q-1)$, and $(P-1,Q+1)$ all appearing in $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ with multiplicity one, when $P,Q \geq 1$. When either $P=0$ or $Q=0$, we exclude from the list above those K types that have negative coordinates.

Proposition 2.11. Using vectors described in formulas (2.10(a)–(d)) with lengths described by formula (2.10e), an $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ decomposition into irreducible components is given by the following equations:

Basis Vectors for $(P+1, Q+1)$ K type:

(2.11a) $$X_{e_1+e_2} \otimes v_P \otimes w_Q = \frac{1}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1} \otimes w_{Q+1}.$$

For $0 \leq i_1 \leq P$,

(2.11b) $$-X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_Q + (P-i_1)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_Q = \frac{1}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1}.$$

For $0 \leq i_2 \leq Q$,

(2.11c) $$X_{e_1-e_2} \otimes v_P \otimes w_{Q-2i_2} + (Q-i_2)X_{e_1+e_2} \otimes v_P \otimes w_{Q-2(i_2+1)} = \frac{1}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}} v_{P+1} \otimes w_{Q+1-2(i_2+1)}.$$

For $0 \leq i_1 \leq P$ and $0 \leq i_2 \leq Q$,

(2.11d)

$$\begin{aligned} X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} &- (Q-i_2)X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2(i_2+1)} \\ &+ (P-i_1)X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2} \\ &+ (P-i_1)(Q-i_2)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2(i_2+1)} \\ &= \frac{1}{((P+1)(Q+1))^{\frac{1}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1-2(i_2+1)}. \end{aligned}$$

Basis Vectors for $(P-1, Q+1)$ K type:

For $0 \leq i_1 \leq P-1$,

(2.11e) $$X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_Q + (i_1+1)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_Q = \frac{P(P+1)^{\frac{1}{2}}}{(Q+1)^{\frac{3}{2}}} v_{P-1-2i_1} \otimes w_{Q+1}.$$

For $0 \leq i_1 \leq P-1$ and $0 \leq i_2 \leq Q$,

(2.11f)

$$- X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} + (Q-i_2)X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2(i_2+1)}$$
$$+ (i_1+1)X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2}$$
$$+ (i_1+1)(Q-i_2)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2(i_2+1)}$$
$$= \frac{P(P+1)^{\frac{1}{2}}}{(Q+1)^{\frac{1}{2}}} v_{P-1-2i_1} \otimes w_{Q+1-2(i_2+1)}.$$

Basis Vectors for $(P-1, Q-1)$ K type:

For $0 \leq i_1 \leq P-1$ and $0 \leq i_2 \leq Q-1$,

(2.11g)

$$X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} - (i_1+1)X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2}$$
$$+ (i_2+1)X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2(i_2+1)}$$
$$+ (i_1+1)(i_2+1)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2(i_2+1)}$$
$$= PQ((P+1)(Q+1))^{\frac{1}{2}} v_{P-1-2i_1} \otimes w_{Q-1-2i_2}.$$

Basis Vectors for $(P+1, Q-1)$ K type:

For $0 \leq i_2 \leq Q_1$,

(2.11h) $$X_{e_1-e_2} \otimes v_P \otimes w_{Q-2i_2} - (i_2+1)X_{e_1+e_2} \otimes v_P \otimes w_{Q-2(i_2+1)}$$
$$= \frac{Q(Q+1)^{\frac{1}{2}}}{(P+1)^{\frac{3}{2}}} v_{P+1} \otimes w_{Q-1-2i_2}.$$

For $0 \leq i_1 \leq P$ and $0 \leq i_2 \leq Q_1$,

(2.11i)

$$X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} + (P-i_1)X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2}$$
$$+ (i_2+1)X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2(i_2+1)}$$
$$- (P-i_1)(i_2+1)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2(i_2+1)}$$
$$= \frac{Q(Q+1)^{\frac{1}{2}}}{(P+1)^{\frac{1}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q-1-2i_2}.$$

PROOF. In each of the K types, a choice of correspondence is required for the highest weight vector. Once a choice is made, the remaining correspondences are forced by the $\mathfrak{k}$ action. If we use the root vectors defined in (2.1a) and (2.1b), then the important bracket relations for our calculations are:

$$(2.11j) \quad \begin{array}{ll} [X_{2e_1}, X_{-e_1+e_2}] = -X_{e_1+e_2} & [X_{2e_2}, X_{e_1-e_2}] = X_{e_1+e_2} \\ [X_{2e_1}, X_{-e_1-e_2}] = X_{e_1-e_2} & [X_{2e_2}, X_{-e_1-e_2}] = -X_{-e_1+e_2} \\ [X_{-2e_1}, X_{e_1+e_2}] = -X_{-e_1+e_2} & [X_{-2e_2}, X_{e_1+e_2}] = X_{e_1-e_2} \\ [X_{-2e_1}, X_{e_1-e_2}] = X_{-e_1-e_2} & [X_{-2e_2}, X_{-e_1+e_2}] = -X_{-e_1-e_2}. \end{array}$$

The remaining bracket relations between $\{X_{\pm 2e_1}, X_{\pm 2e_2}\}$ and the vectors of $\mathrm{Ad}|_{\mathfrak{p}}$ are 0.

The proof for each of the K types is the same, and so we will do the proof for the $(P+1, Q+1)$ K type and state the correspondence of highest weight vectors for the other K types.

In the K type $(P+1, Q+1)$, the sole tensor of weight $(P+1, Q+1)$ is $X_{e_1+e_2} \otimes v_P \otimes w_Q$. From (2.2) and (2.10e), we know its length is $(P!\,Q!)^{\frac{3}{2}}$. We choose the correspondence

$$X_{e_1+e_2} \otimes v_P \otimes w_Q = c\, v_{P+1} \otimes w_{Q+1},$$

where c is an added constant that guarentees both sides have equal length. Because the length of $v_{P+1} \otimes w_{Q+1}$ is $((P+1)!)^{\frac{3}{2}}((Q+1)!)^{\frac{3}{2}}$, the constant c is $\dfrac{1}{((P+1)(Q+1))^{\frac{3}{2}}}$. Substituting this into the equation above, we get

$$X_{e_1+e_2} \otimes v_P \otimes w_Q = \frac{1}{((P+1)(Q+1))^{\frac{3}{2}}}\, v_{P+1} \otimes w_{Q+1},$$

which is formula (2.11a).

Applying X_{-2e_1} to both sides of (2.11a) yields

$$\begin{aligned} -X_{-e_1+e_2} \otimes v_P \otimes w_Q + P\, X_{e_1+e_2} \otimes v_{P-2} \otimes w_Q \\ &= \frac{(P+1)}{((P+1)(Q+1))^{\frac{3}{2}}}\, v_{P+1-2} \otimes w_{Q+1} \\ &= \frac{1}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}\, v_{P+1-2} \otimes w_{Q+1}. \end{aligned}$$

Successive applications of X_{-2e_1}, one can show by induction, give

$$\begin{aligned} -X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_Q + (P-i_1) X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_Q \\ &= \frac{1}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}\, v_{P+1-2(i_1+1)} \otimes w_{Q+1}, \end{aligned}$$

for $0 \leq i_1 \leq P$. This is formula (2.11b).

Applying X_{-2e_2} to (2.11a) gives

$$
\begin{aligned}
X_{e_1-e_2} \otimes v_P \otimes w_Q + Q\, X_{e_1+e_2} \otimes v_P \otimes w_{Q-2} \\
&= \frac{(Q+1)}{((P+1)(Q+1))^{\frac{3}{2}}}\, v_{P+1} \otimes w_{Q+1-2} \\
&= \frac{1}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}}\, v_{P+1} \otimes w_{Q+1-2}.
\end{aligned}
$$

Successive applications of X_{-2e_2} yield the formula

$$
\begin{aligned}
X_{e_1-e_2} \otimes v_P \otimes w_{Q-2i_2} + (Q-i_2) X_{e_1+e_2} \otimes v_P \otimes w_{Q-2(i_2+1)} \\
&= \frac{1}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}}\, v_{P+1} \otimes w_{Q+1-2(i_2+1)},
\end{aligned}
$$

for $0 \leq i_2 \leq Q$. As in the previous formula, this can be shown by induction. This is formula (2.11c).

Now apply X_{-2e_2} to (2.11b), for any fixed i_1. We obtain the equation

$$
\begin{aligned}
X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_Q - Q\, X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2} \\
+(P-i_1) X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_Q \\
+(P-i_1) Q\, X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2} \\
&= \frac{(Q+1)}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}\, v_{P+1-2(i_1+1)} \otimes w_{Q+1-2} \\
&= \frac{1}{((P+1)(Q+1))^{\frac{1}{2}}}\, v_{P+1-2(i_1+1)} \otimes w_{Q+1-2}.
\end{aligned}
$$

Once again by applying X_{-2e_2} successively, one can show inductively that

$$X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} - (Q-i_2)X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_{Q-2(i_2+1)}$$

$$+ (P-i_1)X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2}$$

$$+ (P-i_1)(Q-i_2)X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2(i_2+1)}$$

$$= \frac{1}{((P+1)(Q+1))^{\frac{1}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1-2(i_2+1)},$$

for $0 \leq i_2 \leq Q$. Since i_1 was arbitrarily chosen, this formula is true for $0 \leq i_1 \leq P$. This gives us (2.11d). Alternatively, we could have applied X_{-2e_1} repeatedly to (2.11c) to obtain the same result. This completes the proof for the $(P+1, Q+1)$ K type.

For the other K types, we make a correspondence of highest weights. It should be noted that in order to find a highest weight vector of a K type other than $(P+1, Q+1)$, we must find **all** pure tensors of a given highest weight, and apply both X_{2e_1} and X_{2e_2} to linear combinations of these pure tensors. The highest weight vectors are those nonzero linear combinations for which the X_{2e_1} and X_{2e_2} actions give 0.

For the $(P-1, Q+1)$ K type, the correspondence chosen is:

$$X_{-e_1+e_2} \otimes v_P \otimes w_Q + X_{e_1+e_2} \otimes v_{P-2} \otimes w_Q = \frac{P(P+1)^{\frac{1}{2}}}{(Q+1)^{\frac{3}{2}}} v_{P-1} \otimes w_{Q+1}.$$

For the $(P-1, Q-1)$ K type, the correspondence chosen is:

$$X_{-e_1-e_2} \otimes v_P \otimes w_Q - X_{e_1-e_2} \otimes v_{P-2} \otimes w_Q$$

$$+ X_{-e_1+e_2} \otimes v_P \otimes w_{Q-2} + X_{e_1+e_2} \otimes v_{P-2} \otimes w_{Q-2}$$

$$= PQ((P+1)(Q+1))^{\frac{1}{2}} v_{P-1} \otimes w_{Q-1}.$$

For the $(P+1, Q-1)$ K type, the correspondence is:

$$X_{e_1-e_2} \otimes v_P \otimes w_Q - X_{e_1+e_2} \otimes v_P \otimes w_{Q-2} = \frac{Q(Q+1)^{\frac{1}{2}}}{(P+1)^{\frac{3}{2}}} v_{P+1} \otimes w_{Q-1}.$$

This completes the proof.

Corollary 2.12 Decomposition of pure tensors in $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ into irreducible components is given by the following formulas:

(2.12a)
$$X_{e_1+e_2} \otimes v_P \otimes w_Q = \frac{1}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1} \otimes w_{Q+1}$$

For $0 \le i_1 \le P$,

(2.12b)
$$X_{-e_1+e_2} \otimes v_{P-2i_1} \otimes w_Q = \frac{-(i_1+1)}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1} + \frac{P(P-i_1)}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}} v_{P-1-2i_1} \otimes w_{Q+1}.$$

For $0 \le i_1 \le P-1$,

(2.12c)
$$X_{e_1+e_2} \otimes v_{P-2(i_1+1)} \otimes w_Q = \frac{1}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1} + \frac{P}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}} v_{P-1-2i_1} \otimes w_{Q+1}.$$

For $0 \le i_2 \le Q$,

$$(2.12d)\quad X_{e_1-e_2} \otimes v_P \otimes w_{Q-2i_2} = \frac{(i_2+1)}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1} \otimes w_{Q+1-2(i_2+1)} + \frac{Q(Q-i_2)}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}} v_{P+1} \otimes w_{Q-1-2i_2}.$$

For $0 \le i_2 \le Q-1$,

(2.12e)

$$X_{e_1+e_2} \otimes v_P \otimes w_{Q-2(i_2+1)} = \frac{1}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1} \otimes w_{Q+1-2(i_2+1)} - \frac{Q}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}} v_{P+1} \otimes w_{Q-1-2i_2}.$$

For $0 \le i_1 \le P$ and $0 \le i_2 \le Q$,

$$\begin{aligned}(2.12f)\quad & X_{-e_1-e_2} \otimes v_{P-2i_1} \otimes w_{Q-2i_2} \\ &= \frac{(i_1+1)(i_2+1)}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1-2(i_2+1)} \\ &\quad - \frac{P(P-i_1)(i_2+1)}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}} v_{P-1-2i_1} \otimes w_{Q+1-2(i_2+1)} \\ &\quad + \frac{PQ(P-i_1)(Q-i_2)}{((P+1)(Q+1))^{\frac{1}{2}}} v_{P-1-2i_1} \otimes w_{Q-1-2i_2} \\ &\quad + \frac{Q(i_1+1)(Q-i_2)}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q-1-2i_2}.\end{aligned}$$

For $0 \le i_1 \le P-1$ and $0 \le i_2 \le Q$,

$$\begin{aligned}(2.12g)\quad & X_{e_1-e_2} \otimes v_{P-2(i_1+1)} \otimes w_{Q-2i_2} \\ &= \frac{(i_2+1)}{((P+1)(Q+1))^{\frac{3}{2}}} v_{P+1-2(i_1+1)} \otimes w_{Q+1-2(i_2+1)}\end{aligned}$$

$$+\frac{P(i_2+1)}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}v_{P-1-2i_1}\otimes w_{Q+1-2(i_2+1)}$$

$$-\frac{PQ(Q-i_2)}{((P+1)(Q+1))^{\frac{1}{2}}}v_{P-1-2i_1}\otimes w_{Q-1-2i_2}$$

$$+\frac{Q(Q-i_2)}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}}v_{P+1-2(i_1+1)}\otimes w_{Q-1-2i_2}.$$

For $0\le i_1\le P$ and $0\le i_2\le Q-1$,

(2.12h) $X_{-e_1+e_2}\otimes v_{P-2i_1}\otimes w_{Q-2(i_2+1)}$

$$=-\frac{(i_1+1)}{((P+1)(Q+1))^{\frac{3}{2}}}v_{P+1-2(i_1+1)}\otimes w_{Q+1-2(i_2+1)}$$

$$+\frac{P(P-i_1)}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}v_{P-1-2i_1}\otimes w_{Q+1-2(i_2+1)}$$

$$+\frac{PQ(P-i_1)}{((P+1)(Q+1))^{\frac{1}{2}}}v_{P-1-2i_1}\otimes w_{Q-1-2i_2}$$

$$+\frac{Q(i_1+1)}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}}v_{P+1-2(i_1+1)}\otimes w_{Q-1-2i_2}.$$

For $0\le i_1\le P-1$ and $0\le i_2\le Q-1$,

(2.12i) $X_{e_1+e_2}\otimes v_{P-2(i_1+1)}\otimes w_{Q-2(i_2+1)}$

$$=\frac{1}{((P+1)(Q+1))^{\frac{3}{2}}}v_{P+1-2(i_1+1)}\otimes w_{Q+1-2(i_2+1)}$$

$$+\frac{P}{(P+1)^{\frac{1}{2}}(Q+1)^{\frac{3}{2}}}v_{P-1-2i_1}\otimes w_{Q+1-2(i_2+1)}$$

$$+\frac{PQ}{((P+1)(Q+1))^{\frac{1}{2}}}v_{P-1-2i_1}\otimes w_{Q-1-2i_2}$$

$$-\frac{Q}{(P+1)^{\frac{3}{2}}(Q+1)^{\frac{1}{2}}}v_{P+1-2(i_1+1)}\otimes w_{Q-1-2i_2}.$$

PROOF. This is an immediate consequence of Proposition 2.11 and methods for solving systems of linear equations.

3. Realization of Dual Vectors for $R(K)$

Having accomplished the decomposition of $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$, we move onto our next goal for this chapter, a realization of the dual vectors for the K representation $R(K)$. Recall that $R(K)$ is described in Definition 1.7. For $\sum v_\mu \otimes v_\mu^*$ an element of $V_\mu \otimes V_\mu^*$ and r = right regular representation of K on $R(K)$, we have

$$r(k)(\sum v_\mu \otimes v_\mu^*) = \sum (v_\mu \otimes \pi^*(k) v_\mu^*).$$

This follows from formula (1.7a). The contragredient representation will play a crucial role in determining basis vectors for the K types of $Sp(1,1)$ discrete series. We will see this presently. First, though, we want a more detailed description of the dual vectors.

In the K type (P,Q), we start with the basis vector $v_P \otimes w_Q$ (defined in the previous section). Denoting by $(v_{-P} \otimes w_{-Q})^*$ the linear functional $(\cdot, v_P \otimes w_Q)$, we define linear functionals $(v_{-P+2i} \otimes w_{-Q+2j})^*$ by the relations

$$\text{(2.13a)} \qquad \begin{aligned} X_{2e_1}(v_{-P+2i} \otimes w_{-Q+2j})^* &= (v_{-P+2(i+1)} \otimes w_{-Q+2j})^* \\ X_{2e_2}(v_{-P+2i} \otimes w_{-Q+2j})^* &= (v_{-P+2i} \otimes w_{-Q+2(j+1)})^*, \end{aligned}$$

for $0 \leq i \leq P-1$ and $0 \leq j \leq Q-1$. In addition,

$$\text{(2.13b)} \qquad X_{2e_1}(v_P \otimes w_{-Q+2j})^* = X_{2e_2}(v_{-P+2j} \otimes w_Q)^* = 0.$$

When using the inner product (2.6), we have

$$X(\cdot, v \otimes w) = (\cdot, \bar{X}(v \otimes w)),$$

where conjugation is in $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ relative to $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. This equation, (2.10b), (2.10d), and the equations $\bar{X}_{2e_1} = -X_{-2e_1}$, $\bar{X}_{2e_2} = -X_{-2e_2}$ show that

(2.14a)

$$(v_{-P+2i}\otimes w_{-Q+2j})^* = (-1)^{i+j}\binom{P}{i}\binom{Q}{j}(i!)^2(j!)^2(\cdot,\, v_{P-2i}\otimes w_{Q-2j}),$$

for $0 \le i \le P$ and $0 \le j \le Q$. A variation of this formula that will be particularly useful is derived by substituting $P - i_1$ for i and $Q - i_2$ for j:

(2.14b)

$$(v_{P-2i_1}\otimes w_{Q-2i_2})^* = (-1)^{P+Q-i_1-i_2}\binom{P}{i_1}\binom{Q}{i_2}((P-i_1)!)^2((Q-i_2)!)^2$$
$$\times\, (\cdot,\, v_{2i_1-P}\otimes w_{2i_2-Q}).$$

Here, $0 \le i_1 \le P$ and $0 \le i_2 \le Q$. With these substitutions, we can rewrite (2.13a) and (2.13b) as

(2.13c)
$$X_{2e_1}(v_{P-2i_1}\otimes w_{Q-2i_2})^* = (v_{P-2i_1+2}\otimes w_{Q-2i_2})^*$$
$$X_{2e_2}(v_{P-2i_1}\otimes w_{Q-2i_2})^* = (v_{P-2i_1}\otimes w_{Q-2i_2+2})^*,$$

for $0 \le i \le P-1$ and $0 \le j \le Q-1$. In addition,

(2.13d)
$$X_{2e_1}(v_P\otimes w_{Q-2i_2})^* = X_{2e_2}(v_{P-2i_1}\otimes w_Q)^* = 0.$$

Note that for $K = SU(2)\times SU(2)$, the contragredient of the K type (P,Q) is (P,Q). As a result of this fact, coupled with the formulas (2.5a), (2.5b), (2.8a), (2.8b), and (2.14b), we can write a basis for the K isotypic component of $R(K)$ of type (P,Q) as

(2.15)

$$\{ v_{P-2i_1} \otimes w_{Q-2i_2} \otimes (v_{P-2i'_1} \otimes w_{Q-2i'_2})^* | \ 0 \leq i_1, i'_1 \leq P \text{ and } 0 \leq i_2, i'_2 \leq Q \}.$$

Using the equations (1.7a) it is possible to determine how $l(X_{\pm 2e_1})$, $l(X_{\pm 2e_2})$, $r(X_{\pm 2e_1})$ and $r(X_{\pm 2e_2})$ act on each basis vector, where l and r refer to the differentials of the left and right regular representations on K, respectively. Using formula (2.14b), we can rewrite the element

$$v_{P-2i_1} \otimes w_{Q-2i_2} \otimes (v_{P-2i'_1} \otimes w_{Q-2i'_2})^*$$

as

$$\text{(2.15a)} \quad (-1)^{P+Q-i'_1-i'_2} \binom{P}{i'_1} \binom{Q}{i'_2} ((P-i'_1)!)^2((Q-i'_2)!)^2$$
$$\left(\pi_{(P,Q)}(\cdot)^{-1} v_{P-2i_1} \otimes w_{Q-2i_2}, v_{-P+2i'_1} \otimes w_{-Q+2i'_2}\right).$$

This formula will be useful in Chapters 3 and 4, when reconstructing the $\mathfrak{p}$ action.

4. Irreducible Representations of L

In this section, we provide a concrete realization of certain irreducible representations of $L = S^1 \times SU(2)$. This can be accomplished by finding an irreducible representation of S^1, call it (ψ, V_ψ), and an irreducible representation of $SU(2)$, call it (π_2, W_π), and forming the tensor product. We will be interested in irreducible representations of L with highest weight $(d+4)e_1 + (R-1)e_2$, where d and R are both nonnegative integers, $R \geq 1$ and $d \geq R-1$ (see Chapter 1, Section 3). All irreducible representations of S^1 are one dimensional. Therefore, let $V_\psi = \mathbb{C}_{d+4}$ and let $(\psi, \mathbb{C}_{d+4})$ be the one dimensional representation of S^1 defined by

$$d\psi(H_1)z = (d+4)e_1(H_1)z = (d+4)z, \tag{2.16}$$

where H_1 is given in (2.1c) and $z \in \mathbb{C}_{d+4}$. Also, let (π_2, W_π) be the irreducible representation of $SU(2)$ with highest weight $(R-1)e_2$. This representation is referred to in Section 2 of this chapter as W_{R-1} and it satisfies conditions (2.8(a)-(c)) and (2.9). We have a basis of weight vectors $w_{(R-1-2j)e_2}$, with $0 \leq j \leq R-1$, for the representation (π_2, W_{R-1}). We may hence define a basis of weight vectors for $(\psi \otimes \pi_2, \mathbb{C}_{d+4} \otimes W_{R-1})$ by

$$x_{(d+4)e_1+(R-1-2j)e_2} = 1 \otimes w_{(R-1-2j)e_2}, \tag{2.17}$$

for $0 \leq j \leq R-1$. Here, the weight of each vector is denoted by its subscript. Using the equations (2.8b) and (2.8c), we see that

$$X_{2e_2}(x_{(d+4)e_1+(R-1-2j)e_2}) = x_{(d+4)e_1+(R-1-2j+2)e_2}, \tag{2.18a}$$

for $1 \leq j \leq R-1$. When $j = 0$, the right hand side of this equation is 0. Also we have

(2.18b)

$$X_{-2e_2}(x_{(d+4)e_1+(R-1-2j)e_2}) = (R-1-j)(j+1)\, x_{(d+4)e_1+(R-1-2j-2)e_2},$$

for $0 \leq j \leq R-1$.

CHAPTER 3

KER ∂ AND $\mathfrak{p}$ ACTION FOR $Sp(1,1)$ DISCRETE SERIES

1. The One Row Case

In the first two sections of this chapter, we will determine basis vectors for K types that appear in certain discrete series of $Sp(1,1)$. These basis vectors will be basis vectors for homology. As mentioned in Chapter 1, Sections 1 and 2, these vectors will be elements of a $(\mathfrak{k}, K)$ module. We will then use two propositions, namely Propositions 3.80 and 3.83 of [K-V], in order to reimpose the $\mathfrak{p}$ action on the $(\mathfrak{k}, K)$ module and make it a $(\mathfrak{g}, K)$ module. In this section, we will do this construction for the discrete series $A_{\mathfrak{q}}(\lambda)$, described in Definition 1.18. In the second section of this chapter, we do this construction for the discrete series $\mathcal{L}_1(V_{be_1+e_2})$, defined by (1.3). Here, $V_{be_1+e_2}$ is an irreducible finite dimensional representation of L of highest weight $be_1 + e_2$. The third section begins with a conjecture regarding a nonvanishing vector in homology for K types of the discrete series $\mathcal{L}_1(V_{de_1+(R-1)e_2})$, where $V_{de_1+(R-1)e_2}$ is an irreducible representation of L with highest weight $de_1 + (R-1)e_2$. Using that conjecture, we can prove a theorem that reimposes the $\mathfrak{p}$ action on our $(\mathfrak{k}, K)$ module (1.14b).

To begin this section, we note that only certain choices of λ will yield a discrete series for $A_{\mathfrak{q}}(\lambda)$. The discussion in Chapter 1, Section 2 tells us what those choices are. First, $\lambda = ae_1 + te_2$ must be orthogonal to the members of $\Delta(\mathfrak{l})$. This says that $\lambda = ae_1$. In addition, if τ represents the infinitesimal character of $\mathbb{C}_\lambda$, then $\tau + \delta(\mathfrak{u})$ must be strictly dominant with respect to the members of $\Delta(\mathfrak{u})$ (see (1.4b)). For an irreducible representation of a compact group L with highest weight ρ, the infinitesimal character is given by $\rho + \delta(\mathfrak{l})$, (see [K3, pg.225]). Therefore, we have

$$\tau = \lambda + \delta(\mathfrak{l}) = ae_1 + e_2. \tag{3.1}$$

We add to this $\delta(\mathfrak{u})(= 2e_1)$ and get

$$\lambda + \delta(\mathfrak{g}) = (a + 2)e_1 + e_2. \tag{3.2}$$

This will be strictly dominant with respect to the members of $\Delta(\mathfrak{u})$ precisely when $a \geq 0$. For integrality, we can use the algebraic integrality condition given in [K3, pg. 84], namely that

$$\frac{2\langle ae_1, \alpha\rangle}{\langle \alpha, \alpha\rangle} \text{is in } \mathbb{Z} \text{ for each } \alpha \in \Delta(\mathfrak{g}). \tag{3.2a}$$

Here, we may think of of $\langle \cdot, \cdot \rangle$ as the standard inner product on $\mathbb{R}^2$ with basis vectors e_1 and e_2. We can use this integrality condition since $Sp(1,1)$ is a real group whose complexification is simply connected. Using this algebraic integrality condition, we see that a must be an integer. Therefore, $\lambda = ae_1$, $a \in \mathbb{Z}^+ \cup \{0\}$ are the lambdas of interest for the study of discrete series $A_\mathfrak{q}(\lambda)$.

When $\lambda = ae_1$, the minimal K type of $A_{\mathfrak{q}}(\lambda)$ is $(a+2)e_1$ [K3, Theorem 9.2O]. This corresponds to a Dixmier diagram (Chapter 1, Section 3) for which $R = 1$ and $d = a$. Because $R = 1$, the graph of the Dixmier diagram consists of one row. The Dixmier diagram tells us that every K type in $A_{\mathfrak{q}}(\lambda)$ can be written as

$$(N + a + 2)e_1 + Ne_2,$$

where N is a nonnegative integer. We will use the coordinate notation, namely $(N + a + 2, N)$, thoughout this chapter and the next. For the remainder of this section, a and N will be nonnegative integers.

Our next step is determining a homology basis for each K type that appears in $A_{\mathfrak{q}}(ae_1)$. We begin with some definitions.

Definition 3.3 Suppose (Z, ϕ) is an irreducible, finite dimensional representation of L, $Z^{\#}$ being defined by (1.1), with ϕ extended in a natural way to a representation of $Z^{\#}$. Also, suppose (V_μ, π) and $({V_\mu}^*, \pi^*)$ are the representations mentioned in Definition 1.7 of $R(K)$. Then

$$C_n^*(Z)|_\mu \qquad \text{(3.3a)}$$

is the space of L invariants

$$\left({V_\mu}^* \otimes \bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes Z^{\#}\right)^L$$

for the representation $\pi^* \otimes \mathrm{Ad} \otimes \phi$ (L acts trivially on $\bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k})$). This is isomorphic to the vector space (1.17c), since $V_Z \cong U(\mathfrak{u}) \otimes Z^{\#}$. This is a functor from the category of finite-dimensional $(\mathfrak{l}, L)$ modules to the category of complex vector spaces. When we wish to focus on the map Ker $\partial_n^\natural$ or the map Im $\partial_{n+1}^\natural$ restricted to the K type μ of a discrete series $\mathcal{L}_S(Z)$, we shall use the notation Ker $\partial_n^\natural \subseteq \mathcal{C}_n^*(Z)|_\mu$ or Im $\partial_{n+1}^\natural \subseteq \mathcal{C}_n^*(Z)|_\mu$, respectively. Because $\dim\big(\text{Ker } \partial_S^\natural \subseteq \mathcal{C}_S^*(Z)|_\mu / \text{ Im } \partial_{S+1}^\natural \subseteq \mathcal{C}_S^*(Z)|_\mu\big)$ is the multiplicity of a K type in $\mathcal{L}_S(Z)$, $\big(\text{Ker } \partial_S^\natural \subseteq \mathcal{C}_S^*(Z)|_\mu / \text{ Im } \partial_{S+1}^\natural \subseteq \mathcal{C}_S^*(Z)|_\mu\big)$ is referred to as the **multiplicity space of type μ for $\mathcal{L}_S(Z)$.**

(3.3b) $$\mathcal{C}_{n,K}(Z)|_\mu$$

is the space of L invariants

$$\left(V_\mu \otimes V_\mu{}^* \otimes \textstyle\bigwedge^n((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes Z^\#\right)^L$$

for the representation $r \otimes \text{Ad} \otimes \phi$, where r =right regular representation on $R(K)$. This $(\mathfrak{k}, K)$ module is isomorphic to the module (1.15a) for the same reason as in (3.3a). Also, it is a functor from the category of finite-dimensional $(\mathfrak{l}, L)$ modules to the category of $(\mathfrak{k}, K)$ modules. For $G = Sp(1,1)$, the ordered basis of $\mathfrak{u}$ used in determining a Poincaré-Birkoff-Witt basis for $U(\mathfrak{u})$ is the basis $\{X_{2e_1}, X_{e_1+e_2}, X_{e_1-e_2}\}$. The Poincaré-Birkoff-Witt Theorem [K2, Theorem 2.17] states that a basis for $U(\mathfrak{u})$ is then monomials $X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p$, where m, r, and p are nonnegative integers. Following the example in (3.3a), when we wish to focus on the map Ker ∂_n restricted to the K type μ of a discrete series $\mathcal{L}_S(Z)$, we shall use the notation Ker $\partial_n \subseteq \mathcal{C}_{n,K}(Z)|_\mu$.

For a K type $(N+a+2, N)$ in $A_{\mathfrak{q}}(\lambda)$, we will produce an element in Ker ∂ of $C_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ by first producing an element in Ker $\partial^{\natural}$ of $C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ and then tensoring it with any element of $V_{N+a+2} \otimes W_N$ (defined in Chapter 2, Section 2). In particular, we will choose a set of basis vectors for $V_{N+a+2} \otimes W_N$ and in this way we shall establish a set of basis vectors for Ker $\partial \subseteq C_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$. This follows the procedure discussed in Chapter 1, Section 2. In order to complete the first part, we need some lemmas.

Lemma 3.4 Suppose $0 \leq i \leq N+a+2$, m, r and p are all nonnegative integers. There are no tensors of total weight $0e_1 + 0e_2$ of the form

$$(v_{N+a+2-2i} \otimes w_N)^* \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1$$

in the vector space

$$(V_{N+a+2} \otimes W_N)^* \otimes \bigwedge^0((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+4)e_1}.$$

Proof. The "total weight $= 0$" hypothesis forces the following relations on indices:

$$N + 2 + a - 2i + 2m + r + p + 4 + a = 0$$

$$N + r - p = 0.$$

Adding the two equations and rearranging gives

$$m + r = i - N - a - 3$$

which has no solutions, given the constraints on the indices.

Lemma 3.5 Suppose $0 \leq i \leq N+a+2$, m, r and p are all nonnegative integers. There are no tensors of total weight $0e_1 + 0e_2$ of the form

$$(v_{N+a+2-2i} \otimes w_N)^* \otimes X_{2e_1} \wedge X_{-2e_1} \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1$$

in the vector space

$$(V_{N+a+2} \otimes W_N)^* \otimes \bigwedge^0((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+4)e_1}.$$

PROOF. Since $X_{2e_1} \wedge X_{-2e_1}$ has weight 0, we can apply the same proof as in Lemma 3.4.

Proposition 3.6 $C_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ and $C_2^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ are 0.

PROOF. Let a, N, i, m, r, p be the indices of the previous lemmas. We shall do the proof for $C_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$. The proof for $C_2^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ is completely analogous. A general element in $C_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ looks like

$$(3.7) \qquad \sum_{\substack{\alpha \\ k \geq 1}} c_{\alpha,k}(v_{N+a+2-2i} \otimes w_{N-2k})^* \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1,$$

where each pure tensor in the sum has total weight 0, α is the tuple (i, m, r, p), and each $c_{\alpha,k}$ is a (possibly complex) number. The condition $k \geq 1$ follows from Lemma 3.4 (or Lemma 3.5 for C_2^*). Let k_0 be the smallest positive integer for which w_{N-2k_0} appears as a term in (3.7). Then we can rewrite (3.7) as

$$(3.8) \quad \sum_{\alpha} c_{\alpha,k_0}(v_{N+a+2-2i} \otimes w_{N-2k_0})^* \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1$$
$$+ \sum \{\text{pure tensors whose } w \text{ term has weight at most } (N-2k_0-2)e_2\},$$

where each tensor still has total weight 0. We may assume without loss of generality that the sum (3.8) consists of distinct pure tensors. Applying X_{2e_2} to (3.8) and using formula (2.13c) gives

$$(3.9) \quad \sum_{\alpha} c_{\alpha,k_0}(v_{N+a+2-2i} \otimes w_{N-2k_0+2})^* \otimes X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2} \otimes 1$$
$$+ \sum \{\text{pure tensors whose } w \text{ term has weight at most } (N-2k_0)e_2\}.$$

Each of the pure tensors in the first sum is distinct and nonzero and k_0 is at least 1, so (3.9) will be 0 if and only if c_{α,k_0} is 0 for all α in the sum (3.8). Repeat this argument with k_1, the next smallest positive integer for which w_{k_1} appears in (3.7). Because W_N is finite dimensional, there are only finitely many w_k's, and so all $c_{\alpha,k}$ in (3.7) must be 0.

Corollary 3.10 For each K type $(N+a+2, N)$ of $\mathrm{A}_\mathfrak{q}(ae_1)$, the space $\mathrm{Ker}\ \partial^\natural \subseteq \mathcal{C}_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ is one dimensional.

PROOF. Suppose $\mathrm{Ker}\ \partial^\natural$ and $\mathrm{Im}\ \partial_2^\natural$ are subspaces of $\mathcal{C}_1^*(\mathbb{C}_{ae_1})|_\mu$, for any K type μ in $\mathrm{A}_\mathfrak{q}(ae_1)$. The number dim $(\mathrm{Ker}\ \partial^\natural/\mathrm{Im}\ \partial_2^\natural)$ is the multiplicity of μ in $\mathrm{A}_\mathfrak{q}(ae_1)$ (see Definition 3.3a). We know from the paper [Dix] that the K types $(N+a+2, N)$ appear with multiplicity one in $\mathrm{A}_\mathfrak{q}(ae_1)$. From Proposition 3.6 we have $\mathrm{Im}\ \partial_2^\natural$ is 0 for the K types $(N+a+2, N)$. The result follows.

The next step is determining the constants $c_{\alpha,k}$ for which

$$(3.11) \quad \sum_{\alpha,k} c_{\alpha,k}(v_{N+a+2-2i} \otimes w_{N-2k})^* \otimes X_{-2e_1} \otimes X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2} \otimes 1$$

is a nonzero element of $C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$. As in the proof of Proposition 3.6, each tensor in (3.11) has weight zero. We can substantially reduce the number of pure tensors we consider in (3.11) by means of the next result.

Proposition 3.12 In the sum (3.11), nonzero terms occur only when $i = N + a + 2$.

PROOF. The "total weight = 0" condition in (3.11) forces the following relationship on indices:

$$\begin{aligned} N + a + 2 - 2i - 2 + 2m + r + p + 4 + a &= 0 \\ N - 2k + r - p &= 0. \end{aligned} \tag{3.13}$$

Combining and rearranging these equations gives

$$m + r = i + k - N - a - 2.$$

Since m and r are nonnegative integers, $0 \leq i \leq N + a + 2$, and $0 \leq k \leq N$, then the condition $i < N + a + 2$ forces $k \geq 1$ and we can rewrite (3.11) as

$$\sum_{\alpha} c_{\alpha,k}(v_{-(N+a+2)} \otimes w_{N-2k})^* \otimes X_{-2e_1} \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1$$
$$+ \sum \{\text{pure tensors whose } w \text{ term has weight at most } (N-2)e_2\}.$$

From the proof of Proposition 3.6, we see that the second summation above must be 0 under the L invariance condition. This proves the proposition.

REMARK. In (3.11), we only considered pure tensors with the term X_{-2e_1} from $((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k})$. The reason for this is that the term X_{2e_1} in such a pure tensor alters the relations (3.13) so that no solution is possible, given the constraint on indices.

We can now reduce (3.11) to

$$(3.11a)\quad \sum_{\beta,k} c_{\beta,k}(v_{-(N+a+2)} \otimes w_{N-2k})^* \otimes X_{-2e_1} \otimes X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2} \otimes 1,$$

where β is the tuple α but with the i coordinate equal to $N + a + 2$.

In fact, a systematic study of such pure tensors shows that (3.11a) can be written as

(3.14)

$$\sum_{k=0}^{N} \sum_{j=0}^{\min[k,N-k]} c_{k;j}(v_{-(N+a+2)} \otimes w_{N-2k})^* \otimes X_{-2e_1} \otimes X^j_{2e_1} X^{k-j}_{e_1+e_2} X^{N-k-j}_{e_1-e_2} \otimes 1.$$

The notation $c_{k;j}$ is used to avoid confusion in later sections. We can determine a basis for the space of L invariants (and consequently for Ker $\partial^\natural$, by Proposition 3.6) by using the fact that $X_{2e_2}(v_{N+a+2-2i} \otimes w_{N-2k})^* = (v_{N+a+2-2i} \otimes w_{N-2k+2})^*$ (from (2.13c))æ and knowing how X_{2e_2} acts on pure tensors of the form $X^j_{2e_1} X^{k-j}_{e_1+e_2} X^{N-k-j}_{e_1-e_2} \otimes 1$. We treat the second problem by means of the following result.

Lemma 3.15 Let m, r, and p be nonnegative integers. Various elements of $\mathfrak{sp}(1,1)^{\mathbb{C}}$ act on $U(\mathfrak{u})$ as follows:

$$(3.15a)\quad X_{2e_2}(X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2}) = X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2} X_{2e_2} + pX^m_{2e_1} X^{r+1}_{e_1+e_2} X^{p-1}_{e_1-e_2} + p(p-1) X^{m+1}_{2e_1} X^r_{e_1+e_2} X^{p-2}_{e_1-e_2}$$

(3.15b)

$$X_{-2e_1}(X^m_{2e_1} X^r_{e_1+e_2} X^p_{e_1-e_2}) = rpX^m_{2e_1} X^{r-1}_{e_1+e_2} X^{p-1}_{e_1-e_2}(H_1 - H_2 + p + r - 2)$$

$$+r(r-1)X_{2e_1}^m X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p X_{2e_2} - mX_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^p (H_1+m+r+p-1)$$

$$+rp(r-1)(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^{p-2} - p(p-1)X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-2} X_{-2e_2}$$

$$\text{(3.15c)} \quad X_{-e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p) = mX_{2e_1}^{m-1} X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p$$

$$-2rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p X_{2e_2} - 2rp(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-2}$$

$$+pX_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1}(-H_1+H_2-2r-p+1)$$

$$\text{(3.15d)} \quad X_{-e_1-e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p) = -2pX_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1} X_{-2e_2}$$

$$-rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p (2m+r-1+H_1+H_2) - mX_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^{p+1}$$

$$\text{(3.15e)} \quad X_{-2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p) = X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p X_{-2e_2}$$

$$+rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p+1} + r(r-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p.$$

REMARK. We are interested ultimately in seeing how the various elements of $\mathfrak{sp}(1,1)^{\mathbb{C}}$ mentioned in this lemma act on $U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} \mathrm{ind}_{\mathfrak{l},L}^{\bar{\mathfrak{q}},L}(V)$, V an irreducible representation of L. Therefore, we omit from the equations above any monomial Y of $U(\mathfrak{g})$ that has nontrivial $U(\bar{\mathfrak{u}})$ terms when written out in the ordering coming from the isomorphism $U(\mathfrak{g}) \cong U(\mathfrak{u})U(\mathfrak{l})U(\bar{\mathfrak{u}})$ [K2, Corollary 2.20]. The reason for this is $Y \otimes \mathrm{ind}_{\mathfrak{l},L}^{\bar{\mathfrak{q}},L}(V)$ will be 0.

PROOF. The method used is the same for each of the five equations. Therefore, a detailed proof will be given for the first equation and relevant identities will be given for the other equations.

For the first equation, using bracket relations (2.4), we see that X_{2e_2} commutes with X_{2e_1} and $X_{e_1+e_2}$. Our calculation reduces to the monomial $X_{2e_2}(X_{e_1-e_2})^p$ in $U(\mathfrak{u})$. Following the technique used in the proof of [K3, Lemma 4.38], let

$$\begin{aligned} \mathrm{L}X_{e_1-e_2} &= \text{ left action by } X_{e_1-e_2} \\ \mathrm{R}X_{e_1-e_2} &= \text{ right action by } X_{e_1-e_2}. \end{aligned}$$

Then

$$\mathrm{ad}(X_{e_1-e_2}) = \mathrm{L}X_{e_1-e_2} - \mathrm{R}X_{e_1-e_2}$$

and consequently

$$\mathrm{R}X_{e_1-e_2} = \mathrm{L}X_{e_1-e_2} - \mathrm{ad}(X_{e_1-e_2}).$$

Notice also that the $\mathrm{L}X_{e_1-e_2}$ and $\mathrm{ad}(X_{e_1-e_2})$ commute. By the Binomial Theorem, then,

$$\begin{aligned} X_{2e_2}(X_{e_1-e_2})^p &= (\mathrm{R}X_{e_1-e_2})^p X_{2e_2} \\ &= (\mathrm{L}X_{e_1-e_2} - ad(X_{e_1-e_2}))^p X_{2e_2} \\ &= (\mathrm{L}X_{e_1-e_2})^p X_{2e_2} - p(\mathrm{L}X_{e_1-e_2})^{p-1}\mathrm{ad}(X_{e_1-e_2})X_{2e_2} \\ &\quad + \frac{p(p-1)}{2}(\mathrm{L}X_{e_1-e_2})^{p-2}\mathrm{ad}^2(X_{e_1-e_2})X_{2e_2}, \end{aligned}$$

since $\mathrm{ad}^3(X_{e_1-e_2})X_{2e_2} = 0$. Dropping L from the notation and using bracket relations (2.4), we see that the right hand side can be simplified to

$$(3.16) \qquad X^p_{e_1-e_2}X_{2e_2} + pX^{p-1}_{e_1-e_2}X_{e_1+e_2} - p(p-1)X_{2e_1}(X_{e_1-e_2})^{p-2}.$$

As mentioned in Definition 3.3b, the Poincaré-Birkoff-Witt basis chosen for $U(\mathfrak{u})$ is $\{X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\}$ so we must rewrite $(X_{e_1-e_2})^{p-1}X_{e_1+e_2}$ in terms of this basis:

$$p(\mathrm{L}X_{e_1-e_2})^{p-1}X_{e_1+e_2} = p\big((\mathrm{R}X_{e_1-e_2})^{p-1}X_{e_1+e_2} + (p-1)(\mathrm{R}X_{e_1-e_2})^{p-2}\mathrm{ad}(X_{e_1-e_2})X_{e_1+e_2}\big).$$

Dropping the L and R notation, we can rewrite this as

$$pX_{e_1+e_2}X^{p-1}_{e_1-e_2} + 2p(p-1)X^{p-2}_{e_1-e_2}X_{2e_1},$$

which equals

$$(*) \qquad pX_{e_1+e_2}X^{p-1}_{e_1-e_2} + 2p(p-1)X_{2e_1}X^{p-2}_{e_1-e_2}.$$

Using $(*)$ in (3.16) gives

$$X_{2e_2}X^p_{e_1-e_2} = X^p_{e_1-e_2}X_{2e_2} + pX_{e_1+e_2}X^{p-1}_{e_1-e_2} + p(p-1)X_{2e_1}X^{p-2}_{e_1-e_2}$$

and since X_{2e_1} commutes with $X_{e_1+e_2}$,

$$X_{2e_2}(X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}) = X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}X_{2e_2} + pX^m_{2e_1}X^{r+1}_{e_1+e_2}X^{p-1}_{e_1-e_2} + p(p-1)X^{m+1}_{2e_1}X^r_{e_1+e_2}X^{p-2}_{e_1-e_2}.$$

This is the desired result.

For the second equation, the relevant identities are:

$$X_{-2e_1}X^r_{e_1+e_2} = X^r_{e_1+e_2}X_{-2e_1} - rX^{r-1}_{e_1+e_2}X_{-e_1+e_2} + r(r-1)X^{r-2}_{e_1+e_2}X_{2e_2}$$

$$X_{-2e_1}X^p_{e_1-e_2} = X^p_{e_1-e_2}X_{-2e_1} + pX^{p-1}_{e_1-e_2}X_{-e_1-e_2} - p(p-1)X^{p-2}_{e_1-e_2}X_{-2e_2}$$

$$X_{-e_1+e_2}X^p_{e_1-e_2} = X^p_{e_1-e_2}X_{2e_2} + pX_{e_1+e_2}X^{p-1}_{e_1-e_2} + p(p-1)X_{2e_1}X^{p-2}_{e_1-e_2}$$

$$X_{-2e_1}X^m_{2e_1} = X^m_{2e_1}X_{-2e_1} - mX^{m-1}_{2e_1}(H_1 + m - 1)$$

$$(H_1 + m - 1)X^r_{e_1+e_2} = X^r_{e_1+e_2}H_1 + (m + r - 1)X^r_{e_1+e_2},$$

all proved using bracket relations and the fact that LX and RX commute with ad(X) for all root vectors X.

The relevant identities for the third equation are:

$$X_{-e_1+e_2}X^m_{2e_1} = X^m_{2e_1}X_{-e_1+e_2} + mX^{m-1}_{2e_1}X_{e_1+e_2}$$

$$X_{-e_1+e_2}X^r_{e_1+e_2} = X^r_{e_1+e_2}X_{-e_1+e_2} - 2rX^{r-1}_{e_1+e_2}X_{2e_2}$$

$$X_{-e_1+e_2}X^p_{e_1-e_2} = X^p_{e_1-e_2}X_{-e_1+e_2} + pX^{p-1}_{e_1-e_2}(-H_1 + H_2) - p(p-1)X^{p-1}_{e_1-e_2}$$

$$X_{2e_2}X^p_{e_1-e_2} \text{ as above.}$$

The relevant identities for the fourth equation are:

$$X_{-e_1-e_2}X^m_{2e_1} = X^m_{2e_1}X_{-e_1-e_2} - mX^{m-1}_{2e_1}X_{e_1-e_2}$$

$$X_{-e_1-e_2}X^r_{e_1+e_2} = X^r_{e_1+e_2}X_{-e_1-e_2} - rX^{r-1}_{e_1+e_2}(r + H_1 + H_2 - 1)$$

$$X_{-e_1-e_2}X^p_{e_1-e_2} = X^p_{e_1-e_2}X_{-e_1-e_2} - 2pX^{p-1}_{e_1-e_2}X_{-2e_2}$$

$$(H_1+H_2)X^p_{e_1-e_2} = X^p_{e_1-e_2}(H_1+H_2)$$

$$X_{e_1-e_2}X^r_{e_1+e_2} = X^r_{e_1+e_2}X_{e_1-e_2} + 2rX^{r-1}_{e_1+e_2}X_{2e_1}.$$

Finally, the relevant identity for the fifth equation is:

$$X_{-2e_2}X^r_{e_1+e_2} = X^r_{e_1+e_2}X_{-2e_2} + rX^{r-1}_{e_1+e_2}X_{e_1-e_2} + r(r-1)X^{r-2}_{e_1+e_2}X_{2e_1}.$$

Using Lemma 3.15, we can prove a result that plays a large role in determining the coefficients $c_{k;j}$ of formula (3.14) and ultimately producing a basis element for Ker $\partial^{\natural} \subseteq C^*_1(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$.

Lemma 3.16 X_{2e_2} acts on $U(\mathfrak{u}) \otimes \mathbb{C}_{(a+4)e_1}$ by

$$X_{2e_2}(X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\otimes 1) = pX^m_{2e_1}X^{r+1}_{e_1+e_2}X^{p-1}_{e_1-e_2}\otimes 1$$
$$+ p(p-1)X^{m+1}_{2e_1}X^r_{e_1+e_2}X^{p-2}_{e_1-e_2}\otimes 1.$$

Proof. Consider first the action of X_{2e_2} on $X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\otimes 1$, the latter an element in $U(\mathfrak{g})\otimes_{\bar{\mathfrak{q}}}\mathbb{C}_{(a+4)e_1}(\cong U(\mathfrak{u})\otimes\mathbb{C}_{(a+4)e_1})$. Using Lemma 3.15, we see that the monomial $X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}X_{2e_2}$ vanishes in equivalence, since X_{2e_2} acts as 0 on $\mathbb{C}_{(a+4)e_1}$. Because the other two monomials in (3.15a) are in $U(\mathfrak{u})$, no term can pass through the tensor product. This proves the lemma.

Theorem 3.17 A basis element for Ker $\partial^\natural \subseteq \mathcal{L}_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ is

(3.17a)
$$\sum_{k=0}^{N}\sum_{j=0}^{\min[k,N-k]} c_{k;j}(v_{-(N+a+2)}\otimes w_{N-2k})^*\otimes X_{-2e_1}\otimes X_{2e_1}^{j}X_{e_1+e_2}^{k-j}X_{e_1-e_2}^{N-k-j}\otimes 1,$$

where

$$\text{(3.17b)} \qquad c_{0;0}=1 \qquad \text{and} \qquad c_{k;j}=(-1)^k\binom{k}{j}\prod_{q=0}^{k+j-1}(N-q).$$

We use the convention $\binom{k}{0}=1$ for $k\geq 0$.

PROOF. Because $\partial^\natural$ maps $C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ to $C_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ and because $C_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ is 0 (Proposition 3.6), it is sufficient to produce a basis for $C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$, i.e., we need only consider L invariance.

From Lemma 3.16, it follows that a necessary condition for the vector (3.17a) to be L invariant is that the coefficients $c_{k;j}$ satisfy

$$\text{(3.17c)} \qquad c_{k+1;0}+(N-k)c_{k;0}=0 \qquad \text{for } 0\leq k\leq N-1$$

$$\text{(3.17d)} \quad c_{k+1;k+1}+(N-2k)(N-2k-1)c_{k;k}=0 \qquad \text{for } 0\leq k\leq \begin{cases}\dfrac{N-2}{2}, & N \text{ even}\\[2mm] \dfrac{N-3}{2}, & N \text{ odd}\end{cases}$$

(3.17e)
$$c_{k+1;j}+(N-k-j)c_{k;j} + (N-k-j+1)(N-k-j)c_{k;j-1}=0 \quad \text{for } (k,j) \text{ in (domain 1)},$$

where (domain 1) is the set of $\{(k,j)\}$ satisfying

$$1 \le j \le k \qquad \text{for } 1 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{cases}$$

$$1 \le j \le N-k-1 \qquad \text{for } \left\{ \begin{array}{ll} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{array} \right\} < k \le N-2.$$

If $\vec{x}$ is the vector $(c_{0;0}, c_{1;0}, c_{1;1}, \ldots, c_{N-1;0}, c_{N-1;1}, c_{N;0})$, and $\vec{x}_0$ is the vector $(c_{0;0}, 0, 0, \ldots, 0)$ of same size as $\vec{x}$, then we can write the system of linear equations $\{c_{0;0} = c_{0;0}\} \cup \{3.17(\text{c})\text{–}(\text{e})\}$ in matrix form as

$$A \cdot \vec{x} = \vec{x}_0,$$

where A is a square, lower triangular matrix with all diagonal entries equal to 1. Hence, there is a unique solution for the vector $\vec{x}$. Since

$$c_{0;0} = c_{0;0} \qquad \text{and} \qquad c_{k;j} = (-1)^k \binom{k}{j} \prod_{q=0}^{k+j-1} (N-q) c_{0;0}$$

is a solution to this system, it must be the unique solution. Choosing $c_{0;0} = 1$ gives the coefficients in the statement of the theorem.

REMARK. For certain choices of N, some of the domains listed in the proof of Theorem 3.17 will be empty. For example, when N is 0 or 1, then both (3.17d) and (3.17e) have empty domains. In such cases, we simply remove these equations from consideration. This will also occur in Theorems 3.32 and 3.34.

Definition 3.18 Ker $\partial^{\natural}|_{(N+a+2,N)}$ is the vector (3.17a) with coefficients (3.17b).

Using Theorem 3.17, we can produce a homology basis for each K type $(N+a+2,N)$, namely a basis for each $(\mathfrak{k},K)$ module $\mathcal{C}_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$. We do this simply by attaching a basis of $V_{N+a+2}\otimes W_N$ to the vector Ker $\partial^{\natural}|_{(N+a+2,N)}$.

Definition 3.19 For $0\leq i_1\leq N+a+2$ and $0\leq i_2\leq N$, the vector

$$v_{N+a+2-2i_1}\otimes w_{N-2i_2}\otimes \text{Ker }\partial^{\natural}|_{(N+a+2,N)}$$

is the element

$$\text{(3.19a)}\quad v_{N+a+2-2i_1}\otimes w_{N-2i_2}\otimes\{\sum_{k=0}^{N}\sum_{j=0}^{\min[k,N-k]} c_{k;j}(v_{-(N+a+2)}\otimes w_{N-2k})^* \otimes X_{-2e_1}\otimes X_{2e_1}^{j}X_{e_1+e_2}^{k-j}X_{e_1-e_2}^{N-k-j}\otimes 1\}.$$

Here, $c_{k;j}$ are the coefficients (3.17b). This can be rewritten as

$$\text{(3.19b)}\quad \sum_{k=0}^{N}\sum_{j=0}^{\min[k,N-k]} c_{k;j}\, v_{N+a+2-2i_1}\otimes w_{N-2i_2} \otimes(v_{-(N+a+2)}\otimes w_{N-2k})^*\otimes X_{-2e_1}\otimes X_{2e_1}^{j}X_{e_1+e_2}^{k-j}X_{e_1-e_2}^{N-k-j}\otimes 1.$$

The set of all such vectors, as i_1 ranges between 0 and $N+a+2$ and i_2 ranges between 0 and N, is a basis of Ker $\partial\subseteq\mathcal{C}_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$. The fact that this set is also a homology basis is a special feature of $Sp(1,1)$, since in this case Im $\partial_2^{\natural}$ is 0 for all K types in the discrete series $A_{\mathfrak{q}}(ae_1)$.

Having produced a homology basis, we are ready to reconstruct the full $\mathfrak{g}$ action. Note that reconstruction of the full $\mathfrak{g}$ action is really a reconstruction of the $\mathfrak{p}$ action, since $(\Pi_{\mathfrak{k},L}^{\mathfrak{k},K})_1(\mathcal{F}_{\mathfrak{g},L}^{\mathfrak{k},L}(\mathrm{ind}_{\bar{\mathfrak{q}},L}^{\mathfrak{g},L}(\mathbb{C}_{(a+4)e_1}))$ is a $(\mathfrak{k}, K)$ module.

The key tools in reconstruction of the $\mathfrak{p}$ action are Propositions 3.80 and 3.83 of [K-V]. Proposition 3.83 gives a $\mathfrak{p}$ action on homology by first lifting to cycles, then applying a map α_1, and finally descending again to homology. Proposition 3.80 defines the map α_1. Because the boundary is 0 in our situation, determining the $\mathfrak{p}$ action reduces to computing α_1 on the level of cycles.

To do this, we must interpret α_1. The definition given in Proposition 3.80 is

$$\alpha_1(X \otimes (T \otimes w)) = T \otimes (\mathrm{Ad}(\cdot)^{-1}X)w,$$

where $X \in \mathfrak{g}$, $T \in R(K)$, and $w \in \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes Z^{\#}$. Here, $Z^{\#}$ is the irreducible $\mathfrak{l}$ module defined in (1.1). If $\{X_i\}$ is an orthonormal basis of $\mathfrak{p}$, then by finite dimensional vector space theory,

$$\mathrm{Ad}(\cdot)^{-1}X = \sum_i \langle \mathrm{Ad}(\cdot)^{-1}X, X_i\rangle X_i, \qquad \text{for } \langle\cdot,\cdot\rangle = \text{inner product on } \mathfrak{g}$$

and

$$\begin{aligned} T \otimes (\mathrm{Ad}(\cdot)^{-1}X)w &= \sum_i T \otimes \langle \mathrm{Ad}(\cdot)^{-1}X, X_i\rangle X_i\, w \\ &= \sum_i \langle \mathrm{Ad}(\cdot)^{-1}X, X_i\rangle T \otimes X_i\, w. \end{aligned}$$

The second of these equalities follows from [K-V, B. 18]. This reference produces an isomorphism between a tensor product of spaces over $\mathbb{C}$ and a tensor product of spaces over another space of C^∞ functions. Since $\langle \mathrm{Ad}(\cdot)^{-1}X, X_i\rangle$ is a C^∞ function of K, we are able to pull it through the tensor product.

We will determine the $\mathfrak{p}$ action of homology vectors by calculating $\alpha_1(X \otimes (3.19\text{b}))$, for $X \in \mathfrak{p}$. With X satisfying this condition, we have $\alpha_1(X \otimes (3.19\text{b}))$

$$(3.20\text{a}) \quad = \sum_{\alpha\in\Delta(\mathfrak{g})} \sum_{k=0}^{N} \sum_{j=0}^{\min[k,N-k]} c_{k;j} \langle \mathrm{Ad}(\cdot)^{-1}X, X_\alpha\rangle v_{N+a+2-2i_1} \otimes w_{N-2i_2}$$
$$\otimes \left(v_{-(N+a+2)} \otimes w_{N-2k}\right)^* \otimes X_{-2e_1} \otimes X_\alpha (X_{2e_1}^j X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N-k-j} \otimes 1).$$

Because $\langle \mathrm{Ad}(\cdot)^{-1}X, X_\alpha\rangle = 0$ when $X \in \mathfrak{p}$ and $\alpha \in \Delta(\mathfrak{k})$, we can restrict the first summation above to $\alpha \in \Delta(\mathfrak{p})$. We know from (2.15a) that

$$v_{N+a+2-2i_1} \otimes w_{N-2i_2} \otimes \left(v_{-(N+a+2)} \otimes w_{N-2k}\right)^* = (-1)^{N-k}\binom{N}{k}((N-k)!)^2$$
$$\times \left(\pi(\cdot)^{-1} v_{N+a+2-2i_1} \otimes w_{N-2i_2}, v_{N+a+2} \otimes w_{2k-N}\right),$$

π being the representation $\pi_{(N+a+2,N)}$. Because the product of two matrix coefficients is a matrix coefficient of the tensor product, then using the previous equality, we have

$$\langle \mathrm{Ad}(\cdot)^{-1}X, X_\alpha\rangle v_{N+a+2-2i_1} \otimes w_{N-2i_2} \otimes \left(v_{-(N+a+2)} \otimes w_{N-2k}\right)^*$$
$$= (-1)^{N-k}\binom{N}{k}((N-k)!)^2 \langle (\mathrm{Ad} \otimes \pi)(\cdot)^{-1} X \otimes v_{N+a+2-2i_1} \otimes w_{N-2i_2},$$
$$X_\alpha \otimes v_{N+a+2} \otimes w_{2k-N}\rangle.$$

It follows that (3.20a) equals

$$(3.21)\quad \sum_{\alpha\in\Delta(\mathfrak{p})}\sum_{k=0}^{N}\sum_{j=0}^{\min[k,N-k]} c_{k;j}(-1)^{N-k}\binom{N}{k}((N-k)!)^2$$

$$\times \langle (\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{N+a+2-2i_1}\otimes w_{N-2i_2}, X_\alpha\otimes v_{N+a+2}\otimes w_{2k-N}\rangle$$

$$\otimes X_{-2e_1}\otimes X_\alpha(X_{2e_1}^{j}X_{e_1+e_2}^{k-j}X_{e_1-e_2}^{N-k-j}\otimes 1).$$

If $\alpha_1(\cdot\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2,N)})$ represents the expression

$$\sum_{\alpha\in\Delta(\mathfrak{p})}\sum_{k=0}^{N}\sum_{j=0}^{\min[k,N-k]} c_{k;j}(-1)^{N-k}\binom{N}{k}((N-k)!)^2$$

$$\times \langle \cdot, X_\alpha\otimes v_{N+a+2}\otimes w_{2k-N}\rangle \otimes X_{-2e_1}\otimes X_\alpha(X_{2e_1}^{j}X_{e_1+e_2}^{k-j}X_{e_1-e_2}^{N-k-j}\otimes 1),$$

then (3.21) equals

$$(3.22\text{a})\quad (K\text{ type decomposition of } X\otimes v_{N+a+2-2i_1}\otimes w_{N-2i_2})$$

$$\times (K\text{ type decomposition of } \alpha_1(\cdot\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2,N)})),$$

since $X\otimes v_{N+a+2-2i_1}\otimes w_{N-2i_2}$ is constant in the summation (3.21).

We treat the K type decomposition of $\alpha_1(\cdot\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2,N)})$ first. This procedure can be simplified greatly if we recognize that [K-V, Proposition 3.83] states that α_1 must map Ker $(1\otimes\partial)\to$ Ker ∂. More specifically,

$$(3.23)\quad \alpha_1(\cdot\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2,N)})) = h_1\ \mathrm{Ker}\ \partial^{\natural}|_{(N+a+3,N+1)}$$

$$+ h_2\ \mathrm{Ker}\ \partial^{\natural}|_{(N+a+1,N-1)}$$

for certain numbers h_1 and h_2. We omit from (3.23) the K types vectors for K types that do not appear in the discrete series. Our task is finding these coefficients h_1 and h_2. We begin with h_1. In the vector Ker $\partial^\natural|_{(N+a+3,N+1)}$, there is only one pure tensor with monomial term $X_{e_1-e_2}^{N+1}$, and that tensor is

$$(3.24)\qquad c_{0;0}(v_{-(N+a+3)} \otimes w_{N+1})^* \otimes X_{-2e_1} \otimes (X_{e_1-e_2}^{N+1} \otimes 1).$$

Using the next lemma, we will be able to determine that the only summand of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$ contributing to (3.24) is

(3.25)

$$c_{0;0}(-1)^N(N!)^2\langle \cdot, X_{e_1-e_2} \otimes v_{N+a+2} \otimes w_{-N}\rangle \otimes X_{-2e_1} \otimes X_{e_1-e_2}(X_{e_1-e_2}^N \otimes 1).$$

Lemma 3.26 The Lie algebra $\mathfrak{p}$ acts on $U(\mathfrak{u}) \otimes \mathbb{C}_{(a+4)e_1}$ by means of the following formulas:

$$(3.26a)\qquad X_{e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1) = X_{2e_1}^m X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p \otimes 1$$

(3.26b)

$$X_{e_1-e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1) = X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p+1} \otimes 1 + 2rX_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p \otimes 1$$

(3.26c)

$$X_{-e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes 1) = -p(p+2r+a+3)X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1} \otimes 1 + mX_{2e_1}^{m-1} X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p \otimes 1 - 2rp(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-2} \otimes 1$$

(3.26d) $X_{-e_1-e_2}(X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\otimes 1) = -mX^{m-1}_{2e_1}X^r_{e_1+e_2}X^{p+1}_{e_1-e_2}\otimes 1$

$$- r(2m+r+a+3)X^m_{2e_1}X^{r-1}_{e_1+e_2}X^p_{e_1-e_2}\otimes 1.$$

PROOF. Formula (3.26a) follows from the fact that X_{2e_1} and $X_{e_1+e_2}$ commute. Formula (3.26b) follows from the relation

$$X_{e_1-e_2}(X_{e_1+e_2})^r = X^r_{e_1+e_2}X_{e_1-e_2} + 2rX_{2e_1}X^{r-1}_{e_1+e_2}$$

in $U(\mathfrak{u})$. Formulas (3.26c) and (3.26d) are both consequences of Lemma 3.15.

Applying Lemma 3.26 to $X_\alpha(X^j_{2e_1}X^{k-j}_{e_1+e_2}X^{N-j}_{e_1-e_2}\otimes 1)$, $\alpha\in\mathfrak{p}$, j and k as in (3.17a), we see that $X^{N+1}_{e_1-e_2}\otimes 1$ appears only as $X_{e_1-e_2}(X^N_{e_1-e_2}\otimes 1)$. Using Corollary 2.12, we can rewrite the $(N+a+3, N+1)$ component of (3.25) as

(3.25a) $$\frac{-c_{0;0}}{(N+1)^{\frac{5}{2}}(N+a+3)^{\frac{3}{2}}}\,(v_{-(N+a+3)}\otimes w_{N+1})^*\otimes X_{-2e_1}\otimes X^{N+1}_{e_1-e_2}\otimes 1.$$

This tells us that $h_1 = \dfrac{-c_{0;0}}{(N+1)^{\frac{5}{2}}(N+a+3)^{\frac{3}{2}}}$. To find h_2, we observe that summands of $\alpha_1(\cdot\otimes \text{Ker }\partial^\natural|_{(N+a+2,N)})$ that contribute to the pure tensor in Ker $\partial^\natural|_{(N+a+1,N-1)}$ with monomial term $X^{N-1}_{e_1-e_2}$ can be determined by using Lemma 3.26. The only terms in the tensor product $U(\mathfrak{u})\otimes\mathbb{C}_{(a+4)e_1}$ of the form $X^k_{2e_1}X^{k-j}_{e_1+e_2}X^{N-k-j}_{e_1-e_2}\otimes 1$ that contribute to $X^{N-1}_{e_1-e_2}\otimes 1$ are

$X^N_{e_1-e_2}\otimes 1$	with action from $X_{-e_1+e_2}$
$X_{e_1+e_2}X^{N-1}_{e_1-e_2}\otimes 1$	with action from $X_{-e_1-e_2}$
$X_{2e_1}X^{N-2}_{e_1-e_2}\otimes 1$	with action from $X_{-e_1-e_2}$.

The summands of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$ that correspond are

$$c_{0;0}(-1)^N (N!)^2 \langle \cdot, X_{-e_1+e_2} \otimes v_{N+a+2} \otimes w_{-N} \rangle \otimes X_{-2e_1} \otimes X_{-e_1+e_2}(X^N_{e_1-e_2} \otimes 1)$$

$$c_{1;0}(-1)^{N-1} N((N-1)!)^2 \langle \cdot, X_{-e_1-e_2} \otimes v_{N+a+2} \otimes w_{-N+2} \rangle$$
$$\otimes X_{-2e_1} \otimes X_{-e_1-e_2}(X_{e_1+e_2} X^{N-1}_{e_1-e_2} \otimes 1)$$

$$c_{1;1}(-1)^{N-1} N((N-1)!)^2 \langle \cdot, X_{-e_1-e_2} \otimes v_{N+a+2} \otimes w_{-N+2} \rangle$$
$$\otimes X_{-2e_1} \otimes X_{-e_1-e_2}(X_{2e_1} X^{N-2}_{e_1-e_2} \otimes 1).$$

Using Theorem 3.17, Corollary 2.12, formula (2.14b), and Lemma 3.26, we can expand each of these pure tensors, add together pure tensors involving $X^{N-1}_{e_1-e_2} \otimes 1$, and reassemble using Corollary 2.12 and (2.14b). The result is

$$(3.25b) \quad \frac{N^3(N+a+2)^2(N+a+3)(N+1)}{((N+a+3)(N+1))^{\frac{1}{2}}}(v_{-(N+a+1)} \otimes w_{(N-1)})^*$$
$$\otimes X_{-2e_1} \otimes X^{N-1}_{e_1-e_2} \otimes 1.$$

This tells us that $h_2 = \dfrac{N^3(N+a+2)^2(N+a+3)(N+1)}{((N+a+3)(N+1))^{\frac{1}{2}}}$. Now that we have determined h_1 and h_2 (3.23), we can rewrite the K type decomposition of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$ as

$$(3.27) \quad \frac{-1}{(N+1)^{\frac{5}{2}}(N+a+3)^{\frac{3}{2}}} \text{ Ker } \partial^\natural|_{(N+a+3,N+1)}$$
$$+ N^3(N+a+2)^2((N+a+3)(N+1))^{\frac{1}{2}} \text{ Ker } \partial^\natural|_{(N+a+1,N-1)}$$
$$+ (K \text{ types that do not appear in discrete series}).$$

Next we consider the K type decomposition of $X \otimes v_{N+a+2-2i_1} \otimes w_{N-2i_2}$. We have

$$(3.28) \quad X \otimes v_{N+a+2-2i_1} \otimes w_{N-2i_2}$$

$$= h_1'(\text{ vector}_1 \text{ in the } K \text{ type } (N+a+3, N+1))$$

$$+ h_2'(\text{vector}_2 \text{ in the } K \text{ type } (N+a+1, N+1))$$

$$+ (\text{vectors whose } K \text{ type does not appear in the discrete series}).$$

The values h_1' and h_2' depend on the choice of X, i_1 and i_2, and can be determined by using Corollary 2.12. This is best illustrated by an example. Suppose $X = X_{e_1+e_2}$, and $i_1 = i_2 = 0$. Then Corollary 2.12, namely formula (2.12a), tells us that the decomposition (3.28) is given by

$$X_{e_1+e_2} \otimes v_{N+a+2} \otimes w_N = \frac{1}{((N+a+3)(N+1))^{\frac{3}{2}}} v_{N+a+3} \otimes w_{N+1}.$$

Thus, for $X = X_{e_1+e_2}$ and $i_1 = i_2 = 0$, $h_1' = \frac{1}{((N+a+3)(N+1))^{\frac{3}{2}}}$, $h_2' = 0$, and vector$_1$ is $v_{N+a+3} \otimes w_{N+1}$. The term vector$_2$ will be unimportant, as $h_2' = 0$. Likewise for any choice of X, i_1 and i_2, we can determine a decomposition of the form (3.28). Combining this with (3.27) and substituting into (3.22a), we see that $\alpha_1(X \otimes v_{N+a+2-2i_1} \otimes w_{N-2i_2})$ is

$$\frac{-h_1'}{(N+1)^{\frac{5}{2}}(N+a+3)^{\frac{3}{2}}} \text{vector}_1 \otimes \text{Ker } \partial^{\natural}\big|_{(N+a+3,N+1)}$$

$$+ h_2' N^3 (N+a+2)^2((N+a+3)(N+1))^{\frac{1}{2}} \text{ vector}_2 \otimes \text{Ker } \partial^{\natural}\big|_{(N+a+1,N-1)}$$

$$+ (K \text{ types that do not appear in discrete series}).$$

Each of the formulas presented in the next theorem is determined this way.

Theorem 3.29 For $\mathfrak{g} = \mathfrak{sp}(1,1)^{\mathbb{C}}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition according to the Cartan involution mentioned in Chapter 2. Let $\{X_\alpha\}$ be the set of orthonormal root vectors (2.1b) of $\mathfrak{p}$. Also, let

$$v_{N+a+2-2i_1} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+a+2,N)}$$

be the basis vector for Ker $\partial \subseteq \mathcal{C}_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ stated in Definition 3.19. Then the action of a X_α on any such basis vector is given by the following formulas.

Action of $X_{e_1+e_2}$:

- $X_{e_1+e_2}(v_{N+a+2} \otimes w_N \otimes \text{Ker } \partial^{\natural}|_{(N+a+2,N)})$

$$= \frac{-1}{(N+a+3)^3(N+1)^4} v_{N+a+3} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+a+3,N+1)}.$$

For $0 \leq i_1 \leq N+a+1$,

- $X_{e_1+e_2}(v_{N+a+2-2(i_1+1)} \otimes w_N \otimes \text{Ker } \partial^{\natural}|_{(N+a+2,N)})$

$$= \frac{-1}{(N+a+3)^3(N+1)^4} v_{N+a+3-2(i_1+1)} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+a+3,N+1)}.$$

For $0 \leq i_2 \leq N-1$,

- $X_{e_1+e_2}(v_{N+a+2} \otimes w_{N-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+a+2,N)})$

$$= \frac{-1}{(N+a+3)^3(N+1)^4} v_{N+a+3} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+a+3,N+1)}.$$

For $0 \leq i_1 \leq N+a+2$,

- $X_{-e_1+e_2}(v_{N+a+2-2i_1} \otimes w_N \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$

$$= \frac{(i_1+1)}{(N+a+3)^3(N+1)^4} v_{N+a+3-2(i_1+1)} \otimes w_{N+1} \otimes \text{Ker } \partial^\natural|_{(N+a+3,N+1)}.$$

For $0 \leq i_1 \leq N+a+2$ and $0 \leq i_2 \leq N-1$,

- $X_{-e_1+e_2}(v_{N+a+2-2i_1} \otimes w_{N-2(i_2+1)} \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$

$$= (N+a+2-i_1)N^4(N+a+2)^3 v_{N+a+1-2i_1} \otimes w_{N-1-2i_2} \otimes \text{Ker } \partial^\natural|_{(N+a+1,N-1)}$$

$$+ \frac{(i_1+1)}{(N+a+3)^3(N+1)^4} v_{N+a+3-2(i_1+1)} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^\natural|_{(N+a+3,N+1)}.$$

Action of $X_{-e_1-e_2}$:

For $0 \leq i_1 \leq N+a+2$ and $0 \leq i_2 \leq N-1$,

- $X_{-e_1-e_2}(v_{N+a+2-2i_1} \otimes w_{N-2(i_2+1)} \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$

$$= (N+a+2-i_1)(N-i_2)N^4(N+a+2)^3 \times v_{N+a+1-2i_1} \otimes w_{N-1-2i_2} \otimes \text{Ker } \partial^\natural|_{(N+a+1,N-1)}$$

$$- \frac{(i_1+1)(i_2+1)}{(N+a+3)^3(N+1)^4} v_{N+a+3-2(i_1+1)} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^\natural|_{(N+a+3,N+1)}.$$

REMARK. In the formulas for this theorem, Ker $\partial^\natural|_{(N+a+3,N+1)}$ refers to Definition 3.18 with N replaced by $N+1$. Similarly, Ker $\partial^\natural|_{(N+a+1,N-1)}$ is gotten by replacing N in Definition 3.18 by $N-1$.

2. The Two Row Case

Having reconstructed the full $\mathfrak{g}$ action for Ker $\partial \subseteq \mathcal{C}_{1,K}(\mathbb{C}_{ae_1})|_\mu$, μ being any K type appearing in $A_\mathfrak{q}(\lambda)$, we now turn our attention to the Dixmier diagram with 2 rows (See Chapter 1, Section 3). According to the discussion there, the discrete series we are interested in looking at is $\mathcal{L}_1(V_{be_1+e_2})$, where $V_{be_1+e_2}$ is an irreducible representation of L with highest weight $be_1 + e_2$, b being an integer ≥ 1. The condition on b is precisely what is needed in order for $\mathcal{L}_1(V_{be_1+e_2})$ to be a discrete series (see Chapter 1, inequality (1.4a) or (1.4b)). Throughout this section, b is an integer ≥ 1 and N is a nonnegative integer.

We first obtain a description of Ker $\partial^\natural \subseteq C_1^*(V_{be_1+e_2})|_\mu$ (Definition 3.3a) for each K type μ that appears in the discrete series. We will be interested in those elements in Ker $\partial^\natural \subseteq C_1^*(V_{be_1+e_2})|_\mu$ that do not vanish in the quotient space Ker $\partial^\natural$/Im $\partial_2^\natural$. From there we can immediately determine a basis of Ker $\partial \subseteq \mathcal{C}_{1,K}(V_{be_1+e_2})|_\mu$ (Definition 3.3b). Once we have this basis, we can reconstruct the $\mathfrak{p}$ action. For the two row case, the K types that appear in discrete series look like $(N+b+2, N+1)$ and $(N+b+3, N)$. The row of K types $\{(N+b+2, N+1)\}$, that is to say, the row containing the minimal K type, has no boundary, i.e., Im $\partial_2^\natural$ is 0 for each of these K types. This is the same as the $A_\mathfrak{q}(ae_1)$ case. However, each of K types $\{(N+b+3, N)\}$ has a one dimensional boundary, i.e., (dim(Im $\partial_2^\natural$)=1), hence Ker $\partial^\natural$ is two dimensional for each of these K types.

The K types $(N+b+2, N+1)$ are considered first. As we just mentioned, the boundary is 0 for each one of these; therefore, we will produce a homology basis for Ker $\partial^{\natural} \subseteq C_1^*(V_{be_1+e_2})|_{(N+b+2,N+1)}$.

Proposition 3.30 $C_1^*(V_{be_1+e_2})|_{(N+b+2,N+1)}$ has 0 boundary.

PROOF. We begin with a description of $V_{(b+4)e_1+e_2}$ in terms of basis vectors, since $V_{(b+4)e_1+e_2} = V^{\#}_{be_1+e_2}$ (see (1.1)). Let $x_{(b+4)e_1+e_2}$ and $x_{(b+4)e_1-e_2}$, with weights given by the subscripts, be the vectors defined in Chapter 2, (2.17) with $R = 2$ and $d = b$. Then identity (2.18a) shows that these vectors satisfy the equations

$$\begin{aligned} X_{2e_2}(x_{(b+4)e_1-e_2}) &= x_{(b+4)e_1+e_2} \\ X_{2e_2}(x_{(b+4)e_1+e_2}) &= 0. \end{aligned} \tag{3.30a}$$

Notice that there are no pure tensors of total weight 0 of the form

$$(v_{N+b+2-2i} \otimes w_{N+1})^* \otimes X_{2e_1} \wedge X_{-2e_1} \otimes X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+(2i'-1)e_2}$$

in $(V_{N+b+2} \otimes W_{N+1})^* \bigwedge^2((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes V_{(b+4)e_1+e_2}$, where $0 \leq i \leq N + b + 2$ and $0 \leq i' \leq 1$ are integers. As in the proof of Proposition 3.6, this fact implies that $C_2^*(V_{be_1+e_2})|_{(N+b+2,N+1)}$ is 0, and consequently, the proposition follows.

Lemma 3.31 Let m, r, and p be nonnegative integers. Then the X_{2e_2} action on $U(\mathfrak{u}) \otimes V_{(b+4)e_1+e_2}$ is given by

$$\begin{aligned} &X_{2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2}) \\ &= X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2} + pX_{2e_1}^m X_{e_1+e_2}^{r+1} X_{e_1-e_2}^{p-1} \otimes x_{(b+4)e_1-e_2} \\ &\qquad + p(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^r X_{e_1-e_2}^{p-2} \otimes x_{(b+4)e_1-e_2} \end{aligned}$$

$$X_{2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2})$$
$$= pX_{2e_1}^m X_{e_1+e_2}^{r+1} X_{e_1-e_2}^{p-1} \otimes x_{(b+4)e_1+e_2}$$
$$+ p(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^r X_{e_1-e_2}^{p-2} \otimes x_{(b+4)e_1+e_2}.$$

Proof. This is an immediate consequence of Lemma 3.15 and the equations (3.30a).

Theorem 3.32 A basis for Ker $\partial^\natural \subseteq C_1^*(V_{be_1+e_2})|_{(N+b+2,N+1)}$ is given by

$$\text{(3.32a)}\quad \sum_{k=0}^{N}\sum_{i=0}^{1}\sum_{j=0}^{\min[k,N-k]} c_{k;i;j}(v_{-(N+b+2)} \otimes w_{N+1-2(i+k)})^*$$
$$\otimes X_{-2e_1} \otimes X_{2e_1}^j X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N-k-j} \otimes x_{(b+4)e_1+(2i-1)e_2},$$

where

$$c_{0;0;0} = 1, \qquad c_{0;1;0} = -1 \qquad \text{and}$$
$$\text{(3.32b)}\quad c_{k;i;j} = (-1)^{i+k}\prod_{q=0}^{k+j-1}(N-q)\frac{\prod_{r=1}^{k}(i+r)}{(k-j)!j!} \qquad \text{for } 1 \le k \le N.$$

Proof. Applying X_{2e_2} to (3.32a) and using Lemma 3.31, we see that a necessary condition for L invariance is that the coefficients $c_{k;i;j}$ satisfy the equations

$$\text{(3.32c)}\qquad c_{0;1;0} + c_{0;0;0} = 0$$

$$(3.32\text{d}) \qquad c_{k+1;0;0} + (N-k)c_{k;0;0} = 0 \qquad \text{for } 0 \le k \le N-1$$

$$(3.32\text{e}) \qquad c_{k+1;1;0} + (N-k)c_{k;1;0} + c_{k+1;0;0} = 0 \qquad \text{for } 0 \le k \le N-1$$

$$(3.32\text{f}) \quad c_{k+1;0;k+1} + (N-2k)(N-2k-1)c_{k;0;k} = 0$$

$$\text{for } 2 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{cases}$$

$$(3.32\text{g}) \quad c_{k+1;1;k+1} + (N-2k)(N-2k-1)c_{k;1;k} + c_{k+1;0;k+1} = 0$$

$$\text{for } 2 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{cases}$$

$$(3.32\text{h}) \quad c_{k+1;0;j} + (N-k-j)c_{k;0;j}$$

$$+ (N-k-j+1)(N-k-j)c_{k;0;j-1} = 0 \qquad \text{for } (k,j) \text{ in (domain 1)}$$

$$(3.32\text{i}) \quad c_{k+1;1;j} + (N-k-j)c_{k;1;j}$$

$$+(N-k-j+1)(N-k-j)c_{k;1;j-1} + c_{k+1;0;j} = 0 \qquad \text{for } (k,j) \text{ in (domain 1)},$$

where (domain 1) is the set $\{(k, j)\}$ satisfying

$$1 \le j \le k \qquad \text{for } 1 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{cases}$$

$$1 \le j \le N-k-1 \qquad \text{for } \left\{ \begin{array}{ll} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{array} \right\} < k \le N-2.$$

The rest of the proof is almost identical to the proof of Theorem 3.17, except that the system of equations we use now is $\{3.32(c)\text{-}(i)\} \cup \{c_{0;0;0} = c_{0;0;0}\}$, and the vectors $\vec{x}$ and $\vec{x}_0$ are modified to fit this system. More specifically, $\vec{x}$ is the vector $(c_{0;0;0}, c_{1;0;0}, c_{1;0;1}, \ldots, c_{N;0;0}, c_{0;1;0}, c_{1;1;0}, c_{1;1;1}, \ldots, c_{N;1;0})$ and $\vec{x}_0$ is the vector $(c_{0;0;0}, 0, \ldots, 0)$. If we order the system of equations properly, then we can once again write this system in matrix form as

$$A \cdot \vec{x} = \vec{x}_0,$$

with A a lower triangular matrix having 1's along the diagonal. Since

$$c_{0;1;0} = -c_{0;0;0}$$

$$c_{k;i;j} = (-1)^{i+k}\, c_{0;0;0} \prod_{q=0}^{k+j-1} (N-q) \prod_{r=1}^{k} \frac{(i+r)}{j!(k-j)!} \qquad \text{for } 1 \le k \le N$$

is a solution to this system of equations, it is the unique solution. Choosing $c_{0;0;0} = 1$ gives the coefficients in the statement of the theorem.

Notice that in the previous theorem, as well as Theorem 3.17, L invariance is sufficient to determine an element of Ker $\partial^{\natural}$. This follows because both $\mathcal{C}_0^*(\mathbb{C}_{ae_1})|_{(N+a+2,N)}$ and $\mathcal{C}_0^*(V_{be_1+e_2})|_{(N+b+2,N+1)}$ are 0 for all nonnegative integers N, and $\partial^{\natural}$ maps $\mathcal{C}_1^*$ into $\mathcal{C}_0^*$. However, for the K types $(N+b+3,N)$, $\mathcal{C}_0^*(V_{be_1+e_2})|_{(N+b+3,N)}$ is 0 *only for* $N=0$. Hence, we must work a little in order to produce an element of Ker $\partial^{\natural} \subseteq \mathcal{C}_1^*(V_{be_1+e_2})|_{(N+b+3,N)}$. We begin with the following lemma.

Lemma 3.33 The action of X_{-2e_1} on $U(\mathfrak{u}) \otimes V_{(b+4)e_1+e_2}$ is given by

$$(3.33\text{a})\quad X_{-2e_1}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2})$$

$$= \big(rp(r+p+b+1)X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-1} + rp(r-1)(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^{p-2}$$

$$- m(m+r+p+b+3)X_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^p\big) \otimes x_{(b+4)e_1+e_2}$$

$$- p(p-1)X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-2} \otimes x_{(b+4)e_1-e_2}$$

$$(3.33\text{b})\quad X_{-2e_1}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2})$$

$$= \big(rp(r+p+b+3)X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-1} + rp(r-1)(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^{p-2}$$

$$- m(m+r+p+b+3)X_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^p\big) \otimes x_{(b+4)e_1-e_2}$$

$$+ r(r-1)X_{2e_1}^m X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2}.$$

The action of X_{-2e_2} on $U(\mathfrak{u}) \otimes V_{(b+4)e_1+e_2}$ is given by

$$(3.33\text{c})\quad X_{-2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2})$$

$$= X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2} + rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p+1} \otimes x_{(b+4)e_1+e_2}$$

$$+ r(r-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2}$$

$$(3.33d)\quad X_{-2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2})$$
$$= rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p+1} \otimes x_{(b+4)e_1-e_2}$$
$$+ r(r-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2}.$$

PROOF. Formulas (3.33a) and (3.33b) are an immediate consequence of the application of formula (3.15b) in Lemma 3.15. Formulas (3.33c) and (3.33d) require another fact. Given the definition of weight vectors $x_{(b+4)e_1+e_2}$ and $x_{(b+4)e_1-e_2}$ in (3.30a), it follows from formula (2.18b) that

$$(3.30b)\quad \begin{aligned} X_{-2e_2}(x_{(b+4)e_1+e_2} &= x_{(b+4)e_1-e_2}) \\ X_{-2e_2}(x_{(b+4)e_1-e_2} &= 0). \end{aligned}$$

This, in combination with formula (3.15e) of Lemma 3.15, gives formulas (3.33c) and (3.33d).

Theorem 3.34 An element of Ker $\partial^{\natural} \subseteq C_1^*(V_{be_1+e_2})|_{(N+b+3,N)}$ that does not vanish in homology is

$$(3.34a)\quad \sum_{k=0}^{N+1} \sum_{i=\max[-1,-k]}^{\min[0,N-k]} \sum_{j=0}^{\min[k,N+1-k]} c_{k;i;j}(v_{-(N+b+3)} \otimes w_{N-2(i+k)})^*$$
$$\otimes X_{-2e_1} \otimes X_{2e_1}^j X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+1-k-j} \otimes x_{(b+4)e_1+(2i+1)e_2},$$

where

$$(3.34b)\quad \begin{aligned} c_{0;0;0} &= N+b+3 \\ c_{1;-1;0} &= -(N+b+3) \\ c_{1;-1;1} &= -N(2N+b+5) \end{aligned}$$

$$c_{k;-1;0} = (-1)^{k+1} \prod_{q=0}^{k-2} (N-q) c_{1;-1;0} \qquad \text{for } 2 \leq k \leq N+1$$

$$c_{k;-1;j} = (-1)^{k+1} \prod_{q=1}^{k+j-2} (N-q) \left[N \binom{k-1}{j} c_{1;-1;0} + \binom{k-1}{j-1} c_{1;-1;1} \right]$$

$$\text{for } (k,j) \text{ in (domain 2a)}$$

$$c_{k;-1;k} = (-1)^{k+1} \prod_{q=1}^{2k-2} (N-q) c_{1;-1;1} \qquad \text{for } 2 \leq k \leq \begin{cases} \dfrac{N}{2}, & N \text{ even} \\ \dfrac{N+1}{2}, & N \text{ odd} \end{cases}$$

$$c_{k;0;0} = (-1)^{k+1} \prod_{q=0}^{k-1} (N-q) c_{1;-1;0} \qquad \text{for } 1 \leq k \leq N$$

$$c_{1;0;1} = -((N+1) N c_{0;0;0} + c_{1;-1;1})$$

$$c_{k;0;j} = (-1)^{k+1} \binom{k}{j} \prod_{q=1}^{k+j-2} (N-q) \left[(N+j-k+1) N c_{1;-1;0} - j c_{1;-1;1} \right]$$

$$\text{for } (k,j) \text{ in (domain 2b).}$$

Here, (domain 2a) is the set of $\{(k,j)\}$ satisfying

$$1 \leq j \leq k-1 \qquad \text{for } 2 \leq k \leq \begin{cases} \dfrac{N+2}{2}, & N \text{ even} \\ \dfrac{N+1}{2}, & N \text{ odd} \end{cases}$$

$$1 \leq j \leq N-k+1 \qquad \text{for } \left\{ \begin{array}{ll} \dfrac{N+2}{2}, & N \text{ even} \\ \dfrac{N+1}{2}, & N \text{ odd} \end{array} \right\} < k \leq N-2$$

and (domain 2b) is the set of $\{(k,j)\}$ satisfying

$$1 \le j \le k \qquad \text{for } 2 \le k \le \begin{cases} \dfrac{N}{2}, & N \text{ even} \\ \dfrac{N-1}{2}, & N \text{ odd} \end{cases}$$

$$1 \le j \le N-k+1 \qquad \text{for } \left\{ \begin{matrix} \dfrac{N}{2}, & N \text{ even} \\ \dfrac{N-1}{2}, & N \text{ odd} \end{matrix} \right\} < k \le N.$$

PROOF. We first consider the L invariance condition. Applying X_{2e_2} to the vector (3.34a) and using Lemma 3.31, we see that a necessary condition for the vector (3.34a) to be L invariant is that its coefficients satisfy the relations

$$c_{k+1;-1;0} + (N-k+1)c_{k;-1;0} = 0 \qquad \text{for } 1 \le k \le N \tag{3.34c}$$

$$c_{k+1;-1;j} + (N-k-j+1)c_{k;-1;j} + (N-k-j+2)(N-k-j+1)c_{k;-1;j-1} = 0 \qquad \text{for } (k,j) \text{ in (domain 3)} \tag{3.34d}$$

$$c_{k+1;-1;k+1} + (N-2k+1)(N-2k)c_{k;-1;k} = 0 \qquad \text{for } 1 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd} \end{cases} \tag{3.34e}$$

$$c_{k+1;0;0} + (N-k+1)c_{k;0;0} + c_{k+1;-1;0} = 0 \qquad \text{for } 0 \le k \le N-1 \tag{3.34f}$$

$$c_{N;0;0} + c_{N+1;-1;0} = 0 \tag{3.34g}$$

$$\begin{aligned} c_{k+1;0;j} &+ (N-k-j+1)c_{k;0;j} \\ &+ (N-k-j+2)(N-k-j+1)c_{k;0;j-1} + c_{k+1;-1;j} = 0 \end{aligned} \tag{3.34h}$$

for (k, j) in (domain 3)

$$c_{k+1;0;k+1} + (N-2k+1)(N-2k)c_{k;0;k} + c_{k+1;-1;k+1} = 0 \tag{3.34i}$$

$$\text{for } 1 \le k \le \begin{cases} \dfrac{N-2}{2}, & N \text{ even} \\ \dfrac{N-3}{2}, & N \text{ odd,} \end{cases}$$

where (domain 3) is the set of $\{(k,j)\}$ satisfying

$$1 \le j \le k \qquad \text{for } 1 \le k \le \begin{cases} \dfrac{N}{2}, & N \text{ even} \\ \dfrac{N-1}{2}, & N \text{ odd} \end{cases}$$

$$1 \le j \le N-k \qquad \text{for } \left\{ \begin{array}{ll} \dfrac{N}{2}, & N \text{ even} \\ \dfrac{N-1}{2}, & N \text{ odd} \end{array} \right\} < k \le N-1.$$

The rest of the proof follows the proofs of Theorem 3.17 and Theorem 3.32. For this setting we add the equations

$$c_{0;0;0} = c_{0;0;0}, \quad c_{1;-1;0} = c_{1;-1;0}, \quad \text{and} \quad c_{1;-1;1} = c_{1;-1;1} \tag{3.34j}$$

to the system $\{3.34(c)–(i)\}$.Also, we choose for $\vec{x}$ the vector with ordered coordinates

$$(c_{1;-1;0}, c_{1;-1;1}, c_{2;-1;0}, \ldots, c_{N+1;-1;0}, c_{0;0;0}, c_{1;0;0}, c_{1;0;1}, \ldots, c_{N;0;0})$$

and we let $\vec{x}_0$ be the vector gotten by replacing all coordinates of $\vec{x}$, with the exception of $c_{1;-1;0}$,$c_{1;-1;1}$, and $c_{0;0;0}$, by 0. Through a judicious choice of ordering, the system of equations $\{3.34(c)$-$(j)\}$, and the vectors $\vec{x}_0$ and $\vec{x}$ will form the matrix equation with lower triangular matrix A, where A has 1's on the diagonal. As in the proofs of Theorems 3.17 and 3.32, this shows that the solution vector $\vec{x}$ is unique. Putting aside the specific values given to $c_{0;0;0}$, $c_{1;-1;0}$, and $c_{1;-1;1}$ in (3.34b), we note that the set of coordinates (3.34b) satisfies the system of equations $\{3.34(c)–(j)\}$. By uniqueness, therefore, (3.34b) must be the coordinates of $\vec{x}$.

Before proceeding with the calculation for the kernel, we observe that we have really only used half of the information available to us in order to find L invariance. In fact, the vector (3.34a) with coefficients (3.34b) must also vanish under the X_{-2e_2} action. Using (3.33c) and (3.33d), we can come up with a list of relations that the coefficients $\{c_{k;i;j}\}$ must satisfy in order to be invariant under the X_{-2e_2} action. One necessary relation that aids in our computation of the kernel is

$$c_{0;0;0} = -c_{1;-1;0}. \tag{3.34k}$$

We are now ready to calculate a vector in Ker $\partial^{\natural} \subseteq C_1^*(V_{be_1+e_2})|_{(N+b+3,N)}$. Using formulas (3.33a) and (3.33b) of Lemma 3.33, we can readily verify that a necessary condition for the vector (3.34a) to be in Ker $\partial^\natural$ is that the coefficients $c_{0;0;0}$, $c_{1;-1;0}$, and $c_{1;-1;1}$ satisfy the equation

$$(3.34l) \quad -(N+1)Nc_{0;0;0} + N(N+b+4)c_{1;-1;0} - (N+b+3)c_{1;-1;1} = 0.$$

Because of (3.34k), this can be rewritten as

$$(3.34m) \qquad N(2N+b+5)c_{1;-1;0} = (N+b+3)c_{1;-1;1},$$

and clearly $c_{1;-1;0} = -(N+b+3)$, $c_{1;-1;1} = -N(2N+b+5)$ satisfy (3.34m). From (3.34k) it follows that $c_{0;0;0} = N+b+3$.

We have stated that (3.34l) is a necessary condition, but, in fact, once all the coefficients are written in terms of $c_{0;0;0}$, $c_{1;-1;0}$, and $c_{1;-1;1}$, it is the **only** condition; applying the $\partial^\natural$ operator to the vector (3.34a) with coefficients in terms of $c_{0;0;0}$, $c_{1;-1;0}$, and $c_{1;-1;1}$ will yield only the equation (3.34l), repeatedly.

To see that this vector in the kernel does not vanish in homology, homology for $\partial^\natural$ meaning Ker $\partial^\natural$/Im $\partial_2^\natural$, we make the following observation. The pure tensors that comprise a vector in Im $\partial_2^\natural \subseteq C_1^*(V_{be_1+e_2})|_{(N+b+3,N)}$ have either linear functionals of the form $(v_{N+b+3-2i} \otimes w_{N-2k})^*$, with i satisfying $0 \leq i \leq N+b+2$, or have a monomial term of the form $X_{2e_1}^{m+1} X_{e_1+e_2}^{r} X_{e_1-e_2}^{p}$, with m, r and p all nonnegative integers. Since the tensor

$$c_{0;0;0}(v_{-(N+b+3)} \otimes w_N)^* \otimes X_{-2e_1} \otimes X_{e_1-e_2}^{N+1} \otimes x_{(b+4)e_1+e_2}$$

satisfies neither of these conditions, it is clear the vector (3.34a) is not in Im $\partial_2^\natural$ and therefore does not vanish in homology.

Definition 3.35

(3.35a) $$\text{Ker } \partial^{\natural}\big|_{(N+b+2,N+1)}$$

is the vector (3.32a) with coefficients (3.32b).

(3.35b) $$\text{Ker } \partial^{\natural}\big|_{(N+b+3,N)}$$

is the vector (3.34a) with coefficients (3.34b).

As in the first section, we can now determine a set of vectors for Ker $\partial \subseteq \mathcal{C}_{1,K}(V_{be_1+e_2})|_{\mu}$ (μ being any K type appearing in the discrete series), whose image under the map

(*) $$\text{Ker } \partial \mapsto \text{Ker } \partial/\text{Im } \partial_2$$

is a homology basis for the K type μ in $\mathcal{L}_1(V_{be_1+e_2})$.

Definition 3.36 For $0 \leq i_1 \leq N+b+2$ and $0 \leq i_2 \leq N+1$,

$$v_{N+b+2-2i_1} \otimes w_{N+1-2i_2} \otimes \text{Ker } \partial^{\natural}\big|_{(N+b+2,N+1)}$$

is the vector

(3.36a)
$$\sum_{k=0}^{N}\sum_{i=0}^{1}\sum_{j=0}^{\min[k,N-k]} c_{k;i;j}\, v_{N+b+2-2i_1} \otimes w_{N+1-2i_2} \otimes \left(v_{-(N+b+2)} \otimes w_{N+1-2(i+k)}\right)^{*}$$
$$\otimes X_{-2e_1} \otimes X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N-k-j} \otimes x_{(b+4)e_1+(2i-1)e_2},$$

with the coefficients $c_{k;i;j}$ given by formula (3.32b). If we take all possible i_1 and i_2 satisfying the inequalities above, then this set is a homology basis for the K types $(N+b+2, N+1)$, since the boundary is 0 for these K types.

For $0 \le i_1 \le N+b+3$ and $0 \le i_2 \le N$,

$$v_{N+b+3-2i_1} \otimes w_{N-2i_2} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+3,N)}$$

is the vector

$$\text{(3.36b)} \quad \sum_{k=0}^{N+1} \sum_{i=\max[-1,-k]}^{\min[0,N-k]} \sum_{j=0}^{\min[k,N+1-k]} c_{k;i;j}\, v_{N+b+3-2i_1} \otimes w_{N-2i_2}$$
$$\otimes \left(v_{-(N+b+3)} \otimes w_{N-2(i+k)}\right)^* \otimes X_{-2e_1} \otimes X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+1-k-j}$$
$$\otimes\, x_{(b+4)e_1+(2i+1)e_2},$$

with the coefficients $c_{k;i;j}$ given by formula (3.34b). If we take all i_1 and i_2 satisfying $0 \le i_1 \le N+b+3$ and $0 \le i_2 \le N$, then the image of this set under the map $(*)$ is a homology basis for the K type $(N+b+3, N+1)$.

Now that a homology basis has been given for each K type μ in the discrete series, we proceed to reconstruct the $\mathfrak{p}$ action on a set of basis vectors in $\operatorname{Ker} \partial \subseteq \mathcal{C}_{1,K}(V_{be_1+e_2})\big|_{\mu}$, again using Duflo-Vergne. Proposition 3.83 of [K-V] tells us this is sufficient to determine the $\mathfrak{p}$ action on homology. A distinction between the action of $\mathfrak{p}$ for the one row and two row cases is that in the two row case, action by $\mathfrak{p}$ on a K type can map into three other K types, whereas in the one row case, action by $\mathfrak{p}$ on a K type can map into at most two other K types, as we have seen in Theorem 3.29.

Theorem 3.37 For $\mathfrak{g} = \mathfrak{sp}(1,1)^{\mathbb{C}}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decompsition according to the Cartan involution mentioned in Chapter 2. Let $\{X_\alpha\}$ be the set of orthonormal root vectors (2.1b) of $\mathfrak{p}$. For $0 \leq i_1 \leq N + b + 2$ and $0 \leq i_2 \leq N + 1$, let

$$v_{N+b+2-2i_1} \otimes w_{N+1-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)}$$

be the basis vector for Ker $\partial \subseteq C_{1,K}(V_{be_1+e_2})|_{(N+b+2,N+1)}$ stated in Definition 3.36(a). Then the action of a X_α on any such basis vector is given by the following formulas.

Action of $X_{e_1+e_2}$:

- $X_{e_1+e_2}(v_{N+b+2} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)})$

$$= \frac{-1}{(N+b+3)^3(N+2)^4} v_{N+b+3} \otimes w_{N+2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N+2)}$$

For $0 \leq i_1 \leq N + b + 1$,

- $X_{e_1+e_2}(v_{N+b+2-2(i_1+1)} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)})$

$$= \frac{-1}{(N+b+3)^3(N+2)^4} v_{N+b+3-2(i_1+1)} \otimes w_{N+2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N+2)}.$$

For $0 \leq i_2 \leq N$,

- $X_{e_1+e_2}(v_{N+b+2} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)})$

$$= \frac{-1}{(N+b+3)^3(N+2)^4} v_{N+b+3} \otimes w_{N+2-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N+2)}$$

$$+\frac{(N+1)^3}{(N+b+3)^4(N+2)}v_{N+b+3}\otimes w_{N-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N)}.$$

For $0\leq i_1\leq N+b+1$ and $0\leq i_2\leq N$,

- $X_{e_1+e_2}(v_{N+b+2-2(i_1+1)}\otimes w_{N+1-2(i_2+1)}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+2,N+1)})$

$$=\frac{-1}{(N+b+3)^3(N+2)^4}v_{N+b+3-2(i_1+1)}\otimes w_{N+2-2(i_2+1)}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N+2)}$$

$$+\frac{(N+1)^3}{(N+b+3)^4(N+2)}v_{N+b+3-2(i_1+1)}\otimes w_{N-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N)}$$

$$+\frac{N(N+1)^3(N+b+2)(N+b+4)}{(N+b+3)}v_{N+b+1-2i_1}\otimes w_{N-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+1,N)}.$$

Action of $X_{e_1-e_2}$:

For $0\leq i_2\leq N+1$,

- $X_{e_1-e_2}(v_{N+b+2}\otimes w_{N+1-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+2,N+1)})$

$$=\frac{-(i_2+1)}{(N+b+3)^3(N+2)^4}v_{N+b+3}\otimes w_{N+2-2(i_2+1)}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N+2)}$$

$$-\frac{(N+1-i_2)(N+1)^3}{(N+b+3)^4(N+2)}v_{N+b+3}\otimes w_{N-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N)}.$$

For $0\leq i_1\leq N+b+1$ and $0\leq i_2\leq N+1$,

- $X_{e_1-e_2}(v_{N+b+2-2(i_1+1)}\otimes w_{N+1-2i_2}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+2,N+1)})$

$$=\frac{-(i_2+1)}{(N+b+3)^3(N+2)^4}v_{N+b+3-2(i_1+1)}\otimes w_{N+2-2(i_2+1)}\otimes \text{Ker }\partial^{\natural}\big|_{(N+b+3,N+2)}$$

$$- \frac{(N+1-i_2)(N+1)^3}{(N+b+3)^4(N+2)} v_{N+b+3-2(i_1+1)} \otimes w_{N-2i_2} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+3,N)}$$
$$- \frac{(N+1-i_2)N(N+1)^3(N+b+2)(N+b+4)}{(N+b+3)} v_{N+b+1-2i_1} \otimes w_{N-2i_2}$$
$$\otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+1,N)} .$$

Action of $X_{-e_1+e_2}$:

For $0 \leq i_1 \leq N+b+2$,

- $X_{-e_1+e_2}(v_{N+b+2-2i_1} \otimes w_{N+1} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+2,N+1)})$

$$= \frac{(i_1+1)}{(N+b+3)^3(N+2)^4} v_{N+b+3-2(i_1+1)} \otimes w_{N+2} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+3,N+2)}$$

For $0 \leq i_1 \leq N+b+2$ and $0 \leq i_2 \leq N$,

- $X_{-e_1+e_2}(v_{N+b+2-2i_1} \otimes w_{N+1-2(i_2+1)} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+2,N+1)})$

$$= \frac{(i_1+1)}{(N+b+3)^3(N+2)^4} v_{N+b+3-2(i_1+1)} \otimes w_{N+2-2(i_2+1)}$$
$$\otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+3,N+2)}$$
$$- \frac{(i_1+1)(N+1)^3}{(N+b+3)^4(N+2)} v_{N+b+3-2(i_1+1)} \otimes w_{N-2i_2} \otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+3,N)}$$
$$+ \frac{(N+b+2-i_1)N(N+1)^3(N+b+2)(N+b+4)}{(N+b+3)} v_{N+b+1-2i_1} \otimes w_{N-2i_2}$$
$$\otimes \operatorname{Ker} \partial^{\natural}\big|_{(N+b+1,N)} .$$

Action of $X_{-e_1-e_2}$:

For $0 \leq i_1 \leq N+b+2$ and $0 \leq i_2 \leq N+1$,

- $X_{-e_1-e_2}(v_{N+b+2-2i_1} \otimes w_{N+1-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)})$

$$= -\frac{(i_1+1)(i_2+1)}{(N+b+3)^3(N+2)^4} v_{N+b+3-2(i_1+1)} \otimes w_{N+2-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N+2)}$$

$$- \frac{(i_1+1)(N+1-i_2)(N+1)^3}{(N+b+3)^4(N+2)} v_{N+b+3-2(i_1+1)} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)}$$

$$+ \frac{(N+b+2-i_1)(N+1-i_2)N(N+1)^3(N+b+2)(N+b+4)}{(N+b+3)} \times \quad v_{N+b+1-2i_1} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+1,N)}.$$

For $0 \leq i_1 \leq N+b+3$ and $0 \leq i_2 \leq N$, let

$$v_{N+b+3-2i_1} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)}$$

be the basis vector for Ker $\partial \subseteq \mathcal{C}_{1,K}(V_{be_1+e_2})|_{(N+b+3,N)}$ stated in Definition 3.36(b). Also, suppose η is the number

$$N\big((N+1)(N+b+3)c_{0\,;0\,;0} + 2c_{1\,;-1\,;0}\big) - c_{1\,;0\,;1} - (b+5)c_{1\,;0\,;0},$$

where the coefficients are those of (3.34b). Then the action of an X_α on any such basis vector (3.36b) is given by the following formulas.

Action of $X_{e_1+e_2}$:

- $X_{e_1+e_2}(v_{N+b+3} \otimes w_N \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$

$$= \frac{-(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}$$

For $0 \leq i_1 \leq N+b+2$,

$$\bullet\ X_{e_1+e_2}(v_{N+b+3-2(i_1+1)} \otimes w_N \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$$
$$= \frac{-(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}$$
$$+ \frac{(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4} v_{N+b+2-2i_1} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)}.$$

For $0 \leq i_2 \leq N-1$,

$$\bullet\ X_{e_1+e_2}(v_{N+b+3} \otimes w_{N-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$$
$$= \frac{-(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}.$$

For $0 \leq i_1 \leq N+b+2$ and $0 \leq i_2 \leq N-1$,

$$\bullet\ X_{e_1+e_2}(v_{N+b+3-2(i_1+1)} \otimes w_{N-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$$
$$= \frac{-(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}$$
$$+ \frac{(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4} v_{N+b+2-2i_1} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)}$$
$$+ \frac{N^3(N+b+3)^3\eta}{(N+b+2)(N+b+4)(N+1)} v_{N+b+2-2i_1} \otimes w_{N-1-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N-1)}.$$

Action of $X_{e_1-e_2}$:

For $0 \leq i_2 \leq N$,

- $X_{e_1-e_2}(v_{N+b+3} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$

$$= \frac{-(i_2+1)(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}.$$

For $0 \leq i_1 \leq N+b+2$ and $0 \leq i_2 \leq N$,

- $X_{e_1-e_2}(v_{N+b+3-2(i_1+1)} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$

$$= \frac{-(i_2+1)(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}$$

$$+ \frac{(i_2+1)(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4} v_{N+b+2-2i_1} \otimes w_{N+1-2(i_2+1)} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)}$$

$$- \frac{(N-i_2)N^3(N+b+3)^3\eta}{(N+b+2)(N+b+4)(N+1)} v_{N+b+2-2i_1} \otimes w_{N-2i_2} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N-1)}.$$

Action of $X_{-e_1+e_2}$:

For $0 \leq i_1 \leq N+b+3$,

- $X_{-e_1+e_2}(v_{N+b+3-2i_1} \otimes w_N \otimes \text{Ker } \partial^{\natural}|_{(N+b+3,N)})$

$$= \frac{(i_1+1)(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+4,N+1)}$$

$$+ \frac{(N+b+3-i_1)(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4} v_{N+b+2-2i_1} \otimes w_{N+1} \otimes \text{Ker } \partial^{\natural}|_{(N+b+2,N+1)}.$$

For $0 \le i_1 \le N+b+3$ and $0 \le i_2 \le N-1$,

$$\bullet\ X_{-e_1+e_2}(v_{N+b+3-2i_1} \otimes w_{N-2(i_2+1)} \otimes \text{Ker }\partial^{\natural}|_{(N+b+3,N)})$$

$$= \frac{(i_1+1)(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1-2(i_2+1)}$$

$$\otimes \text{ Ker }\partial^{\natural}|_{(N+b+4,N+1)}$$

$$+ \frac{(N+b+3-i_1)(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4} v_{N+b+2-2i_1} \otimes w_{N+1-2(i_2+1)}$$

$$\otimes \text{ Ker }\partial^{\natural}|_{(N+b+2,N+1)}$$

$$+ \frac{(N+b+3-i_1)N^3(N+b+3)^3\eta}{(N+b+2)(N+b+4)(N+1)} v_{N+b+2-2i_1} \otimes w_{N-1-2i_2}$$

$$\otimes \text{ Ker }\partial^{\natural}|_{(N+b+2,N-1)}.$$

Action of $X_{-e_1-e_2}$:

For $0 \le i_1 \le N+b+3$ and $0 \le i_2 \le N$,

$$\bullet\ X_{-e_1-e_2}(v_{N+b+3-2i_1} \otimes w_{N-2i_2} \otimes \text{Ker }\partial^{\natural}|_{(N+b+3,N)})$$

$$= \frac{-(i_1+1)(i_2+1)(N+b+3)}{(N+b+4)^4(N+1)^4} v_{N+b+4-2(i_1+1)} \otimes w_{N+1-2(i_2+1)}$$

$$\otimes \text{ Ker }\partial^{\natural}|_{(N+b+4,N+1)}$$

$$- \frac{(N+b+3-i_1)(i_2+1)(N+b+3)^3(b+1)(b+3)}{(N+b+4)(N+1)^4}$$

$$\times \quad v_{N+b+2-2i_1} \otimes w_{N+1-2(i_2+1)} \otimes \text{ Ker }\partial^{\natural}|_{(N+b+2,N+1)}$$

$$+ \frac{(N+b+3-i_1)(N-i_2)N^3(N+b+3)^3\eta}{(N+b+2)(N+b+4)(N+1)} v_{N+b+2-2i_1} \otimes w_{N-1-2i_2}$$

$$\otimes \text{ Ker }\partial^{\natural}|_{(N+b+2,N-1)}.$$

PROOF. This follows closely the work done in the one row case, particularly Definition 3.18 through Theorem 3.29. We make use of the following identities, all of which, except for (3.37a) and (3.37b), result from applying Lemma 3.15. The formulas (3.37a) and (3.37b) are proved using the first two lines of the proof of Lemma 3.26.

$$\text{(3.37a)} \quad X_{e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1\pm e_2}) = X_{2e_1}^m X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p \otimes x_{(b+4)e_1\pm e_2}$$

$$\text{(3.37b)} \quad X_{e_1-e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1\pm e_2}) = (X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p+1} + 2rX_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p) \otimes x_{(b+4)e_1\pm e_2}$$

$$\text{(3.37c)} \quad X_{-e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2})$$
$$= \big(-p(p+2r+b+2)X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1} - 2rp(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-2} + mX_{2e_1}^{m-1} X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p\big) \otimes x_{(b+4)e_1+e_2}$$

$$\text{(3.37d)} \quad X_{-e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(b+4)e_1-e_2})$$
$$= \big(-p(p+2r+b+4)X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1} - 2rp(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-2} + mX_{2e_1}^{m-1} X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p\big) \otimes x_{(b+4)e_1-e_2} - 2rX_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p \otimes x_{(b+4)e_1+e_2}$$

(3.37e) $X_{-e_1-e_2}(X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\otimes x_{(b+4)e_1+e_2})$

$$= \big(-r(2m+r+b+4)X^m_{2e_1}X^{r-1}_{e_1+e_2}X^p_{e_1-e_2}$$

$$-mX^{m-1}_{2e_1}X^r_{e_1+e_2}X^{p+1}_{e_1-e_2}\big)\otimes x_{(b+4)e_1+e_2}$$

$$-2pX^m_{2e_1}X^r_{e_1+e_2}X^{p-1}_{e_1-e_2}\otimes x_{(b+4)e_1-e_2}$$

(3.37f) $X_{-e_1-e_2}(X^m_{2e_1}X^r_{e_1+e_2}X^p_{e_1-e_2}\otimes x_{(b+4)e_1-e_2})$

$$= \big(-r(2m+r+b+2)X^m_{2e_1}X^{r-1}_{e_1+e_2}X^p_{e_1-e_2}$$

$$-mX^{m-1}_{2e_1}X^r_{e_1+e_2}X^{p+1}_{e_1-e_2}\big)\otimes x_{(b+4)e_1-e_2}$$

We know from (2.14b) that

(3.37g)

$$(v_{-(N+b+2)}\otimes w_{N+1-2(i+k)})^* = (-1)^{N+1-(i+k)}\binom{N+1}{i+k}((N+1-i-k)!)^2$$

$$\times \quad (\cdot, v_{N+b+2}\otimes w_{2(i+k)-(N+1)}).$$

Our goal in this theorem is an evaluation of the sum of vectors

(3.37h) $\qquad \alpha_1(X\otimes v_{N+b+2-2i_1}\otimes w_{N+1-2i_2}\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+b+2,N+1)}),$

for $0\le i_1\le N+b+2$, $0\le i_2\le N$, X an orthonormal root vectors in (2.1b), and an evaluation of the sum of vectors

(3.37i) $\qquad \alpha_1(X\otimes v_{N+b+3-2i_1}\otimes w_{N-2i_2}\otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+b+3,N)}),$

for $0 \le i_1 \le N + b + 2$, $0 \le i_2 \le N$, and X an orthonormal root vectors in (2.1b). Recall that α_1 was defined in Section 1, right after Definition 3.19. We consider the expression (3.37h) first. Following the discussion in Section 1, particularly Definition 3.19 through decomposition (3.22a), we see that (3.37h) equals

$$\text{(3.37j)} \quad (K \text{ type decomposition of } X \otimes v_{N+b+2-2i_1} \otimes w_{N+1-2i_2}) \\ \times (K \text{ type decomposition of } \alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+b+2,N+1)})),$$

where $\alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+a+2,N+1)}))$ is the expression obtained by replacing $(\mathrm{Ad} \otimes \pi)(\cdot)^{-1} X \otimes v_{N+b+2-2i_1} \otimes w_{N+1-2i_2}$ in the expansion for (3.37h) by a dot (see (3.21) and expression after it for an example). Using (3.37g), the identities (3.37a–f), Theorem 3.32, Corollary 2.12, and arguing as we did in the proof of Theorem 3.29, we see that $\alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+b+2,N+1)}))$ can be decomposed as

$$\text{(3.37k)} \quad \frac{-1}{(N+2)^{\frac{5}{2}}(N+b+3)^{\frac{3}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+3,N+2)} \\ - \frac{(N+1)^2}{(N+b+3)^{\frac{5}{2}}(N+2)^{\frac{1}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+3,N)} \\ + \frac{(N+1)^2(N+b+2)^2 N(N+b+4)(N+2)}{(N+b+3)^{\frac{1}{2}}(N+2)^{\frac{1}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+1,N)} \\ + (K \text{ type that does not appear in discrete series}).$$

Here, $\operatorname{Ker} \partial^\natural|_{(N+b+3,N+2)}$ (respectively, $\operatorname{Ker} \partial^\natural|_{(N+b+1,N)}$) refers to (3.35a) with N replaced by $N+1$ (respectively, $N-1$). Notice in this case that there are three K types of interest, as opposed to the decomposition (3.27), where there are only two. Looking now at (3.37i), we see that it equals

$$(3.37\mathrm{l}) \quad (K \text{ type decomposition of } X \otimes v_{N+b+3-2i_1} \otimes w_{N-2i_2}) \\ \times (K \text{ type decomposition of } \alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+b+3,N)})),$$

where $\alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+b+3,N)})$ is defined analogously to the expression in (3.37j). Using (3.37g), the identities (3.37a–f), Theorem 3.34, Corollary 2.12, and arguing as we did in the proof of Theorem 3.29, we see that the decomposition of $\alpha_1(\cdot \otimes \operatorname{Ker} \partial^\natural|_{(N+b+3,N)})$ can be written as

$$(3.37\mathrm{m}) \quad \frac{-(N+b+3)}{((N+1)(N+b+4))^{\frac{5}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+4,N+1)} \\ + \frac{(N+b+3)^2 N^2 \eta}{(N+b+2)((N+b+4)(N+1))^{\frac{1}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+2,N-1)} \\ + \frac{(N+b+3)^2(b+1)(b+3)}{(N+b+4)^{\frac{1}{2}}(N+1)^{\frac{5}{2}}} \operatorname{Ker} \partial^\natural|_{(N+b+2,N+1)} \\ + (K \text{ type that does not appear in discrete series}).$$

Here, $\operatorname{Ker} \partial^\natural|_{(N+b+4,N+1)}$ (respectively, $\operatorname{Ker} \partial^\natural|_{(N+b+2,N-1)}$) refers to the vector (3.35b) with N replaced by $N+1$ (respectively, $N-1$), and η is defined in the statement of this theorem.

In order to get the formulas in the statement of the theorem, we must decompose the $X \otimes v \otimes w$'s into their various K types by using Corollary 2.12, then multiply by either (3.37h) or (3.37i), and reassemble, again using Corollary 2.12 and (2.14b). For example, suppose we are considering the sum of vectors (3.37i). For a given X, i_1 and i_2, assume that the tensor $X \otimes v_{N+b+2-2i_1} \otimes w_{N-2i_2}$ decomposes under Corollary 2.12 as

$$\text{(3.37n)} \quad \begin{aligned} &h_1'(\text{vector}_1 \text{ in } K \text{ type } (N+b+3, N+1) \\ &\quad + h_2'(\text{vector}_2 \text{ in } K \text{ type } (N+b+3, N) \\ &\quad + h_3'(\text{vector}_3 \text{ in } K \text{ type } (N+b+1, N-1) \\ &\quad + (\text{vector in } K \text{ type that does not appear in discrete series}). \end{aligned}$$

Then, using this and the decomposition (3.37k), we can write (3.37h), following the prescription (3.37l), as

$$\text{(3.37o)} \quad \begin{aligned} &\frac{-h_1'}{(N+2)^{\frac{5}{2}}(N+b+3)^{\frac{3}{2}}}(\text{vector}_1) \otimes \text{Ker } \partial^{\natural}\big|_{(N+b+3,N+2)} \\ &- \frac{h_2'(N+1)^2}{(N+b+3)^{\frac{5}{2}}(N+2)^{\frac{1}{2}}}(\text{vector}_2) \otimes \text{Ker } \partial^{\natural}\big|_{(N+b+3,N)} \\ &+ \frac{h_3'(N+1)^2(N+b+2)^2 N(N+b+4)(N+2)}{(N+b+3)^{\frac{1}{2}}(N+2)^{\frac{1}{2}}} \\ &\qquad \times (\text{vector}_3) \otimes \text{Ker } \partial^{\natural}\big|_{(N+b+1,N)}. \end{aligned}$$

We do not bother with the K type that does not appear in the discrete series since it will be 0 in homology. This is the method used to arrive at all the formulas in the statement of this theorem.

3. The R Row Case

Having produced basis elements of $\text{Ker } \partial \subseteq \mathcal{C}_{1,K}(Z)|_{\mu}$ for K types μ (Z an appropriate irreducible L representation) in the one and two row cases, as well as reconstructing the $\mathfrak{p}$ action in each case, we turn our attention to the R row case, where R is any positive integer. In this case, we state a conjecture about a basis vector in Ker $\partial^\natural$. We then show that such a vector will not vanish in homology. In addition, we reproduce the $\mathfrak{p}$ action by using only a portion of the pure tensors that appear as summands in the Ker $\partial^\natural$ basis vector. This follows the technique used in the one and two row cases. For example, in the one row case, the tensor (3.24) was all we needed to determine the $(N+a+3, N+1)$ component of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$, and three pure tensors that added to give (3.25b) were all we needed to determine the $(N+a+1, N-1)$ component of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(N+a+2,N)})$.

Before we state the conjecture, let us be more precise in our notation so that we can properly identify any K type that appears in the Dixmier R row diagram. To begin, recall that, for R a nonnegative integer, d an integer satisfying $d \geq R-1$, and M an integer satisfying $0 \leq M \leq R-1$, the Dixmier diagram corresponding to R and d is the set of points $\bigcup_{M=0}^{R-1} D_{(R,d,M)}$, where $D_{(R,d,M)}$ is the subset of points

$$\{(N+M+d+2)e_1 + (N+R-M-1)e_2 |\ N \in \mathbb{Z}^+ \cup 0\}$$

of the integer lattice in the first quadrant of the coordinate plane with axes e_1 and e_2 (see Chapter 1, Section 3). Each of the points in this lattice is a K type for the discrete series $\mathcal{L}_1(V_{de_1+(R-1)e_2})$ of $Sp(1,1)$. Here, $V_{de_1+(R-1)e_2}$ is an L irreducible representation with highest weight $de_1 + (R-1)e_2$. The condition $d \geq R-1$ is precisely what is needed in order for $\mathcal{L}_1(V_{de_1+(R-1)e_2})$ to be a discrete series (see (1.4a) or (1.4b)). We have noted previously that when $R = 1$, then $d = a$ of Section 1. Also, when $R = 2$, then $d = b$ of Section 2. Using coordinate notation, we shall write an arbitrary K type of $\mathcal{L}_1(V_{de_1+(R-1)e_2})$ as

$$(N + M + d + 2, N + R - M - 1),$$

where N and M are nonnegative integers, $M \leq R - 1$. If we start from the minimal K type $(d + 2, R - 1)$, then N refers to the number of units moved in the $e_1 + e_2$ direction, and M refers to the number of units moved in the $e_1 - e_2$ direction. We shall use thoughout this section the following abbreviations for K types:

K type	Abbreviation
$(N + M + d + 2, N + R - M - 1)$	$(\vec{N}, \vec{M})$
$(N + M + d + 3, N + R - M)$	$(\vec{N} + 1, \vec{M})$
$(N + M + d + 3, N + R - M - 2)$	$(\vec{N}, \vec{M} + 1)$
$(N + M + d + 1, N + R - M - 2)$	$(\vec{N} - 1, \vec{M})$
$(N + M + d + 1, N + R - M)$	$(\vec{N}, \vec{M} - 1)$.

As a basis for $V_{(d+4)e_1+(R-1)e_2}$, which equals $V^{\#}_{(d+4)e_1+(R-1)e_2}$, we choose weight vectors $x_{(d+4)e_1+(2i-(R-1))e_2}$ (see Chapter 2, Section 4) that satisfy the relations

$$\text{(3.38a)} \quad X_{2e_2} \cdot x_{(d+4)e_1+(2i-(R-1))e_2} = \begin{cases} x_{(d+4)e_1+(2(i+1)-(R-1))e_2}, & \text{for } 0 \leq i \leq R-2 \\ 0, & \text{for } i = R-1. \end{cases}$$

Following formula (2.18b), these weight vectors must satisfy the relations

$$(3.38b)\quad X_{-2e_2}\cdot x_{(d+4)e_1+(2i-(R-1))e_2} = \begin{cases} (R-i)(i)\, x_{(d+4)e_1+(2(i-1)-(R-1))e_2}, & \text{for } 1\le i\le R-1 \\ 0, & \text{for } i=0. \end{cases}$$

Conjecture 3.39 A basis vector in Ker $\partial^{\natural} \subseteq C_1^*(V_{de_1+(R-1)e_2})\big|_{(\vec{N},\vec{M})}$ that does not vanish in homology is

$$(3.39a)\quad \sum_{k=0}^{N+M} \sum_{i=\max[-M,-k]}^{\min[R+N-M-1,R-1-M]} \sum_{j=0}^{\min[k,N+M-k]} c_{k;i;j}$$

$$\times \left(v_{-(N+d+2+M)} \otimes w_{N+R-M-1-2(i+k)}\right)^* \otimes X_{-2e_1} \otimes X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j}$$

$$\otimes\, x_{(d+4)e_1+(2(i+M)-(R-1))e_2},$$

where the coefficients $c_{k;i;j}$ satisfy the sets of relations

(1) $\{c_{k\,;i\,;j} + c_{k\,;i-1\,;j} + (N+M-k-j+1)c_{k-1\,;i\,;j}$

$+ (N+M-k-j+2)(N+M-k-j+1)c_{k-1\,;i\,;j-1} = 0\}_{(k,i,j)\in I_1}$

(2)

$\{(R+N-M-1-i-k)(i+k+1)c_{k\,;i\,;j} + (M+i+1)(R-i-M-1)c_{k\,;i+1\,;j}$

$+ (k+1-j)c_{k+1\,;i\,;j} + (k-j+2)(k-j+1)c_{k+1\,;i\,;j-1} = 0\}_{(k,i,j)\in I_2}$

(3)

$\{(k-j-1)(N+M-k-j+1)(N-M+R+d-2i-2j-1)c_{k-1\,;i+1\,;j}$

$$+(k-j)(N+M-k-j+2)(k-j-1)(N+M-k-j+1)c_{k-1\,;i+1\,;j-1}$$

$$-(N+M-k-j+2)(N+M-k-j+1)(R-i-M-2)(i+M+2)c_{k-2\,;i+2\,;j}$$

$$+(k-j)(k-j-1)c_{k\,;i\,;j}-(j+1)c_{k-1\,;i+1\,;j+1}=0\}_{(k,i,j)\in I_3}.$$

In the set of relations (1), which is determined by the X_{2e_2} action, the index set I_1 consists of those tuples for which at least two of the four coefficients $c_{k\,;i\,;j}$, $c_{k\,;i-1\,;j}$, $c_{k-1\,;i\,;j}$ and $c_{k-1\,;i\,;j-1}$ appear in the summation (3.39a). Similarly , in the set of relations (2) , which is determined by the X_{-2e_2} action, the index set I_2 consists of those tuples for which at least two of the four coefficients $c_{k\,;i\,;j}$, $c_{k\,;i+1\,;j}$, $c_{k+1\,;i\,;j}$ and $c_{k+1\,;i\,;j-1}$ appear in the summation (3.39a). Finally, for the set of relations (3), which are the necessary relations for (3.39a) to be in Ker $\partial^{\natural}$, the author believes that I_3 consists of those tuples for which at least two of the five coefficients $c_{k\,;i\,;j}\ \ldots, c_{k-1\,;i+1\,;j+1}$ appear in the summation (3.39a).

MOTIVATION. For the $\mathfrak{g}$ action on $U(\mathfrak{u})\otimes V_{(d+4)e_1+(R-1)e_2}$ we obtain the following identities as a consequence of Lemma 3.15 and the relations (3.38a) and (3.38b).

For $0\le i\le R-2$,

$$X_{2e_2}(X_{2e_1}^{m}X_{e_1+e_2}^{r}X_{e_1-e_2}^{p}\otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$=X_{2e_1}^{m}X_{e_1+e_2}^{r}X_{e_1-e_2}^{p}\otimes x_{(d+4)e_1+(2(i+1)-(R-1))e_2}$$

$$+p\big(X_{2e_1}^{m}X_{e_1+e_2}^{r+1}X_{e_1-e_2}^{p-1}+(p-1)X_{2e_1}^{m+1}X_{e_1+e_2}^{r}X_{e_1-e_2}^{p-2}\big)$$

$$\otimes\ x_{(d+4)e_1+(2i-(R-1))e_2}.$$

When $i = R-1$, the first term on the right hand side vanishes.

$$X_{-2e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= (R-i)(i)\, X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2(i-1)-(R-1))e_2}$$

$$+ r(X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p+1} + (r-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p)$$

$$\otimes x_{(d+4)e_1+(2i-(R-1))e_2}.$$

For $0 \le i \le R-2$,

$$X_{-2e_1}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= \big(rp(r+p+R+d-2i+1)X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-1}$$

$$- m(d+m+r+p+3)X_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^p$$

$$+ rp(r-1)(p-1)X_{2e_1}^{m+1} X_{e_1+e_2}^{r-2} X_{e_1-e_2}^{p-2}\big) \otimes x_{(d+4)e_1+(2i-(R-1))e_2}$$

$$+ r(r-1)X_{2e_1}^m X_{e_1+e_2}^{r-2} X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2(i+1)-(R-1))e_2}$$

$$- p(p-1)(R-i)i\, X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-2} \otimes x_{(d+4)e_1+(2(i-1)-(R-1))e_2}.$$

When $i = R-1$, the second to last term on the right hand side vanishes.

To see how each of these identities is used in determining relations among coefficients, let us work through an example. We apply X_{2e_2} to a summand of (3.39a):

$$\text{(3.39b)} \quad X_{2e_2}\{c_{k;i;j}(v_{-(N+d+2+M)} \otimes w_{N+R-M-1-2(i+k)})^*$$

$$\otimes X_{-2e_1} X_{2e_1}^j X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j} \otimes x_{(d+4)e_1+(2(i+M)-(R-1))e_2}\}$$

$$= c_{k\,;i\,;j}\big[(w_{N+R-M-1-2(i+k-1)})^* \otimes X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j}$$

$$\otimes\, x_{(d+4)e_1+(2(i+M)-(R-1))e_2}$$

$$+ (w_{N+R-M-1-2(i+k)})^* \otimes X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j}$$

$$\otimes\, x_{(d+4)e_1+(2(i+M+1)-(R-1))e_2}$$

$$+ (w_{N+R-M-1-2(i+k)})^* \otimes \big((N+M-k-j) X_{2e_1}^{j} X_{e_1+e_2}^{k-j+1} X_{e_1-e_2}^{N+M-k-j-1}$$

$$+ (N+M-k-j)(N+M-k-j-1) X_{2e_1}^{j+1} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-2}\big)$$

$$\otimes\, x_{(d+4)e_1+(2(i+M)-(R-1))e_2}\big].$$

Notice we removed $v_{-(N+d+2+M)}$ and X_{-2e_1} from the pure tensors of the right hand side of (3.39b). We may do this since X_{2e_2} does not act on them and therefore they do not affect the equations. Our next step is substituting different values for k, i, and j so that all pure tensors on the right hand side of (3.39b) look the same. For example, if we make all tensors look like the first, then we substitute $i-1$ for i in the second tensor, $k-1$ for k in the third tensor, and $k-1$ for k, $j-1$ for j in the fourth. Making these substitutions in the coefficients of each tensor as well gives us the relation

$$c_{k\,;i\,;j} + c_{k\,;i-1\,;j} + (N+M-k-j+1)c_{k-1\,;i\,;j}$$

$$+ (N+M-k-j+2)(N+M-k-j+1)c_{k-1\,;i\,;j-1} = 0.$$

The choice of index set I_1 comes from constraints on the indices k, i, and j. The requirement that at least two coefficients be nonzero prevents the degenerate case $c_{k\,;i\,;j} = 0$ for one (and subsequently all) coefficient(s) of (3.39a).

The other sets of relations are determined in a like manner, using the X_{-2e_2} action on (3.39a) or applying the $\partial^\natural$ operator to (3.39a) in order to determine the relations necessary so that (3.39a) is in Ker $\partial^\natural$. Notice that, because $X_{-2e_1}(v_{-(N+d+2+M)} \otimes w_\beta)^* = 0$ for any weight β of $W_{N+R-M-1}$, applying the $\partial^\natural$ operator to (3.39a) is equivalent to replacing

$$\text{(3.39c)} \qquad X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j} \otimes x_{(d+4)e_1+(2(i+M)-(R-1))e_2}$$

by

$$X_{-2e_1}(X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j} \otimes x_{(d+4)e_1+(2(i+M)-(R-1))e_2})$$

in each summand of (3.39a).

By studying $C_2^*(V_{de_1+(R-1)e_2})\big|_{(\vec{N},\vec{M})}$ and Im $\partial_2^\natural$, we observe that the pure tensors that comprise an element of Im $\partial_2^\natural \subseteq C_1^*(V_{de_1+(R-1)e_2})\big|_{(\vec{N},\vec{M})}$, with $\bigwedge^1((\mathfrak{u}\oplus\bar{\mathfrak{u}})\cap\mathfrak{k})$ part equal to X_{-2e_1} have either the term $v_{(N+d+2+M)-2q}$ with $q < N+d+2+M$ in $(V_{N+d+2+M} \otimes W_{N+R-M-1})^*$ or have a monomial term $X_{2e_1}^{m+1} X_{e_1+e_2}^{r} X_{e_1-e_2}^{p}$, with m, r, and p all nonnegative integers. Since

$$c_{0\,;0\,;0}(v_{-(N+d+2+M)} \otimes w_{N+R-M-1})^* \otimes X_{-2e_1} \otimes X_{e_1-e_2}^{N+M}$$
$$\otimes\ x_{(d+4)e_1+(2M-(R-1))e_2}$$

satisfies neither of these conditions, it follows that (3.39a) is not an element of Im $\partial_2^\natural \subseteq C_1^*(V_{de_1+(R-1)e_2})\big|_{(\vec{N},\vec{M})}$ and therefore does not vanish in homology.

The author has confirmed that Conjecture 3.39 is true for the case when $M = 0$. For this case, the boundary is 0, and L equivalence is the same as being a member of Ker $\partial^{\natural}$. We have seen this in Sections 1 and 2. In fact, the coefficients in this case are

$$c_{k;i;j} = (-1)^{i+k} \prod_{q=0}^{k+j-1} (N-q) \frac{\prod_{r=1}^{k}(i+r)}{(k-j)!\,j!}\, c_{0;0;0} \qquad \text{for } 1 \le k \le N.$$

This expression reduces to formula (3.17b) for $R = 1$ and formula (3.32b) for $R = 2$.

Now that we have shown the vector (3.39a) survives in homology, we are ready to reconstruct the $\mathfrak{p}$ action. Toward this end, let

$$\alpha_1(X \otimes v_{N+M+d+2-2i_1} \otimes w_{N+R-M-1-2i_2} \otimes \text{Ker } \partial^{\natural}\big|_{(\vec{N},\vec{M})})$$

be the expression

$$\begin{aligned}
(3.39\text{d}) \quad & \sum_{\alpha\in\Delta(\mathfrak{p})} \sum_{k=0}^{N+M} \sum_{i=\max[-M,-k]}^{\min[R+N-M-1,R-1-M]} \sum_{j=0}^{\min[k,N+M-k]} c_{k;i;j} \\
& \times (-1)^{N+R-M-1-(i+k)} \binom{N+R-M-1}{i+k} ((N+R-M-1-(i+k))!)^2 \\
& \times \langle (\text{Ad}\otimes\pi)(\cdot)^{-1} X \otimes v_{N+M+d+2-2i_1} \otimes w_{N+R-M-1-2i_2}, \\
& X_\alpha \otimes v_{N+M+d+2} \otimes w_{2(i+k)-(N+R-M-1)} \rangle \otimes X_{-2e_1} \\
& \otimes X_\alpha (X_{2e_1}^{j} X_{e_1+e_2}^{k-j} X_{e_1-e_2}^{N+M-k-j} \otimes x_{(d+4)e_1+(2(i+M)-(R-1))e_2}),
\end{aligned}$$

where $0 \le i_1 \le N+M+d+2$, $0 \le i_2 \le N+R-M-1$, and the coefficients $c_{k;i;j}$ are those of (3.39a) satisfying relations (1)-(3). Also, let

$\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ be the expression gotten by replacing

$$(\text{Ad} \otimes \pi)(\cdot)^{-1} X \otimes v_{N+M+d+2-2i_1} \otimes w_{N+R-M-1-2i_2}$$

in (3.39d) by a dot. A summand in $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ is completely determined by $\alpha \in \Delta(\mathfrak{p})$, k, i, and j, so it makes sense to abbreviate such a summand (coefficient included) by $\alpha_1(X_\alpha, k; i; j)$. With the next result, we can determine formulas for the $\mathfrak{p}$ action on $\mathcal{L}_{1,K}(V_{de_1+(R-1)e_2})|_{(\vec{N},\vec{M})}$. It turns out that each of these formulas is a generalization of the formulas given in Theorems 3.29 and 3.37. However, we shall not write out explicit formulas here.

Theorem 3.40 Suppose η and ϵ are given by the equations

$$\eta = (N+R-M-1)[c_{0\,;0\,;0}(N+M)(N+R+d+2-M)+2c_{1\,;-1\,;0}] - c_{1\,;0\,;1}$$
$$- 2(N+M)(R-M-1)(M+1)c_{0\,;1\,;0} - (2M+5+d-R)c_{1\,;0\,;0}$$

and

$$\epsilon = 2M(N+M)(R-M)c_{0\,;0\,;0} + (2M+3+d-R)c_{1\,;-1\,;0} + c_{1\,;-1\,;1},$$

where the values $c_{k;i;j}$ come from the expression (3.39a). Then the K type decomposition of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ is given by

$$\text{(3.40a)} \quad \alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$$
$$= \frac{-c_{0\,;0\,;0}}{c^{N+1,M}_{0\,;0\,;0}(N+R-M)^{\frac{5}{2}}(N+M+d+3)^{\frac{3}{2}}} \text{Ker } \partial^\natural|_{(\vec{N}+1,\vec{M})}$$
$$+ \frac{c_{0\,;1\,;0}(N+R-M-1)^2}{c^{N,M+1}_{0\,;0\,;0}(N+R-M)^{\frac{1}{2}}(N+M+d+3)^{\frac{3}{2}}} \text{Ker } \partial^\natural|_{(\vec{N},\vec{M}+1)}$$

$$+\frac{(N+M+d+2)^2(N+R-M-1)^2\,\eta}{c_{0\,;0\,;0}^{N-1,M}((N+R-M)(N+M+d+3))^{\frac{1}{2}}}\ \mathrm{Ker}\ \partial^{\natural}\big|_{(\vec{N}-1,\vec{M})}$$

$$-\frac{(N+M+d+2)^2\,\epsilon}{c_{0\,;0\,;0}^{N,M-1}(N+R-M)^{\frac{5}{2}}(N+M+d+3)^{\frac{1}{2}}}\ \mathrm{Ker}\ \partial^{\natural}\big|_{(\vec{N},\vec{M}-1)}\,,$$

where $c_{k\,;i\,;j}^{N\pm1,M}$(respectively $c_{k\,;i\,;j}^{N,M\pm1}$) is the coefficient $c_{k\,;i\,;j}$ of (3.39a) for the K type $(\vec{N}\pm1,\vec{M})$ (respectively $(\vec{N},\vec{M}\pm1)$), and $c_{k;i;j}$ refers to the coefficient of (3.39a) for the K type $(\vec{N},\vec{M})$. Consequently, for $X\in\mathfrak{p}$,

$$X(\,v_{N+M+d+2-2i_1}\otimes w_{N+R-M-1-2i_2}\otimes \mathrm{Ker}\ \partial^{\natural}\big|_{(\vec{N},\vec{M})})$$

$$=\alpha_1(X\otimes v_{N+M+d+2-2i_1}\otimes w_{N+R-M-1-2i_2}\otimes \mathrm{Ker}\ \partial^{\natural}\big|_{(\vec{N},\vec{M})})$$

can be explicitly calculated.

PROOF. We decompose α_1 one K type at a time. For the time being, we avoid giving specific values for the coefficients $c_{k\,;i\,;j}$. Some remarks about possible choices will be made later.

We can conclude from the following identities that the sole contribution to the monomial $X_{e_1-e_2}^{N+M+1}$ in the vector (3.39a) for the $(\vec{N}+1,\vec{M})$ K type comes from

$$X_{e_1-e_2}(X_{e_1-e_2}^{N+M}\otimes x_{(d+4)e_1+(2M-(R-1))e_2}).$$

It therefore is sufficient to consider only $\alpha_1(X_{e_1-e_2},0;0;0)$ when determining the contribution of $\alpha_1(\cdot\otimes \mathrm{Ker}\ \partial^{\natural}\big|_{(\vec{N},\vec{M})})$ to the $(\vec{N}+1,\vec{M})$ K type. These identities all follow from Lemma 3.15 and actions (3.38a) and (3.38b), unless otherwise stated.

For $0 \leq i \leq R-1$,

$$(3.40b) \quad X_{e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= X_{2e_1}^m X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2},$$

since $X_{e_1+e_2}$ commutes with X_{2e_1}.

For $0 \leq i \leq R-1$,

$$(3.40c) \quad X_{e_1-e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p+1} \otimes x_{(d+4)e_1+(2i-(R-1))e_2}$$

$$+ 2r X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2},$$

since

$$X_{e_1-e_2} X_{e_1+e_2}^r = X_{e_1+e_2}^r X_{e_1-e_2} + 2r X_{2e_1} X_{e_1+e_2}^{r-1}$$

in $U(\mathfrak{u})$.

For $0 \leq i \leq R-2$,

$$(3.40d) \quad X_{-e_1+e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= -2r X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2(i+1)-(R-1))e_2}$$

$$+ \big(m X_{2e_1}^{m-1} X_{e_1+e_2}^{r+1} X_{e_1-e_2}^p - 2rp(p-1) X_{2e_1}^{m+1} X_{e_1+e_2}^{r-1} X_{e_1-e_2}^{p-2}$$

$$+ p(2i-2r-2-d-p-R) X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1}\big) \otimes x_{(d+4)e_1+(2i-(R-1))e_2}.$$

When $i = R-1$, the first term vanishes on the right hand side of (3.40d) vanishes.

For $0 \leq i \leq R-1$,

$$(3.40e) \quad X_{-e_1-e_2}(X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^p \otimes x_{(d+4)e_1+(2i-(R-1))e_2})$$

$$= -2p(R-i)i\, X_{2e_1}^m X_{e_1+e_2}^r X_{e_1-e_2}^{p-1}\, x_{(d+4)e_1+(2(i-1)-(R-1))e_2}$$

$$+ \big(-mX_{2e_1}^{m-1} X_{e_1+e_2}^r X_{e_1-e_2}^{p+1} - r(2m+2i+4+d+r-R) X_{2e_1}^m X_{e_1+e_2}^{r-1} X_{e_1-e_2}^p\big)$$

$$\otimes\, x_{(d+4)e_1+(2i-(R-1))e_2}.$$

By using formulas (3.40b–e) and arguing as in sections 1 and 2 of this chapter, we can conclude that the sole contribution to the tensor of (3.39a) with coefficient $c_{0\,;0\,;0}^{N+1,M}$ is from $\alpha_1(X_{e_1-e_2}, 0; 0; 0)$, and that contribution is

$$\frac{-c_{0\,;0\,;0}}{(N+R-M)^{\frac{5}{2}}(N+M+d+3)^{\frac{3}{2}}}(v_{-(N+d+3+M)} \otimes w_{(N+R-M)})^*$$

$$\otimes X_{-2e_1} \otimes X_{e_1-e_2}^{N+M+1} \otimes x_{(d+4)e_1+(2M-(R-1))e_2}.$$

Hence, the portion of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ that maps to the $(\vec{N}+1, \vec{M})$ K type from $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ is

$$\frac{-c_{0\,;0\,;0}}{c_{0\,;0\,;0}^{N+1,M}(N+R-M)^{\frac{5}{2}}(N+M+d+3)^{\frac{3}{2}}} \text{ Ker } \partial^\natural|_{(\vec{N}+1,\vec{M})}.$$

The contribution to the $(\vec{N}, \vec{M}+1)$ K type can be determined by the summand $\alpha_1(X_{e_1-e_2}, 0; 1; 0)$. Again, using the formulas (3.40b–e), we see that the sole contribution to the tensor in (3.39a) with coefficient $c_{0\,;0\,;0}^{N,M+1}$ is from

$\alpha_1(X_{e_1-e_2}, 0; 1; 0)$, and using Corollary 2.12, formulas (2.14b) and (3.40c), we determine that the contribution is

$$\frac{c_{0\,;1\,;0}(N+R-M-1)^2}{(N+M+d+3)^{\frac{3}{2}}(N+R-M)^{\frac{1}{2}}}(v_{-(N+d+3+M)} \otimes w_{(N+R-M-2)})^* \\ \otimes X_{-2e_1} \otimes X_{e_1-e_2}^{N+M+1} \otimes x_{(d+4)e_1+(2(M+1)-(R-1))e_2}.$$

We may conclude that the $(\vec{N}, \vec{M}+1)$ portion of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$ is

$$\frac{c_{0\,;1\,;0}(N+R-M-1)^2}{c_{0\,;0\,;0}^{N,M+1}(N+R-M)^{\frac{1}{2}}(N+M+d+3)^{\frac{3}{2}}} \text{ Ker } \partial^\natural|_{(\vec{N},\vec{M}+1)}.$$

For the remaining two K types, the relevant summands are given as well as the respective portions of $\alpha_1(\cdot \otimes \text{Ker } \partial^\natural|_{(\vec{N},\vec{M})})$. The interested reader may verify these calculations for him/herself. For the K type $(\vec{N}-1, \vec{M})$, there are, in general, five contributing summands. They are

$$\alpha_1(X_{-e_1+e_2}, 0; 0; 0), \qquad \alpha_1(X_{-e_1+e_2}, 1; -1; 0), \qquad \alpha_1(X_{-e_1-e_2}, 1; 0; 0), \\ \alpha_1(X_{-e_1-e_2}, 0; 1; 0), \qquad \text{and} \qquad \alpha_1(X_{-e_1-e_2}, 1; 0; 1).$$

As in previous calculations, we use Corollary 2.12, and the formulas (2.14b), (3.40d), and (3.40e) to decompose, combine, and reassemble the summand with coefficient $c_{0\,;0\,;0}^{N-1,M}$. We use formulas (3.40b–e) to verify that the five terms listed above are the only contributing terms. The result is

$$\frac{(N+M+d+2)^2(N+R-M-1)^2\,\eta}{c_{0\,;0\,;0}^{N-1,M}((N+R-M)(N+M+d+3))^{\frac{1}{2}}} \text{ Ker } \partial^\natural|_{(\vec{N}-1,\vec{M})},$$

η being defined in the statement of Theorem 3.40. Note that certain coefficients that appear in η do not exist when $M = 0$ or 1. That is the reason we say that there are five contributing summands in general. Sometimes there are less.

For the K type $(\vec{N}, \vec{M} - 1)$, there are, in general, three contributing summands. They are $\alpha_1(X_{-e_1-e_2}, 0; 0; 0)$, $\alpha_1(X_{-e_1-e_2}, 1; -1; 0)$, and $\alpha_1(X_{-e_1-e_2}, 0; 0; 0)$. The $(\vec{N}, \vec{M} - 1)$ portion is

$$\frac{-(N + M + d + 2)^2 \, \epsilon}{c_{0\,;0\,;0}^{N,M-1}(N + R - M)^{\frac{5}{2}}(N + M + d + 3)^{\frac{1}{2}}} \text{ Ker } \partial^{\natural}\big|_{(\vec{N}, \vec{M}-1)},$$

ϵ being defined in the statement of Theorem 3.40 as well. Having accomplished the K type decomposition of $\alpha_1(\cdot \otimes \text{Ker } \partial^{\natural}|_{(\vec{N}, \vec{M})})$, we may decompose

$$X \otimes v_{N+M+d+2-2i_1} \otimes w_{N+R-M-1-2i_2}$$

using Corollary 2.12 and we may retrieve the $\mathfrak{p}$ action in the same manner as was done in sections 1 and 2 of this chapter.

Our attention now turns to a determination of the coefficients $c_{k\,;i\,;j}$ of Ker $\partial^{\natural}|_{(\vec{N}, \vec{M})}$. Notice that in the formula (3.40a), the coefficients needed to describe the $\mathfrak{p}$ action are $c_{0\,;0\,;0}^{P,Q}$, for various P and Q, $c_{0\,;1\,;0}$, $c_{1\,;-1\,;0}$, $c_{1\,;0\,;0}$, $c_{1\,;0\,;1}$, and $c_{1\,;-1\,;1}$. We use the sets of equations (1)-(3) to tell us the relations among these coefficients.

<u>Equation from I_1 set</u>	<u>Tuple in I_1 chosen</u>
$c_{0\,;1\,;0} + c_{0\,;0\,;0} = 0$	$(0, 1, 0)$
$c_{1\,;0\,;0} + c_{1\,;-1\,;0} + (N + M)c_{0\,;0\,;0} = 0$	$(1, 0, 0)$
$c_{1\,;0\,;1} + c_{1\,;-1\,;1} + (N + M)(N + M - 2)c_{0\,;0\,;0} = 0$	$(1, 0, 1)$

Equation from I_2 set	Tuple in I_2 chosen
$M(R-M)c_{0\,;0\,;0} + c_{1\,;-1\,;0} = 0$	$(0,-1,0)$
$(M-1)(R-M+1)c_{1\,;-1\,;0} + 2c_{2\,;-2\,;0} = 0$	$(1,-2,0)$

and from the set of equations I_3 we get

$$-(N+M)(N+M-1)(R-M)Mc_{0\,;0\,;0} - (N+M+d+2)c_{1\,;-1\,;1}$$
$$+(N+M-1)(N+R-M+d+3)c_{1\,;-1\,;0} + 2c_{2\,;-2\,;0} = 0,$$

with the tuple (2,-2,0) being chosen. These equations indicate that the coefficients $c_{0\,;1\,;0}$ through $c_{1\,;-1\,;1}$ above can be written as nonzero multiples of $c_{0\,;0\,;0}$. Note that the $c^{P,Q}_{0\,;0\,;0}$ can always be chosen to be 1. However, in the one and two row cases, the author chose values $c^{P,Q}_{0\,;0\,;0}$ for the various K types appearing in the discrete series so that all $c^{P,Q}_{k\,;i\,;j}$ would be integers. This was done for purely aesthetic reasons. The author has verified that in the R row case, when $M=0$, the coefficients $c^{N\pm1,0}_{k\,;i\,;j}$ will be integers when $c^{N\pm1,0}_{0\,;0\,;0}$ is 1. This generalizes the results found in the one and two row cases.

CHAPTER 4

$Sp(1,n)$ DISCRETE SERIES

In this chapter, we begin with a description of the K types that appear in the $A_{\mathfrak{q}}(\lambda)$ discrete series for $Sp(1,n)$, $n \geq 1$. It turns out that each of these K types has multiplicity one, as we shall see. We also provide a method for reconstructing the $\mathfrak{p}$ action on the $(\mathfrak{k}, K)$ modules (1.15a), where μ refers to a K type that appears in the discrete series $A_{\mathfrak{q}}(\lambda)$ and Z is a one-dimensional irreducible representation of $L = S^1 \times Sp(n)$ with weight λ. We can do this under the assumption that there exists a decomposition into K types for $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$, where $\pi_{(P,Q)}$ is an irreducible finite dimensional representation of $K = Sp(1) \times Sp(n)$ with highest weight $Pe_1 + Qe_2$, P and Q both nonnegative integers. We determined such a decomposition for the case $n = 1$ in Chapter 2, Section 3.

Throughout this chapter the following notation is in force:

$$G = Sp(1,n)$$

$$K = Sp(1) \times Sp(n) = SU(2) \times SP(n)$$

$$L = S^1 \times Sp(n)$$

$$\mathfrak{g}_0 = \mathfrak{sp}(1,n)$$

$$\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$$

$$\mathfrak{k}_0 = \mathfrak{su}(2) \oplus \mathfrak{sp}(n)$$
$$\mathfrak{k} = (\mathfrak{k}_0)^{\mathbb{C}}.$$

Also, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ [K3, pg.1] and $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. Further, we let $\mathfrak{l}_0 = \mathfrak{t}_0 \oplus \mathfrak{sp}(n) \subseteq \mathfrak{k}_0$, where $\mathfrak{t}_0$ is the Cartan subalgebra in $\mathfrak{su}(2)$, and $\mathfrak{l} = (\mathfrak{l}_0)^{\mathbb{C}}$. As mentioned in Chapter 1, we can write

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \bar{\mathfrak{u}},$$

where $\mathfrak{u}$ is a nilpotent lie subalgebra of $\mathfrak{g}$ [K1, (1.1b)]. We can also write $\mathfrak{q} = \mathfrak{u} \oplus \mathfrak{l}$ and $\bar{\mathfrak{q}} = \bar{\mathfrak{u}} \oplus \mathfrak{l}$. For any subspace $\mathfrak{m}$ of $\mathfrak{g}$, we use the notation $\Delta^+(\mathfrak{m})$ to denote the positive roots of $\mathfrak{m}$, and $\Delta(\mathfrak{m})$ to denote all the roots of $\mathfrak{m}$.

We choose

$$\Pi = \{e_1 - e_2,\, e_2 - e_3, \ldots,\, e_n - e_{n+1},\, 2e_{n+1}\}$$

as the system of simple roots for $\mathfrak{g}$. It follows that

$$\Delta^+ = \Delta^+(\mathfrak{g}) = \{e_i \pm e_j \,|\, 1 \leq i < j \leq n+1\} \cup \{2e_k \,|\, 1 \leq k \leq n+1\}.$$

Using the setting of Theorem 1.6(a) as well as the notation there,

$$\beta = 2e_1 = \text{largest root}$$
$$\beta_0 = e_1 - e_2 = \text{unique simple root not } \perp \text{ to } \beta.$$

We know $\mathfrak{l}$ is built from simple roots. In fact,

$$\Pi(\mathfrak{l}) = \Pi - \{e_1 - e_2\} = \text{simple roots of } \mathfrak{l}$$

$$\Delta^+(\mathfrak{l}) = \{e_i \pm e_j \,|\, 2 \le i < j \le n+1\} \cup \{2e_k \,|\, 2 \le k \le n+1\}.$$

Because $\Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u}) = \Delta^+$, then

$$\Delta(\mathfrak{u}) = \{2e_1\} \cup \{e_1 \pm e_j \,|\, 2 \le j \le n+1\}.$$

The set of compact roots, $\Delta(\mathfrak{k})$, is the set $\Delta(\mathfrak{l}) \cup \{\pm 2e_1\}$, and the set of noncompact roots, $\Delta(\mathfrak{p})$, is the set $\Delta(\mathfrak{g}) - \Delta(\mathfrak{k})$.

We are now ready to compute the K types for $A_{\mathfrak{q}}(\lambda)$. The key tool is Theorem 8.29 of [K-V]. The main hypothesis in that theorem is that λ satisfies

$$(*) \qquad \text{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha \rangle \ge 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

Here we may think of $\langle \cdot, \cdot \rangle$ as the standard inner product on $\mathbb{R}^{n+1}$ with coordinate axes labeled by $e_1, \cdots, e_{n+1}$. Recall that $\lambda \perp \Delta(\mathfrak{l})$ (Definition 1.18). If $\lambda = a_1 e_1 + \cdots a_{n+1} e_{n+1}$, then the orthonogality condition implies that $\lambda = a_1 e_1$. We shall replace a_1 by a. As in Chapter 3, we may use the integrality condition (3.2a), since $Sp(1,n)$ is a real group whose complexification is simply connected. In (3.2a), the inner product is the standard inner product on $\mathbb{R}^{n+1}$ mentioned above. This implies $a \in \mathbb{Z}$. Further, the inequality $\langle \lambda + \delta(\mathfrak{g}), \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{g})$, which is the condition for discrete series (see (1.4a)), says that a is nonnegative. Because $\delta(\mathfrak{u}) = (n+1)e_1$ and every $\alpha \in \Delta(\mathfrak{u})$ has $+1$ as coefficient for e_1, it is clear that $(*)$ is satisfied when $A_{\mathfrak{q}}(\lambda)$ is a discrete series for $Sp(1,n)$. For the remainder of this chapter, $\lambda = ae_1$ with a representing a nonnegative integer. Also, N will represent a nonnegative integer. A key fact used in the determination of K types is the decomposition of $S(\mathfrak{u} \cap \mathfrak{p})$ into irreducible $\mathfrak{l}$ modules.

Lemma 4.1 For $G = Sp(1,n)$, $\mathfrak{l}$, $\mathfrak{u}$, $\mathfrak{p}$, $\Delta(\mathfrak{u})$ and $\Delta(\mathfrak{p})$ as above, the decomposition of $S(\mathfrak{u} \cap \mathfrak{p})$ into irreducible $\mathfrak{l}$ modules is given by

$$S(\mathfrak{u} \cap \mathfrak{p}) = \bigoplus_{N \in \{0\} \cup \mathbb{Z}^+} S^N(\mathfrak{u} \cap \mathfrak{p}), \tag{4.1a}$$

with each $S^N(\mathfrak{u} \cap \mathfrak{p})$ being an irreducible $\mathfrak{l}$ module.

PROOF. Since each $S^N(\mathfrak{u} \cap \mathfrak{p})$ is an $\mathfrak{l}$ module and the decomposition (4.1a) is valid as a vector space direct sum, it is sufficient to prove that each $S^N(\mathfrak{u} \cap \mathfrak{p})$ is irreducible as an $\mathfrak{l}$ module. Towards this end, we note that $\Delta(\mathfrak{u} \cap \mathfrak{p}) = \{e_1 \pm e_j \,|\, 2 \leq j \leq n+1\}$. The cardinality of $\Delta(\mathfrak{u} \cap \mathfrak{p})$ is $2n$, so using [K2, Proposition 2.14], which states that for E a finite-dimensional vector space of dimension m the dimension of $S^N(E)$ is $\binom{N+m-1}{m-1}$, we see that $\dim S^N(\mathfrak{u} \cap \mathfrak{p}) = \binom{N+2n-1}{2n-1}$. Meanwhile, the highest weight of $S^N(\mathfrak{u} \cap \mathfrak{p})$ is $N(e_1 + e_2)$ and the Weyl dimension formula [K3, Theorem 4.48] applied to the group $L = Sp(n)$ (the semisimple part of $S^1 \times Sp(n)$), with $\delta(\mathfrak{l}) = \sum_{j=0}^{n-1}(n-j)e_{j+2}$ shows that an irreducible $\mathfrak{l}$ module of highest weight $N(e_1+e_2)$ has dimension $\binom{N+2n-1}{2n-1}$. Therefore, $S^N(\mathfrak{u}\cap\mathfrak{p})$ is an irreducible $\mathfrak{l}$ module of highest weight $N(e_1+e_2)$ and the lemma is proved.

REMARK. The previous lemma is true for $G = Sp(1,n)$, but this result is not true for all groups that appear on Wolf's list [Bes, Table 14.52].

Proposition 4.2 For $G = Sp(1,n)$, let L, $\mathfrak{u}$, $\mathfrak{p}$ be as stated above. Also, suppose $\Delta^+(\mathfrak{k}) = \{e_2 \pm e_j \,|\, 2 \le i < j \le n+1\} \cup \{2e_k \,|\, 1 \le k \le n+1\}$ and $\lambda = ae_1$. Then the K types that appear in the $A_\mathfrak{q}(\lambda)$ discrete series are

$$\text{(4.2a)} \qquad \Lambda' = (N+a+2n)e_1 + Ne_2 \qquad \text{for } N \in \{0\} \cup \mathbb{Z}^+,$$

each occuring with multiplicity one.

PROOF. As mentioned previously, the key tool used in proving this proposition is **Theorem 8.29** of [K-V]. We give the statement of that theorem now:

In the case of $A_\mathfrak{q}(\lambda)$, suppose that the inequality $(*)$ (mentioned above) is satisfied. For $v \in \mathfrak{h}^*$, the space of linear functionals on the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$, define $\mathcal{P}(v)$ to be the multiplicity of v as a weight in $(S(\mathfrak{u}\cap\mathfrak{p}))^{\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{n}}$. Put $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$, and let W^1 be the subset of $W(\mathfrak{k},\mathfrak{h})$, the Weyl group of K,defined by

$$W^1 = \{w \in W(\mathfrak{k},\mathfrak{h}) \,|\, \Delta^+(w) \subseteq \Delta(\mathfrak{u})\},$$

where $\Delta^+(w) = \{\alpha \in \Delta^+(\mathfrak{k}) \,|\, w^{-1}\alpha < 0\}$. If Λ' is $\Delta^+(\mathfrak{k})$ dominant and integral, then the K type Λ' occurs in $A_{\mathfrak{q}}(\lambda)$ with multiplicity

$$\sum_{s \in W^1} (\det s)\mathcal{P}\big(s(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k}))\big).$$

Note for our situation that $\mathfrak{l} \subseteq \mathfrak{k}$, so that $\mathfrak{l} \cap \mathfrak{k} \cap \mathfrak{n} = \mathfrak{l} \cap \mathfrak{n}$. Also, for the Weyl group of K we may use $W(\mathfrak{k}, \mathfrak{h}) = \{$permutations and sign changes of $\{e_1, \cdots, e_{n+1}\}\}$ (see [K3, pg.78]). By Lemma 4.1, the only weights that appear with nonzero multiplicity in $S(\mathfrak{u} \cap \mathfrak{p})^{\mathfrak{l} \cap \mathfrak{n}}$ are $N(e_1 + e_2)$, $N \in \{0\} \cup \mathbb{Z}^+$. Since $\lambda = ae_1$ and $\delta(\mathfrak{u} \cap \mathfrak{p}) = ne_1$, then $\Lambda = (a + 2n)e_1$.

For Λ' a linear functional that is $\Delta^+(\mathfrak{k})$ dominant and integral, $s \in W^1$ with $s \neq 1$, we claim that the multiplicity $\mathcal{P}(s(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k})))$ is 0. To see this, we observe that there is only one other $s \in W^1$ besides the identity, and that is the reflection s_{2e_1} that maps e_1 to $-e_1$ and leaves e_j, $2 \leq j \leq n+1$, fixed. If $\Lambda' = a_1 e_1 + \cdots + a_{n+1} e_{n+1}$ is $\Delta^+(\mathfrak{k})$ dominant, then $a_1 \geq 0$, since $\alpha = 2e_1 \in \Delta^+(\mathfrak{k})$. Also, $\delta(\mathfrak{k}) = \delta(\mathfrak{u} \cap \mathfrak{k}) + \delta(\mathfrak{l}) = e_1 + \sum_{j=0}^{n-1} (n - j)e_{j+2}$ and therefore $\Lambda' + \delta(\mathfrak{k})$ has e_1 coefficient $a_1 + 1 > 0$. It follows that the element $s_{2e_1}(\Lambda' + \delta(\mathfrak{k}))$ has e_1 coefficient $-(a_1 + 1) < 0$. Since $\Lambda + \delta(\mathfrak{k})$ has e_1 coefficient $a + 2n + 1$, the element $s_{2e_1}(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k}))$ has a strictly negative a_1 coefficient. However, we know that the only K types in $S(\mathfrak{u} \cap \mathfrak{p})$ with nonzero multiplicity all have nonnegative e_1 coefficient. Hence, the claim is verified.

We can conclude that any K type Λ' appearing in $A_{\mathfrak{q}}(\lambda)$ has multiplicity $\mathcal{P}(\Lambda' - \Lambda)$. Because of (4.2a), this is equivalent to saying

$$\Lambda' = \Lambda + N(e_1 + e_2) = (N + a + 2n)e_1 + Ne_2,$$

for $N \in \{0\} \cup \mathbb{Z}^+$. Also, because $\mathcal{P}(N(e_1 + e_2)) = 1$ for each N, the multiplicity of each Λ' is one. This finishes the proof.

In the rest of this chapter we will need a description of basis vectors for some K types of $R(K)$, specifically those K types that appear in the discrete series. Using Definition 1.7, we see that such a K type can be written

$$(4.3) \qquad (V_{N+a+2n} \otimes W_{Ne_2}) \otimes (V_{N+a+2n} \otimes W_{Ne_2})^*,$$

where V_{N+a+2n} is an irreducible representation of $SU(2)$ of highest weight $(N + a + 2n)e_1$ (completely described in Chapter 2, Section 2), W_{Ne_2} is an irreducible representation of $Sp(n)$ of highest weight Ne_2, and the contragredient representation is given by $(V_{N+a+2n} \otimes W_{Ne_2})^*$. We already have a detailed description of basis vectors, complete with lengths, for V_{N+a+2n}. For the purposes of determining a basis for Ker $\partial^\natural \subseteq C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$ (Definition 3.3a) and reconstructing the $\mathfrak{p}$ action on the $(\mathfrak{k}, K)$ module , Ker $\partial \subseteq C_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$ (Definition 3.3b), we will need some description of basis vectors for W_{Ne_2}. With this goal in mind, let $\{\beta_i\}$ be the set of weights (with multiplicity) of W_{Ne_2}. We assume there is a basis of weight vectors $\{w_{\beta_i}\}$ and an inner product on W_{Ne_2} so that the length of each w_{β_i} is known. Using the fact that the set of weights of W_{Ne_2}, together with their multiplicities, is closed under action by the Weyl group of K [K-V, Theorem 4.10] and using the description of the Weyl group for K given in the proof of Theorem 4.2, we see that

$$(\text{multiplicity of } \beta_i) = (\text{multiplicity of } -\beta_i).$$

Therefore, we can define a one-one correspondence $w_{\beta_i} \longleftrightarrow w_{-\beta_i}$, and consequently it makes sense to define

$$(v_\alpha \otimes w_{\beta_i})^* = (\cdot, v_{-\alpha} \otimes w_{-\beta_i}), \tag{4.4}$$

where α is any weight of V_{N+a+2n} and the inner product on $V_{N+a+2n} \otimes W_{Ne_2}$ is $(\cdot,\cdot)$, determined in a canonical way from the inner products of V_{N+a+2n} and W_{Ne_2}. This definition differs slightly from (2.13). In that definition, we wanted $(v_\alpha \otimes w_{\beta_i})^*$ to satisfy certain relations under the $\mathfrak{k}$ action. In the remainder of this chapter, we shall drop the subscript i from β_i, since it will not be needed. Thus, for α and α', weights of V_{N+a+2n}, and for β and β', weights of W_{Ne_2}, we can now write any basis vector of (4.3) as

$$v_{\alpha'} \otimes w_{\beta'} \otimes (v_\alpha \otimes w_\beta)^*.$$

This generalizes the formula (2.15). We shall also need a description of root vectors for $\mathfrak{g}$. We choose the vectors $\{X_\alpha\}$ described in the following result:

Cartan's Theorem For each $\alpha \in \Delta(\mathfrak{g})$ a vector $X_\alpha \in \mathfrak{g}^\alpha$ can be chosen such that for all $\alpha, \beta \in \Delta(\mathfrak{g})$

$$[X_\alpha, X_{-\alpha}] = H_\alpha, \qquad [H, X_\alpha] = \alpha(H) X_\alpha \qquad \text{for } H \in \mathfrak{h};$$

$$[X_\alpha, X_\beta] = 0 \qquad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Delta(\mathfrak{g});$$

$$[X_\alpha, X_\beta] = N_{\alpha,\beta}\, X_{\alpha+\beta} \qquad \text{if } \alpha + \beta \in \Delta(\mathfrak{g}),$$

where the constants $N_{\alpha,\beta}$ satisfy

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}.$$

For any such choice

$$N_{\alpha,\beta}^2 = \frac{q(1-p)}{2}\, \alpha(H_\alpha),$$

where $\beta + n\alpha$ $(p \leq n \leq q)$ is the α-series containing β.

For a proof, see [Hel, Theorem 5.5, pg.176]. In this theorem, $\mathfrak{g}^\alpha$ refers to the one dimensional eigenspace of $\mathfrak{g}$ with eigenvalue α (see [Hel, pg.165] or [K3, pg.66]). Also, H_α is the unique vector in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that satisfies the relation

$$B(H, H_\alpha) = \alpha(H) \qquad \text{for all } H \in \mathfrak{h}.$$

B is the Killing form in this relation. Notice that the Killing form differs from the form used in Chapters 2 and 3, namely $\frac{1}{2}$ Trace form. We did not make use of this theorem in those chapters because we used a matrix realization of $\mathfrak{g} = \mathfrak{sp}(1,1)^{\mathbb{C}}$ and we were able to compute directly the bracket relations between various root vectors of $\mathfrak{g}$ (for example, (2.4) and (2.11j)). However, for general $Sp(1,n)$, the matrix realization quickly becomes too cumbersome. With the theorem mentioned above, it is possible to prove a result like Lemma 3.15, describing the action of $\mathfrak{g}$ on $U(\mathfrak{u})$ by means of the bracket relations. However, we will not need to do this.

We now show that the boundary is 0 in the space of L invariants of

$$(4.5\text{a}) \qquad (V_{N+a+2n} \otimes W_{Ne_2})^* \otimes \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+2(n+1))e_1}.$$

The space of L invariants shall hereafter be denoted by $(4.5\text{a})^L$. The spaces (4.5a) and $(4.5\text{a})^L$ are actually the spaces (1.17a) and (1.17c), respectively, of Chapter 1, for the case where $\mu = (N+a+2n)e_1 + Ne_2$ and $Z = \mathbb{C}_{ae_1}$. Following the method of proof in Proposition 3.6, we prove the boundary is 0 by showing that the space of L invariants of

$$(4.5\text{b}) \qquad (V_{N+a+2n} \otimes W_{Ne_2})^* \otimes \bigwedge^2((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+2(n+1))e_1},$$

hereafter referred to as $(4.5\text{b})^L$, is 0. Borrowing notation from Chapter 3, we let $C_j^*(\mathbb{C}_{ae_1})\big|_{(N+a+2n,N)}$ (Definition 3.3a) be the space of L invariants $(4.5\text{a})^L$ with $\bigwedge^1$ replaced by $\bigwedge^j$. Now suppose $\gamma_1 = 2e_1$, $\gamma_2 = e_1 + e_2$, $\gamma_3 = e_1 - e_2$, $\ldots$, $\gamma_{2n} = e_1 + e_{n+1}, \gamma_{2n+1} = e_1 - e_{n+1}$ is an ordering of the roots of $\Delta(\mathfrak{u})$. If $\mathbb{Z}_0{}^+$ is the set of nonnegative integers, we denote by $\gamma = (m, r_1, p_1, r_2, p_2, \ldots, r_n, p_n)$ an element of $(\mathbb{Z}_0{}^+)^{2n+1}$. Let $X_{\mathfrak{u}}^{\gamma}$ be the monomial $X_{\gamma_1}^{m} \ldots X_{\gamma_{2n+1}}^{p_n} \in U(\mathfrak{u})$.

Lemma 4.6 There are no pure tensors of the form

$$(**) \qquad (v_\alpha \otimes w_N)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^{\gamma} \otimes 1$$

in (4.5b), for α a weight of V_{N+a+2n}.

PROOF. If the tensor $(**)$ has total weight 0, then the nonnegative elements r_1 and p_1 in γ must satisfy the relation $r_1 = N + p_1$. However, because the e_1 coefficient of each root of $\Delta(\mathfrak{u})$ is at least 1, then even for $\alpha = -(N + a + 2n)e_1$, no choice of γ for which $r_1 = p_1 + N$ will make the e_1 coefficient of (weight of $(**)$) vanish. Because $\alpha = -(N + a + 2n)e_1$ is the most negative weight of V_{N+a+2n}, the result follows.

CONVENTION. For α a weight of V_{N+a+2n} and β a weight of W_{Ne_2}, the generic tensor

$$(4.7a) \qquad (v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^{\gamma} \otimes 1$$

in (4.5a) is, unless otherwise stated, assumed to be of total weight 0. Similarly, the generic tensor

$$(4.7b) \qquad (v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \otimes X_{\mathfrak{u}}^{\gamma} \otimes 1$$

in (4.5b) is, unless otherwise stated, assumed to be of total weight 0.

From Lemma 4.6, we see that $\beta \neq Ne_2$ in (4.7a). We will use this in the next result.

Proposition 4.8 $\mathcal{C}_2^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$ is 0.

PROOF. This proof is a generalization of the proof given in Proposition 3.6. A general element of $\mathcal{C}_2^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$ looks like

$$\sum c_{\alpha,\beta,\gamma}(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^\gamma \otimes 1. \tag{4.8a}$$

Without loss of generality, we may assume that each of the summands in (4.8a) is distinct. Let β_0 be a largest weight of W_{Ne_2} that appears in (4.8a). Here the term "largest" means β_0 is the largest weight in the set of weights to which it is comparable. According to Lemma 4.6, $\beta_0 \neq Ne_2$. Notice we can rewrite (4.8a) as

$$\sum c_{\alpha,\beta_0,\gamma}(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^\gamma \otimes 1 + \sum_{\beta \neq \beta_0} c_{\alpha,\beta,\gamma}(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^\gamma \otimes 1. \tag{4.8b}$$

For μ any root of $\Delta^+(\mathfrak{l})$, we have

$$X_\mu(4.8\text{b}) = \sum c_{\alpha,\beta_0,\gamma}\, X_\mu(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \wedge X_{2e_1} \otimes X_{\mathfrak{u}}^\gamma \otimes 1$$
$$+ (\text{sums of pure tensors with } (V_{N+a+2n} \otimes W_{Ne_2})^* \text{ weight} \neq \alpha + \beta_0 + \mu).$$

The expression $X_\mu(v_\alpha \otimes w_\beta)^*$ can be 0 even if $\alpha + \beta_0 + \mu$ is a weight of $(V_{N+a+2n} \otimes W_{Ne_2})^*$. However, $X_\mu(v_\alpha \otimes w_\beta)^*$ cannot be 0 for **every** μ

in $\Delta^+(\mathfrak{l})$ since this would imply the vector is of the form $(v_\alpha \otimes w_{Ne_2})^*$ (see [K3, Theorem 4.28(c)]). Therefore, let us choose a μ for which the expression $X_\mu(v_\alpha \otimes w_\beta)^* \neq 0$. As in the proof of Proposition 3.6, this implies that all $c_{\alpha,\beta_0,\gamma}$ are 0. Repeating this argument for every weight that appears in (4.8a) shows that all $c_{\alpha,\beta,\gamma}$ are 0 and we are done.

Corollary 4.9 $\dim\{ C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}\} = 1$.

PROOF. This follows immediately from Propositions 4.2 and 4.8.

The next result will prove useful when reconstructing the $\mathfrak{p}$ action.

Lemma 4.10 If $(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \otimes X_\mathfrak{u}^\gamma \otimes 1$ is a pure tensor that appears as a summand in an element of $C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$, then $\alpha = -(N+a+2n)e_1$.

PROOF. Using the technique of proof found in the previous proposition, we can prove that any vector

$$\sum c_{\alpha,\beta,\gamma}(v_\alpha \otimes w_\beta)^* \otimes X_{-2e_1} \otimes X_\mathfrak{u}^\gamma \otimes 1 \tag{4.10a}$$

with no β equal to Ne_2, is in $(4.5a)^L$ if and only if all $c_{\alpha,\beta,\gamma}$ are 0. With this in mind, note that if $\alpha > -(N+a+2n)e_1$, then the "total weight = 0" condition forces the tensor (4.7b) to have $\beta \neq Ne_2$. Reconsidering the sum (4.10a) as an element of (4.5a), but now allowing $\alpha = -(N+a+2n)e_1$ as a weight we can rewrite (4.10a) as

$$\begin{aligned}(4.10\text{b}) \quad &\big(\text{sum of pure tensors with } \alpha = -(N+a+2n)e_1\big) \\ &\qquad + \big(\text{sum of pure tensors with } \alpha > -(N+a+2n)e_1\big).\end{aligned}$$

From the comments above, the second sum here will have no β equal to Ne_2. The statement made at the beginning of this proof indicates that the second sum is L invariant if and only if it is 0. Hence the vector (4.10b) is L invariant implies that only the first summand is nonzero. Our proof is completed.

REMARK. We observe that in the preceding discussion, tensors of the form

$$(v_\alpha \otimes w_\beta)^* \otimes X_{2e_1} \otimes X_{\mathfrak{u}}^{\gamma} \otimes 1$$

have been ignored. The reason for this is that there are no total weight 0 tensors of this form (this generalizes the remark of Chapter 3, Section 1).

At this point, rather than produce a basis element for the vector space $\mathcal{C}_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$, which incidentally is also a basis element for the kernel of

$$\partial^\natural : \mathcal{C}_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)} \to \mathcal{C}_0^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)},$$

we observe that any basis element must contain as a summand a nonzero multiple of

$$(4.11) \qquad (v_{-(N+a+2n)} \otimes w_{Ne_2})^* \otimes X_{-2e_1} \otimes X_{e_1-e_2}^N \otimes 1.$$

This follows immediately from the first statement in the proof of Lemma 4.10. Using this, we can prove that reconstruction of the $\mathfrak{p}$ action for the space of L invariants of

$$(4.12) \quad (V_{N+a+2n} \otimes W_{Ne_2}) \otimes (V_{N+a+2n} \otimes W_{Ne_2})^* \otimes \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+2(n+1))e_1}$$

is a solvable problem. As in (4.5a) and (4.5b), we will use the notation $(4.12)^L$ to refer to the space of L invariants. Notice that the spaces (4.12) and $(4.12)^L$ are the spaces (1.15b) and (1.15a), respectively, for μ and Z as mentioned in the comments after (4.5a). Before proving the theorem about $\mathfrak{p}$ reconstruction, however, we need one more lemma.

Lemma 4.13 Consider the expression

$$(***) \qquad X_\alpha(X_{2e_1}^m X_{e_1+e_2}^{r_1} X_{e_1-e_2}^{p_1} \cdots X_{e_1+e_{n+1}}^{r_n} X_{e_1-e_{n+1}}^{p_n})$$

in $U(\mathfrak{g})$, where $\alpha \in \Delta(\mathfrak{p})$. In what follows, only the nonzero exponents are listed. Given that $0 \leq p_1 \leq N$,

Case (1): if $(***)$ has total weight $(N+1)(e_1 - e_2)$, then $\alpha = e_1 - e_2$ and $p_1 = N$

Case (2): if $(***)$ has total weight $(N-1)(e_1 - e_2)$, then either

(a) $\alpha = e_1 - e_2$ and $p_1 = N - 2$ OR

(b) $\alpha = -e_1 - e_2$ and $m = 1$ and $p_1 = N - 2$ OR

(c) $\alpha = -e_1 - e_2$ and $r_1 = 1$ and $p_1 = N - 1$ OR

(d) $\alpha = -e_1 - e_2$ and $p_1 = N - 2$ and $r_j = p_j = 1$ for some $j > 1$ OR

(e) $\alpha = -e_1 + e_2$ and $p_1 = N$ OR

(f) For some $j > 1$, $\alpha = -e_1 - e_j$ and $p_1 = N - 1$ and $r_j = 1$ OR

(g) For some $j > 1$, $\alpha = -e_1 + e_j$ and $p_1 = N - 1$ and $p_j = 1$.

Proof. The weight of the expression $(***)$ is

$$\alpha + (m + r_1 + p_1 + \cdots + r_n + p_n)e_1 + \sum_{i=1}^{n}(r_i - p_i)e_{i+1}.$$

Let us consider Case (1) first. The choices for α are e_1+e_j, e_1-e_j, $-e_1+e_j$ and $-e_1-e_j$, $2 \le j \le n+1$. It is easier if we consider $j=2$ and $j>2$ separately. When $j=2$, the choice $\alpha = e_1+e_2$ forces the equations

$$\begin{aligned} r_i &= p_i, \qquad i>1 \\ m+r_1+p_1+\cdots+r_n+p_n+1 &= \quad N+1 \\ r_1-p_1 \qquad\qquad\qquad +1 &= -(N+1). \end{aligned}$$

Combining these gives us the equation

$$m+\sum_{i=1}^{n} r_i+2=0,$$

which is impossible to solve, since all exponents must be nonnegative. With the exception of $\alpha = e_1-e_2$, all other choices of α will force the condition $p_1 \ge N+1$, which is not allowed by assumption. This is true for both $j=2$ and $j>2$. However, when $\alpha = e_1-e_2$, we have the equations

$$\begin{aligned} r_i &= p_i, \qquad i>1 \\ m+r_1+p_1+\cdots+r_n+p_n+1 &= \quad N+1 \\ r_1-p_1 \qquad\qquad\qquad -1 &= -(N+1). \end{aligned}$$

The unique solution to this system is $p_1=N$ (all other exponents being 0). This takes care of Case (1).

For Case (2), we notice that (a) is really Case (1) with N replaced by $N-2$. This observation and the proof of Case (1) shows that for $\alpha = e_1 \pm e_j$,

$j \geq 2$, $\alpha = e_1 - e_2$ and $p_1 = N - 2$ is the only choice that gives a solution. We can therefore concentrate now solely on $\alpha = -e_1 \pm e_j$, $j \geq 2$. As in Case (1), we consider $j = 2$ and $j > 2$ separately. When $j = 2$ the choice $\alpha = -e_1 - e_2$ forces the equations

$$\begin{aligned} r_i &= p_i, \qquad i > 1 \\ m + r_1 + p_1 + \cdots + r_n + p_n - 1 &= \quad N - 1 \\ r_1 - p_1 \qquad\qquad\qquad - 1 &= -(N - 1). \end{aligned}$$

Combining these gives

$$m + \sum_{i=1}^{n} r_i - 1 = 0 \qquad \text{and} \qquad p_1 = r_1 + N - 2$$
$$r_i = p_i \qquad i > 1.$$

The choice $m = 1$ gives (b) of Case (2). The choice $r_1 = 1$ gives (c) of Case (2), and the choice $r_i = 1$ for some $i > 1$ gives (d) of Case (2).

Using the same techniques as above, we verify that (e), (f), and (g) are correct. We leave the proof to the reader.

REMARKS.

1) The lemma just proved will be used in the proof of Theorem 4.14. Case 2, (a) will not be relevant in this proof.

2) Notice that , for β any weight of W_{Ne_2}, the pure tensor

$$(v_{-(N+a+2n)} \otimes w_\beta)^* \otimes X_{-2e_1} \otimes X_{2e_1}^{m} X_{e_1+e_2}^{r_1} X_{e_1-e_2}^{p_1} \cdots X_{e_1+e_{n+1}}^{r_n} X_{e_1-e_{n+1}}^{p_n} \otimes 1$$

in the space

$$(V_{N+a+2n} \otimes W_{Ne_2})^* \otimes \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{(a+2(n+1))e_1}$$

having total weight 0 implies that the exponent p_1 satisfies $p_1 \leq N$. This explains the reason for the hypothesis of Lemma 4.13.

3) In the statement of the next theorem, the term "effectively computable" will mean that we can produce a list of formulas describing the action of X_α, $\alpha \in \mathfrak{p}$, as we have done in Theorems 3.29 and 3.37. We do **not** mean this term in any formal way, such as a logician or philosopher might interpret it.

Theorem 4.14 Let P and Q be nonnegative integers. Under the assumption that there exists a explicit decomposition of $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(P,Q)}$ into K irreducible components, then reconstruction of the $\mathfrak{p}$ action for (4.12) is effectively computable.

PROOF. We begin with some definitions. Borrowing notation from Chapter 3, we let

$$\mathrm{Ker}\ \partial^\natural\big|_{(N+a+2n,N)}$$

be the basis vector of Ker $\partial^\natural \subseteq C_1^*(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$ that has coefficient 1 for the summand (4.11) (see Definition 3.18). If Ker $\partial^\natural\big|_{(N+a+2n,N)}$ is the vector

$$\sum_{\beta,\gamma} c_{\beta,\gamma}(v_{-(N+a+2n)} \otimes w_\beta)^* \otimes X_{-2e_1} \otimes X_{\mathfrak{u}}^\gamma \otimes 1,$$

then for α' a weight in V_{N+a+2n} and β' a weight in W_{Ne_2},

$$(4.14a) \qquad v_{\alpha'} \otimes w_{\beta'} \otimes \text{Ker } \partial^{\natural}|_{(N+a+2n,N)}$$

is the element

$$\sum_{\beta,\gamma} v_{\alpha'} \otimes w_{\beta'} \otimes (v_{-(N+a+2n)} \otimes w_{\beta})^* \otimes X_{-2e_1} \otimes X_{\mathfrak{u}}^{\gamma} \otimes 1$$

of $(4.12)^L$ (see Definition 3.19). Because the boundary is 0 for $(4.12)^L$, then the set of vectors (4.14a), as $v_{\alpha'}$ ranges over the basis of weight vectors of V_{N+a+2n} and $w_{\beta'}$ ranges over the basis of weight vectors of W_{Ne_2}, is a homology basis for Ker $\partial \subseteq C_{1,K}(\mathbb{C}_{ae_1})|_{(N+a+2n,N)}$. We observed this in Chapter 3, Section 1 for $n = 1$. We note that the action of a vector $X \in \mathfrak{p}$ on a basis vector (4.14a) is

$$\alpha_1(X \otimes v_{\alpha'} \otimes w_{\beta'} \otimes \text{Ker } \partial^{\natural}|_{(N+a+2n,N)}),$$

where α_n is the map defined in the statement of the Duflo-Vergne proposition (stated in Chapter 1, Section 2). We also observed this in Chapter 3, Section 1 for $n = 1$. Our goal is therefore calculating

$$\alpha_1(X \otimes v_{\alpha'} \otimes w_{\beta'} \otimes \text{Ker } \partial^{\natural}|_{(N+a+2n,N)}).$$

Using Proposition 4.2, we only need consider those portions in K types $(N+1+a+2n, N+1)$ and $(N-1+a+2n, N-1)$. Using Lemma 4.13, Case (1), we see that the contribution to the K type $(N+1+a+2n, N+1)$ can be determined from

$$\alpha_1(X \otimes v_{\alpha'} \otimes w_{\beta'} \otimes (v_{-(N+a+2n)} \otimes w_{Ne_2})^* \otimes X_{-2e_1} \otimes X_{e_1-e_2}^N \otimes 1),$$

in particular the term

$$\langle \mathrm{Ad}(\cdot)^{-1} X, X_{e_1-e_2} \rangle (\pi(\cdot)^{-1} v_{\alpha'} \otimes w_{\beta'}, v_{N+a+2n} \otimes w_{-N}) \otimes X_{-2e_1} \otimes X_{e_1-e_2}(X_{e_1-e_2}^N \otimes 1),$$

where we have dropped $(N+a+2n, N)$ from $\pi_{(N+a+2n,N)}$. This equals

$$\langle (\mathrm{Ad} \otimes \pi)(\cdot)^{-1} X \otimes v_{\alpha'} \otimes w_{\beta'}, X_{e_1-e_2} \otimes v_{N+a+2n} \otimes w_{-Ne_2} \rangle \otimes X_{e_1-e_2}^{N+1} \otimes 1. \tag{4.14b}$$

By assumption, $X_{e_1-e_2} \otimes v_{N+a+2n} \otimes w_{-Ne_2}$ will decompose as

$$h_1(v_{N+a+2n+1} \otimes w_{-(N+1)e_2}) + (\text{pure tensors in other } K \text{ types}),$$

for some (possibly complex) number h_1. Thus we can rewrite (4.14b) as

$$h_1 \left((N+1+a+2n, N+1) \text{ component of } X \otimes v_{\alpha'} \otimes w_{\beta'} \right) \otimes (v_{-(N+1+a+2n)} \otimes w_{(N+1)e_2})^* \otimes X_{-2e_1} \otimes X_{e_1-e_2}^{N+1} \otimes 1.$$

Suppose now that for our chosen X, α' and β', we have

$$(4.14c) \quad X \otimes v_{\alpha'} \otimes w_{\beta'} = h_1'(\text{vector}_1 \text{ in } K \text{ type } (N+1+a+2n, N+1))$$
$$+ h_2'(\text{vector}_2 \text{ in } K \text{ type } (N-1+a+2n, N-1))$$
$$+ (\text{vectors in } K \text{ types that do not appear in the discrete series})$$

under the assumed $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_{(N+a+2n,N)}$ decomposition. This means that

$$\big((N+1+a+2n, N+1) \text{ component of } X \otimes v_{\alpha'} \otimes w_{\beta'}\big)$$
$$= h_1'(\text{vector}_1 \text{ in } K \text{ type } (N+1+a+2n, N+1))$$

and it follows that the $(N+1+a+2n, N+1)$ portion of

$$\alpha_1(X \otimes v_{\alpha'} \otimes w_{\beta'} \otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2n,N)})$$

is

$$h_1\, h_1'\,(\text{vector}_1 \otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+1+a+2n,N+1)}).$$

This completes the calculation for the K type $(N+1+a+2n, N+1)$.

In order to calculate the portion of

$$\alpha_1(X \otimes v_{\alpha'} \otimes w_{\beta'} \otimes \mathrm{Ker}\ \partial^{\natural}|_{(N+a+2n,N)})$$

for the K type $(N-1+a+2n, N-1)$, we first must have a more detailed description of the indices $c_{\beta,\gamma}$ appearing in $\mathrm{Ker}\ \partial^{\natural}|_{(N+a+2n,N)}$. In the hopes of clearing up confusion rather than creating more confusion, we introduce some alternate notation for certain $c_{\beta,\gamma}$ appearing in Ker $\partial^{\natural}\big|_{(N+a+2n,N)}$. In what follows, $2 < j \leq n+1$:

Monomial in $U(\mathfrak{u})$	Notation for corresponding $c_{\beta,\gamma}$
$X_{e_1+e_2}X_{e_1-e_2}^{N-1}$	c_2
$X_{e_1-e_2}^{N-1}X_{e_1+e_j}$	c_j
$X_{e_1-e_2}^{N-1}X_{e_1-e_j}$	d_j
$X_{2e_1}X_{e_1-e_2}^{N-2}$	e
$X_{e_1-e_2}^{N-2}X_{e_1+e_j}X_{e_1-e_j}$	f_j.

Using this notation, Lemma 4.13, Case (2), and the fact that the coefficient of the pure tensor in Ker $\partial^{\natural}\big|_{(N+a+2n,N)}$ corresponding to the monomial $X_{e_1-e_2}^{N}$ is 1, we see that the the portion of

$$\alpha_1(X \otimes\, v_{\alpha'} \otimes w_{\beta'} \otimes \text{ Ker } \partial^{\natural}\big|_{(N+a+2n,N)})$$

for the K type $(N-1+a+2n, N-1)$ comes from the sum of vectors

$$\text{(4.14d)}\quad \langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1+e_2}\otimes v_{N+a+2n}\otimes w_{-Ne_2}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1+e_2}(X_{e_1-e_2}^{N}\otimes 1)$$

$$+\, c_2\,\langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1-e_2}\otimes v_{N+a+2n}\otimes w_{-(N-2)e_2}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1-e_2}(X_{e_1+e_2}X_{e_1-e_2}^{N-1}\otimes 1)$$

$$+\sum_{j=3}^{n+1} c_j \{\langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1-e_j}\otimes v_{N+a+2n}\otimes w_{-Ne_2+e_2+e_j}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1-e_j}(X_{e_1-e_2}^{N-1}X_{e_1+e_j}\otimes 1)\}$$

$$+\sum_{j=3}^{n+1} d_j \{\langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1+e_j}\otimes v_{N+a+2n}\otimes w_{-Ne_2+e_2-e_j}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1+e_j}(X_{e_1-e_2}^{N-1}X_{e_1-e_j}\otimes 1)\}$$

$$+ e\,\langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1-e_2}\otimes v_{N+a+2n}\otimes w_{-Ne_2+2e_2}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1-e_2}(X_{2e_1}X_{e_1-e_2}^{N-2}\otimes 1)$$

$$+\sum_{j=3}^{n+1} f_j \{\langle(\mathrm{Ad}\otimes\pi)(\cdot)^{-1}X\otimes v_{\alpha'}\otimes w_{\beta'}, X_{-e_1-e_2}\otimes v_{N+a+2n}\otimes w_{-Ne_2+2e_2}\rangle$$

$$\otimes X_{-2e_1}\otimes X_{-e_1-e_2}(X_{e_1-e_2}^{N-2}X_{e_1+e_j}X_{e_1-e_j}\otimes 1)\}.$$

Under the assumed $\mathrm{Ad}|_{\mathfrak{p}}\otimes\pi_{(N+a+2n,N)}$ decomposition, we can write

$$X_{-e_1-e_j}\otimes v_{N+a+2n}\otimes w_{-Ne_2+e_2+e_j} = m_j(\,v_{N-1+a+2n}\otimes w_{-(N-1)e_2})$$

$$+\,(\text{vectors in other } K \text{ types}),$$

$$X_{-e_1+e_j}\otimes v_{N+a+2n}\otimes w_{-Ne_2+e_2-e_j} = n_j(\,v_{N-1+a+2n}\otimes w_{-(N-1)e_2})$$

$$+\,(\text{vectors in other } K \text{ types}),$$

for $2\le j\le n+1$. Also, by means of Cartan's Theorem, we can write

(4.14e)

$$X_{-e_1-e_2}(X_{e_1+e_2}X_{e_1-e_2}^{N-1}\otimes 1) = p_2X_{e_1-e_2}^{N-1}\otimes 1+\sum(\text{other monomials})\otimes 1$$

$$X_{-e_1-e_j}(X_{e_1-e_2}^{N-1}X_{e_1+e_j}\otimes 1) = p_j X_{e_1-e_2}^{N-1}\otimes 1 + \sum(\text{other monomials})\otimes 1$$

$$X_{-e_1+e_j}(X_{e_1-e_2}^{N-1}X_{e_1-e_j}\otimes 1) = q_j X_{e_1-e_2}^{N-1}\otimes 1 + \sum(\text{other monomials})\otimes 1$$

$$X_{-e_1-e_2}(X_{2e_1}X_{e_1-e_2}^{N-2}\otimes 1) = s_2 X_{e_1-e_2}^{N-1}\otimes 1 + \sum(\text{other monomials})\otimes 1$$

and also the formula

$$X_{-e_1-e_2}(X_{e_1-e_2}^{N-2}X_{e_1+e_j}X_{e_1-e_j}) = t_j X_{e_1-e_2}^{N-1}\otimes 1 + \sum(\text{other monomials})\otimes 1.$$

In each on the previous formulas, $2 < j \leq n+1$.

If h_2' is the sum of terms

$$h_2' = n_2q_2 + c_2m_2p_2 + e\,m_2s_2 + \sum_{j=3}^{n+1}(c_jm_jp_j + d_jn_jq_j + f_jm_2t_j),$$

then it is clear from formulas (4.14c), (4.14d), and (4.14e) that the $(N-1+a+2n, N-1)$ portion of

$$\alpha_1(X\otimes v_{\alpha'}\otimes w_{\beta'}\otimes \text{Ker }\partial^{\natural}|_{(N+a+2n,N)})$$

is

$$h_2\,h_2'\,(\text{vector}_2\otimes \text{Ker }\partial^{\natural}|_{(N-1+a+2n,N-1)}).$$

Since $(N+1+a+2n, N+1)$ and $(N-1+a+2n, N-1)$ are the only two K types in the $\mathrm{A}_{\mathfrak{q}}(\lambda)$ discrete series to which $\alpha_1(X\otimes v_{\alpha'}\otimes w_{\beta'}\otimes \text{Ker }\partial^{\natural}|_{(N+a+2n,N)})$ maps, we are done.

CHAPTER 5
RESULTS FOR GENERAL QUATERNIONIC DISCRETE SERIES

In this chapter, we present some results that are true for a general quaternionic discrete series, defined in Chapter 1 (pg. 7). We will consider only the $A_{\mathfrak{q}}(\lambda)$ discrete series (Definition 1.18). In what follows, $\mathfrak{p}$ refers to the subspace of $\mathfrak{g}$ built from noncompact roots. Also, β refers to the highest weight of $\mathfrak{k}$ (see Theorem 1.6(a)), and root vectors X_α are constructed by means of Cartan's Theorem (stated in Chapter 4). The term α_0 refers to the largest noncompact root of $\mathfrak{g}$ (not to be confused with the map α_n referring to the proposition of Duflo-Vergne, stated in Chapter 1).

The first main result is a concrete realization of the multiplicity space of $A_{\mathfrak{q}}(\lambda)$ for the minimal K type $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$, hereafter abbreviated by Λ, of $A_{\mathfrak{q}}(\lambda)$ (see Definition 3.3a). In fact, we shall show that $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_\Lambda$ is 0, and as a result, the multiplicity space mentioned above will be Ker $\partial^{\natural} \subseteq \mathcal{C}_1^*(\mathbb{C}_\lambda)|_\Lambda$. We precede this result with two lemmas; the first is used in the proof of the second, and the second will be used throughout this chapter.

Lemma 5.1 Suppose G_1 and G_2 are compact groups with π_1, π_2 irreducible representations of G_1 and G_2, respectively, on finite-dimensional complex Hilbert spaces V_1 and V_2. Then π_1 and π_2 may be assumed to be unitary, by [K3, Proposition 1.6]. If the representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$

on $V_1 \otimes V_2$ is given by

$$(\pi_1 \otimes \pi_2)(g_1, g_2) = \pi_1(g_1) \otimes \pi_2(g_2) \qquad \text{for } g_1 \in G_1 \text{and } g_2 \in G_2,$$

then $\pi_1 \otimes \pi_2$ is an irreducible representation. Conversely, if π is an irreducible representation of $G_1 \times G_2$, then $\pi \cong \pi_1 \otimes \pi_2$, where π_1 is irreducible for G_1 and π_2 is irreducible for G_2.

PROOF. To show the first statement, namely that $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$, note that it is sufficient to show that if

$$\Phi: V_1 \otimes V_2 \to V_1 \otimes V_2$$

is a linear operator commuting with all $(\pi_1 \otimes \pi_2)(g_1, g_2)$, then Φ is scalar. The reason for sufficiency is:

> If Z is an invariant subspace, then $Z^\perp$ is invariant since $\pi_1 \otimes \pi_2$ is unitary and it follows that $V_1 \otimes V_2 = Z \oplus Z^\perp$. Let Φ be the identity on Z and 0 on $Z^\perp$. One can check that Φ commutes with $(\pi_1 \otimes \pi_2)(g_1, g_2)$.

Let us therefore assume that Φ is a linear operator satisfying the above condition. We wish to show that Φ is scalar. Let us take a basis of operators on $V_1 \otimes V_2$ of the form $A_i \otimes B_j \in \mathrm{End}(V_1) \otimes \mathrm{End}(V_2)$. If $\Phi = \sum_{i,j} c_{i,j}(A_i \otimes B_j)$, we get the following identities by considering $g_2 = 1$ in $(\pi_1 \otimes \pi_2)(g_1, g_2)$:

$$\begin{aligned}
\Phi(\pi_1(g_1) \otimes Id) &= (\pi_1(g_1) \otimes Id)\Phi \\
\sum_{i,j} c_{i,j}(A_i \otimes B_j)(\pi_1(g_1) \otimes Id) &= \sum_{i,j} c_{i,j}(\pi_1(g_1) \otimes Id)(A_i \otimes B_j) \\
\sum_{i,j} c_{i,j}(A_i\, \pi_1(g_1) \otimes B_j) &= \sum_{i,j} c_{i,j}(\pi_1(g_1)A_i \otimes B_j).
\end{aligned}$$

Because $\{B_j\}$ is a basis for $\mathrm{End}(V_2)$, the last equation tells us that

$$\sum_i c_{i,j} A_i\, \pi_1(g_1) = \sum_i c_{i,j}\pi_1(g_1)A_i \qquad \text{for each } j.$$

This equation says that for each j, $\sum_i c_{i,j}A_i$ commutes with $\pi_1(G_1)$. By a corollary of Schur's Lemma [K3, Corollary 1.9], it follows that $\sum_i c_{i,j}A_i = d_j Id$ for each j. Thus, we can write

$$\begin{aligned}
\Phi = \sum_j \sum_i c_{i,j} A_i \otimes B_j &= \sum_j d_j(Id \otimes B_j) \\
&= Id \otimes \sum_j d_j B_j \\
&= Id \otimes B.
\end{aligned}$$

We can repeat this argument with Φ and the linear operators $Id \otimes \pi_2(g_2)$ ($g_1 = 1$ in $\pi_1 \otimes \pi_2(g_1, g_2)$) to conclude that B is also scalar. From this we see that Φ is scalar and therefore $\pi_1 \otimes \pi_2$ is irreducible.

Conversely, if π is irreducible for $G_1 \times G_2$, then the irreducible representations from the $\pi_1 \otimes \pi_2$ construction above already give enough for the expansion of matrix coefficients.

Lemma 5.2 Suppose $K = SU(2) \times L_{ss}$, with L_{ss} being the semisimple part of the compact group L. Let Λ' be $\Delta^+(\mathfrak{k})$ dominant and integral. If $V_{\Lambda'}$ is an irreducible representation of K with highest weight Λ', then the decomposition of $V_{\Lambda'}$ into irreducible L representations is given by the formula

$$V_{\Lambda'} = \sum_{k=0}^{2M} V_{\Lambda' - k\beta}, \tag{5.2a}$$

with M a nonnegative rational number such that $2M \in \mathbb{Z}$.

PROOF. For our situation of quaternionic discrete series, $L = S^1 \times L_{ss}$, $S^1 \subset SU(2)$ being the Cartan subgroup. From Lemma 5.1, it follows that an irreducible representation of K with highest weight Λ' is of the form $V_{\Lambda_1'} \otimes W_{\Lambda_2'}$, where $V_{\Lambda_1'}$ is an irreducible representation of $SU(2)$ with highest weight Λ_1', $W_{\Lambda_2'}$ is analogously defined for L_{ss}, and $\Lambda_1' + \Lambda_2' = \Lambda'$. Similarly, an irreducible representation of L is of the form

(irreducible representation of $S^1 \subset SU(2)$)

$\times$ (irreducible representation of L_{ss}).

If $\{v_{\Lambda_1' - k\beta}\}$, $0 \leq k \leq \dfrac{2\langle \Lambda_1', \beta \rangle}{|\beta|^2}$ is a basis of weight vectors for $V_{\Lambda_1'}$ (weights given by the subscripts), then each $v_{\Lambda_1' - k\beta}$ is a basis for the one-dimensional (therefore irreducible) representation $\mathbb{C}v_{\Lambda_1' - k\beta}$ of $S^1 \subset SU(2)$. Again by Lemma 5.1, the set

$$\mathbb{C}v_{\Lambda_1' - k\beta} \otimes W_{\Lambda_2'}, \qquad 0 \leq k \leq \frac{2\langle \Lambda_1', \beta \rangle}{|\beta|^2}$$

is a collection of irreducible L representations in $V_{\Lambda'}$. If $M = \dfrac{\langle \Lambda_1{}', \beta \rangle}{|\beta|^2}$, then clearly

$$V_{\Lambda'} = \sum_{k=0}^{2M} \mathbb{C}v_{\Lambda_1{}'-k\beta} \otimes W_{\Lambda_2{}'}.$$

Rewriting $\mathbb{C}v_{\Lambda_1{}'-k\beta} \otimes W_{\Lambda_2{}'}$ as $V_{\Lambda'-k\beta}$ gives the statement of the lemma.

Throughout the remainder of this chapter, we shall use the notation v to refer to vectors of V_Λ' or its contragredient rather than $v \otimes w$, as was done in previous chapters (notably Chapters 3 and 4).

Using the notation $V_\Lambda \otimes (V_\Lambda)^*$ as the K isotypic subspace of $R(K)$ (see Definition 1.7), we have

Proposition 5.3 A basis vector for the multiplicity space of $\mathrm{A}_{\mathfrak{q}}(\lambda)$ for the minimal K type Λ is

(5.3a) $$(v_{-\Lambda})^* \otimes X_{-\beta} \otimes 1 \otimes 1,$$

where $(v_{-\Lambda})^*$ is a fixed vector in $(V_\Lambda)^*$ with weight given by its subscript.

Proof. As mentioned in the discussion before the statement of the proposition, we show that $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_\Lambda$ is 0. The proof is similar to the work done in Propositions 3.6 and 4.8 for the special case of the minimal K type. Our first step in proving $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_\Lambda$ is 0 is showing that the weights of $(V_\Lambda)^*$ are

(5.3b) $$\Lambda, \Lambda - \beta, \Lambda - 2\beta, \cdots, -\Lambda.$$

We can show (5.3b) by showing that the weights of V_Λ are those listed in (5.3b) and by noting that the weights of the contragredient representation V^* are exactly the negatives of the weights of a representation V (see [K2, pg.55]). The weight Λ is a multiple of β, since the set of roots $\beta \cup \{\text{simple roots of } \mathfrak{l}\}$ form a basis (over $\mathbb{R}$) for the space of all linear functional of $\mathfrak{k}$, and $\Lambda \perp \Delta(\mathfrak{l})$ (from Definition 1.18). Also, because λ is dominant and algebraically integral with respect to $\Delta^+(\mathfrak{k})$, it is a nonnegative integer multiple of β. Let us therefore write $\Lambda = M_0\beta$ for some nonnegative integer M_0. Humphreys shows [Hum, pg.125, Exercise 1], that if $\alpha \in \Delta(\mathfrak{k})$ and $0 \leq k \leq \dfrac{2\langle\Lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ for some symmetric, positive definite, bilinear form $\langle\cdot,\cdot\rangle$ on the Euclidean space of linear functionals of $\mathfrak{k}$, then

$$(5.3c) \qquad m(\Lambda - k\alpha) = 1,$$

where $m(\alpha)$ refers to the multiplicity of the weight α in V_Λ. Using $\alpha = \beta$ and making the substitution $M_0\beta$ for Λ, we see that

$$m((M_0 - k)\beta) = 1 \qquad \text{for } 0 \leq k \leq 2M_0.$$

This shows that the weights (5.3b) are included in the set of weights of V_Λ. To see that these are the only weights, we note that V_Λ decomposes as the direct sum $V_\Lambda = \bigoplus V_{\Lambda - k\beta}$ with $0 \leq k \leq 2M_0$, by Lemma 5.2. Also, each weight $\Lambda - k\beta$ is orthogonal to $\Delta(\mathfrak{l})$ and hence $V_{\Lambda-k\beta}$ is a one-dimensional representation of L. This statement proves that the only weights of V_Λ are in fact those listed in (5.3b). From the remarks above, the weights of $(V_\Lambda)^*$

are also those listed in (5.3b). Let us fix nonzero vectors $v_{\Lambda-k\beta} \in V_{\Lambda-k\beta}$ and define linear functionals $(v_{-(\Lambda-k\beta)})^*$ by $(v_{-(\Lambda-k\beta)})^* = (\cdot, v_{\Lambda-k\beta})$. Consider now the space

$$(V_\Lambda)^* \otimes \bigwedge^2((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}, \tag{5.3d}$$

where we have used the equality $\Lambda + \beta = \lambda + 2\delta(\mathfrak{u})$. The space of L invariants of (5.3d) is $\mathcal{C}_2^*(\mathbb{C})|_\Lambda$. The possible weights of a pure tensor in (5.3d) are

$$M_0\beta - k\beta + \sum_{\alpha\in\Delta(\mathfrak{u})} n_\alpha \alpha + (M_0+1)\beta, \tag{5.3e}$$

where $0 \leq k \leq 2M_0$ and each n_α is a nonnegative integer. Since $\langle \beta, \alpha \rangle > 0$ for every $\alpha \in \Delta(\mathfrak{u})$, then if we consider the inner product

$$\langle \beta, (2M_0 + 1 - k)\beta + \sum_{\alpha\in\Delta(\mathfrak{u})} n_\alpha \alpha \rangle$$

we will never get 0 ($k \leq 2M_0$). Hence, any sum of weights (5.3e) is nonzero, and consequently, there are no nonzero tensors of total weight 0 in (5.3d). We conclude that $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_\Lambda$ is 0. From this it follows that Ker $\partial^\natural \subseteq \mathcal{C}_1^*(\mathbb{C}_\lambda)|_\Lambda$ is actually an equality of sets. Notice that the possible weights of a pure tensor in

$$(V_\Lambda)^* \otimes \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta} \tag{5.3f}$$

are

$$M_0\beta - k\beta \pm \beta + \sum_{\alpha\in\Delta(\mathfrak{u})} n_\alpha \alpha + (M_0+1)\beta, \tag{5.3g}$$

where k and n_α are as in (5.3e). Again using the fact that $\langle \beta, \alpha \rangle > 0$ for every $\alpha \in \Delta(\mathfrak{u})$, we see that a sum (5.3g) will be 0 if and only if $k = 2M_0$, $n_\alpha = 0$ for each $\alpha \in \Delta(\mathfrak{u})$, and we choose $-\beta$ from the set $\{\pm\beta\}$. It follows that the only possible nonzero elements in $\mathcal{C}_1^*(\mathbb{C}_\lambda)|_\Lambda$ are complex multiples of

$$(v_{-\Lambda})^* \otimes X_{-\beta} \otimes 1 \otimes 1, \tag{5.3h}$$

where $X_{-\beta} \in \bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k})$ with weight $-\beta$. In order to complete our proof, we need only check that the tensor (5.3h) is indeed L invariant. From our discussion prior to (5.3d), we know that $-\Lambda$ is an L highest and an L lowest weight in $(V_\Lambda)^*$. Therefore, by the Theorem of the Highest Weight, $X_\varepsilon(v_{-\Lambda})^* = X_{-\varepsilon}(v_{-\Lambda})^* = 0$ for all $\varepsilon \in \Pi(\mathfrak{l})$. The same is true of the action of $X_{\pm\varepsilon}$ on an element of $\mathbb{C}_{\Lambda+\beta}$. It follows that, for $\varepsilon \in \Delta(\mathfrak{l})$,

$$\begin{aligned} X_\varepsilon((v_{-\Lambda})^* \otimes X_{-\beta} \otimes 1 \otimes 1) & \\ &= (X_\varepsilon(v_{-\Lambda})^*) \otimes X_{-\beta} \otimes 1 \otimes 1 + (v_{-\Lambda})^* \otimes X_{-\beta} \otimes X_\varepsilon \otimes 1 \\ &= 0 + (v_{-\Lambda})^* \otimes X_{-\beta} \otimes 1 \otimes X_\varepsilon 1 \qquad \text{since } \mathfrak{l} \subseteq \bar{\mathfrak{q}} \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

From this calculation we conclude that the vector (5.3h) is L invariant and therefore forms a basis for $\mathcal{C}_1^*(\mathbb{C}_\lambda)|_\Lambda$, which in this case equals the multiplicity space of $\mathrm{A}_\mathfrak{q}(\lambda)$ for the minimal K type Λ. This completes the proof.

From Proposition 5.3 and Definition 3.3b, we can write a basis for Ker $\partial \subseteq C_{1,K}(\mathbb{C}_\lambda)|_\Lambda$ as

$$(5.4) \qquad v_{\Lambda - k\beta} \otimes \text{Ker } \partial^{\natural}\big|_\Lambda \qquad \text{for } 0 \leq k \leq 2M_0.$$

For each k, $v_{\Lambda-k\beta}$ is a fixed nonzero basis element of $V_{\Lambda-k\beta}$, and M_0 is the nonnegative integer satisfying $\Lambda = M_0\beta$. The collection (5.4) is in fact a homology basis for the minimal K type, because we showed in the proof of Proposition 5.3 that the space $C_2^*(\mathbb{C}_\lambda)|_\Lambda$ is 0.

Our ultimate goal in this chapter is a proof of the effective computability of the $\mathfrak{p}$ action on a basis vector in (5.4). In order to prove this, however we must prove two results that are interesting in their own right. The first result gives a general form of any K type appearing with nonzero multiplicity in $A_\mathfrak{q}(\lambda)$, and the proof shows that there is only one non-identity element in the set W^1 (defined in the proof of Proposition 4.2). The second result shows that $C_0^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ and $C_2^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ are 0 (recall α_0 is the largest noncompact root in Δ). The proof of the second result uses characters and it places a restriction on those K types μ in $A_\mathfrak{q}(\lambda)$ for which $C_0^*(\mathbb{C}_\lambda)|_\mu$ and $C_2^*(\mathbb{C}_\lambda)|_\mu$ may be nonzero.

Proposition 5.5 The only non-identity element of W^1 (defined in the proof of Proposition 4.2) is s_β, which maps β to $-\beta$ and leaves all other simple roots of $\mathfrak{k}$ fixed. Consequently, the only $\Delta^+(\mathfrak{k})$ linear functionals Λ' that appear with nonzero multiplicity in $A_\mathfrak{q}(\lambda)$ are of the form

$$(5.5a) \qquad \Lambda' = \Lambda + \eta,$$

where η is an element of $S(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{n}}$. The multiplicity of Λ' is the multiplicity of η in $S(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{n}}$.

PROOF. We know from the definition of W^1 and from the fact that $\Delta(\mathfrak{u}\cap\mathfrak{k}) = \beta$ that the only choices for $\Delta^+(w)$ (also defined in the proof of Proposition 4.2) are the empty set $\varnothing$ and the one element set $\{\beta\}$. For an element w in the Weyl group of K, Knapp and Vogan define [K-V, (4.133)] the **length** of w, denoted $l(w)$, by $l(w) = |\Delta^+(w)|$. We can easily verify that $l(w) = l(w^{-1})$. Knapp shows [K3, pp.80–81] that $l(w)$ is also the smallest number of factors needed to represent $w \in W$ as $w = s_{\alpha_{i_k}} \cdots s_{\alpha_{i_1}}$ with $\alpha_{i_k}, \cdots, \alpha_{i_1}$ in $\Pi(\mathfrak{k})$, a simple root system for $\mathfrak{k}$. Here, s_α is the **root reflection** defined by

$$s_\alpha(\phi) = \phi - \frac{2\langle\phi,\alpha\rangle}{|\alpha|^2}\alpha \qquad \text{for } \alpha \in \Delta \text{and } \phi \in (\mathfrak{h}_0)'_{\mathbb{R}}.$$

In this definition $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$, $(\mathfrak{h}_0)_{\mathbb{R}} = i\mathfrak{h}_0$, and $(\mathfrak{h}_0)'_{\mathbb{R}}$ denotes the (real) dual space of $(\mathfrak{h}_0)_{\mathbb{R}}$. Also, $\langle\cdot,\cdot\rangle$ in the definition of root reflection is some inner product defined on $(\mathfrak{h}_0)'_{\mathbb{R}}$. For a further discussion, see [K3, pg.69]. The definition of s_β in the statement of the proposition follows from the definition of the root reflection and the fact that β is orthogonal to all simple roots of $\mathfrak{l}$ with respect to the inner product mentioned in that definition.

Recall that the simple root system we choose for $\Delta(\mathfrak{k})$ is given by the set $\{\beta\} \cup \Pi(\mathfrak{l})$. From the statement above, the only choice of lengths for $w \in W^1$ is either 0 or 1. If $l(w) = 0$, then w is the identity. If $l(w) = 1$, then $w = s_\alpha$ for $\alpha \in \Pi(\mathfrak{k})$. Because $s_\alpha(\alpha) = -\alpha$ [K3, Lemma 4.8] for each simple root α, then $\alpha = \beta$ is the only choice that gives $\Delta^+({s_\alpha}^{-1}) = \{\beta\}$. Since ${s_\alpha}^{-1} = s_\alpha$ for all roots α, we conclude that s_β is the only non-identity element in W^1..

To prove the second part of the proposition, we recall from the statement of Theorem 8.29 of [K-V] (presented in proof of Proposition 4.2) that the multiplicity of a K type Λ' in $A_{\mathfrak{q}}(\lambda)$ is given by

$$\sum_{s \in W^1} (\det s)\mathcal{P}\big(s(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k}))\big),$$

where $\mathcal{P}(v)$ is the multiplicity of v as a weight in $(S(\mathfrak{u} \cap \mathfrak{p}))^{\mathfrak{l} \cap \mathfrak{n}}$. Let us consider the summand with $s = s_\beta$. We can rewrite this summand as

(5.5b)

$$\begin{aligned} &s_\beta(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k})) \\ &\qquad = s_\beta(\Lambda') + s_\beta(\delta(\mathfrak{k})) - \Lambda - \delta(\mathfrak{k}) \\ &\qquad = \Lambda' - \frac{2\langle \Lambda', \beta\rangle}{|\beta|^2}\beta + s_\beta(\delta(\mathfrak{k})) - \delta(\mathfrak{k}) - \Lambda \qquad \text{from definition of } s_\beta. \end{aligned}$$

Knapp shows [K3, Proposition 4.33] that $s_\beta(\delta(\mathfrak{k})) = \delta(\mathfrak{k}) - \beta$. Therefore, from the equations (5.5b), it follows that

$$s_\beta(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k})) = \Lambda' - \frac{2\langle \Lambda', \beta\rangle}{|\beta|^2}\beta - \Lambda - \beta. \tag{5.5c}$$

Because Λ' and Λ are both $\Delta^+(\mathfrak{k})$ dominant, one can easily verify that the inner product of (5.5c) with β is strictly negative. However, every weight that appears in $S(\mathfrak{u} \cap \mathfrak{p})$ has nonnegative inner product with β. Since the weights of $S(\mathfrak{u} \cap \mathfrak{p})^{\mathfrak{l} \cap \mathfrak{n}}$ are a subset of the weights of $S(\mathfrak{u} \cap \mathfrak{p})$, we see that the only contribution to the sum

$$\sum_{s \in W^1} (\det s) \mathcal{P}\big(s(\Lambda' + \delta(\mathfrak{k})) - (\Lambda + \delta(\mathfrak{k}))\big)$$

comes from $s = 1$, the identity. The second part of the proposition then follows immediately.

Proposition 5.6 $C_0^*(\mathbb{C}_\lambda)|_{\Lambda + \alpha_0}$ and $C_2^*(\mathbb{C}_\lambda)|_{\Lambda + \alpha_0}$ are 0.

PROOF. As mentioned above, this proof uses characters. The first part of the proof will use a general K type and only at the end of the proof will we consider the K type $\Lambda + \alpha_0$.. From Proposition 5.6, we know that a K type appearing in $A_{\mathfrak{q}}(\lambda)$ is of the form $\Lambda + \eta$, for η a weight in $S(\mathfrak{u} \cap \mathfrak{p})^{\mathfrak{l} \cap \mathfrak{n}}$. Consider the space

$$(V_{\Lambda+\eta})^*|_L \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}, \tag{5.6a}$$

where $(V_{\Lambda+\eta})^*|_L$ is the representation $(V_{\Lambda+\eta})^*$ of K restricted to L. The space of L invariants of (5.6a) is $C_0^*(\mathbb{C}_\lambda)|_{\Lambda+\eta}$. Because (5.6a) is an L representation space and L is compact, the Peter-Weyl Theorem says that (5.6a) breaks up as a direct sum of finite-dimensional irreducible L representations. The space $C_0^*(\mathbb{C}_\lambda)|_{\Lambda+\eta}$ is the subset of vectors in (5.6a) for which the action of L is the trivial action. In other words, we have nontrivial L invariance in (5.6a) if and only if the trivial representation of L appears with nonzero multiplicity in the decomposition of (5.6a) into irreducible L representations. Using Lemma 5.2 and the fact that the contragredient functor is additive [K-V, pg.118 and pg.839] we see that

$$(V_{\Lambda+\eta})^* = \sum_{j=0}^{N} (V_{\Lambda+\eta-j\beta})^*,$$

for some nonnegative integer N. Therefore, we can rewrite (5.6a) as

$$\sum_{j=0}^{N} (V_{\Lambda+\eta-j\beta})^* \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}. \tag{5.6b}$$

Also, because $U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta} = \bigoplus_{n \in \mathbb{Z}^+ \cup \{0\}} U^n(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}$ as L representations, then we have reduced the problem to determining whether the trivial representation appears in

$$(V_{\Lambda+\eta-j\beta})^* \otimes U^n(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}, \tag{5.6c}$$

for $0 \leq j \leq N$ (upper bound mentioned earlier) and any nonnegative integer n. This question can be answered by using a result that appears in [Che], namely **Corollary 1** on page 188. We state a slightly modified version here:

The number of times that a finite-dimensional irreducible representation τ_1 of a compact Lie group G appears in a finite dimensional representation τ is equal to

$$\int_G \Theta_\tau(g)\overline{\Theta_{\tau_1}(g)}\,dg \tag{5.6d}$$

where $\Theta_\tau(\cdot)$ is the character function for the representation τ of G.

Chevalley [Che, pg.172] uses the term **matrix representation** $\tau(g)$ to mean a matrix realization of the endomorphism $\tau(g)$ with respect to a basis $v_1, \ldots, v_n$ of V, τ being a representation of G on the finite-dimensional Hilbert space V. He then defines [Che, Definition 1, pg.186] the **character** $\Theta_\tau(g)$ of $\tau(g)$ to be the trace of the matrix $\tau(g)$ with respect to the basis $v_1, \ldots, v_n$ of V. It is well known that the trace of $\tau(g)$ is independent of the choice of basis (see [H-K, Exercise 15, pg.106]). Some useful facts about characters are:

(1) $\Theta_{Id}(g) = 1$

(2) $\Theta_{\tau\otimes\phi}(g) = \Theta_\tau(g)\Theta_\phi(g)$ for representations τ and ϕ

(3) $\Theta_{(\tau_1)^*}(g) = \overline{\Theta_{\tau_1}(g)}$ for irreducible unitary τ_1 and its contragredient $(\tau_1)^*$.

Fact (1) is immediate from the definition of character. Fact (2) is proven in[Che, Proposition 1, pg.187]. Fact (3) is proven in [K3, Lemma 1.11].

Returning now to the question of whether the trivial representation is in (5.6c), we note that Corollary 1 (mentioned above) and Facts (1) and (3) about characters imply that this question is equivalent to determining whether or not the integral

$$\int_L \Theta_{(V_{\Lambda+\eta-j\beta})^* \otimes U^n(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}}(l)\, dl \tag{5.6e}$$

is nonzero. Using Facts (2) and (3), we can rewrite (5.6e) as

$$\int_L \overline{\Theta_{V_{\Lambda+\eta-j\beta}}(l)}\; \Theta_{U^n(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}}(l)\, dl. \tag{5.6f}$$

Since $V_{\Lambda+\eta-j\beta}$ is an irreducible representation of L, we can apply Corollary 1 again and conclude that the number of times that the trivial representation appears in (5.6c) is precisely

(∗) number of times $\Lambda+\eta-j\beta$ can be written as $\gamma+\Lambda+\beta$,

where γ is an L highest weight in the decomposition of $U^n(\mathfrak{u})$ into irreducible L representations. Notice that because $\langle\beta,\alpha\rangle > 0$ for every $\alpha \in \Delta(\mathfrak{u})$, γ is also $\Delta^+(\mathfrak{k})$ dominant. We can rewrite the condition (∗) as

(5.6g) number of times η can be written as $\gamma+(j+1)\beta$, with j and γ as above.

At this point, we specialize to the case where $\eta = \alpha_0$, the largest noncompact root. Because $j \geq 0$, we see that the left hand side of

$$(5.6h) \qquad \alpha_0 - \beta - j\beta = \gamma,$$

which is condition (5.6g) for the case $\eta = \alpha_0$, will not have positive inner product with β. Therefore, no choice of a $\Delta^+(\mathfrak{k})$ dominant γ will give equality in (5.6h). Notice that this conclusion is independent of the values j and n; indeed, we only used that $j \geq 0$. Thus, we conclude that the trivial representation does not appear in

$$(V_{\Lambda+\alpha_0})^*|_L \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta},$$

and so $\mathcal{C}_0^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ must be 0. A completely analogous proof shows that $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ is 0.

Having proven Propositions 5.5 and 5.6, we are ready to determine the action of a root vector $X \in \mathfrak{p}$ on a homology basis vector in (5.4).

Proposition 5.7 Let $v_{\Lambda-k\beta} \otimes \text{Ker } \partial^{\natural}\big|_{\Lambda}$ be one of the basis vectors (5.4). If X is a root vector in $\mathfrak{p}$, then the action

$$(5.7a) \qquad X(v_{\Lambda-k\beta} \otimes \text{Ker } \partial^{\natural}\big|_{\Lambda}) = \alpha_1(X \otimes v_{\Lambda-k\beta} \otimes \text{Ker } \partial^{\natural}\big|_{\Lambda})$$

is effectively computable, in the sense of Proposition 4.14. In fact, the only nonvanishing K type in the decomposition of (5.7a) is $\Lambda + \alpha_0$.

PROOF. Our goal is computing $\alpha_1(X \otimes v_{\Lambda-k\beta} \otimes \operatorname{Ker} \partial^{\mathfrak{h}}|_{\Lambda})$. We can rewrite this expression as

$$(5.7b) \qquad \sum_{\alpha\in\Delta(\mathfrak{p})} \langle(\mathrm{Ad}\otimes\pi_\Lambda)(\cdot)^{-1}X\otimes v_{\Lambda-k\beta}, X_\alpha\otimes v_\Lambda\rangle\otimes X_{-\beta}\otimes X_\alpha(1\otimes 1),$$

and because $X_\alpha(1\otimes 1)=0$ when $\alpha\in\Delta(\bar{\mathfrak{u}})$, we can rewrite (5.7b) as

$$(5.7c) \qquad \sum_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p})} \langle(\mathrm{Ad}\otimes\pi_\Lambda)(\cdot)^{-1}X\otimes v_{\Lambda-k\beta}, X_\alpha\otimes v_\Lambda\rangle\otimes X_{-\beta}\otimes X_\alpha(1\otimes 1).$$

Of primary interest is the summand in (5.7c) for which $\alpha=\alpha_0$, the largest noncompact root of $\Delta(\mathfrak{g})$. We claim that α_0 appears with multiplicity 1 in $S(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{n}}$. It certainly appears with multiplicity 1 in $S^1(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{n}}$. Because every positive noncompact root of $\mathfrak{g}$ is distinguished by the fact that it contains β_0 with coefficient 1 in its simple root expansion (recall β_0 is the unique simple root of $\Delta(\mathfrak{g})$ nonorthogonal to β), we see that α_0 does not appear as a weight in $S^N(\mathfrak{u}\cap\mathfrak{p})$ for $N\geq 2$, consequently it does not appear as a weight in $S^N(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{n}}$ and our claim is proved. We know from Proposition 5.5 that the K type $\Lambda+\alpha_0$ appears with multiplicity 1 in $\mathrm{A}_{\mathfrak{q}}(\lambda)$. Considering the space

$$(5.7d) \qquad (V_{\Lambda+\alpha_0})^*\otimes{\textstyle\bigwedge}^1((\mathfrak{u}\oplus\bar{\mathfrak{u}})\cap\mathfrak{k})\otimes U(\mathfrak{u})\otimes\mathbb{C}_{\Lambda+\beta},$$

we next claim that the some nonzero multiple of the tensor

$$(5.7e) \qquad (v_{-(\Lambda+\alpha_0)})^*\otimes X_{-\beta}\otimes X_{\alpha_0}\otimes 1$$

must appear as a summand of any nonzero vector in $\mathcal{C}_1^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$, which is the space of L invariants of (5.7d). In (5.7e), $(v_{-(\Lambda+\alpha_0)})^*$ is some fixed nonzero vector in $(V_{\Lambda+\alpha_0})^*$ with weight $-(\Lambda+\alpha_0)$. Proposition 5.6 shows that $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0} = \mathcal{C}_0^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0} = 0$. As in previous chapters, this fact shows that a vector in (5.7d) is L invariant if and only if it is in Ker $\partial^\natural$. To prove the claim about vector (5.7e), we note that a nonzero vector in $\mathcal{C}_1^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ looks like

$$\text{(5.7f)} \qquad \sum_{\omega,\gamma} c^-_{\omega,\gamma}(v_\omega)^* \otimes X_{-\beta} \otimes X_{\mathfrak{u}}^\gamma \otimes 1 + \sum_{\omega,\gamma} c^+_{\omega,\gamma}(v_\omega)^* \otimes X_\beta \otimes X_{\mathfrak{u}}^\gamma \otimes 1.$$

In formula (5.7f), ω ranges over all the weights of $(V_{\Lambda+\alpha_0})^*$. Also, if $\gamma_1,\ldots,\gamma_n$ is an ordering of the roots of $\mathfrak{u}$, then γ is the tuple $(\gamma_1,\ldots,\gamma_n)$ and $X_{\mathfrak{u}}^\gamma$ is the monomial $X_{\gamma_1}^{r_1}\ldots X_{\gamma_n}^{r_n}$. Last, $c^-_{\omega,\gamma}$ and $c^+_{\omega,\gamma}$ are complex coefficients. As in the proofs of Propositions 3.6 and 4.8, we may assume that each of the tensor summands in (5.7f) is distinct. It is clear that any nonzero sum of tensors (5.7f) must contain weights ω that are L lowest weights. From Lemma 5.2 and the definition of the contragredient representation, it follows that these weights are necessarily of the form $\omega = -(\Lambda+\alpha_0-k\beta)$, where $0 \le k$ (the upper bound for k is unimportant). Let us consider what pure tensors of total weight 0 in(5.7d) have the form

$$(*) \qquad (v_{-(\Lambda+\alpha_0-k\beta)})^* \otimes X_{-\beta} \otimes X_{\mathfrak{u}}^\gamma \otimes 1 \qquad \text{OR}$$

$$(**) \qquad (v_{-(\Lambda+\alpha_0-k\beta)})^* \otimes X_{\beta} \otimes X_{\mathfrak{u}}^\gamma \otimes 1.$$

In the first case $(*)$, the monomial $X_{\mathfrak{u}}^\gamma$ is forced to have weight $\alpha_0-k\beta$. Because the weight of $X_{\mathfrak{u}}^\gamma$ must be $\Delta^+(\mathfrak{k})$ dominant, k must be 0. In the second case (**), the monomial $X_{\mathfrak{u}}^{\gamma}$ is forced to have weight $\alpha_0 - (k+2)\beta$. Since no choice of k will make this weight be $\Delta^+(\mathfrak{k})$, our claim is proved. Notice that the only tensor in (5.7f) with term $(v_{-(\Lambda+\alpha_0)})^*$ is (5.7e). This fact follows from the observation that the only elements of $U(\mathfrak{u})$ with weight α_0 are multiples of X_{α_0}. Since the multiplicity of $\Lambda + \alpha_0$ is 1 in $A_{\mathfrak{q}}(\lambda)$, we know that the dimension of $\mathcal{C}_{\text{I}}^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ is 1 (Definition 3.3a and Proposition 5.6). We define Ker $\partial^{\natural}|_{\Lambda+\alpha_0}$ to be the nonzero element of $\mathcal{C}_{\text{I}}^*(\mathbb{C}_\lambda)|_{\Lambda+\alpha_0}$ with summand (5.7e), i.e., the coefficient of the pure tensor (5.7e) is 1 in Ker $\partial^{\natural}|_{\Lambda+\alpha_0}$. Having made this definition, we are now ready to determine the value of (5.7a). As in (3.22a) and the proof of Theorem 4.14, the expression $\alpha_1(X \otimes v_{\Lambda-k\beta} \otimes \text{Ker } \partial^{\natural}|_{\Lambda})$ can be written

$$(5.7\text{g}) \quad (K \text{ type decomposition of } X \otimes v_{\Lambda-k\beta}) \times (K \text{ type decomposition of } \alpha_1(\cdot \otimes \text{Ker } \partial^{\natural}|_{\Lambda+\alpha_0})),$$

where $\alpha_1(\cdot \otimes \text{Ker } \partial^{\natural}|_{\Lambda+\alpha_0})$ is the expression gotten by replacing the term $(\text{Ad} \otimes \pi_\Lambda)(\cdot)^{-1} X \otimes v_{\Lambda-k\beta}$ in (5.7c) with a dot (for an example, see formula (3.22a)). Let us consider the K type decomposition of $X \otimes v_{\Lambda-k\beta}$. We claim that, in the K type decomposition of $\text{Ad}|_{\mathfrak{p}} \otimes \pi_\Lambda$, the only K type that appears in the discrete series $A_{\mathfrak{q}}(\lambda)$ is $\Lambda+\alpha_0$. To prove this claim, we begin by noting that the sum of two or more noncompact positive roots of $\mathfrak{g}$ cannot be a noncompact positive root. This follows from the characterization of noncompact positive roots as being those roots whose weight expansions in terms of simple roots have β_0 appearing exactly once. Next, we observe that any K type appearing in $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_\Lambda$ is of the form $\Lambda +$ (noncompact root) ([K3, Exercise 13, pg.111]). Third, Proposition 5.5 says that any K type appearing in $A_{\mathfrak{q}}(\lambda)$ is of the form $\Lambda + \eta$, for η an L highest weight of $S(\mathfrak{u} \cap \mathfrak{p})^{\mathfrak{l} \cap \mathfrak{n}}$. Finally, we observe that β is larger than any noncompact root. These four statements combined prove the claim. Notice this most recent claim proves the statement that the only nonvanishing K type in the K type decomposition of (5.7a) is $\Lambda + \alpha_0$. Using this information, we can write the K type decomposition of $X \otimes v_{\Lambda - k\beta}$ as

(5.7h) $\quad h_1(\text{vector}_1 \text{ whose } K \text{ type is } \Lambda + \alpha_0)$

$$+ (\text{vectors whose } K \text{ types do not appear in discrete series}).$$

The only K type of interest to us in the decomposition of $\alpha_1(\cdot \otimes \mathrm{Ker}\ \partial^{\natural}|_{\Lambda})$ is $\Lambda + \alpha_0$. In order determine the portion with this K type, we use the definition of $\mathrm{Ker}\ \partial^{\natural}|_{\Lambda+\alpha_0}$. It is clear from the formula (5.7c) and the definition of the expression $\alpha_1(\cdot \otimes \mathrm{Ker}\ \partial^{\natural}|_{\Lambda})$ that the sole contribution to the summand (5.7e) of $\mathrm{Ker}\ \partial^{\natural}|_{\Lambda+\alpha_0}$ comes from the term

(5.7i) $$(\cdot, X_{\alpha_0} \otimes v_\Lambda) \otimes X_{-\beta} \otimes X_{\alpha_0} \otimes 1.$$

Under the $\mathrm{Ad}|_{\mathfrak{p}} \otimes \pi_\Lambda$ decomposition, the tensor $X_{\alpha_0} \otimes v_\Lambda$ equals $\overline{h_1{}'}\, v_{\Lambda+\alpha_0}$ for some complex $h_1{}'$. There are no other terms appearing in the decomposition. Assuming that $(v_{-(\Lambda+\alpha_0)})^* = (\cdot, v_{\Lambda+\alpha_0})$, we have $\langle \cdot, X \otimes v_{\Lambda+\alpha_0} \rangle = h_1{}'(v_{-\Lambda+\alpha_0})^*(\langle \cdot, \cdot \rangle$ is conjugate-linear in the second coordinate). Using (5.7i), we can write the K type decomposition of $\alpha_1(\cdot \otimes \text{Ker } \partial^{\natural}|_{\Lambda})$ as

(5.7j) $\quad h_1{}' \text{ Ker } \partial^{\natural}|_{\Lambda+\alpha_0}$

$$+ \text{(vectors whose } K \text{ types do not appear in the discrete series).}$$

From formulas (5.7g), (5.7h), and (5.7j) it follows that

$$\alpha_1(X \otimes v_{\Lambda-k\beta} \otimes \text{Ker } \partial^{\natural}|_{\Lambda}) = h_1 h_1{}' \, \text{vector}_1 \otimes \text{Ker } \partial^{\natural}|_{\Lambda+\alpha_0} .$$

This completes the proof.

REMARK. Formula (5.7c), which implicitly uses the fact that the $(\mathfrak{l}, L)$ module $\bigwedge^1((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k})$ is a trivial module, shows that the coefficients $c^+_{\omega,\gamma}$ in the sum (5.7f) are 0.

At this point, we continue with a discussion about the space $\mathcal{C}^*_2(\mathbb{C}_\lambda)|_{\Lambda'}$, with Λ' a K type of nonzero multiplicity in $\mathrm{A}_{\mathfrak{q}}(\lambda)$. We have seen in Chapters 3 and 4 (notably Propositions 3.6 and 4.8) that $\mathcal{C}^*_2(\mathbb{C}_\lambda)|_{\Lambda'}$ is 0. The benefit of this result is that when computing a basis for homology, we need only consider L invariance. We might ask whether this result is true for the general $\mathfrak{g}$. The answer unfortunately is no. We will show in the next proposition that, for any K type Λ' that is a multiple of the largest root β, the space $\mathcal{C}^*_2(\mathbb{C}_\lambda)|_{\Lambda'}$ is nonzero. Because $\mathcal{L}_2(\mathbb{C}_\lambda)$ is 0 [K-V, Theorem 5.35], we see that $\text{Ker } \partial^{\natural}_2 \subseteq \mathcal{C}^*_2(\mathbb{C}_\lambda)|_{\Lambda'}$ is 0 and hence **any** nonzero element in $\mathcal{C}^*_2(\mathbb{C}_\lambda)|_{\Lambda'}$ will map under $\partial^{\natural}$ to a nontrivial element in $\text{Im } \partial^{\natural}_2 \subseteq \mathcal{C}^*_1(\mathbb{C}_\lambda)|_{\Lambda'}$.

Proposition 5.8 Suppose $\Lambda' = M\beta$ is a K type that appears in the discrete series $A_{\mathfrak{q}}(\lambda)$ for $M \in \mathbb{Z}^+$ satisfying $M \geq M_0 + 1$ ($M_0\beta = \Lambda$). Then $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_{M\beta}$ is a nontrivial vector space.

REMARK. This proposition is not irrelevant. If $G = SO_o(4,4)$, then the Dynkin diagram corresponding to the simple roots of $\mathfrak{g}$ is D_4, and there are examples of K types that are nonzero multiples of β in this case.

PROOF. $\mathcal{C}_2^*(\mathbb{C}_\lambda)|_{M\beta}$ is the subset of L invariant vectors of the space

$$(5.8a) \qquad (V_{M\beta})^* \otimes \bigwedge^2((\mathfrak{u} \oplus \bar{\mathfrak{u}}) \cap \mathfrak{k}) \otimes U(\mathfrak{u}) \otimes \mathbb{C}_{\Lambda+\beta}.$$

From formula (5.3b) we know that the weights of $(V_{M\beta})^*$ are

$$M\beta, (M-1)\beta, (M-2)\beta, \cdots, -M\beta.$$

Let $(v_{(M-k)\beta})^*$, $0 \leq k \leq 2M$, be fixed nonzero elements of $(V_{M\beta})^*$, with weights indicated by the subscripts. Consider the tensor in (5.8a) given by

$$(5.8b) \qquad (v_{-M\beta})^* \otimes X_{-\beta} \wedge X_\beta \otimes X_\beta^{M-M_0-1} \otimes 1.$$

The claim is that this tensor is in fact L invariant. To see this, notice first that (5.8b) has weight 0. Next, let ε be any element of $\Pi(\mathfrak{l})$. Then X_β and X_ε commute in $U(\mathfrak{g})$ (β is the largest root) and $X_\varepsilon(v_{-M\beta})^* = 0$, since $(v_{-M\beta})^*$

is an L highest weight in $(V_{M\beta})^*$ (proof of Proposition 5.3). It follows that

$$
\begin{aligned}
&X_\varepsilon\big((v_{-M\beta})^* \otimes X_{-\beta} \wedge X_\beta \otimes X_\beta^{M-M_0-1} \otimes 1\big) \\
&\quad = \big(X_\varepsilon(v_{-M\beta})^*\big) \otimes X_{-\beta} \wedge X_\beta \otimes X_\beta^{M-M_0-1} \otimes 1 \\
&\qquad\qquad + (v_{-M\beta})^* \otimes X_{-\beta} \wedge X_\beta \otimes X_\varepsilon\big(X_\beta^{M-M_0-1} \otimes 1\big) \\
&\quad = 0 + (v_{-M\beta})^* \otimes X_{-\beta} \wedge X_\beta \otimes X_\beta^{M-M_0-1} \otimes X_\varepsilon 1 \\
&\quad = 0,
\end{aligned}
\tag{5.8c}
$$

since $X_\varepsilon 1 = 0$. With ε as above, we can use (1) $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and (2) all the elements of $\Pi(\mathfrak{k})$ are linearly independent [K3, Proposition 4.6] when applying Cartan's Theorem to $[X_\beta, X_{-\varepsilon}]$ to conclude that the bracket is 0, and hence X_β and $X_{-\varepsilon}$ commute in $U(\mathfrak{g})$. We have $X_{-\varepsilon}(v_{-M\beta})^* = 0$ since $(v_{-M\beta})^*$ is also an L lowest weight, and so by replacing X_ε by $X_{-\varepsilon}$ in (5.8c), we have that the tensor (5.8b) is annihilated by $X_{-\varepsilon}$ for all simple roots ε of $\mathfrak{l}$. This fact, in combination with the fact that vector (5.8b) has weight 0 and is annihilated by X_ε for all simple roots ε of $\mathfrak{l}$, shows that vector (5.8b) is an L invariant tensor in the space (5.8a) and our result is proven.

REMARK. Notice that we can repeat this argument with any pure tensor in (5.8a) that has total weight 0 if the term in $U(\mathfrak{u})$ is of the form X_β^n for some $n \in \mathbb{Z}^+ \cup \{0\}$, since each $(v_{(M-k)\beta})^*$ is both an L highest and L lowest weight.

CHAPTER 6

UNITARY EQUIVALENCE OF TWO REALIZATIONS OF $Sp(1,1)$ DISCRETE SERIES

In this chapter, we shall produce a unitary equivalence between two different spaces of functions, both of which are realizations of an $Sp(1,1)$ discrete series. Throughout this chapter, $G = Sp(1,1)$. All references to page numbers, unless otherwise stated, refer to the paper [Tak].

In order to describe the first space, we use define some notation. Let (ρ, V) be a unitary irreducible representation of $K = SU(2) \times SU(2)$, with V a finite dimensional Hilbert space and $\langle \cdot, \cdot \rangle_V$ the scalar product for V. Then Takahashi defines (pg.392) $L^2_V(G)$ as the Hilbert space of functions $F(g)$ that are square integrable, with values in V, for the scalar product:

$$(F, F') = \int_G \langle F(g), F'(g) \rangle_V \, dg.$$

The representation defined on $L^2_V(G)$ is left regular representation:

$$U_g(F)(h) = F(g^{-1}h) \qquad \text{for } F \in L^2_V(G).$$

He also defines, on the same page, $L^2_{V,\rho}(G)$ to be the subspace of functions F such that

$$F(gk) = \rho(k)^{-1} F(g) \qquad \text{for each } k \in K \text{ and } g \in G.$$

Takahashi points out that $L^2_{V,\rho}(G)$ is stable for $U_g, g \in G$ and hence obtains a representation U on $L^2_{V,\rho}(G)$ by restriction. If D is the set of infinitely differentiable functions in $L^2_{V,\rho}(G)$, then $\{U, D\}$ will be one of the spaces used in defining our unitary equivalence. At this point, we describe another space of functions that Takahashi considers in his thesis. Takahashi proves that this space of functions is unitarily equivalent to $\{U, D\}$. In order to describe Takahashi's second space of functions, we need some information about quaternions and about $Sp(1,1)$ realized as matrices with quaternion entries.

Takahashi (pg.361) defines a quaternion to be the expression

$$x = x_1 + x_2 i + x_3 j + x_4 k,$$

where $x_1, \cdots, x_4 \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. He denotes by $\bar{x}$ the **conjugate of** x, i.e., $\bar{x} = x_1 - x_2 i - x_3 j - x_4 k$. Also, the **norm of** x, denoted $|x|$, is the nonnegative real number $(x\bar{x})^{\frac{1}{2}}$. Takahashi (pg.362) defines B to be the "open unit ball of quaternions", namely B is the set of quaternions whose length, defined by the norm, is less than 1. On the same page, he defines $Sp(1,1)$ to be the subset of 2×2 matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with quaternion entries a, b, c, and d, satisfying the relations

$$(6.1) \qquad \bar{a}b = \bar{c}d \qquad |a|^2 - |c|^2 = 1 \qquad |d|^2 - |b|^2 = 1.$$

Takahashi's second space of functions (pg.392), which will be important in the proof of Theorem 6.7, is constructed as follows:

> For p a half-integer≥ 1 (a rational number p for which $2p \in \mathbb{Z}$), let $H^{\rho,p}$ be the Hilbert space of (equivalence classes of) functions $f(q)$ defined on B, with values in V, and square integrable for the measure $(1-|q|^2)^{2p-2}\,d\mu(q)$, given the scalar product:
>
> $$(f_1, f_2)_{\rho,p} = c\int_B \langle f_1(q), f_2(q)\rangle_V (1-|q|^2)^{2p-2}\,d\mu(q).$$

Here, c is a certain positive constant, and $\langle\cdot,\cdot\rangle_V$ is the scalar product for V. Also, $\mu(q)$ is the Euclidean measure on B, and Takahashi proves (pg.374, Lemma 1.4) that $(1-|q|^2)^{-4}\,d\mu(q)$ is an invariant measure for B.

The representation of G on $H^{\rho,p}$, denoted $T_g^{\rho,p}$, is given by

$$(T_g^{\rho,p} f)(q) = |cq+d|^{-2p-2}\rho(k(g^{-1},q)^{-1})f((aq+b)(cq+d)^{-1})$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $f \in H^{\rho,p}$, and $k(g^{-1},q)^{-1}$ the element

$$\begin{pmatrix} \dfrac{a+b\bar{q}}{|cq+d|} & 0 \\ 0 & \dfrac{cq+d}{|cq+d|} \end{pmatrix} \in K.$$

Hereafter, $T_g^{\rho,p}$ is abbreviated by T. Also on pg. 392, Takahashi shows that $\{T, H^{\rho,p}\}$ and $\{U, L^2_{V,\rho}(G)\}$ are unitarily equivalent. He does this by using the map $J^{\rho,p} : H^{\rho,p} \to L^2_{V,\rho}(G)$, defined on a function $f \in H^{\rho,p}$ and with image $F = J^{\rho,p} \circ f$, where

$$F(g) = F(s(q)k) = c^{\frac{1}{2}} 2^{-2} (1 - |q|^2)^{p+1} \rho(k)^{-1} f(q).$$

The notation $s(q)k$ refers to a decomposition of the group G. Takahashi proves (pg.373) that every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G can be written uniquely in the form $g = s(q)k$, where $q = bd^{-1} \in B$, $k = \begin{pmatrix} a/|a| & 0 \\ 0 & d/|d| \end{pmatrix}$, and $s(q) = \begin{pmatrix} |a| & q|a| \\ \bar{q}|d| & |d| \end{pmatrix}$. The space of infinitely differentiable functions in $H^{\rho,p}$ is denoted $H_0^{\rho,p}$ and Takahashi comments (pg.393) that $J^{\rho,p}$ induces an isometry between $\{T, H_0^{\rho,p}\}$ and $\{U, D\}$.

In order to describe our space of functions, we provide some background. Early attempts by the author to study this question of quaternionic discrete series began with an attempt to imitate the theory of holomorphic discrete series. A detailed study of this theory is given in [K3, Chapter 6]. A construction that is used there is the Harish-Chandra decomposition for G, where G is a linear reductive group assumed to satisfy certain conditions on the Lie algebra level. It turns out that such a decomposition exists for $Sp(1,1)$, although we do not have a theorem analogous to [K3, Theorem 6.3] in this case. We write

(6.2a) $$Sp(1,1) \subseteq P^+ K^{\mathbb{H}} P^-,$$

where

$$(6.2b) \qquad \begin{aligned} P^+ &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{H}) \middle| \, |b| < 1 \right\} \\ K &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a, d \in SP(1) = SU(2) \right\} \\ K^{\mathbb{H}} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a, d \in \mathbb{H}^\times = SP(1) \times \mathbb{R}^+ \right\} \\ P^- &= \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in GL(2,\mathbb{H}) \middle| \, |c| < 1 \right\}. \end{aligned}$$

Here, $\mathbb{H}$ refers to the set of quaternions, and $|\cdot|$ is the norm of a quaternion (defined earlier). By means of an identification of quaternions with certain 2×2 complex matices, Takahashi shows (pg.362) that the group $SU(2)$ may be thought of as those quaternions with norm 1. An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G can be decomposed uniquely according to (6.2a) as

$$(6.3) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.$$

Consider now the collection

(6.4)

$$\Gamma'_{(M,N)}(\rho) = \{ f : P^+ K^{\mathbb{H}} P^- \to V_{(M,N)} | f(p^+ k^{\mathbb{H}} p^-) = \rho_{(M,N)}(k^{\mathbb{H}})^{-1} f(p^+) \quad \text{for } p^+ k^{\mathbb{H}} p^- \in G \},$$

where $p^+ \in P^+$, $k^{\mathbb{H}} \in K^{\mathbb{H}}$, $p^- \in P^-$, and $(\rho_{(M,N)}, V_{(M,N)})$ is a finite dimensional irreducible representation of K with highest weight $Me_1 + Ne_2$.

For instance, using notation of Chapter 2, Section 2, we may use $V_{(M,N)} = V_M \otimes W_N$ and $\rho_{(M,N)} = \pi_{(M,N)}$. The M and N of interest for us in this chapter are those satisfying the inequality $0 \leq M \leq N-2$.

In (6.3) we use an extension of ρ to a representation of $K^{\mathbb{H}}$. For $\begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \in K^{\mathbb{H}}$, this is accomplished by means of the formula

$$\rho_{(M,N)} \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} = |k_1|^M |k_2|^N \rho_{(M,N)} \begin{pmatrix} k_1/|k_1| & \\ & k_2/|k_2| \end{pmatrix}. \tag{6.5}$$

In order to consider discrete series, we require an inner product on the space $\Gamma'_{(M,N)}(\rho)$. We define an inner product on this space by

$$(f, f') = \int_G \langle f(g), f'(g) \rangle_V (1 - |\theta(g)|^2)^2 \, dg \tag{6.6}$$

where $\langle \cdot, \cdot \rangle_V$ is an inner product on $V_{(M,N)}$ that is unitary with respect to $\rho_{(M,N)}$, and $\theta(g) = bd^{-1}$for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Because there is a canonical way to create the inner product on the tensor product of two inner product spaces, we can use (2.6) to give us $\langle \cdot, \cdot \rangle_V$. This brings us to our second space of functions. Let

$$\Gamma_{(M,N)}(\rho) = \left\{ f \in \Gamma'_{(M,N)}(\rho) \middle| f \text{ is square integrable for (6.6)} \right\},$$

with representation $(L(g)f)(h) = f(g^{-1}h)$ for $g, h \in Sp(1,1)$ (left regular representation).

Before stating our theorem of unitary equivalence, we make one more reference to a result in Takahashi's thesis. Recall in the situation where

$G = Sp(1,1)$ that an irreducible representation of K is the tensor product of two irreducible representations of $SU(2)$ (Chapter 2, Section 2). Takahashi denotes such an irreducible representation of K by $\rho_K^{n,n'}$, for nonnegative half-integers n and n' (pg.382). If (ρ^n, V^n) is an irreducible representation of $SU(2)$ on a Hilbert space V^n of dimension $2n+1$ and $(\rho^{n'}, V^{n'})$ is an irreducible representation of $SU(2)$ on a Hilbert space $V^{n'}$ of dimension $2n'+1$, then Takahashi defines $\rho_K^{n,n'}$ on $V^n \otimes V^{n'}$ by

$$\rho_K^{n,n'}(k) = \rho^n(u) \otimes \rho^{n'}(v) \qquad \text{for } k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

Using this definition, he proves **Proposition 3.1** (pg.399), which we state now and shall refer to in the next theorem:

> Let n, p be two half-integers such that $n \geq p \geq 1$ and $n - p \in \mathbb{Z}$, and let ρ be a representation of K of the form $\rho_K^{n,0}$ or $\rho_K^{0,n}$. Then the subspace $S^{\rho,p}$ of $H_0^{\rho,p}$ formed by the functions f for which
>
> $$T_\Omega f = [-n(n+1) - (p+1)(p-2)]f$$
>
> is nontrivial and closed in $H^{\rho,p}$. Here, T_Ω is the Casimir operator.

Having stated this, we are now ready to prove the equivalence.

Theorem 6.7 Let M and N be integers satisfying $0 \leq M \leq N-2$ and assume that every function $F \in L^2_{V,\rho}(G)$ can be extended to a smooth function on $P^+K^{\mathbb{H}}P^-$ in such a way that the transformation law (6.4) holds. Then there exists a map $\tau{:}\Gamma_{(M,N)}(\rho) \to \{U, D\}$ that is a unitary equivalence. The map τ is given on a function $f \in \Gamma_{(M,N)}(\rho)$ by

$$(\tau f)(g) = (1 - |\theta(g)|^2)f(g),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1,1)$ and $\theta(g) = bd^{-1}$. The correspondence of (n,p) in (Chapter II, Proposition 3.1) with (M,N) in $\Gamma_{(M,N)}(\rho)$ is given by

$$\begin{aligned} M &= n - p \\ N &= n + p. \end{aligned} \tag{6.7a}$$

Further, the function f_0 defined on B by

$$f_0(q) = f\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \qquad \text{for } f \in \Gamma_{(M,N)}(\rho) \cap \operatorname{Ker} \mathcal{D}$$

is an element of $H_0^{\rho,p}$ and under the assumption that $f_0 \in \mathcal{S}^{\rho,p}$ (defined above), then the eigenvalues of the respective Casimir operators acting on $\Gamma_{(M,N)}(\rho) \cap \operatorname{Ker} \mathcal{D}$ and $\mathcal{S}^{\rho,p}$ are equal up to a normalization constant.

PROOF. We begin by proving the correspondence (6.7a). In order to get these equations, we look at the corner K type of the Dixmier diagram $\pi^+_{n,p}$ [Dix, pg. 25 fig. 3]. We choose $\pi^+_{n,p}$ because (it turns out) our choice of positive root system for $Sp(1,1)$ is the system with $e_2 - e_1 > 0$. This choice corresponds to $\pi^+_{n,p}$, whereas the choice $e_1 - e_2 > 0$ corresponds to $\pi^-_{n,p}$. The corner K type of $\pi^+_{n,p}$ is given by $(\frac{1}{2}(n-p), \frac{1}{2}(n+p))$, which is the minimal K type. Because (M, N) is the minimal K type of $\Gamma_{(M,N)}(\rho)$, we are tempted to let $M = \frac{1}{2}(n-p)$ and $N = \frac{1}{2}(n+p)$. However, this will not work, since $\frac{1}{2}(n-p)$ and $\frac{1}{2}(n+p)$ need not be integers. Rather, we double the picture, letting $n' = 2n$ and $p' = 2p$. Then

$$M = \frac{1}{2}(n' - p') = (n - p)$$
$$N = \frac{1}{2}(n' + p') = (n + p)$$

is an acceptable correspondence, since both $n-p$ and $n+p$ are both integers.

To show that τ defines a unitary equivalence, we begin by showing that τ maps into $\{U, D\}$. Using the decomposition (6.3) and the fact that $a - bd^{-1}c = \dfrac{a}{|d|^2}$ (from (6.1)), we see that a function f in $\Gamma_{(M,N)}(\rho)$ satifies

$$\begin{aligned} f(g) &= \rho_{(M,N)}\begin{pmatrix} |d|^2 a^{-1} & \\ & d^{-1} \end{pmatrix} f\begin{pmatrix} 1 & \theta(g) \\ 0 & 1 \end{pmatrix} && \text{by (6.4)} \\ (*) \qquad &= |d|^{M-N} \rho_{(M,N)}\begin{pmatrix} a/|d| & \\ & d/|d| \end{pmatrix}^{-1} f\begin{pmatrix} 1 & \theta(g) \\ 0 & 1 \end{pmatrix} && \text{by (6.5)} \\ &= (1 - |\theta(g)|^2)^p \rho_{(M,N)}\begin{pmatrix} a/|d| & \\ & d/|d| \end{pmatrix}^{-1} f\begin{pmatrix} 1 & \theta(g) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where the last equality follows from the identities $M - N = -2p$ (6.7a) and $|d|^2 = (1 - |\theta(g)|^2)^{-1}$. The second of these identities is a result of relations (6.1). We can rewrite $f(g)$ as

$$\text{(6.7b)} \qquad f(g) = (1 - |q|^2)^p \rho_{(M,N)}(k)^{-1} f_0(q),$$

if we use Takahashi's notation $g = s(q)k$ mentioned before, and if we define $f_0 \in H_0^{\rho,p}$ by $f_0(q) = f\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$. To see that $f_0 \in H_0^{\rho,p}$ requires some proof, most notably that it is square integrable for the inner product $(\cdot,\cdot)_{\rho,p}$ mentioned previously in this chapter. We show this by first showing that $f \in \Gamma_{(M,N)}(\rho)$ satisfies

$$\text{(6.7c)} \quad f(g) = \rho_{(M,N)}(k)^{-1} f(s(q)) = \rho_{(M,N)}(k^{\mathbb{H}})^{-1} f(p^+)$$

$$\text{for } g = s(q)k = p^+ k^{\mathbb{H}} p^- \in G.$$

The relationship between q and p^+ in the equation $s(q)k = p^+ k^{\mathbb{H}} p^-$ is given by $p^+ = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$. The formula (6.7c) follows from the series of relations

$$\begin{aligned} f(g) &= \rho_{(M,N)}(k^{\mathbb{H}})^{-1} f(p^+) && \text{for } g = p^+ k^{\mathbb{H}} p^- \text{ by definition} \\ &= \rho_{(M,N)}(k)^{-1}(1 - |q|^2)^p f(p^+) && \text{for } q = \theta(g) \text{ by } (*) \\ &= \rho_{(M,N)}(k)^{-1} f(s(q)k), \end{aligned}$$

since $f(s(q)k) = (1-|q|^2)^p f(p^+)$ for $f \in \Gamma_{(M,N)}(\rho)$ by (6.4) and the definition of $s(q) \in G$. Using (6.7c), we have

$$\begin{aligned} &(f_0(q), f_0(q))_{\rho,p} \\ &= \int_B (1 - |q|^2)^{2p-2} \langle f_0(q), f_0(q) \rangle_V \, d\mu(q) \end{aligned}$$

$$= \int_B \int_K (1 - |q|^2)^{2p-2} \langle f_0(q), f_0(q) \rangle_V \, d\mu(q) \, dk \qquad \text{(see below)}$$

$$= \int_B \int_K (1 - |q|^2)^{-2} \langle \rho_{(M,N)}(k^{\mathbb{H}})^{-1} f(p^+), \rho_{(M,N)}(k^{\mathbb{H}})^{-1} f(p^+) \rangle_V \, d\mu(q) \, dk \qquad \text{by } (*)$$

$$= \int_B \int_K (1 - |q|^2)^{-2} \langle f(s(q)k), f(s(q)k) \rangle_V \, d\mu(q) \, dk \qquad \text{by (6.7c)}$$

$$= \int_G (1 - |\theta(g)|^2)^2 \langle f(g), f(g) \rangle_V \, dg \qquad \text{by (Lemma 1.4, pg.394)}$$

$$< \infty,$$

since $f \in \Gamma_{(M,N)}(\rho)$. In the second equation above, we choose a Haar measure for K and normalize it so that $\int_K 1 \, dk = 1$. The fact that we can find such a Haar measure is well known (see for example [Rud, Theorem 5.14, pg.130]). Hence, $(\tau f)(g) = (1 - |q|^2)^{p+1} \rho(k)^{-1} f_0(q)$ is a function in $\{U, D\}$, being the image under the isometry $J^{\rho,p}$ of the function $4c^{-\frac{1}{2}} f_0(q)$ (c a constant mentioned previously).

The fact that τ is injective follows immediately from the observation that $|\theta(g)| < 1$ for every $g \in Sp(1,1)$ and hence $1 - |\theta(g)|^2$ is nonzero for every $g \in Sp(1,1)$.

We define an inverse for $F \in L^2_{V,\rho}(G)$ in the obvious way:

$$(\tau^{-1}F)(g) = (1 - |\theta(g)|^2)^{-1}F(g).$$

This is well defined and injective, from the comments about $|\theta(g)|$ above. In order to show that this inverse function satifies the transformation law of (6.4), we use the assumption that F can be extended to a function (also denoted by F) on $P^+K^{\mathbb{H}}P^-$ satisfying the transformation law of (6.4). Because $(1 - |\theta(g)|^2)^{-1}$ is a real number and because $\rho(k^{\mathbb{H}})^{-1}$ is real linear, it follows that $\tau^{-1}F \in \Gamma'_{(M,N)}(\rho)$. The fact that $\tau^{-1}F$ is actually an element of $\Gamma_{(M,N)}(\rho)$ follows immediately from the hypothesis that $F \in L^2_{V,\rho}(G)$. Note that the relation (6.7c) shows that any extension of $F \in L^2_{V,\rho}(G)$ satisfying the transformation law of (6.4) also satisfies the transformation law in the definition of $L^2_{V,\rho}(G)$. This condition is a necessary condition for any extension of F.

To show unitary equivalence, we observe that for $f \in \Gamma_{(M,N)}(\rho)$ and $F = \tau f$,

$$\begin{aligned}
(f, f) &= \int_G \langle f(g), f(g) \rangle_V (1 - |\theta(g)|^2)^2 \, dg \\
&= \int_G \langle (1 - |\theta(g)|^2) f(g), (1 - |\theta(g)|^2) f(g) \rangle_V \, dg \\
&= \int_G \langle (\tau f)(g), (\tau f)(g) \rangle_V \, dg \\
&= \int_G (F(g), F(g))_V \, dg \\
&= (F, F).
\end{aligned}$$

Last, we show that the map τ commutes with the G action. For $g, h \in G$, f as above,

$$\begin{aligned}\tau(L(g)f(h)) &= \tau(f(g^{-1}h)) \\ &= (1 - |\theta(g^{-1}h)|^2)f(g^{-1}h) \\ &= U(g)(\tau f)(h).\end{aligned}$$

This completes the proof of the unitary equivalence.

The eigenvalue of the Casimir operator on $\Gamma_{(M,N)}(\rho)$ is given in [K-W, Corollary 3.2]. More precisely, the Casimir operator is constant on those functions that lie in Ker $\mathcal{D}$, where $\mathcal{D}$ is the Schmid operator. The formula for the Casimir operator in [K-W, Corollary 3.2] is given by

$$\Omega f = m(|\Lambda|^2 - |\delta|^2)f,$$

where m is a normalization constant, $|\Lambda|$ is the length of $\Lambda = \rho_{(M,N)} + \delta_K - \delta_n$ and $|\delta|$ is the length of δ. The terms δ, δ_K and δ_n refer to the half-sum of the positive roots of $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{p}$ respectively (recall from Chapter 2 that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$). Our choice of positive systems for $\mathfrak{k}$ and $\mathfrak{g}$ is

$$\Delta^+ = \{e_2 \pm e_1, 2e_1, 2e_2\} \quad \text{and} \quad \Delta^+_K = \{2e_1, 2e_2\} \quad \text{and} \quad \Delta^+_n = \{e_2 \pm e_1\}.$$

Then

$$\delta = e_1 + 2e_2 \quad \text{and} \quad \delta_K = e_1 + e_2 \quad \text{and} \quad \delta_n = e_2.$$

Knapp and Wallach [K-W, pg.166] define the space $C^\infty(G, \rho_{(M,N)})$ to be the space of all C^∞ functions $F: G \to V_{(M,N)}$ such that $F(kg) = \rho_{(M,N)}(k)F(g)$ for all k in K and g in G. Our space of functions is a modification of $C^\infty(G, \rho_{(M,N)})$ in that our functions are defined are a larger set $P^+K^{\mathbb{H}}P^-$ and the transformation law in (6.4) has $\rho_{(M,N)}(\cdot)^{-1}$ rather than $\rho_{(M,N)}(\cdot)$. In order to apply the results from [K-W] to our space of functions $\Gamma_{(M,N)}(\rho)$, we must change the definition of $Xf(g)$ before (2.5) of [K-W] from

$$Xf(g) = \frac{d}{dt} f(\exp(tX)^{-1}g)\Big|_{t=0}$$

to

$$Xf(g) = \frac{d}{dt} f(g\ \exp(tX))\Big|_{t=0}.$$

Returning to our calculation of the Casimir operator, we shall use the normalization constant $m = -\frac{1}{2}$. Using the positive system above and the notation in [K-W, Corollary 3.2], we can write the terms $\Lambda = (M+1)e_1 + Ne_2 = (M+1, N)$ and $\delta = (1,2)$. Hence the eigenvalue of the Casimir action for those functions f in $\Gamma_{(M,N)}(\rho) \cap \operatorname{Ker} \mathcal{D}$ is

$$\begin{aligned} -\frac{1}{2}((M+1)^2 + N^2 - 5) &= -\frac{1}{2}((n-p+1)^2 + (n+p)^2 - 5) \qquad \text{by (6.4)} \\ &= -\frac{1}{2} \cdot 2(n(n+1) + (p+1)(p-2)) \\ &= -(n(n+1) + (p+1)(p-2)). \end{aligned}$$

Meanwhile, for those functions f_0 in $\mathcal{S}^{\rho,p}$ of (Chapter II Proposition 3.1), if $F = J^{\rho,p} \circ f_0$, then equation (8) on pg. 394 shows

$$U_\Omega F = -(n(n+1) + (p+1)(p-2))F,$$

where U_Ω is the Casimir operator on D. For a given (n,p), $f_0 \in H_0^{\rho,p}$ corresponds to $f \in \Gamma_{(M,N)}(\rho) \cap \operatorname{Ker} \mathcal{D}$ via $f_0 = (J^{\rho,p})^{-1} \circ \tau f$. The second assumption made in the statement of the theorem, that $f_0 \in \mathcal{S}^{\rho,p}$, allows us the calculate an eigenvalue of the Casimir operator. The calculation above shows that the eigenvalues of the corresponding Casimir operators are equal for these corresponding functions. We have thus proven the last statement of the theorem.

REMARKS. Takahashi does not give a very explicit description of the space $\mathcal{S}^{\rho,p}$ beyond exhibiting an element in the space (pg.400). In the case where $n = p = 1$, so that $M = 0$ and $N = 2$ by (6.7a), Takahashi shows that the constant function $f_0(q) = v$ for some fixed $v \in V$ is in $\mathcal{S}^{\rho,p}$. The author has verified that the function $f \in \Gamma_{(0,2)}(\rho)$ defined by

$$f(p^+ k^{\mathbb{H}} p^-) = \rho_{(0,2)}(k^{\mathbb{H}})^{-1} f_0(q) = \rho_{(0,2)}(k^{\mathbb{H}})^{-1} v$$

is in $\operatorname{Ker} \mathcal{D}$ when $p^+ k^{\mathbb{H}} p^- = g \in Sp(1,1)$. In this case, the eigenvalue of both Casimir operators is 0.

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INDEX OF NOTATION

In this list, Latin, German and script symbols appear together and are followed by Greek symbols and non-letters.