Irreducible Tempered Representations of the Special Linear Group over a p-adic Field

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Abstract of the Dissertation

Irreducible Tempered Representations of the Special Linear Group over a p-adic Field

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The classification of the irreducible admissible representations of the special linear group $SL_n(F)$ over a p-adic field reduces to the classification of the irreducible tempered representations. These latter representations are known to comprise the irreducible constituents of standard unitary representations parabolically induced from discrete series. The reducibility of these standard unitary induced representations is controlled by the reducibility of standard unitary induced representations of $GL_n(F)$ when restricted to $SL_n(F)$. Moreover the reducibility is measured by a finite abelian group known as the R group.

In this thesis, the discrete series of the standard Levi components of $SL_n(F)$ are classified and, using the R group, divided into Class I and Class

II representations. The Class I representations include those with a cyclic R group. Theorems of Mackey are employed to demonstrate that the reducibility of representations parabolically induced from Class I discrete series actually occurs on an intermediate Levi component. A complete classification of the irreducible constituents of the standard unitary induced representations in the Class I case is then given. Explicit realizations of these constituents are made in terms of the reducibility of some basic representations, called "building block representations", and the reducibility of the discrete series of the general linear group.

For Mummy,

my best friend

and greatest role model

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I. Introduction

Connections with algebraic number theory, automorphic forms, and harmonic analysis motivate the study of the representation theory of reductive groups defined over a local field F. Irreducible admissible representations play a significant role in this theory, and consequently their classification is an important step in understanding the representation theory of reductive groups. Langlands [Lan] originally addressed this classification question for reductive groups defined over $\mathbb{R}$ or $\mathbb{C}$ in 1972, and his ideas have influenced the research in representation theory since that time. In the real or complex case, Langlands reduced the classification of irreducible admissible representations to knowledge of the irreducible "tempered" representations. By the 1977 conference in Corvallis, Casselman had generalized this result to handle reductive groups G defined over any local field.

The first ingredient for the classification is the discrete series representations, i.e., unitary representations with square integrable matrix coefficients modulo the center of G. If $P = MN$ is a parabolic subgroup of a reductive group G and σ is a discrete series representations of the Levi component M, then we can construct a representation of G by parabolic induction. We refer to such a representation as a standard unitary induced representation. The irreducible tempered representations of G are the irreducible constituents of these standard unitary induced representations. Thus a full classification of the irreducible admissible representations arises from an understanding of the discrete series of the Levi components of G and the reducibility of the standard unitary induced representations.

For the case that the field F is $\mathbb{R}$ or $\mathbb{C}$, these steps were completed some time ago. Harish-Chandra [H-C1] parameterized the discrete series, and Knapp-Zuckerman [K-Z] analyzed the reducibility of the standard representations and classified the irreducible tempered representations. Thus the Langlands classification may be regarded as complete for real or complex reductive groups.

The reductive groups defined over a nonarchimedean field F have proved more difficult to handle. It is known [Cas] that the representations induced from parabolic subgroups have a finite composition series; however, most results for reductive groups over a nonarchimedean field pertain only to specific groups. The historical role and seeming familiarity of matrix groups makes them a natural choice for study.

Indeed the classification for the general linear group $GL_n(F)$ is now complete. Jacquet [Jac] proved that the standard unitary induced representations are always irreducible, and Bernstein-Zelevinsky [B-Z] reduced the classification of the discrete series to a classification of the "supercuspidal" representations. Kutzko [Ku] determined the supercuspidals of $GL_2(F)$, and the classification of supercuspidals for $GL_n(F)$ was carried out independently by Bushnell-Kutzko [B-K] and Corwin [Cor].

The classification for $GL_n(F)$, however, does not handle closely related groups like the special linear group $SL_n(F)$, which is the subject of this thesis. We consider the case that F is a nonarchimedean local field of characteristic 0 (though we believe that the restriction on the characteristic is not necessary). Thus F is a finite extension of the p-adic numbers for some prime p, and we shall refer to such a field as a p-adic field. Classification of the discrete series of $SL_n(F)$ amounts to understanding how the discrete series of $GL_n(F)$ reduce under restriction to $SL_n(F)$, and thus the completion of the Langlands classification for $SL_n(F)$ comes down to an understanding of the reducibility of the standard unitary induced representations. It turns out, just as with the discrete series, that the reducibility depends upon the reducibility of unitary induced representations of $GL_n(F)$ when restricted to $SL_n(F)$.

Suppose $P = MN$ is a parabolic subgroup of $SL_n(F)$, and let σ be a discrete series representation of M. Let $\pi = \mathrm{Ind}_{MN}^{SL_n}(\sigma)$ be the standard unitary induced representation of $SL_n(F)$ determined by σ. The reducibility of π is measured by a finite group known as the R group, which quantifies the reducibility of the induced representation by specifying a basis of the commuting algebra. The R group was introduced for groups defined over $\mathbb{R}$ by Knapp-Stein [K-S1] and was adapted to the nonarchimedean case by Muller [Mul] and Winarsky [Win].

For the problem at hand, the first induced representations studied in detail were those of the principal series, i.e., the representations induced from the upper triangular group (a minimal parabolic subgroup), using a unitary character on the diagonal subgroup as the inducing data. Gelbart-Knapp [G-K1] showed that the R group for this situation is always abelian, and they found that the structure of the R group points to some connection with abelian Galois extensions of F. The R group is shown to be canonically isomorphic to a group of characters of the multiplicative group $F^\times$ of F, and subsequently as a certain quotient group of $F^{\times}$. Thus by local class field theory, the R group is identified with the Galois group of a finite abelian Galois extension of F. Goldberg [Gol], building upon work of Shahidi [Shd] and Gelbart-Knapp, showed that the R group for a standard unitary induced representation coming from an arbitrary parabolic subgroup, is canonically isomorphic to a certain quotient group of characters of the multiplicative group of F. In this thesis, we realize R in this fashion and as a related permutation group; see Section 3.3.

Consider the reducibility of the standard unitary induced representations of $SL_n(\mathbb{R})$. If $P = MN$ is a parabolic subgroup of $SL_n(\mathbb{R})$ and σ is in the discrete series of M, then the R group of σ is either trivial or isomorphic to $\mathbb{Z}_2$. The first step in describing the irreducible constituents of the standard unitary induced representations is to parameterize the discrete series of M. These representations are, in fact, induced from $M^{\#}$, the product of the identity component and the center of M. Since the identity component of M is the direct sum of some copies of $SL_2(\mathbb{R})$, we obtain a set of inducing data for the discrete series of M by means of the discrete series of $SL_2(\mathbb{R})$.

The induction to $SL_n(\mathbb{R})$ is then broken into two pieces; first from $M^{\#}$ to M, and then from MN to $SL_n(\mathbb{R})$. The reducibility of $\pi = \mathrm{Ind}_{MN}^{SL_n}(\sigma)$ actually occurs at the level of M, and the irreducible constituents of π are obtained by inducing the irreducible constituents on M to $SL_n(\mathbb{R})$. Understanding the reducibility on M comes down to understanding the reducibility of some basic representations on $SL_2^{\pm}(\mathbb{R})$; see [K-S2] and [K-Z]. This example is the model for our approach to $SL_n(F)$, where F is a p-adic field.

Since the discrete series representations of $SL_n(F)$ arise from the restriction of discrete series representations of $GL_n(F)$, we are able to associate (non-uniquely) a standard unitary induced representation of $GL_n(F)$ to that of $SL_n(F)$. The reducibility of this representation of $GL_n(F)$ when restricted to $SL_n(F)$ controls the reducibility of the standard unitary induced representation we started with. The theory presented in Gelbart-Knapp [G-K1] extends directly to show that any reducibility in the restriction to $SL_n(F)$ actually occurs on a subgroup of finite index in $GL_n(F)$, and this subgroup is pointed to by the R group. This fact allows us to apply "Mackey theory" to our situation.

We are first led to classify the discrete series of the Levi components M for standard parabolic subgroups of $SL_n(F)$ (Theorem 4.2.2). If σ is a discrete series representation of M, we write $\pi = \mathrm{Ind}_{MN}^{SL_n(F)}(\sigma)$ for the standard unitary induced representation determined by σ. We then use the R group of σ to describe the discrete series representations of $\tilde{M}$ that contain σ in their restriction to M, where $\tilde{M}$ is the corresponding Levi component in $GL_n(F)$ (Propositions 4.4.1 and 4.4.2). Combining these two ideas gives us a specific realization for a representation σ' on a certain intermediate subgroup M^X (i.e., $M \subseteq M^X \subseteq \tilde{M}$) whose restriction to M is equivalent to σ. The group M^X is the Levi component for a subgroup G^X of $GL_n(F)$ containing $SL_n(F)$, and we can consider the representation achieved by parabolically inducing σ' to G^X. The reducibility of this representation, when restricted to $SL_n(F)$, is exactly that of the representation achieved by parabolically inducing σ to G. Thus our attention turns to studying the representation

$$\pi' = \mathrm{Ind}_{M^X N}^{G^X}(\sigma') \tag{1}$$

and its restriction to $SL_n(F)$.

We then define so-called "building block representations." To do so, we suppose that M is the "block diagonal" subgroup consisting of exactly k blocks of equal size q. Let $G_n^X(F)$ be a subgroup of finite index in $GL_n(F)$ that contains $SL_n(F)$, and let $M^X = G^X \cap M$. Define a representation of $\prod_{i=1}^k G_q^X$ by

$$\mathrm{Ind}_{\prod G_q^X}^{M^X} \left. \begin{pmatrix} \tilde{\varepsilon}\tau & & & \\ & \eta \otimes \tau & & \\ & & \ddots & \\ & & & \eta^{k-1} \otimes \tau \end{pmatrix} \right|_M , \tag{2}$$

for some discrete series representation τ of G_q^X. Here $\tilde{\varepsilon} = \mathrm{diag}(\varepsilon, 1, \ldots, 1)$ is a specific coset representative for $GL_n(F)/G^X$. Then the representation in (2) can be parabolically induced to G_n^X, and the induced representation is called a building block representation.

Let M be the Levi component for a standard parabolic subgroup of $SL_n(F)$, and let σ be in the discrete series of M. We form the induced representation π' as in equation (1), where σ' satisfies $\sigma'|_M \cong \sigma$. If σ has a cyclic R group or, more generally, σ is a "Class I" representation in the sense of Section 4.4, then we are able to associate to π' a set of building block representations. We break the induction to $SL_n(F)$ into two pieces; first to a larger Levi component M' determined by the building block representations, and then to all of $SL_n(F)$. Mackey theory allows us to see that the reducibility actually occurs on M', and consequently to describe the irreducible constituents of $\pi'|_{SL_n(F)}$ in terms of the reducibility of these building block representations and the reducibility of discrete series representations (Theorems 5.1.2, 5.2.1). Since $\pi'|_{SL_n(F)} \cong \pi = \mathrm{Ind}_{MN}^{SL_n(F)}(\sigma)$, we can determine the irreducible constituents of π.

If σ is a "Class II" representation in the sense of Section 4.4, the situation is more complicated, and the reducibility is not handled solely by the methods discussed in Chapter 5. Instead we are led to define a linear system of equations over the characters of the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{SL_n(F)}(\sigma)$ (again actually written in terms of our representation π' of (1)), and we seek to understand when this system contains an invertible subsystem by which we can compute the characters. This is the subject of Chapter 6.

Goldberg [Gol] studied the "elliptic" representations of $SL_n(F)$, determining the exact requirements, in terms of the R group, for the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{SL_n(F)}(\sigma)$ to be elliptic. The representations with this property are exactly those which can be written as a building block representation. Though the representations covered by Theorems 5.1.2 and 5.2.1 generally do not have elliptic constituents, we see that their constituents are induced from elliptic representations (Proposition 5.3.3). Moreover we can often realize the characters of the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{SL_n(F)}(\sigma)$ when σ is a Class II representation as a linear combination of characters that are induced from elliptic ones.

II. Introductory Material

§2.1. p-adic Fields.

Let p be prime, and for any rational number $\frac{m}{n}$, write $\frac{m}{n} = p^a \frac{m'}{n'}$ with m' and n' not divisible by p. Then we define the **p-adic norm** on $\mathbb{Q}$ by

$$\left|\frac{m}{n}\right|_p = p^{-a} \quad \text{for every } \frac{m}{n} \in \mathbb{Q}. \tag{1}$$

The completion of $\mathbb{Q}$ with respect to this norm is called the p-adic numbers and is written $\mathbb{Q}_p$. Any finite extension of $\mathbb{Q}_p$ is referred to as a **p-adic field**. These are locally compact nondiscrete nonarchimedean fields.

Let F be a p-adic field, and let E be a field extension of F of degree n. Fix a basis of E over F. For any $a \in E^\times$, left multiplication by a defines an invertible linear operator on $E^\times$, and consequently determines an element of $GL_n(F)$. The determinant of this matrix is said to be the **norm** of a (over F), and is written $N_{E/F}(a)$. The subgroup $N_{E/F}(E^\times)$ of $F^\times$ is called the **norm group** of E (in F). If F is a finite extension of a p-adic field K, then $N_{E/K}(a) = N_{F/K}(N_{E/F}(a))$. The following theorem is the fundamental result of local class field theory.

Theorem 2.1.1. (ref. [Ser, Chap XIV]) Let F be a p-adic field. Then there is a one-one correspondence between the closed subgroups H of finite index in $F^\times$ and the finite abelian Galois extensions E over F, the correspondence being that H is the norm group of E over F; moreover the degree of the extension $[E : F]$ is $|F^\times/H|$, and the Galois group of E/F is canonically isomorphic to $F^\times/H$.

Suppose that $[F : \mathbb{Q}_p] = n$. Then there is a unique extension $|\cdot|_F$ of the p-adic norm $|\cdot|_p$, namely

$$|x|_F = |N_{F/\mathbb{Q}_p}(x)|_p^{1/n} \quad \text{for all } x \in F.$$

The **ring of integers** in F is the maximal subring of F, i.e.,

$$\mathcal{O}_F = \{x \in F \mid |x|_F \leq 1\}. \tag{2}$$

Then $\mathcal{O}_F$ is a Dedekind domain and contains a unique prime ideal

$$\mathfrak{P}_F = \{x \in F \mid |x|_F < 1\} = \omega\mathcal{O}_F, \tag{3}$$

for some $\omega \in F$. We call ω a **prime** element of $\mathcal{O}_F$. The additive group F is then written $F = \cup_{n\in\mathbb{Z}} \omega^n \mathfrak{P}_F$. Moreover the subgroups $\{\mathfrak{P}^n\}_{n\geq 0}$ form a neighborhood base of 0, and the subgroups $\{1+\mathfrak{P}^n\}_{n\geq 0}$ form a neighborhood base of 1. If $F = \mathbb{Q}_p$, then $\mathfrak{P}_F = (p)$.

Since $\mathcal{O}_F$ is Dedekind, $\mathfrak{P}_F$ is a maximal ideal. The field $k = \mathcal{O}_F/\mathfrak{P}_F$ is called the **residue class field** of F, and $q = |k|$ is some power of p. If $F = \mathbb{Q}_p$, then $q = p$ and $k \cong \mathbb{F}_p$.

For a complete development of p-adic fields and local class field theory, the reader should consult Cassels-Frohlich [C-F], Serre [Ser], or Hasse [Has]. The role of p-adic fields in representation theory can be found in [Car] or [Sil], among others.

§2.2. Structure Theory.

We are interested in studying certain representations of the closed subgroups of $GL_n(F)$ containing $SL_n(F)$, and the structure of these groups plays a significant role both in defining these representations and in their investigation. Such groups, in general, are not linear algebraic groups. Some of the terms used in connection with linear algebraic groups, however, are meaningful when describing the structure of our groups. Therefore we will define and use these terms in this context.

Let F be a nonarchimedean local field of characteristic 0, i.e., a finite extension of the p-adic numbers for some p. Let $\tilde{G}_n = GL(n, F)$, let $G_n = SL(n, F)$, and let G' be an arbitrary closed subgroup of $\tilde{G}_n$ containing G_n. Let $\tilde{M}_0$ be the subgroup of diagonal matrices in $\tilde{G}_n$, and let N_0 be the group of upper triangular matrices with 1's in the diagonal entries. Then $\tilde{P}_0 = \tilde{M}_0 N_0$ is called the **standard minimal parabolic subgroup** of $\tilde{G}_n$, and any closed subgroup $\tilde{P}$ containing $\tilde{P}_0$ is called a **standard parabolic subgroup** of $\tilde{G}_n$.

The standard parabolic subgroups of $\tilde{G}_n$ are determined in the following way. Suppose that $\{\bar{n}_i\}_{i=1}^{k}$ is a partition of n, and let

$$\tilde{M} \cong GL_{\bar{n}_1} \times GL_{\bar{n}_2} \times \ldots \times GL_{\bar{n}_k} \tag{1}$$

be the "block diagonal" subgroup of $\tilde{G}_n$ corresponding to this partition. Each $GL_{\bar{n}_i}$ is called a **block** of $\tilde{M}$. Let N be the upper triangular matrices of $\tilde{G}_n$ with the identity matrix in each block. Then $\tilde{P} = \tilde{M}N$ is a standard parabolic subgroup of $\tilde{G}_n$, and all standard parabolic subgroups are of this form. The group $\tilde{M}$ is called the **Levi component** of $\tilde{P}$, and N is called the **unipotent radical** of $\tilde{M}$. The subgroup A of $\tilde{M}$ containing scalar matrices in each block is called the **split component** of $\tilde{M}$.

If G' is a closed subgroup of $\tilde{G}_n$ containing G_n and if $\tilde{P} = \tilde{M}N$ is a standard parabolic subgroup of $\tilde{G}_n$, then $P' = G' \cap \tilde{P}$ is a **standard parabolic subgroup** of G'. Moreover $P' = M'N$ with $M' = \tilde{M} \cap G'$. The terms **Levi component**, **block**, etc. have the analogous meaning in G'.

Let M' be a Levi component for a standard parabolic subgroup of G', and let A be the split component of M'. Indicate by $N_{G'}(M')$ the normalizer of M' in G', and indicate by $Z_{G'}(M')$ the centralizer of M' in G'. Then the **Weyl group** $W_{G'}(M') = W_{G'}(G'/A)$ is the quotient of the normalizer by the centralizer: $W_{G'}(M') = N_{G'}(A)/Z_{G'}(A)$. Notice that $W_{G'}(M') = W_{\tilde{G}}(\tilde{M})$. Furthermore for any w in $N_{G'}(M')$ the action on M' given by $g \mapsto w^{-1}gw$ permutes the blocks, and the subgroup yielding the identity permutation is $Z_{G'}(M')$. Hence we may regard $W_{G'}(M')$ as a subgroup of the symmetric group S_k, where k is the number of blocks in M'.

If g is in $\tilde{M}$, with $\tilde{M}$ as in equation (1), we write $g = (g_1, g_2, ..., g_k)$. Also for $a \in F^\times$, we define $\tilde{a} \in \tilde{M}_0$ as the diagonal matrix with determinant a given by $\tilde{a} = \operatorname{diag}(a, 1, \ldots, 1)$.

Let $(F^\times)^\vee$ denote the group of continuous unitary characters of $F^\times$. If κ is in $(F^\times)^\vee$, then κ lifts to a character of $\tilde{G}_n$ by $\kappa(g) = \kappa \circ \det(g)$. For any subgroup H of $\tilde{G}_n$ and any subset $S \subseteq (F^\times)^\vee$, we define

$$H^S = \{h \in H \mid \kappa(h) = 1 \quad \text{for all } \kappa \in S\}.$$

For simplicity, we write G_n^S for $\tilde{G}_n^S$ and M^S for $\tilde{M}^S$, and we denote the corresponding standard parabolic subgroup of G_n^S by $P^S = M^S N$. By continuity of the determinant mapping, any open normal subgroup G' of finite index in $\tilde{G}_n$ is equal to $\tilde{G}_n^S$ for some subset $S \subseteq (F^\times)^\vee$. For any set of characters S in $(F^\times)^\vee$, we set

$$N_S = \bigcap_{\eta \in S} \ker \eta \subseteq F^\times .$$

By local class field theory, N_S corresponds to the norm group of some field extension of F. With these definitions, if $\{a_i\}$ is a complete set of coset representatives for $F^\times / N_S$, then $\{\tilde{a}_i\}$ is a complete set of representatives for $\tilde{G}_n / \tilde{G}_n^S$.

A maximal abelian subgroup of semisimple elements in $\tilde{G}_n$ is a **Cartan subgroup** of $\tilde{G}_n$. To describe the Cartan subgroups of $\tilde{G}_n$, we first relate field extensions of degree n over F with specific Cartan subgroups. Suppose E is such a field extension, and fix a basis of E over F. For any $a \in E^\times$, we have an invertible linear operator on $E^\times$ given by left multiplication by a. Thus $a \in F^\times$ corresponds to a matrix in $\tilde{G}_n$. The collection of these matrices exhibits $E^\times$ as embedded in $\tilde{G}_n$, and the embedded version of $E^\times$ is a Cartan subgroup of $\tilde{G}_n$.

Now let $\{\bar{n}_i\}$ be a partition of n, and write $\tilde{M}$ as in equation (1). Suppose $\{E_i\}$ is a set of extensions of F with $[E_i : F] = \bar{n}_i$. Then $E_i^\times$ embeds in the above fashion into the ith block of $\tilde{M}$, and thus the multiplicative group of the direct sum of the fields E_i embeds into $\tilde{M}$. The resulting subgroup of $\tilde{M}$ is a Cartan subgroup of $\tilde{G}_n$, and so is any $\tilde{G}_n$-conjugate of this subgroup. If G' is a closed subgroup of $\tilde{G}_n$ containing G_n and if $\tilde{\Gamma}$ is a

Cartan subgroup of $\tilde{G}_n$, then $\Gamma' = \tilde{\Gamma} \cap G'$ is a Cartan subgroup of G'.

A **totally disconnected group** is a separable locally compact group whose open compact subgroups form a neighborhood base at the identity. Every closed subgroup of $\tilde{G}_n$ is a totally disconnected group.

In any of the above definitions, when the size n is understood, n may be dropped from the notation.

§2.3. Representations.

Let G be a totally disconnected group. A **representation** of G is a homomorphism $\pi : G \to GL(V)$, where V is a complex vector space. If π is a representation of G, we denote the space on which G acts by V^π.

If W is a subspace of V^π such that $\pi(g)W \subseteq W$ for all $g \in G$, then W is said to be an **invariant** subspace, and we say that π has a **subrepresentation** on W. If the only invariant subspaces of V^π are $\{0\}$ and all of V^π, then π is said to be **irreducible**. Otherwise we say π is **reducible**. Two representations π_1 and π_2 of G are **equivalent** if there exists a vector space isomorphism $E : V^{\pi_1} \to V^{\pi_2}$ such that

$$\pi_2(g)Ev = E\pi_1(g)v \quad \text{for all } g \in G.$$

The notion of equivalent representations determines an equivalence relation on the representations on G, and the equivalence class of a representation π is denoted $[\pi]$.

Let π be an irreducible representation of G, and let H be a subgroup of G. The restriction of π to H is a representation of H and is denoted $\pi|_H$. Suppose W is invariant under $\pi|_H$ such that $V^{\pi|_H} = W \oplus W'$ for some W'. Then the subrepresentation of H on W is said to be a **constituent** of $\pi|_H$, as is any representation ρ of H that is equivalent to this subrepresentation. In this thesis we are concerned solely with irreducible constituents.

If π_1 and π_2 are representations of G and if E is a linear map from V^{π_1} into V^{π_2} satisfying $E\pi_1(g)v = \pi_2(g)Ev$ for all v in V^{π_1}, then E is said to be an **intertwining operator** between π_1 and π_2. The set of self-intertwining operators for a representation π of G forms an algebra $\mathcal{C}(\pi)$, which is known as the **commuting algebra** of π.

If V is a complex vector space, write V' for the space of complex-valued linear functionals on V. Let $\langle v', v \rangle = v'(v)$ for every $v' \in V'$ and $v \in V$. If π is a representation on V, then its **contragredient** is the representation on V' defined by the property $\langle \pi'(g)v', v \rangle = \langle v', \pi^{-1}(g)v \rangle$ for all $v \in V$, $v' \in V'$, and $g \in G$.

Let S be a set. A function $f : G \to S$ is said to be **smooth** or **locally constant** if every $g \in G$ lies in a neighborhood $U(g)$ whose image $f(U(g))$ is a single point of S. The space of smooth functions from G into $\mathbb{C}$ is denoted $C^\infty(G)$; the subspace of functions with compact support is denoted $C_c^\infty(G)$.

Let π be a representation of G on V. An element $v \in V$ is called **smooth** if the mapping $g \mapsto \pi(g)v$ is a smooth function of G into V. Denote the subspace of smooth vectors in V by V_∞. The representation π of G is **smooth** if $V = V_\infty$. In other words, π is smooth if and only if the stabilizer of each vector v in V^π is compact open, if and only if each vector v in V^π is fixed by some compact open subgroup of G. If π is a smooth representation and π' is its contragredient, write $\tilde{V}$ for $(V')_\infty$, and write $\tilde{\pi}$ for the representation of G on $\tilde{V}$. This representation is called the **smooth contragredient** of π. When π is smooth, the term "contragredient" will refer henceforth to the smooth contragredient.

Let G be a totally disconnected group, and let H be a subgroup of G. Suppose that τ is a representation of H, and fix an element g of G. Then we can define a representation $g\tau$ of $H^g = gHg^{-1}$ by $g\tau(h) = \tau(g^{-1}hg)$ for all h in H^g.

Let μ be a left Haar measure on G', and denote integration of functions in $C_c^\infty(G')$ with respect to μ by $\int_{G'} f(g)dg$. Define the **modular function** $\Delta_{G'}$ on G' by

$$\int_{G'} f(g)dg = \Delta_{G'}(g_0)\int_{G'} f(gg_0)dg.$$

If π is a smooth representation of G, let $\pi(f)$ be the operator on V^π defined by

$$\pi(f)v = \int_G \Delta_G^{1/2}(g)f(g)\pi(g)vdg.$$

Let π be a smooth representation of G on V. If v is in V and $\tilde{v}$ is in $\tilde{V}$, the function $f(g) = <\tilde{v}, \pi(g)v>$ is called a **matrix coefficient** of π.

A representation π of G is said to be **pre-unitary** if there exists a Hermitian positive definite form B on V^π such that $B(\pi(g)v_1, \pi(g)v_2) = B(v_1, v_2)$ for all $v_1, v_2 \in V^\pi$ and $g \in G$. The term pre-unitary is used here because V^π is not necessarily a Hilbert space, and therefore the term unitary is not accurate. It should be noted, however, that some authors (e.g., Silberger) use these two terms interchangeably.

If π is a smooth representation such that the K-invariant vectors are finite dimensional for every compact open subgroup K in G, then π is an **admissible** representation. Let π be an irreducible smooth admissible representation, and let $Z(G)$ denote the center of G. If π is unitary on $Z(G)$ and every matrix coefficient of π is in $L^{2+\varepsilon}(G)$ modulo $Z(G)$ for $\varepsilon > 0$, then we say that π is a **tempered** representation. The irreducible tempered representations, up to equivalence, are denoted $\mathcal{E}_t(G)$. The tempered representations whose matrix coefficients are square integrable modulo $Z(G)$ are called **discrete series** representations. The discrete series representations are pre-unitary [Sil, Section 1.11], and we denote by $\mathcal{E}_2(G)$ the set of discrete series representations of G, taken up to equivalence. The discrete series representations whose matrix coefficients are compactly supported modulo $Z(G)$ are called **supercuspidal** representations.

Let π be an irreducible smooth admissible representation of G. Then $\pi(f)$ is of finite rank for every f in $C_c^\infty(G)$; see [Car]. Thus the linear functional

$$\Theta_\pi(f) = \mathrm{Trace}(\pi(f))$$

is well defined for f in $C_c^\infty(G)$ and is called the **character** of π. It is continuous on $C_c^\infty(G)$, and hence is a distribution. The characters of inequivalent irreducible admissible representations are linearly independent; see [Sil].

III. Induced Representations and the R group

The purpose of this chapter is to introduce the representations that are investigated in this thesis, as well as to state the main theorems used in our study of these representations. We are interested in the reducibility of generalized principal series representations π of G_n, as defined in Section 1. We will see that there is a corresponding irreducible representation of $\tilde{G}_n$ that contains π in its restriction to G_n, and Sections 2 and 3 provide results that allow us to take advantage of this fact in our investigation. In the final section of this chapter we combine all of this information to prove results for the generalized principal series that are analogous to those for the principal series found in Gelbart-Knapp [G-K1].

§3.1. Induced Representations.

Let G' be a totally disconnected group, and let H be a closed subgroup of G'. If τ is an irreducible representation of H we want to construct a representation on all of G'. This construction is called **induction**, and it generalizes the notion originally introduced by Frobenius for finite groups; see [Ser2]. Further discussion and proofs of many of the results cited here for infinite groups can be found in Silberger [Sil].

The definition of the induced space for infinite groups involves an extra factor not found in the finite group case. This complication arises from the fact that G'/H may not have a nontrivial left-invariant measure. Let μ be a left Haar measure on G', and denote integration of functions in $C_c^\infty(G')$ with respect to μ by $\int_{G'} f(g)dg$. Recall from Section 2.3 that the modular function of G' is defined by

$$\int_{G'} f(g)dg = \Delta_{G'}(g_0) \int_{G'} f(gg_0)dg.$$

The modular function of H is defined analogously. Define a function δ on H by $\delta(h) = \Delta_H(h)/\Delta_{G'}(h)$ for all $h \in H$. This function δ is used to compensate for the lack of a nontrivial left-invariant measure on G'/H.

Definition. (Induced Representation) If H is a closed subgroup of G' and τ is an irreducible smooth representation of H on V^τ, let V be the space of functions $f : G' \to V^\tau$ satisfying:

(1) $f(g'h) = \delta^{-1/2}(h)\tau^{-1}(h)f(g')$ for all $g' \in G'$ and $h \in H$, and

(2) There exists a compact open subgroup K of G' such that $f(gk) = f(g)$ for g in G and k in K.

Let G' act on V by $(gf)(x) = f(g^{-1}x)$. We write $\pi = \mathrm{Ind}_H^{G'}(\tau)$ for this representation, and we say that π is **induced** from τ. In some cases, when there can be no confusion, we write $\pi = \mathrm{Ind}(\tau)$.

REMARKS.

(1) The functions in V are smooth, however, condition (2) is generally a stronger assumption than smoothness of the functions in V. The representation $\pi = \mathrm{Ind}_H^{G'}(\tau)$ is smooth. If τ is pre-unitary and H is cocompact, then $\pi = \mathrm{Ind}_H^{G'}(\tau)$ is also pre-unitary; see [Sil, Cor. 1.7.9].

(2) Suppose G' is a closed normal subgroup of $\tilde{G}_n$ containing G_n, and suppose $H = M'N$ is a proper standard parabolic subgroup of G'. Let τ be

in $\mathcal{E}_2(M')$, and extend τ to all of H by putting the trivial representation on N. The representation $\pi = \mathrm{Ind}_{M'N}^{G'}(\tau)$ is said to be **parabolically induced** from τ. In this case, G'/H does not have a left-invariant Haar measure.

(3) If H is a closed subgroup of finite index in G' then $\delta \equiv 1$. The representation $\pi = \mathrm{Ind}_H^{G'}(\tau)$ is said to be **finitely induced** from τ. Representations of this type are the focus of Mackey theory, which is discussed in the next section.

The next lemma makes precise the concept that "induced of induced is induced," and is referred to as **double induction**. Since it is an integral part of the theory of induced representations, it will frequently be used without explicit reference.

Lemma 3.1.1 (Double Induction). Let G be a totally disconnected group, and let G' and H be closed subgroups of G with $H \subseteq G'$. Suppose τ is a smooth representation of H. Then

$$\mathrm{Ind}_{G'}^{G}\left(\mathrm{Ind}_H^{G'}(\tau)\right) \cong \mathrm{Ind}_H^G(\tau).$$

Lemma 3.1.2. Let σ be a smooth representation of H, and let η be a quasi-character of G'. Then $\eta \otimes \mathrm{Ind}_H^{G'}(\sigma) \cong \mathrm{Ind}_H^{G'}(\eta|_H \otimes \sigma)$.

PROOF. The operator $(Ef)(x) = \eta^{-1}(x)f(x)$ for $f \in V^{\mathrm{Ind}(\sigma)}$ implements the equivalence from left to right. Q.E.D.

Lemma 3.1.3. Let σ be a smooth representation of H, and fix g in G'. Let $H^g = gHg^{-1}$, and let $g\sigma$ be the representation of H^g defined in Section 2.3. Then $\mathrm{Ind}_{H^g}^{G'}(g\sigma) \cong \mathrm{Ind}_H^{G'}(\sigma)$.

PROOF. The operator $(Ef)(x) = f(xg^{-1})$ implements the equivalence from left to right. Q.E.D.

We are concerned with representations parabolically induced from discrete series. Let M be a Levi component of G_n written in block diagonal form, and recall that $\mathcal{E}_2(M)$ contains the discrete series representations of M, up to equivalence. Let σ be in $\mathcal{E}_2(M)$. Extend σ to all of MN by putting the trivial representation on N. The representation $\pi = \mathrm{Ind}_{MN}^{G_n}(\sigma)$ is said to be in the **generalized principal series** of G_n. These representations are pre-unitary, since discrete series representations are pre-unitary and MN is cocompact in G_n.

A special case arises when $M = M_0$ is the Levi component for the standard minimal parabolic of G_n. Then $\mathcal{E}_2(M)$ consists of unitary characters of M, and we say that $\pi = \mathrm{Ind}_{M_0 N}^{G_n}(\sigma)$ is in the **principal series** of G_n. The reducible members of the principal series are the object of study by Gelbart-Knapp [G-K1], and the characters of the irreducible constituents are determined in Assem [As] when n is prime.

Similarly we can define generalized principal series representations of $\tilde{G}_n$, written $\tilde{\pi} = \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$ with $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$. These representations play a significant role in our study of the generalized principal series of G_n, as we shall see in the next few sections. In particular we shall see that to each of the generalized principal representations π of G_n we can associate a generalized principal representation $\tilde{\pi}$ of $\tilde{G}_n$, and it is crucial to the rest of our study that $\tilde{\pi}$ is irreducible.

Theorem 3.1.4. (ref. Jacquet [Jac]) Let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$. Then $\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$ is irreducible.

§3.2. Restriction Theorem and Mackey Theory.

The following theorems are used in an essential way throughout our development of results on reducibility. The first regards the restriction of irreducible admissible representations of a totally disconnected group to an open normal subgroup of finite index. The results due to Mackey deal specifically with representations induced from subgroups of finite index; proofs of these results can be found in [K-V, Section 2.4].

Theorem 3.2.1 (Restriction Theorem). [G-K2, Section 2] Let π be an irreducible admissible representation of a totally disconnected group G, and let H be an open normal subgroup of G such that G/H is a finite abelian group. Then

(a) $\pi|_H$ is the finite direct sum of $\mathcal{M}$ irreducible admissible representations of H, all occurring with the same multiplicity m.

(b) The number of one-dimensional characters ν in

$$X_H(\pi) = \{\nu \in G^\vee \mid \nu|_H = 1 \text{ and } \pi \otimes \nu \cong \pi\}$$

is $m^2\mathcal{M}$. In particular if $m = 1$, then $\dim \mathcal{C}(\pi|_H) = |X_H(\pi)|$, where $\mathcal{C}(\pi|_H)$ is the commuting algebra of $\pi|_H$ defined in Section 2.3.

(c) If π_1 is a constituent of $\pi|_H$, define

$$G_{\pi_1} = \{g \in G \mid g\pi_1 \cong \pi_1\}.$$

Then G/G_{π_1} acts simply transitively on the constituents of $\pi|_H$. Furthermore if each of the constituents of $\pi|_H$ occurs with multiplicity one, then the set

$$G^{X_H} = \{g \in G \mid \nu(g) = 1 \text{ for all } \nu \in X_H(\pi)\}$$

is equal to G_{π_1}.

(d) Let π be an irreducible admissible representation of H. Suppose that $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity free and contain π. Then $\tilde{\pi} \cong \kappa \otimes \tilde{\pi}'$ for some character κ of G, which is trivial on H.

REMARK. Recall that the finite subgroups of $(F^\times)^\vee$ are in one-one correspondence with the closed subgroups H of $\tilde{G}_n$ containing G_n. The characters of $\tilde{G}$ can then be taken as elements of $(F^\times)^\vee$ composed with the determinant mapping. If π has the multiplicity one property, then (c) can be written as

$$\tilde{G}^{X_H} = \{g \in \tilde{G}_n \mid \det g \in N_X\},$$

where $N_X = \cap \ker \xi$, the intersection being taken over $\xi \in X_H(\pi)$. By local class field theory, N_X is the norm subgroup in $F^\times$ of some finite abelian Galois extension of F.

Theorem 3.2.2 (Frobenius Reciprocity). Let G be a totally disconnected group, and let G_1 be an open subgroup of finite index. Suppose π is a smooth representation of G_1, and suppose φ is a smooth representation of G. Let $\tilde{\pi} = \mathrm{Ind}_{G_1}^{G}(\pi)$.

(1) Define

$$e_\pi : V^{\tilde{\pi}} \to V^\pi$$

to be evaluation at 1. Then

$$\mathrm{Hom}_G\left(V^\varphi, V^{\tilde{\pi}}\right) \cong \mathrm{Hom}_{G_1}\left(V^\varphi|_{G_1}, V^\pi\right)$$

under the map $\Phi \to e_\pi \circ \Phi$, and this isomorphism is natural in φ and π.

(2) Define

$$j_\pi : V^\pi \to V^{\tilde{\pi}}$$

as the map sending $v \in V^\pi$ to the function on G given by

$$j_\pi(v)(g) = \begin{cases} \pi(g^{-1})v & \text{for } g \in G_1 \\ 0 & \text{for } g \notin G_1. \end{cases}$$

Then

$$\mathrm{Hom}_G\left(V^{\tilde{\pi}}, V^\varphi\right) \cong \mathrm{Hom}_{G_1}\left(V^\pi, V^\varphi|_{G_1}\right)$$

under the map $\Phi \to \Phi \circ j_\pi$, and this isomorphism is natural in π and φ.

Theorem 3.2.3 (Mackey). Let G be a totally disconnected group, and let G_1 and G_2 be subgroups of finite index in G. Let π be a smooth representation of G_1, and let $g\pi$ denote the representation of $G_2 \cap gG_1g^{-1}$ given by $h \to \pi(g^{-1}hg)$ for g in G. Then

$$\mathrm{Ind}_{G_1}^{G}(\pi)\big|_{G_2} \cong \bigoplus_{\substack{\text{double cosets} \\ G_2 g G_1}} \mathrm{Ind}_{G_2 \cap gG_1g^{-1}}^{G_2}(g\pi).$$

Corollary 3.2.4. Let G be a totally disconnected group, and let G_1 and G_2 be subgroups of finite index in G. If π_1 and π_2 are smooth representations of G_1 and G_2 respectively, then

$$\mathrm{Hom}_G\left(\mathrm{Ind}_{G_1}^{G}(\pi_1), \mathrm{Ind}_{G_2}^{G}(\pi_2)\right) \cong \bigoplus_{\substack{\text{double cosets} \\ G_2 g G_1}} \mathrm{Hom}_{G_2 \cap gG_1g^{-1}}(g\pi_1, \pi_2).$$

Corollary 3.2.5. Let G be a totally disconnected group, and let H be an open normal subgroup of finite index in G. Let π be an irreducible admissible representation of H. Then there exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π.

§3.3. Theory of the R group.

Let $P = MN$ be a standard parabolic subgroup of $G = G_n$, and let $\tilde{M}$ be a Levi component of $\tilde{G} = \tilde{G}_n$ such that $M = \tilde{M} \cap G$. If σ is in $\mathcal{E}_2(M)$, then σ lifts to a representation σ' of $M' = Z(\tilde{G})M$ by putting the trivial representation on $Z(\tilde{G})$. By Corollary 3.2.5 there exists a representation $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_{M'} \supseteq \sigma'$, and therefore σ is contained in $\tilde{\sigma}|_M$.

If $w \in N_G(M)$, let $\bar{w}$ be the coset of $W_G(M)$ containing w. Let σ be in $\mathcal{E}_2(M)$, and denote by $[\sigma]$ the equivalence class of σ. As in Section 2.3, define $w\sigma$ by $w\sigma(m) = \sigma(w^{-1}mw)$. Then the class of $w\sigma$ depends only on $\bar{w}$.

Let $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$. The R group of σ, adapted by Muller and Winarsky from the notion introduced by Knapp-Stein [K-S] for groups defined over the real numbers, quantifies the reducibility of π by specifying a basis of the commuting algebra. In its original form, the R group comes out of the theory of intertwining operators. We can define intertwining operators between π and $\pi_w = \mathrm{Ind}_{MN}^{G}(w\sigma)$ for $w \in N_G(M)$. With certain normalizations in place, these operators span the commuting algebra of π. The R group corresponds to the normalized operators that are not scalar, and hence truly indicates reducibility of π. The reader who is interested in the details of this theory should consult [K-S] or [Shd]. It is known for our situation that the R group is a finite abelian group [Ke, H-S].

There is an alternate realization of the R group written in terms of the $\tilde{\sigma}$'s that contain σ in their restriction to M. Let

$$W(\sigma) = \{\bar{w} \in W_G(M) \mid [w\tilde{\sigma}] = [\eta \otimes \tilde{\sigma}] \quad \text{for some } \eta \in (F^\times)^\vee\}, \qquad (1)$$

and let

$$W'(\sigma) = \{\bar{w} \in W_G(M) \mid [w\tilde{\sigma}] = [\tilde{\sigma}]\}. \tag{2}$$

Then $W(\sigma)$ corresponds to intertwining operators of π [Gol, Shd], and $W'(\sigma)$ corresponds to the subset of scalar operators [Shd]. Thus the R group $R(\sigma)$ is isomorphic to $W(\sigma)/W'(\sigma)$. We use this realization to view the R group as a permutation group.

Another realization of $R(\sigma)$ as a certain quotient group of characters is due to Goldberg [Gol]. Let

$$\bar{L}(\tilde{\sigma}) = \{\, \eta \in (F^\times)^\vee \mid \eta \otimes \tilde{\sigma} \cong w\tilde{\sigma} \quad \text{for some } \bar{w} \in W_G(M)\,\}, \tag{3}$$

and recall that $X_M(\tilde{\sigma}) = \{\eta \in (F^\times)^\vee \mid \eta \otimes \tilde{\sigma} \cong \tilde{\sigma}\}$. Write $X(\tilde{\sigma}) = X_M(\tilde{\sigma})$. Then the R group of σ is canonically isomorphic to $\bar{L}(\tilde{\sigma})/X(\tilde{\sigma})$. This fact generalizes the analogous result of Gelbart-Knapp [G-K1] for the reducible principal series, i.e., when $P = MN$ is the standard minimal parabolic.

§3.4. Reducibility of the Generalized Principal Series.

Let $G = G_n$, and let M be the Levi component for a standard parabolic subgroup of G. Let $\tilde{G} = \tilde{G}_n$, and take $\tilde{M}$ to be the Levi component of $\tilde{G}$ corresponding to M. If σ is in $\mathcal{E}_2(M)$, let $\tilde{\sigma}$ be an element of $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$.

We want to see that $\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})\big|_G$ is equivalent to $\mathrm{Ind}_{MN}^{G}(\tilde{\sigma}|_M)$ and that any reducibility actually occurs on a subgroup of finite index in $\tilde{G}$.

The latter fact was originally proved for the principal series by Gelbart-Knapp [G-K1]. Unlike the principal series, however, $\tilde{\sigma}|_M$ may have more than one irreducible constituent, and therefore the reducibility of $\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})\big|_G$ is generally not the same as that of $\mathrm{Ind}_{MN}^{G}(\sigma)$. Consequently when we turn to describe the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ in Chapter 5, it will be necessary to utilize the classification of $\mathcal{E}_2(M)$ completed in Chapter 4, in addition to the results below.

Lemma 3.4.1. Let G' be an open normal subgroup of $\tilde{G}$, with standard parabolic subgroup (P', M'), and let τ be in $\mathcal{E}_2(M')$. If H is an closed normal subgroup of G', then

$$\mathrm{Ind}_{M'N'}^{G'}(\tau)\Big|_H \cong \mathrm{Ind}_{(M'\cap H)N'}^{H}(\tau|_{M'\cap H})$$

under restriction of functions from G' to H. In particular if $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$, then

$$\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})\big|_G \cong \mathrm{Ind}_{MN}^{G}(\tilde{\sigma}|_M)\,.$$

PROOF. Let $\pi_1 = \mathrm{Ind}_{M'N'}^{G'}(\tau)$ and $\pi_2 = \mathrm{Ind}_{(M'\cap H)N'}^{H}(\tau|_{M'\cap H})$. To see that the restriction map is one-one, notice that every $x \in G'$ can be written $x = hg'$ for some $h \in H$ and $g' \in M'$. This follows from the fact that closed normal subgroups correspond to sets of unitary characters, and therefore the determinant mapping can be used to get diagonal coset representatives, as in Section 2.3. If f is in V^{π_1}, then $f(x) = f(hg') = \mu^{-1/2}(g')\tau^{-1}(g')f(h)$. Thus if $f|_H = 0$, then $f = 0$.

To see that restriction is onto, let $\tilde{f}$ be in V^{π_2}, and define

$$f(x) = f(hg') = \mu^{-1/2}(h)\tau^{-1}(g')\tilde{f}(h)$$

for all $x \in G'$. Then f is in V^{π_1} and restricts to $\tilde{f}$ on H. We need to show that f is well-defined.

Let $x = h_1 g_1$, and suppose that we can also write $x = h_2 g_2$. Then $h_1^{-1}h_2 = g_1 g_2^{-1}$ is in M', and

$$\begin{aligned} f(x) = f(h_1 g_1) &= \mu^{-1/2}(g_1)\tau^{-1}(g_1)\tilde{f}(h_1) \\ &= \mu^{-1/2}(g_1)\tau^{-1}(g_1)\tilde{f}(h_2 h_2^{-1} h_1) \\ &= \mu^{-1/2}(g_1)\tau^{-1}(g_1)\tilde{f}(h_2(g_2 g_1^{-1})) \\ &= \mu^{-1/2}(g_2)\tau^{-1}(g_2 g_1^{-1})\tau^{-1}(g_1)\tilde{f}(h_2) \\ &= f(h_2 g_2). \end{aligned}$$

Lastly H equivariance is clear, and the equivalence follows. Q.E.D.

Recall from Section 3 that

$$\bar{L}(\tilde{\sigma}) = \{\, \eta \in (F^{\times})^{\vee} \mid \eta \otimes \tilde{\sigma} \cong w\tilde{\sigma} \quad \text{for some } w \in W_G(M) \,\}.$$

This is a finite subgroup of $(F^{\times})^{\vee}$, and consequently corresponds to a subgroup $\tilde{G}^{\bar{L}} = \tilde{G}^{\bar{L}(\tilde{\sigma})}$ of finite index in $\tilde{G}$. The following proposition shows that all of the reducibility of $\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})\big|_G$ actually occurs on $\tilde{G}^{\bar{L}}$.

Proposition 3.4.2. If σ is in $\mathcal{E}_2(M)$ and $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M$ contains σ, then the reducibility of $\mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$ restricted to G is all accounted for on $G^{\bar{L}}$. Precisely if $\pi = \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$, then

$$\dim \mathcal{C}(\pi|_{G^{\bar{L}}}) = \dim \mathcal{C}(\pi|_G),$$

where $\mathcal{C}$ indicates the commuting algebra.

PROOF. The proof given by Gelbart-Knapp [G-K1] for the principal series carries over to the case of an arbitrary standard parabolic subgroup of $\tilde{G}$.

Define $H = ZG$, with Z being the scalar matrices. Apply the Restriction Theorem 3.2.1 to π with the subgroup H of $\tilde{G}$. Then

$$X_H(\pi) = \{\nu \in G^\vee \mid \nu|_H = 1 \text{ and } \pi \otimes \nu \cong \pi\},$$

and $\dim \mathcal{C}(\pi|_H) = |X_H(\pi)| = \dim \mathcal{C}(\pi|_G)$, since Z acts by scalars. Now define

$$H' = \{g \in \tilde{G} \mid \nu(g) = 1 \quad \text{for all } \nu \in X_H(\pi)\}.$$

Then $H \subseteq H'$, and we apply the Restriction Theorem 3.2.1 again, this time with H'. By definition $X_H = X_{H'}$, so $\dim \mathcal{C}(\pi|_{H'}) = \dim \mathcal{C}(\pi|_H)$. We need to show that $H' = G^{\bar{L}}$. For any character η in $G^\vee$,

$$\eta \in X_H(\pi) \quad \text{if and only if} \quad \eta \otimes \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma}) \cong \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma}). \qquad (1)$$

The left hand side of the equivalence in (1) is equivalent to $\mathrm{Ind}(\tilde{\sigma} \otimes \eta)$ (Lemma 3.1.2), and therefore we want η such that $\mathrm{Ind}(\tilde{\sigma}) \cong \mathrm{Ind}(\tilde{\sigma} \otimes \eta)$. This occurs exactly when $\tilde{\sigma} \otimes \eta \cong w\tilde{\sigma}$, for some $w \in W_G(M)$. Thus

$$X_H(\pi) = \bar{L}(\tilde{\sigma}).$$

Passing to kernels, we have our result. Q.E.D.

IV. Discrete Series of M and the R group

§4.1. Discrete Series Results.

Let $G = G_n$, and suppose M is the Levi component for a standard parabolic subgroup of G. Let σ be a discrete series representation of M. We presently investigate the discrete series of M, and the purpose of this section is to provide certain results that are necessary for this investigation. The lemmas that follow regard equivalences between various discrete series representations and between certain induced representations.

Let $\tilde{G} = \tilde{G}_n$, and let $\tilde{M}$ be the Levi component of $\tilde{G}$ corresponding to M. Write

$$\tilde{M} \cong \tilde{G}_{\tilde{n}_1} \times \tilde{G}_{\tilde{n}_2} \times \ldots \times \tilde{G}_{\tilde{n}_k}.$$

If σ is in $\mathcal{E}_2(M)$, we have seen that there exists some $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. The work of Flath [Fl] shows that $\tilde{\sigma}$ can be written as $\tilde{\sigma} \cong \otimes_{i=1}^{k} \tilde{\sigma}_i$, where $\tilde{\sigma}_i \in \mathcal{E}_2(\tilde{G}_{\tilde{n}_i})$. Henceforth we write $\tilde{\sigma}$ in this fashion without further reference to its justification.

Lemma 4.1.1. (ref. Flath [Fl]) Let $\tilde{M}$ be a Levi component for a standard parabolic subgroup of $\tilde{G}$. Let π and ρ be in $\mathcal{E}_2(\tilde{M})$, and write $\pi \cong \pi_1 \otimes \ldots \otimes \pi_k$ and $\rho \cong \rho_1 \otimes \ldots \otimes \rho_k$. If $\pi \cong \rho$, then $\pi_i \cong \rho_i$ for each i with $1 \leq i \leq k$.

Our second result is very important. It says if σ is in $\mathcal{E}_2(M)$ and we form $\bar{P} = M\bar{N}$, where $\bar{N}$ is the transpose of N, then the standard unitary induced representations of G obtained by inducing σ from P or $\bar{P}$ are equivalent. Combining this result with Lemma 3.1.3, we have $\mathrm{Ind}_{MN}^{G}(\sigma) \cong \mathrm{Ind}_{MN}^{G}(w\sigma)$ for any $w \in N_G(M)$.

Theorem 4.1.2 (cf. Silberger [Sil Theorems 2.5.8, 5.4.4.1]). Let G' be a closed subgroup of $\tilde{G}_n$ containing G_n, and let $P' = M'N$ be a standard parabolic subgroup of G'. Suppose σ is in $\mathcal{E}_2(M')$. Then

$$\operatorname{Ind}_{M'N}^{G'}(\sigma) \cong \operatorname{Ind}_{M'\bar{N}}^{G'}(\sigma).$$

Lemma 4.1.3. Let $\tilde{M}$ be a Levi component of $\tilde{G}$. Suppose that M' and H are closed normal subgroups of finite index in $\tilde{M}$ with the property $H \subseteq M'$. Let g be in $\tilde{M}$. If τ is an irreducible representation of H, then

$$\operatorname{Ind}_H^{M'}(g\tau) \cong g \operatorname{Ind}_H^{M'}(\tau).$$

Proof. The representation $g \operatorname{Ind}_H^{M'}(\tau)$ by definition acts on the same space as $\operatorname{Ind}_H^{M'}(\tau)$, written $V^{\operatorname{Ind}(\tau)}$. Let the map E on $V^{\operatorname{Ind}(\tau)}$ be given by

$$(Ef)(x) = f^g(x) = f(g^{-1}xg).$$

Then if f is in $V^{\operatorname{Ind}(\tau)}$,

$$\begin{aligned}(Ef)(mh) &= f^g(mh)\\ &= f(g^{-1}mhg)\\ &= \tau(g^{-1}h^{-1}g)f(g^{-1}mg)\\ &= (g\tau)(h^{-1})f^g(m)\\ &= (g\tau)(h^{-1})(Ef)(m),\end{aligned}$$

for any $m \in M'$ and $h \in H$. Thus E maps $V^{\mathrm{Ind}(\tau)}$ into $V^{\mathrm{Ind}(g\tau)}$.

To see that E is one-one and onto, first suppose $Ef = 0$. Then $f(x) = (Ef)(gxg^{-1}) = 0$ for all $x \in M'$, and E is one-one. Moreover if $\tilde{f}$ is in $V^{\mathrm{Ind}(g\tau)}$, then $\tilde{f}^{g^{-1}}$ is in $V^{\mathrm{Ind}(\tau)}$ and maps to $\tilde{f}$ under E.

To check that the action of M' is preserved, let $g\pi = g\mathrm{Ind}_H^{M'}(\tau)$, and let $\pi_g = \mathrm{Ind}_H^{M'}(g\tau)$. Then

$$\begin{aligned}
\pi_g(g')(Ef)(x) &= (Ef)({g'}^{-1}x) \\
&= f^g({g'}^{-1}x) \\
&= f(g^{-1}{g'}^{-1}xg) \\
&= \big(\pi(g^{-1}g'g)f\big)(g^{-1}xg) \\
&= (g\pi(g')f)^g(x) \\
&= E(g\pi(g')f)(x).
\end{aligned}$$

Thus E is an equivalence operator between $g\pi$ and π_g. Q.E.D.

Lemma 4.1.4. Let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$, and write $\tilde{\sigma} \cong \otimes_{i=1}^k \tilde{\sigma}_i$. Let $\bar{w}$ be in $W_{\tilde{G}}(\tilde{M})$, and suppose r is the corresponding permutation on $\{1, \ldots, k\}$. Let $\tilde{\sigma}' \cong \otimes_{i=1}^k \tilde{\sigma}_{r(i)}$. If $V^{\tilde{\sigma}} = V_1 \otimes \ldots \otimes V_k$, then the tensor product isomorphism

$$V_1 \otimes \ldots \otimes V_k \cong V_{r(1)} \otimes \ldots \otimes V_{r(k)} \tag{1}$$

induces the equivalence $w\tilde{\sigma} \cong \tilde{\sigma}'$.

PROOF. As vector spaces, we have $V_1 \times \ldots \times V_k \cong V_{r(1)} \times \ldots \times V_{r(k)}$, the isomorphism being given left to right by $(v_1, \ldots, v_k) \mapsto (v_{r(1)}, \ldots, v_{r(k)})$. Lift for any $m \in M'$ and $h \in H$. Thus E maps $V^{\mathrm{Ind}(\tau)}$ into $V^{\mathrm{Ind}(g\tau)}$.

To see that E is one-one and onto, first suppose $Ef = 0$. Then $f(x) = (Ef)(gxg^{-1}) = 0$ for all $x \in M'$, and E is one-one. Moreover if $\tilde{f}$ is in $V^{\mathrm{Ind}(g\tau)}$, then $\tilde{f}^{g^{-1}}$ is in $V^{\mathrm{Ind}(\tau)}$ and maps to $\tilde{f}$ under E.

To check that the action of M' is preserved, let $g\pi = g\mathrm{Ind}_H^{M'}(\tau)$, and let $\pi_g = \mathrm{Ind}_H^{M'}(g\tau)$. Then

$$\begin{aligned}
\pi_g(g')(Ef)(x) &= (Ef)({g'}^{-1}x) \\
&= f^g({g'}^{-1}x) \\
&= f(g^{-1}{g'}^{-1}xg) \\
&= \big(\pi(g^{-1}g'g)f\big)(g^{-1}xg) \\
&= (g\pi(g')f)^g(x) \\
&= E(g\pi(g')f)(x).
\end{aligned}$$

Thus E is an equivalence operator between $g\pi$ and π_g. Q.E.D.

Lemma 4.1.4. Let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$, and write $\tilde{\sigma} \cong \otimes_{i=1}^k \tilde{\sigma}_i$. Let $\tilde{w}$ be in $W_{\tilde{G}}(\tilde{M})$, and suppose r is the corresponding permutation on $\{1, \ldots, k\}$. Let $\tilde{\sigma}' \cong \otimes_{i=1}^k \tilde{\sigma}_{r(i)}$. If $V^{\tilde{\sigma}} = V_1 \otimes \ldots \otimes V_k$, then the tensor product isomorphism

$$V_1 \otimes \ldots \otimes V_k \cong V_{r(1)} \otimes \ldots \otimes V_{r(k)} \tag{1}$$

induces the equivalence $w\tilde{\sigma} \cong \tilde{\sigma}'$.

PROOF. As vector spaces, we have $V_1 \times \ldots \times V_k \cong V_{r(1)} \times \ldots \times V_{r(k)}$, the isomorphism being given left to right by $(v_1, \ldots, v_k) \mapsto (v_{r(1)}, \ldots, v_{r(k)})$. Lift this map to the tensor products through the universal mapping property, and call the lifting E. If $g = (g_1, \ldots, g_k)$, then

$$\begin{aligned} E\big((w\tilde{\sigma})(g)(v_1 \otimes \ldots \otimes v_k)\big) &= E\big(\tilde{\sigma}(w^{-1}(g_1, \ldots, g_k)w)(v_1 \otimes \ldots \otimes v_k)\big) \\ &= E\big(\tilde{\sigma}(g_{r^{-1}(1)}, \ldots, g_{r^{-1}(k)})(v_1 \otimes \ldots \otimes v_k)\big) \\ &= \tilde{\sigma}_{r(1)}(g_1)v_{r(1)} \otimes \ldots \otimes \tilde{\sigma}_{r(k)}(g_k)v_{r(k)} \\ &= \tilde{\sigma}'(g_1, \ldots, g_k)(v_{r(1)} \otimes \ldots \otimes v_{r(k)}) \\ &= \tilde{\sigma}'(g_1, \ldots, g_k)\big(E(v_1 \otimes \ldots \otimes v_k)\big). \end{aligned}$$

Therefore E is an equivalence operator between $w\tilde{\sigma}$ and $\tilde{\sigma}'$. Q.E.D.

The rest of this section is not used until Section 3, in which we begin the process of describing the discrete series in terms of the R group. If $\tilde{\sigma} \cong \otimes_{i=1}^{k} \tilde{\sigma}_i$, we will see that the R group provides a nice expression for some rearrangement of the $\tilde{\sigma}_i$ (or equivalently, the blocks of $\tilde{M}$). The representation $\tilde{\sigma}'$ achieved by this rearrangement is not generally equivalent to $\tilde{\sigma}$; in fact, rearranging the blocks of $\tilde{M}$ usually produces a new (though isomorphic) group, and then the notion of equivalence is not valid. We would like to say, however, that these representations are strongly related, and this desire motivates the following definition.

Definition. Let H and H' be isomorphic groups, and let $\varphi : H \to H'$ be an isomorphism. Suppose τ and τ' are representations of the respective groups. Then we say that τ and τ' are **similar** if there exists a vector space isomorphism $A : V^{\tau} \to V^{\tau'}$ such that

$$A\tau(h)v = \tau'(\varphi(h))Av$$

for all $h \in H$ and $v \in V^\tau$. When τ and τ' are similar, we write $\tau \approx \tau'$.

If $\tilde{M}$ is a Levi component of $\tilde{G}$ written in block diagonal form and if $\tilde{M}'$ is a reordering of the blocks of $\tilde{M}$, then $\tilde{M}' = s^{-1}\tilde{M}s$, where s is some standard permutation matrix in $\tilde{G}$. The map $\varphi : \tilde{G} \to \tilde{G}$ given by $g \mapsto s^{-1}gs$ is an automorphism of $\tilde{G}$, and maps $N_{\tilde{G}}(\tilde{M})$ to $N_{G'}(\tilde{M}')$. This fact will be used extensively in Sections 3 and 4.

EXAMPLE. Let $\tilde{M} = \tilde{G}_n \times \tilde{G}_{n'} \times \tilde{G}_n$, and suppose that $\tilde{\sigma} \cong \tau \otimes \tau' \otimes \eta\tau$ for some τ in $\mathcal{E}_2(\tilde{G}_n)$, τ' in $\mathcal{E}_2(\tilde{G}_{n'})$, and $\eta \in (F^\times)^\vee$. Let $\tilde{M}' = \tilde{G}_n \times \tilde{G}_n \times \tilde{G}_{n'}$. Then $\tilde{M} \cong \tilde{M}'$, and $\tilde{\sigma} \approx \tau \otimes \eta\tau \otimes \tau'$.

§4.2. The Discrete Series of M.

Let M be the Levi component for a standard parabolic subgroup of $G = G_n$, and let σ be in $\mathcal{E}_2(M)$. Let $\tilde{M}$ be the corresponding Levi component in $\tilde{G} = \tilde{G}_n$, and let $\tilde{\sigma}$ be a representation in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Define

$$X(\tilde{\sigma}) = X_M(\tilde{\sigma}) = \{\eta \in (F^\times)^\vee \mid \eta \otimes \tilde{\sigma} \cong \tilde{\sigma}\}.$$

Mackey theory allows us to use the group $X(\tilde{\sigma})$ to classify the discrete series of M.

Let $M' = Z(\tilde{G})M$. Then M' has finite index in $\tilde{M}$, and the Restriction Theorem (Theorem 3.2.1) says that $X_{M'}(\tilde{\sigma})$ is a finite group. This group describes the commuting algebra of $\tilde{\sigma}|_{M'}$, which is also the commuting algebra of $\tilde{\sigma}|_M$ since $Z(\tilde{G})$ acts as scalars. Thus $X(\tilde{\sigma})$ is a finite group. This fact, combined with the multiplicity-one property below, allows us to parameterize the constituents of $\tilde{\sigma}|_M$ by means of $X(\tilde{\sigma})$ and, consequently describe the elements of $\mathcal{E}_2(M)$.

Lemma 4.2.1 (Multiplicity One of $\tilde{\sigma}$). Suppose $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$. Then the representation

$$\mathrm{Ind}_{MN}^{G}(\tilde{\sigma}|_M) \cong \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})\,|_G,$$

which is the finite direct sum of irreducible representations, is multiplicity free. As a consequence, every constituent of $\tilde{\sigma}|_M$ appears with multiplicity one.

Remarks.

(1) The isomorphism cited in the lemma is Lemma 3.4.1.

(2) The proof is basically that found in Labesse-Langlands [L-L] for $SL(2)$; see also Shahidi [Shd].

(3) It follows from this lemma and Lemma 3.4.1 that the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ occur with multiplicity one. This result was originally proved by Howe and Silberger [H-S] by different means.

Proof. We write $\pi = \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$. Recall that this representation is irreducible (Theorem 3.1.4), and note that the reducibility of $\pi|_G$ actually occurs on a subgroup G' of finite index in $\tilde{G}$, by Proposition 3.4.2. Assume $\tau \subseteq \pi|_{G'}$. We are to show that

$$\dim\ \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) = 1. \qquad (1)$$

This is accomplished by using the theory of Whittaker Models. Let N_0 be the unipotent radical of the standard minimal parabolic subgroup of G_n. A character ψ of N_0 is said to be non-degenerate if for $s \in N_G(M_0)$ with $s \notin M_0$, we have

$$\psi|_{N_0 \cap s^{-1}\bar{N}_0 s} \neq 1.$$

A representation π of $\tilde{G}$ is said to have a Whittaker Model for ψ if π embeds into $\mathrm{Ind}_N^{\tilde{G}}(\psi)$. The work of Jacquet [Jac] shows that π has a Whittaker model for each nondegenerate character ψ of N_0. Fix such a character ψ, and let φ be a nonzero $\tilde{G}$ map of π into $\mathrm{Ind}_N^{\tilde{G}}(\psi)$. It is known that Whittaker models have the multiplicity one property [Shk, G-Ka], and thus φ is unique up to a scalar:

$$\dim\ \mathrm{Hom}_{\tilde{G}}(\pi, \mathrm{Ind}_N^{\tilde{G}}(\psi)) = 1. \tag{2}$$

For any representation ρ of G', Mackey theory gives

$$\mathrm{Ind}_{G'}^{\tilde{G}}(\rho)\big|_{G'} \cong \bigoplus_j g_j\rho,$$

the sum taken over a complete set of coset representatives g_j for $\tilde{G}/G'$. Applying this formula to $\rho = \mathrm{Ind}_N^{G'}(\psi)$, we have

$$\mathrm{Ind}_N^{\tilde{G}}(\psi)\Big|_{G'} \cong \mathrm{Ind}_{G'}^{\tilde{G}}\left(\mathrm{Ind}_N^{G'}(\psi)\right)\Big|_{G'} \cong \bigoplus_j g_j \mathrm{Ind}_N^{G'}(\psi). \tag{3}$$

The map φ above is one-one since π is irreducible. When φ is composed with an inclusion $\tau \subseteq \pi|_{G'}$, it induces a nonzero G' map of τ into $\mathrm{Ind}_N^{\tilde{G}}(\psi)$. By

(3), we obtain a nonzero G' map φ' of τ into $g_j \mathrm{Ind}_N^{G'}(\psi)$ for some $g_j \in \tilde{G}$. The map φ' is one-one since τ is irreducible. Then we have

$$\begin{aligned}
\dim\ \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) &= \dim\ \mathrm{Hom}_{\tilde{G}}(\pi, \mathrm{Ind}_{G'}^{\tilde{G}}(\tau)) \quad \text{by Frobenius Recipocity} \\
&\leq \dim\ \mathrm{Hom}_{\tilde{G}}\Big(\pi, \mathrm{Ind}_{G'}^{\tilde{G}}\Big(g_j \mathrm{Ind}_N^{G'}(\psi)\Big)\Big) \\
&\qquad \text{via composition with } \mathrm{Ind}(\varphi') \\
&= \dim\ \mathrm{Hom}_{\tilde{G}}\Big(\pi, \mathrm{Ind}_{G'}^{\tilde{G}}\Big(\mathrm{Ind}_N^{G'}(\psi)\Big)\Big) \quad \text{by Lemma 3.1.3} \\
&= \dim\ \mathrm{Hom}_{\tilde{G}}\Big(\pi, \mathrm{Ind}_N^{\tilde{G}}(\psi)\Big) \\
&= 1 \qquad \text{by (2)},
\end{aligned}$$

and (1) follows. Q.E.D.

If σ is in $\mathcal{E}_2(M)$, then the Restriction Theorem 3.2.1d shows that the representations $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ that contain σ in their restriction to M differ from one another only by a character of $F^\times$. We define an equivalence relation on $\mathcal{E}_2(\tilde{M})$: $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$ if and only if $\tilde{\sigma}_1 \cong \tilde{\sigma}_2 \otimes \eta$ for some $\eta \in (F^\times)^\vee$. As a result, the constituents of $\tilde{\sigma}_1|_M$ and $\tilde{\sigma}_2|_M$ match when $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$, and are otherwise disjoint.

The irreducible constituents of $\tilde{\sigma}|_{M^{X(\tilde{\sigma})}}$, according to the lemma above (Lemma 4.2.1) and the Restriction Theorem (Theorem 3.2.1) restrict irreducibly to distinct elements of the discrete series of M. This one-one correspondence between the irreducible constituents of $\tilde{\sigma}|_{M^{X(\tilde{\sigma})}}$ and the irreducible constituents of $\tilde{\sigma}|_M$ allows us to classify $\mathcal{E}_2(M)$ by explicitly describing the constituents of $\tilde{\sigma}|_{M^{X(\tilde{\sigma})}}$ for all $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$. The crucial concepts in this process are illustrated in the example below, in which $\tilde{M}$ has only two blocks and $X(\tilde{\sigma})$ is cyclic. This example is chosen because it is the simplest situation for which Mackey theory sheds new light on the irreducible constituents of $\tilde{\sigma}|_M$; if $\tilde{M}$ is a single block, the restriction of $\tilde{\sigma}$ to M, for $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$, produces the discrete series of SL_n, and the relationship among constituents of a fixed $\tilde{\sigma}$ is already seen in the Restriction Theorem.

Example. Let $\tilde{G} = GL_n$, and let $\tilde{M}$ be a Levi component of $\tilde{G}$ of the form $\tilde{M} \cong GL_{\bar{n}_1} \times GL_{\bar{n}_2}$. To simplify our notation, we write $\tilde{M} \cong \tilde{G}_1 \times \tilde{G}_2$. Suppose that $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$ and that $\tilde{\sigma} \cong \tilde{\sigma}_1 \otimes \tilde{\sigma}_2$. Let

$$X = X(\tilde{\sigma}) = \{\eta \in (F^\times)^\vee \mid \eta \otimes \tilde{\sigma} \cong \tilde{\sigma}\}.$$

Assume that X is cyclic with generator ζ. Let

$$G_i^X = \{g \in \tilde{G}_i \mid \zeta(g) = 1\}.$$

Then $\zeta \in X(\tilde{\sigma}_i)$ for $i = 1, 2$, by Lemma 4.1.1. Multiplicity One and the Restriction Theorem show that $\tilde{\sigma}_i|_{G_i^X}$ contains exactly $r = |X|$ irreducible constituents, and they are inequivalent. As a result, $\tilde{\sigma}|_{G_1^X \times G_2^X}$ contains exactly r^2 irreducible constituents, and they are inequivalent.

On the other hand, $\tilde{\sigma}|_{M^X}$ has $r = |X|$ irreducible inequivalent constituents, and the index of $G_1^X \times G_2^X$ in M^X is r. The constituents of $\tilde{\sigma}|_{G_1^X \times G_2^X}$ are partitioned into sets of r representations that induce equivalently and irreducibly to M^X, thereby recovering the irreducible constituents of $\tilde{\sigma}|_{M^X}$. In fact, if ρ is an irreducible constituent of $\tilde{\sigma}|_{M^X}$, the representations in $\rho|_{G_1^X \times G_2^X}$ form one of the aforementioned sets. To see this, apply part (c) of the Restriction Theorem to ρ. If τ is a constituent of $\tilde{\sigma}|_{G_1^X \times G_2^X}$, then the set containing τ consists of $\{g_i\tau\}$, where the g_i are a complete set of coset representatives for $M^X/(G_1^X \times G_2^X)$. By Multiplicity One, the representations in $\{g_i\tau\}$ are inequivalent, and thus by Corollary 3.2.4, $\rho \cong \mathrm{Ind}_{G_1^X \times G_2^X}^{M^X}(g_i\tau)$ for any g_i. Consequently inducing one representation from each set recovers the irreducible constituents contained in $\tilde{\sigma}|_{M^X}$.

Fix a constituent ρ of $\tilde{\sigma}|_{M^X}$. By the Restriction Theorem 3.2.1c,

$$\tilde{\sigma} \cong \rho \oplus \tilde{\varepsilon}\rho \oplus \cdots \oplus \tilde{\varepsilon}^{r-1}\rho,$$

where ε is a member of $F^\times$ whose coset generates $F^\times/N_X$ and $\tilde{\varepsilon} \in \tilde{M}$ is defined as in Section 2.2 by $\tilde{\varepsilon} = \mathrm{diag}(\varepsilon, 1, \ldots, 1)$. Let ρ_1 and ρ_2 be irreducible constituents of $\tilde{\sigma}_1|_{G_1^X}$ and $\tilde{\sigma}_2|_{G_2^X}$, respectively, with the property that $\rho_1 \otimes \rho_2 \subseteq \rho|_{G_1^X \times G_2^X}$. The above remarks show that $\tilde{\sigma}|_{M^X}$ can be written explicitly as

$$\tilde{\sigma}|_{M^X} \cong \bigoplus_{\ell=0}^{r-1} \tilde{\varepsilon}^\ell \, \mathrm{Ind}_{G_1^X \times G_2^X}^{M^X}(\rho_1 \otimes \rho_2). \tag{1}$$

The representation $\tilde{\varepsilon}^\ell \, \mathrm{Ind}_{G_1^X \times G_2^X}^{M^X}(\rho_1 \otimes \rho_2)$ on the right hand side of (1) is actually equivalent to $\mathrm{Ind}_{G_1^X \times G_2^X}^{M^X}(\tilde{\varepsilon}^\ell(\rho_1 \otimes \rho_2))$, by Lemma 4.1.3. Moreover since $\tilde{\varepsilon}$ is diagonal, $\tilde{\varepsilon}^\ell(\rho_1 \otimes \rho_2) \cong (\tilde{\varepsilon}^\ell\rho_1) \otimes \rho_2$, where the $\tilde{\varepsilon}$ on the right is understood to be in G_1. Therefore

$$\tilde{\sigma}|_{M^X} \cong \bigoplus_{\ell=0}^{r-1} \mathrm{Ind}_{G_1^X \times G_2^X}^{M^X}\left(\tilde{\varepsilon}^\ell\rho_1 \otimes \rho_2\right). \tag{2}$$

Suppose now that $\sigma \in \mathcal{E}_2(M)$ and that $\sigma \subseteq \tilde{\sigma}|_M$. We know that the reducibility of $\tilde{\sigma}|_M$ occurs already in the restriction to M^X. Therefore each term on the right hand side of (1) is irreducible when restricted to M, and σ must be equivalent to one of these restrictions. Consequently we have determined the form of all the members of $\mathcal{E}_2(M)$ contained within $\tilde{\sigma}|_M$.

Theorem 4.2.2 below generalizes this example and provides a classification of $\mathcal{E}_2(M)$, introducing the notation we will use to parameterize these representations. First we show that for any finite subgroup X of $(F^\times)^\vee$ and set of representations satisfying some conditions, we can construct elements of $\mathcal{E}_2(M)$. Then we start with an element σ of $\mathcal{E}_2(M)$ and an element $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ that contains σ upon restriction to M. We consider the restriction of $\tilde{\sigma}$ to M^X. Using Mackey theory, we are able to obtain an explicit description of $\tilde{\sigma}|_{M^X}$, which yields a set of parameters for the constituents of $\tilde{\sigma}|_M$.

Theorem 4.2.2 (Classification of the Discrete Series of M). Let $\tilde{G} = \tilde{G}_n$, and let $\tilde{M}$ be a Levi component for $\tilde{G}$ of the form $\tilde{M} \cong GL_{\tilde{n}_1} \times \ldots \times GL_{\tilde{n}_k}$. Set $\tilde{G}_i = GL_{\tilde{n}_i}$. If X is a finite subgroup of $(F^\times)^\vee$, let G_i^X be the subgroup of $\tilde{G}_i$ defined as $G_i^X = \{g \in \tilde{G}_i \mid \eta(g) = 1 \text{ for all } \eta \in X\}$. Let $\Pi = \{\pi_i \mid 1 \leq i \leq k\}$ be a set of representations with $\pi_i \in \mathcal{E}_2(G_i^X)$, and suppose $\cap_{i=1}^k X(\pi_i) = \{1\}$. If ε is a coset representative of $F^\times / N_X$ and $\tilde{\varepsilon} = \operatorname{diag}(\varepsilon, 1, \ldots, 1)$ is in $\tilde{G}_1$, then

$$\tau_\varepsilon = \operatorname{Ind}_{G_1^X \times G_2^X \times \ldots \times G_k^X}^{M^X} \begin{pmatrix} \tilde{\varepsilon}\pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_k \end{pmatrix}$$

is in $\tilde{\sigma}|_{M^X}$ for some $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$, and τ_ε restricts irreducibly to a member of $\mathcal{E}_2(M)$. Conversely assume σ is in $\mathcal{E}_2(M)$ and $\tilde{\sigma}$ is an element of $\mathcal{E}_2(\tilde{M})$ such that $\sigma \subseteq \tilde{\sigma}|_M$. Let

$$X = X(\tilde{\sigma}) = \{\zeta \in (F^\times)^\vee \mid \zeta \otimes \tilde{\sigma} \cong \tilde{\sigma}\}.$$

Then there exists $\pi_i \in \mathcal{E}_2(GL_i^X)$ for $1 \leq i \leq k$ such that

$$\tilde{\sigma}|_{M^X} \cong \bigoplus_{i=1}^{r} \mathrm{Ind}_{G_1^X \times \ldots \times G_k^X}^{M^X} \begin{pmatrix} \tilde{\varepsilon}_j \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_k \end{pmatrix}, \qquad (1)$$

where $\{\varepsilon_j\}$ is a set of coset representatives for $F^\times/N_X$, $\tilde{\varepsilon}_j = \mathrm{diag}(\varepsilon_j, 1, \ldots, 1)$ in $\tilde{G}_1$, $r = |X|$, and $\cap_{i=1}^k X(\pi_i) = \{1\}$. Each summand on the right hand side of (1) restricts irreducibly to M. Consequently σ is determined by the set $\{\pi_i\}$ and some ε_j.

REMARKS.

(1) The condition that $\cap X(\pi_i) = \{1\}$ ensures that all of the reducibility of $\tilde{\sigma}$ is accounted for on M^X.

(2) As in the example, the $\tilde{\varepsilon}_j$ factors can, equivalently, be written in several places. This fact follows from Lemma 4.1.3 and the diagonal expression for $\tilde{\varepsilon}_j$.

(3) In the case of a single block ($\tilde{M} \cong GL_n$), the irreducible constituents of $\tilde{\sigma}|_{M^X}$ restrict to give members of $\mathcal{E}_2(SL_n)$, as remarked earlier. In this case, the theorem is a restatement of parts (a) and (c) of the Restriction Theorem, with $G = GL_n$, $H = G^X$, and $\pi = \tilde{\sigma}$.

(4) The second half of the theorem does not determine exactly which ε_i is associated with σ. Instead the classification provides a form for members of $\mathcal{E}_2(M)$ by describing the unordered set of constituents of $\tilde{\sigma}|_M$; the set of these constituents is often referred to as a **packet** or ***L*-packet**.

PROOF. Let X be a finite subgroup of $(F^\times)^\vee$, and let Π a set of representations as in the hypotheses. By Corollary 3.2.5, there exists ρ in $\mathcal{E}_2(\tilde{G}_1)$ such that $\pi_1 \subseteq \rho|_{G_1^X}$. The Restriction Theorem 3.2.1c and Multiplicity One of ρ imply that $\rho|_{G_1^X} \cong \oplus\tilde{\varepsilon}\pi_1$, the direct sum taken over coset representatives ε of $F^\times/N_X$, and that $\pi_1 \not\cong \tilde{\varepsilon}\pi_1$ for any $\varepsilon \notin N_X$. Using Corollary 3.2.4, we see that

$$\tau_\varepsilon = \mathrm{Ind}_{G_1^X \times \ldots \times G_k^X}^{M_X} \begin{pmatrix} \tilde{\varepsilon}\pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_k \end{pmatrix}$$

is irreducible. Moreover $\tau_\varepsilon|_M$ is irreducible by our condition on the $X(\pi_i)$. Hence $\tau_\varepsilon|_M \in \mathcal{E}_2(M)$.

Now let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Write $\tilde{\sigma} \cong \otimes\tilde{\sigma}_i$ for $\tilde{\sigma}_i \in \mathcal{E}_2(\tilde{G}_i)$. Let $X = X(\tilde{\sigma})$. If $r = |X(\tilde{\sigma})|$, then $\tilde{\sigma}|_{M^X}$ contains r constituents, each occurring with multiplicity one, by the Restriction Theorem 3.2.1b. For any $\zeta \in X$, Lemma 4.1.1 says that ζ is also in $X(\tilde{\sigma}_i)$, and then $\tilde{\sigma}_i|_{G_i^X}$ has r constituents. Consequently the restriction of $\tilde{\sigma}$ to $G_1^X \times G_2^X \times \ldots \times G_k^X$ yields r^k representations, and a counting argument gives us the desired result. Specifically the index of $G_1^X \times G_2^X \times \ldots \times G_k^X$ in M^X is r^{k-1}, and the representations of $\tilde{\sigma}|_{G_1^X \times \ldots G_k^X}$ are partitioned into sets of r^{k-1} constituents that induce equivalently to M^X. Inducing one of the representations from each set produces r inequivalent irreducible representations on M^X, which are the constituents of $\tilde{\sigma}|_{M^X}$. Fix a constituent τ of $\tilde{\sigma}|_{M^X}$ and a constituent $\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_k$ of $\tau|_{G_1^X \times \ldots \times G_k^X}$. Applying the Restriction Theorem 3.2.1c allows us to write

$$\tilde{\sigma}|_{M^X} \cong \bigoplus_{i=1}^{r} \tilde{\varepsilon}_j \operatorname{Ind}_{G_1^X \times \ldots \times G_k^X}^{M^X} \begin{pmatrix} \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_k \end{pmatrix}, \tag{1}$$

where $\{\varepsilon_j\}$ is a complete set of coset representatives for $F^\times/N_X$. As in Remark (2), the $\tilde{\varepsilon}_j$ pass through to the inducing representation by Lemma 4.1.3. We know that all of the reducibility of $\tilde{\sigma}$ occurs on M^X (Lemma 3.4.2), and therefore each member of $\mathcal{E}_2(M)$ contained in $\tilde{\sigma}|_M$ is obtained by restricting some representation on the right hand side of (1) to M. Q.E.D.

§4.3. Discrete Series with a Given R group: the Case $M = M_0$.

Suppose that $M = M_0$ is the Levi component for the standard minimal parabolic of G_n and that $\tilde{M} = \tilde{M}_0$ is the corresponding Levi component in $\tilde{G}_n$. Let $G = G_n$, and let $\tilde{G} = \tilde{G}_n$. If σ is in $\mathcal{E}_2(M)$, we want to use the R group of σ to describe the elements of $\mathcal{E}_2(\tilde{M})$ that contain σ in their restriction to M.

Since M is the diagonal subgroup, σ is a unitary character of M, and if $\tilde{\sigma}$ is an element of $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$, then $\tilde{\sigma}|_M = \sigma$. Thus $\tilde{\sigma}$ can be written

$$\tilde{\sigma} \cong \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n,$$

with ξ_i in $(F^\times)^\vee$. Moreover $X(\tilde{\sigma}) = \{1\}$ since $\tilde{\sigma}$ is one-dimensional, and therefore $R = R(\sigma) \cong \bar{L}(\tilde{\sigma})$.

By Lemma 3.1.3 we may reorder the blocks of $\tilde{M}$ and consequently the ξ_i without affecting the class of the induced representation. Since $R = R(\sigma)$ is contained in $W_G(M) = W_{\tilde{G}}(\tilde{M})$, it is a subgroup of the permutations on the blocks of $\tilde{M}$. Let $\bar{w} \in R$, and let r be the corresponding permutation on $\{1, 2, \ldots, n\}$. In particular the (1×1) blocks can be reordered so that r is the product of disjoint cycles that are in increasing numerical order, with the 1-cycles of r all occurring at the end. Such a reordering is the restriction of an automorphism φ on $\tilde{G}$. Let $\varphi(\tilde{\sigma})$ denote the representation achieved by the same reordering of the ξ_i. Then $\varphi(N_{\tilde{G}}(\tilde{M}))$ acts on $\varphi(\tilde{\sigma})$.

Choose as the coset representative for $\varphi(\bar{w})$ the standard permutation matrix corresponding to r. Specifically suppose

$$r = (1 \ \cdots \ n_1)(n_1 + 1 \ \cdots \ n_2) \cdots (n_{\ell-1} + 1 \ \cdots \ n_\ell), \tag{1}$$

and let w_i be the matrix in $\tilde{G}_{n_i - n_{i-1}}$ defined by

$$w_i = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

where we take n_0 to be 0. Then the block diagonal matrix

$$w = (w_1, w_2, \ldots, w_\ell, 1, \ldots, 1)$$

is a coset representative of $\varphi(\bar{w})$ and is the permutation matrix corresponding to r.

Let $\tilde{\sigma}' = \varphi(\tilde{\sigma})$. Recall from the definition in Section 4.1 that $\tilde{\sigma}$ is "similar" to $\tilde{\sigma}'$, and we write $\tilde{\sigma} \approx \tilde{\sigma}'$. For simplicity, rewrite $\tilde{\sigma}' \cong \otimes_{i=1}^{k} \xi_i$.

Using w as defined above, we compute that

$$\begin{aligned} w\tilde{\sigma}'(g_1, g_2, \ldots, g_n) &= \tilde{\sigma}'(g_{r^{-1}(1)}, g_{r^{-1}(2)}, \ldots, g_{r^{-1}(n)}) \\ &= \xi_1(g_{r^{-1}(1)}) \otimes \xi_2(g_{r^{-1}(2)}) \otimes \ldots \otimes \xi_n(g_{r^{-1}(n)}). \end{aligned}$$

Since $\tilde{\sigma}'$ is one-dimensional, this is

$$\begin{aligned} &= \xi_1(g_{r^{-1}(1)})\xi_2(g_{r^{-1}(2)}) \ldots \xi_n(g_{r^{-1}(n)}) \\ &= \xi_{r(1)}(g_1) \otimes \xi_{r(2)}(g_2) \otimes \ldots \otimes \xi_{r(n)}(g_n). \end{aligned}$$

If η is the character in $\bar{L}(\tilde{\sigma})$ associated to $\bar{w}$, then $\varphi^{-1}(w)$ is in $\bar{w}$, and the condition that $\eta \otimes \tilde{\sigma} \cong \varphi^{-1}(w)\tilde{\sigma}$ forces $\eta \otimes \tilde{\sigma}' \cong w\tilde{\sigma}'$ under the automorphism φ. Therefore Lemma 4.1.4 implies that

$$\begin{aligned} (\eta \otimes \xi_1(g_1)) \otimes \ldots \otimes (\eta \otimes \xi_n(g_n)) &= (\eta \otimes \tilde{\sigma}')(g_1, \ldots, g_n) \\ &= w\tilde{\sigma}'(g_1, \ldots, g_n) \\ &= \xi_{r(1)}(g_1) \otimes \ldots \otimes \xi_{r(n)}(g_n), \end{aligned}$$

and thus

$$\eta\xi_i = \xi_{r(i)} \quad \text{for all } i. \tag{2}$$

Proposition 4.3.1. Let $M = M_0$ be the Levi component for the standard minimal parabolic of G. Let σ be in $\mathcal{E}_2(M)$. Let $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ have the property that $\tilde{\sigma}|_M = \sigma$. Suppose $\bar{w}$ in R has order m for some $m > 1$, and suppose η is the corresponding character of order m in $\bar{L}(\tilde{\sigma})$. Let the permutation r associated to $\bar{w}$ be written as in equation (1). Then r is the product of ℓ disjoint m cycles, where $\ell = n/m$. Consequently there exist characters $\kappa_i \in (F^\times)^\vee$ for $1 \le i \le \ell$ such that

$$\tilde{\sigma} \approx \Big(\kappa_1 \otimes \ldots \otimes (\eta^{m-1}\kappa_1)\Big) \otimes \ldots \otimes \Big(\kappa_\ell \otimes \ldots \otimes (\eta^{m-1}\kappa_\ell)\Big).$$

PROOF. Let $\tilde{\sigma}'$ be the representation on $\varphi(\tilde{M})$ determined by reordering the blocks of $\tilde{M}$ so that r can be written as in (1), and write $\tilde{\sigma}' \cong \otimes_{i=1}^{\ell}\xi_i$. By equation (2), we can write $\tilde{\sigma}'$ as

$$\tilde{\sigma}' \cong \bigotimes_{h=1}^{\ell} \Big(\xi_{n_{h-1}+1} \otimes \ldots \otimes (\eta^{n'_h}\xi_{n_{h-1}+1})\Big) \otimes \xi_{n_\ell+1} \otimes \ldots \otimes \xi_n,$$

where $n_0 = 0$, and $n'_h = n_h - n_{h-1}$. We claim that each $n'_h = m$ and that $n_\ell = n$. In fact since $\eta \otimes (\eta^{n'_h - 1}\xi_{n_h+1}) = \eta^{n'_h}\xi_{n_h+1}$, equation (2) implies that $\eta \otimes (\eta^{n'_h-1}\xi_{n_{h-1}+1}) = \eta^{n'_h}\xi_{n_{h-1}+1} = \xi_{n_{h-1}+1}$, but this implies that $\eta^{n'_h} = 1$. Thus m divides n'_h. But the permutation r has order m; so n'_h divides m, and we must have $m = n'_h$. Similarly equation (2) implies that $\eta\xi_{n_\ell+j} = \xi_{n_\ell+j}$ for each j with $1 \le j \le n - n_\ell$, which is not possible since $\eta \ne 1$, and therefore $n_\ell = n$. Let $\kappa_h = \xi_{n_{h-1}+1}$, and the result follows. Q.E.D.

REMARK. When the R group is cyclic, this proposition fully describes the possibilities for the $\tilde{\sigma}$'s in $\mathcal{E}_2(\tilde{M})$ containing σ in their restriction to M, up to similarity.

We use the proposition to describe fully the elements of $\mathcal{E}_2(\tilde{M})$ that contain σ in their restriction to M. The following definition simplifies the final expression for $\tilde{\sigma}$ determined in Proposition 4.3.2 below. When studying the generalized principal series, we shall make an analogous definition.

Definition. Suppose $\eta_1, \ldots, \eta_\ell$ are characters in $(F^\times)^\vee$ with respective orders $m_1, \ldots, m_\ell$. Let $m(i) = m_1 m_2 \cdots m_i$. Fix κ in $(F^\times)^\vee$, and define the representation $\Omega(\eta_1, \ldots, \eta_i, \kappa)$ for $1 \leq i \leq \ell$ on $\prod_{j'=1}^{m(i)} F^\times$ recursively by

$$\Omega(\eta_1, \kappa) = \kappa \otimes \eta_1 \kappa \otimes \ldots \otimes \eta_1^{m_1 - 1} \kappa$$

and

$$\Omega(\eta_1, \ldots, \eta_i, \kappa) = \bigotimes_{j=0}^{m_i - 1} \eta_i^j \, \Omega(\eta_1, \ldots, \eta_{i-1}, \kappa).$$

Proposition 4.3.2. Let $M = M_0$ be the Levi component of the standard minimal parabolic subgroup of G_n, and let σ be in $\mathcal{E}_2(M)$. Suppose that $R = R(\sigma)$, and write

$$R \cong \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_v\mathbb{Z},$$

with each $m_j > 1$. Write $|R| = m$. Let $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_v$ be a set of generators for the respective factors of R, and let $\eta_1, \ldots, \eta_v$ be the associated generators of $\bar{L}(\tilde{\sigma})$. Then

$$\tilde{\sigma} \approx \Omega(\eta_1, \ldots, \eta_v, \zeta_1) \otimes \ldots \otimes \Omega(\eta_1, \ldots, \eta_v, \zeta_s),$$

where $s = n/m$ and ζ_i is in $(F^\times)^\vee$ for $1 \le i \le s$.

Proof. Let S' be an ordered subset of S, and proceed by induction on the number of characters in S'. Proposition 4.3.1 handles the case that S' is a single character. Suppose the result holds for $S' = \{\eta_1, \eta_2, \dots, \eta_{v'-1}\}$. Then for some subsets C_i' of S' with $1 \le i \le \ell$, and sets of discrete series representations $\{\tau_i\}_{i=1}^{\ell}$ and $\{\rho_{\ell'}\}_{l'=1}^{s}$, we can write

$$\tilde{\sigma} \approx \Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_1) \otimes \dots \otimes \Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_\ell), \tag{3}$$

where $\ell = n/m(v'-1)$. The right hand side of (3) is a representation of some reordering of the blocks of $\tilde{M}$, say $\tilde{M}'$.

Now suppose that $S' = \{\eta_1, \eta_2, \dots, \eta_{v'}\}$. The character $\eta_{v'}$ in $\bar{L}(\tilde{\sigma})$ induces a permutation $r_{v'}$ on the blocks of $\tilde{M}'$.

Fix j with $1 \le j \le \ell$. We show that for each such j the orbit of $\Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_j)$ under the powers of $r_{v'}$ is as claimed. Since equation (2) implies that $\eta_{v'}\kappa_j$ is equal to its image under $r_{v'}$, we have the following two possibilities:

(1) $\eta_2\kappa_j = \zeta\kappa_j$, where $\zeta\kappa_j$ is contained in $\Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_j)$. But this implies that $\eta_2 = \zeta$, which contradicts the assumption that $\eta_{v'}$ generates a distinct factor of $\bar{L}(\tilde{\sigma})$. Therefore this can not happen.

(2) $\eta_2\kappa_j = \zeta\kappa_{j'}$ for some $j' \ne j$ and for some ζ such that $\zeta\kappa_{j'}$ is contained in $\Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_{j'})$. Then we must also have

$$\eta_{v'}\zeta'\kappa_j = \zeta\zeta'\kappa_{j'} \in \Omega(\eta_1, \dots, \eta_{v'-1}, \kappa_{j'})$$

for any $\zeta'\kappa_j$ contained in $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j)$. Thus the length of the representation $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j)$ is the same as that of $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_{j'})$, and $r_{v'}$ sends $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j)$ to $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_{j'})$. The other powers of $r_{v'}$ force $\tilde{\sigma}$ to contain

$$\Omega(S',\kappa_j)=\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j)\otimes\ldots\otimes\eta_{v'}^{m_{v'}-1}\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j),$$

which by definition is similar to $\Omega(\eta_1,\ldots,\eta_{v'},\kappa_j)$.

We conclude that the orbit of $\Omega(\eta_1,\ldots,\eta_{v'-1},\kappa_j)$ gives rise to the representation $\Omega(\eta_1,\ldots,\eta_{v'},\kappa_j)$ for each j. This representation has length $m(v')$, hence $\tilde{\sigma}$ is similar to

$$\Omega(\eta_1,\ldots,\eta_{v'},\kappa'_1)\otimes\ldots\otimes\Omega(\eta_1,\ldots,\eta_{v'},\kappa'_{\ell}),$$

where $\ell'=n/m(v')$ and κ'_i is in $(F^\times)^\vee$. The final result follows by induction. Q.E.D.

We have completely described the possibilities for $\tilde{\sigma}$ in $\mathcal{E}_2(\tilde{M})$ in terms of the R group. When M is not the Levi component for the standard minimal parabolic subgroup of G_n, the argument for determining the form of $\tilde{\sigma}$ is similar. The results are slightly more complicated to formulate, however, because we are unable to state exact equalities (as in equation (2)) when the representations are not one-dimensional. Instead we must write things in terms of equivalences, and more variation is possible. Details for this general case are carried out in the next section.

§4.4. Discrete Series with a Given R group: the General Case.

We now generalize the results of the previous section to encompass the case that M is the Levi component for an arbitrary standard parabolic subgroup of $G = G_n$. The arguments are parallel to those in the case of the principal series. Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. We describe $\tilde{\sigma}$ in terms of the R group of σ.

Write $\tilde{M}$ in block diagonal form as

$$\tilde{M} \cong GL_{\tilde{n}_1} \times GL_{\tilde{n}_2} \times \ldots \times GL_{\tilde{n}_k}, \qquad (1)$$

and write $\tilde{\sigma} \cong \otimes_{i=1}^{k} \tilde{\sigma}_i$. By Lemma 3.1.3 we can reorder the blocks of $\tilde{M}$ and consequently the $\tilde{\sigma}_i$ without affecting the class of the induced representation. Since $R = R(\sigma)$ is contained in $W_G(M) = W_{\tilde{G}}(\tilde{M})$, it is a subgroup of the permutations on the blocks of $\tilde{M}$. Let $\bar{w}$ be in R, and let r be the corresponding permutation on $\{1, 2, \ldots, k\}$. As in the previous section, rearrange the blocks of $\tilde{M}$ so that r is the product of disjoint cycles that are in increasing numerical order, and so that the 1-cycles of r all occur at the end. Such a reordering is the restriction of an automorphism φ on $\tilde{G}$, and consequently φ can be seen as acting on R. Suppose

$$r = (1 \ \cdots \ n_1)(n_1 + 1 \ \cdots \ n_2) \cdots (n_{\ell-1} + 1 \ \cdots \ n_\ell), \qquad (2)$$

with $1 < n_1 < \cdots < n_\ell \leq k$, and set $n_0 = 1$. Let w_i be the matrix in

$\tilde{G}_{(n_i - n_{i-1})\bar{n}_i}$ defined by

$$w_i = \begin{pmatrix} 0 & 0 & \dots & 0 & I_{\bar{n}_i} \\ I_{\bar{n}_i} & 0 & \dots & 0 & 0 \\ 0 & I_{\bar{n}_i} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_{\bar{n}_i} & 0 \end{pmatrix},$$

where $\bar{n}_i$ is the common size of the blocks permuted by $(n_{i-1}+1 \ \cdots \ n_i)$. Then the block diagonal matrix $w = (w_1, \dots, w_\ell, 1, \dots, 1)$ is in $\varphi(\bar{w})$ and is the permutation matrix corresponding to r.

The automorphism φ induces a reordering of the $\tilde{\sigma}_i$ and produces a representation on $\varphi(\tilde{M})$. Denote this representation by $\tilde{\sigma}'$. Recall from Section 4.1 that under this automorphism $\tilde{\sigma}$ is similar to $\tilde{\sigma}'$, and we write $\tilde{\sigma} \approx \tilde{\sigma}'$. For simplicity, rewrite $\tilde{\sigma}' \cong \otimes_{i=1}^{k} \tilde{\sigma}_i$.

Moreover the coset representative w of $\varphi(\bar{w})$ acts on $\tilde{\sigma}'$ since w is in $N_{\tilde{G}}(\varphi(\tilde{M}))$. We compute that

$$\begin{aligned} w\tilde{\sigma}'(g_1, g_2, \dots, g_k) &= \tilde{\sigma}'(g_{r^{-1}(1)}, g_{r^{-1}(2)}, \dots, g_{r^{-1}(k)}) \\ &= \tilde{\sigma}_1(g_{r^{-1}(1)}) \otimes \tilde{\sigma}_2(g_{r^{-1}(2)}) \otimes \dots \otimes \tilde{\sigma}_k(g_{r^{-1}(k)}), \end{aligned}$$

and Lemma 4.1.4 implies that

$$w\tilde{\sigma}'(g_1, g_2, \dots, g_k) = \tilde{\sigma}_{r(1)}(g_1) \otimes \tilde{\sigma}_{r(2)}(g_2) \otimes \dots \otimes \tilde{\sigma}_{r(k)}(g_k).$$

In this sense, we may think of w or r as acting on the set of representations $\{\tilde{\sigma}_i\}_{1 \le i \le k}$. If η is the character in $\bar{L}(\tilde{\sigma})$ associated to $\bar{w}$ taken modulo

$X(\tilde{\sigma})$, then $\varphi^{-1}(w)$ is in $\bar{w}$, and the condition that $\eta \otimes \tilde{\sigma} \cong \varphi^{-1}(w)\tilde{\sigma}$ forces $\eta \otimes \tilde{\sigma}' \cong w\tilde{\sigma}'$ under the automorphism φ. Therefore Lemma 4.1.1 implies that

$$\eta \otimes \tilde{\sigma}_i \cong \tilde{\sigma}_{r(i)} \qquad \text{for } 1 \leq i \leq k. \tag{3}$$

More explicitly, we have the following equivalences among the $\tilde{\sigma}_i$:

$$\begin{array}{llr}
\eta \otimes \tilde{\sigma}_1 \cong \tilde{\sigma}_2, & \eta \otimes \tilde{\sigma}_{n_1+1} \cong \tilde{\sigma}_{n_1+2}, & \\
\eta \otimes \tilde{\sigma}_2 \cong \tilde{\sigma}_3, & \eta \otimes \tilde{\sigma}_{n_1+2} \cong \tilde{\sigma}_{n_1+3}, & \\
\vdots \quad \vdots \;\; \vdots & \vdots \quad \vdots \quad \vdots & \text{etc.} \\
\eta \otimes \tilde{\sigma}_{n_1} \cong \tilde{\sigma}_1, & \eta \otimes \tilde{\sigma}_{n_2} \cong \tilde{\sigma}_{n_1+1}, &
\end{array}$$

The first column shows that $\tilde{\sigma}_i \cong \eta^{i-1} \otimes \tilde{\sigma}_1$ for $2 \leq i \leq n_1$ and that η^{n_1} is in $X(\tilde{\sigma}_1)$. The second column shows that $\tilde{\sigma}_i \cong \eta^{i-n_1-1} \otimes \tilde{\sigma}_{n_1+1}$ for $n_1 + 2 \leq i \leq n_2$ and that $\eta^{n_2-n_1}$ is in $X(\tilde{\sigma}_{n_1+1})$. And so on. In other words, the hth nontrivial cycle in r corresponds to a set of representations of the form $\{\eta^j \otimes \tilde{\sigma}_{n_{h-1}+1}\}$ for all j with $0 \leq j < n_h - n_{h-1}$.

Definition. Let H be a closed subgroup of $GL(n', F)$, and let τ be in $\mathcal{E}_2(H)$. If η is a character of $F^\times$, let $\text{ord}(\eta) = \text{ord}_\tau(\eta)$ be the order of η modulo $X(\tau)$. Define a representation $\Omega(\eta, \tau)$ of $\prod_{i=1}^{\text{ord}(\eta)} H$ by

$$\Omega(\eta, \tau) = \tau \otimes (\eta \otimes \tau) \otimes \ldots \otimes (\eta^{\text{ord}(\eta)-1} \otimes \tau).$$

We say that $\Omega(\eta, \tau)$ is **generated** by τ and that it has **length** $\text{ord}(\eta)$.

Proposition 4.4.1. Let M be the Levi component for a standard parabolic subgroup of G such that $M = \tilde{M} \cap G$, where $\tilde{M}$ is written as in (1). Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Suppose $\bar{w}$ is an element of R with order $m > 1$, let η be the corresponding character in $\bar{L}(\tilde{\sigma})$ taken modulo $X(\tilde{\sigma})$. Rearrange the blocks of $\tilde{M}$ so that the associated permutation r can be written as in (2). Grouping together the blocks of $\tilde{M}$ that are in the same orbit of r, we rewrite

$$\tilde{M} \cong \Big(\prod_{t=1}^{n_1} GL_{\tilde{n}_1}\Big) \times \ldots \times \Big(\prod_{t=1}^{n_\ell} G_{\tilde{n}_\ell}\Big) \times G_{\tilde{n}'_1} \times \ldots \times G_{\tilde{n}'_s}.$$

Then there exist representations $\tau_i \in \mathcal{E}_2(GL_{\tilde{n}_i})$ for $1 \leq i \leq \ell$ and representations $\rho_{j'} \in \mathcal{E}_2(GL_{n'_i})$ for $1 \leq j' \leq s$ such that the n_i's in the expression for r satisfy $n_i - n_{i-1} = \mathrm{ord}_{\tau_i}(\eta)$. Consequently

$$\tilde{\sigma} \approx \Omega(\eta, \tau_1) \otimes \ldots \otimes \Omega(\eta, \tau_\ell) \otimes \rho_1 \otimes \ldots \otimes \rho_s,$$

and $\rho_i \cong \eta \otimes \rho_i$.

Proof. Let $\tilde{\sigma}'$ be the representation similar to $\tilde{\sigma}$ that is determined by reordering the blocks of $\tilde{M}$. The restrictions of equation (3) show that $\tilde{\sigma}'$ can be written

$$\Big(\tilde{\sigma}_1 \otimes \ldots \otimes \eta^{n_1 - 1}\tilde{\sigma}_1\Big) \otimes \ldots \otimes \Big(\tilde{\sigma}_{n_{\ell-1}+1} \otimes \ldots \otimes \eta^{n_\ell - n_{\ell-1}}\tilde{\sigma}_{n_{\ell-1}+1}\Big) \otimes \rho_1 \otimes \ldots \otimes \rho_s,$$

where the $\rho_{j'}$ are fixed by r. Let $\tau_i = \tilde{\sigma}_{n_{i-1}+1}$. We claim that $n_i - n_{i-1}$ is actually the order of η modulo $X(\tau_i)$. Without loss of generality, assume that the order of η modulo $X(\tau_1)$ is less than n_1, say n'_1. Then $\eta^{n'_1} \otimes \tau_1 \cong \tau_1$, and n'_1 divides n_1. Write $n_1 = jn'_1$ for some j.

For simplicity denote by $W'(\sigma)$, etc. again the image of $W'(\sigma)$ under φ, where φ is the automorphism implementing this reordering of blocks. By the realization of $W'(\sigma)$ in equation 2 of Section 3.3, the elements of $W_G(M)$ corresponding to the transpositions $(1 \quad n'_1)$, $(n \quad 2n'_1), \ldots$, and $((j-2)n'_1 \quad (j-1)n'_1)$ are all in $W'(\sigma)$. Therefore the cosets $\bar{w}W'$ and $\bar{w}'W'$ are equal, where $\bar{w}'$ corresponds to the permutation

$$\Big(\prod_{h=0}^{j-1}(hn'_1+1 \ \ldots \ (h+1)n'_1)\Big)\big((n_1+1 \ \cdots \ n_2)\cdots(n_{\ell-1}+1 \ \cdots \ n_\ell)\big).$$

Thus we can take $\bar{w}'$ to be the coset representative in $W(\sigma)/W'$. Therefore if $\bar{w}$ is taken modulo $W'(\sigma)$, then $n_i - n_{i-1} = \mathrm{ord}_{\tau_i}(\eta)$, and consequently the permutation r corresponding to $\bar{w}$ in R can be written in the asserted form. The conclusion follows. Q.E.D.

REMARK. When R is cyclic, the proposition describes all of the possible $\tilde{\sigma}$'s (up to similarity) in $\mathcal{E}_2(\tilde{M})$ that contain σ in their restriction to M.

Definition. Let H be a closed subgroup of $GL(n', F)$, let τ be in $\mathcal{E}_2(H)$, and suppose $\eta_1, \ldots, \eta_\ell$ are characters in $(F^\times)^\vee$ with respective orders $m_1, \ldots, m_\ell$ modulo $X(\tau)$. Set $m(i) = m_1 m_2 \cdots m_i$. Let $\Omega(\eta_1, \tau)$ be as in the previous definition, and define the representation $\Omega(\eta_1, \ldots, \eta_i, \tau)$ of $\prod_{j'=1}^{m(i)} H$ recursively by

$$\Omega(\eta_1, \ldots, \eta_i, \tau) = \bigotimes_{j=0}^{m_i-1} \eta_i^j \, \Omega(\eta_1, \ldots, \eta_{i-1}, \tau).$$

We say that $\Omega(\eta_1, \ldots, \eta_i, \tau)$ has **length** $m(i)$.

Definition. Let $M' \cong \prod_{i=1}^{k} \tilde{G}_{\bar{n}_i}$ for some k, and suppose that we can also write $M' \cong M_1 \times M_2$. Let τ be in $\mathcal{E}_2(M')$, and let ρ be in $\mathcal{E}_2(M_1)$. Then τ is said to **contain** the representation ρ if the isomorphism $\tilde{M} \cong M_1 \times M_2$ induces

$$\tau \approx \rho \otimes \tau',$$

for some τ' in $\mathcal{E}_2(M_2)$. Also if τ_i is in $\mathcal{E}_2(\tilde{G}_{\bar{n}_i})$ for $1 \leq i \leq s$, then we say that $\{\tau_i\}$ **contains** ρ if ρ is contained in $\otimes\tau_i$.

EXAMPLE. The context in which we use the above definition is as follows. Suppose that in the situation we are considering $\tilde{M} \cong \tilde{G}_n \times \tilde{G}_{n'} \times \tilde{G}_n$, and assume that $\tilde{\sigma} \cong \tau \otimes \tau' \otimes \eta\tau$ for some τ in $\mathcal{E}_2(\tilde{G}_n)$, τ' in $\mathcal{E}_2(\tilde{G}_{n'})$, and η in $(F^\times)^\vee$. Let $M_1 = \tilde{G}_n \times \tilde{G}_n$, and let $M_2 = \tilde{G}_{n'}$. Then $\tilde{M} \cong M_1 \times M_2$, and under this isomorphism $\tilde{\sigma} \approx \tau \otimes \eta\tau \otimes \tau'$. Thus we say that $\tilde{\sigma}$ contains $\tau \otimes \eta\tau$. Moreover the set $\{\tau, \tau', \eta\tau\}$ is said to contain $\tau \otimes \eta\tau$.

Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ with $\tilde{\sigma}|_M \supseteq \sigma$. We know that R is a finite abelian group and therefore is the product of cyclic groups. Write

$$R \cong \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \ldots \times \mathbb{Z}/m_v\mathbb{Z},$$

with each $m_i > 1$, and let $\bar{w}_1, \ldots, \bar{w}_v$ be elements of R that generate the respective factors of R. Let $S = \{\eta_1, \ldots, \eta_v\}$ be a set of characters in $\bar{L}(\tilde{\sigma})$ taken modulo $X(\tilde{\sigma})$ such that η_i is associated with $\bar{w}_i$.

Proposition 4.4.2. In the above situation, there exist subsets C_i of S with $1 \leq i \leq k'$ for some k', and sets of discrete series representations $\{\tau_i\}_{i=1}^{k'}$ and $\{\rho_{\ell'}\}_{\ell'=1}^{s'}$ such that

$$\tilde{\sigma} \approx \Omega(C_1, \tau_1) \otimes \Omega(C_2, \tau_2) \otimes \ldots \otimes \Omega(C_{k'}, \tau_{k'}) \otimes \rho_1 \otimes \ldots \otimes \rho_{s'},$$

where every η_j is in some C_i. The representations $\rho_{\ell'}$ and τ_i are characterized by the properties:

(a) η_t is in $X(\rho_{\ell'})$ for all t and ℓ' with $1 \leq t \leq v$ and $1 \leq \ell' \leq s'$.

(b) If η_t is not in C_i for some t with $1 \leq t \leq v$, then η_t is in $X(\tau_i)$.

REMARK. This proposition reduces to Proposition 4.4.1 when R is a cyclic group. In that situation, $C_i = S = \{\eta\}$ for all i.

PROOF. Let S' be an ordered subset of S, and proceed by induction on the number of characters in S'. Proposition 4.4.1 handles the case that S' is a single character. Suppose the result holds for $S' = \{\eta_1, \eta_2, \ldots, \eta_{v'-1}\}$. Then for some subsets C_i' of S' with $1 \leq i \leq \ell$, and sets of discrete series representations $\{\tau_i\}_{i=1}^{\ell}$ and $\{\rho_{\ell'}\}_{l'=1}^{s}$, we can write

$$\tilde{\sigma} \approx \Omega(C_1', \tau_1') \otimes \ldots \otimes \Omega(C_\ell, \tau_\ell') \otimes \rho_1 \otimes \ldots \otimes \rho_s, \tag{4}$$

such that properties (a) and (b) are satisfied, and every η_j in S' is in some C_i'. Let φ be the automorphism of $\tilde{G}$ implementing this similarity.

Now suppose that $S' = \{\eta_1, \eta_2, \ldots, \eta_{v'}\}$. The character $\eta_{v'}$ in $\bar{L}(\tilde{\sigma})$ corresponds to a permutation $r_{v'}$ on the blocks of $\varphi(\tilde{M})$. We do not choose a specific form for $r_{v'}$ because we are trying to refine further the expression in equation (4). We want to show that the orbit of each of the $\Omega(C'_i, \tau'_i)$ and each of the $\rho_{\ell'}$ under the powers of $r_{v'}$ contains representations only of the asserted form.

First fix j with $1 \leq j \leq \ell$. We show that for each such j the orbit of $\Omega(C'_j, \tau'_j)$ is as claimed. Since equation (3) implies that $\eta_{v'}\tau'_j$ is equivalent to its image under $r_{v'}$, we have the following four possibilities:

(1) $\eta_{v'} \otimes \tau'_j \cong \zeta \otimes \tau'_{i'}$ for some character ζ such that $\zeta \otimes \tau'_{i'}$ is contained in $\Omega(C'_{i'}, \tau'_{i'})$ (i.e., ζ is some product of the characters in $C'_{i'}$), and with $i \neq j$. Suppose $\zeta' \otimes \tau'_j$ be contained in $\Omega(C'_j, \tau'_j)$. Then $\eta_{v'} \otimes \tau'_j \cong \zeta \otimes \tau'_{i'}$ forces

$$\eta_{v'} \otimes (\zeta' \otimes \tau'_j) \cong \zeta\zeta' \otimes \tau'_{i'} \in \Omega(C'_{i'}, \tau'_{i'}).$$

Since permutations are one-one and onto, the length of $\Omega(C'_j, \tau'_j)$ is equal to the length of $\Omega(C'_{i'}, \tau'_{i'})$. Thus $\Omega(C'_{i'}, \tau'_{i'}) \cong \eta_{v'} \otimes \Omega(C'_j, \tau'_j)$, and therefore $r_{v'}$ sends $\Omega(C'_j, \tau_j)$ to $\Omega(C'_{i'}, \tau'_{i'})$. The other powers of $r_{v'}$ force $\tilde{\sigma}$ to contain

$$\Omega(\eta_{v'} \cup C'_j, \tau'_j) = \Omega(C'_j, \tau'_j) \otimes \left(\eta_{v'} \otimes \Omega(C'_j, \tau'_j)\right) \otimes \ldots \otimes \left(\eta_{v'}^{u-1} \otimes \Omega(C'_j, \tau'_j)\right),$$

where $u = \mathrm{ord}_{\tau'_j}(\eta_{v'})$. Property (b) holds by the induction hypothesis.

(2) $\eta_{v'} \otimes \eta'_j \cong \zeta \otimes \tau'_j$ for some $\zeta \otimes \tau'_j$ contained in $\Omega(C'_j, \tau'_j)$. Then $\zeta^{-1} \otimes \tau'_j$ is also contained in $\Omega(C'_j, \tau'_j)$, where ζ^{-1} is taken modulo $X(\tau'_j)$. Therefore

$$\eta_{v'} \otimes (\zeta^{-1} \otimes \tau'_j) \cong \tau'_j,$$

and $\eta_{v'}$ is in $X(\zeta^{-1} \otimes \tau'_j)$. Since each of the $\eta_\iota \in C'_j$ implements a permutation, we have

$$\eta_{v'} \otimes \Omega(C'_j, \tau'_j) \approx \Omega(C'_j, \tau'_j) \approx \Omega(C'_j, \zeta^{-1} \otimes \tau'_j),$$

and $\eta_{v'}$ satisfies property (b) for $\Omega(C'_j, \zeta^{-1} \otimes \tau'_j)$.

(3) $\eta_{v'} \otimes \tau'_j \cong \tau'_j$. Then $\eta_{v'}$ is in $X(\tau'_j)$, and property (b) is satisfied by $\eta_{v'}$.

(4) $\eta_{v'} \otimes \tau'_j \cong \rho_{\ell'}$ for some ℓ' with $1 \leq \ell' \leq s$. This, in fact, can not happen. If such an equivalence were true, then we would also have $\eta_{v'} \otimes (\eta_{i'} \otimes \tau'_j) \cong \rho_{\ell'}$ for any $\eta_{i'} \in C'_j$, since $\eta_{i'} \in X(\rho_{i'})$ by the induction hypothesis. But then $\eta_{v'}$ can not implement a permutation.

We conclude from each of these possibilities that the orbit of $\Omega(C'_j, \tau'_j)$ contains only representations of the asserted type for each j with $1 \leq j \leq \ell$. Now fix j' with $1 \leq j' \leq s$. We want to see that the representations contained in the orbit of $\rho_{j'}$ under the powers of $r_{v'}$ also satisfy the proposition. For $\rho_{j'}$ there are two possibilities:

(1) $\eta_{v'} \otimes \rho_{j'} \cong \rho_{i'}$ for some $i' \neq j'$. Considering the full orbit of ρ_i under the powers of $r_{v'}$ shows that $\tilde{\sigma}$ contains the representation $\Omega(\eta_{v'}, \rho_{j'})$, and property (b) is satisfied by the induction hypothesis.

(2) $\eta_{v'} \otimes \rho_{j'} \cong \rho_{j'}$. In this case, $\eta_{v'}$ is in $X(\rho_{j'})$, and property (a) is satisfied.

We have shown that $\tilde{\sigma}$ contains only representations of the required form for $S' = \{\eta_1, \eta_2, \ldots, \eta_{v'}\}$. If $\eta_{v'}$ is not in any of the C_i, then it fixes all of the τ_i and $\rho_{\ell'}$ by properties (a) and (b). Hence $\eta_{v'}$ has order 1 modulo

$X(\tilde{\sigma})$, which contradicts the hypothesis that it generates a distinct factor in $\bar{L}(\tilde{\sigma})$. The final result follows by induction. Q.E.D.

Later we will want to distinguish between two distinct types of discrete series. For the first type, we are able to give explicit realizations for the irreducible constituents (Section 5.2). The second type is more complicated and is studied in Chapter 6.

Definition. Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$ be such that $\tilde{\sigma}|_M \supseteq \sigma$. The discrete series representation σ or $\tilde{\sigma}$ is said to be a **Class I** representation if either

a) the R group of σ is cyclic, or

b) when $\tilde{\sigma}$ is written as in Proposition 4.4.2, all of the C_i contain a single character.

Otherwise we say that σ or $\tilde{\sigma}$ is a **Class II** representation.

V. Constituents of the Generalized Principal Series

§5.1. Cyclic R group.

We are interested in gaining information about the irreducible constituents of the generalized principal series representations with a cyclic R group. To do so, we combine the classification of the discrete series with Mackey theory, and express the constituents in terms of the constituents of some basic representations, called "building block representations".

To define these representations, first suppose that M is the Levi component of $G_{n'}$ given by a product of k blocks of equal size q. Moreover assume that σ is in $\mathcal{E}_2(M)$ and that $R(\sigma) \cong \mathbb{Z}/k\mathbb{Z}$, and let $X = X(\tilde{\sigma})$. Let $\bar{\eta}$ be a generator for $\bar{L}(\tilde{\sigma})/X(\tilde{\sigma})$. Then we know from Theorem 4.2.2 and Proposition 4.4.1 that

$$\sigma \cong \operatorname{Ind}_{\prod G_q^X}^{M^X} \left.\begin{pmatrix} \tilde{\varepsilon}\tau & & & \\ & \eta \otimes \tau & & \\ & & \ddots & \\ & & & \eta^{k-1} \otimes \tau \end{pmatrix}\right|_M , \tag{1}$$

for some $\tau \in \mathcal{E}_2(G_q^X)$ and some coset representative ε of $F^\times/N_X$. The representation on the right hand side of (1) can be parabolically induced to $G_{n'}^X$, and the restriction of the resulting representation to $G_{n'}$ is equivalent to $\operatorname{Ind}_{MN}^G(\sigma)$ by Lemma 3.4.1. This motivates the following definition.

Definition. Let X be a subset of $(F^\times)^\vee$, and let η be in $(F^\times)^\vee$ with $\eta \notin X$. If τ is in $\mathcal{E}_2(G_q^X)$ and ε is a coset representative for $F^\times/N_X$, let $\ell = \operatorname{ord}_\tau(\eta)$, and define the representation $\Omega_\eta(\tilde{\varepsilon},\tau)$ of $\prod_{i=1}^{\ell} G_q^X$ by

$$\Omega_\eta(\tilde{\varepsilon},\tau) = \tilde{\varepsilon}\tau \otimes (\eta \otimes \tau) \otimes \ldots \otimes (\eta^{\ell-1} \otimes \tau).$$

The representation Ω of $G_{\ell q}^X$ achieved by parabolically inducing $\Omega_\eta(\tilde{\varepsilon},\tau)$ is called a **building block representation**:

$$\Omega = \mathrm{Ind}_{M^X N}^{G_{\ell q}^X}\left(\mathrm{Ind}_{\prod G_q^X}^{M^X}(\Omega_\eta(\tilde{\varepsilon},\tau))\right).$$

REMARKS.

(1) Notice that $\Omega_\eta(\tilde{\varepsilon},\tau)$ is a generalization of the representation $\Omega(\eta,\tau)$ introduced in Chapter 4. The $\tilde{\varepsilon}$ is a necessary factor arising from the classification of the discrete series of M (Theorem 4.2.2).

(2) The representation σ as in equation (1) produces a building block representation in the obvious fashion, and by Theorem 4.2.2 and Lemma 3.4.1 the reducibility of $\mathrm{Ind}_{MN}^{G_{n'}}(\sigma)$ is the same as that of $\Omega|_G$, where Ω is the respective building block representation. We shall see in Section 3 that these representations are exactly the generalized principal series whose constituents are "elliptic". Notice, however, that the definition does not allow us to go backwards; that is to say, not every building block representation restricts to a generalized principal series representation. This is because $X(\tau)$ is not necessarily trivial, as is required by Theorem 4.2.2 in order to lead to an element of the generalized principal series.

(3) Our main theorems describe the reducibility of the $\mathrm{Ind}_{MN}^{G}(\sigma)$ in terms of the reducibility of the building block representations when σ has

a cyclic R group or, more generally, when σ is a Class I representation as defined in Section 4.4. We shall see that each such generalized principal series gives rise to some set of building block representations.

Now let M be the Levi component for a standard parabolic subgroup of $G = G_n$, and let σ be in $\mathcal{E}_2(M)$. Suppose that $R(\sigma) \cong \mathbb{Z}/m\mathbb{Z}$ as a permutation group and that $\bar{\eta}$ is a generator of $\bar{L}(\sigma)/X(\tilde{\sigma})$. Let $\tilde{M}$ be the corresponding Levi component of $\tilde{G} = \tilde{G}_n$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Recall from Proposition 4.4.1 that

$$\tilde{\sigma} \approx \Omega(\eta, \tau_1) \otimes \ldots \otimes \Omega(\eta, \tau_k) \otimes \rho_1 \otimes \ldots \otimes \rho_s, \tag{2}$$

where $m = \mathrm{LCM}(\{\mathrm{ord}_\eta(\tau_i)\}_{i=1}^k)$. Moreover we have a classification of the discrete series representations in the packet of σ (Theorem 4.2.2). If r is the permutation on the blocks of $\tilde{M}$ induced by η, we can write r as the product of k disjoint cycles with the ith cycle permuting the blocks corresponding to $\Omega(\eta, \tau_i)$. We first study the case that r is a single m cycle (i.e., $k = 1$). The general case of a cyclic R group then follows from similar arguments.

Let σ be in $\mathcal{E}_2(M)$ with $R(\sigma) \cong \mathbb{Z}/m\mathbb{Z}$, and suppose that r is a single m cycle. Then we can write

$$\tilde{M} \cong \left(\prod_{i=1}^{m} \tilde{G}_q\right) \times \tilde{G}_{n_1} \times \ldots \times \tilde{G}_{n_s},$$

and expression (2) can be rewritten

$$\tilde{\sigma} \approx \Omega_\eta(\tau) \otimes \rho_1 \otimes \ldots \otimes \rho_s, \tag{3}$$

with $\tau \in \mathcal{E}_2(\tilde{G}_q)$, $\rho_i \in \mathcal{E}_2(GL_{n_i})$, and $\rho_i \cong \eta \otimes \rho_i$. Furthermore from the description of the discrete series of M, if $X = X(\tilde{\sigma})$, we know that there exists σ' on M^X such that $\sigma'|_M = \sigma$. In particular expression (3) says that for some coset representative ε of $F^\times/N_X$ we must have

$$\sigma' \approx \operatorname{Ind}_{\prod_{i=1}^m G_q^X \times \prod_{i=1}^s G_{n_i}^X}^{M^X} \begin{pmatrix} \Omega_\eta(\tilde{\varepsilon}, \tau') & & & \\ & \rho_1' & & \\ & & \ddots & \\ & & & \rho_s' \end{pmatrix}, \tag{4}$$

where σ_1' and ρ_i' are irreducible constituents of $\tau|_{G_q^X}$ and $\rho|_{G_{n_i}^X}$, respectively.

Then associated to σ' is the representation $\Omega_\eta(\tilde{\varepsilon}, \tau')$. Let Ω be the respective building block representation on G_{mq}^X. Since $[G_{mq}^{\bar{L}} : G_{mq}^X] = m$, there are m irreducible constituents in $\Omega|_{G_{mq}^{\bar{L}}}$. Write this reduction as

$$\Omega|_{G_{\bar{L}}} = \Phi_1 \oplus \cdots \oplus \Phi_m.$$

We will implement double induction and Mackey theory to express the reducibility of $\operatorname{Ind}_{M^X N}^{G^X}(\sigma')\big|_G \cong \operatorname{Ind}_{MN}^G(\sigma)$ in terms of the Φ_i and constituents of $\rho_j|_{G_{n_j}}$. The following example displays all of the necessary ingredients.

Example. Let M be a Levi component of G_n whose corresponding $\tilde{M}$ is of the form $\tilde{M} \cong \tilde{G}_m \times \tilde{G}_m \times \tilde{G}_r$. Suppose σ is in $\mathcal{E}_2(M)$ with $R(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$, and suppose $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$ with $\tilde{\sigma}|_M \supseteq \sigma$. In this case, (4) can be rewritten

$$\sigma' \approx \operatorname{Ind}_{G_m^X \times G_m^X \times G_r^X}^{M_n^X} \begin{pmatrix} \tilde{\varepsilon}\pi_1 & & \\ & \zeta\pi_1 & \\ & & \pi_2 \end{pmatrix},$$

with $\zeta \neq 1 \in (F^\times)^\vee$ and $\zeta^2 = 1$. We want to describe explicitly the irreducible constituents of $\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\,|_{G}^{\bar{L}}$. Form $\Omega_\eta(\tilde{\varepsilon}, \pi_1)$, and let

$$\Omega = \mathrm{Ind}_{M^X N}^{G_{\ell q}^X} \mathrm{Ind}_{\prod G_q^X}^{M^X} (\Omega_\eta(\tilde{\varepsilon}, \tau))$$

be the corresponding building block representation.

Let $\bar{L} = \bar{L}(\tilde{\sigma})$. Since $\bar{L}/X$ is generated by ζ, we must have $G_n^{X,\zeta} = G_n^{\bar{L}}$ for any size n. By the Restriction Theorem and the definition of $X(\Omega)$,

$$\Omega|_{G^{\bar{L}}} = \mathrm{Ind}_{G_m^X \times G_m^X}^{G_{2m}^X} \begin{pmatrix} \tilde{\varepsilon}\pi_1 & \\ & \zeta\pi_1 \end{pmatrix}\Bigg|_{G^{\bar{L}}} \cong A \oplus B.$$

If we set $M' = \tilde{G}_{2m} \times \tilde{G}_r$, then

$$\mathrm{Ind}_{M^X N}^{G_n^X}(\sigma')\Bigg|_{G^{\bar{L}}}$$

$$\cong \mathrm{Ind}_{M'^X N'}^{G_n^X} \left(\mathrm{Ind}_{M^X N}^{M'^X} \mathrm{Ind}_{G_m^X \times G_m^X \times G_r^X}^{M^X} \begin{pmatrix} \tilde{\varepsilon}\pi_1 & & \\ & \zeta\pi_1 & \\ & & \pi_2 \end{pmatrix} \right)\Bigg|_{G^{\bar{L}}} \quad (5)$$

$$\cong \mathrm{Ind}_{M'^X N'}^{G_n^X} \left(\mathrm{Ind}_{G_{2m}^X \times G_r^X}^{M'^X} \begin{pmatrix} \mathrm{Ind}_{G_m^X \times G_m^X}^{G_{2m}^X} (\Omega_\zeta(\tilde{\varepsilon}, \pi)) & \\ & \pi_2 \end{pmatrix} \right)\Bigg|_{G^{\bar{L}}} \quad (6)$$

$$\cong \mathrm{Ind}_{M'^{\bar{L}} N'}^{G_n^{\bar{L}}} \left(\mathrm{Ind}_{G_{2m}^X \times G_r^X}^{M'^X} \begin{pmatrix} \Omega & \\ & \pi_2 \end{pmatrix}\Bigg|_{M'^{\bar{L}}} \right). \quad (7)$$

are irreducible and not equivalent, and hence by equation(7), induce to the two irreducible constituents of $\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\,|_{G^{\bar{L}}}$. Therefore when restricted to M, they are the constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$, i.e,

$$\mathrm{Ind}_{MN}^{G_n}(\sigma) \cong \mathrm{Ind}_{M'^{\bar{L}}N}^{G_n^{\bar{L}}}\left(\mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^1 \end{pmatrix} \bigoplus \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} B & \\ & \pi_2^1 \end{pmatrix}\right)\Bigg|_{G_n}$$

$$\cong \mathrm{Ind}_{MN}^{G}\left(\mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^1 \end{pmatrix}\Bigg|_{M} \bigoplus \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} B & \\ & \pi_2^1 \end{pmatrix}\Bigg|_{M}\right)$$

REMARKS.

(1) Notice that if we restrict Φ all the way down to M_2, we get four inequivalent representations, written

$$\begin{pmatrix} A & \\ & \pi_2^1 \end{pmatrix}, \quad \begin{pmatrix} A & \\ & \pi_2^2 \end{pmatrix}, \quad \begin{pmatrix} B & \\ & \pi_2^1 \end{pmatrix}, \text{ and } \begin{pmatrix} B & \\ & \pi_2^2 \end{pmatrix}.$$

According to Mackey theory, these matrices induce to $M'^{\bar{L}}$ in pairs i.e.,

$$\mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^1 \end{pmatrix} \cong \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} B & \\ & \pi_2^2 \end{pmatrix}$$

and

$$\mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^2 \end{pmatrix} \cong \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} B & \\ & \pi_2^1 \end{pmatrix}.$$

This shows two things. First our pieces are not achieved by inducing the constituents of $\Phi|_{M_2}$; this actually gives us twice our representation. Second we could also write

$$\Phi \cong \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^1 \end{pmatrix} \bigoplus \mathrm{Ind}_{M_2}^{M'^{\bar{L}}}\begin{pmatrix} A & \\ & \pi_2^2 \end{pmatrix}.$$

(2) Also notice that if $X(\pi_1) \neq \{1\}$, then $\Omega|_{G_{2m}}$ is not equivalent to a generalized principal series representation, by Theorem 4.2.2.

Let us return the situation that r is a single m cycle. Let σ' be the representation described in equation (4). Form $\Omega_\eta(\tilde{\varepsilon}, \tau')$, and let Ω be the associated building block representation. The proposition below describes the reducibility of $\mathrm{Ind}_{MN}^{G}(\sigma)$ in a fashion analogous to the example.

Theorem 5.1.1. In the above situation, if $\Omega|_{G_{mq}^{\bar{L}}} \cong \Phi_1 \oplus \cdots \oplus \Phi_m$ and $M' \cong \tilde{G}_{mq} \times \tilde{G}_{n_1} \times \cdots \times \tilde{G}_{n_s}$, then

$$\mathrm{Ind}_{P^X}^{G^X}(\sigma')\Big|_{G^{\bar{L}}} \cong \bigoplus_{j=1}^{m} \mathrm{Ind}_{P'^{\bar{L}}}^{G^{\bar{L}}} \mathrm{Ind}_{G_{mq}^{\bar{L}} \times \prod_{i=1}^{s} G_{n_i}^{\bar{L}}}^{M'^{\bar{L}}} \begin{pmatrix} \Phi_j & & & \\ & \varphi_1 & & \\ & & \ddots & \\ & & & \varphi_s \end{pmatrix},$$

where $M'^{\bar{L}} = M' \cap G^{\bar{L}}$ and each φ_j is an irreducible constituent of $\rho_j'|_{G_{n_j}^{\bar{L}}}$. The representations on the right hand side of this equation restrict irreducibly to G and are equivalent to the constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$.

Proof. Our goal is to show that all of the reducibility of $\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\big|_G$ actually occurs on the Levi component $M'^{\bar{L}}$ of $G^{\bar{L}}$. We have the following series of equivalences:

$$\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\big|_{G^{\bar{L}}} \cong \mathrm{Ind}_{M'^X N'}^{G^X}\left(\mathrm{Ind}_{M^X N}^{M'^X}(\sigma')\right)\Big|_{G^{\bar{L}}} \qquad (1)$$

$$\cong \operatorname{Ind}_{P'^X}^{G^X} \operatorname{Ind}_{P^X}^{M'^X} \operatorname{Ind}_{\prod_{i=1}^{m} G_q^X \times \prod_{i=1}^{s} G_{n_i}^X}^{M^X} \left.\begin{pmatrix} \Omega_\eta(\tilde{\varepsilon}, \tau') & & & \\ & \rho_1' & & \\ & & \ddots & \\ & & & \rho_s' \end{pmatrix}\right|_{G^{\bar{L}}} \quad (2)$$

$$\cong \operatorname{Ind}_{M'^{\bar{L}} N'}^{G^{\bar{L}}} \left(\left. \operatorname{Ind}_{G_{mq}^X \times \prod_{i=1}^{s} G_{n_i}^X}^{M'^X} \begin{pmatrix} \Omega & & & \\ & \rho_1' & & \\ & & \ddots & \\ & & & \rho_s' \end{pmatrix} \right|_{M'^{\bar{L}}} \right) \quad (3)$$

$$\cong \operatorname{Ind}_{M'^{\bar{L}} N'}^{G^{\bar{L}}} \left(\bigoplus_{j=1}^{m} \operatorname{Ind}_{G_{mq}^{\bar{L}} \times \prod_{i=1}^{s} G_{n_i}^{\bar{L}}}^{M'^{\bar{L}}} \begin{pmatrix} \Phi_j & & & \\ & \varphi_1 & & \\ & & \ddots & \\ & & & \varphi_s \end{pmatrix} \right) \quad (4)$$

Equivalences (1) and (2) are given by double induction, and (3) follows from Lemma 3.4.1. The final equivalence, given in (4), will come from Mackey theory and the group $X(\operatorname{Ind}(\sigma'))$.

Let the inducing representation in (3) be written

$$\tau = \operatorname{Ind}_{G_{mq}^X \times \prod_{i=1}^{s} G_{m_i}^X}^{M'^X} \begin{pmatrix} \Omega & & & \\ & \rho_1' & & \\ & & \ddots & \\ & & & \rho_s' \end{pmatrix}.$$

If we restrict τ down to $G_{mq}^{\bar{L}} \times \prod_{i=1}^{s} G_{n_i}^{\bar{L}}$, the Restriction Theorem tells us that we have m^{s+1} irreducible constituents. Mackey theory says that these constituents must come from the restriction of the constituents of $\tau|_{M'^{\bar{L}}}$, and therefore the irreducible constituents of $\tau|_{M'^{\bar{L}}}$ can be recovered by inducing constituents of $\tau|_{(G^{\bar{L}}_{mq}\times\prod_{i=1}^{s} G^{\bar{L}}_{n_i})}$. In particular the irreducible constituents on the smaller group induce equivalently in sets of m^s to $M^{\bar{L}}$ since the index of $G^{\bar{L}}_{mq} \times \prod_{i=1}^{s} G^{\bar{L}}_{n_i}$ in $M'^{\bar{L}}$ is m^s. Moreover the Restriction Theorem tells us that if τ' is an irreducible constituent of $\tau|_{M'^{\bar{L}}}$, then the other constituents are given by $\tilde{\varepsilon}\tau'$ where ε runs through a complete set of coset representatives for $F^\times/N_{\bar{L}}$. By Lemma 4.1.3, the $\tilde{\varepsilon}$ factor passes through to the inducing representation and can be seen as acting on Φ_1. Thus if we fix φ_i in $\rho'_i|_{G^{\bar{L}}_{n_i}}$, we have

$$\tau|_{M'^{\bar{L}}} \cong \bigoplus_{j=1}^{m} \mathrm{Ind}_{G^{\bar{L}}_{mq}\times\prod_{i=1}^{s} G^{\bar{L}}_{n_i}}^{M'^{\bar{L}}} \begin{pmatrix} \Phi_j & & & \\ & \varphi_1 & & \\ & & \ddots & \\ & & & \varphi_s \end{pmatrix},$$

and the result follows. Q.E.D.

REMARK. Notice that we can write our final result in several ways by choosing a different fixed constituent of $\tau|_{G^{\bar{L}}_{mq}\times\prod_{i=1}^{s} G^{\bar{L}}_{n_i}}$. For example

$$\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\Big|_{G^L} \cong \bigoplus_{j=1}^{m} \mathrm{Ind}_{M'^{\bar{L}} N'}^{G^{\bar{L}}} \mathrm{Ind}_{G^{\bar{L}}_{mq}\times\prod_{i=1}^{s} G^{\bar{L}}_{n_i}}^{M'^{\bar{L}}} \begin{pmatrix} \Phi_1 & & & \\ & \varphi_1 & & \\ & & \ddots & \\ & & & \varphi_s^j \end{pmatrix},$$

where Φ_1 is a fixed constituent of $\Omega|_{G^{\bar{L}}}$ and φ_s^j ranges through the m constituents of $\varphi_s|_{G^{\bar{L}}_{n_s}}$.

Now consider the case in which r may contain more than one cycle. Then we can write

$$\tilde{M} \cong \Big(\prod_{i=1}^{\ell_1} \tilde{G}_{q_1}\Big) \times \ldots \times \Big(\prod_{i=1}^{\ell_k} \tilde{G}_{q_k}\Big) \times \tilde{G}_{n_1} \times \ldots \times \tilde{G}_{n_s},$$

and we write $\tilde{\sigma}$ as in equation (2). Let $X = X(\tilde{\sigma})$. Then there exists σ' such that $\sigma'|_M \cong \sigma$ by Theorem 4.2.2. Let $H = \bigoplus_{i=1}^{k} \big(\prod_{j=1}^{\ell_j} G_{q_i}^X\big) \times \prod_{i=1}^{s} G_{n_i}^X$. Then using the expression above, we can write σ' as

$$\sigma' \approx \mathrm{Ind}_H^{M^X} \begin{pmatrix} \Omega_\eta(\tilde{\varepsilon}, \tau_1') & & & & & & \\ & \Omega(\eta, \tau_2') & & & & & \\ & & \ddots & & & & \\ & & & \Omega(\eta, \tau_k') & & & \\ & & & & \rho_1' & & \\ & & & & & \ddots & \\ & & & & & & \rho_s' \end{pmatrix},$$

for some coset representative ε of $F^\times/N_X$. Here $\tau_i' \subseteq \tau_i|_{G_{q_i}^X}$ and $\rho_i' \subseteq \rho|_{G_{n_i}^X}$. Notice that $\Omega(\eta, \tau_i') \cong \Omega_\eta(\tilde{1}, \tau_i')$, and let Ω_i be the building block representation on $G_{\ell_i q_i}^X$ with $\ell_i = \mathrm{ord}_\eta(\tau_i')$. Each of the Ω_i has m constituents when restricted to $G_{\ell_i q_i}^L$. By a proof analogous to the proposition, we have the following result.

Theorem 5.1.2. In the above situation, assume $\Omega_1|_{G^L_{\ell_1 q_1}} \cong \Phi_1 \oplus \cdots \oplus \Phi_m$, and let

$$M' = \prod_{i=1}^{k} \tilde{G}_{\ell_i q_i} \times \prod_{i=1}^{s} \tilde{G}_{n_i}.$$

Then the irreducible constituents of $\mathrm{Ind}_{M^X N}^{G_n^X}(\sigma')|_{G^L}$ are given by

$$\mathrm{Ind}_{M^X N}^{G_n^X}(\sigma')\big|_{G_n^L}$$

$$\cong \bigoplus_{j=1}^{m} \mathrm{Ind}_{P'^L}^{G^L} \mathrm{Ind}_{\prod_{i=1}^{k} G^L_{\ell_i q_i} \times \prod_{i=1}^{s} G^L_{n_i}}^{M'^L} \begin{pmatrix} \Phi_j & & & & & & \\ & \omega_2 & & & & & \\ & & \ddots & & & & \\ & & & \omega_k & & & \\ & & & & \varphi_1 & & \\ & & & & & \ddots & \\ & & & & & & \varphi_s \end{pmatrix},$$

where φ_i is a fixed constituent of $\rho'_i|_{G^L_{n_i}}$ and ω_i is a fixed constituent of $\Omega_i|_{G^L_{\ell_i n_i}}$. Consequently the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ are given by the irreducible restriction of these representations to G.

REMARK. Once again there are several different ways we can write the same result.

§5.2. Class I Representations.

Let M be a Levi component of $G = G_n$, and suppose σ in $\mathcal{E}_2(M)$ is a Class I representation. By Proposition 4.4.1 this means that if $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$ with $\tilde{\sigma}|_M \supseteq \sigma$, then

$$\tilde{\sigma} \approx \Omega(C_1, \tau_1) \otimes \Omega(C_2, \tau_2) \otimes \ldots \otimes \Omega(C_k, \tau_v) \otimes \rho_1 \otimes \ldots \otimes \rho_s, \qquad (1)$$

where each C_i is a single character. Write

$$R \cong \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_v\mathbb{Z} \qquad (2)$$

with each $m_i > 1$, and let $\bar{w}_1, \ldots, \bar{w}_v$ be generators for the distinct factors of R. Then the nontrivial orbits of the $\bar{w}_i$ are disjoint. This fact allows us to give an explicit realization for the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$ in the same way as for the cyclic R group case.

Let $X = X(\tilde{\sigma})$, and let $\eta_1, \ldots, \eta_v$ be characters in $\bar{L} = \bar{L}(\tilde{\sigma})$ corresponding to $\bar{w}_1, \ldots, \bar{w}_v$, respectively. By the classification of discrete series (Theorem 4.2.2), there exists $\sigma' \in \mathcal{E}_2(M^X)$ such that $\sigma'|_M \cong \sigma$. Moreover using (1), we can write

$$\sigma' \approx \begin{pmatrix} \Omega_{C_1}(\tilde{\varepsilon}, \tau_1') & & & & & & \\ & \Omega(C_2, \tau_2') & & & & & \\ & & \ddots & & & & \\ & & & \Omega(C_k, \tau_k') & & & \\ & & & & \rho_1' & & \\ & & & & & \ddots & \\ & & & & & & \rho_s' \end{pmatrix}, \qquad (3)$$

where ε is some coset representative for $F^\times/N_X$. We can write

$$\tilde{M} \cong \Big(\prod_{i=1}^{\ell_1} \tilde{G}_{q_1}\Big) \times \ldots \times \Big(\prod_{i=1}^{\ell_k} \tilde{G}_{q_k}\Big) \times \tilde{G}_{n_1} \times \ldots \times \tilde{G}_{n_s}.$$

Let Ω_i be the building block representation corresponding to $\Omega(C_i, \tau_i')$ for $2 \le i \le k$, and let Ω_1 be the building block representation corresponding to $\Omega_{C_1}(\tilde{\varepsilon}, \tau_1')$. Suppose $C_j = \eta_i$. Then $\eta_{i'}$ is in $X(\tau_j)$ for all $i' \neq i$ by Proposition 4.4.1, and therefore there are $m_{i'} = \mathrm{ord}_{\eta_{i'}}(\tilde{\sigma})$ constituents when Ω_j is restricted to $G^{\eta_{i'}}_{\ell_j q_j}$, using equation (2). Moreover when Ω_j is restricted to $G^{\eta_i}_{\ell_j q_j}$ it has m_i irreducible constituents, according to the theory of the R group and the Restriction Theorem. Thus $\Omega_j|_{G^{\bar{L}}_{q_j \ell_j}}$ has $|R|$ irreducible constituents. Denote the reducibility of Ω_1 on $G^{\bar{L}}_{q_1 \ell_1}$ by

$$\Omega_1|_{q_1 \ell_1} \cong \Phi_1 \oplus \cdots \oplus \Phi_m,$$

where $|R| = m$.

Theorem 5.2.1. In the above situation, we can express irreducible constituents of $\mathrm{Ind}^{G^X}_{M^X N}(\sigma')\big|_{G^{\bar{L}}}$ by

$$\mathrm{Ind}^{G^X}_{M^X N}(\sigma')\big|_{G^{\bar{L}}}$$

$$\cong \bigoplus_{j=1}^{m} \mathrm{Ind}^{G^{\bar{L}}}_{P'^{\bar{L}}} \mathrm{Ind}^{M'^{\bar{L}}}_{\prod_{i=1}^{k} G^{\bar{L}}_{q_i \ell_i} \times \prod_{i=1}^{s} G^{\bar{L}}_{n_i}} \begin{pmatrix} \Phi_j & & & & & & \\ & \omega_2 & & & & & \\ & & \ddots & & & & \\ & & & \omega_k & & & \\ & & & & \varphi_1 & & \\ & & & & & \ddots & \\ & & & & & & \varphi_s \end{pmatrix},$$

where $M' = \prod_{i=1}^{k} \tilde{G}_{q_i \ell_i} \times \prod_{i=1}^{s} \tilde{G}_{n_i}$, the representations ω_i are a fixed constituents of $\Omega_i|_{G^L_{q_i \ell_i}}$, and $\varphi_{i'}$ is a fixed constituent of $\rho'_{i'}|_{G^L_{n_{i'}}}$ for $1 \leq i' \leq s$. Therefore the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ are given by the irreducible restriction of the representations on the right hand side of this equation to G.

PROOF. The key fact in this result is that if $C_j = \eta_i$, then $\eta_{i'}$ is in $X(\tau_j)$ for all $i' \neq i$. Then the proof is essentially the same as that for Theorem 5.1.2. By the Restriction Theorem, $\mathrm{Ind}_{M^X N}^{G^X}(\sigma')\big|_{G^L}$ has $m = |R|$ irreducible constituents. As in Theorem 5.1.2, we enlarge the Levi component to M' through double induction and apply Lemma 3.4.1. Thus we are interested in the reducibility of

$$\mathrm{Ind}_{\prod_{i=1}^{k} G^X_{q_i \ell_i} \times \prod_{i=1}^{s} G^X_{n_i}}^{M'^X} \begin{pmatrix} \Omega_1 & & & & & \\ & \ddots & & & & \\ & & \Omega_k & & & \\ & & & \rho'_1 & & \\ & & & & \ddots & \\ & & & & & \rho'_s \end{pmatrix} \Bigg|_{M'^L} .$$

This representation also has m constituents by the Restriction Theorem, and if we restrict all the way down to $\prod_{i=1}^{k} G^{\bar{L}}_{q_i \ell_i} \times \prod_{i=1}^{s} G^{\bar{L}}_{n_i}$ there are m^{k+s} irreducible constituents. We then apply Mackey theory, using the fact that the index of $\prod_{i=1}^{k} G^{\bar{L}}_{q_i \ell_i} \times \prod_{i=1}^{s} G^{\bar{L}}_{n_i}$ in $M'^{\bar{L}}$ is equal to m^{k+s-1}, to see that the constituents on the smaller group induce equivalently in sets of m^{k+s-1} to give the constituents on $M'^{\bar{L}}$. By the Restriction Theorem, if τ is one constituent of this representation on ${M'}^{\bar{L}}$, then the other constituents are given by $\tilde{\varepsilon}\tau$, where ε runs through a complete set of coset representatives for $F^\times/N_{\bar{L}}$. Fix the constituent

$$\tau = \mathrm{Ind}_{\prod_{i=1}^{k} G^{L}_{q_i \ell_i} \times \prod_{i=1}^{s} G^{\bar{L}}_{n_i}}^{{M'}^{L}} \begin{pmatrix} \Phi_1 & & & & & & \\ & \omega_2 & & & & & \\ & & \ddots & & & & \\ & & & \omega_k & & & \\ & & & & \varphi_1 & & \\ & & & & & \ddots & \\ & & & & & & \varphi_s \end{pmatrix},$$

and the result follows since the $\tilde{\varepsilon}$ factor passes through to the inducing representation by Lemma 4.1.3.   Q.E.D.

§5.3. Elliptic Representations.

Let G be a totally disconnected group. An element x of G is said to be **elliptic** if its centralizer is compact, modulo $Z(G)$. The set of all elliptic elements within G is denoted G^e. Suppose that π is a tempered representation of G and that its global character is Θ_π. It is known that Θ_π is a locally integrable function [H-C2], and we can therefore consider its restriction to G^e. If $\Theta_\pi|_{G^e} \neq 0$, we say that π is **elliptic**.

Let $P = MN$ be a standard parabolic subgroup of G, and let A be the split component of M. Suppose that σ is in $\mathcal{E}_2(M)$ and that $R = R(\sigma)$

is the R group of σ. Let $X(M)$ be the group of F-rational characters of M. Define the **real lie algebra** of A as $\mathfrak{a} = \mathrm{Hom}(X(M), \mathbb{Z}) \otimes \mathbb{R}$. The group $W_G(A) = W_G(M)$ acts on $X(A)$ and consequently on $\mathfrak{a}$. If $\bar{w}$ is in $R(\sigma)$, let $\mathfrak{a}_{\bar{w}} = \{H \in \mathfrak{a} \mid wH = H\}$. Furthermore let $\mathfrak{a}_R = \cap a_{\bar{w}}$, the intersection taken over $\bar{w} \in R(\sigma)$. Let $\mathfrak{z}$ denote the real lie algebra of Z. If $G = G_n$, then $\mathfrak{z} = \{0\}$. Write $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$.

Theorem 5.3.1 (Arthur [Ar]). Suppose that $\mathcal{C}(\pi) \cong \mathbb{C}[R]$ and R is abelian. Then the following are equivalent:

(a) $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$ has an elliptic constituent,

(b) all of the constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ are elliptic,

(c) There is a $\bar{w} \in R$ with $\mathfrak{a}_{\bar{w}} = \mathfrak{a}_R$.

Theorem 5.3.2 (Herb [H]). Suppose that $\mathcal{C}(\pi) \cong \mathbb{C}[R]$ and R is abelian. If ρ be an irreducible constituent of $\mathrm{Ind}_{MN}^{G}(\sigma)$, then $\rho \cong \mathrm{Ind}_{M'N'}^{G}(\tau)$ for some proper Levi subgroup M' and some irreducible tempered representation τ of M' if and only if $\mathfrak{a}_R \neq \mathfrak{z}$. Moreover M' and τ can be chosen with τ elliptic if and only if there is a $\bar{w}_0 \in R$ with $\mathfrak{a}_R = \mathfrak{a}_{\bar{w}_0}$.

Goldberg [Gol] has made an extensive study of the role of the R group in the theory of elliptic representations for $SL_n(F)$. His theorems rely on the results of Arthur and Herb cited above. Throughout this section we consider the following situation. Let $P = MN$ be a standard parabolic subgroup of $G_n = SL_n(F)$, and write the corresponding $\tilde{M}$ in "block diagonal" form.

Consequently

$$\tilde{M} \cong \tilde{G}_{m_1} \times \ldots \times GL_{m_k}.$$

Let $\sigma \in \mathcal{E}_2(M)$ and $\tilde{\sigma} \in \mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Suppose $R = R(\sigma)$ be the R group of σ, and write $R \cong \bar{L}(\tilde{\sigma})/X(\tilde{\sigma})$.

Goldberg showed that the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$ are elliptic if and only if $\tilde{\sigma} \approx \Omega_\eta(\tau)$ and that the constituents are neither elliptic nor induced from elliptic if $\tilde{\sigma}$ is a Class II representation (see Section 5.4); see [Gol].

Proposition 5.3.3. Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Then the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ are induced from elliptic representations if and only if either

(1) $R(\sigma)$ is cyclic and $\tilde{\sigma} \not\approx \Omega_\eta(\tau)$ for any discrete series representation τ, or

(2) $R(\sigma)$ is not cyclic and $\tilde{\sigma}$ is a Class I representation.

PROOF. The proof follows from calculating $\mathfrak{a}_R$ and applying Theorem 5.3.2. In the first case, $\mathfrak{a}_R = \mathfrak{a}_{\bar{w}}$ where $\bar{w}$ generates $R(\sigma)$. In the second case, $\mathfrak{a}_R = \mathfrak{a}_{\bar{w}_1 \ldots \bar{w}_k}$, where $\{\bar{w}_i\}$ is a complete set of generators for the distinct factors of $R(\sigma)$, since the nontrivial orbits of the $\bar{w}_i$ are disjoint. Let $\bar{w}$ be in $R(\sigma)$, and let r be the corresponding permutation on the blocks of M. Then the Lie algebra $\mathfrak{a}_{\bar{w}}$ consists of the diagonal $(n \times n)$ matrices of trace 0 that have the same scalar on all of the blocks within each orbit of r. Therefore if we write $\tilde{\sigma}$ as in Proposition 4.4.1 or Proposition 4.4.2, respectively, then

$$\mathfrak{a}_R = \left\{ \begin{pmatrix} d_1 I_{\ell_1 n_1} & & & & & & \\ & d_2 I_{\ell_2 n_2} & & & & & \\ & & \ddots & & & & \\ & & & d_k I_{\ell_k n_k} & & & \\ & & & & d'_1 & & \\ & & & & & \ddots & \\ & & & & & & d'_s \end{pmatrix} \right\},$$

where ℓ_i is the length of the ith cycle in r and n_i is the common size of the blocks permuted by the ith cycle. Since $s \geq 1$, we have $\mathfrak{a}_R \neq \mathfrak{z} = \{0\}$. Thus the irreducible constituents are induced from elliptic representations. Q.E.D.

Theorems 5.1.2 and 5.2.1 give concrete realizations of this proposition, and we can check, in fact, that the inducing representations of the constituents are elliptic. This follows by combining the fact that discrete series are elliptic (ref. [Clo]), that finite induction preserves ellipticity, and Goldberg's result on elliptic constituents (ref. [Gol]). We shall see in the next chapter that while the irreducible constituents of generalized principal series induced from Class II representations are neither elliptic nor induced from elliptic representations, the characters in many cases can be written as a linear combination of characters that are induced from elliptic characters.

VI. Class II Representations: An Approach Through Characters

Let M be the Levi component for a standard parabolic subgroup of $G = G_n$, and let $\tilde{M}$ be the corresponding Levi component of $\tilde{G} = \tilde{G}_n$. Let σ be in $\mathcal{E}_2(M)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Suppose σ is a Class II representation, i.e., $R = R(\sigma)$ is not cyclic and $\tilde{\sigma}$ has the property that if we write

$$\tilde{\sigma} \approx \Omega(C_1, \tau_1) \otimes \Omega(C_2, \tau_2) \otimes \ldots \otimes \Omega(C_k, \tau_k) \otimes \rho_1 \otimes \ldots \otimes \rho_s, \qquad (1)$$

according to Proposition 4.4.2, then not all of the C_i are single characters. In this case, we are unable to determine the constituents of $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$ using only the methods of Mackey theory and the Restriction Theorem as in Chapter 5. Instead we approach the problem through characters, and we are led to a system of linear equations defined over the characters of the irreducible constituents of $\pi = \mathrm{Ind}_{MN}^{G}(\sigma)$. This alternate approach proves to be partially successful.

We introduce the notions of a "bottom layer" and an "intermediate layer" of subgroups within $\tilde{G}$. If $C_G(\sigma)$ is the set of characters of the constituents of π, then we shall see that the bottom layer provides a linear system of equations over $C_G(\sigma)$. To solve for the characters in $C_G(\sigma)$ we need a subset of $|R|$ linearly independent equations. Dependency within the equations occurs because of character identities on subgroups in the intermediate layer. These relations may make it impossible to solve for $C_G(\sigma)$ by using only the equations from the bottom layer. We find, however, that in many cases the system is solvable, and other cases illustrate the problems faced in trying to solve such a system. A series of examples are found in Section 3.

We have seen in Section 5.3 that the characters in $C_G(\sigma)$ are not elliptic, nor are they induced from elliptics. In the situations we can handle, however, these characters are a linear combinations of characters induced from elliptics.

§6.1. System of Linear Equations.

Let $R = R(\sigma)$ be the R group of σ, and write

$$R \cong \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \ldots \times \mathbb{Z}/m_v\mathbb{Z},$$

with each $m_i > 1$. Let $\bar{w}_1, \ldots, \bar{w}_v$ be elements of R that generate the respective factors of R, and let $S = \{\eta_1, \ldots, \eta_v\}$ be a set of characters in $\bar{L}(\tilde{\sigma})$ taken modulo $X(\tilde{\sigma})$ such that η_i is associated with $\bar{w}_i$. Then the sets C_i in equation (1) are subsets of S. Let $X = X(\tilde{\sigma})$. By the classification of discrete series (Theorem 4.2.2), there exists a representation σ' of M^X such that $\sigma'|_M \cong \sigma$. Using (1), we can write

$$\sigma' \approx \tilde{\varepsilon}\,\Omega(C_1, \tau_1') \otimes \Omega(C_2, \tau_2') \otimes \ldots \otimes \Omega(C_k, \tau_k') \otimes \rho_1' \otimes \ldots \otimes \rho_s', \qquad (2)$$

where the τ_i' are irreducible constituents of $\tau_i|_{G_{q_i}^X}$ and the ρ_i' are irreducible constituents of $\rho_i|_{G_{n_i}^X}$. Here $\tilde{\varepsilon}$ is the matrix $\mathrm{diag}(\varepsilon, 1, \ldots, 1)$ in $\tilde{G}_{\ell_1 q_1}$ for some coset representative ε of $F^\times/N_X$. Then Lemma 3.4.1 says that

$$\operatorname{Ind}_{M^X N}^{G^X}(\sigma')\big|_G \cong \operatorname{Ind}_{MN}^{G}(\sigma),$$

and Proposition 3.4.2 says that the reducibility on the left hand side of this equivalence actually occurs on $G^{\bar{L}}$, where $\bar{L} = \bar{L}(\tilde{\sigma})$. Let $\pi' = \operatorname{Ind}_{M^X N}^{G^X}(\sigma')$. To simplify notation, the phrase "characters of $\pi'|_H$," where H is a subgroup of G^X, refers to the characters of the irreducible constituents of $\pi'|_H$. Denote the set of characters of $\pi'|_H$ by $C_H(\sigma)$.

According to the Restriction Theorem and the theory of the R group, we have $G^X/G^{\bar{L}} \cong R$. Subgroups R' of R correspond in the following way to closed normal subgroups G' of G^X such that $G^{\bar{L}} \subseteq G'$. Using the isomorphism $R(\sigma) \cong \bar{L}(\sigma)/X(\tilde{\sigma})$, consider R' as a subgroup of characters. Then

$$G' = \{g \in G^X \mid \det(g) \in \ker\xi, \quad \text{for all } \xi \in R'\}$$

is a closed normal subgroup of G^X containing $G^{\bar{L}}$, and $G^X/G' \cong R'$. Consequently the subgroup lattice of R, as a finite abelian group, determines a similar subgroup lattice inside of G^X. The structure of this lattice determines whether or not equations can be found for the characters of $\pi'|_{G^{\bar{L}}}$.

Definition. A **bottom layer subgroup** H, within the subgroup lattice determined by R, is one such that G^X/H is cyclic and for any other closed normal subgroup G' with $G^{\bar{L}} \subseteq G' \subseteq H$ the group G^X/G' is not cyclic. The set of bottom layer subgroups is called the **bottom layer** and is denoted $\mathcal{B}$.

The subgroups in $\mathcal{B}$ correspond to maximal cyclic subgroups of R. The rationale behind choosing $\mathcal{B}$ in this manner will be made clear in the next section. First we show how to obtain a linear system of equations over $C_G(\sigma)$ from the groups in $\mathcal{B}$, and the following example is used as a template for the theory.

EXAMPLE. Let $G = SL_4(F)$, and let κ and η be nontrivial characters of $F^\times$ such that $\kappa^2 = \eta^2 = 1$. Define a representation of M_0 by $\sigma = 1 \otimes \kappa \otimes \eta \otimes \eta\kappa$, where M_0 is the Levi component of the standard minimal parabolic subgroup of G. Then $R = R(\sigma)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and σ is a Class II representation. If $\tilde{\sigma}$ is in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$, then $\tilde{\sigma}|_M \cong \sigma$ and $X(\tilde{\sigma}) = \{1\}$. Thus $R \cong \bar{L}(\tilde{\sigma})$. For this particular σ, we can choose $\tilde{\sigma} = \sigma$. We have the group lattices

$$\begin{array}{ccc} & \bar{L}(\tilde{\sigma}) & \\ / & | & \backslash \\ \{1,\kappa\} & \{1,\eta\kappa\} & \{1,\eta\} \\ \backslash & | & / \\ & \{1\} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \tilde{G} & \\ / & | & \backslash \\ G^\kappa & G^{\eta\kappa} & G^\eta \\ \backslash & | & / \\ & G^{\bar{L}} & \end{array} .$$

Then $\mathcal{B} = \{G^\kappa, G^\eta, G^{\eta\kappa}\}$. Let $\tilde{\pi} = \mathrm{Ind}_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma})$, and denote the irreducible constituents of $\tilde{\pi}|_{G^{\bar{L}}}$ by

$$\tilde{\pi}|_{G^{\bar{L}}} = \tau \oplus \varepsilon_\kappa \tau \oplus \varepsilon_\eta \tau \oplus \varepsilon_\eta \varepsilon_\kappa \tau,$$

where ε_κ and ε_η are coset representatives for $\tilde{G}/G^\kappa$ and $\tilde{G}/G^\eta$, respectively. If we write $\tilde{\pi}|_{G^\kappa} = \pi_\kappa^1 \oplus \pi_\kappa^2$, then by the Restriction Theorem $\pi_\kappa^2 = \varepsilon_\kappa \pi_\kappa^1$.

Denote the characters of these constituents by Θ^1_κ and $\varepsilon_\kappa\Theta^1_\kappa$. If θ_τ is the character of τ, then we have the equations

$$\Theta^1_\kappa = \theta_\tau + \varepsilon_\eta\theta_\tau$$

and

$$\varepsilon_\kappa\Theta^1_\kappa = \varepsilon_\kappa\theta_\tau + \varepsilon_{\eta\kappa}\theta_\tau.$$

The groups G^η and $G^{\eta\kappa}$ determine four more equations, which we write as

$$\begin{aligned} \Theta_\eta &= \theta_\tau + \varepsilon_\kappa\theta_\tau & \qquad \Theta_{\eta\kappa} &= \theta_\tau + \varepsilon_\eta\varepsilon_\kappa\theta_\tau \\ \varepsilon_\eta\Theta_\eta &= \varepsilon_\eta\theta_\tau + \varepsilon_\eta\varepsilon_\kappa\theta_\tau & \qquad \varepsilon_\eta\Theta_{\eta\kappa} &= \varepsilon_\eta\theta_\tau + \varepsilon_\kappa\theta_\tau, \end{aligned}$$

where we take ε_η as the nontrivial coset representative of $\tilde{G}/G^{\eta\kappa}$ (note that $\varepsilon_\eta\varepsilon_\kappa$ is *not* a nontrivial coset representative). This linear system of 6 equations has rank 4, and therefore we can solve for θ_τ, etc. We find, for example, that

$$\theta_\tau = \frac{1}{2}(\Theta_\kappa + \Theta_\eta - \varepsilon_\eta\Theta_{\eta\kappa}),$$

and we have an understanding of Θ_κ, Θ_η, and $\Theta_{\eta\kappa}$ from Theorem 5.1.2. Each of these characters is induced from elliptic ones (see Section 5.3), and therefore we can express θ_τ, etc. as a linear combination of characters that are induced from elliptics.

Now return to the general situation. Write σ' as in equation (2), and form the subgroup lattice in G^X related to R. Let $\pi' = \mathrm{Ind}_{M^X N}^{G^X}(\sigma')$. If H is in the bottom layer $\mathcal{B}$, then write $|G^X/H| = b$ and $R_H \subseteq R$ for the group of characters associated to H. By multiplicity one and the Restriction Theorem, $\pi'|_H$ has b irreducible constituents. From our results on cyclic R groups, we understand the constituents of $\pi'|_H$, and consequently their characters. Our equations come from the relationships between these characters and those for the constituents of $\pi'|_{G^{\bar{L}}}$. Since all of the reducibility of $\pi'|_G$ occurs on the group $G^{\bar{L}}$, we are interested in studying $\pi'|_{G^{\bar{L}}}$. Write

$$G^X/H = \{\bar{1}, \bar{\zeta}, \bar{\zeta}^2, \ldots, \bar{\zeta}^{b-1}\}$$

and

$$H/G^{\bar{L}} = \{\bar{\varepsilon}_0 = \bar{1}, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_{m-1}\},$$

where $|R| = mb$.

If Φ is an irreducible constituent of $\pi'|_H$, then the Restriction Theorem says that

$$\Phi|_{G^{\bar{L}}} = \bigoplus_{i=0}^{m-1} \varepsilon_i \tau, \tag{3}$$

where τ is a fixed constituent of $\Phi|_{G^{\bar{L}}}$. As usual $\varepsilon_i\tau$ refers to the representation of $G^{\bar{L}}$ given by $\varepsilon_i\tau(g) = \tau(\varepsilon_i^{-1} g \varepsilon_i)$. With equation (3) in mind, to each $\Phi \subseteq \pi'|_H$ we associate a set of constituents in $\pi'|_{G^{\bar{L}}}$. More importantly if Θ is the character of Φ, we associate the corresponding set $S_H(\Theta)$ of characters of $\pi'|_{G^L}$. Let $\varepsilon_i\theta_\tau$ be the character of the representation $\varepsilon_i\tau$, and write this set as

$$S_H(\Theta) = \{\varepsilon_i\theta_\tau\}_{0\leq i\leq m-1}. \tag{4}$$

The relationship of equation (3) implies that

$$\Theta|_{G^L} = \sum_{i=0}^{m-1} \varepsilon_i\theta_\tau. \tag{5}$$

According to the Restriction Theorem, the other constituents of $\pi'|_H$ are given by $\zeta^j\Phi$ for $1 \leq j \leq b-1$, and consequently

$$S_H(\zeta^j\Theta) = \{\varepsilon_i\zeta^j\theta_\tau\}_{0\leq i\leq m-1}. \tag{6}$$

Let $C(\sigma) = C_{\bar{L}}(\sigma)$ denote the set of characters for the constituents of $\pi'|_{G^L}$. Then $C(\sigma) = \bigcup_{j=0}^{b-1} S_H(\zeta^j\Theta)$ and $S_H(\zeta^i\Theta) \cap S_H(\zeta^j\Theta) = \emptyset$ whenever $i \neq j$. Thus the $S_H(\Theta)$ form a partition of $C(\sigma)$ that is independent of the choice of τ.

Every irreducible constituent Φ of $\pi'|_H$ determines a linear equation over $C(\sigma)$ as in equation (5). By multiplicity one, the characters of the irreducible constituents are linearly independent [Sil], and therefore the linear equations given by (5) are necessarily independent. Considering the restriction of π to each of the bottom layer subgroups, we obtain a linear system of equations. If we can find a linearly independent subset of $|R| = mb$ equations within this collection, we can obtain an expression for the characters of $\pi'|_{G^L}$, and consequently for the characters of the irreducible constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$.

The subgroup lattices in our earlier example are very simple and there is no real "intermediate layer" inhibiting the independence of our equations. In general this intermediate layer plays a significant role, which is explained in the next section.

§6.2. Intermediate Layer $\mathcal{M}$.

Retain the notation of the previous section. If G' is a subgroup of G^X with $G^{\bar{L}} \subseteq G'$, then we can consider the restriction of π' to G'. Let H be in $\mathcal{B}$ with $H_1 \subseteq G'$. As in the previous section, we can write the characters of the constituents of $\pi'|_{G'}$, when restricted to G^{H_1}, as a sum of characters for certain constituents of $\pi'|_{H_1}$. Thus G' determines a partition of the characters in $\pi'|_{H_1}$. Suppose that $H_2 \neq H_1$ is in $\mathcal{B}$ and that H_2 is contained in G'. Then we can also write the characters of the constituents of $\pi'|_{G'}$, when restricted to G^{H_2}, as a sum of certain characters for constituents of $\pi'|_{H_2}$. Therefore if Φ is an irreducible constituent of $\pi'|_{G'}$ with character Θ we have

$$\Theta|_{H_1} = \theta_1 + \varepsilon_1\theta_1 + \cdots + \varepsilon_k\theta_1 \tag{1}$$

and

$$\Theta|_{H_2} = \theta_2 + \rho_1\theta_2 + \cdots + \rho_\ell\theta_2, \tag{2}$$

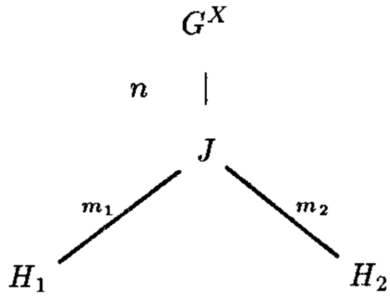
where θ_i is the character for an irreducible constituent of $\Phi|_{H_i}$, the ε_i are a complete set of coset representatives of G/H_1, and the ρ_i are a complete set of coset representatives of G/H_2. The restriction of equations (1) and (2) to $G^{\bar{L}}$ give character identities relating the characters in $\pi'|_{H_1}$ and $\pi'|_{H_2}$, and consequently serves to limit the independence of the equations determined by H_1 and H_2. These relationships are maximized by the groups in the "intermediate layer" defined below.

Definition. For any two elements H_1 and H_2 in $\mathcal{B}$, the smallest subgroup of G^X in the lattice of R containing both H_1 and H_2 is said to be an **intermediate group**. The set of such groups is referred to as the **intermediate layer** and is denoted $\mathcal{M}$.

Let σ be in $\mathcal{E}_2(M)$, and form the subgroup lattice in G^X determined by the R group of σ. Let J be in the intermediate layer, and let H_1 and H_2 be in the bottom layer. We have $|G^X/J|$ character identities relating the characters of $\pi'|_{H_1}$ and the characters of $\pi'|_{H_2}$. These identities consequently limit the number of independent equations arising from H_1 and H_2. Dependence among equations occurs in two principal situations, described below as (R-1) and (R-2) restrictions.

Fix the following notation. Let σ in $\mathcal{E}_2(M)$ be a Class II representation, and let R be the R group of σ. Let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. Write $X = X(\tilde{\sigma})$ and $\bar{L} = \bar{L}(\tilde{\sigma})$. Form the subgroup lattice within G^X determined by R. Let σ' be as in equation (2) of Section 1, and let $\pi' = \mathrm{Ind}_{M^X N}^{G^X}(\sigma')$.

(R-1) RESTRICTIONS. Suppose we have a sublattice within the lattice determined by R with the diagram:



(R-1)

where H_1 and H_2 are in $\mathcal{B}$, and J is in $\mathcal{M}$. Then H_1 provides $m_1 n$ linear equations and H_2 provides $m_2 n$ equations, as described in Section 1. The character identities on J, however, limit the independence of these equations. Specifically let R_1 and R_2 be the subgroups of R such that $G^X/H_1 \cong R_1$ and $G^X/H_2 \cong R_2$. Also let $G^X/J \cong R' \subseteq R$. Since $H_1 \subseteq J$ and G^X/H_1 is cyclic, so is G^X/J. Suppose ξ is a coset representative for a generator of G^X/J. Then

$$\pi'|_J \cong \Phi \oplus \xi\Phi \oplus \cdots \oplus \xi^{n-1}\Phi$$

for any constituent Φ of $\pi'|_J$. We can fix φ and $\tilde{\varphi}$ in $\Phi|_{H_1}$ and $\Phi|_{H_2}$, respectively, and write

$$\Phi|_{H_1} \cong \varphi \oplus \varepsilon_1\varphi \oplus \cdots \oplus \varepsilon_1^{m_1-1}\varphi$$

and

$$\Phi|_{H_2} \cong \tilde{\varphi} \oplus \varepsilon_2\tilde{\varphi} \oplus \cdots \oplus \varepsilon_2^{m_2-1}\tilde{\varphi},$$

where $\bar{\varepsilon}_1$ generates J/H_1, and $\bar{\varepsilon}_2$ generates J/H_2. Each of the $\varepsilon_1^k\varphi$ and $\varepsilon_2^\ell\tilde{\varphi}$ provides an equation over $C(\sigma)$, as described in Section 1. Let θ be the character of φ, and let $\tilde{\theta}$ be the character of $\tilde{\varphi}$. If Θ is the character of Φ then we have the character identity

$$\begin{aligned}\Theta|_{G^{\mathcal{L}}} &= \theta|_{G^{\mathcal{L}}} + \varepsilon_1\theta|_{G^{\mathcal{L}}} + \cdots + \varepsilon_1^{m_1-1}\theta|_{G^{\mathcal{L}}} \\ &= \tilde{\theta}|_{G^{\mathcal{L}}} + \varepsilon_2\tilde{\theta}|_{G^{\mathcal{L}}} + \cdots + \varepsilon_2^{m_2-1}\tilde{\theta}|_{G^{\mathcal{L}}}. \qquad (3)\end{aligned}$$

The relationship in (3) limits the independence of the equations associated to $\varepsilon_1^k\theta$ and $\varepsilon_2^\ell\tilde{\theta}$ to at most m_1+m_2-1 independent equations. By considering all n of the irreducible constituents in $\pi'|_{\mathcal{I}}$, we see that there are at most $n(m_1+m_2-1)$ independent equations in those determined by H_1 and H_2.

(R-2) Restrictions. This situation must occur in conjunction with an (R-1) restriction and is significantly more subtle. It shows the relationship between character identities on certain subgroups in the intermediate layer. Suppose we have the sublattice

$$\begin{array}{ccccc} & & G^X & & \\ & {\scriptstyle m_1}\diagup & & \diagdown{\scriptstyle m_2} & \\ J_1 & & & & J_2 \\ & {\scriptstyle n/m_1}\diagdown & & \diagup{\scriptstyle n/m_2} & \\ & & H & & \end{array} \qquad (R\text{-}2)$$

with J_1, J_2 in $\mathcal{M}$ and $H \in \mathcal{B}$. Suppose $R_H = <\bar{\eta}>$, and choose $\xi \in G^X$ such that $\eta(\xi)$ is a primitive nth root of unity. Then $G^X/H \cong <\bar{\xi}>$. Consider the constraints on the possibilities for J_1 and J_2. Assume that $m_2 > m_1$. We write $R_1 = R_{H_1} \cong <\bar{\eta}^{n/m_1}>$ and $R_2 = R_{H_2} \cong <\bar{\eta}^{n/m_2}>$. We want $J_2 \not\subseteq J_1$ and this is true if and only if $R_1 \not\subseteq R_2$. This happens exactly when m_1 does not divide m_2.

Write our quotient groups as:

$$G^X/J_1 \cong <\bar{\xi}> = \{\bar{1}, \bar{\xi}, \ldots, \bar{\xi}^{m_1-1}\},$$
$$G^X/J_2 \cong <\bar{\xi}> = \{\bar{1}, \bar{\xi}, \ldots, \bar{\xi}^{m_2-1} = \bar{\xi}^{n-m_1}\},$$
$$J_1/H \cong <\bar{\xi}^{m_1}> = \{\bar{1}, \bar{\xi}^{m_1}, \ldots, \bar{\xi}^{(n/m_1-1)m_1} = \bar{\xi}^{n-m_2}\}, \text{ and}$$
$$J_2/H \cong <\bar{\xi}^{m_2}> = \{\bar{1}, \bar{\xi}^{m_2}, \ldots, \bar{\xi}^{(n/m_2-1)m_2}\}.$$

Let $d = \gcd(m_1, m_2)$, and write $m_1 = de$ and $m_2 = df$. Since m_1 does not divide m_2, we know that $f > e > 1$.

Recall that $C_H(\sigma)$ is the set of characters for the constituents of $\pi'|_H$. The above remarks show that there are two partitions of $C_H(\sigma)$ given by J_1 and J_2. Fix $\varphi \subseteq \pi'|_H$ with character θ. Without loss of generality, we can choose $\Phi_1 \subseteq \pi'|_{J_1}$ and $\Phi_2 \subseteq \pi'|_{J_2}$ so that their respective characters Θ_1 and Θ_2 satisfy

$$\Theta_1|_H = \theta + \xi^{m_1}\theta + \cdots + \xi^{n-m_1}\theta$$

and

$$\Theta_2|_H = \theta + \xi^{m_2}\theta + \cdots + \xi^{n-m_2}\theta. \tag{4}$$

Write $S_{J_i}(\Theta_i)$ for the characters in $C_H(\sigma)$ that are given by $\Theta_i|_H$. In relation to the character identities given in (4), we write $S_1(\Theta_1) = S_{J_1}(\Theta_1)$ and $S_2(\Theta_2) = S_{\mathcal{I}_2}(\Theta_2)$. Then $S_i(\Theta_i)$ contains n/m_i characters in $C_H(\sigma)$.

The other elements of $C_{J_1}(\sigma)$ are given by $\xi^k\Theta_1$ for $1 \leq k \leq m_2 - 1$, while those in $C_{\mathcal{I}_2}(\sigma)$ are $\xi^\ell\Theta_2$ for $1 \leq \ell \leq m_1 - 1$. We want to see the connection between these two sets that prevents some degree of independence among the equations determined by $\mathcal{B}$. Define

$$U_0 = S_1(\Theta_1) \cup S_1(\xi^d\Theta_1) \cup \cdots \cup S_1(\xi^{(e-1)d}\Theta_1).$$

This contains $e(n/m_1) = f(n/m_2)$ characters in $C_H(\sigma)$. If r is the smallest positive integer such that $(r+1)e > f$, then $\xi^{m_2}\theta = \xi^{d(f-re)}\xi^{rm_1}\theta$ is in U_0 since $n/m_1 \geq r$. Using multiples of r we see that, in fact,

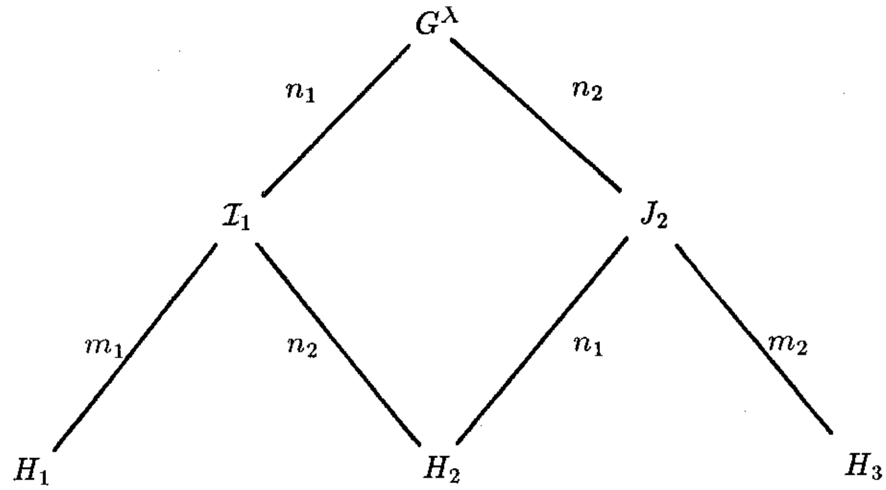
$$U_0 = S_2(\Theta_2) \cup S_2(\xi^d\Theta_2) \cup \cdots \cup S_2(\xi^{(f-1)d}\Theta_2),$$

and more generally, if we define $U_i = \bigcup S_1(\xi^{i+kd}\Theta_1)$ for $0 \leq i \leq d-1$, then

$$\begin{aligned} U_i &= S_1(\xi^i\Theta_1) \cup S_1(\xi^{i+d}\Theta_1) \cup \cdots \cup S_1(\xi^{i+(e-1)d}\Theta_1) \\ &= S_2(\xi^i\Theta_2) \cup S_2(\xi^{i+d}\Theta_2) \cup \cdots \cup S_2(\xi^{i+(f-1)d}\Theta_2). \end{aligned} \tag{5}$$

Thus $C_H(\sigma) = \bigcup_{i=0}^{d-1} U_i$. As in Section 1, each $\xi^\ell\theta$ in $C_H(\sigma)$ determines a linear equation over $C(\sigma)$, and the U_i give another partition of $C(\sigma)$. Because of the relations in (5), the U_i translate to relations between the equations arising from the $\xi^\ell\theta$, and therefore inhibit their independence. Since the exact nature of this restriction is not obvious, we provide a few detailed examples.

Example of (R-2) Restriction. Consider the situation where the lattice in G^X determined by R contains the following sublattice:



with $\gcd(m_1, m_2) = 1$, $\{H_1, H_2, H_3\} \subseteq \mathcal{B}$ and $\{J_1, J_2\} \subseteq \mathcal{M}$. Let us count the maximal number of independent equations coming from this sublattice.

We can get $m_1 n_1$ equations from H_1. This gives all the character identities determined by J_1 (as in equation (3)), and by (R-1) we have at most $n_1(n_2 - 1)$ independent equations from H_2. From (R-2) we see that these equations give $n_2 - 1$ of the character identities determined by J_2, as in equation (4), and we are limited to $(n_2 - 1)(m_2 - 1)$ equations from H_3. Therefore the maximum number of independent equations determined by $\{H_1, H_2, H_3\}$ is $m_1 n_1 + n_1(n_2 - 1) + (n_2 - 1)(m_2 - 1)$. Notice that we could have started by taking $n_2 m_2$ equations on H_3 or $n_1 n_2$ equations on H_2.

These relationships make the process of determining a linearly independent subset of equations significantly more complex. The previous example illustrates the simplest case. Now consider the situation where

$R \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, which provides very nice insight into the complexity that can occur.

Another Example of (R-2) Restriction. Let $\sigma \in \mathcal{E}_2(M)$ with $R(\sigma) \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Under the realization $R(\sigma) \cong \bar{L}(\sigma)/X(\tilde{\sigma})$, write the coset representatives for generators of the distinct factors of $R(\sigma)$ as η and κ. Form the subgroup lattice of R, and form the associated one inside of G^X. There are twelve subgroups in $\mathcal{B}$ corresponding to maximal cyclic subgroups of R and seven subgroups in $\mathcal{M}$. Fix $\xi, \rho \in G^X$ such that $\eta(\xi)$ and $\kappa(\rho)$ are primitive 6th roots of unity, and let

$$G_{(i,j)} = G^{\eta^i \kappa^j} = \{g \in G^X \mid \eta^i \kappa^j (\det g) = 1\}.$$

Then we have

$$\mathcal{B} = \{G_{(1,0)}, G_{(3,2)}, G_{(1,2)}, G_{(1,4)}, G_{(3,1)}, G_{(1,3)}, G_{(1,5)}, G_{(1,1)}, G_{(0,1)}, G_{(2,1)}, G_{(2,3)}, G_{(4,1)}\}$$

and

$$\mathcal{M} = \{G_{(3,0)}, G_{(3,3)}, G_{(0,3)}, G_{(0,2)}, G_{(2,0)}, G_{(2,2)}, G_{(2,4)}\}.$$

Each subgroup in $\mathcal{B}$ is contained in two elements of $\mathcal{M}$, once with index 2 and once with index 3. The elements of $\mathcal{M}$ have index 2 or 3 in G^X and they correspond to subgroups of R which are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$. Consequently we have two character identities on those intermediate subgroups corresponding to a $\mathbb{Z}/2\mathbb{Z}$, and we have three character identities on those intermediate subgroups corresponding to a $\mathbb{Z}/3\mathbb{Z}$. Moreover the elements of $\mathcal{B}$ correspond to subgroups of R which are isomorphic to $\mathbb{Z}/6\mathbb{Z}$, and thus we have six equations determined by each bottom layer subgroup, giving a total of 72 equations over $C(\sigma)$. We want to see that 36 of these are independent.

Begin by taking six equations from $G_{(1,0)} \in \mathcal{B}$. As in equation (3), these six equations give us the two character identities on $G_{(3,0)} \in \mathcal{M}$ and they also give the three character identities on $G_{(2,0)} \in \mathcal{M}$. The subgroups $G_{(3,2)}$, $G_{(1,2)}$, and $G_{(1,4)}$ are also contained in $G_{(3,0)}$, and they are of index 3 in $G_{(3,0)}$. Since we already have the character identities on $G_{(3,0)}$, we can take only 4 equations from each of these subgroups; specifically we take two equations within each of the character identities. At this point, we have 18 independent equations.

Now use the (R-2) restriction. The group $G_{(3,2)}$ is in $G_{(0,2)}$, as well as $G_{(3,0)}$, and the four equations we took on $G_{(3,2)}$ sum to give two of the three character identities on $G_{(0,2)}$. This fact limits the number of equations we can take from any other $G' \subseteq G_{(0,2)}$. Similarly the equations from $G_{(1,2)}$ and $G_{(1,4)}$ complete two character identities on $G_{(2,4)}$ and $G_{(2,2)}$, respectively.

The subgroup $G_{(3,1)} \in \mathcal{B}$ is contained in $G_{(0,2)}$ with index 2, and our equations from $G_{(3,2)}$ have given us two of the character identities on $G_{(0,2)}$. Hence we can take only 3 equations from $G_{(3,1)}$, each being part of a different character identity on $G_{(0,2)}$. By the (R-2) relation, they sum to give one character identity on $G_{(3,3)}$. The intermediate group $G_{(3,3)}$ also contains the bottom layer subgroups $G_{(1,3)}$, $G_{(1,5)}$, and $G_{(1,1)}$. By the constraints of this containment, and their containment within a subgroup in $\mathcal{M}$ with index 2, there are only two additional independent equations from each of these subgroups in $\mathcal{B}$. Therefore the bottom layer subgroups within $G_{(3,3)}$ contribute 9 equations, giving 27 equations thus far.

The elements of $\mathcal{B}$ that are contained in $G_{(0,3)}$ are in an analogous situation to those contained in $G_{(3,3)}$. There are four such groups, providing an additional 9 equations. As a result, we have 36 linearly independent equations over $C(\sigma)$ that we can write down explicitly and use to determine the characters of the irreducible constituents in $\mathrm{Ind}_{MN}^{G}(\sigma)$.

Since each $H \in \mathcal{B}$ provides $|G^X/H|$ equations, it may appear more constructive to take $\mathcal{B}$ to correspond to maximal proper subgroups of R, rather than only maximal cyclic subgroups. We define $\mathcal{B}$ in this way because in order to make use of the equations given by a bottom layer subgroup in the lattice for R, they need to be genuine character identities. Even when we are able to obtain useful character identities on an H with G^X/H not cyclic, the identities do not provide any additional independent equations, as the following lemma shows.

Lemma 6.2.1. Let $\sigma \in \mathcal{E}_2(M)$ be a Class II representation of M, and let $\pi' = \mathrm{Ind}_{M^X N}^{G^X}(\sigma')$, where σ' is defined as in Section 1 (equation (2)). Let R be the R group of σ, and let $\mathcal{B}$ be the bottom layer in the subgroup lattice determined by R. Suppose H is a subgroup of G^X containing $G^{\bar{L}}$ and such that G^X/H is not cyclic. Assume that the characters of the irreducible constituents in $\pi'|_H$ are understood. Then the number of independent equations determined by $\mathcal{B} \cup H$ is the same as the number determined by $\mathcal{B}$.

PROOF. Let R_H be the subgroup of R associated to H, and form the group lattice within G^X determined by R_H. Then we can consider the bottom layer $\mathcal{B}'$ of this lattice. Since we know the characters of $\pi'|_H$, the bottom layer subgroups in $\mathcal{B}'$ provide $|G^X/H|$ independent character identities in terms of the characters of $\pi'|_H$ and, consequently in terms of the characters of $\pi|_{G^{\mathcal{L}}}$. Thus all the equations over $C(\sigma)$ coming from H can, in fact, be given by equations coming from $\mathcal{B}'$, which is contained in $\mathcal{B}$ by construction. Q.E.D.

§6.3. Examples.

Let F be a p-adic field, and recall that F is a finite extension of the p-adic number field $\mathbb{Q}_p$; see Section 2.1. Suppose that $[F : \mathbb{Q}_p] = m$, and let σ be a Class II discrete series representation of M, where M is the Levi component for a standard parabolic subgroup of $G = SL_n(F)$. Form the corresponding Levi component $\tilde{M}$ of $\tilde{G} = GL_n(F)$, and let $\tilde{\sigma}$ be in $\mathcal{E}_2(\tilde{M})$ such that $\tilde{\sigma}|_M \supseteq \sigma$. The character group $\bar{L}(\tilde{\sigma})$ corresponds to a finite abelian Galois extension of F in the following way. Let

$$N_{\bar{L}} = \bigcap_{\eta \in \bar{L}(\tilde{\sigma})} \ker \eta.$$

This is a closed subgroup of finite index in $F^\times$ and, according to the fundamental theorem of Local Class Field theory, is therefore the norm group of a finite Galois extension K of F. Since the R group of σ is a quotient group of $\bar{L}(\tilde{\sigma})$, we can determine the possible R groups by determining the closed subgroups of finite index in $F^\times$.

Theorem 6.3.1 (Ref. Hasse [Has]). In the above situation, the following group isomorphism holds

$$F^\times \cong \mathbb{Z} \times M \times H_0,$$

where M is the roots of unity in $F^\times$ and H_0 is a free abelian group of rank $n = [F : \mathbb{Q}_p]$.

Retain the notation of the previous sections. Let σ be a Class II representation of M, and let R be the R group of σ. Recall that $C(\sigma)$ is the set of characters for the irreducible constituents of $\pi'|_{G^L}$, and form the linear system of equations determined by the bottom layer $\mathcal{B}$ over $C(\sigma)$. We seek a linearly independent subset of $|R|$ equations. The remainder of this section consists of examples, three of which are solvable and one that is not solvable.

Example 1. Suppose $R \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ with $n \geq 1$. Let η and κ be characters of $F^\times$ whose cosets generate the distinct factors of R and such that $\eta^p = 1$ and $\kappa^{p^n} = 1$ modulo $X(\tilde{\sigma})$. Define subgroups $G_{(i,j)}$ of G^X by

$$G_{(i,j)} = G^{\eta^i \kappa^j} = \{g \in G^X \mid \eta^i \kappa^j(\det g) = 1\}.$$

Then $\mathcal{B}$ is the set of distinct subgroups whose quotient groups in G^X are isomorphic to $\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}, \ldots, \mathbb{Z}/p^n\mathbb{Z}$. Specifically

$$\mathcal{B} = \{G_{(1,0)}, G_{(1,p^{n-1})}, G_{(1,2p^{n-1})}, \ldots, G_{(1,(p-1)p^{n-1})}, G_{(1,p^{n-2})}, \ldots,$$
$$G_{(1,(p-1)p^{n-2})}, \ldots, G_{(1,p)}, \ldots, G_{(1,(p-1)p)}, G_{(0,1)}, G_{(1,1)}, \ldots, G_{(p-1,1)}\},$$

and $\mathcal{M} = \{G^X, G_{(0,p)}, G_{(0,p^2)}, \ldots, G_{(0,p^{n-1})}\}$. Each $G_{(0,p^i)} \in \mathcal{M}$ contains those groups in $\mathcal{B}$ whose quotients in G^X are isomorphic to $\mathbb{Z}/p^{(n+1-i)}\mathbb{Z}$, and G^X is the intermediate subgroup for those bottom layer subgroups whose quotients in G^X are isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We can then compute that the number of independent equations from $\mathcal{B}$ is

$$p^n + p^{n-1}(p-1)(p-1) + p^{n-2}(p-1)(p-1) + \cdots + p(p-1)(p-1) + p(p-1) = p^{n+1},$$

and therefore we can compute the characters of $\pi'|_{G^t}$, and consequently the characters of $\mathrm{Ind}_{MN}^{G}(\sigma)$.

Example 2. Suppose $R \cong \mathbb{Z}/pq\mathbb{Z} \times \mathbb{Z}/pq\mathbb{Z}$ with p and q distinct primes, and take $p < q$. By counting arguments, we have $(p+1)(q+1)$ subgroups of R isomorphic to $\mathbb{Z}/pq\mathbb{Z}$; the associated subgroups in G^X comprise $\mathcal{B}$. Moreover R contains $p+1$ subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and $q+1$ subgroups isomorphic to $\mathbb{Z}/q\mathbb{Z}$; the associated subgroups in G^X comprise $\mathcal{M}$. To count independent equations, notice that the elements of $\mathcal{B}$ are given by nontrivial intersections of those in $\mathcal{M}$. We consider the equations on these intersections, as we vary through the $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ subgroups of $\mathcal{M}$. The number of independent equations is then

$$pq + \sum_{i=1}^{q+1} p(q-1) + \sum_{i=2}^{p+1} \left(q(p-1) + \sum_{j=2}^{q+1} (p+1)(q+1) \right)$$
$$= p^2 q^2 = |R|,$$

and we can solve for the characters of $\pi'|_{G^L}$, and consequently for those of $\mathrm{Ind}_{MN}^{G}(\sigma)$.

Example 3. Suppose $R \cong \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$. Let η and κ be in $(F^\times)^\vee$ such that their cosets generate the distinct factors of R. Define

$$G_{(i,j)} = G^{\kappa^i \eta^j} = \{g \in G^X \mid \kappa^i \eta^j(\det g) = 1\}.$$

Then the bottom layer and intermediate layer in the subgroup lattice determined by R are equal to

$$\mathcal{B} = \{G_{(1,i)} \text{ for } i \in \mathbb{Z}/p^2\mathbb{Z}; G_{(pj,1)} \text{ for } j \in \mathbb{Z}/p\mathbb{Z}\} \quad \text{and}$$
$$\mathcal{M} = \{G_{(p,pi)} \text{ for } i \in \mathbb{Z}/p\mathbb{Z}; G_{(0,p)}; G^X\}.$$

The bottom layer subgroups are isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$, while the intermediate layer subgroups are isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Each intermediate layer subgroup contains p bottom layer subgroups. Starting with $G_{(0,p)}$ we can obtain $p^2 + (p-1)^2 p$ equations from the p groups in $\mathcal{B}$ that are also contained in $G_{(0,p)}$. The bottom layer groups contained in each of the remaining p intermediate groups provide $(p^2-1)+(p-1)^2 p$ equations. Thus we have

$$p^2 + (p-1)^2 p + p((p^2-1) + (p-1)^2 p) = p^4$$

independent equations, and we can solve for the characters in $\pi'|_{G^{\bar{L}}}$.

Example 4. It is not true that being able to determine character equations on every sublattice of the lattice for R implies that the character equations for $\mathrm{Ind}(\sigma)$ can be determined. An example of this phenomenon occurs when $R \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Consider R as a group of characters, generated by the cosets of κ and η where $\kappa^4 = 1$ and $\eta^6 = 1$ modulo $X(\tilde{\sigma})$. If we define $G_{(i,j)}$ as above, then

$$\mathcal{B} = \{G_{(1,3)}, G_{(1,2)}, G_{(0,1)}, G_{(2,3)}\}$$

and $\mathcal{M} = \{G_{(2,0)}, G_{(0,2)}, G^X\}$. We can compute that $\mathcal{B}$ provides only 18 independent equations over $C(\sigma)$. Consequently we can not determine the characters of the constituents of $\mathrm{Ind}_{MN}^{G}(\sigma)$ by this method alone. One the other hand, we can compute the characters of $\pi'|_H$ for every $H \subsetneq G^X$ properly containing $G^{\bar{L}}$. This example shows that while the R group is isomorphic to a group of characters, it is in fact intricately attached to both $\tilde{\sigma}$ and G^X.

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FREQUENTLY USED NOTATION

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