An Index Theorem on Foliated Bundles

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Abstract of the Dissertation
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A leafwise elliptic operator on a foliation determines an index map from the K-theory of the ambient manifold to that of the space of leaves. In this thesis, we focus on certain operators on foliated bundles, study this index map in the context of Kasparov's KK-theory, and give a cohomological formula for this index map as it is detected by certain homology classes on the space of leaves.

To all my mentors, especially my parents;

And to Li-Hua.

Contents

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Introduction

In this thesis we shall study an index problem for certain leafwise elliptic operators on foliated bundles. We begin by reviewing some developments which have motivated our research.

Index theory deals with the interplay between analytic and topological invariants of certain operators on manifolds. In general, these operators are naturally associated to the geometric structures under consideration. In the tradition of de Rham and Hodge, index theory reveals the underpinning analytical structures of certain topological invariants of the manifold.

The simplest index problem appears on the unit circle $\mathbf{T}$. Let $L^2(\mathbf{T})$ denote the space of complex-valued, square-integrable functions on $\mathbf{T}$, $H^2(\mathbf{T})$ the closed subspace spanned by $\{e^{2\pi i \cdot n \cdot \theta}\}_{n \geq 0}$, and P the orthonormal projection from $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$. Any continuous function f on $\mathbf{T}$ gives rise to a Toeplitz operator T_f on $H^2(\mathbf{T})$, defined to be

$$T_f(h) = P(f \cdot h)$$

for any h in $H^2(\mathbf{T})$. A result of Gohberg and Krein [29] in the late 50's states that T_f is a Fredholm operator if and only if its symbol f is nowhere vanishing on $\mathbf{T}$, and the Fredholm index of T_f is minus the winding number of f.

Elliptic differential operators on closed Riemannian manifolds are known to be Fredholm and thus have an index. The celebrated Atiyah-Singer index theorem [6] of the early 60's calculates this index in terms of topological data of the manifold and the principal symbol of the operator. Its importance was recognized immediately and has become better appreciated since, partly because of its growing number of far-reaching generalizations and unexpected, deep applications in various fields of mathematics and theoretical physics.

One important line of development, which has inspired our research, originates from the above index result of Gohberg-Krein [29] on $\mathbf{T}$. Influenced by the Atiyah-Singer index theorem, in the late 60's Coburn, Douglas, Schaeffer and Singer [14] sought to develop index theory for more general Toeplitz operators, and they succeeded for a couple of classes of operators, which serve as the prototypes for later developments. Particularly pertinent to this thesis is the analogue of Gohberg-Krein's work for the real line $\mathbf{R}$ instead of the unit circle $\mathbf{T}$. In this case, the space is the non-negative spectral subspace of the operator $-i(d/dx)$ on $\mathbf{R}$, and the symbols are almost periodic functions on $\mathbf{R}$. They considered the algebra generated by the Toeplitz operators, and showed that every Fredholm operator in this algebra is necessarily invertible, and therefore has index zero. However, using an appropriate representation of this algebra, they were able to characterize the Toeplitz operators which are generalized Fredholm operators in the sense of Breuer, and to calculate the real-valued Breuer index in terms of the mean-motion of the function.

These works heralded the introduction of language and ideas from homological algebra into operator theory and operator algebra, which would soon revolutionize the latter subjects. Among the next major triumphs along this line was the work of Brown, Douglas and Fillmore [13] on the classification of essentially normal operators. This great leap forward in operator theory gave, in return, an explicit analytic realization of K-homology that Atiyah had proposed a few years before. Meanwhile it was realized that C^*-algebras provide a natural framework for K-cohomology. Out of these developments the algebraic topology for C^*-algebras, or "non-commutative topology", came into being and has been an active and fruitful new field (see, for example, [38], [3], [1] and [40]). In particular, Kasparov unified K-homology and K-cohomology into the powerful KK-theory (see [10]) in his pursuit of the Novikov conjecture; Baum and Douglas [8] developed the odd analogue of the Atiyah-Singer index theorem, which exemplifies the importance of Toeplitz operators in this field; Connes developed the index theory for measured foliations (see [46]) and initiated the study of non-commutative differential geometry [16] by introducing cyclic cohomology of algebras, which is a de Rham theory for operator algebras; Connes and Moscovici [19] proved the Novikov conjecture for hyperbolic groups; and localizations of topological Pontryagin classes have been obtained, independently, by Connes, Sullivan and Teleman [22] and by Moscovici and Wu [45].

Along with the development of such new high technologies, several authors have taken another look at those classical results on Toeplitz operators. It turns out that the work of CDSS [14] lends itself naturally to geometrically interesting situations, which have led to the beautiful work ([26]) of Douglas, Hurder and Kaminker on index theorems for leafwise elliptic operators on a foliation.

Let $\mathcal{F}$ be a smooth foliation structure on a closed manifold W. Connes [15] first introduces a C^*-algebra $\mathcal{C}$, which recaptures the topology of the space of leaves. Let $D_{\mathcal{F}}$ be a leafwise elliptic differential operator for the foliation. Then it is "invertible" module $\mathcal{C}$ and defines a KK-element $[D_{\mathcal{F}}]$ in $KK_*(C(W), \mathcal{C})$, where $C(W)$ is the algebra of continuous functions on W (cf. [15] [19]). The fundamental problem is to study this KK-element, or the map $[D_{\mathcal{F}}] : K^*(W) \to K_*(\mathcal{C})$ that it induces, and to relate this study with the geometry and topology of the foliation.

One basic tool is Connes' cyclic cohomology ([16]). Any cyclic cocycle τ on an appropriate subalgebra of $\mathcal{C}$ induces a group homomorphism τ_*: $K_*(\mathcal{C}) \to \mathbf{C}$ in K-theory. And an important way to understand the map $[D_{\mathcal{F}}]$ is to study the composite $\tau_* \circ [D_{\mathcal{F}}]$: $K^*(W) \to \mathbf{C}$ for various cyclic cocycles τ.

For example, a holonomy invariant transverse measure μ on the foliation induces a trace τ_μ on $\mathcal{C}$, which in turn induces a map $[\tau_\mu] : K_0(\mathcal{C}) \to \mathbf{R}$. The composite $(\tau_\mu)_* \circ [D_{\mathcal{F}}]$ in this case can be achieved by a longitudinal cyclic cocycle defined on the algebra $C^\infty(W)$ of smooth functions on W in an explicit, analytic way, and has been calculated by Connes (see [15], [46], [25]) in terms of the characteristic class of the symbol of $D_{\mathcal{F}}$ and the Ruelle-Sullivan class of the measure μ. When the operator is leafwise self-adjoint, Douglas, Hurder and Kaminker [25] show that the pairing can be interpreted as the Breuer index of a family of leafwise Toeplitz operators.

Much more has been accomplished for foliations coming from a suspension construction. Let M be a closed oriented Riemannian manifold, Γ a discrete group and $p : \widetilde{M} \to M$ a principal Γ-bundle over M. Let Γ act on a closed Riemannian manifold V isometrically and "essentially freely" (see §2.6). Then there is a natural foliation structure on $W = \widetilde{M} \times_\Gamma V$, each leaf being the image of $\widetilde{M} \times v$ for some $v \in V$. Any (self-adjoint) elliptic differential operator D on M lifts to become a leafwise (self-adjoint) elliptic operator $D_{\mathcal{F}}$ for the foliation. In [26], under certain conditions which we shall not specify, the longitudinal cyclic cocycle of $D_{\mathcal{F}}$ is related, through a (surprising) renormalization scheme, to a "sharp transverse" cyclic cocycle defined on a smooth subalgebra of the C^*-algebra for the fibration $\widetilde{M} \times_\Gamma V \to M$. This renormalization scheme provides a connection between Breuer indices of families of leafwise Toeplitz operators and the relative η−invariants of Atiyah-Patodi-Singer [6], which had been long sought after (cf. [57]).

Our motivation is to try to understand and extend the work of DHK. The trace τ_μ on $\mathcal{C}$ defined by a holonomy-invariant transverse measure μ for the foliation is but a 0-cyclic cocycle for the algebra. So a natural program is to construct other cyclic cocycles τ on $\mathcal{C}$, to calculate $\tau \circ [D_{\mathcal{F}}]$ and to relate this pairing to the transverse geometry and spectral data.

In some important cases, Douglas [23] [24] has constructed some higher cyclic cocycles for the foliation algebra using partially elliptic operators and renormalization schemes. This is an interesting subject in its own light and deserves further study. The cyclic cocycles that we shall study in this thesis, however, are more topological in nature. We began with an observation about Connes' work [17] on the fundamental transverse class, that his approach works not only for the fundamental transverse class, but for any invariant closed form on V as well. (This is also pointed out in a recent paper by Connes, Gromov and Moscovici [18].) This is very nice since we can put some (sometimes all) cohomology classes of V into the picture in a natural way. From this we also realized that it is convenient to shift our focus from the foliation algebra $\mathcal{C}$ to the reduced crossed-product C^*-algebra $C(V) \rtimes_r \Gamma$, which is strongly Morita equivalent to $\mathcal{C}$. It has been known (cf. [49]) that a cyclic cocycle $\tau_{\omega,\rho}$ on the algebraic crossed-product $C^\infty(V) \rtimes_{alg} \Gamma$ can be constructed from a closed, Γ-invariant form ω on V and a group cocycle ρ for Γ. We shall generalize some ideas of Jolissant [36] to extend such a cyclic cocycle continuously to a smooth subalgebra of $C(V) \rtimes_r \Gamma$ under certain conditions, thus making it possible to pair $\tau_{\omega,\rho}$ with $[D_{\Gamma,V}] = [D_{\mathcal{F}}] \otimes [\text{Morita}] \in KK_1(C(W), C(V) \rtimes_r \Gamma)$.

Conceivably, the best framework to carry out the computation of such a pairing is the bivariant Chern character theory in bivariant cyclic theory. Indeed, this has been introduced and developed in recent years and a very satisfactory theory seems within reach thanks to the work of Kassel, Wang, and especially Nistor (cf. [37], [58], [59], [50], and [51]). However, detailed calculations still appear difficult.

Fortunately our present situation is very special in that the holonomy covering of each leaf is canonically diffeomorphic to $\widetilde{M}$ and the lifting of the leafwise elliptic operator on each holonomy covering is the lifting of D on $\widetilde{M}$. This observation translates into a simple (and perhaps known) and illuminating decomposition of $[D_{\Gamma,V}]$ into two much simpler KK-elements:

Theorem 0.1. $[D_{\Gamma,V}] = [\mathcal{V}] \otimes_{C(M)} [D]$, *where* $[D] \in K_1(C(M), \mathbf{C})$ *is the K-homology element determined by the elliptic operator* D *on* M, $\otimes_{C(M)}$ *is the Kasparov product, and* $[\mathcal{V}] \in KK_0(C(W), C(M) \otimes C(V) \rtimes_r \Gamma)$ *is determined purely by the covering structure and the action of* Γ *on* V.

This result seems to be related to the renormalization scheme in DHK's work and we hope to have more to say about this in the near future. But for the purpose of this thesis, its main advantage is that the bivariant characters of $[D]$ and $[\mathcal{V}]$ are either known or simple. This enables us to calculate the pairing $\tau_{\omega,\rho} \circ [D_{\Gamma,V}]$. The main result of this thesis is the following:

Theorem 0.2. *For any* $[u] \in K^1(W)$*:*

$$(\tau_{\omega,\rho} \circ [D_{\Gamma,V}])([u]) = c \cdot \int_W ch^*([u]) \wedge \Phi^*(\omega_\rho \wedge \omega \wedge ch_*(D)),$$

where c *is a constant,* ω_ρ *is a form on* M *determined by the covering structure and the group cocycle* ρ *(see §4.5), and* Φ^* *is the natural map sending* Γ*-invariant forms on* $M \times V$ *to forms on* $W = \widetilde{M} \times_\Gamma V$.

This thesis is organized as follows: In §1 we outline certain aspects of Kasparov's KK-theory and Connes' cyclic cohomology. We give details only to those that can not be readily found in the existing literature. Although our focus in this thesis is on foliated bundles, it turns out to be both convenient and useful to work in the context of group actions. Therefore, in §2, we mimic the construction of the longitudinal KK-element to define a KK-element $[D_{\Gamma,V}] \in KK_\sigma(C(W), C(V) \rtimes_r \Gamma)$, where σ is the parity of D. This is done without the "essential freeness" condition on the action of Γ on V. We then compare this construction with the construction of the higher Γ-index of D by Miscenko and Fomenko (see [44], [19] and [42]). On the other hand, with the "essential freeness" condition, $[D_{\Gamma,V}]$ is equivalent to the longitudinal KK-element on the foliated bundle. We then prove Theorem 0.1 (see Proposition 2.10 and Theorem 2.11) and discuss the element $[\mathcal{V}]$ in some detail. In §3 we review the construction of cyclic cocycles $\tau_{\omega,\rho}$ on $C^\infty(V) \rtimes_{alg} \Gamma$ and generalize ideas in [36], [19] and [34] to show that, under certain conditions, these cyclic cocycles can be extended to some smooth subalgebras of $C(V) \rtimes_r \Gamma$ and hence can be paired with the K-theory of $C(V) \rtimes_r \Gamma$. In §4 we calculate the pairing $\tau_{\omega,\rho} \circ [D_{\Gamma,V}]$ and prove the main result, Theorem 0.2 (see Theorem 4.7). In the special case where V consists of one single point, our main result recovers the work of Connes and Moscovici [19] on the Novikov conjecture (see, however, Remark 5.5 of [19]). See also [42] and [60]. We then conclude this thesis with some remarks on what remains to be done.

§1 Preliminaries

In this section we shall outline aspects of KK-theory of Kasparov and cyclic cohomology theory of Connes that we shall use in this thesis.

We take Blackadar's book [10] as the standard reference for KK-theory.

1.1 Definition. *Let A be a C^*-algebra. A pre-Hilbert module for A is a right module H_c over a dense *-subalgebra A_c of A, equipped with an A-valued "inner-product" $< -, - >: \mathsf{H}_c \times \mathsf{H}_c \to \mathsf{A}$, such that:*

(1) $< -, - >$ is linear in the second variable;

(2) $< h_1, h_2 >^* = < h_2, h_1 >$, for any h_1, $h_2 \in \mathsf{H}_c$;

(3) $< h_1, h_2 \cdot a > = < h_1, h_2 > \cdot a$, for any h_1, $h_2 \in \mathsf{H}_c$ and any $a \in \mathsf{A}_c$;

(4) for any $h \in \mathsf{H}_c$, $< h, h > \geq 0$ as an element in A; and if $< h, h > = 0$, then $h = 0$.

It is then easy to show that $\|h\| = \| < h, h > \|_{\mathsf{A}}^{1/2}$ defines a norm on H_c (cf. [10] §13.1.3). H_c is called a Hilbert module over A if it is complete with respect to this norm. Note that in this case it is actually a right A-module and (3) holds for any $a \in \mathsf{A}$.

As one can expect, the completion of a pre-Hilbert module for A is a Hilbert module over A. *In this thesis all Hilbert modules are separable as* *normed spaces.*

A Hilbert module over the complex numbers $\mathbf{C}$ is just a Hilbert space. On the other hand, any C^*-algebra A is a Hilbert module over itself with the inner product: $< a, b >= a^*b$. Furthermore, if ℓ^2_{A} is the space of all sequences (a_i) in A such that $\sum_i a^*a < \infty$, then it is a Hilbert module over A with the inner product $< (a_i), (b_i) >= \sum_i a_i^* b_i$. ℓ^2_{A} is the universal separable Hilbert module over A in the sense that any separable Hilbert module over A can be embedded into ℓ^2_{A} (Absorption Theorem, cf. [10] §13.6.2).

1.2 Hilbert modules over $C(X) \rtimes_r \Gamma$. Let X be a compact Hausdorff space on which a discrete group Γ acts by homeomorphisms and let $C(X) \rtimes_r \Gamma$ be the reduced crossed product C^*-algebra, which can be constructed by introducing a Borel measure on X and representing $C(X)$ and Γ on the Hilbert space $L^2(X) \otimes \ell^2(\Gamma)$, as follows:

$$\pi(f)(\xi \otimes e_\gamma) = \gamma^{-1}(f)\xi \otimes e_\gamma,$$

$$\pi(\gamma_1)(\xi \otimes e_\gamma) = \xi \otimes e_{\gamma_1^{-1}\gamma},$$

for any $f \in C(X)$ and any γ, $\gamma_1 \in \Gamma$, where $\{e_\gamma\}_{\gamma\in\Gamma}$ is the standard basis for $\ell^2(\Gamma)$. $C(X) \rtimes_r \Gamma$ is the C^*-algebra generated by $\{\pi(f) : f \in C(X)\}$ and $\{\pi(\gamma) : \gamma \in \Gamma\}$.

To construct Hilbert modules over $C(X) \rtimes_r \Gamma$, we consider a Γ-equivariant Hilbert bundle $H \to X$. Thus, each fibre H_x is a Hilbert space and each map $\gamma : H_x \to H_{x\cdot\gamma}$ is unitary. A continuous field of Hilbert spaces for the system

(X, Γ) (cf. [15]) is such a bundle together with a linear space $\mathbb{S}$ of sections of the bundle such that:

(1) $\mathbb{S}$ is Γ-invariant: $\gamma(\xi) \in \mathbb{S}$ for any $\xi \in \mathbb{S}$ and any $\gamma \in \Gamma$, where $\gamma(\xi)(x) \stackrel{\text{def}}{=} \gamma^{-1}(\xi(x \cdot \gamma))$;

(2) $\mathbb{S}$ is a $C(X)$-bimodule: $f \cdot \xi \in \mathbb{S}$ for any $\xi \in \mathbb{S}$ and any $f \in C(X)$, where $(f \cdot \xi)(x) \stackrel{\text{def}}{=} f(x)\xi(x)$;

(3) $\ll \xi, \eta \gg \stackrel{\text{def}}{=} \sum_\gamma < \xi, \gamma(\eta) > \cdot \gamma \in C(X) \rtimes_r \Gamma$ for any ξ, $\eta \in \mathbb{S}$, where $< \xi, \gamma(\eta) > (x) = < \xi(x), \gamma^{-1}(\eta(x \cdot \gamma)) >_{H_x}$, which implies, in particular, that the function $x \hookrightarrow \|\xi(x)\|_{H_x}$ is continuous for any $\xi \in \mathbb{S}$;

(4) if K is the completion of $\mathbb{S}$ with respect to the following scalar inner-product:

$$[\xi, \eta] = \int_X < \xi(x), \eta(x) >_{H_x} dx,$$

then the map $T_\xi(f \otimes e_\gamma) = f \cdot \gamma(\xi)$ extends to a bounded operator from $L^2(X) \otimes \ell^2(\Gamma)$ to K.

We now make $\mathbb{S}$ into a pre-Hilbert module for $C(X) \times_r \Gamma$. It follows from (1) and (2) that $\mathbb{S}$ is a right module over $C(X) \times_{alg} \Gamma$ if we define

$$\xi \cdot (f \cdot \gamma) = \gamma^{-1}(f \cdot \xi),$$

for any $\xi \in \mathbb{S}$, $f \in C(X)$ and any $\gamma \in \Gamma$. It is routine to show that $\ll -, - \gg$ defined in (3) satisfies conditions (1), (2) and (3) in Definition 1.1. On the other hand, by a straightforward calculation, we have

$$T_\xi^*(\eta) = \sum_\gamma < \gamma^{-1}(\xi), \eta > \otimes e_\gamma,$$

and hence,

$$\ll \xi, \eta \gg = T_\xi^* T_\eta,$$

which verifies condition (4) in Definition 1.1. Therefore, S gives a pre-Hilbert module for $C(X) \rtimes_r \Gamma$. Its completion is a Hilbert module over $C(X) \rtimes_r \Gamma$. In fact, any Hilbert module over $C(X) \rtimes_r \Gamma$ can be constructed in this fashion (see [15], where the discussion applies to any topological groupoid).

1.3. We now consider certain classes of operators between Hilbert modules. If H_1, H_2 are two Hilbert modules over A, then $\mathcal{B}(\mathsf{H}_1, \mathsf{H}_2)$ denotes the space of all operators $T : \mathsf{H}_1 \to \mathsf{H}_2$ which have an adjoint $T^* : \mathsf{H}_2 \to \mathsf{H}_1$ such that $< T(h_1), h_2 >=< h_1, T^*(h_2) >$ for any $h_i \in \mathsf{H}_i$. Such operators are automatically bounded module homomorphisms. However, unlike the Hilbert space situation, a bounded module homomorphism between two Hilbert modules over A does not necessarily have an adjoint.

For any $h_1 \in \mathsf{H}_1$ and $h_2 \in \mathsf{H}_2$, we can define a "rank one" operator $T_{h_2,h_1} : \mathsf{H}_1 \to \mathsf{H}_2$ by letting

$$T_{h_2,h_1}(h) = h_2 \cdot < h_1, h > .$$

One can easily check that $T^*_{h_2,h_1} = T_{h_1,h_2}$; therefore, $T_{h_2,h_1} \in \mathcal{B}(\mathsf{H}_1, \mathsf{H}_2)$. The closed linear space spanned by all such operators will be denoted by $\mathcal{K}(\mathsf{H}_1, \mathsf{H}_2)$. This is the space of compact operators. In particular, let $\mathcal{B}(\mathsf{H}) = \mathcal{B}(\mathsf{H}, \mathsf{H})$ and $\mathcal{K}(\mathsf{H}) = \mathcal{K}(\mathsf{H}, \mathsf{H})$. Then $\mathcal{B}(\mathsf{H})$ is a C^*-algebra and $\mathcal{K}(\mathsf{H})$ is a two-sided ideal in $\mathcal{B}(\mathsf{H})$.

A graded Hilbert module H over A is a Hilbert module H over A together with an operator $g \in \mathcal{B}(\mathsf{H})$ which is a symmetry (that is, $g = g^*$ and $g^2 = 1$). In other words, $\mathsf{H} = \mathsf{H}^+ \oplus \mathsf{H}^-$, where $\mathsf{H}^\pm$ are Hilbert modules over A, $g = 1$ on H^+ and $g = -1$ on H^-. This induces a grading on $\mathcal{B}(\mathsf{H})$: $T \in \mathcal{B}_+(\mathsf{H})$ if $T \cdot g = g \cdot T$, and $T \in \mathcal{B}_-(\mathsf{H})$ if $T \cdot g = -g \cdot T$.

1.4 Kasparov modules. Let A_1, A_2 be two (separable) C^*-algebras. An odd Kasparov module for $(\mathsf{A}_1, \mathsf{A}_2)$ is a triple (H, ϕ, F), where H is a Hilbert module over A_2, ϕ is a C^*-homomorphism from A_1 into $\mathcal{B}(\mathsf{H})$, $F = F^* \in \mathcal{B}(\mathsf{H})$, and $\phi(a)F - F\phi(a) \in \mathcal{K}(\mathsf{H})$, $(F^2 - 1)\phi(a) \in \mathcal{K}(\mathsf{H})$ for any $a \in \mathsf{A}_1$. The set of all odd Kasparov modules for $(\mathsf{A}_1, \mathsf{A}_2)$ will be denoted by $\mathsf{E}_1(\mathsf{A}_1, \mathsf{A}_2)$.

An even Kasparov module for $(\mathsf{A}_1, \mathsf{A}_2)$ is a quadruple (H, g, ϕ, F), where $(\mathsf{H}, \phi, F) \in \mathsf{E}_1(\mathsf{A}_1, \mathsf{A}_2)$, and g is a grading on H, under which $\phi(a) \in \mathcal{B}_+(\mathsf{H})$ for any $a \in \mathsf{A}_1$ and $F \in \mathcal{B}_-(\mathsf{H})$. Let $\mathsf{E}_0(\mathsf{A}_1, \mathsf{A}_2)$ denote the set of all even Kasparov modules for $(\mathsf{A}_1, \mathsf{A}_2)$.

The KK-groups are defined by: $KK_*(\mathsf{A}_1, \mathsf{A}_2) = \mathsf{E}_*(\mathsf{A}_1, \mathsf{A}_2)/\sim$, (see [10], §17 for the definition of the equivalence relation $\sim$).

In particular, if X is a compact metrizable space, then $KK_*(\mathbf{C}, C(X))$ is isomorphic to $K^*(X)$, the K-theory of Atiyah-Hirzebruch, constructed from vector bundles over X, while $KK_*(C(X), \mathbf{C})$ is isomorphic to $K_*(X)$, the K-homology of X. The analytic realization of the latter theory, that is, the use of elliptic operators as cycles for that theory, was proposed by Atiyah [4], accomplished by Brown-Douglas-Fillmore[13] and has played a central role in index theory. We now very briefly discuss a class of elliptic operators.

1.5 Dirac operators and K-homology. Let M be a closed Riemannian manifold. A (smooth) hermitian vector bundle E over M is called a Clifford bundle, if

(1) each fibre E_m of E is a left module over the complexified Clifford algebra $CL(T_mM \otimes \mathbf{C})$;

(2) the action of a vector $v \in T_mM$ on E_m is skew-adjoint: that is, $< v \cdot \xi_1, \xi_2 > + < \xi_1, v \cdot \xi_2 >= 0$ for any $\xi_i \in E_m$; and

(3) there is a connection ∇ on E which is compatible with the Levi-Civita connection on the tangent bundle TM of M, that is,

$$\nabla_X(Y \cdot \xi) = \nabla_X(Y) \cdot \xi + Y \cdot \nabla_X(\xi),$$

for any vector fields X, Y on M and for any smooth section ξ of E.

Given a Clifford bundle E over M, the associated Dirac operator D is the first order differential operator on $C^\infty(E)$ given by:

$$(D\xi)(m) = \sum_\alpha e_\alpha \cdot (\nabla_{e_\alpha}\xi)(m),$$

where $\{e_\alpha\}$ is any orthonormal basis for T_mM.

It can be shown (cf. for example [54]) that D is formally self-adjoint. Using the Hermitian metric on E and the Riemannian metric on M, we construct the Hilbert space $L^2(E)$. Let φ be the representation of the algebra $C(M)$ on $L^2(E)$ as multiplication operators and set $\lambda(D) = D(1 + D^2)^{-1/2}$ using the functional calculus of D. Then it can be shown (cf. [4]) that the triple $(L^2(E), \varphi, \lambda(D))$ is an odd Kasparov module over $(C(M), \mathbf{C})$. When E has a(n often natural) grading under which D is odd, $L^2(E)$ inherits the grading under which $\lambda(D)$ is odd, and we get an even Kasparov module.

1.6 Kasparov product. Let A_i be C^*-algebras. Then there is the Kasparov product:

$$KK_{\sigma_1}(\mathsf{A}_1, \mathsf{A}_2) \times KK_{\sigma_2}(\mathsf{A}_2, \mathsf{A}_3) \stackrel{\otimes_{\mathsf{A}_2}}{\to} KK_{\sigma_1+\sigma_2}(\mathsf{A}_1, \mathsf{A}_3),$$

for $\sigma_i \in \mathbf{Z}_2$. In most references (cf. [10], [19]), this is treated in detail for more general (graded) C^*-algebras but only for the case when $\sigma_i = 0$. The general case follows from that discussion but we believe it is of interest to give some details here. We now discuss the two cases where $\sigma_1 = 0$.

If $(\mathsf{H}_1, g_1, \phi_1, F_1) \in \mathsf{E}_0(\mathsf{A}_1, \mathsf{A}_2)$, and $(\mathsf{H}_2, \phi_2, F_2) \in \mathsf{E}_1(\mathsf{A}_2, \mathsf{A}_3)$, then the product $(\mathsf{H}, g_1, \phi_1, F_1) \otimes_{\mathsf{A}_2} (\mathsf{H}_2, \phi_2, F_2)$ is given by any $(\mathsf{H}, \phi, F) \in \mathsf{E}_1(\mathsf{A}_1, \mathsf{A}_3)$ satisfying:

(1) $\mathsf{H} = \mathsf{H}_1 \otimes_{\phi_2} \mathsf{H}_2$, which is constructed from the algebraic tensor product $\mathsf{H}_1 \otimes_{\mathsf{A}_2} \mathsf{H}_2$ together with the following inner-product:

$$< h_1 \otimes h_2, h_1' \otimes h_2' >=< h_2, \phi_2(< h_1, h_1' >_1) h_2' >_2,$$

where $< -, - >_i$ is the inner product on H_i;

(2) $\phi = \phi_1 \otimes_{\phi_2} 1$;

(3) $\phi(a)[(F_1 \otimes 1) \cdot F + F \cdot (F_1 \otimes 1)]\phi(a)^* \geq 0 \bmod \mathcal{K}(\mathsf{H})$ for any $a \in \mathsf{A}_1$;

(4) $F \cdot T_{h_1} - T_{g_1(h_1)} \cdot F \in \mathcal{K}(\mathsf{H}_2, \mathsf{H})$ for any $h_1 \in \mathsf{H}_1$, where for any $h_1 \in \mathsf{H}_1$, $T_{h_1} \in \mathcal{B}(\mathsf{H}_2, \mathsf{H})$ is defined by:

$$T_{h_1}(h_2) = h_1 \otimes_{\phi_2} h_2.$$

If $(\mathsf{H}_1, g_1, \phi_1, F_1) \in \mathsf{E}_0(\mathsf{A}_1, \mathsf{A}_2)$ and $(\mathsf{H}_2, g_2, \phi_2, F_2) \in \mathsf{E}_0(\mathsf{A}_2, \mathsf{A}_3)$, then $(\mathsf{H}_1, g_1, \phi_1, F_1) \otimes_{\mathsf{A}_2} (\mathsf{H}_2, g_2, \phi_2, F_2)$ is given by any $(\mathsf{H}, g, \phi, F) \in \mathsf{E}_0(\mathsf{A}_1, \mathsf{A}_3)$ such that (H, ϕ, F) satisfies the same conditions (1) - (4) as above, and $g = g_1 \otimes_{\phi_2} g_2$, that is, the corresponding grading on H is given by

$$\mathsf{H}^+ = \mathsf{H}_1^+ \otimes_{\phi_2} \mathsf{H}_2^+ \oplus \mathsf{H}_1^- \otimes_{\phi_2} \mathsf{H}_2^-,$$

and

$$\mathsf{H}^- = \mathsf{H}_1^+ \otimes_{\phi_2} \mathsf{H}_2^- \oplus \mathsf{H}_1^+ \otimes_{\phi_2} \mathsf{H}_2^-.$$

The Kasparov product is well-defined, enjoys some functorial properties, and has many powerful applications. See [10] for a full treatment of KK-theory.

We now turn to cyclic cohomology, which was introduced by Connes [16] as a non-commutative analogue of the de Rham theory.

Let A be a locally convex topological algebra. This means that there is a family $\{p_\mu\}$ of semi-norms on A which defines its (Hausdorff) topology, under which the product on A is jointly-continuous, that is, for any $p \in \{p_\mu\}$, there is $p' \in \{p_\mu\}$ such that

$$p(a \cdot b) \leq p'(a)p'(b),$$

for any $a, b \in A$. If, furthermore, the family $\{p_\mu\}$ is countable and the topology is complete, then A is called a Fréchet algebra. All topological algebras that we shall encounter in this thesis will be Fréchet.

For $m \geq 0$, let $C_\lambda^m(A)$ be the space of all continuous multi-linear maps τ: $\underbrace{A \times A \times \cdots \times A}_{m+1} \to \mathbf{C}$ which are cyclic, that is:

$$\tau(a_0, a_1, \cdots, a_m) = (-1)^m \tau(a_m, a_0, \cdots, a_{m-1})$$

for any $a_i \in A$. For any $\tau \in C_\lambda^m(A)$, we define

$$(b\tau)(a_0, a_1, \cdots, a_{m+1}) = \sum_{i=0}^{m} (-1)^i \tau(a_0, a_1, \cdots, a_i a_{i+1}, \cdots, a_{m+1}) +$$

$$+(-1)^{m+1}\tau(a_{m+1}a_0, a_1, \cdots, a_m)$$

for any $a_i \in A$. Then it is easy to show that $b\tau \in C_\lambda^{m+1}(A)$ and $b^2 = 0$.

1.7 Definition. *The cyclic cohomology of A, denoted by $HC^*(A)$, is the homology of the following cochain complex:*

$$0 \longrightarrow C_\lambda^0(A) \xrightarrow{b} C_\lambda^1(A) \xrightarrow{b} C_\lambda^2(A) \xrightarrow{b} \cdots.$$

The space of cyclic cocycles of degree m will be denoted by $Z_\lambda^m(A)$.

Consider the algebra $\mathbf{C}$ of complex numbers. It follows from the cyclicity that $C_\lambda^{2m}(\mathbf{C}) \cong \mathbf{C}$, and $C_\lambda^{2m+1}(\mathbf{C}) \cong \{0\}$. Therefore, $HC^{2m}(\mathbf{C}) \cong \mathbf{C}$ and $HC^{2m+1}(\mathbf{C}) \cong \{0\}$. For $a_i \in \mathbf{C}$, let

$$\sigma(a_1, a_2, a_3) = 2\pi i a_1 a_2 a_3.$$

Then σ is a generator for $HC^2(\mathbf{C})$, which will be used to define the suspension map (see §1.8 below).

There is another basic cyclic cocycle that we shall use later. Let Tr_k be the trace on $M_k(\mathbf{C})$:

$$Tr_k(a) = \sum_{i=1}^{k} a_{ii} \text{ for any } a = (a_{ij}) \in M_k(\mathbf{C}).$$

Then it is easy to show that $Tr_k \in Z_\lambda^0(M_k(\mathbf{C}))$.

In fact, for any algebra A, $Z_\lambda^0(A)$ consists of traces on A. For this reason, higher degree cyclic cocycles are called higher traces on A. One way to make this more precise is as follows:

Let $\Omega(A) = \sum_{m \geq 0} \Omega^m(A)$ be the (topological) universal graded differential algebra over A (see [15] and [27] for details). A continuous linear functional $\hat{\tau}$ on $\Omega^m(A)$ is called a closed graded trace of degree m if:

1). $\tau(da_1 da_2 \cdots da_m) = 0$ for any $a_i \in A$; and

2). $\tau(\omega_1 \cdot \omega_2) = (-1)^{\partial(\omega_1)\cdot\partial(\omega_2)}\tau(\omega_2 \cdot \omega_1)$, for any $\omega_i \in \Omega(A)$ such that $\omega_1 \cdot \omega_2 \in \Omega^m(A)$.

For any closed graded trace $\hat{\tau}$ of degree m, we define

$$\tau(a_0, a_1, \cdots, a_m) = \hat{\tau}(a_0 da_1 \cdots da_m)$$

for any $a_i \in A$. It is easy to show that $\tau \in Z_\lambda^m(A)$. Conversely, any $\tau \in Z_\lambda^m(A)$ induces a closed graded trace of degree m in an obvious way. Therefore, we shall use these two descriptions interchangeably.

1.8 Cup product. Let A, B be locally convex topological algebras, and $A \otimes B$ their projective tensor product ([55]). Then $A \otimes B$ is again a locally convex topological algebra. If $\phi \in Z_\lambda^m(A)$ and $\varphi \in Z_\lambda^n(B)$, let $\hat{\phi} : \Omega^m(A) \to \mathbf{C}$ and $\hat{\varphi} : \Omega^n(B) \to \mathbf{C}$ be the corresponding closed graded traces. Let $\hat{\phi}\#\hat{\varphi}$ be the composite of the following maps:

$$\Omega^{m+n}(A \otimes B) \longrightarrow \Omega^m(A) \otimes \Omega^n(B) \xrightarrow{\hat{\phi}\otimes\hat{\varphi}} \mathbf{C},$$

where the first map comes from the universality of $\Omega(A \otimes B)$. Then $\hat{\phi}\#\hat{\varphi}$ is a closed graded trace and determines a cyclic cocycle $\phi\#\varphi \in Z_\lambda^{m+n}(A \otimes B)$. This construction defines the cup product in cyclic cohomology:

$$\# : HC^m(A) \times HC^n(B) \longrightarrow HC^{m+n}(A \otimes B).$$

See [16] for details.

Recall, from §1.7, that $\sigma \in Z^2_\lambda(\mathbf{C})$, determined by $\sigma(1,1,1) = 2\pi i$, is a generator for $HC^2(\mathbf{C})$. Therefore, $S(\tau) \stackrel{\text{def}}{=} \tau \# \sigma$ induces a map:

$$S : HC^m(A) \longrightarrow HC^{m+2}(A \otimes \mathbf{C}) \equiv HC^{m+2}(A).$$

This is the suspension map in cyclic cohomology.

1.9 Pairing with K-theory. Suppose that A is unital. Let $K_0(A)$ be the algebraic K_0-group of A, that is, the (Grothendieck) group associated to the semi-group of algebraically equivalent idempotents in matrix algebras over A. Let $K_1(A)$ be the quotient of $GL_\infty(A)$ by the equivalence relation $u \sim v$ which holds when u can be connected to v by a piecewise linear path in $GL_\infty(A)$ (cf. [17], [48]). If A is a C^*-algebra, then $K_i(A)$ is naturally isomorphic to $KK_i(\mathbf{C}, A)$ (see [10]).

For any idempotent $e \in M_k(A)$ and any $\tau \in Z^{2m}_\lambda(A)$, we define:

$$< e, \tau >= \frac{1}{(2\pi i)^m \cdot m!} \cdot (\tau \# Tr_k)(e, e, \cdots, e).$$

And if $u \in GL_k(A)$ and $\tau \in Z^{2m+1}_\lambda(A)$, we define:

$$< u, \tau >= \frac{m!}{(2\pi i)^{m+1} \cdot (2m+1)!} \cdot (\tau \# Tr_k)(u^{-1} - 1, u - 1, u^{-1} - 1, \cdots, u - 1).$$

1.10 Theorem. **(Connes[17])** *(1) These maps induce well-defined pairings:*

$$K_0(A) \times HC^{2m}(A) \longrightarrow \mathbf{C},$$

and

$$K_1(A) \times HC^{2m+1}(A) \longrightarrow \mathbf{C}.$$

(2) $< x, \tau > = < x, S(\tau) >$ *for any* $x \in K_*(A)$ *and* $\tau \in HC^*(A)$.

1.11 Remarks. (1) When A is not unital, $K_0(A)$ is defined to be the kernel of the natural map $K_0(A^+) \to K_0(\mathbf{C})$, where A^+ is the unitization of A, and $K_1(A)$ is defined to be $K_1(A^+)$. On the other hand, for any $\tau \in Z^k_\lambda(A)$, we define

$$\tau^+(a_0 + c_0 \cdot I, \cdots, a_k + c_k \cdot I) = \tau(a_0, \cdots, a_k),$$

where $a_i \in A$, $c_i \in \mathbf{C}$, and I is the unit in A^+. Then it follows easily that $\tau^+ \in Z^k_\lambda(A^+)$. The pairings in Theorem 1.10 then extend to this case.

(2) For any algebra A, we can repeat the discussions in §1.7 -§1.10, dropping all continuity conditions along the way, and get the algebraic cyclic cohomology of A, which pairs with the algebraic K-theory $K_0(A)$, $K_1(A)$ of A.

But in this thesis, we shall focus on the (topological) cyclic cohomology. In fact, the topological algebras we shall encounter will be certain special subalgebras of C^*-algebras. Recall that (cf. [10], [17], [48]) a dense subalgebra A^∞ of a C^*-algebra A is called smooth, if

(1) for each $k > 0$, $M_k(\mathsf{A}^\infty)$ is closed under the holomorphic functional calculus in $M_k(\mathsf{A})$; and

(2) A^∞ is a Fréchet algebra under a certain topology which is finer than the norm topology it inherits from A.

1.12 Theorem. *If* A^∞ *is a smooth subalgebra of* A*, then the natural inclusions* $K_i(\mathsf{A}^\infty) \to K_i(\mathsf{A})$, $i = 0, 1$, *are isomorphisms. Therefore, we have the* *following pairings:*

$$K_i(\mathfrak{A}) \times HC^{2m+i}(\mathfrak{A}^\infty) \to \mathbf{C},$$

where $i = 0,\ 1$ *and* $m \geq 0$.

Proof. For the proof of the first statement, see [10], [17]. The second statement then follows from Theorem 1.10. □

We now look at an example. Let X be a smooth closed manifold, $\mathfrak{A} = C(X)$, and $\mathfrak{A}^\infty = C^\infty(X)$ with the usual C^∞ topology to make it a Fréchet algebra. For any closed differential form ω on X, let $n = dim(X) - deg(\omega)$, and define

$$\tau_\omega(f_0, f_1, \cdots, f_n) = \int_X f_0 df_1 \wedge \cdots \wedge df_n \wedge \omega,$$

for $f_i \in C^\infty(X)$. Then $\tau_\omega \in Z^n_\lambda(C^\infty(X))$, and

1.13 Proposition. *For any* $x \in K^*(X) = K_*(C(X))$,

$$< x, \tau_\omega >= \int_X ch^*(x) \wedge \omega,$$

where $ch^* : K^*(X) \to H^*_{dR}(X)$ *is the Chern character map.*

Proof. Suppose first that $n = 2m$. For any idempotent $e \in M_k(C^\infty(M))$, it is well-known (cf. [17]) that

$$ch^*(e) = \sum_{l \geq 0} \frac{1}{(2\pi i)^l l!} \cdot Tr_k(e \cdot de \cdot de)^l.$$

Since $e \cdot de \cdot de \cdot e = e \cdot de \cdot de$, it follows immediately that

$$\int_X ch^*(e) \wedge \omega = \frac{1}{(2\pi i)^m m!} \cdot Tr_k(e \cdot de \cdot de)^m \wedge \omega =< e, \tau_\omega > .$$

Now suppose that $n = 2m + 1$. For any $g \in GL_k(C^\infty(X))$, we have

$$ch^*(g) = \sum_{l \geq 0} \frac{(-1)^l \cdot l!}{(2\pi i)^l (2l+1)!} Tr_k (g^{-1} dg)^{2l+1}.$$

See [28] for a very nice discussion. It is easy to show that $(dg \cdot g^{-1})^2 = -dg \cdot d(g^{-1})$, from which it follows that

$$\int_X ch^*(g) \wedge \omega = \int_X \frac{(-1)^m \cdot m!}{(2\pi i)^m (2m+1)!} Tr_k (g^{-1} dg)^{2m+1} \wedge \omega$$

$$= \int_X \frac{m!}{(2\pi i)^m (2m+1)!} Tr_k [g^{-1} \cdot (dg \wedge dg^{-1})^m] \wedge \omega = < g, \tau_\omega > .$$

This completes the proof. □

Finally we study the relation between tensor product in K-theory and cup product in cyclic cohomology. Let $\mathbb{A}$, $\mathbb{B}$ be C^*-algebras and $\mathbb{A} \otimes \mathbb{B}$ their tensor product with the maximum C^*-norm. Then there is a product:

$$K_i(\mathbb{A}) \otimes K_j(\mathbb{B}) \xrightarrow{\otimes} K_{i+j}(\mathbb{A} \otimes \mathbb{B}),$$

which can be defined by using the Kasparov product:

$$\begin{aligned} K_i(\mathbb{A}) \otimes K_j(\mathbb{B}) \quad &\cong \quad KK_i(\mathbf{C}, \mathbb{A}) \otimes KK_j(\mathbf{C}, \mathbb{B}) \\ &\xrightarrow{\text{Kasparov}} \quad KK_{i+j}(\mathbf{C}, \mathbb{A} \otimes \mathbb{B}) \cong K_{i+j}(\mathbb{A} \otimes \mathbb{B}). \end{aligned}$$

1.14 Theorem. *Let $\mathbb{A}^\infty$, $\mathbb{B}^\infty$ be smooth subalgebras of $\mathbb{A}$, $\mathbb{B}$ respectively. If $\mathbb{A}^\infty \otimes \mathbb{B}^\infty$ is smooth in $\mathbb{A} \otimes \mathbb{B}$, then for any $x \in K_i(\mathbb{A})$, $y \in K_j(\mathbb{B})$ and any $\phi \in Z_\lambda^{2m+i}(\mathbb{A}^\infty)$, $\varphi \in Z_\lambda^{2n+j}(\mathbb{B}^\infty)$, we have:*

$$< x \otimes y, \phi \# \varphi > = < x, \phi > \cdot < y, \varphi > .$$

The proof of Theorem 1.14 will be achieved in several steps.

1.15 Lemma. *Theorem 1.14 holds when $i = j = 0$.*

Proof. When $\mathbb{A}$ and $\mathbb{B}$ are both unital, the tensor product

$$K_0(\mathbb{A}) \otimes K_0(\mathbb{B}) \to K_0(\mathbb{A} \otimes_{max} \mathbb{B})$$

is induced by the following obvious map:

$$M_k(\mathbb{A}) \otimes M_l(\mathbb{B}) \to M_{k\cdot l}(\mathbb{A} \otimes \mathbb{B}).$$

Therefore, to prove Lemma 1.15 in this case, let $e \in M_k(\mathbb{A}^\infty)$, $f \in M_l(\mathbb{B}^\infty)$ be idempotents. Without loss of generality we assume $k = l = 1$. Note that

$$e \cdot de \cdot e = 0, e \cdot de \cdot de \cdot e = e \cdot de \cdot de,$$

and

$$f \cdot df \cdot f = 0, f \cdot df \cdot df \cdot f = f \cdot df \cdot df.$$

Therefore,

$$\begin{aligned} < e \otimes f \quad , \quad \phi \# \varphi >&= \frac{1}{(2\pi i)^{m+n} \cdot (m+n)!} \cdot (\phi \# \varphi)(\underbrace{e \otimes f, e \otimes f, \cdots, e \otimes f}_{2m+2n+1}) \\ &= \frac{1}{(2\pi i)^{m+n} \cdot (m+n)!} \cdot C^n_{m+n} \cdot (\phi \# \varphi)(e \cdot \underbrace{de \cdots de}_{2m} \otimes f \cdot \underbrace{df \cdots df}_{2n}) \\ &= < e, \phi > \cdot < f, \varphi > . \end{aligned}$$

This proves Lemma 1.15 for this case.

When $\mathbb{A}$ is unital but $\mathbb{B}$ is not unital, recall that $K_0(\mathbb{B})$ is defined to be the kernel of the natural map $q : K_0(\mathbb{B}^+) \to K_0(\mathbf{C})$. Therefore, any $y \in K_0(\mathbb{B})$ is of the form $y = y_1 - y_2$, where $y_i \in K_0(\mathbb{B}^+)$ and $q(y_1) = q(y_2)$ (cf. [10]). Then for any $x \in K_0(\mathbb{A})$ and $y = y_1 - y_2 \in K_0(\mathbb{B})$, we define

$$x \otimes y = x \otimes y_1 - x \otimes y_2.$$

Therefore, to check Lemma 1.15 for this case, it suffices to show that:

$$< x \otimes y, \phi \# \varphi^+ > = < x, \phi > \cdot < y, \varphi_1 >$$

for $y \in K_0(\mathbb{B}^+)$, where $\varphi^+ \in Z_\lambda^{2n}(\mathbb{B}^{\infty +})$ is as defined in Remark 1.11. But this follows immediately from the first case. The same argument takes care of the case when $\mathbb{B}$ is unital but $\mathbb{A}$ is not.

When both $\mathbb{A}$ and $\mathbb{B}$ are non unital, the proof is very similar to that of the second case, but we shall compare things in $K_0(\mathbb{A}^+ \otimes \mathbb{B}^+)$. Note that each row or column in the following diagram is exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0(\mathbb{A} \otimes \mathbb{B}) \arrow[r] \arrow[d] & K_0(\mathbb{A}^+ \otimes \mathbb{B}) \arrow[r] \arrow[d] & K_0(\mathbb{B}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(\mathbb{A} \otimes \mathbb{B}^+) \arrow[r] \arrow[d] & K_0(\mathbb{A}^+ \otimes \mathbb{B}^+) \arrow[r, "q_2"] \arrow[d, "q_1"'] & K_0(\mathbb{B}^+) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_0(\mathbb{A}) \arrow[r] \arrow[d] & K_0(\mathbb{A}^+) \arrow[r] \arrow[d] & K_0(\mathbb{C}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence, $K_0(\mathbb{A} \otimes \mathbb{B})$ can be identified with the subgroup $kernel(q_1) \cap kernel(q_2)$ in $K_0(\mathbb{A}^+ \otimes \mathbb{B}^+)$. Then the map $K_0(\mathbb{A}) \otimes K_0(\mathbb{B}) \to K_0(\mathbb{A} \otimes \mathbb{B})$ can be defined and the proof of this case is almost identical to the proof of the second case. We omit the details. $\square$

The proof of the remaining cases of Theorem 1.14 can be reduced to Lemma 1.15, by applying a well-known result in K-theory and its analogue in cyclic cohomology. We first recall these results.

It is well-known that, for any C^*-algebra $\mathcal{A}$, there are natural isomorphisms $t_1 : K_1(\mathcal{A}) \to K_0(\mathcal{A} \otimes C_0(\mathbf{R}))$ (cf. [10], Theorem 8.2.2) and $t_0 : K_0(\mathcal{A}) \to K_1(\mathcal{A} \otimes C_0(\mathbf{R}))$ (Bott periodicity, cf. [10], Theorem 9.2.1). We now recall the analogue of this result in cyclic cohomology.

Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}$. With pointwise multiplication as its product, it is a subalgebra of $C_0(\mathbf{R})$. In fact it is a smooth subalgebra, since it carries a natural Fréchet topology, which is determined by the following sequence of semi-norms: for any $k, l \in \mathbb{N}$,

$$p_{k,l}(f) = Sup_{x\in\mathbf{R}}\|(1 + \|x\|)^l \cdot (\frac{d^k f}{dx^k})(x)\|$$

for any $f \in \mathcal{S}(\mathbf{R})$. For any $f_0, f_1 \in \mathcal{S}(\mathbf{R})$, we define:

$$\epsilon(f_0, f_1) = \int_{\mathbf{R}} f_0 df_1.$$

Then $\epsilon \in Z^1_\lambda(\mathcal{S}(\mathbf{R}))$. Note that $\mathcal{S}(R)$ is nuclear in the category of locally convex topological spaces (cf. [55]). Therefore, for any smooth subalgebra $\mathcal{A}^\infty$ of $\mathcal{A}$, the projective tensor product $\mathcal{A}^\infty \otimes \mathcal{S}(\mathbf{R})$ can be realized as an algebra of certain functions on $\mathbf{R}$ with values in $\mathcal{A}^\infty$. With this interpretation, it is not difficult to show that $\mathcal{A}^\infty \otimes \mathcal{S}(\mathbf{R})$ is smooth in $\mathcal{A} \otimes C_0(\mathbf{R})$. Therefore, for any $\phi \in Z^{2m+i}_\lambda(\mathcal{A}^\infty)$, $\phi\#\epsilon$ can be paired with $K_{i+1}(\mathcal{A} \otimes C_0(\mathbf{R}))$. The following result is due to Elliott, Natsume and Nest [27]:

1.16 Theorem. **([27])** *For any $x \in K_i(\mathcal{A})$ and any $\phi \in Z^{2m+i}_\lambda(\mathcal{A})$, we have:*

$$< x, \phi > = < t_i(x), \phi\#\epsilon > .$$

Proof of Theorem 1.14. We can now complete the proof of Theorem 1.14. We will treat in detail only the case where $i = j = 1$, since the proofs of the other two cases are similar but easier. For any $x \in K_1(\mathsf{A})$, $y \in K_1(\mathsf{B})$ and any $\phi \in Z_\lambda^{2m+1}(\mathsf{A}^\infty)$, $\varphi \in Z_\lambda^{2n+1}(\mathsf{B}^\infty)$, we have:

$$
\begin{aligned}
& < x, \phi > \cdot < y, \varphi > \\
= \; & < t_1(x), \phi \# \epsilon > \cdot < t_1(y), \varphi \# \epsilon > \\
= \; & < t_1(x) \otimes t_1(y), (\phi \# \epsilon) \# (\varphi \# \epsilon) >,
\end{aligned}
$$

where the first equation follows from Theorem 1.16 and the second from Lemma 1.15. Let β be the isomorphism from $\mathsf{A} \otimes C_0(\mathbf{R}) \otimes \mathsf{B} \otimes C_0(\mathbf{R})$ onto $\mathsf{A} \otimes \mathsf{B} \otimes C_0(\mathbf{R}) \otimes C_0(\mathbf{R})$, determined by

$$\beta(a \otimes f \otimes b \otimes g) = a \otimes b \otimes g \otimes f.$$

Then it can be shown that:

$$\beta_*(t_1(x) \otimes t_1(y)) = (t_1 \circ t_0)(x \otimes y),$$

and

$$\beta^{-1^*}(\phi \# \epsilon \# \varphi \# \epsilon) = \phi \# \varphi \# \epsilon \# \epsilon.$$

Therefore, using Theorem 1.16 twice, we have:

$$
\begin{aligned}
& < x, \phi > \cdot < y, \varphi > \\
= \; & < (t_1 \circ t_0)(x \otimes y), \phi \# \varphi \# \epsilon \# \epsilon > \\
= \; & < x \otimes y, \phi \# \varphi > .
\end{aligned}
$$

This completes the proof of Theorem 1.14. $\square$

§2 K-theory for an index map

In this section we shall construct an interesting KK-element, modeled closely after the construction of longitudinal KK-elements on foliations. It is the KK-theoretic version of the construction in Miscenko and Fomenko [44]. As we shall see, this KK-element, or the index map it induces, plays a fundamental role in certain index problems. We shall then decompose this index map into the product of two simpler factors. This decomposition will be basic in calculating the pairing of the index map with cyclic cocycles in this thesis. Finally, we shall study in some detail a KK-element in the decomposition which is determined by a covering structure.

We start with the following input data:

M	is a closed oriented Riemannian manifold;
$p : \widetilde{M} \to M$	is a Galois covering with deck transformation group Γ;
D	is a geometric (first order elliptic differential) operator, operating on sections $C^\infty(E)$ of a Clifford bundle $E \to M$;
V	is a closed manifold on which Γ acts smoothly on the right.

The covering map p induces canonically a map $\pi : \widetilde{M} \times V \to M$. If $\widetilde{E} = \pi^*(E)$ is the pull-back bundle on $\widetilde{M} \times V$, then D has a canonical lifting $\widetilde{D}$ acting on smooth sections of $\widetilde{E}$. Let $W = \widetilde{M} \times_\Gamma V$. For $X = \widetilde{M}$, $M \times V$ or W, let E_X, D_X denote the pull-back (lifting) on X of E and D, respectively.

Some maps that we shall need later are named in the following diagram:

$$\begin{array}{ccc} \widetilde{M} \times V & \xrightarrow{p_0} & M \times V \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle q_0} \\ W = \widetilde{M} \times_\Gamma V & \xrightarrow{q_1} & M. \end{array}$$

The orientation and the Riemannian metric on M determine a volume form $dvol_M$ on M and a volume form $dvol_{\widetilde{M}}$ on $\widetilde{M}$. It is also convenient to fix a Borel measure on V.

We now begin to construct a Kasparov module $(\mathcal{E}, \Psi, \lambda(\widetilde{D}))$ representing an element in $KK_\sigma(C(W), \mathcal{A})$, where $\sigma \in \mathbf{Z}_2$ is the parity of D, and where $\mathcal{A} = C(V) \rtimes_r \Gamma$ is the reduced crossed-product C^*-algebra. For convenience, we shall treat only the odd case in detail, so we assume that D is self-adjoint. The even case is similar (see Remark 2.3 below).

The Hilbert module $\mathcal{E}$ over $\mathcal{A}$ will be constructed from the Γ-equivariant bundle $\widetilde{E} \to V$ (see §1.2).

2.1 Lemma. *Let $\mathcal{E}_c = C_c^\infty(\widetilde{E})$ be the space of smooth sections with compact support of the bundle $\widetilde{E}$ over $\widetilde{M} \times V$. Then $\mathcal{E}_c$ constitutes a continuous fields of Hilbert spaces for the system (V, Γ) (see §1.2), and hence gives rise to a Hilbert module $\mathcal{E}$ over $\mathcal{A}$.*

Proof. Recall, from §1.2, that

$$\gamma(\xi)(\tilde{m}, v) = \xi(\tilde{m} \cdot \gamma, v \cdot \gamma)$$

for any $\xi \in \mathcal{E}_c$ and for any $\gamma \in \Gamma$. Obviously $\gamma(\xi) \in \mathcal{E}_c$ and $\mathcal{E}_c$ is Γ-invariant. Thus, condition (1) in §1.2 is satisfied. Conditions (2) and (3) are also very easy to check. We now turn to condition (4).

In our present situation $K = L^2(\tilde{E}, \widetilde{M} \times V)$. For any $\xi \in \mathcal{E}_c$, the operator $T_\xi : L^2(V) \otimes \ell^2(\Gamma) \to K$ is defined by:

$$T_\xi(\sum_{\gamma\in\Gamma} f_\gamma \otimes e_\gamma)(\tilde{m}, v) = \sum_\gamma f_\gamma(v)\xi(\tilde{m}\cdot\gamma, v\cdot\gamma),$$

where $f_\gamma \in L^2(V)$ and $\{e_\gamma\}$ is the standard basis for $\ell^2(\Gamma)$. We need to show that T_ξ is a bounded operator. Without loss of generality, we assume that the support of ξ is contained in the interior of $M_0 \times V$, where M_0 is a fundamental domain for the covering $\widetilde{M} \to M$. In particular, $\{Supp(\gamma(\xi))\}_{\gamma\in\Gamma}$ is pairwise disjoint. Therefore,

$$\begin{aligned} & \|T_\xi(\sum_\gamma f_\gamma \otimes e_\gamma)\|_K^2 \\ = \ & \|\sum_\gamma f \cdot \gamma(\xi)\|_K^2 = \sum_\gamma \|f \cdot \gamma(\xi)\|_K^2 \\ \leq \ & (Vol(M) \cdot max\{\|\xi(\tilde{m}, v)\|\})^2 \cdot \sum_\gamma \|f_\gamma\|_{L^2(V)}^2. \end{aligned}$$

This establishes condition (4). □

For reference we recall, from §1.2, that the right action of $C^\infty(V) \rtimes_{alg} \Gamma$ on $\mathcal{E}_c$ is given by:

$$\xi * (f \cdot \gamma) = \gamma^{-1}(M_{f^\Uparrow}(\xi))$$

for any $\xi \in \mathcal{E}_c$, $f \in C^\infty(V)$ and $\gamma \in \Gamma$, where $f^\Uparrow \in C^\infty(\widetilde{M} \times V)$ is the lifting of f from V to $\widetilde{M} \times V$ (that is, $f^\Uparrow(\tilde{m}, v) = f(v)$) and $M_{f^\Uparrow}$ is the pointwise (scalar) multiplication operator. The inner product on $\mathcal{E}_c$ is given by:

$$\ll \xi, \eta \gg = \sum_\gamma \int_{\widetilde{M}} < \xi(\tilde{m}, v), \gamma(\eta)(\tilde{m}, v) >_{\tilde{E}} dvol_{\widetilde{M}}(\tilde{m}) \cdot \gamma$$

for any $\xi, \eta \in \mathcal{E}_c$, where $< \cdot, \cdot >_{\widetilde{E}}$ is the Hermitian metric on the bundle $\widetilde{E}$. Note that $\ll \xi, \eta \gg \in C^\infty(V) \rtimes_{alg} \Gamma$. $\mathcal{E}$ is the completion of $\mathcal{E}_c$ with respect to the norm induced by this inner product.

The representation Ψ of $C(W)$ on $\mathcal{E}$ is determined by:

$$\Psi(f)(\xi) = M_{f^\Uparrow}(\xi)$$

for any $f \in C^\infty(W)$ and any $\xi \in \mathcal{E}_c$, where $f^\Uparrow = p_1^*(f) \in C^\infty(\widetilde{M} \times V)$ is the pull-back of f and $M_{f^\Uparrow}$ is again the multiplication operator.

To show that $\Psi(f)$ is bounded, again we use the operator T_ξ defined in §1.2. Note that for any $\xi \in \mathcal{E}_c$, $\|\xi\|_\varepsilon = \|T_\xi\|$ (see §1.2). It is easy to check that $T_{\Psi(f)\xi} = M_{f^\Uparrow} \cdot T_\xi$ for any $\xi \in \mathcal{E}_c$ and any $f \in C^\infty(W)$, where $M_{f^\Uparrow}$ is the multiplication operator on K. Therefore,

$$\|\Psi(f)\xi\|_\varepsilon = \|T_{\Psi(f)\xi}\|$$

$$= \|M_{f^\Uparrow} \cdot T_\xi\| \leq \|f\| \cdot \|T_\xi\| = \|f\| \cdot \|\xi\|_\varepsilon.$$

So $\Psi : C^\infty(W) \to \mathcal{B}(\mathcal{E})$ is a bounded homomorphism. On the other hand, since $f^\Uparrow$ is Γ-invariant,

$$\ll \eta, \Psi(f)(\xi) \gg = \ll \Psi(\bar{f})(\eta), \xi \gg$$

for any $\xi, \eta \in \mathcal{E}_c$, and $f \in C^\infty(W)$. Therefore, Ψ extends to be a C^*-algebra homomorphism.

To define the "Fredholm" operator $\lambda(\widetilde{D})$, note that the lifting $\widetilde{D}$, when restricted to each leaf $\widetilde{M} \times v$ for any $v \in V$, enjoys very pleasant properties (cf. for example, [5], [9]); for example, it is essentially self-adjoint. Let $\lambda(t) = t \cdot (1 + t^2)^{-1/2}$ and let $\lambda(\widetilde{D})$ be the leafwise functional calculus of $\widetilde{D}$.

2.2 Proposition. *$\lambda(\widetilde{D})$ is a well-defined operator in $\mathcal{B}(\mathcal{E})$, and the triple $(\mathcal{E}, \Psi, \lambda(\widetilde{D}))$ defines a KK-element in $KK_1(C(W), \mathcal{A})$.*

Proof. See [53], [25]. A slightly different way to say this is that $\widetilde{D}$ is a regular unbounded operator on $\mathcal{E}$, and $(\mathcal{E}, \Psi, \widetilde{D})$ is an (odd) unbounded Kasparov module in the sense of Baaj and Julg [7] (see also [10], §17.11). Therefore, $\lambda(\widetilde{D}) = \widetilde{D}(1 + \widetilde{D}^2)^{-1/2}$ makes sense as an operator in $\mathcal{B}(\mathcal{E})$ and the triple $(\mathcal{E}, \Psi, \lambda(\widetilde{D}))$ defines a KK-element in $KK_1(C(W), \mathcal{A})$. □

2.3 Remark: the even case. If $D^+ : C^\infty(E^+) \to C^\infty(E^-)$ is a geometric (first order elliptic differential) operator, a typical trick is to set $E = E^+ \oplus E^-$ and

$$D = \begin{pmatrix} 0 & (D^+)^* \\ D^+ & 0 \end{pmatrix}.$$

Then one repeats the above constructions (2.1 - 2.2). The canonical $\mathbf{Z}_2$ grading on E induces a grading g on $\mathcal{E}$, under which $\lambda(\widetilde{D})$ is odd. Therefore, the quadruple $(\mathcal{E}, g, \Psi, \lambda(\widetilde{D}))$ defines an element in $KK_0(C(W), \mathcal{A})$.

2.4 The index map. In either case, this KK-element will be denoted by $[D_{\Gamma,V}]$. Through the Kasparov product it induces a map in K-theory:

$$K^*(W) \xrightarrow{\otimes[D_{\Gamma,V}]} K_{*+\sigma}(\mathcal{A}),$$

where σ is the parity of D. We shall abuse the notation and denote this map also by $[D_{\Gamma,V}]$. The goal of this thesis is to study this map. To see that this map is interesting, we compare this map with two well-known constructions.

First, we recall the construction of a higher Γ-index of elliptic operators by Miscenko and Fomenko [44]. Thus we deal with the even case (see Remark 2.3).

Note that Γ acts on $\mathcal{A}$ in an obvious way: for any $a \in \mathcal{A}$ and any $\gamma \in \Gamma$, $\gamma(a) = \gamma \cdot a$, where the right hand side is the product in $\mathcal{A}$. Let $\widetilde{M} \times \mathcal{A}$ be the trivial $\mathcal{A}$-bundle over $\widetilde{M}$. It is Γ-equivariant and descends to an $\mathcal{A}$-bundle $\mathcal{V} = \widetilde{M} \times_\Gamma \mathcal{A}$ over M. Then D^+ extends to an operator

$$D^+ \otimes 1 : C^\infty(M, E^+ \otimes \mathcal{V}) \to C^\infty(M, E^- \otimes \mathcal{V}).$$

According to Miscenko and Fomenko [44], this operator is $\mathcal{A}$-elliptic and has an index in $K_0(\mathcal{A})$. We denote this index element by $Ind_{MF}(D^+, \Gamma, V)$. Then we have the following:

2.5 Proposition. *$Ind_{MF}(D^+, \Gamma, V) = [D_{\Gamma,V}]([1_W])$, where the right hand side is the index map in §2.4 applied to the the class $[1_W]$ of the trivial line bundle on W.*

Proof. Let $\mathcal{E}'_c = C^\infty(\widetilde{M}, E_{\widetilde{M}} \otimes \mathcal{A})^\Gamma$ denote the space of smooth Γ-invariant sections of the bundle $E_{\widetilde{M}} \otimes \mathcal{A}$ over $\widetilde{M}$. $\mathcal{E}'_c$ can be identified with $C^\infty(M, E \otimes \mathcal{V})$ in an obvious way. Upon this identification, $D \otimes 1$ becomes $D_{\widetilde{M}} \otimes 1$ (recall that D is defined from D^+ as in Remark 2.3 and $D_{\widetilde{M}}$ is the lifting of D on $\widetilde{M}$). Let $\mathcal{E}'$ be the Hilbert module over $\mathcal{A}$ made from $\mathcal{E}'_c$ which is isometrically isomorphic to $L^2(M, E \otimes \mathcal{V})$ under the same identification. Then, as an element in $K_0(\mathcal{A}) \cong KK_0(\mathbf{C}, \mathcal{A})$, $Ind_{MF}(D^+, \Gamma, V)$ can be represented by the even Kasparov module $(\mathcal{E}', g', 1, \lambda(D_{\widetilde{M}} \otimes 1))$, where g' is the canonical grading inherited from the grading of E, and 1 is the obvious unital map from $\mathbf{C}$ to $\mathcal{B}(\mathcal{E}')$.

On the other hand, it is easy to see that $[D_{\Gamma,V}]([1_W])$ can be represented by $(\mathcal{E}, g, 1, \lambda(\widetilde{D}))$, where $\mathcal{E}$ and $\widetilde{D}$ are as in §2.1 and §2.2, g the canonical grading inherited from the grading of E, and 1 is the obvious unital map from $\mathbf{C}$ to $\mathcal{B}(\mathcal{E})$. Given any $\xi \in \mathcal{E}_c$ as in §2.1, let

$$\xi'(\tilde{m}) = \sum_{\gamma} \xi(\tilde{m} \cdot \gamma, v \cdot \gamma) \cdot \gamma,$$

which we regard as a smooth section of the bundle $E_{\widetilde{M}} \otimes \mathcal{A}$ over $\widetilde{M}$. Note that for any given $\tilde{m}$, the sum in the definition of ξ' is in fact a finite sum since the support of ξ is compact. Then with a careful bookkeeping of the induced Γ-actions, ξ' can be shown to be Γ-invariant, and therefore, $\xi' \in \mathcal{E}'$. It is not difficult to show that this map extends to an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$, under which $(\mathcal{E}, g, 1, \lambda(\widetilde{D}))$ is identified with $(\mathcal{E}', g', 1, \lambda(D_{\widetilde{M}} \otimes 1))$. This completes the proof. □

Of particular interest is the case where $V = \{*\}$ consists of one single point, $\Gamma = \pi_1(M)$ is the fundamental group of an even dimensional manifold M, and $D = d + d^*$ is the signature operator on M. In this case the pairing of $Ind_{MF}(d + d^*, \pi_1(M), \{*\})$ with appropriate cyclic cocycles on the group algebra $\mathbf{C}[\Gamma]$ plays a fundamental role in Connes-Moscovici's approach to the Novikov conjecture on the homotopy invariance of higher signatures. See [19], [44] for more details.

The main result of this thesis will recover that of Connes-Moscovici [19] on the Novikov conjecture. However, our motivation and approach are different. The focus of this thesis is on longitudinal KK-elements on foliated bundles. Note that there are two natural product foliation structures on $\widetilde{M} \times V$ which are both Γ-equivariant and thus induce two foliation structures on W: The first one, whose leaves are copies of V, is in fact given by the fibration $q_1 : W \to M$. The second foliation, denoted by $(W, \mathcal{F})$, whose leaves are images of $\widetilde{M} \times v$ for $v \in V$, is more interesting and has attracted the attention of several authors (see, for example, [26], [47]). We will focus on the second foliation in this thesis.

D_W is a leafwise elliptic (self-adjoint) differential operator for the second foliation. It is well-known (cf. [15] [19]) that D_W induces an element in $KK_1(W, \mathcal{C})$, where $\mathcal{C}$ is the foliation algebra with coefficients in E_W. However, to relate this longitudinal KK-element to the index element $[D_{\Gamma,V}]$ defined in §2.2, we need to put one additional condition on the group action, which is:

2.6 Condition. *If the fixed point set of $\gamma \in \Gamma$ contains a nonempty open set, then $\gamma = 1$. See [46], [33] and [47].*

Under this condition, the graph $\mathcal{G}$ (cf. [47]) of the foliation $(W, \mathcal{F})$ is $(\widetilde{M} \times \widetilde{M} \times V)/\Gamma$. Recall that Γ acts on $\widetilde{M} \times \widetilde{M} \times V$ diagonally, that is,

$$(\tilde{m}_r, \tilde{m}_s, v) \cdot \gamma = (\tilde{m}_r \cdot \gamma, \tilde{m}_s \cdot \gamma, v \cdot \gamma).$$

The image of $(\tilde{m}_r, \tilde{m}_s, v)$ in $\mathcal{G}$ will be denoted by $[\tilde{m}_r, \tilde{m}_s, v]$.

The source map s and the range map r from $\mathcal{G}$ to W are obvious from our choice of notation.

Let $\mathcal{C} = C^*(W, \mathcal{F}, E_W)$ be the foliation algebra with coefficients in E_W. It is constructed from the space $C_c^\infty(\mathcal{G}, r^*(E_W) \otimes (s^*(E_W))^*)$ of smooth sections with compact support. (Cf. [47] for details in this case.) A well-known result of Hilsum and Skandalis [32] (see also [46], Theorem 6.14) states that $\mathcal{C}$ is strongly Morita equivalent to the reduced crossed product $\mathcal{A}$. We need to make this more concrete. Note that there is a representation Ψ_2 of $\mathcal{C}$ on $\mathcal{E}$, defined by: for any $k \in C_c^\infty(\mathcal{G}, r^*(E_W) \otimes (s^*(E_W))^*)$ and any $\xi \in \mathcal{E}_c$:

$$(\Psi_2(k)\xi)(\tilde{m}, v) = \int_{\widetilde{M}} k[\tilde{m}, \tilde{m}_s, v]\xi(\tilde{m}_s, v)dvol_{\widetilde{M}}(\tilde{m}_s).$$

2.7 Proposition. **([47], Proposition 2.4)** *Ψ_2 is an isomorphism between $\mathcal{C}$ and the C^*-algebra $\mathcal{K}(\mathcal{E})$ of compact operators on $\mathcal{E}$. In particular, $\mathcal{C}$ is strongly Morita equivalent to $\mathcal{A}$.*

Therefore, the triple $(\mathcal{E}, \Psi_2, 0)$ defines an element in $KK_0(\mathcal{C}, \mathcal{A})$. ($\mathcal{E}$ is trivially graded!) Recall ([10], §19.1) that a KK-element $x \in KK_i(\mathsf{A}, \mathsf{B})$ is a KK-equivalence if there is a KK-element $y \in KK_i(\mathsf{B}, \mathsf{A})$ such that $x \otimes_{\mathsf{B}} y = 1_{\mathsf{A}}$, and $y \otimes_{\mathsf{A}} x = 1_{\mathsf{B}}$. If x is a KK-equivalence, then the map $K_j(\mathsf{A}) \stackrel{\otimes x}{\to} K_{i+j}(\mathsf{B})$ is an isomorphism.

2.8 Proposition. *$(\mathcal{E}, \Psi_2, 0)$ defines a KK-equivalence.*

Proof. Let $\{\eta_n\}_{n \in \mathbf{N}}$ be an orthonormal basis for $L^2(E)$ consisting of smooth sections. This can be chosen from, for example, eigensections of D. We then fix a fundamental domain M_0 for the covering $\widetilde{M} \to M$, and define:

$$\tilde{\eta}_n(\tilde{m}, v)) = \begin{cases} \eta_n(p(\tilde{m})) & \text{if } \tilde{m} \in M_0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

In general, $\tilde{\eta}_n \notin \mathcal{E}_c$. But a standard approximation argument shows that $\tilde{\eta}_n \in \mathcal{E}$. It is then straightforward to show that the following map:

$$\xi \in \mathcal{E} \longrightarrow (\ll \eta_n, \xi \gg)_{n \in \mathbb{N}} \in \ell^2_{\mathcal{A}}$$

induces an isometric isomorphism between $\mathcal{E}$ and $\ell^2_{\mathcal{A}}$. Therefore, $\mathcal{K}(\mathcal{E}) \cong \mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on the Hilbert space $\ell^2_{\mathbb{C}}$ (cf. [10], §13.2.4). Under this identification, $(\mathcal{E}, \Psi_2, 0)$ is the KK-element associated to the identity map $i : \mathcal{A} \otimes \mathcal{K} \to \mathcal{A} \otimes \mathcal{K}$, which is well-known to be a KK-equivalence (cf. [10], §17.8.2(c) and §19.1). □

We now recall the construction of the KK-element associated to the leafwise elliptic operator D_W, which is the central ingredient for analysis on the foliated bundle. See [25] for a more detailed treatment. For any $T \in \mathcal{B}(\mathcal{E})$ and any $k \in \mathcal{C}$, we define:

$$\Psi_3(T)k = \Psi_2^{-1}(T \cdot \Psi_2(k)) \in \mathcal{C}.$$

A direct computation shows that Ψ_3 is a C^*-monomorphism from $\mathcal{B}(\mathcal{E})$ to $\mathcal{B}(\mathcal{C})$, where $\mathcal{C}$ is regarded as a Hilbert module over itself. In fact Ψ_3 is also onto, since the multiplier algebra $\mathcal{M}(\mathcal{K}(\mathcal{E}))$ is canonically isomorphic to $\mathcal{B}(\mathcal{E})$ (cf. [10], Theorem 13.4.1, although in our case this can be proved directly).

Set $\Psi' = \Psi_3 \circ \Psi$ and $\lambda(\widetilde{D})' = \Psi_3(\lambda(\widetilde{D}))$. Then the longitudinal KK-element $[D_W]$ associated to the leafwise elliptic operator D_W is given by the triple $(\mathcal{C}, \Psi', \lambda(\widetilde{D})')$, where $\mathcal{C}$ is thought of as a Hilbert module over itself.

2.9 Remarks. (1) Although each leaf of the foliation may or may not be diffeomorphic to $\widetilde{M}$, the holonomy covering of it always is. And when we construct a KK-element from a leafwise elliptic operator, only the lifting on the holonomy covering matters. This explains the (apparent) simplicity of the construction of the longitudinal KK-element in our case.

(2) This construction is slightly different from the one in [25]. In [25] the foliation algebra $\mathcal{C}$ is represented faithfully on $L^2(\mathcal{G})$. In this thesis the same algebra is represented, again faithfully, on $\mathcal{E}$. It is clear that either way the induced representations of $C(W)$ on $\mathcal{C}$ are identical. The same is true for $\lambda(\widetilde{D})'$. Therefore, the resulting Kasparov modules are identical.

2.10 Proposition. $[D_{\Gamma,V}] = [D_W] \otimes_{\mathcal{C}} [\mathcal{E}, \Psi_2, 0]$, *where* $\otimes_{\mathcal{C}}$ *is the Kasparov product in KK-theory.*

Proof. Straightforward. $\square$

This, together with Proposition 2.8, shows that, as far as K-theory is concerned, $[D_{\Gamma,V}]$ and $[D_W]$ are equivalent.

Now if V admits a Γ-invariant (nontrivial) measure μ, then μ induces a holonomy-invariant transverse measure for the foliation $(W, \mathcal{F})$, which in turn induces a trace on $\mathcal{C}$ and a group homomorphism $Tr_\mu : K_0(\mathcal{C}) \to \mathbf{R}$. The study of the composition $Tr_\mu \circ [D_W] : K^1(W) \to \mathbf{R}$, as expressed in terms of a longitudinal cyclic cocycle on $C^\infty(W)$, is very important in the work of DHK (cf. [26] and references therein).

As the discussions above clearly suggest, one important way to study $[D_{\Gamma,V}]$ is to study the pairing of this element (map) with cyclic cocycles on "nice" subalgebras of $\mathcal{A}$. The work in this thesis is one step along this line.

Our approach is based on the following simple observation. Note that $\widetilde{D}$, as a Γ-invariant, leafwise elliptic operator on $\widetilde{M} \times V$, is very special: the restriction of $\widetilde{D}$ to each leaf $\widetilde{M} \times v$ does not depend upon $v \in V$, and is the same as the lifting of D on $\widetilde{M}$. KK-theory provides the right framework where we can express this observation more succinctly. This is the content of Theorem 2.11 below.

The Miscenko-Fomenko construction (cf. §2.5) provides yet another clue. Recall that the higher index is constructed from two "seemingly unrelated" pieces: an elliptic operator D on M and an $\mathcal{A}$-bundle $\mathcal{V}$ on M. It turns out to be fruitful to look at $\mathcal{V}$ from a slightly different point of view, by turning it into a Kasparov module $(\mathcal{H}, \Phi, 0)$ for $(C(W), C(M \times V) \rtimes_r \Gamma)$.

For brevity, let $\mathcal{B} = C(M \times V) \rtimes_r \Gamma$ and $\mathcal{B}_c = C^\infty(M \times V) \rtimes_{alg} \Gamma$. It is well-known that $C(M \times V) \rtimes_r \Gamma$ is canonically isomorphic to $C(M) \otimes C(V) \rtimes_r \Gamma$, that is, $\mathcal{B} \cong C(M) \otimes \mathcal{A}$ (cf. [52]).

The Hilbert module $\mathcal{H}$ over $\mathcal{B}$ will be constructed from the principal Γ-bundle $\widetilde{M} \times V \to M \times V$. From this principal bundle we construct an infinite dimensional Hermitian bundle over $M \times V$: for any $(m, v) \in M \times V$, the Hilbert space over (m, v) is $\ell^2(\{(\tilde{m}, v) : p(\tilde{m}) = m\})$. Let $\mathcal{H}_c = C_c^\infty(\widetilde{M} \times V)$ be the space of smooth functions on $\widetilde{M} \times V$ with compact support. The proof of Lemma 2.1 can then be repeated to show that $\mathcal{H}_c$ determines a continuous field of Hilbert spaces for the system $(M \times V, \Gamma)$, and gives rise to a Hilbert module $\mathcal{H}$ over $\mathcal{B}$.

For reference, we recall that $\mathcal{H}_c$ is a right $\mathcal{B}_c$-module: for any $h \in \mathcal{H}_c$, any $g \in C^\infty(M \times V)$ and any $\gamma \in \Gamma$:

$$(h * (g \cdot \gamma))(\tilde{m}, v) = g(p(\tilde{m}), v \cdot \gamma^{-1}) h(\tilde{m} \cdot \gamma^{-1}, v \cdot \gamma^{-1}).$$

To define the inner product on $\mathcal{H}_c$, we first note that any $h \in \mathcal{H}_c$ has a canonical push-forward $h^\Downarrow$ in $C^\infty(M \times V)$:

$$h^\Downarrow(m, v) = \sum_{p(\tilde{m})=m} h(\tilde{m}, v),$$

which enjoys the following basic property:

$$\int_{\widetilde{M}} h \cdot dvol_{\widetilde{M}} = \int_M h^\Downarrow \cdot dvol_M$$

as functions in $C^\infty(V)$. Then the inner product on $\mathcal{H}_c$ is defined as follows: for any $h_1, h_2 \in \mathcal{H}_c$,

$$\ll h_1, h_2 \gg = \sum_\gamma (\bar{h}_1 \cdot \gamma(h_2))^\Downarrow \cdot \gamma,$$

where $\bar{h}_1$ is the complex conjugate of h_1. $\mathcal{H}$ is the completion of $\mathcal{H}_c$ with respect to the norm induced by this inner product. It is a finitely generated, projective Hilbert module over $\mathcal{B}$ (we shall justify this in detail in Lemma 2.14).

We now define a representation Φ of $C(W)$ on $\mathcal{H}$. For any $f \in C^\infty(W)$, and any $h \in \mathcal{H}_c$:

$$\Phi(f)h = M_{f^\Uparrow}(h),$$

where $f^{\Uparrow} = p_1^*(f) \in C^\infty(\widetilde{M} \times V)$ is the lifting of f and $M_{f^{\Uparrow}}$ is the scalar-multiplication operator.

The KK-element $[\mathcal{V}]$ for $\mathcal{V}$ is given by the triple $(\mathcal{H}, \Phi, 0)$, together with the trivial grading on $\mathcal{H}$.

Note that this construction is purely topological: We could have used continuous functions instead.

Let $[D] \in K^1(M) = KK_1(C(M), \mathbf{C})$ be the K-homology element associated to D (cf. [8] or §1.5). Then we have the following basic decomposition:

2.11 Theorem. $[D_{\Gamma,V}] = [\mathcal{V}] \otimes_{C(M)} [D]$, *where* $\otimes_{C(M)}$ *denotes the Kasparov product.*

Proof. Recall that $[D] \in KK_1(C(M), \mathbf{C})$ is represented by the Kasparov module $(L^2(E), \psi, \lambda(D))$, where $\lambda(D) = D(1 + D^2)^{-1/2}$ and ψ is the obvious representation of $C(M)$ on $L^2(E)$ by scalar multiplications. Therefore, the triple $(L^2(E) \otimes \mathcal{A}, \psi \otimes 1, \lambda(D) \otimes 1)$ represents $[D] \otimes 1_{\mathcal{A}}$ in $KK_1(\mathcal{B}, \mathcal{A})$. Again for brevity, we let $\mathcal{E}_2 = L^2(E) \otimes \mathcal{A}$ and $\widetilde{\mathcal{E}} = \mathcal{H} \otimes_{\mathcal{B}} \mathcal{E}_2$ as Hilbert modules over $\mathcal{A}$.

By definition, $\widetilde{\mathcal{E}}$ is the module for the Kasparov product $[\mathcal{V}] \otimes_{C(M)} [D]$. We claim that there is a natural module morphism $\Theta : \widetilde{\mathcal{E}} \to \mathcal{E}$. Indeed, for any $h \in \mathcal{H}_c$, any $\xi \in C^\infty(E)$ and any $b \in C^\infty(V) \rtimes_{alg} \Gamma$, since $C^\infty(V) \rtimes_{alg} \Gamma$ embeds into $\mathcal{B}_c$ in a canonical way, $h * b \in \mathcal{H}_c$ makes sense, and we define:

$$\Theta(h \otimes \xi \otimes b) = M_{h*b}(\xi^{\Uparrow}) = (h * b) \cdot \xi^{\Uparrow},$$

where $\xi^{\Uparrow} = \pi^*(\xi) \in C^\infty(\tilde{E})$ is the lifting of ξ from a section of E over M to a section of $\tilde{E}$ over $\widetilde{M} \times V$. Note that Θ is obviously a $C^\infty(V) \rtimes_{alg} \Gamma$ module morphism. Therefore, to check that Θ is isometric, it suffices to check it for any $h_1, h_2 \in \mathcal{H}_c$ and any $\xi_1, \xi_2 \in C(E)$. This can be done as follows:

$$
\begin{aligned}
& \ll \Theta(h_1 \otimes \xi_1), \Theta(h_2 \otimes \xi_2) \gg \\
= & \ll h_1 \cdot \xi_1^{\Uparrow}, h_2 \cdot \xi_2^{\Uparrow} \gg_{\mathcal{E}} \qquad \text{(by def. of } \Theta) \\
= & \sum_\gamma \int_{\widetilde{M}} < h_1 \cdot \xi_1^{\Uparrow}, \gamma(h_2 \cdot \xi_2^{\Uparrow}) >_{\tilde{E}} dvol_{\widetilde{M}} \cdot \gamma \quad \text{(by def. of } \ll \cdot, \cdot \gg_{\mathcal{E}}) \\
= & \sum_\gamma \int_{M} < h_1 \cdot \xi_1^{\Uparrow}, \gamma(h_2 \cdot \xi_2^{\Uparrow}) >_{\tilde{E}}^{\Downarrow} dvol_{M} \cdot \gamma \\
= & \sum_\gamma \int_{M} (\bar{h}_1 \gamma(h_2))^{\Downarrow} \cdot < \xi_1, \xi_2 >_E dvol_{M} \cdot \gamma \\
= & \sum_\gamma \int_{M} \ll \xi_1, (\bar{h}_1 \gamma(h_2))^{\Downarrow} \cdot \xi_2 \gg_E dvol_{M} \cdot \gamma \\
= & \ll \xi_1, \ll h_1, h_2 \gg_{\mathcal{H}} \cdot \xi_2 \gg_{\mathcal{E}_2} \qquad \text{(by def. of } \ll \cdot, \cdot \gg_{\mathcal{H}}) \\
= & \ll h_1 \otimes \xi_1, h_2 \otimes \xi_2 \gg .
\end{aligned}
$$

Therefore, Θ extends to be an isometric isomorphism. It is also surjective since $C_c^\infty(\tilde{E}) \otimes_{alg} C(V)$ is dense in $\mathcal{E}$. From now on, we use Θ to identify $\tilde{\mathcal{E}}$ and $\mathcal{E}$.

It is then easy to see that under this identification of modules, the representation Ψ of $C(W)$ on $\mathcal{E}$ is identical to the representation $\Phi \otimes_{\mathcal{B}} 1$ of $C(W)$ on $\tilde{\mathcal{E}}$.

Since the "Fredholm" operator for $[\mathcal{V}]$ is 0, to prove the theorem it remains to show that $\lambda(\widetilde{D})$ is a $\lambda(D) \otimes 1$-connexion, in other words,

$$\lambda(\widetilde{D}) \circ T_h - T_h \circ (\lambda(D) \otimes 1) \in \mathcal{K}(\mathcal{E}_2, \mathcal{E})$$

for any $h \in \mathcal{H}$, where $T_h \in \mathcal{B}(\mathcal{E}_2, \mathcal{E})$ is defined by $T_h(\xi) = h \otimes \xi$ for any $\xi \in \mathcal{E}_2$ (see §1.6, or [10]). Since $C_c^\infty(\widetilde{M}) \otimes_{alg} C^\infty(V)$ is dense in $\mathcal{H}$, it suffices to check this statement for $h = \tilde{f} \cdot g$, where $g \in C^\infty(V)$, $\tilde{f} \in C_c^\infty(\widetilde{M})$, and $Supp(\tilde{f})$ is contained in an open ball $U \subset \widetilde{M}$ sufficiently small that $p|_U : U \to M$ is one-to-one. Let $f \in C^\infty(M)$ be the push-forward of $\tilde{f}$: $f(m) = \sum_{p(\tilde{m})=m} \tilde{f}(\tilde{m})$. Then, following [9], we make some choices:

Choose $\delta > 0$, such that $dist(Supp(\tilde{f}), \partial U) > 2\delta$.

Choose $\chi \in C^\infty(\widetilde{M})$, such that

$$\chi(\tilde{m}) = \begin{cases} 1 & \text{if dist(Supp}(\tilde{f}), \tilde{m}) \leq \delta \\ \\ 0 & \text{if dist(Supp}(\tilde{f}), \tilde{m}) \geq 2\delta. \end{cases}$$

Now choose smooth functions λ_1 and λ_2 on $\mathbf{R}$ such that λ_1 is in the Schwartz space, $Supp(\hat{\lambda}_0) \subset (-\delta, \delta)$ (where $\hat{\lambda}_0$ is the Fourier transform of λ_0) and $\lambda_0 + \lambda_1 = t(1+t^2)^{-1/2}$.

Then we have the following basic facts:

(1) $\lambda_1(D)$ and $[\lambda_0(D), M_f]$ are in $\mathcal{K}(L^2(E))$ for any $f \in C^\infty(M)$;

(2) $\lambda_1(\widetilde{D}) \in \mathcal{C} = \mathcal{K}(\mathcal{E})$ (cf. [53], [9]);

(3) $\lambda_1(\widetilde{D}) M_{\tilde{f}} = M_\chi \lambda_1(\widetilde{D}) M_{\tilde{f}} = M_\chi p^*(\lambda_1(D) M_f)$ on $C_c^\infty(\widetilde{E}|_{\widetilde{M}\times v})$ for each $v \in V$ (see (2.25) and (2.26) in [9]);

(4) $T_h \circ (\lambda_0(D) \otimes 1) = T_{\chi g} \circ (M_f \lambda_0(D) \otimes 1)$; and

(5) $\lambda_0(\widetilde{D}) \circ T_h = T_{g\chi} \circ (\lambda_0(D) M_f \otimes 1)$.

(1) is well-known. We now check (5) in detail, the proof for (4) is similar. Note that both sides of (5) are bounded morphisms from the Hilbert $\mathcal{A}$-module

$\mathcal{E}_2$ to $\mathcal{E}$. Therefore, it suffices to check (5) on $C^\infty(E)\otimes 1$. For any $\xi \in C^\infty(E)$, we have:

$$
\begin{aligned}
&\lambda_0(\widetilde{D}) \circ T_h(\xi \otimes 1) = \lambda_0(\widetilde{D})(h \otimes \xi \otimes 1) \qquad \text{(by def. of } T_h) \\
= \;& \lambda_0(\widetilde{D})(M_h \xi^{\Uparrow}) \qquad\qquad\qquad \text{(by the map } \Theta) \\
= \;& \lambda_0(\widetilde{D})(M_{\tilde{f}g}\xi^{\Uparrow}) = M_g \lambda_0(\widetilde{D})(M_{\tilde{f}}\xi^{\Uparrow}),
\end{aligned}
$$

which, by (3) and the fact that $\tilde{f}\cdot\gamma(\tilde{f}) \equiv 0$ if $1 \neq \gamma \in \Gamma$, implies that

$$
\begin{aligned}
\lambda_0(\widetilde{D}) \circ T_h(\xi \otimes 1) &= M_{g\chi}(\lambda_0(D)M_f\xi)^{\Uparrow} \\
&= (T_{g\chi} \circ (\lambda_0(D)M_f \otimes 1))(\xi \otimes 1).
\end{aligned}
$$

This establishes (5). And we are ready to prove that $\lambda(\widetilde{D})$ is a $\lambda(D)\otimes 1$-connexion:

$$
\begin{aligned}
&\lambda(\tilde{D}) \circ T_h - T_h \circ (\lambda(D) \otimes 1) \\
\equiv \;& \lambda_0(\widetilde{D}) \circ T_h - T_h \circ (\lambda_0(D) \otimes 1) \quad \operatorname{Mod} \mathcal{K}(\mathcal{E}_2, \mathcal{E}) \qquad \text{(by (1) and (2))} \\
= \;& T_{g\chi} \circ (\lambda_0(D)M_f \otimes I) - T_{g\chi} \circ (M_f \lambda_0(D) \otimes 1) \qquad \text{(by (4) and (5))} \\
= \;& T_{g\chi} \circ ([\lambda_0(D), M_f] \otimes 1) \\
\equiv \;& 0 \qquad \operatorname{Mod} \mathcal{K}(\mathcal{E}_2, \mathcal{E}) \qquad\qquad \text{(by (1))}.
\end{aligned}
$$

This completes the proof. □

The even version of Theorem 2.11 is also true and can be proved in exactly the same way.

2.12 Remark. We believe that this theorem is closely related to the renormalization scheme in the work of DHK [26]. We plan to explore this point further in the future.

The element $[D]$ has been the focal point of index theory. In the rest of this section we shall study the KK-element $[\mathcal{V}] \in KK_0(C(W), \mathcal{B})$. One of the main points we want to make in this thesis is that this element deserves closer attention. This study also paves the road for the detailed calculations in §4.

For this purpose we choose, once and for all, a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^N$ for M such that each U_i is contractable. Then for each i, $\widetilde{M}|_{U_i} = p^{-1}(U_i) \to U_i$ is a trivial Γ-bundle over U_i, and we fix a trivialization by choosing one connected component $\widetilde{U}_i$ of $p^{-1}(U_i)$. Therefore, for each i, p maps $\widetilde{U}_i$ diffeomorphically onto U_i and $\{\widetilde{U}_i \cdot \gamma\}_\gamma$ is a partition of $p^{-1}(U_i)$, that is,

$$p^{-1}(U_i) = \cup_\gamma \widetilde{U}_i \cdot \gamma \quad \text{and} \quad \widetilde{U}_i \cdot \gamma_1 \cap \widetilde{U}_i \cdot \gamma_2 = \emptyset \quad \text{if} \quad \gamma_1 \neq \gamma_2.$$

Let $\{\varphi_i\}$ be a partition of unity on M subordinate to $\mathcal{U}$. We require that $\sqrt{\varphi_i}$ be smooth for each i (otherwise we take $\varphi_i' = \varphi_i^2 / \sum_{n=1}^N \varphi_n^2$ as the new partition of unity). Then we "partially lift" these functions by defining

$$\tilde{\varphi}_i(\tilde{m}, v) = \begin{cases} \varphi_i(p(\tilde{m})) & \text{if } \tilde{m} \in \widetilde{U}_i \\ \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, for each i, $Supp(\tilde{\varphi}_i)$ is compact and is contained in $\widetilde{U}_i \times V$; thus $\tilde{\varphi}_i \in \mathcal{H}_c$.

2.13 Lemma. *For any $h \in \mathcal{H}$,*

$$h = \sum_i \sqrt{\tilde{\varphi}_i} * \ll \sqrt{\tilde{\varphi}_i}, h \gg_{\mathcal{H}} .$$

Proof. Since $\{\widetilde{U}_i \cdot \gamma\}_\gamma$ is a partition of $p^{-1}(U_i)$ for each i, it is clear that $\{\gamma(\varphi_i)\}_{\gamma,i}$ is a partition of unity for $\widetilde{M} \times V$. Therefore, for any $h \in \mathcal{H}_c$, we have:

$$h = \sum_{i,\gamma} \gamma^{-1}(\tilde{\varphi}_i) \cdot h,$$

which, by the fact that $Supp(\sqrt{\tilde{\varphi}_i} \cdot \gamma(h)) \Subset \widetilde{U}_i \times V$, implies that

$$\begin{aligned} h &= \sum_{i,\gamma} \gamma^{-1}\{\sqrt{\tilde{\varphi}_i} * (\sqrt{\tilde{\varphi}_i} \cdot \gamma(h))^{\Downarrow}\} \\ &= \sum_i \sqrt{\tilde{\varphi}_i} * (\sum_\gamma (\sqrt{\tilde{\varphi}_i} \cdot \gamma(h))^{\Downarrow} \cdot \gamma) \\ &= \sum_i \sqrt{\tilde{\varphi}_i} * \ll \sqrt{\tilde{\varphi}_i}, h \gg . \end{aligned}$$

The general case follows from the continuity. □

Recall that $\mathcal{B} = C(M \times V) \rtimes_r \Gamma$. Let

$$\mathcal{B}^N = \underbrace{\mathcal{B} \oplus \mathcal{B} \oplus \cdots \oplus \mathcal{B}}_{N\, times}.$$

Then $\mathcal{B}^N$ becomes a trivial Hilbert module over $\mathcal{B}$ in a canonical way (cf. [10]). There is a Hilbert module morphism $\Theta_1 : \mathcal{H} \to \mathcal{B}^N$, defined as follows: for any $h \in \mathcal{H}$,

$$\Theta_1(h) = (\ll \sqrt{\tilde{\varphi}_1}, h \gg, \cdots, \ll \sqrt{\tilde{\varphi}_N}, h \gg).$$

2.14 Lemma. *Θ_1 is isometric and $\mathcal{H}$ is a finitely generated projective module over $\mathcal{B}$.*

Proof. For any $h, h' \in \mathcal{H}_c$:

$$\ll h, h' \gg \;=\; \ll \underbrace{\sum_{i=1}^N \sqrt{\tilde{\varphi}_i} * \ll \sqrt{\tilde{\varphi}_i}, h \gg}_{h}, h' \gg \qquad \text{(by Lemma 2.13)}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \ll \sqrt{\tilde{\varphi}_i}, h \gg^{*} \cdot \ll \sqrt{\tilde{\varphi}_i}, h' \gg \\
&= \ll \Theta_1(h), \Theta_1(h') \gg_{\mathcal{B}^N} .
\end{aligned}
$$

Therefore, Θ_1 is isometric and $\mathcal{H}$ can be thought of as a submodule of $\mathcal{B}^N$. On the other hand, there is a natural projection Θ_2 from $\mathcal{B}^N$ onto $\mathcal{H}$:

$$\Theta_2(b_1, \cdots, b_N) = \sum_{i=1}^{N} \sqrt{\tilde{\varphi}_i} * b_i.$$

Therefore, $\mathcal{H}$ is a finitely generated projective module over $\mathcal{B}$. □

Under the standard basis for $\mathcal{B}^N$, Θ_2 is given by a matrix $\theta = (\theta_{ij})_{N\times N}$ in $M_N(\mathcal{B})$, where $\theta_{ij} = \ll \sqrt{\tilde{\varphi}_i}, \sqrt{\tilde{\varphi}_j} \gg$. It is easy to check that $\theta = \theta^* = \theta^2$. Clearly $[\theta]$ can be identified with $[1_W] \otimes [\mathcal{V}]$.

2.15 Example. We now consider a very simple yet interesting example. Let $V = \{*\}$ consist of one single point, $M = \mathbf{T} = \mathbf{R}/\mathbf{Z}$ the unit circle, and $\Gamma = \pi_1(\mathbf{T}) \cong \mathbf{Z}$. We take $U_1 = (\frac{1}{6}, \frac{5}{6})/\mathbf{Z}$, $U_2 = (-\frac{1}{3}, \frac{1}{3})/\mathbf{Z}$ as an open cover for $\mathbf{T}$ and take any partition of unity $\{\varphi_1, \varphi_2\}$ subordinate to this cover. Set $\widetilde{U}_1 = (\frac{1}{6}, \frac{5}{6})$ and $\widetilde{U}_2 = (-\frac{1}{3}, \frac{1}{3})$, and then define $\tilde{\varphi}_i$ accordingly. Then an easy calculation shows that:

$$\theta = \begin{pmatrix} \varphi_1 & \sqrt{\varphi_1\varphi_2} \cdot (\chi_{(0,1/2)} + \chi_{(1/2,1)} \cdot u) \\ \sqrt{\varphi_1\varphi_2} \cdot (\chi_{(0,1/2)} + u^{-1} \cdot \chi_{(1/2,1)}) & \varphi_2 \end{pmatrix},$$

where $\chi_{(a,b)}$ is the characteristic function on $(a,b)/\mathbf{Z} \subseteq \mathbf{T}^1$ and $u = 1 \in \mathbf{Z}$. It can be shown that $[1]$ and θ generate $K_0(C(\mathbf{T}) \otimes C_r^*(\mathbf{Z})) \cong K^0(\mathbf{T}^2) \cong \mathbf{Z}^2$, where $[1]$ corresponds to the trivial line bundle on $\mathbf{T}^2$, the two-torus.

2.16 Remark. In general, since θ is actually in $M_N(C(M) \otimes C_r^*(\Gamma))$, it is clear that $[D_{\Gamma,V}]([1_W])$, which (by Proposition 2.5) equals $Ind_{MF}(D, \Gamma, V)$, lies in the image of the natural inclusion map: $K^*(C_r^*(\Gamma)) \hookrightarrow K^*(\mathcal{A})$. However, by introducing the Γ action on the manifold V, we gain at least two important advantages. We can pair $[D_{\Gamma,V}]$ with more interesting elements in $K^*(W)$, which is crucial for the work of Douglas, Hurder and Kaminker [26]. And we can pull back cyclic cocycles, for example cocycles corresponding to secondary classes, on smooth subalgebras of $\mathcal{A}$ to get new, interesting cocycles on smooth subalgebras of $C_r^*(\Gamma)$ (cf. [18]).

From Lemma 2.14, the representation $\Phi : C(W) \to \mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ can be rewritten as a C^* -algebra homomorphism $\Phi : C(W) \to M_N(\mathcal{B})$. To make this more precise, we need to define a "partial transfer" operation, which depends upon the map ϕ_i defined in the following diagram to make it commute:

$$\begin{array}{ccc} \widetilde{U}_i \times V & = & \widetilde{U}_i \times V \\ {\scriptstyle p_1}\downarrow{\scriptstyle \cong} & & {\scriptstyle p_0}\downarrow{\scriptstyle \cong} \\ \widetilde{U}_i \times_\Gamma V & \xrightarrow{\phi_i} & U_i \times V. \end{array}$$

We then define, $\Phi_i(f) = (\phi_i^{-1})^*(f)$ for any $f \in C(W)$. In other words, given any function f on W, we first lift it to $\widetilde{M} \times V$ and restrict it to $\widetilde{U}_i \times V$, then push forward to get a function $\Phi_i(f)$ on $U_i \times V$. The same definition works for differential forms and we have $\Phi_i : \Omega^*(W) \to \Omega^*(U_i \times V)$ as a morphism between two graded differential algebras and $\varphi_i \Phi_i(\omega) \in \Omega^*(M \times V)$ for any $\omega \in \Omega^*(W)$.

From now on in this thesis we require that the cover $\mathcal{U}$ for M be good. Recall ([11]) that a cover $\mathcal{U} = \{U_i\}$ is good if for any multi-index $i_1, i_2, \cdots,$

i_k, the set $\cap_{j=1}^k U_{i_j}$ is either empty or contractible. A finite good (open) cover exists on any closed manifold. It follows that if $U_i \cap U_j \neq \emptyset$, then there exists a unique element $\gamma \in \Gamma$ such that $(\widetilde{U}_i \cdot \gamma) \cap \widetilde{U}_j \neq \emptyset$. Denote this element by γ_{ij}. Then $\{\gamma_{ij}\}$ satisfies the cocycle condition:

$$\gamma_{ij} \cdot \gamma_{jk} = \gamma_{ik} \text{ if } U_i \cap U_j \cap U_k \neq \emptyset.$$

2.17 Lemma. *(1) If $h_1, h_2 \in \mathcal{H}_c$, $Supp(h_1) \subset \widetilde{U}_i \times V$ and $Supp(h_2) \subset \widetilde{U}_j \times V$, then for any $f \in C(W)$:*

$$\ll h_1, \Phi(f) h_2 \gg = \begin{cases} h_1^{\Downarrow} \cdot \Phi_i(f) \cdot \gamma_{ij} \cdot h_2^{\Downarrow} & \text{if } U_i \cap U_j \neq \emptyset \\ \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\Phi(f) = (\Phi_{ij})_{N \times N}$ is given by:

$$\Phi_{ij}(f) = \sqrt{\varphi_i} \cdot \Phi_i(f) \cdot \sqrt{\varphi_j} \cdot \gamma_{ij}.$$

(2) For any function f on W, if $U_i \cap U_j \neq \emptyset$, then on $(U_i \cap U_j) \times V$:

$$\gamma_{ij}(\Phi_j(f)) \equiv \Phi_i(f).$$

The same is true for differential forms.

(3) For any Γ invariant form $\omega \in \Omega^*(M \times V)$, let $\Phi^*(\omega) = \sum_i \phi_i^*(\varphi_i \omega)$. Then $p_\Gamma^*(\Phi^*(\omega)) = p_0^*(\omega)$ and $\Phi^*(\omega)$ is the only form satisfying this condition.

Proof. (1) follows from a straightforward calculation. Note that Γ acts trivially on $(\sqrt{\varphi_i})^{\Downarrow} = \sqrt{\varphi_i}$. Each $\sqrt{\varphi_i}$ is a function on M, but here and in what follows, we use the same notation to denote a function on M and its pull-back function on $M \times V$ via the map q_0. Also note that if $U_i \cap U_j = \emptyset$, then $\Phi_{ij}(f) = 0$. Although in this case γ_{ij} is not defined, we will still write:

$$\Phi_{ij}(f) = \sqrt{\varphi_i} \cdot \Phi_i(f) \cdot \sqrt{\varphi_j} \cdot \gamma_{ij},$$

since in this case $\varphi_i \cdot \varphi_j = 0$ anyway.

(2) It follows from the following commutative diagram:

$$\begin{array}{ccccccc}
(U_i \cap U_j) \times V & \xrightarrow{\text{inclusion}} & U_i \times V & \xrightarrow{p_0^{-1}} & \widetilde{U}_i \times V & \xrightarrow{p_1} & \widetilde{M} \times_\Gamma V \\
\Big\downarrow{\scriptstyle \gamma_{ij}} & & & & & & \Big\downarrow{\text{identity}} \\
(U_i \cap U_j) \times V & \xrightarrow{\text{inclusion}} & U_j \times V & \xrightarrow{p_0^{-1}} & \widetilde{U}_j \times V & \xrightarrow{p_1} & \widetilde{M} \times_\Gamma V.
\end{array}$$

That is, if $m \in U_i \cap U_j$, then there is a unique $\tilde{m} \in \widetilde{U}_i$ such that $p(\tilde{m}) = m$. It follows that $\tilde{m}\gamma_{ij} \in \widetilde{U}_j$. Therefore, for any $v \in V$,

$$\gamma_{ij}(\Phi_j(f))(m, v) = \Phi_j(f)(m, \gamma_{ij}^{-1}v)$$

$$= f(p_1(\tilde{m}\gamma_{ij}, \gamma_{ij}^{-1}v)) = f(p_1(\tilde{m}, v)) = \Phi_i(f)(m, v).$$

(3) If Ω is a Γ-invariant differential form on $M \times V$, then $P_0^*(\Omega)$ is a Γ-invariant form on $\widetilde{M} \times V$, which can be pushed down to get a differential form on W. The content of (3) is to identify this form using local data. To prove it, note that by the definition of ϕ_i, we have $p_0 = \phi_i \circ p_1$ on $\widetilde{U}_i \times V$, from which the conclusion follows readily. □

For the calculation in §4, another representation Ξ of $C(W)$ on $M_N(\mathcal{A})$ seems more convenient. It corresponds to another realization of $\mathcal{H}$ as a direct summand of $\mathcal{B}^N$. (As far as K-theory is concerned, we are free to do this.)

To construct this, we need a family of auxiliary functions on M, which are basically smooth variations of characteristic functions on $Supp(\varphi_i)$. So for each i, χ_i is a smooth function on M, $0 \leq \chi_i \leq 1$, $Supp(\chi_i) \subset U_i$, and $\chi_i \equiv 1$ on $Supp(\varphi_i)$. And again we will partially lift these functions to get elements in $\mathcal{H}_c$:

$$\tilde{\chi}_i(\tilde{m}, v) = \begin{cases} \chi_i(p(\tilde{m})) & \text{if } \tilde{m} \in \widetilde{U}_i \\ \\ 0 & \text{otherwise.} \end{cases}$$

The construction and main properties of the new representation Ξ are summarized in the following lemma:

2.18 Lemma. *(1) For any $h \in \mathcal{H}$,*

$$h = \sum_{i=1}^{N} \tilde{\chi}_i * \ll \tilde{\varphi}_i, h \gg .$$

(2) The map $\Theta_3 : \mathcal{H} \to \mathcal{B}^N$, given by:

$$\Theta_3(h) = (\ll \tilde{\varphi}_1, h \gg, \cdots, \ll \tilde{\varphi}_N, h \gg),$$

is an embedding of $\mathcal{H}$ into $\mathcal{B}^N$ as a direct summand.

(3) The corresponding representation Ξ of $C(W)$ on $\mathcal{B}^N$, is given by $\Xi(f) = (\Xi_{ij}(f))_{N\times N}$ where $\Xi_{ij}(f) = \varphi_i \cdot \Phi_i(f) \cdot \chi_j \cdot \gamma_{ij}$.

Proof. (1) It is easy to check that $\ll \tilde{\varphi}_i, h \gg = \sqrt{\varphi_i} \cdot \ll \sqrt{\tilde{\varphi}_i}, h \gg$, and therefore,

$$\sum_{i=1}^{N} \tilde{\chi}_i * \ll \tilde{\varphi}_i, h \gg = \sum_{i=1}^{N} \tilde{\chi}_i * (\sqrt{\varphi_i} \cdot \ll \sqrt{\tilde{\varphi}_i}, h \gg)$$

$$= \sum_{i=1}^{N} (\tilde{\chi}_i * \sqrt{\varphi_i}) * \ll \sqrt{\tilde{\varphi}_i}, h \gg = \sum_{i=1}^{N} \sqrt{\tilde{\varphi}_i} * \ll \sqrt{\tilde{\varphi}_i}, h \gg,$$

which, by Lemma 2.13, equals h.

(2) For any $h \in \mathcal{H}$:

$$\begin{aligned}
\ll \Theta_3(h), \Theta_3(h) \gg &= \sum_{i=1}^{N} \ll \tilde{\varphi}_i, h \gg^* \cdot \ll \tilde{\varphi}_i, h \gg \\
&= \sum_{i=1}^{N} \ll \sqrt{\tilde{\varphi}_i}, h \gg^* \cdot \varphi_i \cdot \ll \sqrt{\tilde{\varphi}_i}, h \gg \\
&\leq \sum_{i=1}^{N} \ll \sqrt{\tilde{\varphi}_i}, h \gg^* \cdot \ll \sqrt{\tilde{\varphi}_i}, h \gg \qquad (\text{since } \varphi_i \leq 1) \\
&= \ll h, h \gg_{\mathcal{B}} .
\end{aligned}$$

Therefore, $\|\Theta_3(h)\| \leq \|h\|$ for any h. Furthermore, it has a partial inverse: $\Theta_4 : \mathcal{B}^N \to \mathcal{H}$ defined by

$$\Theta_4(b_1, b_2, \cdots, b_N) = \sum_{i=1}^{N} \tilde{\chi}_i * b_i.$$

And we have

$$\begin{aligned}
\|\Theta_4(b_1, b_2, \cdots, b_N)\| &= \|\sum_{i=1}^{N} \tilde{\chi}_i * b_i\| \\
&\leq \sum_{i=1}^{N} \|\tilde{\chi}_i * b_i\| \leq \sum_{i=1}^{N} \|\tilde{\chi}_i\| \cdot \|b_i\| \\
&= \sum_{i=1}^{N} \|b_i\| \qquad (\text{since } \|\tilde{\chi}_i\| = 1\ ) \\
&\leq N \cdot \|(b_1, b_2, \cdots, b_N)\|.
\end{aligned}$$

Since $\Theta_4 \circ \Theta_3 = 1_{\mathcal{H}}$, we have $\|\Theta_3\| \geq N^{-1}$ and Θ_3 has closed range. Note also that Θ_3 and Θ_4 are $\mathcal{B}$-module morphisms and $\mathcal{B}^N = \Theta_3(\mathcal{H}) \oplus Ker(\Theta_4)$. This proves (2).

(3) By Lemma 2.21:

$$\Xi_{ij}(f) = \ll \tilde{\varphi}_i, M_{f\Uparrow}(\tilde{\chi}_j) \gg = \varphi_i \Phi_i(f) \chi_j \cdot \gamma_{ij}.$$

$\square$

Note that Ξ is a bounded representation but not a *-representation.

§3 Extending some cyclic cocycles on $C^\infty(V) \rtimes_{alg} \Gamma$

In this section we shall review the construction of some cyclic cocycles on $\mathcal{A}_c = C^\infty(V) \rtimes_{alg} \Gamma$ from closed Γ-invariant differential forms on V and group cocycles for Γ (see [49], [34] for more discussions). We then generalize some ideas in Jolissaint [36] (see also [19] and [34]) to show that, under certain conditions, these cocycles can be extended to some smooth subalgebras of $\mathcal{A} = C(V) \rtimes_r \Gamma$. In §4 we shall calculate the pairing of these (extended) cyclic cocycles with $[D_{\Gamma,V}]$.

3.1 Construction of the cyclic cocycle $\tau_{\omega,\rho}$ on $\mathcal{A}_c$. Recall that the group cohomology [12] of Γ is the homology of the following cochain complex:

$$0 \longrightarrow \mathbf{C} \xrightarrow{\delta} C^1(\Gamma) \xrightarrow{\delta} C^2(\Gamma) \xrightarrow{\delta} ...,$$

where $C^n(\Gamma)$ is the set of all maps $\rho : \underbrace{\Gamma \times \Gamma \times ... \times \Gamma}_{n+1} \to \mathbf{C}$ which are both Γ-invariant and antisymmetric, and where

$$(\delta\rho)(\gamma_0, \gamma_1, ..., \gamma_{n+1}) = \sum_{i=0}^{n+1}(-1)^i \rho(\gamma_0, \gamma_1, ..., \hat{\gamma}_i, ..., \gamma_{n+1})$$

for any $\gamma_i \in \Gamma$. Here, as usual, $\hat{\gamma}_i$ means deleting γ_i. By definition, ρ is Γ-invariant if

$$\rho(\gamma\gamma_0, \gamma\gamma_1, ..., \gamma\gamma_n) = \rho(\gamma_0, \gamma_1, ..., \gamma_n)$$

for any $\gamma, \gamma_i \in \Gamma$; ρ is antisymmetric if

$$\rho(\gamma_{\sigma(0)}, \gamma_{\sigma(1)}, ..., \gamma_{\sigma(n)}) = (-1)^{\sigma}\rho(\gamma_0, \gamma_1, ..., \gamma_n)$$

for any $\gamma_i \in \Gamma$ and any σ in the permutation group $\mathcal{S}_{n+1}$ of $\{0, 1, \cdots, n\}$.

There is a canonical morphism τ from this cochain complex $C^\bullet(\Gamma)$ to the cochain complex $C^\bullet(\mathbf{C}[\Gamma])$ for the cyclic cohomology of the group algebra $\mathbf{C}[\Gamma]$:

$$\tau_\rho(\gamma_0, \gamma_1, ..., \gamma_n) = \begin{cases} \rho(1, \gamma_1, \gamma_1\gamma_2, ..., \gamma_1\gamma_2...\gamma_n) & \text{if } \gamma_0\gamma_1...\gamma_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

for any $\rho \in C^n(\Gamma)$. This morphism $\rho \to \tau_\rho$ induces a map from the group cohomology of Γ to the cyclic cohomology of $\mathbf{C}[\Gamma]$. It is not surprising that if ρ is a group cocycle, then τ_ρ enjoys some interesting antisymmetry properties. We shall be more specific about this in §4.

Similarly, any closed differential form ω on V produces, in a canonical way, a cyclic cocycle on $C^\infty(V)$, as follows:

$$\tau_\omega(g_0, g_1, ..., g_m) = \int_V g_0 dg_1 ... dg_m \wedge \omega$$

for any $g_i \in C^\infty(V)$, where $m = dim(V) - deg(\omega)$. Therefore, given any group cocycle ρ and any closed differential form ω on V, we can construct a cup-product $\tau_\omega \# \tau_\rho$ on $C^\infty(V) \otimes_{alg} \mathbf{C}[\Gamma]$ (see §1.8). Then a multilinear functional

τ on $(\mathcal{A}_c)^{\otimes(m+n+1)}$ can be defined as follows: for any $g_0, g_1, ..., g_{m+n} \in C^\infty(V)$ and any $\gamma_0, \gamma_1, ..., \gamma_{m+n} \in \Gamma$, let

$$\tau(g_0\gamma_0, g_1\gamma_1, ..., g_{m+n}\gamma_{m+n}) = \tau_\omega \# \tau_\rho(h_0 \otimes \gamma_0, h_1 \otimes \gamma_1, ..., h_{m+n} \otimes \gamma_{m+n}),$$

where

$$h_0 = g_0, \quad h_1 = \gamma_0(g_1), \quad \cdots, \quad h_k = \gamma_0\gamma_1 \cdots \gamma_{k-1}(g_k) \text{ for any } k > 0.$$

When ω is Γ-invariant, it is straightforward to check that the τ thus defined is a cyclic $(m+n)$-cocycle on $\mathcal{A}_c$. We shall denote this cyclic cocycle by $\tau_{\omega,\rho}$ when it is desirable to specify ω and ρ.

However, as we have discussed in §1.9, to do the pairing with K-theory, we have to extend $\tau_{\omega,\rho}$ to a suitable smooth subalgebra of the reduced crossed product $\mathcal{A}$. The purpose of this section is to generalize some ideas in [36] (cf. also [19] [34]) to show that, under certain conditions (see (3.5), (3.10) and (3.11)), the cyclic cocycle $\tau_{\omega,\rho}$ can be extended to a smooth subalgebra of $\mathcal{A}$. In §4, we shall calculate the pairing of the extended cyclic cocycle with the index map $[D_{\Gamma,V}]$ defined in §2.4.

3.2 Closable derivations and smooth subalgebras. We now review a few well-known facts concerning closable operators and closable derivations.

Let ∇ be a closable operator on a Banach space B and $\bar{\nabla}$ be its closure. Then by definition,

$$Dom(\bar{\nabla}) = \{b \in B : \exists b_n \in Dom(\nabla), b_n \to b, \text{and } \{\nabla(b_n)\} \text{ is a Cauchy sequence}\}.$$

In other words, $Dom(\bar{\nabla})$ is the completion of $Dom(\nabla)$ under the graph norm p:

$$p(b) = \|b\| + \|\nabla(b)\|.$$

Furthermore, if ∇ is densely defined, then a derivation D can be defined on $\mathcal{B}(B)$, the algebra of bounded linear operators on B, as follows: Let $Dom(D)$ be the set of all operators $S \in \mathcal{B}(B)$ such that $S(Dom(\nabla)) \subseteq Dom(\nabla)$ and $\nabla S - S\nabla$ extends to an element of $\mathcal{B}(B)$; and for any $S \in Dom(D)$, let $D(S) \in \mathcal{B}(B)$ be the extension of $\nabla S - S\nabla$. Then D is a closable derivation on $\mathcal{B}(B)$. If $\bar{D}$ is its closure, then $Dom(\bar{D})$ is the completion of $Dom(D)$ under the graph norm:

$$p(S) = \|S\| + \|D(S)\|.$$

3.3 Convention: ordered product. *In the rest of this thesis, in particular in §4, we shall use the following convention: If $\{D_i\}$ is a sequence of elements in a (non-commutative) unital algebra, and if I is a finite subset of $\mathbb{Z}$, then we define:*

$$D_I \stackrel{\text{def}}{=} \prod_I D_i \stackrel{\text{def}}{=} \begin{cases} 1 & \textit{if } I = \emptyset \\ \\ D_{i_1} \cdot D_{i_2} \cdots D_{i_k} & \textit{if } I = \{i_1, ..., i_k\} \textit{ and } i_1 < i_2 < ... < i_k. \end{cases}$$

Now let $\mathcal{A}$ be a unital C^*-algebra, and let $D_1, D_2, \ldots, D_n$ be closed derivations on $\mathcal{A}$. We always assume that each $Dom(D_i)$ is a unital subalgebra of $\mathcal{A}$. Let

$$\mathcal{D} = \bigcap_{I \subseteq \{1,2,...,n\}} Dom(D_I).$$

Then we have the following known fact (compare Theorem 1.2 in [34]):

3.4 Proposition. *$\mathcal{D}$ is a unital Banach algebra under the graph norm:*

$$p(a) = \sum_{I \subseteq \{1,2,...,n\}} \|D_I(a)\|_{\mathcal{A}},$$

and it is a smooth subalgebra of its closure in $\mathcal{A}$.

Proof. In this proof, "an index set I" means a subset I of $\{1, 2, ..., n\}$. Let us first show that $\mathcal{D}$ is complete under the graph norm. Indeed, if a sequence $\{a_i\}$ is Cauchy with respect to the graph norm, then for each index set I, there is an element b_I in $\mathcal{A}$, such that:

$$\lim_{i \to \infty} \|D_I(a_i) - b_I\|_{\mathcal{A}} = 0.$$

By induction it is easy to see that $b_\emptyset \in \mathcal{D}$ and $b_I = D_I(b_\emptyset)$. So $\{\mathcal{D}, p\}$ is a Banach space.

It is in fact a Banach algebra. Obviously $1 \in \mathcal{D}$ and $p(1) = 1$. If a, b are in $\mathcal{D}$, we have, for any index set I, that

$$D_I(a \cdot b) = \sum_{J \subseteq I} D_J(a) D_{I \setminus J}(b).$$

Therefore,

$$\begin{aligned} p(a \cdot b) &= \sum_I \|D_I(a \cdot b)\|_{\mathcal{A}} \\ &= \sum_I \| \sum_{J \subseteq I} D_J(a) \cdot D_{I \setminus J}(b)\|_{\mathcal{A}} \\ &\leq \sum_I \sum_J \|D_I(a)\|_{\mathcal{A}} \cdot \|D_J(b)\|_{\mathcal{A}} = p(a) \cdot p(b). \end{aligned}$$

This shows that $\mathcal{D}$ is a unital Banach algebra.

Now we follow a standard way to show that $\mathcal{D}$ is smooth in its closure in $\mathcal{A}$. Suppose a is an element in $\mathcal{D}$ and $\|a\|_{\mathcal{A}} < 1$. We need to show that $1-a$ is invertible in $\mathcal{D}$. For this purpose, we note that for any index set I, and for any integer N bigger than n, we have:

$$D_I(a^N) = \sum_{J_1,\ldots,J_N \text{ a partition of } I} D_{J_1}(a) \cdot D_{J_2}(a) \cdot \cdots \cdot D_{J_N}(a),$$

where $J_i = \emptyset$ for at least $N - |I|$ of the set $\{J_i\}$. Hence,

$$\begin{aligned} \|D_I(a^N)\|_{\mathcal{A}} &\leq \sum_{J_1,\ldots,J_N \text{ a partition of } I} \|D_{J_1}(a)\| \cdot \|D_{J_2}(a)\| \cdots \|D_{J_N}(a)\| \\ &\leq \sum_{J_1,J_2,\cdots,J_N} p(a)^k \cdot \|a\|^{N-k} = (Np(a))^k \cdot \|a\|^{N-k}. \end{aligned}$$

Therefore,

$$\sum_{N=0}^{\infty} \|D_I(a^N)\| \leq \sum_{N=0}^{\infty} (Np(a))^k \cdot \|a\|^{N-k} < \infty,$$

and

$$\sum_{N=0}^{\infty} p(a^N) < \infty.$$

So $\sum_{N=0}^{\infty} a^N$ converges in $\mathcal{D}$ in p-norm, and $(1-a)^{-1} \in \mathcal{D}$.

From this it is routine to check that for any $a \in \mathcal{D}, \sigma_{\mathcal{D}}(a) = \sigma_{\bar{\mathcal{D}}}(a)$, and $\mathcal{D}$ is closed under the holomorphic functional calculus (see, for example, [34]).

To show that $\mathcal{D}$ is smooth in $\bar{\mathcal{D}}$, consider, for each integer $m > 0$, a new derivation:

$$\tilde{D}_k(a_{ij}) = (D_k(a_{ij})) \text{ on } \mathcal{A} \otimes M_m(C)$$

and repeat the previous argument. $\square$

Of course these D_i's do not have to be different. And by taking intersection, one can restate Proposition 3.4 in a slightly more general form. In particular, taking $D_n = D$ for any $n \in \mathbb{N}_+$, one recovers Theorem 1.2 in [34].

In the rest of this thesis, we assume that the following condition holds.

3.5 Condition. *V is a closed oriented Riemannian manifolds, and the action of Γ on V is isometric and orientation-preserving.*

We now represent $\mathcal{A} = C(V) \rtimes_r \Gamma$ on a Hilbert space H_Γ and construct two derivations on $\mathcal{B}(H_\Gamma)$. Let:

$\mathcal{M}^0 = C^\infty(V)$ be the space of smooth functions on V;

$\mathcal{M} = \Omega^1(V)$ be the space of smooth 1-forms on V, which is a bi-module over $\mathcal{M}^0$ in a canonical way;

$\mathcal{M}^n = \underbrace{\mathcal{M} \otimes_{\mathcal{M}^0} \mathcal{M} \otimes_{\mathcal{M}^0} \cdots \otimes_{\mathcal{M}^0} \mathcal{M}}_{\text{n times}}$; and $\mathcal{M}^\infty = \sum_{n\geq 0}^{\oplus} \mathcal{M}^n$.

Now using the Riemannian metric and the volume form on V, we get some L^2-spaces:

$$H_n = \text{the } L^2\text{-completion of } \mathcal{M}^n,$$

and

$$H = \sum_{n\geq 0}^{\oplus} H_n, \quad H_\Gamma = H \otimes_{\mathbf{C}} \ell^2(\Gamma).$$

Note that $C(V)$ can be canonically represented on H as multiplication operators, which induces a canonical faithful representation $\mathcal{T}$ of $\mathcal{A}$ on K, namely:

$$\mathcal{T}(f)(\omega \otimes e_{\gamma'}) = {\gamma'}^{-1}(f)\omega \otimes e_{\gamma'},$$

and

$$\mathcal{T}(\gamma)(\omega \otimes e_{\gamma'}) = \omega \otimes e_{\gamma^{-1}\gamma'}$$

for any $f \in C(V)$, any $\omega \in H$, and any γ, $\gamma' \in \Gamma$, where $\{e_\gamma : \gamma \in \Gamma\}$ is the standard basis for $\ell^2(\Gamma)$. This is a so-called regular representation of $\mathcal{A}$ (cf. [52]).

In fact, $\mathcal{T}$ has a natural extension. Note that with tensor product as multiplication, $\mathcal{M}^\infty$ is an algebra. The action of Γ on V induces a canonical action on $\mathcal{M}^\infty$. On the other hand, the natural tensor product on $\mathcal{M}^\infty$: $\mathcal{M}^\infty \times \mathcal{M}^\infty \xrightarrow{\otimes} \mathcal{M}^\infty$ induces a canonical representation of $\mathcal{M}^\infty$ on H. The corresponding regular representation of $\mathcal{M}^\infty \rtimes_{alg} \Gamma$ on the Hilbert space H_Γ extends the regular representation $\mathcal{T}$ of $\mathcal{A}$. We shall denote this extension again by $\mathcal{T}$.

We now start to construct derivations on $\mathcal{B}(H_\Gamma)$. The first derivation comes from the canonical connection ∇_1 on the cotangent bundle of V,

$$\mathcal{M} \xrightarrow{\nabla_1} \mathcal{M} \otimes_{\mathcal{M}^0} \mathcal{M},$$

which is dual to the Levi-Civita connection on the tangent bundle. It also has a natural extension:

$$\mathcal{M}^n \xrightarrow{\nabla_1} \mathcal{M}^{n+1}$$

for any $n \geq 0$, which satisfies the Leibnitz rule:

$$\nabla_1(\omega_1 \otimes \omega_2) = \nabla_1(\omega_1) \otimes \omega_2 + \omega_1 \otimes \nabla_1(\omega_2).$$

Note that when $n = 0$, $\nabla_1 : \mathcal{M}^0 \to \mathcal{M}^1$ is by definition the exterior derivative d on V. Moreover, since Γ acts on V isometrically, ∇_1 is Γ-invariant. In other words, the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{M}^n & \xrightarrow{\nabla_1} & \mathcal{M}^{n+1} \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\
\mathcal{M}^n & \xrightarrow{\nabla_1} & \mathcal{M}^{n+1}
\end{array}
$$

for any $\gamma \in \Gamma$.

∇_1 is a closable operator on H, and we extend it to a closable operator on H_Γ by declaring:

$$\nabla_1(\omega \otimes e_\gamma) = \nabla_1(\omega) \otimes e_\gamma.$$

Let D_1 be the closable derivation associated to ∇_1 (see §3.2), and $\bar{D}_1$ be its closure.

Since ∇_1 is Γ-invariant and satisfies the Leibnitz rule, we have the following easy facts:

3.6 Lemma. *(1) For any $\gamma \in \Gamma$, $\mathcal{T}(\gamma) \in Dom(D_1)$, and $D_1(\mathcal{T}(\gamma)) = 0$.*

(2) For any $\omega \in \mathcal{M}^\infty$, $\mathcal{T}(\omega) \in Dom(D_1)$, and $D_1(\mathcal{T}(\omega)) = \mathcal{T}_{\nabla_1(\omega)}$.

(3) $\mathcal{T}(\mathcal{M}^\infty \rtimes_{alg} \Gamma) \subseteq Dom(D_1^k)$ for any k.

Proof. Straightforward calculations. □

The second derivation D_2 comes from the group Γ and the construction is very much the same as that of D_1. (Cf. [17, 34].) We first fix a length function (cf. [36], [34]), called ι, on Γ. This is a nonnegative function on Γ satisfying $\iota(1) = 0$ and the triangular inequality $\iota(\gamma_1 \cdot \gamma_2) \leq \iota(\gamma_1) + \iota(\gamma_2)$. It can be used to define an operator ∇_2 on H_Γ, as follows:

$$\nabla_2(\omega \otimes e_\Gamma) = \iota(\gamma) \cdot \omega \otimes e_\gamma$$

Let D_2 be the closable derivation associated to ∇_2, and $\bar{D}_2$ be its closure. Then we have:

3.7 Lemma. *$\mathcal{T}(\mathcal{M}^\infty \rtimes_{alg} \Gamma) \subseteq Dom(D_2^k)$ for any k. And for any $\omega \in \mathcal{M}^\infty$ and any $\gamma \in \Gamma$,*

$$D_2^k(\mathcal{T}(\omega \cdot \gamma)) = \mathcal{T}(\omega) \cdot \theta(\gamma)^k \cdot \mathcal{T}(\gamma),$$

where $\theta(\gamma)$ is the operator on H_Γ defined by:

$$\theta(\gamma)(\omega \otimes e_{\gamma'}) = [\iota(\gamma') - \iota(\gamma \cdot \gamma')] \cdot \omega \otimes e_{\gamma'}.$$

Proof. A bounded operator L on H_Γ will be called *block-diagonal*, if L maps the subspace $H \otimes e_\gamma$ into itself for each $\gamma \in \Gamma$, that is, if we can write $L = diag(L_\gamma)_{\gamma \in \Gamma}$, where $L_\gamma \in \mathcal{B}(H)$ for each $\gamma \in \Gamma$. It is easy to see that for any block-diagonal operator L on H_Γ, $L \in Dom(D_2)$ and $D_2(L) = 0$.

In particular, for any $\omega \in \mathcal{M}^\infty$, $\mathcal{T}(\omega)$ is block-diagonal. Therefore, $\mathcal{T}(\omega)$ is in $Dom(D_2)$ and $D_2(\mathcal{T}(\omega)) = 0$. On the other hand, it is straightforward to show that if $\gamma \in \Gamma$, then $\mathcal{T}(\gamma) \in Dom(D_2)$ and

$$D_2(\mathcal{T}(\gamma)) = \theta(\gamma) \cdot \mathcal{T}(\gamma).$$

Note that for any $\gamma \in \Gamma$, $\theta(\gamma)$ is block-diagonal. The lemma now follows from the Leibnitz rule. □

Now for any $f \in C^\infty(V)$, we define:

$$\|f\| = max\{|f(v)| : v \in V\},$$

and

$$\|df\| = max\{\|df(v)\| : v \in V\},$$

where $\|df(v)\|$ is the metric on T_v^*V determined by the Riemannian metric on V. And, of course, for any $\omega \in \mathcal{M}^n$, $\|\omega\|_{L^2}$ shall denote the norm of ω in the L^2-space H_n.

3.8 Global Sobolev Lemma. *For any $l > 1+\frac{1}{2}dim(V)$, there is a constant c, such that*

$$\|f\| + \|df\| \le c\sum_{i=0}^{l} \|\nabla_1^i f\|_{L^2}$$

for any $f \in C^\infty(V)$.

Proof. See [30]. Note that $\sum_{i=0}^{l} \|\nabla_1^i f\|_{L^2}$ is a norm used to define the global Sobolev space on V. □

Now let $\|\cdot\|_{k,l}$ be the graph norm in Proposition 3.4 associated to $\bar{D}_2$, $\bar{D}_2, \cdots, \bar{D}_2, \bar{D}_1, \bar{D}_1, \cdots, \bar{D}_1$ with k copies of $\bar{D}_2$ and l copies of $\bar{D}_1$. Let $\mathcal{A}_{k,l}$ be the completion of $\mathcal{A}_c$ with respect to the graph norm $\|\cdot\|_{k,l}$.

3.9 Proposition. *For any integers k, $l > 0$, $\mathcal{A}_{k,l}$ is a smooth subalgebra of $\mathcal{A}$. And for any given integer t, one can choose k, l and a constant $c > 0$ large enough, such that:*

$$\sum_\gamma (\|f_\gamma\| + \|df_\gamma\|)^2 \cdot (1 + \iota(\gamma))^{2t} \le c \cdot \|f\|_{k,l}^2$$

for any $f \in \mathcal{A}_{k,l}$.

Proof. The first statement follows directly from Proposition 3.4. On the other hand, combining Lemmas 3.6 and 3.7, we have: $\mathcal{T}(\omega\cdot\gamma) \in Dom(\bar{D}_2^k \cdot \bar{D}_1^l)$ and

$$(\bar{D}_2^k \cdot \bar{D}_1^l)(\mathcal{T}(\omega\cdot\gamma)) = \mathcal{T}(\nabla^l(\omega)) \cdot \theta(\gamma)^k \cdot \mathcal{T}(\gamma),$$

for any integers k, $l \geq 0$, and for any $\omega\cdot\gamma \in \mathcal{M}^\infty \rtimes_{alg} \Gamma$. In particular, if $f = \sum_{\gamma\in\Gamma} f_\gamma \cdot \gamma \in \mathcal{A}_c$, then

$$\begin{aligned}\|\bar{D}_2^k \bar{D}_1^l(\mathcal{T}(f))\|_{B(H_\Gamma)} &\geq \frac{1}{\sqrt{Vol(V)}} \cdot \|\bar{D}_2^k \bar{D}_1^l(f)(1\otimes e_1)\|_{H_\Gamma} \\ &= \frac{1}{\sqrt{Vol(V)}} \cdot \sqrt{\sum_\gamma \|\nabla^l(f_\gamma)\|_{L^2}^2 \cdot \iota(\gamma)^{2k}}.\end{aligned}$$

This, together with Lemma 3.8, implies that for any fixed integer t, we can always take k, l large enough such that for any f,

$$\sum_\gamma (\|f_\gamma\| + \|df_\gamma\|)^2 \cdot (1+\iota(\gamma))^{2t} \leq c\|f\|_{k,l}^2$$

for some constant c independent of f. This completes the proof. $\square$

Recall that in §3.1 a cyclic cocycle $\tau_{\omega,\rho}$ on $\mathcal{A}_c$ was constructed from a group cocycle ρ and a closed, Γ-invariant differential form ω on V. We are now ready to show that, under certain conditions, this cyclic cocycle can be extended to $\mathcal{A}_{k,l}$ for some k, l. These conditions are:

3.10 Condition. *Γ is rapidly decaying: that is, there exists a length function ι on Γ and fixed constants c and r such that for any $\sum_\gamma a_\gamma \cdot \gamma \in \mathbf{C}[\Gamma]$:*

$$\|\sum_\gamma a_\gamma \cdot \gamma\|_{C^*(\Gamma)}^2 \leq c\sum_\gamma |a_\gamma|^2 (1+\iota(\gamma))^{2r}.$$

3.11 Condition. *The group cocycle ρ is of polynomial growth, that is, there are fixed constants c and s such that:*

$$|\tau_\rho(\gamma_0, \gamma_1, \cdots, \gamma_n)| \leq c \prod_{i=1}^{n} (1 + \iota(\gamma_i))^s$$

for any $\gamma_0, \cdots, \gamma_n \in \Gamma$, where τ_ρ is defined from ρ as in §3.1 and ι is the length function as in Condition 3.10.

Keep in mind that the action of Γ on V is assumed to be isometric and orientation-preserving (cf. Condition 3.5) throughout this section and §4.

The main result in this section is the following:

3.12 Theorem. *If Conditions 3.5, 3.10 and 3.11 are satisfied, then the cyclic cocycle $\tau_{\omega,\rho}$ constructed in §3.1 can be extended to the smooth subalgebra $\mathcal{A}_{k,l}$ of $\mathcal{A}$ when k, l are large enough.*

Before the proof, we recall that each element of the reduced group C^*-algebra $C_r^*(\Gamma)$ can be written uniquely in the form $\sum_\gamma a_\gamma \cdot \gamma$, where $a_\gamma \in \mathbf{C}$, and we have:

$$|a_1| \leq \| \sum_\gamma a_\gamma \cdot \gamma \|_{C_r^*(\Gamma)}.$$

The map: $\sum_\gamma a_\gamma \cdot \gamma \to a_1$ is the so-called conditional expectation on $C_r^*(\Gamma)$. See [52].

Proof of Theorem 3.12. For the integers r in Condition 3.10 and s in Condition 3.11, we choose integers k, $l > 0$ such that the inequality in Proposition 3.9 holds for $t = r + s$. Then for any $f_i = \sum_\gamma f_{i,\gamma} \cdot \gamma \in \mathcal{A}_c$, we have

$$|\tau_{\omega,\rho}(f_0, f_1, \cdots, f_{m+n})|$$

$$\leq \sum_{\gamma_0,\cdots,\gamma_{m+n}} |\tau_{\omega,\rho}(f_{0;\gamma_0}\cdot\gamma_0, f_{1;\gamma_1}\cdot\gamma_1,\cdots,f_{m+n;\gamma_{m+n}}\cdot\gamma_{m+n})|.$$

By the definition of $\tau_{\omega,\rho}$ in §3.1, we have

$$\begin{aligned}
&|\tau_{\omega,\rho}(f_0, f_1,\cdots,f_{m+n})| \\
= &\sum_{\gamma_0,\cdots,\gamma_{m+n}} |\tau_{\omega\otimes\rho}(g_{0;\gamma_0}\otimes\gamma_0, g_{1;\gamma_1}\otimes\gamma_1,\cdots,g_{m+n;\gamma_{m+n}}\otimes\gamma_{m+n})| \\
\leq &\sum_{\gamma_0,\gamma_1,\cdots,\gamma_{m+n}} \sum_{1\leq j_1<\cdots<j_n\leq m+n} \Big|\int_V g_{o,\gamma_0}\cdot[\prod_{j\in J} g_{j;\gamma_j}]\cdot[\prod_{j\notin J} d(g_{j;\gamma_j})]\wedge\omega\Big|\cdot \\
&\cdot|\tau_\rho(\gamma_0\gamma_1\cdots\gamma_{j_1-1}(d\gamma_{j_1})\gamma_{j_1+1}\cdots\cdots)| \qquad \text{(Convention (3.9))} \\
\leq &\ const. \sum_{\gamma_0\cdot\gamma_1\cdot\gamma_{m+n}=1} \|g_{0;\gamma_0}\| \prod_{j=1}^{m+n} [(\|g_{j;\gamma_j}\| + \|d(g_{j;\gamma_j})\|)\cdot(1+l(\gamma_j))^s]. \\
= &\ const. \sum_{\gamma_0\cdot\gamma_1\cdot\gamma_{m+n}=1} \|f_{0;\gamma_0}\| \prod_{j=1}^{m+n} [(\|f_{j;\gamma_j}\| + \|d(f_{j;\gamma_j})\|)\cdot(1+l(\gamma_j))^s],
\end{aligned}$$

since the Γ action is isometric. Now we define $a_0 = \sum_\gamma \|f_{o,\gamma}\|\cdot\gamma \in \mathbf{C}[\Gamma]$, and

$$a_j = \sum_\gamma [(\|f_{j,\gamma}\| + \|d(f_{j,\gamma})\|)\cdot(1+\iota(\gamma))^s]\cdot\gamma \in \mathbf{C}[\Gamma]$$

for $j > 0$. Then

$$\sum_{\gamma_0\cdot\gamma_1\cdot\gamma_{m+n}=1} \|f_{0;\gamma_0}\| \prod_{j=1}^{m+n} [(\|f_{j;\gamma_j}\| + \|d(f_{j;\gamma_j})\|)\cdot(1+l(\gamma_j))^s]$$

is the conditional expectation of the product $a_0\cdot a_1\cdots a_{m+n} \in C_r^*(\Gamma)$. Therefore,

$$\begin{aligned}
&|\tau(f_0, f_1,\cdots,f_{m+n})| \\
\leq &\ const. \sum_{\gamma_0\cdot\gamma_1\cdot\gamma_{m+n}=1} \|f_{0;\gamma_0}\| \prod_{j=1}^{m+n} [(\|f_{j;\gamma_j}\| + \|d(f_{j;\gamma_j})\|)\cdot(1+l(\gamma_j))^s] \\
\leq &\ const.\|a_0\cdot a_1\cdots a_{m+n}\|_{C^*(\Gamma)} \\
\leq &\ const.\|a_0\|_{C^*(\Gamma)}\cdot\|a_1\|_{C^*(\Gamma)}\cdots\|a_{m+n}\|_{C^*(\Gamma)} \\
\leq &\ const.(\sum_\gamma \|f_{0;\gamma}\|^2(1+l(\gamma))^{2(t+s)})^{1/2}.
\end{aligned}$$

$$\cdot\{\prod_{j=1}^{m+n}([\|f_{j,\gamma}\| + \|d(f_{j,\gamma})\|]^2(1+l(\gamma))^{2(t+s)})^{1/2}\} \qquad \text{(by 3.10)}$$
$$\leq \; const.\|f_0\|_{k,l}\cdot\|f_1\|_{k,l}\cdots\|f_{m+n}\|_{k,l} \qquad \text{(by 3.9)}.$$

Therefore, $\tau_{\omega,\rho}$ can be extended to $\mathcal{A}_{k,l}$. □

3.13 Remark. If Γ is word hyperbolic in the sense of Gromov, then Γ is rapidly decaying and each group cohomology class of Γ has a representative ρ which is of polynomial growth (cf., for example, [19]). Therefore, if the action of Γ on V is isometric, the cyclic cocycle $\tau_{\rho,\omega}$, for that chosen representative ρ and any closed, Γ-invariant form ω on V, can be extended to a smooth subalgebra of $\mathcal{A}$.

§4 An Index Theorem

Recall, from §1.9, that any cyclic cocycle τ on a smooth subalgebra of $\mathcal{A} = C(V) \rtimes_r \Gamma$ induces a map:

$$\tau_* : K_*(\mathcal{A}) \to \mathbf{C}.$$

Furthermore, from §2 we have come to realize that it is important to calculate the composite:

$$K^*(W) \xrightarrow{[D_{\Gamma,V}]} K_*(\mathcal{A}) \xrightarrow{\tau_*} \mathbf{C},$$

where $W = \widetilde{M} \times_\Gamma V$. In this section we shall calculate this composite for those cyclic cocycles $\tau_{\omega,\rho}$ discussed in §3. The calculations we have to perform here are largely algebraic and should be useful in more general situations.

Let us remark that, although a natural way to calculate this composite would be to calculate the bivariant character of $[D_{\Gamma,V}]$ in the bivariant cyclic theory (cf. [41], [58], [59] and [50] [51]), it appears difficult to do that (see, however, [60]). To demonstrate the simplicity of our situation, in this thesis we shall work within the framework of the ordinary topological cyclic cohomology.

We assume that Γ is rapidly decreasing (cf. §3.10), the action of Γ on V is isometric and orientation-preserving, ω is a closed Γ-invariant differential form of degree $\deg \omega$ on V, and ρ is a group n-cocycle for Γ which is of polynomial growth (cf. §3.11). By Proposition 3.12, the cyclic cocycle $\tau_{\omega,\rho}$ defined in §3.1 can be extended to the subalgebra $\mathcal{A}_{k,l}$ of $\mathcal{A}$ when k, l are large enough.

The geometric operator D on M determines a K-homology element $[D]$ in $K^\sigma(M) \cong KK_\sigma(C(M), \mathbf{C})$, where σ is the parity of D. Using the Kasparov product $[D]$ induces the map:

$$K^\sigma(M) \cong KK_\sigma(\mathbf{C}, C(M)) \overset{\otimes[D]}{\to} KK_0(\mathbf{C}, \mathbf{C}) \cong \mathbf{Z}.$$

The classical Atiyah-Singer index theorem [6] and its odd analogue due to Baum-Douglas [8] have identified this map in a more concrete way:

4.1 Theorem. *There is a formal sum $Ch(D)$ of closed differential forms on M of different degrees, such that*

$$x \otimes [D] = \int_M ch^*(x) \wedge Ch(D)$$

for any $x \in K^\sigma(M)$, where $ch^*(x)$ is the Chern character of x in the de Rham cohomology of M.

4.2 Remarks. (1) In fact, the Atiyah-Singer index theorem and the Baum-Douglas index theorem are much stronger than Theorem 4.1. Besides giving better analytic interpretations of the index, they actually give the recipe to construct $Ch(D)$ from the principal symbol of D, the Thom isomorphism in topological K-theory and the characteristic classes of the tangent bundle on M. See [6], [8] for more details.

(2) $Ch(D)$, or more precisely its Poincaré dual, is the character of D in the de Rham theory of M. Its character in the cyclic cohomology of $C^\infty(M)$

has also been developed: D determines a finitely-summable Fredholm module over $C^\infty(M)$, and hence has a character in the cyclic cohomology of $C^\infty(M)$ (cf. [16] [20]), which has been calculated, at least for some special but most typical cases, by several authors (see, for example, [16], and the paper by Fox and Haskell in [40]).

Let $\tau_{Ch(D)}$ be the formal sum of cyclic cocycles on $C^\infty(M)$ corresponding to $Ch(D)$ (see §1.12), then Theorem 4.1 translates into:

4.3 Lemma. *For any $x \in K^\sigma(M)$,*

$$x \otimes [D] = < ch^*(x), \tau_{Ch(D)} > .$$

By Theorem 2.11, $[D_{\Gamma,V}]$ is the composite of the following maps:

$$K^*(W) \overset{[V]}{\to} K_*(\mathcal{B}) \overset{\otimes[D]}{\to} K_{\sigma+*}(\mathcal{A}).$$

(Recall that $\mathcal{B} \overset{\text{def}}{=} C(M \times V) \rtimes_r \Gamma = C(M) \otimes \mathcal{A}$.) Now given $\tau_{\omega,\rho}$, our first step is to calculate the composition $\tau_{\omega,\rho} \circ [D]$: $K_*(\mathcal{B}) \to \mathbf{C}$. We have the following:

4.4 Proposition. *Let $\tau_{\omega\wedge Ch(D),\rho}$ be the formal sum of cyclic cocycles on $C^\infty(M \times V) \rtimes_{alg} \Gamma$ defined as in §3.1. Then $\tau_{\omega\wedge Ch(D),\rho}$ extends to a smooth subalgebra of $\mathcal{B}$ and*

$$\tau_{\omega,\rho} \circ [D] = (-1)^{dim(M)\cdot dim(V)+\sigma\cdot(n+deg\omega)} \tau_{\omega\wedge Ch(D),\rho}$$

as maps from $K_*(\mathcal{B})$ to $\mathbf{C}$.

Proof. Note that Γ acts on M trivially. Therefore, if the action of Γ on V is isometric and orientation-preserving, so is the action of Γ on $M \times V$. This, together with conditions (3.10) and (3.11), implies that $\tau_{\omega \wedge Ch(D),\rho}$ extends to a smooth subalgebra of $\mathcal{B}$ by Theorem 3.12.

Now for any $x \in K_*(\mathcal{A})$ and any $y \in K^*(M)$, since $y \otimes [D]$ is an integer, we have

$$\begin{aligned} & \tau_{\omega,\rho}(x \otimes y \otimes [D]) = \tau_\omega(x) \cdot (y \otimes [D]) \\ = \ & \tau_\omega(x) \cdot < ch^*(y), \tau_{Ch(D)} > = < x \otimes y, \tau_{\omega,\rho} \# \tau_{Ch(D)} >, \end{aligned}$$

where the last equality follows from Theorem 1.14. Recall (see §1.14) that the Kasparov product defines a natural map:

$$K_*(\mathcal{A}) \otimes K_*(C(M)) \to K_*(\mathcal{B}).$$

It follows, for example, from Schochet's Künneth Theorem for tensor product (cf. [10]) that this map becomes an isomorphism after tensoring both sides by $\mathbf{C}$. Therefore, for any $z \in K_*(\mathcal{B})$, we have

$$\tau_{\omega,\rho}(z \otimes [D]) = < z, \tau_{\omega,\rho} \# \tau_{Ch(D)} > .$$

Then by a direct but tedious calculation, we have

$$\tau_{\omega,\rho} \# \tau_{Ch(D)} = (-1)^{dim(M) \cdot dim(V) + \sigma \cdot (n + deg\omega)} \tau_{\omega \wedge Ch(D),\rho}$$

This completes the proof. $\square$

Our next step is to calculate $\tau_{\omega \wedge Ch(D),\rho} \circ \Xi_*$, where $\Xi : C(W) \to M_N(\mathcal{B})$ is as in Lemma 2.24 and $\Xi_* : K^*(W) \to K_*(\mathcal{B})$ is the induced map in K-theory.

For this purpose we now assume that Ω is any closed, Γ-invariant form on $M \times V$. Repeating the discussion in §3.1, we construct a cyclic cocycle $\tau_{\Omega,\rho}$ on $\mathcal{B}_c = C^\infty(M \times V) \rtimes_{alg} \Gamma$. (With conditions 3.5, 3.10 and 3.11, it can be extended to a smooth algebra of $\mathcal{B}$. But we shall ignore this for awhile.) Then we define a pull-back cyclic cocycle $\Xi^*(\tau)$ on $C^\infty(W)$, as follows:

$$\Xi^*(\tau_{\Omega,\rho})(f_0, \cdots, f_k) = \tau_{\Omega,\rho}(\Xi(f_0), \cdots, \Xi(f_k))$$

for any $f_i \in C^\infty(W)$. Note that $\Xi^*(\tau_{\Omega,\rho})$ is well-defined as a cyclic cocycle on $C^\infty(W)$ since Ξ maps $C^\infty(W)$ into $M_N(\mathcal{B}_c)$.

In the rest of this section we shall seek a better formula for $\Xi^*(\tau_{\Omega,\rho})$. To formulate the result, we first review the characteristic map $\check{c}$ (cf. [31]) from the group cohomology of Γ to the Čech cohomology of M,

$$\check{c} : C^\bullet(\Gamma) \to C^\bullet(\mathcal{U}, C),$$

where the right hand side is the Čech complex associated with the cover $\mathcal{U}$. For any group cochain ρ, the corresponding Čech cochain is given by

$$\check{c}(\rho)(i_0, i_1, ..., i_n) = \rho(\gamma_{i_0 i_0}, \gamma_{i_0 i_1}, ..., \gamma_{i_0 i_n})$$

for any $i_0, i_1, ..., i_n$ such that $U_{i_0} \bigcap U_{i_1} \bigcap ... \bigcap U_{i_n} \neq \emptyset$. Note that $\check{c}(\rho)$ is antisymmetric since ρ is. Then $\check{c}$ induces a map: $H^*(\Gamma) \to \check{H}^*(M)$ which is determined by the covering structure and is independent of the specific trivialization of it. Now using an explicit collating formula and with the help of a partition of unity, we can identify Čech cohomology with de Rham cohomology(cf. [11]). For the Čech cocycle constructed above, the corresponding differential form is:

$$\omega_\rho = (-1)^n \sum_{i_0, i_1, \ldots, i_n} \varphi_{i_0} d\varphi_{i_1} \ldots d\varphi_{i_n} \rho(\gamma_{i_0 i_0}, \gamma_{i_0 i_1}, \ldots, \gamma_{i_0 i_n}).$$

To summarize, for each group cocycle ρ, we can construct a cyclic cocycle τ_ρ (see §3.1) for the group algebra $\mathbf{C}[\Gamma]$ and a differential form ω_ρ on M. They are intimately related. For later reference, we write down one formula which relates them:

4.5 Lemma. *For each i_0, the restriction of ω_ρ on U_{i_0} is given by:*

$$\omega_\rho|_{U_{i_0}} = [\sum_{i_1, \ldots, i_n = 1}^{N} d\varphi_{i_1} \ldots d\varphi_{i_n} \tau_\rho(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \gamma_{i_2 i_3}, \ldots, \gamma_{i_n i_0})]|_{U_{i_0}}.$$

Proof. A direct calculation. See [11]. □

The main result of this section is the following:

4.6 Theorem. *Suppose that ρ is a group n-cocycle of Γ and Ω is a closed, Γ-invariant differential form on $M \times V$. Let $\tau_{\Omega,\rho}$ be the cyclic cocycle on $\mathcal{B}_c$, constructed from the recipe in §3.1, then:*

$$\Xi^*(\tau_{\Omega,\rho}) = \frac{(-1)^{n(n+1)/2}}{(2\pi i)^n \cdot n!} \cdot S^n(\tau_{\Phi^*(\omega_\rho \wedge \Omega)}),$$

where Φ^* is as defined in Lemma 2.17, S is the suspension operator in cyclic cohomology (cf. §1.8 or [16]) and $\tau_{\Phi^*(\omega_\rho \wedge \Omega)}$ is the cyclic cocycle on $C^\infty(W)$ given by:

$$\tau_{\Phi^*(\omega_\rho \wedge \Omega)}(f_0, \cdots, f_k) = \int_W f_0 df_1 \cdots df_k \wedge \Phi^*(\omega_\rho \wedge \Omega).$$

Our main result of this thesis then follows immediately:

4.7 Theorem. *If the group Γ is rapidly decaying and its action on V isometric and orientation-preserving, if ω is any closed Γ-invariant differential form on V, and if ρ is a group n-cocycle of polynomial growth, then the cyclic cocycle $\tau_{\omega,\rho}$ on $\mathcal{A}_c$, constructed from ρ and ω in a canonical way (see §3.1), can be extended to a smooth subalgebra of $\mathcal{A}$, and for any $x \in K^*(W)$,*

$$< [D_{\Gamma,V}] \circ x, \tau_{\omega,\rho} >= c \cdot \int_W ch^*(x) \wedge \Phi^*(\omega_\rho \wedge \omega \wedge Ch(D)),$$

where

$$c = \frac{(-1)^{dim(M)\cdot dim(V)+\sigma\cdot(n+deg\omega)+n(n+1)/2}}{(2\pi i)^n \cdot n!}.$$

Proof of Theorem 4.7. With all these conditions, by Theorem 3.12, $\tau_{\omega,\rho}$ and $\tau_{\omega\wedge Ch(D),\rho}$ can be continuously extended to a smooth subalgebra of $\mathcal{A}$ and $\mathcal{B}$, respectively. On the other hand, $\tau_{\Phi^*(\omega_\rho\wedge\omega\wedge Ch(D))}$ is certainly a cyclic cocycle on $C^\infty(W)$.

Now for any $x \in K^*(W)$, we have,

$$\begin{aligned}
& < [D_{\Gamma,V}] \circ x, \tau_{\omega,\rho} > \\
= \; & < ([\mathcal{V}] \otimes [D]) \circ x, \tau_{\omega,\rho} > && \text{(Theorem 2.11)} \\
= \; & c_1 < [\mathcal{V}] \circ x, \tau_{\omega\wedge Ch(D),\rho} > && \text{(Prop. 4.4)} \\
= \; & c_1 < \Xi_*(x), \tau_{\omega\wedge Ch(D),\rho} >
\end{aligned}$$

$$\begin{aligned} &= c_1 < x, \Xi^*(\tau_{\omega\wedge Ch(D),\rho}) > \\ &= c \cdot \int_W ch^*(x) \wedge \Phi^*(\omega_\rho \wedge \omega \wedge Ch(D)) \qquad \text{(Theorem 4.6)} \end{aligned}$$

This proves the theorem. □

4.8 Remarks. (1) Recall that $q_1 : W = \widetilde{M} \times_\Gamma V \to M$. It is easy to show that:

$$\Phi^*(\omega_\rho \wedge Ch(D) \wedge \omega) = q_1^*(\omega_\rho \wedge Ch(D)) \wedge \Phi^*(\omega),$$

where $q_1^* : \Omega^*(M) \to \Omega^*(X)$ is the pull-back map of differential forms. $\Phi^*(\omega)$ is the push-forward (on W) of the Γ-invariant form ω on $\widetilde{M} \times V$ (cf. Lemma 2.17).

(2) When ω is a volume form on V, it induces a holonomy-invariant transverse measure for the foliation $(W, \mathcal{F})$, and the pairing in this case has been calculated in [19], [25].

(3) When V consists of one single point, Theorem 4.7 is the higher Γ-index theorem of Connes and Moscovici ([19], Theorem 5.4). See also [42]. Recently, and independently, F. Wu [60] has given another proof of the same theorem. His calculation and the one to be presented below have some overlap.

The rest of this section is devoted to the proof of Theorem 4.6. First we shall point out some antisymmetry properties of the cyclic cocycle τ_ρ, which will be used to simplify the calculation. To be more specific, it is convenient to introduce a standing convention for this section:

In the following calculation, we shall use the differential algebra notation *for cyclic cohomology (cf. §1.7 or [16]). Note that by definition:*

$$\tau\{a_0 \cdot da_1 \cdots da_k\} = \tau(a_0, \cdots, a_k).$$

Now recall that we have fixed the connecting data $\{\gamma_{ij}\}$ for the covering space in §2. We then have:

4.9 Lemma. *Given any group cocycle ρ, for any $i_0, i_1, ..., i_n \in \{1, 2, ..., N\}$:*

(1) $\tau_\rho\{\gamma_{i_0 i_1} \cdot [\prod_{j=1}^n d(\gamma_{i_j i_{j+1}})]\}$ is antisymmetric in $\{0, 1, ..., n\}$ where $i_{n+1} = i_0$.

(2) $\tau_\rho\{[\prod_{j=1}^n d(\gamma_{i_j i_{j+1}})]\} = 0$.

(3) For any integer K, any index $i_0, \cdots, i_K$, and any $\sigma_0, \cdots, \sigma_K \in \{0, 1\}$. Let $i_{K+1} = i_0$. Then for any n, the following

$$\tau_\rho\{[\prod_{j=0}^{n-1} d^{\sigma_j}(\gamma_{i_j i_{j+1}})] \cdot d(\gamma_{i_n i_{n+1}}) \cdot d(\gamma_{i_{n+1} i_{n+2}}) \cdot [\prod_{j=n+2}^{K} d^{\sigma_j}(\gamma_{i_j i_{j+1}})]\}$$

is anti-symmetric with respect to the pair (i_n, i_{n+1}). The same is true for:

$$\tau_\rho\{[\prod_{j=0}^{n-1} d^{\sigma_j}(\gamma_{i_j i_{j+1}})] \cdot d(\gamma_{i_n i_{n+1}}) \cdot \gamma_{i_{n+1} i_{n+2}} \cdot [\prod_{j=n+2}^{K} d^{\sigma_j}(\gamma_{i_j i_{j+1}})]\}.$$

Proof.

$$\begin{aligned} & \tau_\rho(\gamma_{i_0 i_1}, ..., \gamma_{i_n i_0}) \\ = \;& \rho(1, \gamma_{i_1 i_2}, \gamma_{i_1 i_3}, ..., \gamma_{i_1 i_n}) && \text{(by definition)} \\ = \;& \rho(\gamma_{i_1 i_1}, \gamma_{i_1 i_2}, \gamma_{i_1 i_3}, ..., \gamma_{i_1 i_0}) && (\gamma_{i_1 i_1} = 1\ ) \\ = \;& \rho(\gamma_{i_k i_1}, \gamma_{i_k i_2}, ..., \gamma_{i_k i_0}) && \text{(for any } k\text{, by } \Gamma \text{ invariance of } \rho) \end{aligned}$$

which is obviously antisymmetric on $\{0,1,...,n\}$ since ρ is antisymmetric. This proves (1). To prove (2), note that $\gamma_{i_1i_2}\gamma_{i_2i_3}...\gamma_{i_ni_0} = \gamma_{i_1i_0}$. If $\gamma_{i_1i_0} \neq 1$, then $\tau_\rho(1,\gamma_{i_1i_2},...,\gamma_{i_ni_0}) = 0$ by definition of τ_ρ. On the other hand, if $\gamma_{i_1i_0} = 1$, then (2) follows from (1).

(3) Since τ_ρ is a *cyclic cocycle*, it suffices to consider the following special case:

$$\tau_\rho\{[\prod_{j=0}^{n-1} d^{\sigma_j}(\gamma_{i_ji_{j+1}})]\cdot d(\gamma_{i_ni_{n+1}})\cdot d(\gamma_{i_{n+1}i_0})\}$$

But this is a linear combination of terms in (1) and (2); hence the result follows. $\square$

From now on we let $m = dim(M\times V) - deg(\Omega)$, let $\mathsf{I} = (i_0,\cdots,i_{m+n})$ be any index set such that $1\leq i_j\leq N$ for each j, and let $\mathsf{J} = (j_1,j_2,\cdots,j_n)$ be any subset of $\{1,2,\cdots,m+n\}$ such that $j_1<j_2<...<j_n$. For any given J, Λ_J denotes the characteristic function for J, $\sigma_1(\mathsf{J}) = (-1)^{m\cdot n+n(n+1)/2+j_1+\cdots+j_n}$ and $\sigma_2(\mathsf{J}) = (-1)^{m\cdot n+j_1+\cdots+j_n}$.

For convenience, we introduce some auxiliary indices: $i_{m+n+1}\stackrel{\text{def}}{\equiv} i_0$ for any given I and $j_0\stackrel{\text{def}}{\equiv}0$, $j_{n+1}\stackrel{\text{def}}{\equiv}j_1$ for any given J. And in any graded differential algebra (Ω^*,d), we define $d^0\equiv 1$, that is, $d^0(\omega)=\omega$ for any $\omega\in\Omega^*$.

For any $f_0,f_1,...,f_{m+n}\in C^\infty(W)$,

$$\Xi^*(\tau)(f_0,f_1,...,f_{m+n}) = \tau(\Xi(f_0),\Xi(f_1),...,\Xi(f_{m+n}))$$

$$=\sum_{i_0,...,i_{m+n}=1}^{N}\tau(\Xi_{i_0i_1}(f_0),\Xi_{i_1i_2}(f_1),...,\Xi_{i_{m+n}i_0}(f_{m+n})).$$

Note that each $\Xi(f_i)$ is an $N\times N$ matrix. Let

$$\mathcal{S}(\mathsf{I}) = \tau(\Xi_{i_0i_1}(f_0),\Xi_{i_1i_2}(f_1),...,\Xi_{i_{m+n}i_0}(f_{m+n})).$$

Using the graded differential algebra notation, we have:

$$\mathcal{S}[\mathbf{I}] = \tau\{\Xi_{i_0 i_1} \cdot \prod_{j=1}^{m+n} d(\Xi_{i_j i_{j+1}})\}$$

$$= \tau\{\varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \chi_{i_1} \cdot \gamma_{i_0 i_1} \cdot \prod_{j=1}^{m+n} d(\varphi_{i_j} \cdot \Phi_{i_j}(f_j) \cdot \chi_{i_{j+1}} \cdot \gamma_{i_j i_{j+1}})\},$$

which, by the definition of $\tau_{\Omega,\rho}$ (see §3.1), equals to

$$\begin{aligned}
&\tau_{\Omega\otimes\rho}\{\varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \chi_{i_1} \otimes \gamma_{i_0 i_1} \cdot \prod_{j=1}^{m+n} d[\varphi_{i_j} \cdot \gamma_{i_0 i_j}(\Phi_{i_j}(f_j)) \cdot \chi_{i_{j+1}} \otimes \gamma_{i_j i_{j+1}}]\} \\
= \;& \tau_{\Omega\otimes\rho}\{\varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \chi_{i_1} \otimes \gamma_{i_0 i_1} \cdot \\
& \cdot \sum_{\mathbf{J}} \prod_{j=1}^{m+n} [d^{1-\Lambda_{\mathbf{J}}(j)}(\varphi_{i_j} \cdot \gamma_{i_0 i_j}(\Phi_{i_j}(f_j)) \cdot \chi_{i_{j+1}}) \otimes d^{\Lambda_{\mathbf{J}}(j)}(\gamma_{i_j i_{j+1}})]\} \\
= \;& \sum_{\mathbf{J}} \sigma_1(\mathbf{J}) \cdot \{\tau_\rho[\prod_{j=0}^{m+n} d^{\Lambda_{\mathbf{J}}(j)}(\gamma_{i_j j_{j+1}})] \cdot \\
& \cdot \tau_\Omega[\varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \chi_{i_1} \cdot \prod_{j=1}^{m+n} d^{1-\Lambda_{\mathbf{J}}(j)}(\varphi_{i_j} \cdot \gamma_{i_0 i_j}(\Phi_{i_j}(f_j)) \cdot \chi_{i_{j+1}})]\} \\
= \;& \sum_{\mathbf{J}} \sigma_1(\mathbf{J}) \cdot \{\tau_\rho[\prod_{j=0}^{m+n} d^{\Lambda_{\mathbf{J}}(j)}(\gamma_{i_j j_{j+1}})] \cdot \\
& \cdot \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \chi_{i_1} \cdot [\prod_{j=1}^{m+n} d^{1-\Lambda_{\mathbf{J}}(j)}(\varphi_{i_j} \cdot \gamma_{i_0 i_j}(\Phi_{i_j}(f_j)) \cdot \chi_{i_{j+1}})] \wedge \Omega\}.
\end{aligned}$$

Note that the integration is, in fact, over $\cap_{j=0}^{m+n} U_{i_j}$, and hence we can apply Lemma 2.17(2) to get $\gamma_{i_0 i_j}(\Phi_{i_j}(f_j)) = \Phi_{i_0}(f_j)$. On the other hand, since φ_i and χ_i appear in pairs, $\varphi_i \cdot \chi_i = \varphi_i$, and $\varphi_i \cdot d(\chi_i) \equiv 0$, it is easy to see that we can delete these χ_i's without changing anything at all. Therefore, we have:

$$\begin{aligned}
S(\mathbf{I}) \;=\; & \sum_{\mathbf{J}} \sigma_1(\mathbf{J}) \cdot \{\tau_\rho[\prod_{j=0}^{m+n} d^{\Lambda_{\mathbf{J}}(j)}(\gamma_{i_j j_{j+1}})] \cdot \\
& \cdot \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot [\prod_{j=1}^{m+n} d^{1-\Lambda_{\mathbf{J}}(j)}(\varphi_{i_j} \cdot \Phi_{i_0}(f_j))] \wedge \Omega\}
\end{aligned}$$

Now let

$$\rho(\mathbf{I},\mathbf{J}) = \tau_\rho[\prod_{j=0}^{m+n} d^{\Lambda_{\mathbf{J}}(j)}(\gamma_{i_j j_{j+1}})],$$

and

$$\Omega(\mathbf{I},\mathbf{J}) = \int_{M\times V} \varphi_{i_0}\cdot\Phi_{i_0}(f_0)\cdot[\prod_{j=1}^{m+n} d^{1-\Lambda_{\mathbf{J}}(j)}(\varphi_{i_j}\cdot\Phi_{i_0}(f_j))]\wedge\Omega.$$

Then we claim:

4.10 Claim. *For any fixed* $\mathbf{J}$, *if* $j_{k+1} = j_k + 1$ *for some* k *or if* $j_n = m+n$, *then* $\sum_{\mathbf{I}}\{\rho(\mathbf{I},\mathbf{J})\cdot\Omega(\mathbf{I},\mathbf{J})\} = 0$.

Proof of the Claim. First we deal with the case where $j_{k+1} = j_k + 1$ for some k. In this case $\Omega(\mathbf{I},\mathbf{J})$ is symmetric with respect to (i_{n-1}, i_n) but $\rho(\mathbf{I},\mathbf{J})$ is anti-symmetric by Lemma 4.9. Therefore, for any fixed $i_j, j = 0, 1, \cdots, ..., n-2, n+1, \cdots, m+n$, we have:

$$\sum_{i_{n-1}, i_n = 1}^{N}\{\rho(\mathbf{I},\mathbf{J})\cdot\Omega(\mathbf{I},\mathbf{J})\} = 0.$$

Hence, the conclusion follows.

When $j_n = m+n$, $\Omega(\mathbf{I},\mathbf{J})$ is again symmetric with respect to (i_0, i_{m+n}), because:

$$\begin{aligned} & \Omega(\mathbf{I},\mathbf{J}) \\ = \quad & \int_{\cap_{j=0}^{m+n}U_{i_j}} \varphi_{i_0}\cdot\Phi_{i_0}(f_0)\cdot\{\textstyle\prod_{j=1}^{m+n-1} d^{1-\Lambda_{\mathbf{J}}(j)}[\varphi_{i_j}\cdot\Phi_{i_0}(f_j)]\}\cdot \\ & \cdot\varphi_{i_{m+n}}\cdot\Phi_{i_0}(f_{m+n})\wedge\Omega \\ = \quad & \int_{\cap_{j=0}^{m+n}U_{i_j}} \varphi_{i_0}\cdot\Phi_{i_{m+n}}(f_0)\cdot\{\textstyle\prod_{j=1}^{m+n-1} d^{1-\Lambda_{\mathbf{J}}j}[\varphi_{i_j}\cdot\Phi_{i_{m+n}}(f_j)]\}\cdot \end{aligned}$$

$$\cdot \varphi_{i_{m+n}} \cdot \Phi_{i_{m+n}}(f_{m+n}) \wedge \Omega.$$

(In the last step we applied $\gamma_{i_{m+n}i_0}$ to the integrand without changing the integral.) On the other hand,

$$\rho(\mathsf{I},\mathsf{J}) = \tau_\rho\{\gamma_{i_0 i_1} \cdot [\prod_{j=1}^{m+n-1} d^{\Lambda_{\mathsf{J}}(j)}(\gamma_{i_j i_{j+1}})] \cdot d(\gamma_{i_{m+n}i_0})\}$$

is, again by Lemma 4.9, antisymmetric with respect to the pair (i_0, i_{m+n}). This verifies the claim. □

Therefore, from now on we consider *only* those J where $j_k + 1 < j_{k+1}$ for each k, and $j_n < m+n$. Note that for such a J,

$$\rho(\mathsf{I},\mathsf{J}) = \tau_\rho\{\prod_{k=1}^{n} d(\gamma_{i_{j_k} i_{j_k+1}}) \cdot \gamma_{i_{j_k+1} i_{j_{k+1}}}\}$$

is independent of the values of i_j unless $j \in \mathsf{J}\cup(\mathsf{J}+1)$, and it is anti-symmetric with respect to the pair (i_{j_k}, i_{j_k+1}) for any k. Therefore, if we fix any such J, we have

$$\begin{aligned}
& \sum_{\mathsf{I}}(\rho(\mathsf{I},\mathsf{J}) \cdot \Omega(\mathsf{I},\mathsf{J})) \\
= \; & \sum_{\mathsf{I}} \rho(\mathsf{I},\mathsf{J}) \cdot \int_{M\times V} \Big(\prod_{k=0}^{n} (\varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot \{ \prod_{j_{k-1}<j<j_k} d[\varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k})]\}) \Big) \wedge \Omega \\
= \; & \sum_{\mathsf{I}} \rho(\mathsf{I},\mathsf{J}) \cdot \sigma_2(\mathsf{J}) \cdot \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot (\prod_{j \notin \mathsf{J}\cup(\mathsf{J}+1)} d[\varphi_{i_j} \cdot \Phi_{i_0}(f_j)]) \wedge \\
& \wedge \Big(\prod_{k=1}^{n} (\varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot d[\varphi_{i_{j_k+1}} \cdot \Phi_{i_0}(f_{j_k+1})]) \Big) \wedge \Omega \\
= \; & \sigma_2(\mathsf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \Big(\prod_{j \notin \mathsf{J}\cup(\mathsf{J}+1)} d(\sum_{i_j=1}^{N} \varphi_{i_j} \cdot \Phi_{i_0}(f_j)) \Big) \\
& \wedge \Big(\sum_{j \in \mathsf{J}\cup(\mathsf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathsf{I},\mathsf{J}) \cdot (\prod_{k=1}^{n} \{\varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot d[\varphi_{i_{j_k+1}} \cdot \Phi_{i_0}(f_{j_k+1})]\}) \Big) \wedge \Omega
\end{aligned}$$

$$
\begin{aligned}
= \;& \sigma_2(\mathbf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \Big(\prod_{j\notin \mathbf{J}\cup(\mathbf{J}+1)} d(\Phi_{i_0}(f_j)) \Big) \wedge \\
& \wedge \Big(\sum_{j\in \mathbf{J}\cup(\mathbf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathbf{I},\mathbf{J}) \cdot (\prod_{k=1}^{n} \{ \varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot d[\varphi_{i_{j_k+1}} \cdot \Phi_{i_0}(f_{j_k+1})]\}) \Big) \wedge \Omega \\
= \;& \sigma_2(\mathbf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \Big(\prod_{j\notin \mathbf{J}\cup(\mathbf{J}+1)} d(\Phi_{i_0}(f_j)) \Big) \wedge \Big(\sum_{j\in \mathbf{J}\cup(\mathbf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathbf{I},\mathbf{J}) \cdot \\
& \cdot (\prod_{k=1}^{n} \{ \varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot [d(\varphi_{i_{j_k+1}}) \cdot \Phi_{i_0}(f_{j_k+1}) + \varphi_{i_{j_k+1}} \cdot d(\Phi_{i_0}(f_{j_k+1}))]\}) \Big) \wedge \Omega \\
= \;& \sigma_2(\mathbf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \Big(\prod_{j\notin \mathbf{J}\cup(\mathbf{J}+1)} d(\Phi_{i_0}(f_j)) \Big) \wedge \Big(\sum_{j\in \mathbf{J}\cup(\mathbf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathbf{I},\mathbf{J}) \cdot \\
& \cdot (\prod_{k=1}^{n} \{ \varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot d(\varphi_{i_{j_k+1}}) \cdot \Phi_{i_0}(f_{j_k+1}) + \\
& + \underbrace{\varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot \varphi_{i_{j_k+1}} \cdot d[\Phi_{i_0}(f_{j_k+1})]}_{\text{symmetric w.r.t. } (i_{j_k}, i_{j_k+1})}\}) \Big) \wedge \Omega \\
= \;& \sigma_2(\mathbf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \cdot \Phi_{i_0}(f_0) \cdot \Big(\prod_{j\notin \mathbf{J}\cup(\mathbf{J}+1)} d(\Phi_{i_0}(f_j)) \Big) \wedge \\
& \wedge \Big(\sum_{j\in \mathbf{J}\cup(\mathbf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathbf{I},\mathbf{J}) \cdot (\prod_{k=1}^{n} \{ \varphi_{i_{j_k}} \cdot \Phi_{i_0}(f_{j_k}) \cdot d(\varphi_{i_{j_k+1}}) \cdot \Phi_{i_0}(f_{j_k+1})\}) \Big) \wedge \Omega
\end{aligned}
$$

Once again, this follows because $\rho(\mathbf{I},\mathbf{J})$ is anti-symmetric. Before we conclude the calculation, we need to establish a Lemma:

4.11 Lemma. *If $\Omega^n(\Gamma)$ is the space of "forms of degree n" in the universal graded differential algebra $\Omega^*(\mathbf{C}(\Gamma))$ and τ any linear functional on $\Omega^n(\Gamma)$, then*

$$
\sum_{i_1,\ldots,i_{2n}=1}^{N} (\prod_{j=1,3,\ldots,2n-1} \varphi_{i_j}) \cdot (\prod_{j=2,4,\ldots,2n} d\varphi_{i_j}) \cdot
$$

$$
\cdot \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_3}(d\gamma_{i_3 i_4})\ldots(d\gamma_{i_{2n-1} i_{2n}})\gamma_{i_{2n} i_{2n+1}})
$$

$$= (-1)^n \sum_{i_2,i_4,\ldots,i_{2n}} \Big(\prod_{j=2,4,\ldots,2n} d\varphi_{i_j} \Big) \cdot \tau(\gamma_{i_0 i_2} d\gamma_{i_2 i_4} d\gamma_{i_4 i_6} \ldots d\gamma_{i_{2n-2} i_{2n}} d\gamma_{i_{2n} i_{2n+1}}).$$

Note that, when $\tau = \tau_\rho$ and $i_{2n-1} = i_0$, the right hand side is exactly $\omega_\rho|_{U_{i_0}}$ (see Lemma 4.5).

Proof. This can be proved by induction on τ and n. Instead of giving a formal proof, we will look at the situation when $n = 2$.

$$\begin{aligned}
LHS &= \sum_{i_1,\cdots,i_4} \varphi_{i_1}\varphi_{i_3} d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_3}(d\gamma_{i_3 i_4})\gamma_{i_4 i_5}) \\
&= \sum_{i_1,\cdots,i_4} \varphi_{i_1}\varphi_{i_3} d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_3}[d\gamma_{i_3 i_5} - \gamma_{i_3 i_4} d(\gamma_{i_4 i_5})]) \\
&= \sum_{i_1,i_2,i_3} \varphi_{i_1}\varphi_{i_3} d\varphi_{i_2} (d\sum_{i_4} \varphi_{i_4}) \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_3}(d\gamma_{i_3 i_5})) \\
&- \sum_{i_1,i_2,i_4} \varphi_{i_1}(\sum_{i_3} \varphi_{i_3}) d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_4}(d\gamma_{i_4 i_5})) \\
&= 0 - \sum_{i_1,i_2,i_4} \varphi_{i_1} d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_2})\gamma_{i_2 i_4}(d\gamma_{i_4 i_5})) \quad (\text{since } \textstyle\sum_j \varphi_j = 1) \\
&= -\sum_{i_1,i_2,i_4} \varphi_{i_1} d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_1}[d\gamma_{i_1 i_4} - \gamma_{i_1 i_2}(d\gamma_{i_2 i_4})](d\gamma_{i_4 i_5})) \\
&= -\sum_{i_1,i_4} \varphi_{i_1}(d\sum_{i_2} \varphi_{i_2}) d\varphi_{i_4} \tau(\gamma_{i_0 i_1}(d\gamma_{i_1 i_4})(d\gamma_{i_4 i_5})) \\
0 &+ \sum_{i_2,i_4} (\sum_{i_1} \varphi_{i_1}) d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_2}(d\gamma_{i_2 i_4})(d\gamma_{i_4 i_5})) \\
&= \sum_{i_2,i_4} d\varphi_{i_2} d\varphi_{i_4} \tau(\gamma_{i_0 i_2}(d\gamma_{i_2 i_4})(d\gamma_{i_4 i_5})) = RHS.
\end{aligned}$$

□

With Lemma 4.11, we can now conclude the computation:

$$\begin{aligned}
&\sum_{\mathsf{I}} (\rho(\mathsf{I},\mathsf{J}) \cdot \Omega(\mathsf{I},\mathsf{J})) \\
= \;& \sigma_2(\mathsf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0} \Phi_{i_0}(f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot [\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} d(f_j)]) \wedge \\
&\wedge \Big(\sum_{j\in \mathsf{J}\cup(\mathsf{J}+1)} \sum_{i_j=1}^{N} \rho(\mathsf{I},\mathsf{J}) (\prod_{k=1}^{n} (\varphi_{i_{j_k}} d\varphi_{i_{j_k+1}})) \Big) \wedge \Omega
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_2(\mathsf{J}) \cdot \sum_{i_0=1}^{N} \int_{M\times V} \varphi_{i_0}\Phi_{i_0}(f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j)) \wedge \omega_\rho \wedge \Omega \\
&= \sigma_2(\mathsf{J}) \cdot \sum_{i_0} \int_W \phi_{i_0}^* \Big(\varphi_{i_0}\Phi_{i_0}[f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j)] \wedge \omega_\rho \wedge \Omega\Big) \\
&= \sigma_2(\mathsf{J}) \cdot \int_W f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j) \wedge \Big(\sum_{i_0} (\phi_{i_0}^*(\varphi_{i_0}\omega_\rho \wedge \Omega))\Big) \\
&= \sigma_2(\mathsf{J}) \cdot \int_W f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j) \wedge \Phi^*(\omega_\rho \wedge \Omega)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\Xi^*(\tau)(f_0, \cdots, f_{m+n}) \\
&= \sum_{\mathsf{J}} \sigma_1(\mathsf{J}) \cdot \sigma_2(\mathsf{J}) \cdot \int_W f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j) \wedge \Phi^*(\omega_\rho \wedge \Omega) \\
&= (-1)^{n(n+1)/2} \cdot \sum_{\mathsf{J}} \int_W f_0 \cdot (\prod_{j\in \mathsf{J}\cup(\mathsf{J}+1)} f_j) \cdot (\prod_{j\notin \mathsf{J}\cup(\mathsf{J}+1)} df_j) \wedge \Phi^*(\omega_\rho \wedge \Omega) \\
&= \frac{(-1)^{n(n+1)/2}}{(2\pi i)^n \cdot n!} \cdot S^n(\tau_{\Phi^*(\omega_\rho\wedge\Omega)})(f_0, f_1, \cdots, f_{m+n}),
\end{aligned}
$$

by the definition of S. This completes the proof of Theorem 4.6. □

Concluding Remarks

1. In DHK [26], a longitudinal cyclic cocycle τ on $C^\infty(\widetilde{M}\times_\Gamma V)$ is constructed from $D_{\mathcal{F}}$, such that the following diagram commutes:

$$\begin{array}{ccc} K^1(\widetilde{M}\times_\Gamma V) & \xrightarrow{[D_{\mathcal{F}}]} & K_0(\mathcal{C}) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau_\mu} \\ \mathbf{C} & = & \mathbf{C}. \end{array}$$

On the other hand, a "sharp transverse" cyclic cocycle τ^s is defined on a smooth subalgebra of $C(M)\otimes\mathcal{K}$, which is the C^*-algebra for the fibration $\widetilde{M}\times_\Gamma V\to M$. Then a renormalization scheme transfers the transverse cocycle τ^s into a cyclic cocycle on $C^\infty(\widetilde{M}\times_\Gamma V)$, which is in the same cyclic cohomology class as the longitudinal cocycle τ. This should be compared with Theorem 2.11. The character of D in the cyclic cohomology of $C^\infty(M)$ should be related to the transverse cocycle τ^s of $D_{\mathcal{F}}$, and the KK-element $[\mathcal{V}]$, or the corresponding C^*-homomorphism Φ (see §2), somehow echoes the transfer map in [26]. We believe this is interesting and it would be useful to make this point more precise.

2. In this thesis we have calculated the pairing of $[D_{\Gamma,V}]$ with certain natural, extendable cyclic cocycles on $C^\infty(V)\rtimes_{alg}\Gamma$. There are other natural cyclic cocycles to consider. For example, if Γ acts on the unit circle $\mathbf{T}$ and preserves the orientation on $\mathbf{T}$, Connes [17] has constructed a cyclic 2-cocycle on a smooth subalgebra of $C(\mathbf{T}) \rtimes_r \Gamma$, which corresponds to the Godbillion-Vey class. The pairing of this cocycle with $Ind(D, \Gamma, V)$ has also been carried out by Connes [17] and Moriyoshi and Natsume [47]. Our approach should work for this cocycle. However, the problem is how to deal with more general secondary invariants of the foliation.

On the other hand, the algebraic cyclic cohomology of $C^\infty(V) \rtimes_{alg} \Gamma$ is largely known (see Nister[49]). However, it is in general very difficult to decide whether a cyclic cocycle on $C^\infty(V) \rtimes_{alg} \Gamma$ can be extended. New techniques have to be developed based on a better understanding of the dynamics of the action of Γ on V. The work of Ji [35] might offer some hope.

3. An index theorem becomes infinitely more satisfying and powerful when the significance of the analytic index is understood in a concrete way. The work of Douglas, Hurder and Kaminker [26] provides a model. In [26], the corresponding cyclic cocycle in question is $\tau = \tau_{\omega,\rho}$, where ω is a Γ-invariant volume form on V and ρ is the trivial group 0-cocycle on Γ: $\rho(\Gamma) = 1$ for any $\gamma \in \Gamma$. In this case [26], the pairing of $\tau \circ [D_{\Gamma,V}]$ with a certain element in $K^1(\widetilde{M} \times_\Gamma V)$ gives rise to the relative η-invariant of D on M. The most important task for us is to search for similar geometric/topological interpretations for the more general cyclic cocycles that we have discussed in this thesis. This should be related to works on higher η-invariants (cf. [43] [60]).

And eventually, the challenge is to put everything one can do for foliated bundles in a form which lends itself naturally to treating more general foliations. It seems certain that renormalization schemes and super-connexions will be indispensable tools in this quest. But, only time will tell.

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