Some Examples of Spaces with Non-positively and Negatively Curved Exotic Triangulations

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Abstract of the Dissertation

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We show that, in general, non-positive curvature does not imply piecewise linear rigidity, in two cases: differentiable manifolds and piecewise linear stratified spaces.

Contents

Introduction

In this work we show that, in general, non-positive curvature does not imply piecewise linear rigidity. We do this in two cases.

First Case: Differentiable Manifolds.

A fundamental problem in the geometry and topology of manifolds is the following.

0.1. When do two homotopically equivalent manifolds are diffeomorphic, PL homeomorphic or homeomorphic?

When both manifolds in (0.1.) are closed, hyperbolic and of dimension greater than 2, Mostow's Rigidity theorem says that they are isometric, in particular diffeomorphic. When both manifolds have strictly negative curvature, results of Eells and Sampson [8], Hartman [13] and Al'ber [1] show that if $f : M_1 \to M_2$ is a homotopy equivalence then it is homotopic to a unique harmonic map. Lawson and Yau conjectured that this harmonic map is always a diffeomorphism (see problem 12 of a list of problems presented by Yau in [21] which asks to prove (0.1.), differentiably, for strictly negative curved manifolds). Farrel and Jones [9] gave counterexamples to this conjecture by proving the following. If M is a real hyperbolic manifold and Σ is an exotic sphere, then given $\epsilon > 0$, M has a finite covering $\tilde{M}$ such that the connected sum $\tilde{M}\#\Sigma$ is not diffeomorphic to $\tilde{M}$ and admits a riemannian metric with all sectional curvatures in the interval $(-1-\epsilon, -1+\epsilon)$. Because there are exotic spheres only in dimensions 7 and up this does not give counterexamples to

Lawson-Yau conjecture in dimension less than 7. The constructions here give counterexamples in dimension 6. Explicitely, we have the following theorem, that is a consequence of theorem (I.3.1.) and construction (I.3.2.):

0.2. Theorem. *There are closed real hyperbolic manifolds M of dimension 6, such that the following holds. Given $\epsilon > 0$, M has a finite cover $\tilde{M}$ that supports an exotic (smoothable) PL structure that admits a riemannian metric with sectional curvatures in the interval* $(-1-\epsilon, -1+\epsilon)$.

These manifolds are the ones that appear at the end of [17], for the real hyperbolic case.

Also, in [10] Farrell and Jones proved that (0.1.) holds topologically when one manifold is non-positively curved and has dimension greater than 4. And, again by [9], (0.1.) does not hold, diffeomorphically, for dimensions greater than 6. Then it is natural to ask if (0.1.) holds PL homeomorphically for non-positively curved manifolds. (Note that [9] does not answer this because connected sum with spheres does not change PL structures.) In section 4 we show that, in general, this is not the case (for dimensions greater than 5). In fact, we obtain the following

0.3. Corollary. *For $n \geq 6$, there are closed non-positively curved manifolds of dimension n that support exotic (smoothable) PL structures admitting riemannian metrics with non-positive sectional curvatures.*

These manifolds are simply the product of the manifolds in (0.2.) with the m-torus.

Here is a short outline of Chapter I. First, in section 1, we show how to change (concordance classes of) triangulations (modulo some closed subset) by cutting along a hypersurface and glueing back with a twist. Then, in section 3, we take this hypersurface to be totally geodesic and search for one with a large tubular neighborhood width, so that we can use the same method as [9] (see section 2) to provide this exotic triangulation with a riemannian metric with sectional curvatures in $(-1-\epsilon, -1+\epsilon)$.

Second Case: Piecewise Linear Stratified Spaces.

Two triangulations on a space that agree on each stratum are not necessarily PL equivalent, because we can glue these strata in different ways. Indeed, Anderson and Hsiang [2] gave obstructions for this problem, which depend on the lower K groups of the links of the strata. Also, Farrell and Jones [10] proved that the lower K groups of a non-positively curved manifold are zero. These two facts motivate the following problem about the uniqueness of triangulations on non-positively curved simplicial complexes. Explicitly, given two triangulations on a compact space that agree in each stratum that are piecewise flat, non-positively curved (in the sense of Gromov [12]) and complete (as geodesic spaces; that is, every geodesic can always be extended to the real line), we ask if they are PL equivalent. Here we give an example of a compact space with two non-equivalent piecewise flat non-positively curved complete triangulations that agree on each stratum. Also we give an example showing another space with two non-equivalent triangulations, one being piecewise flat non-positively curved and the other not admitting a piecewise flat non-positively curved subdivision. This last example is related to problem 14 of the list of problems presented by Yau in [22]. There he asks when a compact simplicial complex admit a non-positively curved metric. Then our second example says that we can not decide if a simplicial complex admits a non-positively curved metric just by looking at topological information, we need information of the triangulation itself.

Chapter I

Examples of Non-Positively Curved Manifolds with Non-Positively Curved Exotic Triangulations.

1. Triangulation Lemmas. Recall that if M is a PL manifold and $C \subset M$, a closed subset (assume $m = dimM \geq 6$ or $dimM \geq 5$ and $\partial M \subset C$) then there is a one to one correspondance between $\check{H}^3(M, C; \mathbf{Z}_2)$ (this is Čech cohomology) and the set of concordance classes of PL structures on M that agree with the given one on a neighborhood of C. We can choose this correspondance to be such that it sends the given PL structure to 0. Next we sketch how this correspondance is given (see [14]).

Denote by τ_0 the given PL structure on M. Also denote by B_{TOP} and B_{PL} the stable classifying spaces for TOP and PL microbundle structures and $TOP/PL \to B'_{PL} \to B_{TOP}$ the fibration we obtain from the forgetful map $B_{PL} \to B_{TOP}$. Let τ be other PL structure on M that agree with τ_0 on a neighborhood of C. Then there is an n such that $\tau \times \mathbf{R}^n$ is concordant to a PL structure θ, that makes $M \times \mathbf{R}^n$ a PL microbundle (trivial over a neighborhood of C) over M_{τ_0}. This gives a correspondance between concordance classes of PL structures on M that agree with τ_0 on a neighborhood of C and $TOP/PL(\epsilon(M)\ rel\ C)$, the set of stable concordance classes ($rel\ C$) of PL microbundle structures of the trivial bundle $\epsilon(M)$ over M_{τ_0} (see [14] p.176). But $TOP/PL(\epsilon(M)\ relC)$ is also in corespondance with $Lift(f\ relC,\ F_o)$, the set of vertical homotopy classes of liftings of f to B'_{PL}, where $f : M \to B_{TOP}$ classify $\epsilon(M)$ and $F_0 : \{neighborhood\ of\ C\} \to B'_{PL}$ is a given lifting of $f\mid_{neighborhood\ of\ C}$ (it classify $\tau_0 \mid_{neighborhood\ of\ C}$). But $\epsilon(M)$ is a trivial bundle so that we can choose f to be a constant map (and F_0 also constant because our PL microbundle structures are trivial over a neighborhoood of C), hence $TOP/PL(\epsilon(M)\ rel\ C)$ is in correspondance with $[M, C; TOP/PL]$, the set of homotopy classes of maps from M to TOP/PL that send a neighborhood of C to a previously fixed point. But TOP/PL is a Eilenberg-Maclane space of type $(3, \mathbb{Z}_2)$, so that $[M, C; TOP/PL]$ is in correspondance with $\check{H}^3(M, C; \mathbb{Z}_2)$. Note that this correspondance depends on which PL structure we are sending to zero in $\check{H}^3(M, C; \mathbb{Z}_2)$ and is also completely determined by this choice.

Given a concordance class of triangulations $[\tau]$ denote by $c_{[\tau]} = c_\tau \in \check{H}^3(M, C; \mathbb{Z}_2)$ the corresponding cohomology class and also given a cohomology class c write $[\tau_c] = [\tau]_c$ for the corresponding concordance class of triangulations.

We have the following

1.1. Lemma. *Let $p : \tilde{M} \to M$ be a covering, $C \subset M$ closed and $m = dimM \geq 6$ (or $dimM \geq 5$ and $\partial M \subset C$). Suppose M has a PL stucture τ_0 and denote by $\tilde{\tau}_0$ the pullback $p^*\tau_0$ of τ_0 and make these two triangulations correspond to zero in $\check{H}^3(\tilde{M}, p^{-1}(C); \mathbb{Z}_2)$ and $\check{H}^3(M, C; \mathbb{Z}_2)$ respectively. Then $[\tau]_{p^*c} = [p^*\tau_c]$ for all $c \in \check{H}^3(M, C; \mathbb{Z}_2)$. Equivalently, $c_{p^*\tau} = p^*c_\tau$ for every PL structure τ on M.*

Note that if τ_1 and τ_2 are concordant PL structures on M, then $p^*\tau_1$ and

$p^*\tau_2$ are also concordant.

Proof. Let τ be a PL structure on M $(rel\ C)$. If θ is a PL structure that makes $M \times \mathbb{R}^n$ (for some n) a PL microbundle over M_{τ_0} concordant $(rel\ C)$ to $\tau \times \mathbb{R}^n$, then $\tilde{p}^*\theta$ is a PL structure that makes $\tilde{M} \times \mathbb{R}^n$ a PL microbundle over $\tilde{M}_{\tilde{\tau}_0}$ concordant $(rel\ p^{-1}C)$ to $p^*\tau \times \mathbb{R}^n$, where $\tilde{p} = (p, Id_{\mathbb{R}^n})$. Then if $h : M \to TOP/PL \subset B'_{PL}$ classify θ, hp classify $\tilde{p}^*\theta$. So, pulling back PL structures gives a map $[M, C; TOP/PL] \to [\tilde{M}, p^{-1}(C); TOP/PL]$ given by $h \longmapsto hp$. This completes the prove of the lemma.

Now, given a PL manifold M, we show how to change PL structures by cutting along a closed hypersurface N of M, and glueing back with a twist.

Denote by M_χ the $CAT(= PL$ or $DIFF)$ manifold obtained by cutting along N (a CAT closed hypersurface), and identifying by χ the two copies of N we get, where N is a CAT closed hypersurface and $\chi : N \to N$ is a CAT isomorphism. In what follows we assume that the relative set is nice enough (for example, deformation retract of a subcomplex) so that we replece Čech cohomology by singular cohomology.

1.2. Lemma. *Let M be a PL oriantable n-manifold, $n \geq 6$, N a closed PL hypersurface with a tubular neighbourhood $g : W \cong_{PL} N \times [-1, 1]$ of N in M, where $g(N) = N \times \{0\}$, and $J \subset N$ open with $\bar{J}$ compact. Then for every $c \in H^3(M, M \setminus J; \mathbb{Z}_2)$, there is a PL isomorphism $\chi : N \to N$, such that M_χ (that is, its PL structure) corresponds to c (by the correspondance that sends the given PL structure to 0) and χ is the identity outside a compact neighborhood of $\bar{J}$.*

Note that J is not open in M but $g^{-1}(J \times (-\delta, \delta))$ is, where $\delta < 1$, and $(M, M \setminus g^{-1}(J \times (-\delta, \delta))$ is a deformation retract of $(M, M \setminus J)$.

Proof. Denote by τ_0 the given PL structure on M and make it correspond to $0 \in H^3(M, M \setminus J; \mathbb{Z}_2)$. Now, τ_c (a PL structure that corresponds to c) is a PL structure on W that agrees with τ_0 outside $g^{-1}(J \times (-\delta, \delta))$. In particular they agree on $g^{-1}((N \setminus J) \times [-1, 1])$, so that W_{τ_c} is a PL product there (bacause W_{τ_0} is). By the s-cobordism theorem (and the fact that the torsion of a homeomorphism is zero), we have that there is a PL homeomorphism $h : (W, \tau_c) \to N \times [-1, 1]$, such that

$$hg^{-1} \mid_{N \times \{-1\} \cup (\{N \setminus V\} \times [-1,1])} = Id_{N \times \{-1\} \cup (\{N \setminus V\} \times [-1,1])}$$

where $\bar{J} \subset V \subset \overline{V} \subset N$, $\overline{V}$ compact and V open. Let $\chi = (h^{-1}g) \mid_{N \times \{1\}}$. Then we see that M_χ corresponds to τ_c (here to obtain M_χ we are cutting along $g^{-1}(N \times \{1\}) \subset W$), for we can define a PL homeomorphism $H : M_{\tau_c} \to M_\chi$ by

$$H(x) = \begin{cases} g^{-1}h(x) & x \in W \\ \\ x & x \in M \setminus W \end{cases}$$

Note that $\chi \mid_{N \setminus V} = Id_{N \setminus V}$.

This completes the proof of lemma (1.2.).

1.3. Remark. Note that if τ is smoothable, then, using now the differentiable s-cobordism theorem, we can choose χ to be smooth.

2. Geometric Lemma. Let M be a differentiable manifold and consider metrics A on $M \times I$, where $I = [1,2]$ is a closed interval, satisfying (recall that the tangent space at a point $(x,t) \in M \times I$ is isomorphic to $T_xM \oplus \mathbf{R}(\frac{\partial}{\partial t}) \mid_t$)

2.1. (a)For any $v \in T_xM, A(v, \frac{\partial}{\partial t}) = 0$.

(b)$A(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 1$.

Equivalently, $A = S_t + dt^2$, where S_t is a metric on M depending on t.

2.2. Lemma *Let M be compact and $A = S_t + dt^2$ a metric on $M \times I$ satisfying (2.1.). Then given $\epsilon > 0$ there is an L such that for $\alpha > L$ all the sectional curvatures of A_α lie in $(-1-\epsilon, -1+\epsilon)$, where A_α is the metric on $M \times I$ given by $A_\alpha(x,t) = cosh^2(\alpha t)S_t + \alpha^2 dt^2$.*

The proof is the same as the proof of lemma (3.5.) of [9], just replace the function sinh by cosh and the $m-1$ sphere by any compact manifold.

3. Construction of the examples. First we proof the following

3.1. Theorem. *Consider the following data. For each $k = 1, 2, 3, \ldots$ we have closed hyperbolic manifolds $M_0(k), M_1(k), M_2(k), M_3(k)$ such that*

(a) $dim M_0(k) = 6$, $dim M_1(k) = 5$, $dim M_2(k) = 3$, $dim M_3(k) = 3$.

(b) $M_2(k) \subset M_1(k) \subset M_0(k)$ and $M_3(k) \subset M_0(k)$. All the inclusions are totally geodesic.

(c) $M_2(k)$ and $M_3(k)$ intersect in one point transversally.

(d) For each k there is a finite covering map $p(k) : M_0(k) \to M_0(1)$ such that

$p(k)(M_i(k)) = M_i(1)$, *for* $i = 0, 1, 2, 3$.

(e) $M_1(k)$ *has a tubular neighborhoood in* $M_0(k)$ *of width* $r(k)$ *and* $r(k) \to \infty$ *as* $k \to \infty$.

Then, given $\epsilon > 0$, *there is a* K *such that all* $M_0(k)$, $k \geq K$, *have exotic (smoothable) triangulations admitting riemannian metrics with all sectional curvatures in the interval* $(-1-\epsilon, -1+\epsilon)$.

Proof. Denote by $\sigma(k)$ the triangulation on $M_0(k)$ induced by the hyperbolic structure and make it correspond to zero in $H^3(M_0(k), \mathbb{Z}_2)$. Also, denote by $g(k)$ the restriction of the hyperbolic metric on $M_0(k)$ to the totally geodesic submanifold $M_1(k)$. Then the tubular neighborhood of width $r(k)$ of $M_1(k)$ in $M_0(k)$ is isometric to $M_1(k) \times [-r(k), r(k)]$ with metric, at a point (x, t), given by $(cosh^2 t)g(k) + dt^2$ (note that hyperbolic n-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^{n-1} \times \mathbb{R}$ with metric $(cosh^2 t)g + dt^2$, where g is the hyperbolic metric on $\mathbb{H}^{n-1}$).

Take now a tubular neighborhood $W(k)$ of $M_2(k)$ in $M_1(k)$. We can suppose that $p(k) |_{W(k)}: W(k) \to W(1)$ is a covering. Let the open set $U(k)$ be such that $\overline{U(k)}$ is a compact neighborhood of $M_2(k) \times \{2\}$ in $W(k) \times (0, 3)$.

Consider the cohomology class $c(k) \in H^3(W(k) \times (0,3), W(k) \times (0,3) \setminus U(k); \mathbb{Z}_2) \cong H^3(W(k) \times (0,3), W(k) \times (0,3) \setminus M_2(k) \times \{2\}; \mathbb{Z}_2)$ dual to $M_2(k) \times \{2\} \subset W(k) \times (0,3)$.

Denote by $\tau(k)$ the triangulation, modulo the complement of $U(k)$, on $W(k) \times (0,3)$ corresponding to $c(k)$.

Let $f(k) : W(k) \to W(k)$ be the PL isomorphism corresponding to $c(k)$ given by lemma (1.2.), so that the triangulation of $(W(k) \times (0,3))_{f(k)}$, obtained by identifying $(x,2) \in W(k) \times (0,2]$ with $(f(k)(x),2) \in W(k) \times [2,3)$, corresponds to $c(k)$ (it is concordant to $\tau(k)$).

3.1.1. We have the following claims.

(1) $(p(k) \mid_{W(k)} \times Id_{(0,3)})^* c(1) = c(k)$ and $\tau(k) = (p(k) \mid_{W(k)} \times Id_{(0,3)})^* \tau(1)$.

(2) We can choose $f(k)$ such that it covers $f(1)$.

(3) We can suppose $\tau(k)$ to be smoothable and $f(k)$ a diffeomorphism.

(4) We can take $f(k)$ to be the identity outside a neighborhood $V(k)$ of $M_2(k)$, with $\overline{V(k)} \subset W(k)$ compact.

Proof of the claims. (3) is true because in dimension six there is no obstruction for a PL structure to be smooth so that we can suppose $f(k)$ smooth (see remark (1.3.)). (4) follows from the fact that $\tau(k)$ and $\sigma(k)$ coincide outside $U(k)$ (see proof of lemma (1.2.)). (1) is because $(p \mid_{W(k)})^{-1}(M_2(1)) = M_2(k)$ (the pullback of the dual of a cycle is the dual of the inverse image (to see this just consider a suficiently fine triangulation and its dual cell decomposition and pullback everything)). The second part of (1) follows from lemma (1.1.). For (2) note that the triangulation of $(W(k) \times (0,3))_{f(k)}$ is $\tau(k) = (p(k) \mid_{W(k)} \times Id_{(0,3)})^* \tau(1)$ and if $f(k)$ covers $f(1)$ then $(W(k) \times (0,3))_{f(k)}$ covers $(W(1) \times (0,3))_{f(1)}$ by a PL covering. Hence we can take as $f(k)$ a lifting of $f(1)$ (indeed, we could have defined $f(k)$ in this way). This completes the proof of the claims.

Consider the metric $A(1)$ on $W(1) \times [1,2]$ defined by

$$A(1) = [\delta(t) f(1)^*(g(1)) + (1 - \delta(t)) g(1)] + dt^2$$

where δ is a smooth real function such that $0 \leq \delta(t) \leq 1,\ \delta(1) = 0,\ \delta(2) = 1$

and is constant near 1,2.

For $\epsilon > 0$ let L be the constant given by lemma (2.2.), so that all sectional curvatures of $A(1)_\alpha$ lie in $(-1-\epsilon, -1+\epsilon)$, for $\alpha \geq L$. Note that, because $f(1)$ is the identity outside $V(1)$, we have that $A(1) = g(1)+dt^2$ outside $V(1)\times[1,2]$ and then also $A(1)_\alpha = (cosh^2(\alpha t))g(1)+\alpha^2 dt^2$ outside $V(1)\times[1,2]$. Note that we cannot apply lemma (2.2.) directly because $W(1)\times[1,2]$ is not compact but we can apply the lemma to $M_1(1)\times[1,2]$ because we can extend $A(1)$ to it. Define now a metric $B(1)$ on $(W(k)\times(0,3))_{f(1)}$ (that is $W(k)\times(0,3)$ with triangulation $\tau(1)$) by

$$B(1) = \begin{cases} A(1) & t \in [1,2] \\ \\ g(1)+dt^2 & t \in (0,1] \cup [2,3) \end{cases}$$

Note that this metric is well defined since both definitions coincide on a neighbohood of $t = 1,2$.

Thus $(W(1)\times(0,3))_{f(1)}$ admits riemannian metrics (the metrics $B(1)_\alpha$ for $\alpha \geq L$) with all sectional curvatures in $(-1-\epsilon, -1+\epsilon)$. Remark that $B(1)_\alpha = (cosh^2(\alpha t))g(1) + \alpha^2 dt^2$ outside a compact subset of $W(1)\times(0,3)$ containing $M_2(1)\times\{2\}$.

Also, by defining $B(k) = p(k)^* B(1)$, we have that $(W(k)\times(0,3))_{f(k)}$ (i.e. $W(k)\times(0,3)$ with triangulation $\tau(k)$) admits riemannian metrics (the metrics $B(k)_\alpha$ for $\alpha \geq L$) with all sectional curvatures in $(-1-\epsilon, -1+\epsilon)$. Note that we also have $B(k)_\alpha = (cosh^2(\alpha t))g(k) + \alpha^2 dt^2$ outside a compact subset of $W(k)\times(0,3)$ containing $M_2(k)\times\{2\}$. We try now to fit these models (i.e.

$(W(k) \times (0,3))_{f(k)}$ with the metrics $B(k)_\alpha$) on the $M_0(k)$, for large enough k.

Let K be such that $r(k) > 3L$ for $k \geq K$ (use hypothesis (e) here). We prove that $M_0(k)$ has exotic triangulations with riemannian metrics with sectional curvatures in the interval $(-1-\epsilon, -1+\epsilon)$.

Because of (e) of the statement of the theorem, $M_1(k) \subset M_0(k)$ has a tubular neighborhood of width $r(k)$ isometric to $M_1 \times [-r(k), r(k)]$ with metric $(cosh^2(t))g(k) + dt^2$. In what follows we make no distinction between the tubular neighborhood and $M_1(k) \times [-r(k), r(k)]$.

Consider

$$h(k) : W(k) \times (0,3) \to W(k) \times (0,3L) \subset W(k) \times (-r(k), r(k))$$
$$\subset M_1(k) \times (-r(k), r(k)) \subset M_0(k)$$

given by $(x,t) \longmapsto (x, Lt)$.

Note that $h(k)$ is an isometry, where we are considering $W(k) \times (0,3)$ with metric $cosh^2(Lt)g(k) + L^2dt^2$ and $W(k) \times (0,3L)$ with metric induced by the hyperbolic metric on $M_0(k)$.

Because the triangulation $(h(k)^{-1})^*\tau(k)$ coincide with $\sigma(k)$ outside a compact in $W(k) \times (0,3L)$ we can extend it to all $M_0(k)$ by defining it to be $\sigma(k)$ outside $W(k) \times (0,3L)$. Call this triangulation on $M_0(k)$, $\bar{\tau}(k)$. This (smoothable) triangulation corresponds to the cohomology class $\bar{c}(k) \in H^3(M_0(k), M_0(k) \backslash M_2(k) \times \{2L\}; \mathbb{Z}_2)$ dual to $M_2(k) \times \{2L\} \subset W(k) \times (0,3L) \subset M_1(k) \times (-r(k), r(k)) \subset M_0(k)$ (the correspondance between PL structures and the third cohomology group is natural for restrictions to open sets, see [14] p.195). Define also a metric $\bar{B}(k)$, compatible with $\bar{\tau}(k)$, on $M_0(k)$ to be $(h(k)^{-1})^*B(k)_L$ on $W(k) \times (0,3L)$ and the hyperbolic metric outside

$W(k) \times (0, 3L)$. Note that all sectional curvarures of $\bar{B}(k)$ lie in $(-1-\epsilon, -1+\epsilon)$ (all sectional curvatures are -1 outside a compact subset of $W(k) \times (0, 3L)$).

So, given $\epsilon > 0$, there is a K such that for $k \geq K$, $\bar{\tau}(k)$ is a triangulation on $M_0(k)$ that admits the riamannian metric $\bar{B}(k)$ with all sectional curvatures in the interval

$(-1-\epsilon, -1+\epsilon)$ and $\bar{\tau}(k)$ corresponds (by the correspondance that sends $\sigma(k)$ to zero) to $\bar{c}(k) \in H^3(M_0(k), M_0(k) \setminus M_2(k) \times \{2L\}; \mathbb{Z}_2)$ dual to $M_2(k) \times \{2L\}$.

But $\bar{c}(k)$ is not zero in $H^3(M_0(k); \mathbb{Z}_2)$. That is, if

$$i_3 : H^3(M_0(k), M_0(k) \setminus M_2(k) \times \{2L\}; \mathbb{Z}_2) \to H^3(M_0(k); \mathbb{Z}_2)$$

is the inclusion, then $i_3(\bar{c}(k))$ is not zero because $M_2(k) \times \{2L\}$ is homologous to $M_2(k)$ and it intersects $M_3(k)$ transversally in one point (by hypotesis (c)). This means that $\sigma(k)$ and $\bar{\tau}(k)$ are non-concordant.

Finally we have to prove that $\bar{\tau}(k)$ is indeed not equivalent to $\sigma(k)$.

So suppose

$f : (M_0(k), \bar{\tau}(k)) \to (M_0(k), \sigma(k))$ is a PL homeomorphism. We have two cases:

3.1.2. First Case. *Suppose f is homotopic to the identity.* Let H_t, $0 \leq t \leq 1$, $H_0 = f$, $H_1 = Id$ be a homotopy between f and the identity. Then the map $\overline{H} : M_0(k) \times [0,1] \to M_0(k) \times [0,1]$, defined by $\overline{H}(x,t) = (H_t(x), t)$ is homotopic to $Id_{M_0(k) \times [0,1]}$ and because it is already a homeomorphism on $\partial(M_0(k) \times [0,1])$ we may apply (1.6.1.) of [10] to get a homotopy (which is constant on $\partial(M_0(k) \times [0,1])$) of $\overline{H}$ to a homeomorphism $\tilde{H} : M_0(k) \times [0,1] \to M_0(k) \times [0,1]$.

Because $\tilde{H}_1 = Id$ and $\tilde{H}_0 = f$, by pulling back the triagulation $\sigma(k) \times I$ of $M_0(k) \times [0,1]$ using $\tilde{H}$, we obtain a concordance between $\bar{\tau}(k)$ and $\sigma(k)$. A contradiction because $\sigma(k)$ and $\bar{\tau}(k)$ are non-concordant.

3.1.3. Second Case. By Mostow Rigidity theorem, every homeomorphism from a compact hyperbolic manifold to itself is homotopic to a diffeomorphism, so that we have that $f \sim g$, where $g : (M_0(k), \sigma(k)) \to (M_0(k), \sigma(k))$ is a diffdeomorphism. Then the second case follows by applying the first case to $fg^{-1} \sim Id_{M_0(k)}$ (note that $\sigma(k)$ is equivalent to $(g^{-1})^*\sigma(k)$). This completes the proof of theorem (3.1.).

Remark. Theorem (3.1.) does not work for dimension 5 because the triangulation lemma (1.2.) holds only for dimensions 6 and above. This is because the s-cobordism theorem is not true for dimension 5, so that we do not know if exotic triangulations on $M^4 \times [0,1]$, modulo the boundary, are products, where M^4 is a 4-manifold. Also, in theorem (3.1.) we needed dimension less than seven to assure that the triangulations we obtained are smoothable.

3.2. We construct now, for every $n \geq 4$, manifolds $M_i(k), i = 0,1,2,3$ and $k = 1,2,3,...$ with $dim M_0(k) = n$, $dim M_1(k) = n-1$, $dim M_2(k) = n-3$, $dim M_3(k) = 3$ satisfying (b),(c),(d) and (e) of the theorem. When $n = 6$ they will also satisfy (a).

Fix a positive prime number m and write $E = \mathbf{Q}(\sqrt{m})$. Denote by O_E the set of integers of E. Fix $l \in O_E$ and define, for $k = 1,2,3,...$ the quadratic form $Q(k)$ on $\mathbb{R}^{n+1}$ by

$$Q(k)(x_1, ..., x_{n+1}) = l^{2(k-1)} x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 - \sqrt{m}\, x_{n+1}^2$$

Define now groups

$$G_0 = \{g \in GL(n+1, \mathbb{R}) : \; gH = H\} \quad \textit{where} \;\; H = \{x \in \mathbb{R}^{n+1} \; : \; x_{n+1} > 0\}$$

$$G_1 = \{g \in G_0 : ge_1 = e_1\}$$

$$G_2 = \{g \in G_0 : \; ge_i = e_i, \; i = 1, 2, 3\}$$

$$G_3 = \{g \in G_0 : \; ge_i = e_i, \; i = 4, 5, ..., n\}$$

and

$$H_0(k) = \{g \in G_0 : Q(k)(gx) = Q(k)(x) \;\; \forall x \in \mathbb{R}^{n+1}\}$$

$$H_i(k) = H_0(k) \cap G_i \quad i = 1, 2, 3$$

$$\Gamma_i(k) = H_i(k)_{O_E} \quad i = 0, 1, 2, 3.$$

where the subindex O_E means that the entries of the matrices are in O_E and e_i is the vector in $\mathbb{R}^{n+1}$ whose j–th coordinate is δ_i^j. Note that for all k, $H_i(k) = H_i(1)$ and $\Gamma_i(k) = \Gamma_i(1)$ for $i = 1, 2$ and write just H_1, H_2 and Γ_1, Γ_2, respectively.

Define also

$$X_0 = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : Q(1)(x) = -\sqrt{m}, \;\; x_{n+1} > 0\}$$

$$X_1 = X_0 \cap \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = 0\}$$

$$X_2 = \{(x_1, ..., x_{n+1}) \in X_0 : \; x_1 = x_2 = x_3 = 0\}$$

$$X_3 = \{(x_1, ..., x_{n+1}) \in X_0 : \; x_4 = x_5 = ... = x_n = 0\}$$

and we consider X_0 with the metric, at a point $x \in X_0$, that is the restriction of $Q(1)$ to the hyperplane tangent to X_0 at x. This riemannian metric is of constant curvature $-\frac{1}{\sqrt{m}}$. Note that $X_2 \subset X_1 \subset X_0$, $X_3 \subset X_0$ and all the inclusions are totally geodesic. Note also that $X_2 \cap X_3 = e_{n+1}$.

Consider the $n+1$ by $n+1$ diagonal matrices

$$D(k) = diag\{l^{k-1}, 1, 1, ..., 1\}$$

and note that $D(k)H_i(k)D(k)^{-1} = H_i(1) \quad i = 0, 1, 2, 3.$

Note that $H_i(1)$ acts on X_i and $D(k)\Gamma_i(k)D(k)^{-1} \subset H_i(1)$ for $i = 0, 1, 2, 3$ so define

$$Y_i(k) = X_i / D(k)\Gamma_i(k)D(k)^{-1} \quad i = 0, 1, 2, 3$$

Note that $Y_i(k) = Y_i(1)$ for all k and $i = 1, 2$ so write just Y_1 and Y_2.

3.2.1. Now for an ideal $\mathcal{I}$ of O_E consider the congruence subgroups

$$\Gamma_i(k)_{\mathcal{I}} = \{g \in \Gamma_i(k) : \; g = Id \;\; mod\, \mathcal{I}\}$$

for $i = 0, 1, 2, 3$. Also write

$$Y_i(k)_{\mathcal{I}} = X_i / D(k)\Gamma_i(k)_{\mathcal{I}} D(k)^{-1} \quad i = 0, 1, 2, 3.$$

3.2.2. We have the following facts

1. For any non-trivial ideal $\mathcal{I}$ of O_E, $\Gamma_i(k)_{\mathcal{I}}$ is a subgroup of finite index of $\Gamma_i(k)$ because $O_E/\mathcal{I}$ is finite.

2. $\Gamma_i(k)$ is discrete (see the proof of step 1 of lemma (3.2.3) or [16] p.239).

3. $Y_i(k)$ is compact (see [19] or [16] p.238).

4. For all but finite ideals $\mathcal{I}$, $GL(n+1, O_E)_{\mathcal{I}}$ is torsion free (see [4] p.113), so that all $\Gamma_i(k)_{\mathcal{I}}$ are also torsion free. Then all $Y_i(k)_{\mathcal{I}}$ are compact manifolds . Furthermore, for all but finite ideals $\mathcal{I}$, we have that if

$$\pi(k) : X_0 \to X_0/D(k)\Gamma_0(k)_{\mathcal{I}}D(k)^{-1} = Y_0(k)_{\mathcal{I}}$$

is the projection, then

$$(*) \qquad \pi(k)X_i = X_i/D(k)\Gamma_i(k)_{\mathcal{I}}D(k)^{-1} = Y_i(k)_{\mathcal{I}} \quad i = 0, 1, 2, 3$$

so that the $(Y_i(k))_{\mathcal{I}}$ are (totally geodesic) submanifolds of $Y_0(k)_{\mathcal{I}}$ (see proposition (2.2.) of [17]).

Remark. To be able to apply (2.2.) of [17] we need some remarks. Let σ_i $i = 1, 2, 3$ be the following involutions.

$\sigma_1(x_1, x_2, ..., x_{n+1}) = (-x_1, x_2, ..., x_{n+1})$,

$\sigma_2(x_1, x_2, x_3, x_4..., x_{n+1}) = (-x_1, -x_2, -x_3, x_4..., x_{n+1})$ and

$\sigma_3(x_1, x_2, x_3, x_4, x_5, ..., x_n, x_{n+1}) = (x_1, x_2, x_3, -x_4, -x_5, ..., -x_n, x_{n+1})$. Note that X_i is the fixed point set of σ_i. We alsohave that $\sigma_i\Gamma_0(k)_{\mathcal{I}}\sigma_i = \Gamma_0(k)_{\mathcal{I}}$ $i = 1, 2, 3$ and the following two facts hold

1. $\Gamma_0(k)_{\mathcal{I}}$ acts freely, because is discrete and torsion free.

2. $\Gamma_i(k)_{\mathcal{I}} = \{g \in \Gamma_0(k)_{\mathcal{I}} :\ gX_i = X_i\} = \{g \in \Gamma_0(k)_{\mathcal{I}} :\ \sigma_i g \sigma_i = g\}$ $i = 1, 2, 3$. To see the first equality note that a group of ortogonal matrices with coefficients in O_E is finite. We can apply now (2.2.) of [17] to obtain $(*)$.

3.2.3.Lemma. *The widths $r(k)$ of tubular neighborhoods of $(Y_1)_{\mathcal{I}}$ in $Y_0(k)_{\mathcal{I}}$ can be chosen such that $r(k) \to \infty$ as $k \to \infty$.*

Proof. We have three steps.

Step 1. We prove that

$$(X_0)_{O_E} = X_0 \cap O_E^{n+1} = \{(x_1, ..., x_{n+1}) : \; x_1^2 + ... + x_n^2 - \sqrt{m}\, x_{n+1}^2 = -\sqrt{m}, \; x_i \in O_E\}$$

is closed and discrete.

The proof of this is similar to the proof of the fact that $\Gamma_0(k)$ is discrete (see [3] p.190). So, to prove step 1 note first that O_E is not discrete in $\mathbb{R}$, but the map $\phi : O_E \to \mathbb{R} \times \mathbb{R}$ defined by $x \longmapsto (x, \bar{x})$, where $\bar{x}$ is the conjugate (i.e. $\overline{a + \sqrt{m}\, b} = a - \sqrt{m}\, b$) is a bijection of O_E in $\mathbb{R}^2$ whose image is closed and discrete.

Then $\bar{\phi} : (X_0)_{O_E} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a bijection and also has closed and discrete image. But $proj_2(\bar{\phi}((X_0)_{O_E}))$ is compact (because $x_1^2 + ... + x_n^2 - \sqrt{m}\, x_{n+1}^2 = -\sqrt{m}$ implies $\bar{x}_1^2 + ... + \bar{x}_n^2 + \sqrt{m}\, \bar{x}_{n+1}^2 = \sqrt{m}$) so that $(X_0)_{O_E} = proj_1(\bar{\phi}((X_0)_{O_E}))$ is closed and discrete.

Step 2. We prove that for all $s \in \mathbb{R}^+$ there is a K such that $\| \gamma e_{n+1} \| > s$ for $k > K$ and $\gamma \in D(k)\Gamma_0(k)D(k)^{-1} \setminus \Gamma_1$, where the bars denote the euclidean norm in $\mathbb{R}^{n+1}$.

So, take γ as before, then $\gamma = D(k)\beta D(k)^{-1}$ for some $\beta \in \Gamma_0(k)$. Thus $\gamma e_{n+1} = D(k)\beta D(k)^{-1} e_{n+1} = D(k)\beta e_{n+1} = (l^{k-1} a_1, a_2, a_3, ..., a_{n+1})$ with $a_i \in O_E$ and $a_1 \neq 0$ (because γ is not in Γ_1 and also because $(*)$ implies that

$\Gamma_1 = \{g \in \Gamma_0(k) : gX_1 = X_1\} = \{g \in \Gamma_0(k) : gX_1 \cap X_1 \neq \phi\})$ and note that $(l^{k-1}a_1)^2 + ... + a_n^2 - \sqrt{m}\, a_{n+1}^2 = -\sqrt{m}$ so that $\gamma e_{n+1} = (l^{k-1}a_1, a_2, ..., a_{n+1}) \in (X_0)_{O_E}$. Now $(X_0)_{O_E}$ is closed and discrete by step 1, consequently $(X_0)_{O_E} \cap B(0,s)$ is finite, where $B(0,s)$ is the ball in $\mathbb{R}^{n+1}$ with center in the origin and radius s. Then the set $proj_1((X_0)_{O_E} \cap B(0,s))$ is finite, where $proj_1(x_1, ..., x_{n+1}) = x_1$. By taking K large enough we have that , for $k > K$, l^{k-1} does not divide any of the non-zero elements of $proj_1((X_0)_{O_E} \cap B(0,s))$ so that $\gamma e_{n+1} = (l^{k-1}a_1, a_2, ..., a_{n+1})$ does not belong to $B(0,s)$ which means that $\| \gamma e_{n+1} \| > s$ for $k > K$.

Step 3. We complete the proof. Because of step 2, we have that

$$d\left(e_{n+1}\ ,\ \left\{\gamma e_{n+1}\ :\ \gamma \in D(k)\Gamma_0(k)D(k)^{-1} \setminus \Gamma_1\right\}\ \right) \to \infty$$

as $k \to \infty$, where d is the euclidean distance. Then is easy to see that the same happens with the riemannian metric of X_0 (both induce the same topology), so that the lengths of closed geodesics, not in Y_1, at the point $o = \pi(k)(e_{n+1})$ go to infinity as k goes to infinity, which completes, using the triangular inequality, the proof of the lemma.

We have found manifolds satisfying condition (e) of the theorem. We now pass to finite coverings to find manifolds satisfying (b), (c) and (d). We use now the following result from [17] p.122.

3.2.4. *There are infinitely many ideals $\mathcal{I}$ of O_E such the following two conditions hold*

1. $X_i/\Gamma_i(1)_{\mathcal{I}}$ *is orientable,* $i = 0, 1, 2, 3$.

2.*If* $\gamma \in \Gamma_0(1)_{\mathcal{I}}$ *and* $\gamma x \in X_2$, *for some* $x \in X_3$, *then* $\gamma = g_2 g_3$,

where

$g_i \in \Gamma_i(1)_{\mathcal{I}}$, $i = 2, 3$.

Then we can suppose that the ideal $\mathcal{I}$ we chose before lemma (3.2.3.) satisfy (3.2.4.).

Remark. Statement 2. of (3.2.4.) holds if and only if $Y_2(1)_{\mathcal{I}} \cap Y_3(1)_{\mathcal{I}}$ is one point.

Let now $l \in \mathcal{I}$ and consider $\Gamma_i(k)_{(l^k)}$, $i = 0, 1, 2, 3$, where (l^k) is the principal ideal generated by l^k. Note that $(l^k) \subset \mathcal{I}$ so that $\Gamma_i(k)_{(l^k)} \subset \Gamma_i(k)_{\mathcal{I}}$. Denote by

$$\Sigma_i(k) = D(k)\Gamma_i(k)_{(l^k)}D(k)^{-1} \quad i = 0, 1, 2, 3$$

and note that $\Sigma_i(k)$ is a subgroup of $\Gamma_i(1)_{\mathcal{I}}$ for $i = 0, 1, 2, 3$. Moreover, because $\Gamma_i(1)_{(l^{2k})} \subset \Sigma_i(k) \subset \Sigma_i(1) \subset \Gamma_i(1)_{\mathcal{I}}$, we have that $\Sigma_i(k)$ has finite index in $\Gamma_i(1)_{\mathcal{I}}$ and $\Sigma_i(1)$. Write

$$M_i(k) = X_i/\Sigma_i(k) \quad i = 0, 1, 2, 3.$$

We prove that these manifolds satisfy (b), (c), (d) and (e) of the theorem.

Note that all $M_i(k)$ are compact orientable manifolds because they are finite covers of the $X_i/\Gamma_i(1) = Y_i(1)_{\mathcal{I}}$ (for orientability use 1. of (3.2.4.)). This also imply (b) and the fact that the dimensions are right. Next we prove (d). Remark that $\Sigma_i(k) = \Sigma_0(k) \cap G_i = \Sigma_0(k) \cap \Gamma_i(k)_{\mathcal{I}}$ $i = 1,2,3$. This fact together with $(*)$ implies that if $\bar{\pi}(k) : X_0 \to M_0(k)$ is the projection then $\bar{\pi}(k)(X_i) = M_i(k)$ $i = 1,2,3$. If $p(k)$ denotes the projection $p(k) : M_0(k) \to M_0(1)$, then (d) follows because $\bar{\pi}(1) = p(k)\bar{\pi}(k)$.

Note that $(M_0(k), M_1(k))$ covers $(Y_0(k)_{\mathcal{I}}, (Y_1)_{\mathcal{I}})$ then lemma (3.2.3.) implies that (e) holds.

Finally we prove (c). Let $\gamma \in \Sigma_0(k)$ be such that $\gamma x \in X_2$ for some $x \in X_3$. Then by (3.2.4.) $\gamma = g_2 g_3$, $g_i \in \Gamma_i(1)_{\mathcal{I}}$ $i = 2,3$.

Because $\gamma \in \Sigma_0(k) = D(k)\Gamma_0(k)_{(l^k)}D(k)^{-1}$ there is a $\beta \in \Gamma_0(k)_{(l^k)}$ such that $\gamma = D(k)\beta D(k)^{-1}$. Then

$$\beta = [D(k)^{-1} g_2 D(k)][D(k)^{-1} g_3 D(k)].$$

But

$$D(k)^{-1} g_2 D(k) = g_2$$

so that

$$\beta = g_2[D(k)^{-1} g_3 D(k)]$$

which implies that $D(k)^{-1} g_3 D(k)$ has entries in O_E (because β and g_2 do). But then, because $\beta = Id \ mod\,(l^k)$, we have also that $g_2 = Id \ mod\,(l^k)$ (note that $\beta e_i = g_2 e_i$, $i = 4, ..., n$ and $\beta e_i = e_i \ \ mod\,(l^k)$ and also that g_2 has determinant one (because of 1 of (3.2.4.))). Then also $D(k)^{-1} g_3 D(k) = Id \ mod\,(l^k)$. This means that $g_2 \in \Gamma_2(k)_{(l^k)}$ and $D(k)^{-1} g_3 D(k) \in \Gamma_3(k)_{(l^k)}$. Consequently $g_i \in D(k)\Gamma_i(k)_{(l^k)}D(k)^{-1} = \Sigma_i(k) \subset \Sigma_0(k)$ $i = 2,3$. If $\bar{\pi}(k)$ denote the projection $X_0 \to M_0(k)$ then $\bar{\pi}(k)(x) = \bar{\pi}(k)(\gamma x) = \bar{\pi}(k)(g_2 g_3 x) = \bar{\pi}(k)(g_3 x)$. But

$g_3x \in X_3$ (because $x \in X_3$) and $g_3x \in X_2$ (because $g_2(g_3x) = \gamma x \in X_2$), so that $g_3x = X_2 \cap X_3 = e_{n+1}$, which means that $\bar{\pi}(k)(x) = \bar{\pi}(k)(e_{n+1}) = o$.

4. Non-positive curvature case in higher dimensions. Denote by T^m the m-torus $S^1 \times ... \times S^1$ with the canonical differentiable structure and induced PL structure τ_{T^m}. We prove here that if we take one of the examples of section 3, and product it with T^m, we still have exotic non-positively curved triangulations. To see this note that if (M, τ_0) and (M, τ_1) are two non-positively curved triangulations on M, then $(M \times T^m, \tau_0 \times \tau_{T^m})$ and $(M \times T^m, \tau_1 \times \tau_{T^m})$ are also non-positively curved. Moreover, if (M, τ_0) and (M, τ_1) are non-concordant, then $(M \times T^m, \tau_0 \times \tau_{T^m})$ and $(M \times T^m, \tau_1 \times \tau_{T^m})$ are also non-concordant, for the Kunneth formula tells us that $\mathbb{Z}_2$-cohomology classes do not vanish when we take products. So, it remains to prove that these triangulations are not equivalent. To see this is enough to prove the following (see (3.1.2.) and (3.1.3.)).

4.1. **Proposition.** *Let $f : M \times T^m \to M \times T^m$ be a homeomorphism, where M is a compact orientable hyperbolic manifold. Then $f \sim g$, where g is a diffeomorphism.*

Proof. Because $\pi_1(M)$ has trivial center (see [15]), we have that if $\phi : \pi_1(M \times T^m) \to \pi_1(M \times T^m)$ is an isomorphism, then there are isomorphisms $\phi_1 : \pi_1(M) \to \pi_1(M)$, $\phi_2 : \pi_1(T^m) \to \pi_1(T^m)$ and a homomorphism $\psi : \pi_1(M) \to \pi_1(T^m) \cong \mathbb{Z}^m$ such that

$$\phi = \phi_1 \oplus \phi_2 + 0 \oplus \psi.$$

4.1.1. Lemma. *Let M be a compact oriented differentiable manifold and $\lambda : \pi_1(M) \to \mathbb{Z}^m$ a homomorphism. Then there is a diffeomorphism $h : M \times T^m \to M \times T^m$ such that for $h_* : \pi_1(M \times T^m) \to \pi_1(M \times T^m) \cong \pi_1(M) \oplus \mathbb{Z}^m$, we have*

$$h_* = Id_{\pi_1(M \times T^m)} + 0 \oplus \lambda$$

.

Proof. Because $H_1(M, \mathbb{Z})$ is the abelianization of $\pi_1(M)$ we have that λ factors through it:

$$\pi_1(M) \overset{abelianization}{\longrightarrow} H_1(M, \mathbb{Z}) \overset{\overline{\lambda}}{\to} \mathbb{Z}^m$$

i.e. the composite of these two maps is λ. Let $\rho_i,\ i = 1, .., s$ be a base for the free abelian group $H^1(M, \mathbb{Z}) \cong Hom(H_1(M, \mathbb{Z}), \mathbb{Z})$. Then there are elements $a_i = (n_{i1}, ..., n_{im}) \in \mathbb{Z}^m$ such that $\overline{\lambda} = \sum a_i \rho_i$.

M is compact and oriented, so by Poincare duality, there are $N_i \in H_{n-1}(M, \mathbb{Z})$ dual to ρ_i. We can represent N_i by an embeded $n-1$ closed submanifold (we denote it also by N_i). These N_i have tubular nieghbourhoods $U_i \cong [0, 1] \times N_i$ and we make no distinction between U_i and their images. Define $g_i : U_i \times T^m \to U_i \times T^m$ by

$$g_i(t, x, \theta_1, ..., \theta_m) = (t, x, \theta_1 + 2\pi n_{1i}\delta(t), ..., \theta_m + 2\pi n_{mi}\delta(t))$$

where δ is smooth such that $\delta' \geq 0,\ \delta(0) = 0,\ \delta(1) = 1$ and it is constant near

0,1. Define also $h_i : M \times T^m \to M \times T^m$ by

$$h_i(x) = \begin{cases} g_i(x) & x \in U_i \times T^m \\ \\ x & x \in (M \times T^m) \setminus (U_i \times T^m) \end{cases}$$

These are well defined diffeomorphisms (because the two definitions agree on a neighborhood of ∂U_i). Finally put $h = h_1 ... h_s$. This completes the proof of the lemma because $h_* = Id_{\pi_1(M \times T^m)} + 0 \oplus \lambda$.

We complete now the proof of proposition (4.1.). Let $f : M \times T^m \to M \times T^m$ be a homeomorphism. Let ϕ_1, ϕ_2 and ψ be such that $f_* = \phi_1 \oplus \phi_2 + 0 \oplus \psi$. Let h be as in lemma (4.1.2.) where we take $\lambda = \psi \phi_1^{-1}$. Then $(h^{-1} f)_* = \phi_1 \oplus \phi_2$. By Mostow's rigidity theorem there is diffeomorphism r_1 inducing ϕ_1. Also there is diffeomorphism r_2 inducing ϕ_2. Then $(h^{-1} f)_* = (r_1 \times r_2)_*$ and by (1.6.) of [10] $h^{-1} f \sim r_1 \times r_2$ or $f \sim h(r_1 \times r_2)$, which is a diffeomorphism.

Chapter II

Examples of Non-Positively Curved Simplicial Complexes with Non-Positively Curved Exotic Triangulations

1. Preliminaries.

In this chapter all simplicial complexes are finite dimensional and locally finite.

1.1 Let X be a metric space with metric d_X. A *geodesic segment* is an isometry from an interval into X. A metric space is called *geodesic* if every two points can be joined by a geodesic segment. A *geodesic* is a local isometry from $\mathbb{R}$ to X. A geodesic space is said to be *complete* if every geodesic segment can be extended to a geodesic.

1.2 Let X be a convex linear cell complex. For every cell c_i choose a riemannian metric d_i on it (i.e. there is an isometry from (c_i, d_i) into some riemannian manifold) in such a way that the metrics coincide on each intersection of cells. This gives a way to define the length of a path in X and this determines a metric d on X by defining $d(x, y) = inf\{$lengths of paths joining x to y $\}$. X together with such a metric is called a *piecewise riemannian convex linear cell complex* or a *piecewise riemannian simplicial complex* in case X is a simplicial complex. We have that every piecewise riemannian simplicial complex is always a geodesic space (see [12]).

A piecewise riemannian simplicial complex is piecewise flat, hyperbolic or spherical, if every simplex is isometric to a simplex in $\mathbb{R}^n$ (euclidean space), $\mathbb{H}^n$ (hyperbolic space) or $\mathbb{S}^n$ (a sphere with constant sectional curvature equal to one), respectively, for some n.

1.3 Let X be a geodesic space. A triangle $[x_0, x_1, x_2]$ in X (i.e. the three points x_0, x_1, x_2 together with geodesic segments $[x_i, x_j]$) satisfies the $CAT(\delta)$ inequality (see [12]) if the following holds. For every $y \in [x_1, x_2]$ we form its comparison triangle: a triangle, on the simply connected surface M_δ of constant curvature δ, with vertices x'_0, x'_1, x'_2 such that $d_{M_\delta}(x'_i, x'_j) = d(x_i, x_j)$ and $y' \in [x'_1, x'_2]$ with $d(y, x_i) = d_{M_\delta}(y', x'_i)\, i = 1, 2$. Then $d(x_0, y) \leq d_{M_\delta}(x'_0, y')$.

1.4 A geodesic space *has curvature* $\leq \delta$, if every point has a neighborhood in which every triangle satisfies the $CAT(\delta)$ inequality. A geodesic space is *non-positively curved* if it has curvature ≤ 0. A geodesic space is *negatively curved* if it has curvature $\leq \delta$, for $\delta < 0$. We say that a space has a triangulation with some property if it is homeomorphic to a simplicial complex with that property. For example a space has a *non-positively curved piecewise flat triangulation* K if it is homeomorphic to a non-positively curved piecewise flat simplicial complex K.

1.5 Pasting Spaces. First, we say that a subspace A of a geodesic space X is locally convex if for all $a \in A$ and any two points $x, y \in A$ close enough to a, every (distance minimizing) geodesic segment joining x to y is contained in A.

Let X_1 and X_2 be two geodesic spaces with curvature $\leq \delta$ and $A_i \subset X_i$, $i = 1,2$ locally convex. Suppose there is an isometry f between A_1 and A_2. Then we can form the space X from $X_1 \bigcup X_2$ by identifying A_1 with A_2 using f. We have that X is also a geodesic space with curvature $\leq \delta$ (see [12]).

It is easy to see that if $B_i \subset X_i$, $i = 1,2$, are locally convex such that $A_i \bigcup B_i$, $i = 1,2$ are also locally convex, then the image of $B_1 \bigcup B_2$ in X is also locally convex.

Let K be a piecewise flat simplicial complex and $\mathcal{S}$ be any set of simplices of K. Define $T(K, \mathcal{S})$ to be the piecewise flat simplicial complex obtained by identifying each simplex Δ^{n_i} in $\mathcal{S}$ with a n_i-simplex of a flat torus T^{n_i+1}. By the pasting rule above, if K is non-positively curved then $T(K, \mathcal{S})$ also is, for any $\mathcal{S}$.

1.6 Products. Let X_1 and X_2 be two piecewise flat cell complexes with metrics d_i, $i = 1,2$. This determines a (euclidean) metric on each cell of $X_1 \times X_2$ (the metric $d = \sqrt{d_1^2 + d_2^2}$), and this gives a piecewise flat metric d on $X_1 \times X_2$ in the same way as in **(1.2)**. We have the following *1.6.1. Claim.*

$d = \sqrt{d_1^2 + d_2^2}$ *globally.*

Proof. Denote by $proj_i : X_1 \times X_2 \to X_i$ the projection. Denote also by $\ell(\varphi), \ell_1(\varphi), \ell_2(\varphi)$ the length of a path φ in $X_1 \times X_2, X_1, X_2$, respectively. Also if φ is a path in $X_1 \times X_2$ denote by φ^i the path $proj_i(\varphi)$ in X_i. Note that if φ is a geodesic segment in $X_1 \times X_2$ then φ^i is a geodesic segment in X_i (because geodesic segments minimize distance).

Now, write a geodesic segment φ in $X_1 \times X_2$ as $\varphi = \varphi_1 * \varphi_2 * ... * \varphi_k$, where φ_i is a straight segment on a cell $a_i \times b_i$ of $X_1 \times X_2$ (a_i is a cell of X_1 and b_i is a cell of X_2) and $*$ denotes the join of paths. Because $d = \sqrt{d_1^2 + d_2^2}$ on each cell $a_i \times b_i$ we have $\ell(\varphi_i) = \sqrt{[\ell_1(\varphi_i^1)]^2 + [\ell_2(\varphi_i^2)]^2}$.

Now we proof that the slopes of the φ_i are constant. That is, define $m_i = \frac{\ell_2(\varphi_i^2)}{\ell_1(\varphi_i^1)}$. We show that $m_i = m_{i+1}$. To see this note first that because $\varphi_i * \varphi_{i+1}$ is a geodesic segment then $\varphi_i^j * \varphi_{i+1}^j$ is a geodesic segment in X_j. It is not difficult to see that we can always embbed isometrically $a_i \cup a_{i+1}$ and $b_i \cup b_{i+1}$ in some $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, such that the image of $\varphi_i^j * \varphi_{i+1}^j$ in $\mathbb{R}^{n_j}$ is a straight segment. Then $(a_i \times b_i) \cup (a_{i+1} \times b_{i+1})$ embbeds in $\mathbb{R}^{n_1+n_2}$ by an embbeding that sends $\varphi_i * \varphi_{i+1}$ to a straight segment whose projection in $\mathbb{R}^{n_j}$ is the image of $\varphi_i^j * \varphi_{i+1}^j$ (that is a straight segment too). It follows that $m_i = m_{i+1}$. This means that $m_1 = m_2 = ... = m_k$ and call this number m. Note that also we have $m = \frac{\sum \ell_2(\varphi_i^2)}{\sum \ell_1(\varphi_i^1)} = \frac{\ell_2(\varphi^2)}{\ell_1(\varphi^1)}$.

Now we finish the proof

$$\begin{aligned}\ell(\varphi) &= \sum \ell(\varphi_i) = \sum \sqrt{[\ell_1(\varphi_i^1)]^2 + [\ell_2(\varphi_i^2)]^2} = (\sqrt{m^2+1}) \sum \ell_1(\varphi_i^1) \\ &= (\sqrt{m^2+1})\ell_1(\varphi^1) = \sqrt{[\ell_1(\varphi^1)]^2 + [\ell_2(\varphi^2)]^2}\end{aligned}$$

This proofs the claim.

Let X_1 and X_2, with metrics d_1 and d_2, be two non-positively curved piecewise flat simplicial complexes, then we have that $X_1 \times X_2$ is also a non-positively curved piecewise flat simplicial complex. To see this take a triangle in $X_1 \times X_2$ and apply the $CAT(0)$ inequality to the projections of the triangle and then use claim **(1.6.1)**.

1.7 Cones. Let K be a simplicial complex. Define the *cone* CK over K to be the join $p * K$ for some point p that we call the *vertex* of the cone. We have that cones always admit a subdivision that is non-positively curved. This follows from **(1.5)** and the following fact.

Every simplex $\Delta^n = p * \Delta^{n-1}$ can be given a canonical non-positively curved triangulation τ_n such that the following two conditions are satisfied.

(a_n) $\tau_n \mid_{\Delta_k} = \tau_k$, for every $\Delta_k = p * \Delta^{k-1}$, where Δ^{k-1} is a simplex of $\partial\Delta^{n-1}$.

(b_n) Every subcomplex of $p * \partial\Delta^{n-1}$ is locally convex.

To obtain this define τ_n by induction. For $n = 0$ we have no choice. Suppose τ_{n-1} is defined. Then, because of **(1.5)**, by glueing simplices Δ_{n-1} with metric τ_{n-1} we obtain a non-positively curved triangulation of $p_n * \partial\Delta^{n-1}$ satisfying (a_n) and (b_n). Then, because $(p * \partial\Delta^{n-1}) \times [0,1] \cong_{PL} \Delta^n$, we can obtain τ_n just by crossing with $[0,1]$.

1.8 Completeness. Let K be a simplicial complex. Recall that the star $star(s, K)$ of a simplex s in K is the subcomplex of K formed by all simplices that contain s, together with their faces. Also the link $link(s, K)$ is formed by all simplices in $star(s, K)$ that do not contain s. We have $s * link(s, K) = star(s, K)$. If K is piecewise flat then, by considering solid angles, $link(s, K)$ has a naturally defined piecewise spherical metric (see [12],[7]).

A piecewise spherical polyhedron L, with metric d, is said to be large if any two points x and y with $d(x,y) < \pi$ can be joined by a unique geodesic. This implies that if L is large and $x \in K$, then $B_x(\pi)$ is contractible, where $B_x(\pi)$ is the open ball of radius π and center x. We have the following (see [12])

1.8.1. Lemma. *Let K be piecewise flat simplicial complex. Then the following are equivalent.*

(1) K is non-positively curved.

(2) $link(s, K)$ is large, for every simplex s in k.

(3) $link(s, K)$ satisfies the $CAT(1)$ inequality for all triangles of perimeter less than 2π, and for every simplex s in k.

Let K be a piecewise flat simplicial complex and $x \in K$. Define for $v \in link(x, K)$ the infinitesimal shadow $shad(x, v)$ to be the complement of $B_v(\pi)$ in $link(x, K)$, where $B_v(\pi)$ is the open ball of radius π and center v (we consider here $link(x, K)$ with the spherical metric).

Let $\varphi : [a, b] \to K$ be a geodesic segment in K. Then φ defines a point $v_\varphi \in link(\varphi(b), K)$ (given by the direction of φ at $\varphi(b)$, see [7]). We have the following lemma that is a consequence of **(2d.1)** of [D].

1.8.2. Lemma. *φ can be continued at $\varphi(b)$ if and only if $shad(\varphi(b), v_\varphi) \neq \phi$.*

We say that a simplicial complex K *has a free face* Δ^n if Δ^n is a simplex of K that is the face of exactly one $n+1$ simplex Δ^{n+1} of K.

1.8.3. Lemma. *Let K be a piecewise flat simplicial complex. Consider the following statements.*

(1) K is complete.

(2) $link(x, K')$ is not simplicialy collapsible, for all $x \in K$, where K' is a subdivision of K.

(3) $link(x, K)$ does not have a free face, for all $x \in K$.

(4) K does not have a free face.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). *Furthermore, if K is non-positively curved then the four statements are equivalent.*

Proof. See the appendix.

1.9 Let X be a simply connected space with a piecewise flat non-positivel curved triangulation K. We say that $A \subset X$ is *star shaped* if there is a point x_0 (called the *center*) such that for all $a \in A$, the (unique, see [12]) geodesic segment joining a to x_0 is contained in A. In the following lemma the symbol $\bigwedge$ denotes simple homotopy, in the sense of [6].

1.9.1 Lemma. *Let K be a piecewise flat non-positively curved triangulation of X. Let also $x_0 \in K$ and $A \subset X$ be star shaped with center x_0, such that $star(x_0, K) \subset int(A)$. Then $\overline{A \setminus star(x_0, K)} \bigwedge link(x_0, K) \quad (rel\ link(x_0, K))$.*

Proof. We use the groups $Wh_\epsilon(link(x_0, K))$ of spaces with ϵ-controlled deformation retracts onto $link(x_0, K)$ defined in [5]. Note that contraction along geodesics emanating from x_0 defines a deformation retract r of $\overline{A \setminus star(x_0, K)}$ in $link(x_0, K)$ (see [12]). We have that r is 0-controlled (see [5], p.13), so that $r \in Wh_\epsilon(link(x_0, K))$, for all ϵ. We use the following result of [5] (see corolary 1 on page 73)

For every ϵ there is a $\delta < \epsilon$ such that the map

$Wh_\delta(link(x_0, K)) \to Wh_\epsilon(link(x_0, K))$ is zero.

(The map here is the inclusion map: every δ-controlled deformation retract is also ϵ-controlled, for $\delta < \epsilon$.)

Then $r = 0 \in Wh_\epsilon(link(x_0, K))$. This implies

$$\overline{A \setminus star(x_0, K)} \bigwedge link(x_0, K) \quad (rel\ link(x_0, K)).$$

This completes the proof of the lemma. .

2. First example. Here we give an example of a compact space with two non-equivalent non-positvely curved piecewise flat complete triangulations.

Let H be a PL h-cobordism (of dimension ≥ 6) between M_1 and M_2, such that M_1 is not PL homeomorphic to M_2 (see [11]).

Define

$$Y_i = CM_i \times \mathbb{S}^1 \qquad i = 1, 2$$

where $\mathbb{S}^1$ is the 1-sphere. Call p_i the vertex of CM_i. Note that, by construction, Y_1 and Y_2 come equipped with PL structures.

2.1 Lemma. *There is a homeomorphism $f : Y_1 \to Y_2$, such that $f\mid_{Y_1 \setminus (\{p_1\} \times \mathbb{S}^1)}$ and $f\mid_{\{p_1\} \times \mathbb{S}^1}$ are PL.*

Proof. First, denote by $(CM_2) \bigcup_\alpha (M_2 \times [0, 1])$ the space obtained by identifying $m \in M_2 \subset CM_2$ with $\alpha(m) = (m, 0) \in M_2 \times \{0\} \subset M_2 \times [0, 1]$.

Now, we have

2.2 $Y_2 = CM_2 \times \mathbb{S}^1 \cong_{PL} [(CM_2) \bigcup_\alpha (M_2 \times [0,1])] \times \mathbb{S}^1$
$= (CM_2 \times \mathbb{S}^1) \bigcup_{\alpha \times \iota} (M_2 \times [0,1] \times \mathbb{S}^1)$

where $\iota : \mathbb{S}^1 \to \mathbb{S}^1$ is the identity.

Because the euler characteristic of $\mathbb{S}^1$ is zero we have that $H \times \mathbb{S}^1$ is trivial (see [6]), that is $H \times \mathbb{S}^1 \cong_{PL} (M_2 \times \mathbb{S}^1) \times [0,1]$ (and we assume that the PL homeomorphism satisfies $(m,\theta) \mapsto (m,\theta,0)$, for $(m,\theta) \in M_2 \times \mathbb{S}^1 \subset H \times \mathbb{S}^1$).

Then, from **(2.2)** we obtain

$$Y_2 \cong_{PL} (CM_2 \times \mathbb{S}^1) \bigcup_{\alpha' \times \iota} (H \times \mathbb{S}^1) = (CM_2 \bigcup_{\alpha'} H) \times \mathbb{S}^1$$

where α' is the obvious map.

But $CM_2 \bigcup_{\alpha'} H$ is homeomorphic to CM_1 by a homeomorphism that is PL outside the vertex (see [18]). So Y_2 is homeomorphic to $CM_1 \times \mathbb{S}^1 = Y_1$. This completes the proof of the lemma.

Note that because of **(1.6)** and **(1.7)**, both Y_1 and Y_2 admit non-positively curved piecewise flat triangulations K_i, $i = 1,2$, compatible with the PL structures. Remark that because of the lemma there is a PL homeomorphism between $M_1 \times \mathbb{S}^1$ and $M_2 \times \mathbb{S}^1$. Let L_i be a subdivision of K_i such that this PL homeomorphism is simplicial with respect to the $L_i\,|_{M_i \times \mathbb{S}^1}$, $i = 1,2$. Let $\mathcal{L}_i$ be the set of all simplices of $L_i\,|_{M_i \times \mathbb{S}^1}$ of highest dimension.

Finally define (see **(1.5)**)

$$X_i = T(Y_i, \mathcal{L}_i) \qquad i = 1,2$$

Then both X_i admit piecewise flat non-positively curved triangulations. We have that these spaces are complete (see **(1.8.3)**). Also X_1 and X_2 are homeomorphic by a homeomorphism that is PL outside $\{p_i\} \times \mathbb{S}^1$ and also when restricted to $\{p_i\} \times \mathbb{S}^1$. Note that these spaces are not PL equivalent because the link pair in $(X_i, \{p_i\} \times \mathbb{S}^1)$ of a point in $\{p_i\} \times \mathbb{S}^1$ is PL equivalent to $(M_i, \{two\ \ points\})$ (see p.50 of [20]).

Remark. Note that the only facts about $\mathbb{S}^1$ we used were that the euler characteristic is zero and that is non-positively curved. Then we can replace $\mathbb{S}^1$ by any compact manifold satisfying these two conditions, for example, $M \times \mathbb{S}^1$ when M is compact and non-positively curved.

2.3. Alternative Construction. Instead of using the construction in **(1.5)** we can use other method that will give us a complete simplicial complex with "less singularities". For this we use relative hyperbolization [12]. Recall that the relative hyperbolization $h(K, L)$ of a pair (K, L) of simplicial complexes is obtained by replacing the simplices of the baricentric subdivision of K that are not in L by a canonical hyperbolized simplex. The result has the property that if L is non-positively curved (with respect to a piecewise flat metric defined on K) then $h(K, L)$ is also non-positively curved (the metric in K determines a metric in $h(K, L)$).

Now let W, compact, be such that $\partial W = M_1$ (see **(1.8)**). If M_1 is not a boundary we can take $W = CM_1$. Define

$$Y_i = (CM_i \times \mathbb{S}^1) \bigcup_{\beta_i} (W \times \mathbb{S}^1) \qquad i = 1, 2$$

where β_1 is the identity and β_2 is a homeomorphism that is a simplicial map (with respect to some subdivision of $K_2 |_{M_2 \times \mathbb{S}^1}$, see **(2.1)**). Finally define

$$X_i = h(Y_i, CM_i \times \mathbb{S}^1) \qquad i = 1, 2.$$

Then, again, both X_i are homeomorphic by a homeomorphism that is PL outside $\{p_i\} \times \mathbb{S}^1$. also both admit piecewise flat complete non-positively curved triangulations. These spaces are not PL equivalent because of the same reason as before. Note that if W is a manifold then the X_i are manifolds outside the one dimensional strata $\{p_i\} \times \mathbb{S}^1$. In general we can get a one dimensional strata that is PL equivalent to two $\mathbb{S}^1$.

3. Second Example. Here we give an example of a compact space with two non-equivalent triangulations, one being piecewise flat complete non-positively curved, and the other not admitting a piecewise flat non-positively curved subdivision (i.e. the space with this second triangulation is not PL equivalent to a space with a non-positively curved piecewise flat triangulation).

Let H be a PL h-cobordism (of dimension ≥ 6) between M_1 and M_2. Choose a non-positively curved triangulation K_1 for CM_1 (see **(1.7)**) and a triangulation K_2 of H such that $K_1 |_{M_1} = K_2 |_{M_1}$. Let $\mathcal{K}_i$ be the set of all simplices of $K_i |_{M_1 \times S^1}$ of highest dimension.

Define

$$X_1 = T(CM_1, \mathcal{K}_1)$$

$$X_2 = T(CM_2 \bigcup_\alpha H, \mathcal{K}_2)$$

where α is as in **(2.2)**. Note that both X_1 and X_2, by contruction, come equipped with PL structures.

Then X_1 can be given a non-positively curved complete piecewise flat triangulation (see **(1.5)**,**(1.7)** and **(1.8.3)**). Also, because of [18], X_1 is homeomorphic to X_2 by a homeomorphism that is PL outside the vertex. The following lemma implies that if H is not a product then X_2 does not admit a non-positively curved piecewise flat subdivision.

3.1 Lemma. *If X_2 admits a non-positively curved subdivision then H is trivial (i.e. is a product).*

Proof. If X_2 admits a non-positively curved subdivision then the same happens to its universal cover $\tilde{X}_2$. Denote by $\pi : \tilde{X}_2 \to X_2$ the covering projection and choose a point q such that $\pi(q) = p$, where p is the vertex of $CM_2 \subset X_2$. Let N be the lifting of $CM_2 \bigcup_\alpha H$, with vertex q. Let $S \subset \tilde{X}_2$ be star shaped with center q such that $N \subset S$ (take for example a large ball with center q). We will still denote by M_1, M_2 and H the liftings, contained in N, of M_1, M_2 and H respectively.

Note that $\tilde{X}_2 \setminus N$ has ℓ components, one for each simplex Δ_i of highest dimension in $K_2 \mid_{M_1}$. Write $\tilde{X}_2 \setminus N = Z_1 \bigcup \ldots \bigcup Z_\ell$ (disjoint union) and define $S_i = S \bigcap \overline{Z_i}$.

3.2 Assertion. $S_i \bigwedge \Delta_i \quad (rel\ \Delta_i)$.

Proof. Because of **(1.9.1)**, $\overline{S \setminus star(q, \tilde{X}_2)} \bigwedge link(q, \tilde{X}_2)$, which means that

$$\pi_n(\overline{S \setminus star(q, \tilde{X}_2)}, link(q, \tilde{X}_2)) = 0$$

for all n.

On the other hand we have (see [20], p.21)

$$\textbf{3.3} \qquad (\overline{S \setminus star(q, \tilde{X}_2)}, link(q, \tilde{X}_2)) \cong_{PL} (\overline{S \setminus CM_2}, M_2).$$

Also ($\sim$ denotes homotopy equivalence of pairs)

$$(\overline{S \setminus CM_2}, M_2) \sim (\overline{S \setminus N}, M_1) \sim (\overline{S \setminus N}/\Delta_i, M_1/\Delta_i)$$

where the first equivalence is because H is an h-cobordism and the second because simplices are contractible. Then $\pi_n(\overline{S \setminus N}/\Delta_i, M_1/\Delta_i) = 0$ for all n.

But $(S_i/\Delta_i, \Delta_i/\Delta_i)$ is a retract of $(\overline{S \setminus N}/\Delta_i, M_1/\Delta_i)$ and

$$(S_i/\Delta_i, \Delta_i/\Delta_i) \sim (S_i, \Delta_i)$$

so that also $\pi_n(S_i, \Delta_i) = 0$, for all n. Then, because Δ_i is simply connected, **(8.5)** of [6] implies the asertion.

Note that from the assertion follows that $N \bigwedge S \ \ (rel\ N)$,
so $H \bigwedge \overline{S \setminus CM_2} \ \ (rel\ H)$, which implies $H \bigwedge \overline{S \setminus CM_2} \ \ (rel\ M_2)$.

Then

$$H \bigwedge \overline{S \setminus CM_2} \bigwedge M_2 \quad (rel\ M_2)$$

where the last relation follows from **(1.9.1)** and **(3.3)**.
All this implies $H \bigwedge M_2 \ \ (rel\ M_2)$. This means that H is trivial (see [6]) and completes the proof of the lemma.

In fact the space X_1 has some kind of rigidity, as the next corollary shows.

3.2. Corollary. *Let X be a non-positively curved piecewise flat simplicial complex. Suppose X is homeomorphic to X_1 by a homeomorphism $f : X_1 \to X$ that is PL on each strata. Assume that the triangulation of X restricted to each attached torus $f(T^n)$ and to $f(CM_1 \setminus \{vertex\})$ is PL. Then $X \cong_{PL} X_1$.*

Proof. We prove that X is a space contructed in the same way as X_2 before. Consider $f\,|_{T^{n_i+1}}$, where T^{n_i+1} is one of the torus we glued to CM_1 along the simplex Δ^{n_i}. We have that $f\,|_{T^{n_i+1}\setminus\Delta^{n_i}}$ and $f\,|_{\Delta^{n_i}}$ are PL. Recall that T^{n_i+1} has a canonical flat PL structure and compare it with the PL structure pulled back, by f, from $f(T^{n_i+1})$. These two PL structures coincide outside Δ^{n_i}. Then because Δ^{n_i} is contractible and obstructions to PL structures on a manifold lie on the third comology group of the manifold, we have that in fact these PL structures coincide. This means that we can suppose $f\,|_{T^{n_i+1}}$ to be PL.

Let $M = link(q, X)$ where q is the vertex of X. We have that M is a manifold and $H = X \setminus \bigcup f(T^{n_i+1}) \setminus int(star(q, X))$ is an h-cobordism between M and M_1. Lemma **(3.1)** implies that H is trivial, so that $H \cong_{PL} CM$. Also $f\,|_{M_1}\colon M_1 \to M$ is PL, so that $f\,|_{M_1}$ is PL and $M_1 \cong_{PL} M$. This completes the proof of the corollary.

Remark. For a similar example with riemannian manifolds, consider the torus T^n, $n > 5$. [15] implies that the only differentiable structure that admits a non-positively curved riemannian metric is the canonical one.

4. Third Example. Here we give an example of a compact simplicial complex with two negatively curved piecewise riemannian complete non-equivalent triangulations.

Let $\epsilon > 0$. Let $N_i \subset M_i \;\; i = 1,2$ be closed riemannian manifolds such that

(1) N_i is totally geodesic in M_i.

(2) M_1 is hyperbolic.

(3) All sectional curvatures of M_2 lie in $(-1-\epsilon, -1+\epsilon)$.

(4) There is a homeomorphism $f : M_1 \to M_2$ such that $f\,|_{M_1 \setminus N_1}$ and $f\,|_{N_1}\colon N_1 \to N_2$ are diffeomorphisms.

(5) M_1 and M_2 have non-equivalent PL structures (induced by the differentiable structures).

For examples of such manifolds (in dimension 6) see chapter **I** . Let X_i be the space obtained by identifying two copies of M_i along N_i, or let X_i be M_i considered as a stratified space with strata N_i and $M_i \setminus N_i$. In any case, both, X_1 and X_2, are negatively curved and complete. Also they are homeomorphic but not PL homeomorphic. Remark that the Anderson-Hsiang obstructions for the PL structures to be equivalent come from Kirby-Siebenman manifold obstructions.

Appendix

Here we give the proof of

1.8.3. Lemma. *Let K be a piecewise flat simplicial complex. Consider the following statements.*

(1) K is complete.

(2) $link(x, K')$ is not simplicialy collapsible, for all $x \in K$, where K' is a subdivision of K.

(3) $link(x, K)$ does not have a free face, for all $x \in K$.

(4) K does not have a free face.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). *Furthermore, if K is non-positively curved then the four statements are equivalent.*

Proof. (3) $\Leftrightarrow$ (4). This is because $x * link(x, K) = star(x, K) \subset K$. (2) $\Rightarrow$ (4). Take a point in the interior of the free face. (3) $\Rightarrow$ (2). If a complex is collapsible it has a free face (we need some piece to begin the collapse), and if K' has a free face, so does K. (1) $\Rightarrow$ (4). If K has a free face Δ^n, take a straight segment transversal to it. That geodesic can not be continued.

Assume now K to be non-positively curved. We prove (3) $\Rightarrow$ (1). Suppose K is not complete. By **(1.8.2)** there is a point $x \in K$, such that $shad(x, v) = \phi$, for some $v \in link(x, K)$, which means that $B_v(\pi) = link(x, K)$. We have the following

a.1. Claim. $link(x, K)$ has a free face.

Proof. Write $L = link(x, K)$. Denote by d the metric of L. Given $a \in K$ denote by $\theta_a : [a, d(v,a)] \to L$ the (unique) geodesic segment joining a to v, parametrized by arc length. Consider $\Theta : L \times [0,1] \to L$ given by $\Theta(a,t) = \theta_a(t\, d(v,a))$. Θ is a deformation retract. In fact we have

Step 1. Θ *is lipschitz.*

Proof. In the t direction Θ is lipschitz because it is an isometry of [0,1] in to K. We prove Θ_t is lipschitz. Let $\rho = \pi - max\{d(x,v) : x \in L\}$. Because L is finite we have $\rho > 0$. Let w, z be the north and south pole, respectively, of the two-sphere $\mathbb{S}^2$ and consider the geodesic retraction, Θ', to the north pole w defined on $\mathbb{S}^2 \setminus B_z(\rho)$. Because this function is smooth on a compact, it is lipschitz and call k its constant.

By **(1.8.1)** we have that $L = link(x, K)$ satisfies the $CAT(1)$ inequality, for all triangles of perimeter less than 2π. Let $a \in L$ and $\epsilon > 0$ be small enough so that all triangles with vertices $a, v, b \in L$, where $b \in B_a(\epsilon)$, have perimeter less than 2π (recall that $d(v,a) < \pi$). We have that these triangles satisfy the $CAT(1)$ inequality and let a', b', w be the vertices of the comparison triangle in $\mathbb{S}^2$ (we make v correspond to the north pole w). Let a', c, w be the vertices of the comparison triangle of the triangle with vertices $a, \Theta_t(b), v$. Then $d(\Theta_t(a), \Theta_t(b)) \le d_{\mathbb{S}^2}(\Theta'_t(a'), c)$ (note that $\Theta'_t(a')$ corresponds to $\Theta_t(a)$). Note also that, by comparing the triangle with vertices a, b, v with its comparison triangle, $d_{\mathbb{S}^2}(a', c) = d(a, \Theta_t(b)) \le d_{\mathbb{S}^2}(a', \Theta'_t(b'))$ (here $\Theta_t(b)$ corresponds to $\Theta'_t(b')$). Then $< a'wc \le < a'w\Theta'_t(b')$. It follows that

$$d(\Theta_t(a), \Theta_t(b)) \le d_{\mathbb{S}^2}(\Theta'_t(b'), c) \le d_{\mathbb{S}^2}(\Theta'_t(a'), \Theta'_t(b')) \le k d_{\mathbb{S}^2}(a', b') = kd(a, b)$$

This means that all Θ_t are lipschitz with common constant k. This completes the proof of step 1.

Step 2. *There is a siplex Δ^n of L, where n is the dimension of L, containing an end point in its interior or in the interior of one of its $n-1$ faces.*

An end point is the end point of a geodesic segment, begining at v, that can not be continued.

Proof. Consider $\Theta(L_{n-2} \times [0,1])$, where L_{n-2} is the $(n-2)$-skeleton of L. Because Θ is lipschitz $\Theta(L_{n-2} \times [0,1])$ has dimension at most $n-1$ so that $L \setminus \Theta(L_{n-2} \times [0,1]) \neq \phi$. This proves step 2 because every θ_a can be continued until it has an end point (recall that $L = link(x, K)$ is finite).

Step 3. *If $a \in L$ is an end point, then $link(a, L)$ is contractible.*

Proof. Let $a \in L$ be an end point. Consider $star(a, L) = a * link(a, L) = [0,1] \times link(a, L)/(\{0\} \times link(a, L))$. Denote by r the radial retraction $r : star(a, L) \to \{\frac{1}{2}\} \times link(a, L) \subset star(a, L)$, given by $r(t, a) = (\frac{1}{2}, a)$. Define, for $t < 1$, $\Psi_t = r\Theta_t : star(a, L) \to \{\frac{1}{2}\} \times link(a, L)$. Note that Ψ_t is well defined because a is not in $\Theta(L, t)$, for $t < 1$ (recall that a is an end point).

Let $t < 1$ be close enough to 1, such that $\Theta(\{\frac{1}{2}\} \times link(a, L)) \subset star(a, L)$. We have

$$\{\frac{1}{2}\} \times link(a, L) \xrightarrow{i} star(a, L) \xrightarrow{\Psi_t} \{\frac{1}{2}\} \times link(a, L)$$

where i is the inclusion. But $\Psi_t \mid_{\{\frac{1}{2}\} \times link(a,L)}$ is homotopic to the identity in $\{\frac{1}{2}\} \times link(a, L)$ and at the same time factors through a contractible space (i.e. $star(a, L)$). This happens only if $link(a, L)$ is contractible.

Step 4. We complete the proof. Suppose L does not have a free face. Because of step 2, there is an end point $a \in L$ such that it belongs either to the interior of a Δ^n or to the interior of one of its $n-1$ faces (recall n is the dimension of L). In any case we have that $link(a, L)$ is not contractible, which contradicts step 3. This completes the proof of the claim and the proof of the lemma.

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