Darboux Theorems for Pairs of Submanifolds

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1994

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation

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A variety of fundamental results in symplectic geometry provide for a local characterization of various geometric objects: symplectic manifolds, submanifolds, foliations, etc., the most fundamental and elementary of which is Darboux's Theorem. This thesis examines the extent to which the interior geometry of a pair of submanifolds in a symplectic manifold determines the exterior geometry. Complete results are given for a general case where the submanifolds are symplectic.

To my wife Karmen, and my friend E. C. Taylor.

Contents

Acknowledgements

First and foremost I thank Professor Dusa McDuff for the opportunity to work with her. Long after the dust has settled, her advice and encouragement will be remembered. I would also like to thank Wladek Derek Lorek, Simon Richard, and Professor Leon Takhtajan for their time and insight.

I owe a debt of gratitude to Joe Schaeffer, Brian Yarrington and Leroy Wenstrom for their warm friendship and support over the years, but most especially am I indebted to Edward Taylor, whose unending enthusiasm, companionship, and keen mind will be sorely missed.

Elizabeth Maracle and Kristin and Meg Johnson were nothing less than angels to me. Without them my days would have been longer and darker.

Finally, I want to thank my family for their help and understanding, and my wife Karmen for her love and friendship.

Chapter 1

Introduction

Definition 1.0.1 A **symplectic manifold** (V^{2n}, ω) is an even-dimensional real manifold V with a closed, nondegenerate 2-form ω called the **symplectic form.**

Example 1.0.2 $V = \mathbf{R}^{2n}$ with the symplectic form

$$dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

is a symplectic manifold. This example will appear throughout this thesis; so for convenience we label this space as $(\mathbf{R}^{2n}, \tau_{2n})$ (or simply $(\mathbf{R}^{2n}, \tau)$).

Remark 1.0.3 Henceforward we assume V to be compact and connected (except when $(V, \omega) = (\mathbf{R}^{2n}, \tau)$).

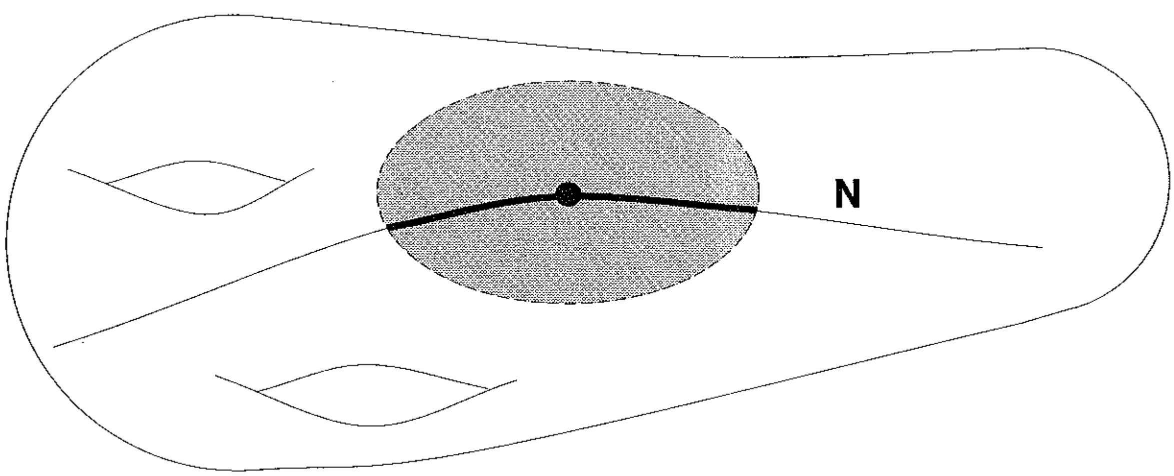
Symplectic manifolds play an important role in classical mechanics, geometrical optics, representation theory, and Kähler geometry. A foundational result is the celebrated Darboux's Theorem, which gives a local characterization for symplectic geometry.

Theorem 1.0.4 (Darboux's Theorem) *Every point p of a symplectic manifold (V^{2n}, ω) has local coordinates (x_i) $(i = 1, \ldots, 2n)$ so that $\omega = \tau_{2n}$.*

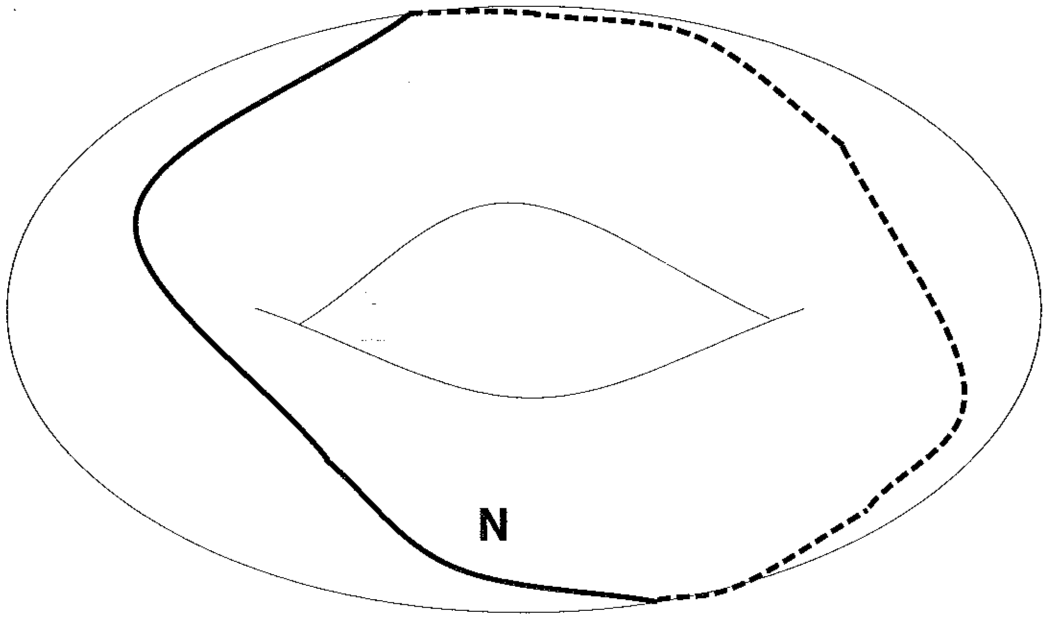
Definition 1.0.5 U is a **Darboux neighborhood,** (x_i) are **Darboux coordinates**, and τ is the **standard symplectic form** on $\mathbf{R}^{2n}$.

Darboux's Theorem shows that all $2n$-dimensional symplectic manifolds locally look like Example 1.0.1, and therefore dimension is the only local invariant. This gives symplectic geometry a markedly different flavor from that of Riemannian geometry, where curvature is a local invariant.

Viewing a point in V as a submanifold, Darboux's Theorem tells us what the symplectic geometry is like near that submanifold. Two perspectives on extending Darboux's Theorem are: Characterize how the geometry of a submanifold near a point (the intrinsic geometry) influences the geometry nearby (the exterior geometry),



or characterize how the geometry of an entire submanifold (intrinsic) influences the geometry nearby (exterior).

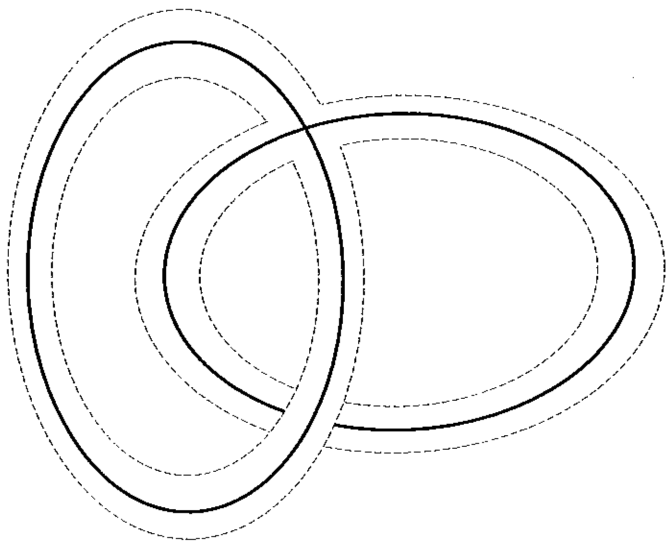


We will label such characterizations as Local and Global Darboux Theorems, respectively. Again, it is interesting to contrast the symplectic case with the Riemannian one, where the second fundamental form is an invariant.

Definition 1.0.6 If $\phi : U \to V$ is a diffeomorphism between symplectic manifolds (U, ω) and (V, τ), then ϕ is a **symplectomorphism** iff $\phi^* \tau = \omega$. If such a symplectomorphism exists, then (U, ω) and (V, τ) are **symplectomorphic** and have **equivalent symplectic stuctures.**

The most fundamental problem in symplectic geometry is determining when two symplectic manifolds are symplectomorphic. Darboux's Theorem gives a partial answer by showing that any point of a symplectic manifold (V, ω) is locally symplectomorphic to a neighborhood of the origin in $(\mathbf{R}^{2n}, \tau)$.

In this thesis we investigate conditions under which two germs of pairs of submanifolds (in general position) are transformable one into the other via a symplectomorphism of tubular neighborhoods.



A necesssary condition for the existence of this equivalence is for the intrinsic geometry of each pair to be symplectomorphic, and this condition will be assumed throughout the thesis (we will say that the pairs are **weakly allowable**). The problem of sufficiency naturally arises, a problem which is the driving force behind our work. The attack on this problem is broken-up into two chapters, Chapters 2 and 3. Chapter 2 further assumes, in addition to weak allowability, that the germs of the symplectic structures are equivalent along their intersection: we will say in this case that the pairs are **allowable**. Our main results then are

Local Darboux Theorem II *Allowability locally defines a pair of submanifolds (in general position) of a symplectic manifold V in a neighborhood of any of its points up to a local symplectomorphism. (See Theorem 2.2.3)*

and

Global Darboux Theorem II *If two germs of pairs of submanifolds of the symplectic manifold V can be smoothly deformed one to the other via allowable pairs in such a way that the induced forms have constant rank, then there is a smooth family of symplectomorphisms between neighborhoods of the pairs. (See Theorem 2.2.6)*

Chapter 3 solves the local equivalence problem (for a generic case) when the pairs are only weakly allowable.

Local Classification Theorem for Pairs *Any pair in generic position is locally symplectomorphic to a neighborhood of a standard model. (See Corollary 3.5.6)*

For simplicity we will assume that we are given one germ with two symplectic structures inducing identical closed 2-forms on each submanifold. We tackle the problem of equivalence for pairs in the next chapter, but for now let us look at the analagous question for submanifold germs.

Definition 1.0.7 Write (V, ω_0, ω_1) (or simply V) for an even-dimensional manifold with two symplectic forms ω_0 and ω_1. If N is a submanifold in V, then N is **allowable** iff the forms induced on N by ω_0 and ω_1 equal one another (i.e. $i_N^*\omega_0 = i_N^*\omega_1$ where $i_N : N \hookrightarrow V$ is the inclusion map). If $f : U \to W$ is a diffeomorphism between subsets U and W of V, then f is **allowable with respect to N** (or just **allowable**) iff $f\mid_{U\cap N}$ is the identity map.

Theorem 1.0.8 (Local Darboux Theorem I) *([A], [AG]) The restriction* *of the symplectic structure to a submanifold N of a symplectic manifold V defines N locally in a neighborhood of any of its points up to a local symplectomorphism.*

More specifically Theorem 1.0.8 says the following: If N is an allowable submanifold of (V, ω_0, ω_1), then for any point $p \in N$ there is an allowable symplectomorphism $\phi : (U_0, \omega_0) \to (U_1, \omega_1)$ between two neighborhoods U_0 and U_1 of p in V.

Remark 1.0.9 If N is a point (Darboux's Theorem) then $i_N^*\omega_0 = i_N^*\omega_1 = 0$ for all symplectic forms ω_0 and ω_1 and so any point is an allowable submanifold (since its interior geometry is always trivial).

Local Darboux Theorem I (LDT I) shows that there are no local exterior invariants of submanifolds, that the interior geometry controls the exterior one. For the global case, however, the interior geometry does not always determine the exterior.

Theorem 1.0.10 (Global Darboux Theorem I) *([A], [AG]) Let N be an allowable submanifold of (V, ω_0, ω_1) (see Definition 1.0.7). Suppose further that ω_0 and ω_1 are smoothly deformable into one another in the class of symplectic structures on V whose induced forms on N always equal one another (i.e. $i_N^*\omega_0 = i_N^*\omega_1$ where $i_N : N \hookrightarrow V$ is the inclusion map). Then there exists an allowable symplectomorphism $\phi : (\mathcal{U}_0, \omega_0) \to (\mathcal{U}_1, \omega_1)$ between neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in V.*

Definition 1.0.11 A smooth path of symplectic forms ω_t, $0 \le t \le 1$, on V whose cohomology classes $[\omega_t]$ are constant ($[\omega_t] = k \;\; \forall t \in [0,1]$) is an **isotopy** of symplectic forms on V. An **allowable isotopy** of symplectic forms ω_t, $0 \le t \le 1$, on (V, N) is an isotopy of symplectic forms on V whose induced forms on N, $i_N^*\omega_t$, are equal (i.e. $i_N^*\omega_t = i_N^*\omega_0 \;\; \forall t \in [0,1]$).

Theorem 1.0.12 (Global Darboux Theorem I) *(revisited) If, in addition to the hypothesis given for Global Darboux Theorem I above, we assume that the deformation of symplectic forms is, in fact, an allowable isotopy, then there is an allowable symplectomorphism* $\phi : V \to V$.

In general it is difficult to know when ω_0 is homotopic to ω_1 as required by Global Darboux Theorem I (GDT I) because the forms induced on N can vary in rank along N in complicated ways (see [Ma]). If the condition $i_N^*\omega_0 = i_N^*\omega_1$ is strengthened to $\omega_0 = \omega_1$ on T_NV, then $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ gives an acceptable homotopy. The problem then arrises of how to go about finding an allowable diffeomorphism $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\varphi^*\omega_1 = \omega_0$ along N. If we assume not only that $i_N^*\omega_0 = i_N^*\omega_1$ but that $i_N^*\omega_a$ has constant rank along N as well, then the next theorem gives the answer. But first some notation.

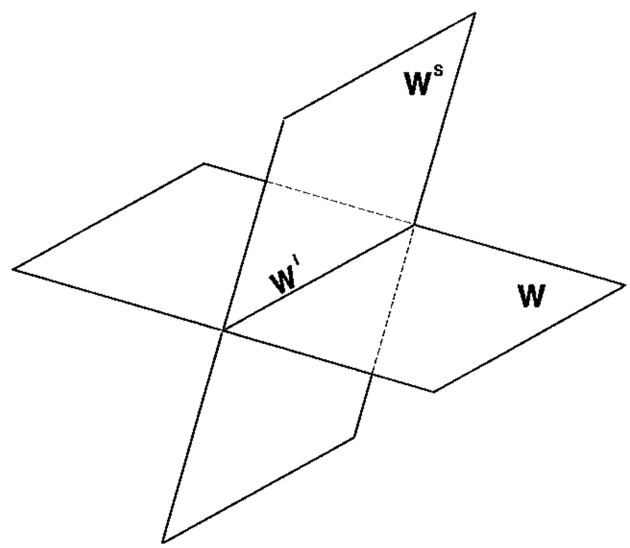
Definition 1.0.13 A **symplectic vector space** (V, Ω) is a vector space V with a non-degenerate, skew-symmetric bilinear form Ω called the **symplectic form**. If $W \subset V$ is a subspace of V then

$$W^s \stackrel{d}{=} \{v \in V \mid \forall w \in W,\ \Omega(v,w) = 0\}$$

is the **skew-orthogonal** (or **symplectically orthogonal**) **subspace of W**. The subspace $W^i \subset W$ defined by

$$W^i \stackrel{d}{=} W \cap W^s$$

is the **radical of W**. Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are **symplectomorphic** iff there is a linear isomorphism $\Phi : V_1 \longrightarrow V_2$ such that $\Omega_1(v, w) = \Omega_2(\Phi(v), \Phi(w))\ \ \forall v, w \in V_1$.



Remark 1.0.14 If p is a point in a symplectic manifold (V, ω), then T_pV (the tangent space of V at p) is a symplectic vector space with symplectic form $\omega(p)$. A symplectic manifold (V, ω) is then a smooth manifold V with a smooth field ω of symplectic forms defined on the fibers of the tangent bundle TV.

Corollary 1.0.15 *([M]) Suppose that N is an allowable submanifold (see Definition 1.0.7) of (V, ω_0, ω_1) such that the forms $i_N^*\omega_0$ and $i_N^*\omega_1$ have constant* *rank along N. Then there is an allowable symplectomorphism $\phi : (\mathcal{U}_0, \omega_0) \to (\mathcal{U}_1, \omega_1)$ between neighborhoods $\mathcal{U}_a$ $(a = 0, 1)$ of N in V iff $\mathbf{1}_N$ (the identity map on N) can be lifted to a vector bundle symplectomorphism $\widehat{\mathbf{1}_N} : T^{s_0}N/T^{i_0}N \to T^{s_1}N/T^{i_1}N$.*

Remark 1.0.16 $T^{s_a}N$ denotes the vector bundle over N whose fibers at points $p \in N$ consist of the skew-orthogonal subspaces (with respect to ω_a) of T_pN in T_pV, and $T^{i_a}N$ is the subbundle of $T^{s_a}N$ whose fibers are the radical subspaces of T_pN. Each fiber of $T^{s_a}N/T^{i_a}N$ is a symplectic vector space.

Sketch of Proof. The existence of the vector bundle symplectomorphism $\widehat{\mathbf{1}_N}$ allows us to locally deform ω_0 so that it equals ω_1 on T_NV (use the tubular neighborhood theorem to equate the vector bundles $T^{s_a}N/T^{i_a}N$ with tubular neighborhoods of N). Putting $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ allows us to apply GDT I (Theorem 1.0.10) and finish the proof. ∎

In general it is difficult to know when $\widehat{\mathbf{1}_N}$ exists. One particularly nice case is when N is a Lagrangian submanifold (i.e. $T^{s_a}N = T^{i_a}N$ for $a = 0, 1$). In this case $\widehat{\mathbf{1}_N}$ always exists since $T^{s_a}N/T^{i_a}N = 0$. As a result some neighborhood of any Lagrangian submanifold in any symplectic manifold is symplectomorphic to some neighborhood of this Lagrangian submanifold in any other symplectic manifold. In general $T^{i_a}N$ is a proper subbundle of $T^{s_a}N$ and there may be obstructions to the existence of $\widehat{i_N}$.

According to Melrose and Arnold ([A], [Me]) Riemannian geometry is a special case of the symplectic geometry of pairs in symplectic manifolds.

Gompf's construction of pairwise symplectic sums ([Go]) depends on a symplectic neighborhood theorem for pairs. For these and other reasons it is advantageous to develop Darboux-type theorems for pairs, and in the next chapter we state and prove results analagous to the ones given above for submanifolds. We will see that for pairs there is a rich local structure akin to the Riemannian case.

Note to the reader. The adjective **allowable** is used frequently throughout this thesis, refering to special maps and symplectic forms. A map is allowable iff it induces the identity map on any and all marked submanifolds. Two symplectic forms ω_0 and ω_1 are **allowable** iff

1. they induce the same intrinsic geometry on each submanifold, and

2. $\omega_0(p) = \omega_1(p)$ if p lies in the intersection of two or more submanifolds.

Weakly allowable maps are those which preserve each submanifold, and **weakly allowable** forms ω_0 and ω_1 are those which necessarily only satisfy condition 1.

Allowable and **weakly allowable** have the obvious meaning for smooth 1-parameter families of maps or forms.

Chapter 2

Darboux Theorems for Pairs of Submanifolds

2.1 The Homotopy Method

Moser [M] developed a powerful, elegant technique showing that a symplectic structure can not be perturbed within its cohomology class.

Theorem 2.1.1 *([MS]) Let ω_t be an isotopy of symplectic forms on V (see Definition 1.0.11). Then there is a smooth family ϕ_t of diffeomorphisms of V such that ${\phi_t}^*\omega_t = \omega_0$.*

Definition 2.1.2 The identity map will always be denoted by **1**.

Proof. This proof will illustrate Moser's technique.

We show below that the family ϕ_t exists if we can find a smooth family of 1-forms σ_t such that

$$\frac{d}{dt}\omega_t = d\sigma_t.$$

Finding the forms σ_t: Since the forms ω_t, $0 \le t \le 1$, are cohomologous to each other, each form $\omega_t - \omega_0$ is exact, and so then are the forms

$$\tau_t = \frac{d}{dt}\omega_t.$$

We are looking for a smooth family of 1-forms ρ_t such that

$$d\rho_t = \tau_t$$

which we can find by Hodge theory: Let g be any Riemannian metric on V and let $d^* : A^2 \to A^1$ denote the associated L^2 adjoint operator of d. According to Hodge theory [G] d restricts to an isomorphism from the range of d^* to the exact 2-forms on N. Thus we can choose $\rho_t \in \text{range}\, d^*$ such that $d\rho_t = \tau_t$.

The diffeomorphisms ϕ_t are now determined by representing them as the time-dependent flow of a smooth family of vector fields X_t on V,

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t, \quad \phi_0 = \mathbf{1}$$

so that ${\phi_t}^*\omega_t$ is constant, i.e. $\frac{d}{dt}{\phi_t}^*\omega_t = 0$ (so $\phi_0 = \mathbf{1} \Rightarrow {\phi_t}^*\omega_t = \omega_0$). Differentiating ${\phi_t}^*\omega_t$ with respect to time yields ([V])

$$\begin{aligned} 0 = \frac{d}{dt}{\phi_t}^*\omega_t &= {\phi_t}^*\frac{d}{dt}\omega_t + i_{X_t}d\omega_t + d(i_{X_t}\omega_t) \\ &= {\phi_t}^*d\sigma_t + 0 + d(i_{X_t}\omega_t) \end{aligned}$$

(here $i_{X_t}\omega_t$ denotes the contraction of ω_t by plugging X_t into the first slot) and so we only require that $d(i_{X_t}\omega_t) = -{\phi_t}^*d\sigma_t$. Now define X_t by the equation

$$i_{X_t}\omega_t = -\sigma_t.$$

The non-degeneracy of ω_t guarantees that X_t is uniquely defined, and the proof is finished. ■

Applications of Moser's technique rely on deforming one symplectic form to another in a "nice" way. For example:

Lemma 2.1.3 *Suppose that ω_0, ω_1 are symplectic forms on (V, N) so that $\omega_0 = \omega_1$ on $T_N V$. Then there is an allowable symplectomorphism (see Definition 1.0.7) $\phi : \mathcal{U}_0 \to \mathcal{U}_1$ between neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in V.*

Proof. On a tubular neighborhood $\mathcal{U}_0$ of N define a local isotopy ω_t, $0 \leq t \leq 1$, connecting ω_0 with ω_1 by setting

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0).$$

ω_t is then a smooth family of closed 2-forms, and shrinking $\mathcal{U}_0$ if necessary we can assume ω_t to be non-degenerate for every t (since $\omega_t = \omega_0$ on $T_N V$). The Relative Poincaré Lemma (Appendix A) guarantees the existence of a smooth family of 1-forms σ_t defined on $\mathcal{U}_0$ (shrink again if necessary) where

$$\omega_t = d\sigma_t$$

and $\sigma_t = 0$ on $T_N V$. σ_t can be chosen to vary smoothly since the homotopy operator H used in the proof of the Relative Poincaré Lemma is a linear (and therefore smooth) operator from $A^k(\mathcal{U}_0) \to A^{k-1}(\mathcal{U}_0)$.

Thus ω_t is an isotopy of symplectic forms defined on $\mathcal{U}_0$. Defining the vector fields X_t by

$$i_{X_t}\omega_t = -\sigma_t,$$

these fields are seen to vanish on N since the 1-forms σ_t vanish there. Moser's method now constructs the allowable symplectomorphisms ϕ_t. [Note: We may have to shrink $\mathcal{U}_0$ again in order for ϕ_t to be defined for all t.] ∎

The above Lemma is a special case of GDT I (see below). It also implies Darboux's Theorem since we can deform any symplectic form to any other at a specified point (this follows from the Linear Darboux Theorem, Theorem C.0.17). Let's go ahead now and prove:

Theorem 2.1.4 Global Darboux Theorem I *[MS] If ω_t, $0 \le t \le 1$, is a smooth family of symplectic forms on (V, N) which induce equal forms on N, then there is an allowable family G_t of embeddings (see Definition 1.0.7) of a tubular neighborhood $\mathcal{U}$ of N in V such that $G_t^*\omega_t = \omega_0$. Furthermore, if the cohomology classes $[\omega_t]$ are constant (i.e. ω_t, $0 \le t \le 1$ is an isotopy of symplectic forms), then $\mathcal{U}$ can be taken as all of V.*

Proof. The family of 2-forms $\omega_t - \omega_0$ satisfies

- $d(\omega_t - \omega_0) = 0$
- $i_N^*(\omega_t - \omega_0) = 0$

so the Relative Poincaré Lemma guarantees the existence of a smooth family of 1-forms σ_t such that

- $\omega_t - \omega_0 = d\sigma_t$
- $\sigma_t = 0$ on $T_N V$.

Moser's method can now be applied to construct G_t with $G_0 = \mathbf{1}$.

For the case where ω_t is an isotopy we begin as above and then extend the family of embeddings G_t to a smooth family of diffeomorphisms H_t of V (via the isotopy extension theorem, see [H]) so that $H_0 = \mathbf{1}$. Put $\Omega_t = H_t^* \omega_t$. Then Ω_t is an isotopy of symplectic forms where $\Omega_t = \Omega_0$ near N.

Remark 2.1.5 Since $[\omega_t] = c$ and H_t is homotopic to $H_0 = \mathbf{1}$, $[\Omega_t] = [H_t^* \omega_t] = [H_0^* \omega_t] = [\omega_t] = c$ (see [BT]). Thus Ω_t is an isotopy of symplectic forms as claimed.

Hodge theory (see proof of Theorem 2.1.1) allows us to write $d\rho_t = \tau_t$ where $\tau_t = \frac{d}{dt}\Omega_t$ and ρ_t is a smooth family of 1-forms (defined on V) which vanish on N. Moser's method now gives the desired family of symplectomorphisms of V. ∎

Moser's method can also be put to good use solving various extension problems.

Theorem 2.1.6 (V) *Let N be an embedded submanifold of a manifold V, and let ω be a skewsymmetric bilinear form on $T_N V$ whose induced form on N is closed. Then ω extends to a closed 2-form on a neighborhood $\mathcal{U}$ of N in V. Moreover, if ω is a symplectic form, then so is its extension.*

Remark 2.1.7 The proof of this theorem is similar to the proof of Theorem 2.2.4 below.

Definition 2.1.8 Let $N \subset V$ be an embedded submanifold. A smooth family of embeddings $f_t : N \to V$, $0 \leq t \leq 1$, which start at the inclusion is an **isotopy** of N in V. If f_t preserves a symplectic form it is a **symplectic isotopy**.

Lemma 2.1.9 *Let f_t be a symplectic isotopy of N in (V, ω). Then f_t can be extended to a symplectic isotopy F_t of a tubular neighborhood $\mathcal{U}$ of N in V.*

Proof. Extend f_t to an isotopy $\tilde{f}_t$ of V. By assumption, $\tilde{f}_t^{\,*}\omega_t$ is a smooth family of symplectic forms on (V, N) which induce equal forms on N. By GDT I (Theorem 2.1.4) there is an allowable isotopy G_t of a tubular neighborhood $\mathcal{U}$ of N in V such that $(G_t^* \circ \tilde{f}_t^{\,*})\omega_t = \omega_0$. Set $F_t = \tilde{f}_t \circ G_t$ to finish the proof. ∎

Lemma 2.1.9 raises the question: When is a symplectic isotopy of N extendable to a symplectic isotopy of all of V? Such an extension may not be possible as there are cohomological obstructions. For example, if N is the unit circle of $(\mathbf{R}^2, \tau)$ and f_t is a dilation of N, then since $f_t(N)$ is a lagrangian submanifold for all values of t, f_t extends to a symplectic isotopy F_t of some annulus of N. If f_t could in fact be extended to a symplectic isotopy of all of $\mathbf{R}^2$, then the area inside $f_t(N)$ would necessarilly remain constant, contradicting the fact that f_t is a dilation of N.

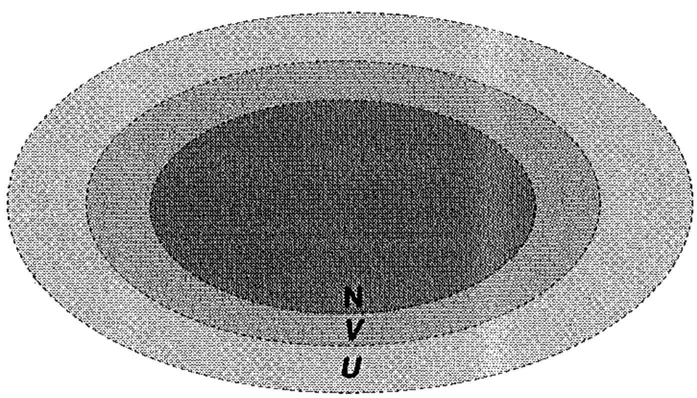
Theorem 2.1.10 *([B]) Given (V,ω), suppose that $f_t : \mathcal{U} \to V$ is a symplectic isotopy of an open neighborhood $\mathcal{U}$ of a compact neighborhood N, and also suppose that*

$$H^2(V, N; \mathbf{R}) = 0.$$

Then there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of N and a symplectic isotopy $F_t : V \to V$ such that

$$F_t = f_t \text{ on } \mathcal{V}$$

for every t.



Proof. (See [MS]) Let $\mathcal{V}$ be a neighborhood contained by $\mathcal{U}$ which retracts onto N. So $H^*(\mathcal{V}, N; \mathbf{R}) = 0$ and the cohomology exact sequence of the triple $(N, \mathcal{V}, V)$ shows that $H^*(V, \mathcal{V}; \mathbf{R}) = 0$. Choose any extension G_t of f_t to all of V and define $\omega_t = G_t^*\omega$, noting that ω_t agrees with ω in $\mathcal{V}$. Hence the forms $\tau_t = \frac{d}{dt}\omega_t$ vanish in $\mathcal{V}$ and so represent relative cohomology classes

$$[\tau_t] \in H^*(V, \mathcal{V}; \mathbf{R}) = 0.$$

Apply relative Hodge theory to the compact manifold $V \setminus \mathcal{V}$ to construct 1-forms σ_t on V vanishing on $\mathcal{V}$ and satisfying $d\sigma_t = \tau_t$. By Moser's method these forms σ_t give rise to diffeomorphisms H_t of V restricting to the identity on $\mathcal{V}$ and satisfying $H_t^*\omega_t = \omega_0 = \omega$ for all t. Thus $H_t \circ G_t$ is the desired extension F_t. ■

I hope the above applications of Moser's technique simply illustrate the power and versatility of his method. We turn our attention now to the more complicated case of submanifold pairs.

2.2 Pairs

Definition 2.2.1 (V, Γ) denotes a manifold V containing two embedded, transversal submanifolds M and N, where $\Gamma \stackrel{d}{=} M \cup N$. Furthermore M, N, and $K \stackrel{d}{=} M \cap N$ are assumed connected. If ω_0 and ω_1 are symplectic forms on V, then (V, Γ) is **weakly allowable** iff ω_0 and ω_1 induce equal forms on both M and on N. (V, Γ) is **allowable** iff ω_0 and ω_1 are weakly allowable and $\omega_0 = \omega_1$ on $T_K V$. A diffeomorphism $\phi : \mathcal{U}_0 \to \mathcal{U}_1$ between subsets of V is **weakly allowable** iff ϕ preserves both M and N, and **allowable** iff $\phi = \mathbf{1}$ on $\mathcal{U}_0 \cap \Gamma$.

Definition 2.2.2 An **allowable** family of symplectic forms on (V, Γ) is a smooth family ω_t, $0 \leq t \leq 1$, of symplectic forms on V where $\omega_t = \omega_0$ on $T_K V$ and where ω_t induces equal forms on both M and on N. An **allowable isotopy** of symplectic forms on (V, Γ) is an allowable family whose cohomology classes $[\omega_t]$ are constant. If the condition $\omega_t = \omega_0$ on $T_K V$ is removed, then allowable is replaced by **weakly allowable**.

We are now in a position to prove a version of LDT I for pairs.

Theorem 2.2.3 (Local Darboux Theorem II) *Let $\mathcal{A} = (M, N)$ be an ordered set of subspaces in $V = \mathbf{R}^{2n}$ ($\mathcal{A}$ is an arrangement in V) and suppose that ω_t is an allowable family of symplectic forms on V. Then there is a smooth, allowable family ϕ_t of embeddings of a tubular neighborhood $\mathcal{U}$ of Γ such that $\phi_t^* \omega_t = \omega_0$.*

Proof. The closed forms $\omega_t - \omega_0$ induce zero forms on TM and on TN, and $\omega_t - \omega_0 = 0$ on $T_K V$. By the Local Poincaré Lemma (see Appendix B) there exists a smooth family of 1-forms σ_t defined on V such that

- $\frac{d}{dt}(\omega_t - \omega_0) = \frac{d}{dt}(\omega_t) = d\sigma_t$
- $\sigma_t = 0 \mathrm{on} T_\Gamma V$

Moser's method finishes the proof. ∎

We also have a local extension theorem for pairs.

Theorem 2.2.4 *Let $\mathcal{A} = (M, N)$ be an ordered set of subspaces in $V = \mathbf{R}^{2n}$ ($\mathcal{A}$ is an arrangement in V) with $\Gamma = M \cup N$. Suppose that ω is a smooth skew-symmetric 2-form on $T_\Gamma V$ whose restriction to TM (and to TN) is a closed form of M (and of N), and suppose that $d\omega = 0$ on $T_K V$. Then ω*

extends to a closed 2-form on V. Moreover, if ω is a symplectic form then so is the extension, at least in some tubular neighborhood of Γ.

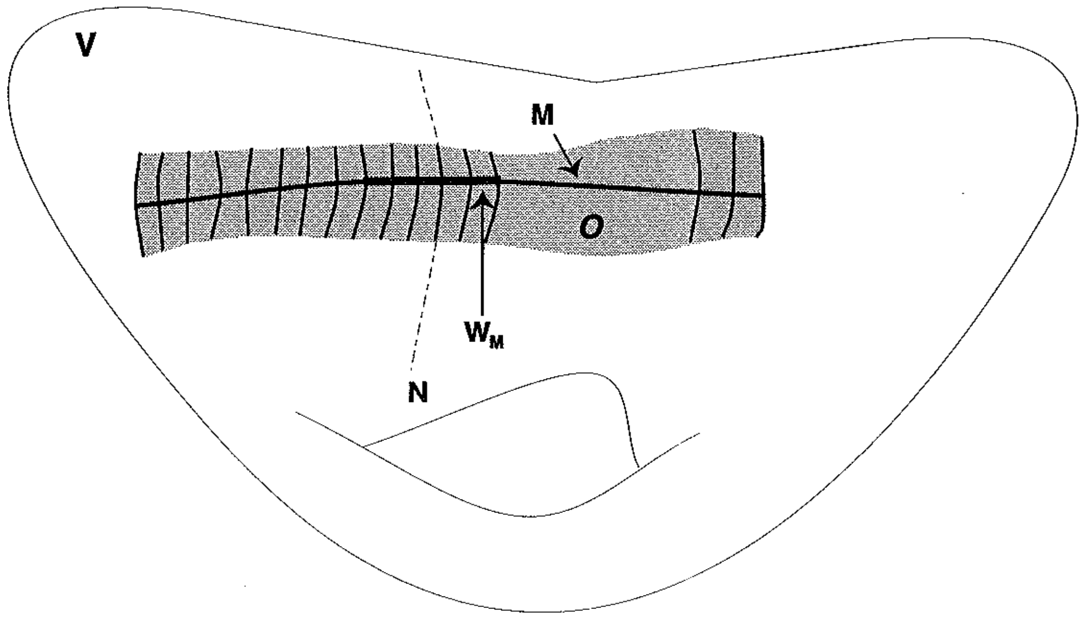
Proof. Let $\tilde{\omega}$ denote a smooth extension of ω to all of V such that $d\tilde{\omega} = 0$ on $T_K V$. The Local Poincaré Lemma guarantees the existence of a smooth 2-form α which vanishes on $T_\Gamma V$ such that $d\alpha = d\tilde{\omega}$ everywhere. Define $\Omega = \tilde{\omega} - \alpha$. Then Ω is the desired extension of ω. If ω is symplectic, then since Ω is a **smooth** extension of ω, Ω must also be symplectic, at least near Γ. ∎

Our next goal is to prove a version of GDT I (Theorem 1.0.10) for pairs.

2.2.1 Global Darboux Theorem I for Pairs

Definition 2.2.5 Given (V, Γ) we can find a bundle neighborhood $\mathcal{O}$ of M in V whose fibers over K lie in N: $\mathcal{O}$ is called an **allowable neighborhood** of M in V. Allowable neighborhoods can always be constructed as follows: By the tubular neighborhood theorem there exists a neighborhood $\mathcal{O}$ of M in V which is diffeomorphic to the normal bundle of M. By choosing a metric such that N is totally geodesic near K we may choose the diffeomorphism so that the fibers over K lie in N. Similarly we define an **allowable neighborhood** of N in V. (See figure on top of next page.)

Theorem 2.2.6 (Global Darboux Theorem II) *Let ω_t, $0 \leq t \leq 1$, be an allowable family of symplectic forms on (V, Γ) (see Definition 2.2.2) which induce forms of constant rank on N (or on M) near K. Then there exists a*



smooth, allowable family ϕ_t of embeddings (see Definition 2.2.1) of a tubular neighborhood $\mathcal{U}$ of Γ such that $\phi_t^* \omega_t = \omega_0$. If the cohomology classes $[\omega_t]$ are constant, then $\mathcal{U}$ can be taken as all of V.

If we could find a smooth family of 1-forms γ_t defined on some neighborhood $\mathcal{U}$ of Γ wherein

- $\dfrac{d}{dt}\omega_t = d\gamma_t$
- $\gamma_t = 0$ *on* $T_\Gamma V$

then the usual homotopy argument would finish the proof. What we need here is a Relative Poincaré Lemma for pairs, which as far as I know is nonexistent. Below we construct γ_t, at least near K (which is all we really need), in a series of steps, reducing the general case to one handled by the Main Lemma below. In effect we prove a special case of the Relative Poincaré Lemma for pairs.

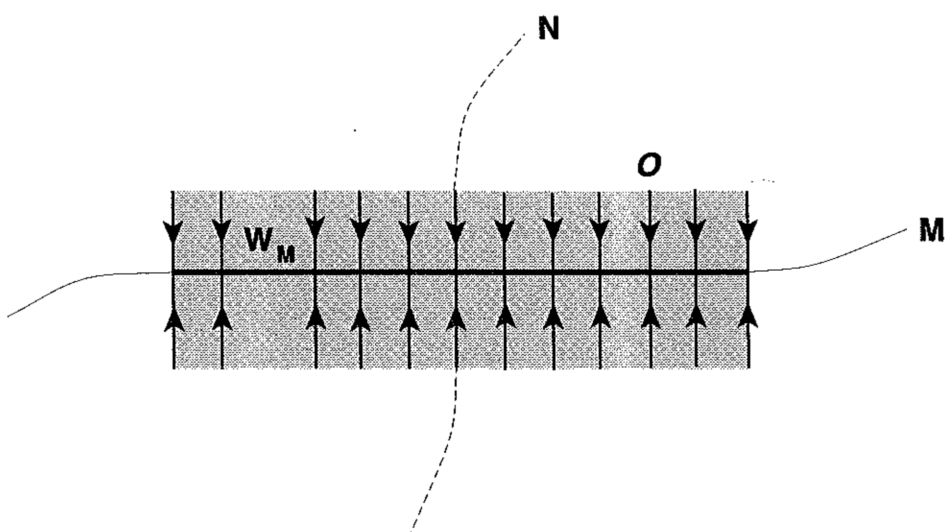
The following technical result proves GDT II for the special case when $\omega_t = \omega_0$ on $T_N V$ (at least near K).

Lemma 2.2.7 (Main Lemma) *Suppose that ω_t, $0 \leq t \leq 1$, is an allowable family of symplectic forms on (V, Γ) where*

$$\omega_t = \omega_0 \textit{ on } T_N V$$

near K. Then there exists a smooth, allowable family ϕ_t of embeddings of a tubular neighborhood $\mathcal{U}$ of Γ such that $\phi_t^* \omega_t = \omega_0$. If the family is also an isotopy then $\mathcal{U}$ can be taken as all of V.

Proof. Let W_M be a tubular neighborhood of K in M, and let $\mathcal{O}$ be an allowable neighborhood of W_M in V (see Definition 2.2.5). Define a smooth deformation retraction f_t of $\mathcal{O}$ onto W_M by contracting along the fibers, where f_1 is the identity map and f_0 is a smooth retraction onto W_M.



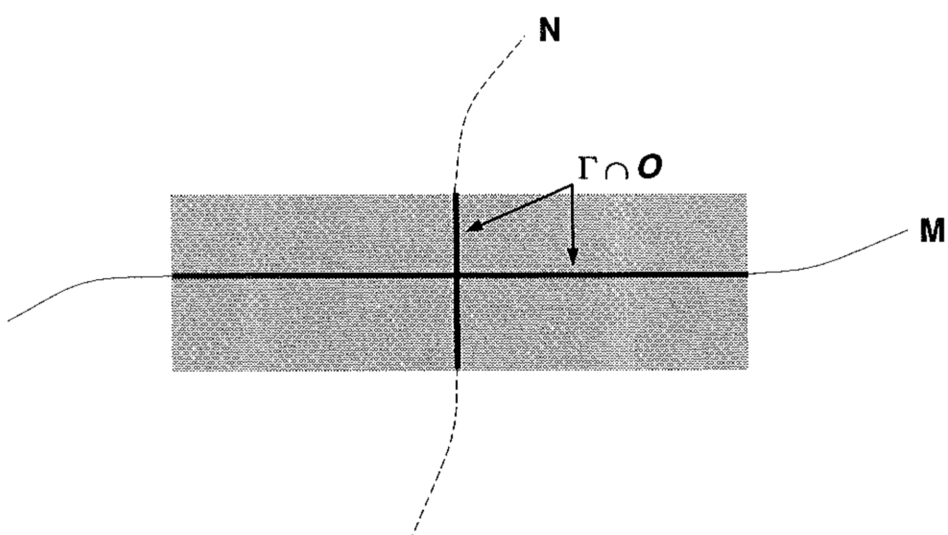
Note that $\omega_t - \omega_0$ is a smooth family of closed 2-forms whose pull-backs to W_M vanish (i.e. they vanish on the subspace of vectors tangent to W_M). By the Relative Poincaré Lemma (see Appendix A) there exists a smooth family of 1-forms γ_t defined on $\mathcal{O}$ which vanish on $W_M \cap \mathcal{O}$ such that $\omega_t - \omega_0 = d\gamma_t$ (on $\mathcal{O}$).

Let's rexamine how the 1-forms γ_t are defined (see Appendix A): Let $\dot{f}_s(f_s(p))$ (or simply $\dot{f}_s$) denote the time-dependent vector field $\frac{d}{d\theta} f_\theta(p) \mid_{\theta=s}$. Then for $p \in \mathcal{O}$ and $v \in T_pV$,

$$\langle \gamma_t(p), v \rangle = \int_0^1 i_{\dot{f}_s}(\omega_t - \omega_0)_{f_s(p)}((\mathrm{d}_p f_s)(v))\, ds$$

Since the forms $\omega_t - \omega_0$ vanish on T_pV for $p \in N \cap \mathcal{O}$, γ_t also vanish on $N \cap \mathcal{O}$. We can now define a time dependent vector field X_t on $\mathcal{O}$ by setting

$$i_{X_t}\omega_t = -\gamma_t$$



and so $X_t = 0$ on $\Gamma \cap \mathcal{O}$. Since time dependent vector fields can be integrated locally just like usual vector fields, and $X_t = 0$ on $\Gamma \cap \mathcal{O}$ $\forall t \in [0,1]$, we can say that the integral curves of X_t are defined for all $t \in [0,1]$ on a perhaps smaller neighborhood $\mathcal{O}$ of the origin. Let ψ_t be the corresponding reduced flow. $X_t = 0$ on $\Gamma \cap \mathcal{O} \Rightarrow \psi_t(p) = p$ for all $p \in \Gamma \cap \mathcal{O}$. Taking the time derivative of $\psi_t^* \omega_t$ yields

$$\begin{aligned} \frac{d}{dt}\psi_t^*\omega_t &= \psi_t^*\frac{d\omega_t}{dt} + (i_{X_t}d\omega_t) + d(i_{X_t}\omega_t) \\ &= \psi_t^*(\omega_1 - \omega_0) + 0 - \psi_t^* d\gamma_t \\ &= 0 \end{aligned}$$

and so $\int_0^s \frac{d}{dt}\psi_t^*\omega_t = \psi_s^*\omega_s - \psi_0^*\omega_0 = 0$ or $\psi_s^*\omega_s = \omega_0$, and thus ψ_s is a smooth, allowable family of symplectic embeddings of $\mathcal{O}$ into V.

The isotopy extension theorem ([H]) allows us to extend ψ_t to an allowable isotopy Ψ_t of all of V. Putting $\Omega_t = \Psi_t^*\omega_t$ gives us an allowable family (or isotopy) of symplectic forms on V such that $\Omega_t = \Omega_0$ near W_M. Two applications of GDT I (Theorem 1.0.10), once for M and once for N, allow us to deform Ω_t in an allowable fashion so that $\Omega_t = \Omega_0$ near Γ. If Ω_t is also an isotopy (which it will be if ω_t was an isotopy) then the usual homotopy argument applies. ∎

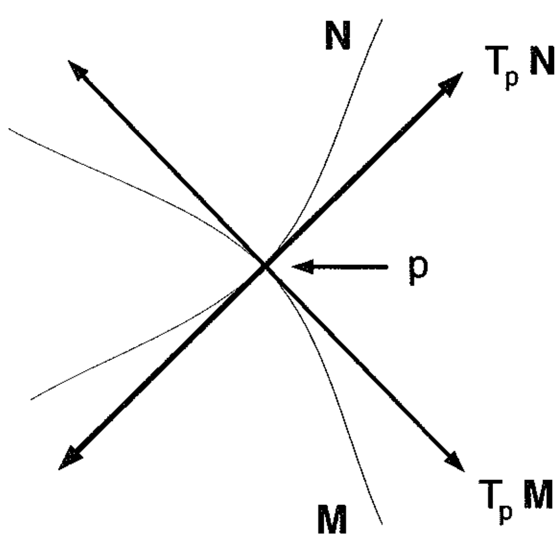
The idea now is to transform the general situation into one where the symplectic forms ω_t are equal on $T_N V$, at least near K, and then to apply the Main Lemma (Lemma 2.2.7). We will first do this for the case where N is symplectic.

Symplectic Pairs

Definition 2.2.8 A pair (V, Γ) is called **symplectic** if K, M, and N are all symplectic submanifolds.

Remark 2.2.9 If M and N are symplectic it does not necessarilly follow that K is symplectic. For example, let M and N be the symplectic subspaces of the symplectic vector space $(\mathbf{R}^6, \tau)$ defined by letting M be those points satisfying $x_2 - x_5 = x_4 - x_6 = 0$ and N those satisfying $x_5 = x_6 = 0$. Then M and N are symplectic but $K = M \cap N$ is isotropic.

Definition 2.2.10 Given a symplectic pair (V, Γ, ω) we say that N is **ω-orthogonal** to M if $\forall p \in K$, $T_p^s M \subset T_p N$. [Note: ω-orthogonality is a reflexive property.]



The next result is the first step toward generalizing the Main Lemma.

Lemma 2.2.11 *Let ω_0 and ω_1 be allowable symplectic forms on the symplectic pair (V,Γ) so that M is ω_0- and ω_1-orthogonal to N. Then there exists an allowable diffeomorphism $\phi : V \to V$ so that $(\phi^*\omega_1)(p) = \omega_0(p)$ for all $p \in N$ near K.*

Remark 2.2.12 Since ϕ is allowable, $(\mathbf{d}\phi)(p) = I_{2n}$ for all $p \in K$.

Proof. Let W_N denote a neighborhood of K in N which is smoothly contractible onto K. The symplectic normal bundles $T^{s_0}W_N$ and $T^{s_1}W_N$ can be identified with bundle neighborhoods of W_N via the tubular neighborhood theorem, and so any bundle isomorphism gives rise to a diffeomorphism between these two tubular neighborhoods. ω-orthogonality allows us to choose these bundle neighborhoods so that the fibers of the symplectic normal bundles over points in K are mapped into M. This fact is used below to guarantee the allowability of the map ϕ. (See figure on top of next page.)

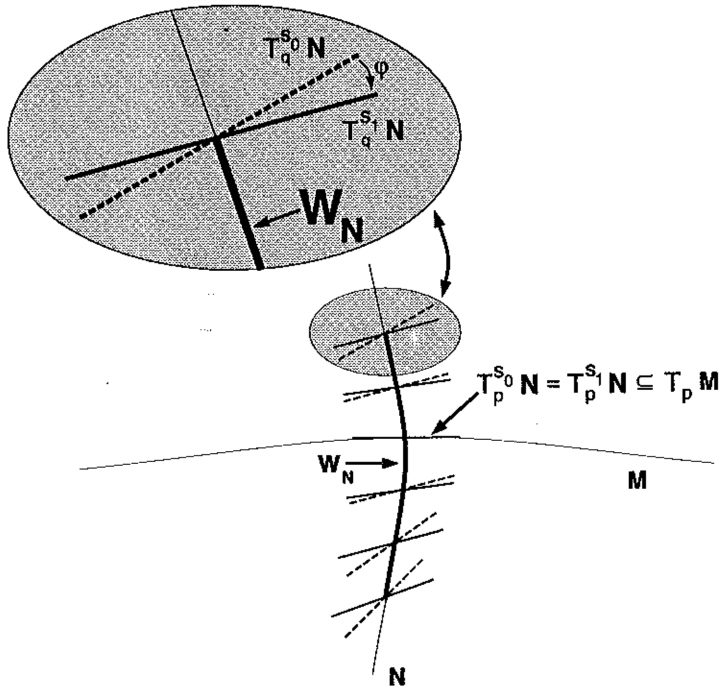
By assumption W_N has codimension equal to 2r in V, and thus $T^{s_a}W_N$ are 2r-dimensional vector bundles over W_N. Let $ESp(2r) \to BSp(2r)$ denote the classifying space for 2r-dimensional symplectic vector bundles. Then there are maps $h_0, h_1 : W_N \to BSp(2r)$ so that

$$T^{s_0}W_N = h_0^*(ESp(2r))$$

and

$$T^{s_1}W_N = h_1^*(ESp(2r))$$

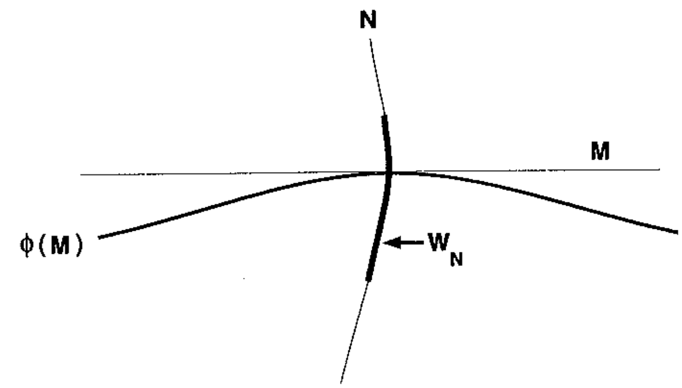
Since $\omega_0 = \omega_1$ on $T_K V$ and W_N is smoothly contractible onto K, we can assume that $h_0 = h_1$ on K and that h_0 is homotopic to h_1. Thus $T^{s_0}W_N \cong T^{s_1}W_N$ and



so there is an allowable diffeomorphism φ (see remarks in the first paragraph) between neighborhoods of K so that $(\varphi^*\omega_1)(p) = \omega_0(p)$ for all $p \in N$ near K. By the isotopy extension theorem ([H]) there is a smooth extension ϕ of φ to all of V. ∎

In general, when M and N are not symplectically orthogonal, the map ϕ constructed in Lemma 2.2.11 will not be allowable since M may not be preserved ($T_p^{s_a}N \not\subset M$). However, we still have $\phi(M)$ tangent to M along K since $\mathbf{d}\phi(p) = I_{2n}$ (for $p \in K$).

To solve this general case we begin as in Lemma 2.2.11 but then bend $\phi(M)$ back onto M (details follow). This gives us Corollary 2.2.15, which is precisely Global Darboux Theorem II (GDT II) for symplectic pairs.



Remark 2.2.13 In constructing ϕ we only require that ω_0 and ω_1 induce symplectic forms on N near K.

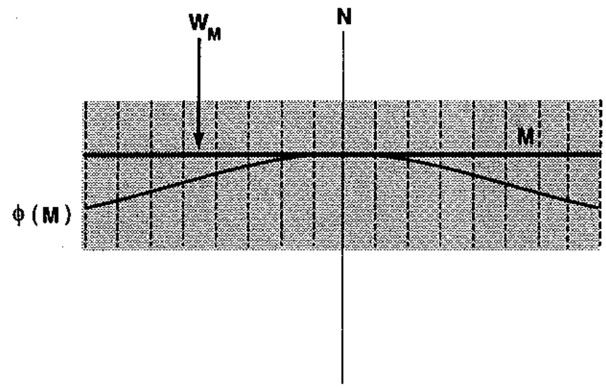
Lemma 2.2.14 *Let ω_0 and ω_1 be allowable symplectic forms on (V, Γ) such that the forms induced on N near K are symplectic. Then there is an allowable symplectomorphism between neighborhoods of Γ.*

Proof. Construct ϕ as in Lemma 2.2.11. Let $W_M \subset M$ be a tubular neighborhood of K, and let $\mathcal{O}$ be an allowable neighborhood of W_M in V, chosen small enough so that ϕ is defined on $\mathcal{O}$. We can choose local coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ for $\mathcal{O}$ so that the linear fiber coordinates are $\mathbf{z}$ and the 0-section (identified with W_M) is given by $\mathbf{z} = \mathbf{0}$. By the Implicit Function Theorem there are smooth locally defined functions $f_1, \ldots, f_{2(n-r-s)}$ such that

$$\phi(M) \cap \mathcal{O} = (\mathbf{x}, \mathbf{y}, f_1(\mathbf{x}, \mathbf{y}), \ldots, f_{2(n-r-s)}(\mathbf{x}, \mathbf{y}))$$

(shrink $\mathcal{O}$ if necessary).

As stated above the plan is to bend $\phi(M)$ back onto M in a "nice" way. To this end locally define a map Θ on $\mathcal{O}$ by



$$(\mathbf{x},\mathbf{y},\mathbf{z}) \mapsto (\mathbf{x},\mathbf{y},z_1 - f_1(\mathbf{x},\mathbf{y}),\ldots,z_{2(n-r-s)} - f_{2(n-r-s)}(\mathbf{x},\mathbf{y})).$$

Θ is actually well-defined globally, at least on a smaller $\mathcal{O}$. Put $\mathcal{O}' = \phi^{-1}(\mathcal{O})$ (shrink $\mathcal{O}$ if necessary), put $\mathcal{O}'' = \Theta(\mathcal{O})$, and define a map $\tilde{\phi}$ by setting it equal to $\Theta \circ \phi : \mathcal{O}' \to \mathcal{O}''$. Then $\tilde{\phi}$ is a local diffeomorphism near K which preserves Γ and induces the identity map on $N \cap \mathcal{O}'$, and which has the nice property that $(\tilde{\phi}^*\omega_1)(p) = \omega_0(p)$ for $p \in N \cap \mathcal{O}$. This last equality is justified as follows:

By construction, $(\phi^*\omega_1)(p) = \omega_0(p)$ for $p \in N \cap \mathcal{O}'$. It only remains to show that the map Θ does not destroy this property. To this end, note that if $p \in \mathcal{O}$ then

$$\mathbf{d}\Theta(p) = \begin{pmatrix} I_{2(r+s)} & 0 \\ \frac{-\partial \mathbf{f}}{\partial x\, \partial y} & I_{2(n-r-s)} \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. Since $\phi(M)$ is tangent to M along K, $\mathbf{d}\Theta(p)$ is the identity matrix for $p \in N \cap \mathcal{O}$. Thus $\mathbf{d}\tilde{\phi}(p) = \mathbf{d}(\Theta \circ \phi)(p) =$

$\mathbf{d}\phi(p)$ (recall that $\phi(p) = p$), and that's why $(\tilde{\phi}^*\omega_1)(p) = \omega_0(p)$ for $p \in N \cap \mathcal{O}'$.

The only drawback to $\tilde{\phi}$ is that, although it induces the inclusion map on N, it may not even preserve M. We can correct this by composing $\tilde{\phi}$ with another local diffeomorphism Φ constructed as follows.

Let ψ denote the map $\tilde{\phi}$ restricted to M. Then ψ maps M into itself (near K) and induces the identity map on K. Since $\mathbf{d}\tilde{\phi}(p)$ is the identity matrix for $p \in K \cap \mathcal{O}'$, then so is $\mathbf{d}\psi(p)$ and therefore $\mathbf{d}\psi^{-1}(p)$ also (recall that $\tilde{\phi}$ induces the identity map on N, and so $\tilde{\phi}(p) = p$ and therefore $\psi(p) = p$). Now locally define Ψ by

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\psi_1^{-1}(\mathbf{x}, \mathbf{y}), \ldots, \psi_{2(r+s)}^{-1}(\mathbf{x}, \mathbf{y}), \mathbf{z}).$$

Ψ is well-defined globally and, for $p \in N \cap \mathcal{O}'$, $\mathbf{d}\Psi(p)$ is the identity matrix. Thus $((\Psi \circ \tilde{\phi})^*\omega_1)(p) = \omega_0(p)$.

In conclusion $\Psi \circ \tilde{\phi}$ is a local, allowable diffeomorphism around K, constructed so that the forms $(\Psi \circ \tilde{\phi})^*\omega_1$ and ω_0 equal on $T_{N \cap \mathcal{O}}V$ and induce the same forms on $M \cap \mathcal{O}$ (since $\Psi \circ \tilde{\phi} \circ i_M = \mathbf{1}$). The Main Lemma (Lemma 2.2.7) allows us to finish the proof. ■

Corollary 2.2.15 *Suppose that ω_t, $0 \leq t \leq 1$, is an allowable family of symplectic forms on (V, Γ) which induce equal symplectic forms on N near K. Then there is a smooth, allowable family ϕ_t, embedding a tubular neighborhood $\mathcal{U}$ of Γ into V so that $\phi^*\omega_t = \omega_0$. If the family is also an isotopy, then $\mathcal{U}$ can be taken as all of V.*

The General Case

GDT II (Theorem 2.2.6) has now been proved for the case where ω_t induce **symplectic** forms on N near K. To finish the proof we need to construct the map ϕ used to prove Lemma 2.2.14 for the case where ω_t induce forms of **constant rank** on N near K. The construction of ϕ below is analogous to the one given previously.

Proof of GDT II. Let W_N (W for simplicity) denote a tubular neighborhood of K in N. If W is sufficiently small then we can decompose the tangent bundle $T_N V$ into a smooth family of direct-sum decompositions (see Appendix C for an explanation of the notation used below)

$$T_W V = (T_W N)^{\sigma_t} \oplus_s B^t(T_W N) \oplus_s C^t(T_W N)$$

where

$$(T_W N)^{\sigma_t} \simeq (T_W N)^{i_t} \oplus Q^t(T_W N),$$

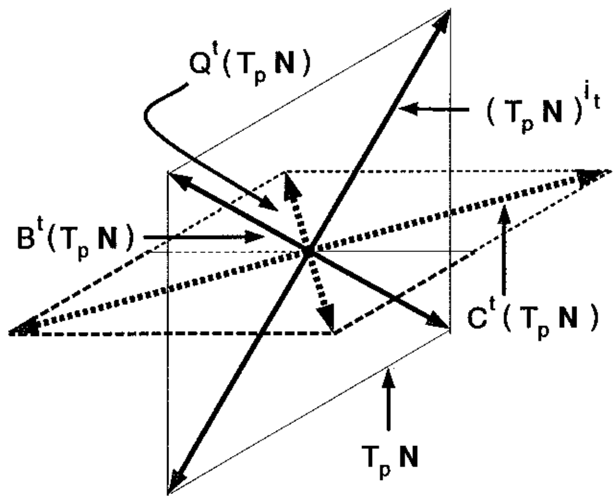
$$C^t(T_W N) \simeq (T_W N)^{s_t} \setminus (T_W N)^{i_t},$$

$$B^t(T_W N) \simeq T_W N \setminus (T_W N)^{i_t},$$

and where $Q^t(T_W N)$ is a smooth family of isotropic, supplementary subbundles of $T_W N + (T_W N)^{s_t}$ in $T_W V$.

There is a smooth family of symplectic bundle isomorphisms

$$f_t : T^0_W V \to T^t_W V$$



which preserve the direct-sum decompositions. Because the forms ω_t induce identical constant forms on N, we may assume $B^0N = B^tN$ and $T_W^{i_0}N = T_W^{i_t}N \quad \forall t \in [0,1]$. So the bundle isomorphisms f_t induce a smooth family of symplectic bundle isomorphisms

$$\widehat{f}_t : C^0(T_W N) \oplus Q^0(T_W N) \to C^t(T_W N) \oplus Q^t(T_W N)$$

descending to a smooth family $F_t : \mathcal{U} \to V$ of embeddings of a tubular neighborhood $\mathcal{U}$ of W in V, a family which induces the identity map on $N \cap \mathcal{U}$ and whose derivative $\mathbf{d}F_t(p)$ is the identity transformation for $p \in K$ (since $\omega_0(p) = \omega_t(p)$). Additionally, $F_0 = \mathbf{1}$. $F_t(N \cap \mathcal{U})$ is tangent to M and the type of argument used to prove Lemma 2.2.14 can now be used to construct an allowable family of embeddings $G_t : \mathcal{U} \to V$ where $G_t^*\omega_t = \omega_0$ on $\mathcal{U} \cap \Gamma$ with $G_0 = \mathbf{1}$. By the isotopy extension theorem ([H]), G_t is extendable to an allowable family H_t of diffeomorphisms of V with $H_0 = \mathbf{1}$. Putting $\Omega_t = H_t^*\omega_t$ gives us an allowable family (isotopy) of symplectic forms Ω_t upon which the usual homotopy argument can now be applied, finishing the proof. ∎

Definition 2.2.16 An **isotopy** of Γ in V is a smooth family of maps $f_t : \Gamma \to V$ which may be extended to a smooth family of embeddings $\tilde{f}_t$ of a neighborhood $\mathcal{U}$ of Γ in V. If $\tilde{f}_t^*\omega$ is an allowable family of symplectic forms on $(\mathcal{U}, \Gamma)$, then f_t is an **allowable isotopy**.

Does Definition 2.2.16 depend on the choice of extension? That is, can one extension be allowable while another one is not? The answer is no, because $(\tilde{f}_t)_*(p)$ is uniquely defined for any $p \in K$.

Corollary 2.2.17 *Suppose f_t is an allowable isotopy of Γ in V. Then f_t can be extended to a symplectic isotopy of a tubular neighborhood of Γ in V (see Definition 2.1.8).*

Proof. The proof is similar to that given for Lemma 2.1.9. ∎

Conjecture 2.2.18 *GDT II (Theorem 2.2.6) is still valid if the constant rank conditions are removed.*

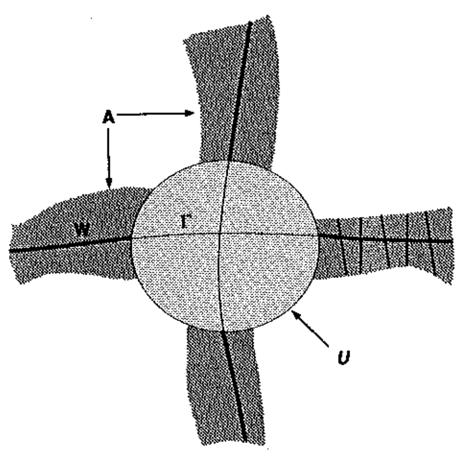
Is GDT II still true if allowable is replaced by weakly allowable? The answer is definitely no since M and N can be symplectically orthogonal with respect to ω_0 but not ω_1 (see Example 3.3.4). The next section gives two reasonable versions of GDT II for the weakly allowable scenario: Either Γ is preserved **or** there is a smooth family of symplectomorphisms, but not both.

2.2.2 Weakly Allowable Pairs

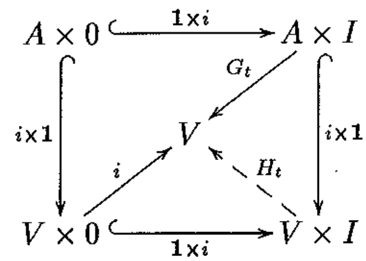
Theorem 2.2.19 *Suppose that Ω_t, $0 \le t \le 1$, is a weakly allowable isotopy of symplectic forms on (V, Γ) (see Definition 2.2.2). For every tubular neighborhood $\mathcal{U}$ of K there is an isotopy $F_t : V \to V$ such that*

- $F_t(\Gamma) = \Gamma$ *outside* $\mathcal{U}$
- $F_t^*(\Omega_t) = \Omega_0$

Proof. Let $W = \Gamma \setminus \mathcal{U}$. Equate $T_W^{st}M \cup T_W^{st}N$ with some tubular neighborhood A of W in V, and define a family of embeddings G_t of A into V via a smooth family of bundle isomorphisms (shrink A if necessary).



By the isotopy extension theorem ([H]) the isotopy of $A \times I$ can be extended to $V \times I$ or, in other words, the following commutatative diagram can be filled in.



Note that $H_t^*(\Omega_t) = \Omega_0$ on W and that $H_t(\Gamma) = \Gamma$ outside $\mathcal{U}$. In fact, we can assume (GDT I, Theorem 2.1.4) that $H_t^*(\Omega_t) = \Omega_0$ on A. Putting $\Sigma_t = H_t^*(\Omega_t)$ gives us an isotopy of symplectic forms inducing equal forms near W. Moser's method can now be applied to construct F_t. ■

Conjecture 2.2.20 *Suppose that Ω_t, $0 \leq t \leq 1$, is a weakly allowable isotopy of symplectic forms on (V, Γ) (see Definition 2.2.2). For every tubular neighborhood $\mathcal{U}$ of K there is a smooth family of diffeomorphisms $F_t : V \to V$ such that*

- $F_t(\Gamma) = \Gamma$
- $F_t^*(\Omega_t) = \Omega_0$ *outside* $\mathcal{U}$

McDuff and Polterovich [MP] have proved this conjecture under some mild restrictions, and Gompf [Go] has a similar result.

We have seen that stronger results are obtainable when ω_t is assumed to be allowable as opposed to merely weakly allowable. The next chapter begins to answer the question: When can a weakly allowable family of symplectic forms be perturbed to an allowable one?

Chapter 3

Arrangements

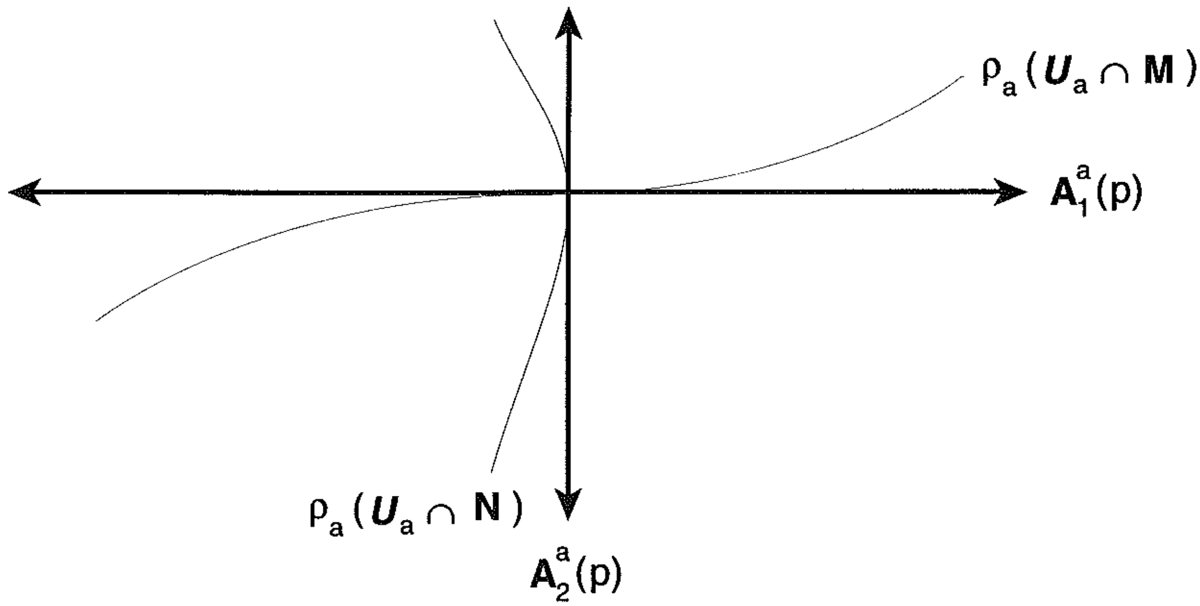
3.1 Introduction

When can a weakly allowable pair (V, Γ) (see Definition 2.2.1) be deformed (preserving both M and N) into an allowable one? This turns out to be a difficult question to answer, and there are many obstructions to the existence of such a deformation. In order to make the problem more tractable we first linearize the problem and then answer the question locally.

Given any points $p, q \in K$ we may choose Darboux neighborhoods $(\mathcal{U}_0, \rho_0)$ containing p and $(\mathcal{U}_1, \rho_1)$ containing q so that $\rho_0(p) = \rho_1(q) = 0$.

Problem 3.1.1 *When does a symplectomorphism $\phi : (\mathcal{U}_0, p) \to (\mathcal{U}_1, q)$, preserving both M and N, exist?*

The tangent space of $\rho_a(\mathcal{U}_a)$ at the origin is the symplectic vector space $(\mathbf{R}^{2n}, \tau_{2n})$. Name the tangent planes of $\rho_0(\mathcal{U}_0 \cap M)$ and $\rho_0(\mathcal{U}_0 \cap N)$ at the origin by $A_1^0(p)$ and $A_2^0(p)$ respectively. Similarly define $A_1^1(p)$ and $A_2^1(p)$.



Let $Sp(2n;\mathbf{R})$ denote the real matrix group consisting of all linear symplectomorphisms of $(\mathbf{R}^{2n}, \tau_{2n})$ (see Definition C.0.12). Now supposing that ϕ does indeed exist, evaluating the Jacobian of $\rho_1 \circ \phi \circ \rho_0^{-1}$ at 0 gives an element of $Sp(2n;\mathbf{R})$ mapping $A_1^0(p)$ onto $A_1^1(q)$ and $A_2^0(p)$ onto $A_2^1(q)$. (Example 3.3.4 shows that there may be an obstruction to the existence of ϕ).

An ordered set of tangent planes such as $\mathcal{A}^a(p) = (A_1^a(p), A_2^a(p))$ in $(\mathbf{R}^{2n}, \tau_{2n})$ above is called a 2-arrangement in $(\mathbf{R}^{2n}, \tau_{2n})$ (see Definition 3.1.2). We are looking for necessary and sufficient conditions for the existence of an element of $Sp(2n;\mathbf{R})$ which maps the 2-arrangement $\mathcal{A}^0(p) = (A_1^0(p), A_2^0(p))$ onto $\mathcal{A}^1(q) = (A_1^1(q), A_2^1(q))$. Note that the existence of such a linear equivalence is independent of the Darboux coordinates choosen, but not the points p and q.

Definition 3.1.2 An **m-arrangement** $\mathcal{A} = (A_1, A_2, \ldots, A_m)$ in $(\mathbf{R}^{2n}, \tau_{2n})$ is a finite collection of subspaces A_j $(j = 1, 2, \ldots, m)$ in $\mathbf{R}^{2n}$ where $A_i \not\subset A_j$ for all $i \neq j$. Each A_j is an **element** of $\mathcal{A}$. Two arrangements $\mathcal{A}_0 = (A_1^0, A_2^0, \ldots, A_{m_0}^0)$ and $\mathcal{A}_1 = (A_1^1, A_2^1, \ldots, A_{m_1}^1)$ in $(\mathbf{R}^{2n}, \tau_{2n})$ are said to be **equivalent** if $m_0 = m_1 = m$ and there is an element of $Sp(2n; \mathbf{R})$ mapping A_j^0 onto A_j^1 for each $j \in \{1, 2, \ldots, m\}$.

For any m-arrangement $\mathcal{A} = (A_1, A_2, \ldots, A_m)$, define $\Gamma_{\mathcal{A}} = \cup_{i=1}^{m} A_i$ (or just Γ) and $K_{\mathcal{A}} = \cup_{i<j}(A_i \cap A_j)$ (or just K). K is called the **singular set** of Γ.

Problem 3.1.3 *Given two arrangments $\mathcal{A}_0$ and $\mathcal{A}_1$ in $(\mathbf{R}^{2n}, \tau_{2n})$, under what conditions are they equivalent?*

Remark 3.1.4 Another perspective on this problem is: Let $G_{s,n}^t$ denote the space of s-dimensional subspaces of $(\mathbf{R}^{2n}, \tau_{2n})$ with rank t. Problem 3.1.3 is equivalent to the classification of the orbits of $Sp(2n; \mathbf{R})$ acting on $G_{s_1,n}^{t_1} \times G_{s_2,n}^{t_2} \times \cdots \times G_{s_m,n}^{t_m}$. Unfortunately very little is known about the spaces $G_{s,n}^t$, the Lagrangian Grassmanian $G_{n,n}^0$ being the only widely studied example.

If we remove the symplectic geometry from this setup we get an old problem that is largely unsolved. In fact, until recently there was no effective way to even compute the homology of the complement of an arrangement. (See [GP]. Also see [O] and [FR] for general surveys on arrangements.)

The equivalence problem **has** been solved for some elementary arrangements: For 1-arrangements the solution is given by Witt's Theorem.

Theorem 3.1.5 (Witt's Theorem) *Let $\mathcal{A}_0 = A_1^0$ and $\mathcal{A}_1 = A_1^1$ be two k-dimensional subspaces of the symplectic vector space $(\mathbf{R}^{2n}, \tau_{2n})$. Then $\mathcal{A}_0$ and $\mathcal{A}_1$ are equivalent iff they have the same rank.*

Proof. See [V], for example. ∎

Some other partial solutions to Problem 3.1.3 include: Any two transversal Lagrangian subspaces (i.e. transversal subspaces of dimension n in $(\mathbf{R}^{2n}, \tau_{2n})$ for which the induced form vanishes) can be mapped to any other such pair. For three or more transversal Lagrangian subspaces there are obstructions ([GS1]). If each arrangement contains exactly two distinct hyperplanes then there are two equivalence classes, distinguished by whether or not the Hamiltonian subspace of each element is contained in the intersection of the two elements.

Solving Problem 3.1.3 is the first step toward finding Darboux-type theorems for unions $\Gamma = \cup_{i=1}^{m} M_i$ of submanifolds M_i in general position in a symplectic manifold V. Typical results to date have been for special cases of 2-arrangements ([Me] and Theorem 16).

Theorem 3.1.6 *([LM]) Let (V, ω_0, ω_1) be a symplectic manifold of dimension $2n$, and let k and l be integers $(0 \leq k \leq l \leq n)$. Suppose F is a completely integrable vector subbundle of TV with dimension $2n - k - l$. Assume further that F^{s_a} is completely integrable for each $a \in \{0, 1\}$ and that $F \cap F^{s_a}$ is of constant dimension $l - k$. Then for every point $p \in V$ there are neighborhoods U_a of p and a symplectomorphism $\phi : (U_0, \omega_0) \to (U_1, \omega_1)$ such that $\phi(p) = p$,* *where ϕ maps each leaf of the foliation of U_0 defined by F (respectively, by F^{s_0}, $F \cap F^{s_0}$, and $F + F^{s_0}$) onto a leaf of U_1 defined by F (respectively, by F^{s_1}, $F \cap F^{s_1}$, and $F + F^{s_1}$).*

Remark 3.1.7 This shows that the foliation of V defined by F (which has constant rank $2n-2l$) may by locally identified with the foliation of $(\mathbf{R}^{2n}, \tau_{2n})$ whose leaves are the affine subspaces of dimension $2n-k-l$ defined by equations of the form

$$x_1 = a_1, x_3 = a_2, x_5 = a_3, \ldots, x_{2k-1} = a_k,$$

$$x_2 = a_{k+1}, x_4 = a_{k+2}, x_6 = a_{k+3}, \ldots, x_{2l} = a_{k+l}$$

where $a_1, a_2, a_3, \ldots, a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+l}$ are constants.

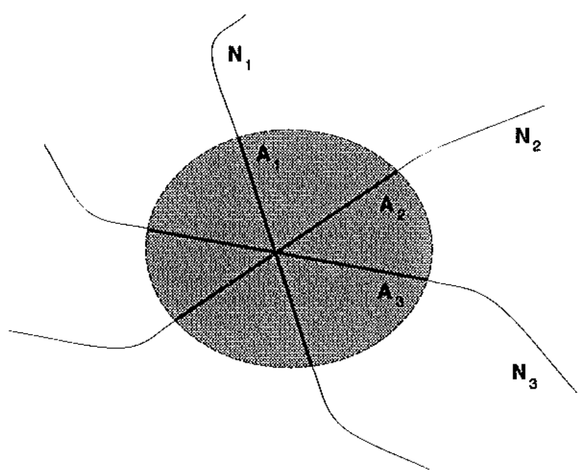
Once Problem 3.1.3 (the linearization of Problem 3.1.1) has been solved there may still be difficulties in finding a weakly allowable symplectomorphism ϕ between $(\mathcal{U}_0, p)$ and $(\mathcal{U}_1, q)$: This difficulty is dealt with in the last section of this chapter (see Corollary 3.5.6).

Conjecture 3.1.8 *([MP2]) Suppose that $\mathcal{A}$ is an arrangement in $\mathbf{R}^{2n}$ (see Definition 3.1.2) and that ω_0 and ω_1 are symplectic forms on $\mathbf{R}^{2n}$ which induce equal forms on each element of $\mathcal{A}$ as well as being equal on $T_K V$. Then there is an allowable symplectomorphism $\phi : (\mathcal{U}_0, \omega_0) \to (\mathcal{U}_1, \omega_1)$ between neighborhoods $\mathcal{U}_a$ of Γ in $\mathbf{R}^{2n}$.*

Remark 3.1.9 This conjecture would be an easy corollary to a Local Poincaré Lemma for arrangements.

Lemma 3.1.10 *([E]) Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an arrangement of $(\mathbf{R}^{2n}, \tau_{2n})$, and let $N_1, N_2, \ldots, N_m$ be germs of submanifolds such that $T_0 N_i = A_i$, $\forall i \in \{1, \ldots, m\}$. Then there exists a smooth diffeomorphism $\phi : \mathcal{U} \to \mathcal{V}$ between neighborhoods of 0 such that $\mathbf{d}\phi(0) = \mathbf{1}$ and $\phi(A_i) = N_i$.*

Definition 3.1.11 The above lemma shows that, given a finite collection $\{N_i\}_{i=1}^m$ of submanifolds in general position, it is always possible to choose local coordinates so that the collection looks like an m-arrangement. Call such a coordinate neighborhood **normal**.



Problem 3.1.12 *Let $\{N_i\}_{i=1}^m$ be a finite collection of allowable submanifolds in general position in the compact manifold (V, ω_0, ω_1). Let K denote the singular set of $\Gamma = \cup_{i=1}^m N_i$. Suppose further that ω_t is an allowable family of symplectic forms which deforms ω_0 into ω_1. Is there an allowable symplectomorphism $\phi : (\mathcal{U}_0, \omega_0) \to (\mathcal{U}_1, \omega_1)$ between neighborhoods $\mathcal{U}_a$ of Γ in V?*

Remark 3.1.13 Assuming Conjecture 3.1.8 to be true, an argument by McDuff and Polterovich [MP] proves the existence of ϕ assuming $H^1(M_i, \mathbf{R}) = 0 \; \forall i \in 1, \ldots, m$.

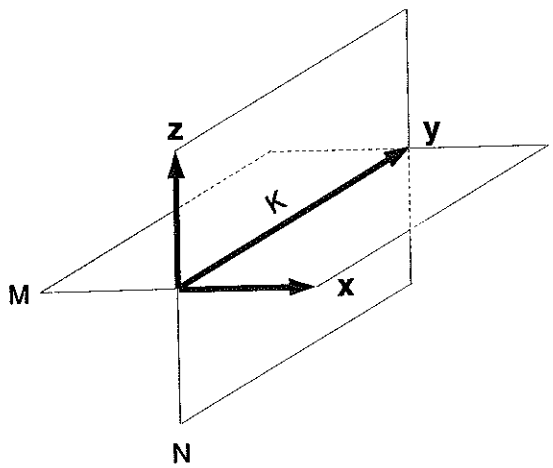
This thesis is focused on exploring intrinsic/extrinsic symplectic geometries of **pairs** of submanifolds, an exploration which has led to the problem of equivalence for 2-arrangements. The next section takes a closer look at this problem.

3.2 2-Arrangements

Suppose $\mathcal{A}_0 = (A^0_1, A^0_2)$ and $\mathcal{A}_1 = (A^1_1, A^1_2)$ are 2-arrangements in $(\mathbf{R}^{2n}, \tau_{2n})$. Equivalence implies that the dimensions and ranks of the subspaces A^0_0, A^0_1, K^0 equal those of the subspaces A^1_0, A^1_1, K^1 [Note: Weak allowability alone gives us these equalities]. Example 3.3.4 below shows that these conditions alone are not sufficient. However they do guarantee the existence of an $L \in GL(2n, \mathbf{R})$ such that $L(A^0_a) = A^1_a$ and such that $L^*\tau_{2n}$ induces the same form on A^0_a as τ_{2n}.

The equivalence problem for 2-arrangements can now be reduced in several short steps to an equivalence problem for symplectic forms:

1. Define $\mathcal{A}^{n,\mu,\nu} = (M, N)$ (or just $\mathcal{A}$) to be the **standard** 2-arrangement in $\mathbf{R}^{2n} = (x_1, \ldots, x_\mu; y_1, \ldots, y_\nu; z_1, \ldots, z_{2n-\mu-\nu}) = (\mathbf{x}, \mathbf{y}, \mathbf{z})$, i.e., $M = (\mathbf{x}, \mathbf{y}, \mathbf{0})$, $N = (\mathbf{0}, \mathbf{y}, \mathbf{z})$, and $K = (\mathbf{0}, \mathbf{y}, \mathbf{0})$.



2. Let $\mathcal{G}_{\mathcal{A}}^{k_0,k_1,k_2}$ (or just $\mathcal{G}$) denote the set of all linear symplectic forms on the vector space $\mathbf{R}^{2n}$ inducing standard forms on K, M, N (of $\mathcal{A}$ above) with ranks k_0, k_1, k_2 respectively. Call these forms **standard**.

3. Let $\mathcal{H}^{k_0,k_1,k_2} \subset GL(2n, \mathbf{R})$ (or just $\mathcal{H}$) denote the subgroup preserving $\mathcal{A}$ and $\mathcal{G}$, i.e., $\mathcal{H}(M) = M$, $\mathcal{H}(N) = N$, and $\mathcal{H}^*(\mathcal{G}) = \mathcal{G}$.

4. The problem then is this: Classify the orbits of the action of $\mathcal{H}$ on $\mathcal{G}$ for a given $\mathcal{A}$.

Definition 3.2.1 Suppose ω_0 and ω_1 belong to $\mathcal{G}$. These symplectic forms are said to be **equivalent** if and only if they lie on the same orbit.

For some set-ups this orbit classification problem is simple to solve:

Example 3.2.2 Suppose that M and N are Lagrangian: this means that $\dim M = \dim N = n$, $K = 0$, and that $k_1 = k_2 = 0$. $\mathcal{G}^{0,0,0}$ is then the set of all skew-symmetric bilinear forms with matrix representations (w/r to the standard, ordered basis)

$$\begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}$$

where C is any element of $GL(n, \mathbf{R})$, and $\mathcal{H}$ is the set of all linear transformations whose matrix representations are given by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A, B are any elements of $GL(n, \mathbf{R})$. If $\omega_0, \omega_1 \in \mathcal{G}$ are written as

$$\omega_0 = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} 0 & D \\ -D^T & 0 \end{pmatrix}$$

then defining $L \in \mathcal{H}$ by

$$L = \begin{pmatrix} C^T & 0 \\ 0 & D^{-1} \end{pmatrix}$$

shows that $L^*\omega_1 = \omega_0$ since

$$\begin{pmatrix} C & 0 \\ 0 & (D^{-1})^T \end{pmatrix} \bullet \begin{pmatrix} 0 & D \\ -D^T & 0 \end{pmatrix} \bullet \begin{pmatrix} C^T & 0 \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}.$$

Thus there is one and only one orbit of $\mathcal{H}$ in $\mathcal{G}$, i.e., all forms in $\mathcal{G}$ are equivalent to one another. ■

Remark 3.2.3 This last example is no surprise since it is a well known fact (as stated earlier) that any two transversal Lagrangian subspaces of $(\mathbf{R}^{2n}, \tau_{2n})$ can be mapped to any other such pair by a linear symplectomorphism.

Similarly we could show that for the same 2-arrangement of Example 3.2.2 there is one and only one orbit in $\mathcal{G}^{0,0,2k}$ (here $n = 2k$, M is Lagrangian, N is symplectic, and $K = 0$).

As n increases, the variety of possible groups $\mathcal{G}_{\mathcal{A}}^{k_0,k_1,k_2}$ grows arbitrarily large, and the problem of establishing equivalence for each such group begs for a general attack plan. Actually, it's not clear that it's possible in any given case to find a finite number of invariants classifying the orbits of $\mathcal{H}$.

The most difficult case should also be the generic one, because then M and N are symplectic. Generically K will also be symplectic, and this is the set-up which will concern us for the rest of this thesis.

3.3 Symplectic 2-Arrangements

This section is concerned with solving the equivalence problem for two 2-arrangements $\mathcal{A}_0 = (A_1^0, A_2^0)$ and $\mathcal{A}_1 = (A_1^1, A_2^1)$ in $(\mathbf{R}^{2n}, \tau_{2n})$ where A_1^a, A_2^a, K^a are symplectic subspaces (such arrangements are called **symplectic 2-arrangements**). This is equivalent to classifying the orbits of $\mathcal{H}$ on $\mathcal{G}^{2s,2(r+s),2(n-r)}$ for the standard arrangement $\mathcal{A}^{n,2r,2s}$. We will give a complete classification of the orbit structure on an open, dense, invariant subset $\mathcal{W}$ of $\mathcal{G}$: To any element in $\mathcal{G}$ we associate a linear operator and show that any two elements in $\mathcal{W}$ are equivalent iff their respective operators have identical rational canonical forms. (Please refer to [HK] for a review of rational canonical forms.)

Let us now classify the standard linear symplectic forms that comprise $\mathcal{G}$.

Definition 3.3.1 Let J_{2k} denote the $2k$ by $2k$ matrix whose diagonal 2×2 blocks are the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with 0's elsewhere. Let I_{2k} denote the $2k$ by $2k$ identity matrix.

Remark 3.3.2 Note that $(J_{2k})^2 = -I_{2k}$.

Then in standard (ordered) coordinates the matrix representation of ω is

$$\begin{pmatrix} J_{2r} & 0 & C \\ 0 & J_{2s} & 0 \\ -C^T & 0 & J_{2(n-r-s)} \end{pmatrix}$$

where C is a $2r$ by $2(n-r-s)$ matrix.

What are the possible choices for C?

To answer this question, break C up into 2×2 blocks and write $C = (\mathbf{c}_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n-r-s$. WLOG $r \geq n-r-s$. Let S^l_μ denote the set of all subsets of $\{1, 2, 3, \ldots, l\}$ with length μ. For $I \in S^r_\mu$ and $J \in S^{n-r-s}_\mu$ with $1 \leq \mu \leq n-r-s$, let $C_{I,J}$ be the $2\mu \times 2\mu$ matrix

$$C_{I,J} = \begin{pmatrix} \mathbf{c}_{I_1,J_1} & \cdots & \mathbf{c}_{I_1,J_\mu} \\ \vdots & & \vdots \\ \mathbf{c}_{I_\mu,J_1} & \cdots & \mathbf{c}_{I_\mu,J_\mu} \end{pmatrix}$$

Now define

$$\kappa_C = \sum_{\mu=1}^{n-r-s} \sum_{\substack{I \in S^r_\mu \\ J \in S^{n-r-s}_\mu}} \det C_{I,J}.$$

Lemma 3.3.3 *ω is symplectic if and only if $\kappa_C \neq 1$.*

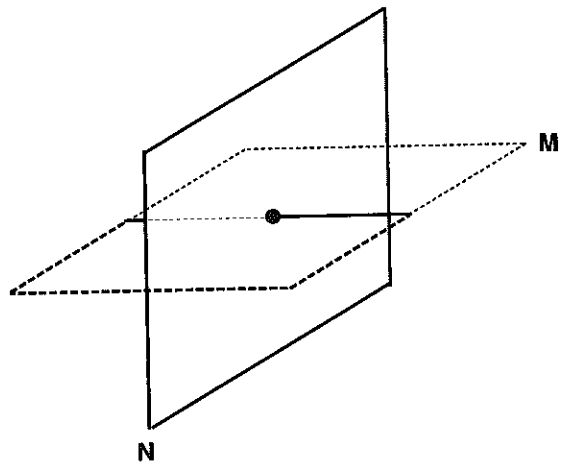
Proof. ω is symplectic if and only if

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n} = \omega^n$$

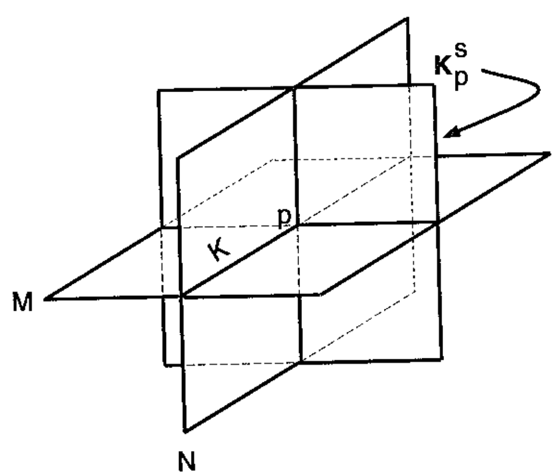
is a volume form. But $\omega^n = n!(1 - \kappa_C)\tau_{2n}$. ∎

Lemma 3.3.3 classifies $\mathcal{G}$ exactly.

Before we continue with the problem of classifying $\mathcal{H}$-orbits we can make a simplification: If $p = (\mathbf{0}, \mathbf{y}_p, \mathbf{0}) \in K$, then for any $\omega \in \mathcal{G}$ the skew-orthogonal plane to K at the point $(\mathbf{0}, \mathbf{y}_p, \mathbf{0})$ is the affine plane given by $(\mathbf{x}, \mathbf{y}_p, \mathbf{z})$. Call this plane K_p^s. Since any element $L \in \mathcal{H}$ must preserve all planes K_p^s, we can assume that $K = 0$ (i.e. $s = 0$).



Now the situation at hand is this: (V, Γ) is the standard 2-arrangement $\mathcal{A}^{n,2r,0} = (M, N)$ of $\mathbf{R}^{2n} = (\mathbf{x}, \mathbf{z})$ $(M^{2r} = (\mathbf{x}, \mathbf{0})$ and $N^{2(n-r)} = (\mathbf{0}, \mathbf{z}))$



and we have two 2-forms ω_0 and ω_1 inducing standard linear symplectic forms on both M and N. Let us write

$$\omega_0 = \begin{pmatrix} J_{2r} & C \\ -C^T & J_{2(n-r)} \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} J_{2r} & D \\ -D^T & J_{2(n-r)} \end{pmatrix}$$

where C and D are any constant matrices such that κ_C and κ_D are not equal to 1.

The following simple example shows that ω_0 may not be equivalent to ω_1.

Example 3.3.4 Put

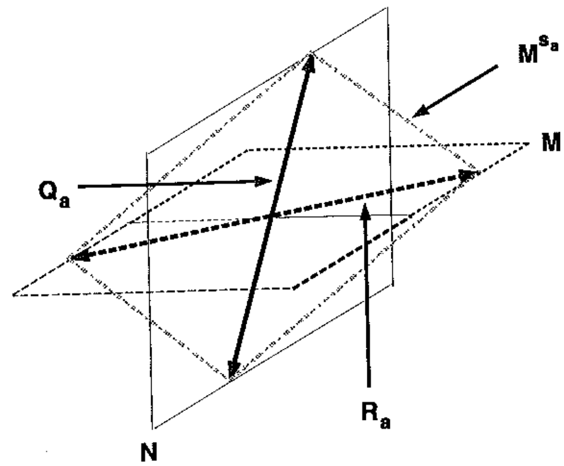
$$\begin{aligned} \omega_0 &= dx_1 \wedge dx_2 + dz_1 \wedge dz_2, \\ \omega_1 &= dx_1 \wedge dx_2 + dz_1 \wedge dz_2 + dx_1 \wedge dz_1. \end{aligned}$$

Thus M and N are both symplectic 2-planes in $\mathbf{R}^4$ with respect to either ω_0 or ω_1. If ω_0 was equivalent to ω_1, then there would be an element L of $GL(4, \mathbf{R})$ mapping M^{s_0} onto M^{s_1}. Since L would also map M onto M and N onto N, $M^{s_0} = N$ would map onto $M^{s_1} \neq N$, a contradiction. ■

Remark 3.3.5 Note that if we change the setup so that L mapped N onto M and M onto N, then an obstruction would still exist: M^{s_0} would have to map onto both $M^{s_1} \neq M$ **and** onto M.

Let $\pi : \mathbf{R}^{2n} \to M$ denote the projection map defined by

$$\pi(x_1, \ldots, x_{2r}, z_1, \ldots, z_{2(n-r)}) = (x_1, \ldots, x_{2r}).$$



Put $Q_a \equiv M^{s_a} \cap N$ and $R_a \equiv \pi(M^{s_a})$.

Necessary conditions for the existence of L would then include

$$(*) = \begin{cases} (i) & \dim R_0 = \dim R_1 \\ (ii) & \text{rank} R_0 = \text{rank} R_1 \\ (iii) & \dim Q_0 = \dim Q_1 \\ (iv) & \text{rank} Q_0 = \text{rank} Q_1 \end{cases}$$

However, conditions $(*)$ alone are not generally sufficient.

Recalling that $L \in \mathcal{H}$ must preserve both M and N, writing L in matrix notation (with respect to the standard coordinates) as

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

(with $A \in GL(2r, \mathbf{R})$ and $B \in GL(2(n-r), \mathbf{R})$) allows us to express the condition $L^*\omega_1 = \omega_0$ as

$$\begin{pmatrix} A^T & 0 \\ 0 & B^T \end{pmatrix} \bullet \begin{pmatrix} J_{2r} & D \\ -D^T & J_{2(n-r)} \end{pmatrix} \bullet \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} J_{2r} & C \\ -C^T & J_{2(n-r)} \end{pmatrix}$$

or

$$\begin{pmatrix} A^T J_{2r} A & A^T D B \\ -B^T D^T A & B^T J_{2(n-r)} B \end{pmatrix} = \begin{pmatrix} J_{2r} & C \\ -C^T & J_{2(n-r)} \end{pmatrix}$$

or

$$(**) = \begin{cases} (i) & A^T J_{2r} A = J_{2r} \\ (ii) & B^T J_{2(n-r)} B = J_{2(n-r)} \\ (iii) & A^T D B = C \end{cases}$$

Conditions $(**)$ are necessary and sufficient for the equivalence of ω_0 and ω_1.

Conditions (i) and (ii) of $(**)$ are equivalent to $A \in Sp(2r, \mathbf{R})$ and $B \in Sp(2(n-r), \mathbf{R})$ respectively. This is to be expected since L induces linear symplectomorphisms with respect to the standard symplectic forms on M and on N. Condition (iii) of $(**)$ is more problematical, and its solution will occupy us for the rest of this section.

Theorem 3.3.6 *Two elements of $\mathcal{G}$ lie on the same orbit iff there is a linear symplectomorphism*

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

(where $A \in Sp(2r, \mathbf{R})$ and $B \in Sp(2(n-r), \mathbf{R})$) mapping M^{s_0} onto M^{s_1}.

Proof. We will prove this theorem by first establishing conditions $(***)$ (see below) which are equivalent to those of $(**)$.

Remark 3.3.7 From here on we will drop subscripts from notation such as J_{2r} or $I_{2(n-r)}$. The setting will make clear the dimensions of these matrices.

Lemma 3.3.8 *The column vectors of the $2n \times 2r$ matrix*

$$\begin{pmatrix} JC \\ I \end{pmatrix}$$

span M^{s_0}. (A similar statement holds for M^{s_1}.)

Proof. Write

$$\begin{pmatrix} JC \\ I \end{pmatrix} = (v_1, \cdots, v_{2(n-r)})$$

where the set of column vectors $\{v_j\}, 1 \leq j \leq 2(n-r)$, are linearly independent. Since $\dim M^{so} = 2n - \dim M = 2(n-r)$, all that remains to be shown is that $v_j \in M^{so}$ for each j.

Definition 3.3.9 If $i \in \mathbf{N}$ then define

$$[i] = \begin{cases} i+1 & \textit{if } i \textit{ is odd} \\ i-1 & \textit{if } i \textit{ is even} \end{cases}$$

Note that $[[i]] = i$.

Definition 3.3.10 For $(i,k) \in \mathbf{N} \times \mathbf{N}$ define the Kronecker function δ on $\mathbf{N} \times \mathbf{N}$ to be

$$\delta(i,k) = \delta_{i,k} = \begin{cases} 0 & \textit{if } i \neq k \\ 1 & \textit{if } i = k \end{cases}$$

Let w_k be the $2n \times 1$ vector with a 1 in the k-th slot and 0's elsewhere (so the set of vectors $\{w_k\}, 1 \leq k \leq 2r$, span M). For $1 \leq i, k \leq 2r$, $\omega_0(w_i, w_k) = \delta_{[i],k}$, so $\omega_0(w_{[i]}, w_k) = \delta_{i,k}$.

For all $1 \leq j \leq 2(n-r)$ and $1 \leq k \leq 2r$,

$$
\begin{aligned}
\omega_0(v_j, w_k) &= \omega_0(w_{2r+j} + \sum_{1\leq i\leq 2r} (-1)^i c_{i,j} w_{[i]}, w_k) \\
&= \omega_0(w_{2r+j}, w_k) + \sum_{1\leq i\leq 2r} (-1)^i c_{i,j}\, \omega_0(w_{[i]}, w_k) \\
&= -c_{k,j} + c_{k,j} \\
&= 0
\end{aligned}
$$

This shows that each vector $v_j \in M^{s_0}$, and this completes the proof. ∎

Lemma 3.3.11 *Conditions (i), (ii), and (iii) of* $(**)$ *are equivalent to*

$$
(***) = \begin{cases} (i) & A^T J A = J \\ (ii) & B^T J B = J \\ (iii) & AJCB^{-1} = JD \end{cases}
$$

Proof. I. $(**) \Leftrightarrow (***)$

This follows from the following equivalent formulations. (Note: Condition (i) of $(**) \Longleftrightarrow A = -JA^{-1}J$.)

$$
\begin{aligned}
C &= A^T DB \\
C &= -JA^{-1}JDB \\
AJCB^{-1} &= JD
\end{aligned}
$$

∎

Although it seems more complicated to replace conditions (**) by (* * *), condition (iii) of (* * *) has an interesting and simple geometrical meaning: If the map L does indeed exist, L would necessarily map M^{s_0} onto M^{s_1}. Thus, M^{s_1} is not only spanned by the column vectors of the $2n \times 2r$ matrix

$$\begin{pmatrix} JD \\ I \end{pmatrix}$$

but also by those of

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \bullet \begin{pmatrix} JC \\ I \end{pmatrix} = \begin{pmatrix} AJC \\ B \end{pmatrix}.$$

Since

$$\begin{pmatrix} AJC \\ B \end{pmatrix} \bullet B^{-1} = \begin{pmatrix} AJCB^{-1} \\ I \end{pmatrix}$$

the column vectors of

$$\begin{pmatrix} AJCB^{-1} \\ I \end{pmatrix}$$

must also span M^{s_1}. Since the column vectors of each of the two matrices

$$\begin{pmatrix} AJCB^{-1} \\ \\ I \end{pmatrix}, \begin{pmatrix} JD \\ \\ I \end{pmatrix}$$

are in fact linearly independent (and so form a basis for M^{s_1}) we must have

$$AJCB^{-1} = JD$$

In other words, to find the desired linear symplectomorphism

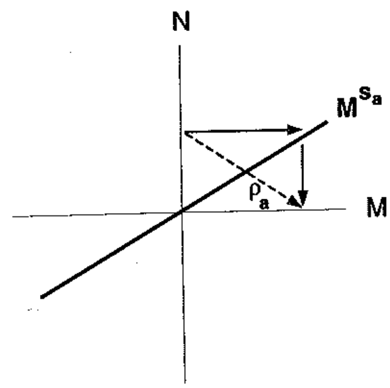
$$L = \begin{pmatrix} A & 0 \\ \\ 0 & B \end{pmatrix}$$

(where $A \in Sp(2r, \mathbf{R})$ and $B \in Sp(2(n-r), \mathbf{R})$) we only need to find such an L that maps M^{s_0} onto M^{s_1}.

This completes the proof of Theorem 3.3.6.

We can now reformulate conditions $(***)$ in terms of an equivalent set of conditions involving the classification of skew-symmetric bilinear forms, a problem which has been solved. To this end define a map $\rho_a : N \to M$ by first mapping N onto M^{s_a} and then projecting onto M. Specifically, if e_j denotes the $2(n-r) \times 1$ vector with a 1 in the j-th row and 0's elsewhere, then $\rho_a(e_j)$ equals the j-th column vector of the matrix JC. Thus, M^{s_a} is the graph of p_a.

We will assume from now on that $Q_a = 0$, i.e. that M^{s_a} has trivial intersection with N: In this case we say that $(\mathcal{A}, \omega_0, \omega_1)$ is **generic.** WLOG



we may also assume that $2r \geq 2(n-r)$ (simply interchange M and N if it isn't).

Definition 3.3.12 The **spectral data** of $(\mathcal{A}, \omega_0, \omega_1)$ consists of the pair of skew-symmetric bilinear forms $(\tau_{2(n-r)}, \rho_a^*(\tau_{2r})) \stackrel{d}{=} (\tau, \beta^a)$.

Proposition 3.3.13 *Given* $(\mathcal{A}, \omega_0, \omega_1)$*,* ω_0 *and* ω_1 *are equivalent if and only if there is an endomorphism* $B \in Sp(2(n-r), \mathbf{R})$ *such that* $(B^{-1})^*\beta^1 = \beta^0$.

Definition 3.3.14 In this case we say that the spectral data are **equivalent.**

Proof. Since ρ_a is nonsingular (due to genericity), $\{(JCB^{-1})(e_j)\}_{j=1}^{j=2(n-r)}$ is a basis for R_0 and $\{(JD)(e_j)\}_{j=1}^{j=2(n-r)}$ is a basis for R_1, for any $B \in Sp(2(n-r), \mathbf{R})$. By Witt's Theorem, $AJCB^{-1} = JD$ (for some $A \in Sp(2r, \mathbf{R})$) iff $\tau_{2r}((JCB^{-1})(e_j), (JCB^{-1})(e_k)) = \tau_{2r}((JD)(e_j), (JD)(e_k))$ for all $1 \leq j, k \leq 2(n-r)$ iff $(B^{-1})^*\beta^1 = \beta^0$. ∎

Thus, two forms ω_0 and ω_1 are equivalent iff there is a linear automorphism B of $\mathbf{R}^{2(n-r)}$ such that

- $B^*\tau = \tau$
- $B^*\beta^1 = \beta^0$

and so we are interested in the simultaneous investigation of pairs of skew-symmetric bilinear forms. The problem at hand is to simultaneously reduce a pair of skew-symmetric bilinear forms (α, β) defined on a vector space V to an elementary, canonical form ([GZ]).

Definition 3.3.15 Call the pair (α, β) of skew-symmetric bilinear forms on V **decomposable** if there exists two supplementary non-zero subspaces V_1 and V_2 of V such that both forms α and β are direct sums of their restrictions to V_1 and V_2, i.e., the subspaces V_1 and V_2 are skew-orthogonal with respect to both forms. The pair is **indecomposable** if it cannot be represented as a direct sum.

We want to write V as a direct sum of indecomposable subspaces $V = \oplus_{i=1}^{k} V_i$ where V_i are skew-orthogonal with respect to α and also β, and for this decomposition to be unique up to an isomorphism of V. To do this it is useful to describe the pair (α, β) as a pair of linear mappings, and to classify pairs of linear mappings instead.

Theorem 3.3.16 *([GZ])*

1. *The list of indecomposable components (up to an isomorphism) of a pair of skew-symmetric bilinear forms is uniquely defined, and the same is true for a pair of linear mappings;*

2. *If a pair of skew-symmetric bilinear forms α, β in a (finite-dimensional) vector space V is indecomposable, then the vector space V can be represented as a direct sum of two subspaces W_1 and W_2, where*

   (a) *Both W_1 and W_2 are isotropic with respect to both forms α and β;*

   (b) *The pairings α and β determine two mappings $\tilde{\alpha}, \tilde{\beta} : W_1 \to W_2^*$, and this pair of mappings from one vector space to another is indecomposable in the above sense;*

3. *On the other side, any indecomposable pair of mappings $\tilde{\alpha}, \tilde{\beta} : W_1 \to W_2^*$ determines an indecomposable pair of skewsymmetric bilinear forms α, β in the vector space $W_1 \oplus W_2$ by the rule*

$$\alpha((w_1, w_2), (w_1', w_2')) = \langle \tilde{\alpha}(w_1), w_2' \rangle - \langle \tilde{\alpha}(w_1'), w_2 \rangle,$$

$$\beta((w_1, w_2), (w_1', w_2')) = \langle \tilde{\beta}(w_1), w_2' \rangle - \langle \tilde{\beta}(w_1'), w_2 \rangle.$$

4. *Any indecomposable pair of mappings from a vector space X_1 to a vector space X_2 is isomorphic to exactly one pair from the list:*

   (a) *The Jordan case $J_k^\lambda, k \geq 1$ with eigenvalue λ: here $X_1 = X_2$, $dim X_1 = k, \tilde{\alpha} = i_{X_1}$, and $\tilde{\beta}$ is a mapping from X_1 to X_1 with exactly one Jordan block (of size k) with eigenvalue λ;*

   (b) *The Jordan case $J_k^\infty, k \geq 1$ with eigenvalue ∞: here $X_1 = X_2$, $dim X_1 = k, \tilde{\alpha}$ is a mapping from X_1 to X_1 with exactly one Jordan block (of size k) with eigenvalue 0, and $\tilde{\beta} = i_{X_1}$;*

(c) *The Kronecker case* $K_k^+, k \geq 1$*: here* $X_1 = S^{k-1}R, X_2 = S^kR$ *(i.e., the symmetrical powers),* R *is a 2-dimensional vector space with a basis* $\{r_1, r_2\}, \tilde{\alpha} = M_{r_1}$*, and* $\tilde{\beta} = M_{r_2}$*, where* M_r *is the mapping of multiplication by* r *from* $S^{k-1}R$ *to* S^kR*;*

(d) *The Kronecker case* $K_k^-, k \geq 1$*: here* $X_1 = S^kR, X_2 = S^{k-1}R$ *(i.e., the symmetrical powers),* R *is a 2-dimensional vector space with a basis* $\{r_1, r_2\}, \tilde{\alpha} = D_{r_1}$*, and* $\tilde{\beta} = D_{r_2}$*, where*

$$D_{r_1} = \frac{\partial}{\partial r_1}, D_{r_2} = \frac{\partial}{\partial r_2} : S^kR \to S^{k-1}R;$$

(e) *The trivial Kronecker case* $K_0^+ : dim X_1 = 0, dim X_2 = 1$*, and* $\tilde{\alpha} = \tilde{\beta} = 0$*;*

(f) *The trivial Kronecker case* $K_0^- : dim X_1 = 1, dim X_2 = 0$*, and* $\tilde{\alpha} = \tilde{\beta} = 0$*;*

5. *If a pair of skew-symmetric bilinear forms is in general position, then*

   (a) *if the space* V *is even dimensional all the indecomposable components are 2-dimensional, canonically defined and correspond to the pairs of mappings* $J_1^\lambda, \lambda \in \mathbf{C} \cup \{\infty\}$*;*

   (b) *if the space* V *is odd dimensional,* $dim V = 2k - 1$*, then there is only one indecomposable component (so the pair is indecomposable), corresponding to the Kronecker case* K_k^- *(or* K_k^+*, since* K_k^- *and* K_k^+ *lead to isomorphic pairs of skew-symmetric bilinear forms);*

6. *If an indecomposable pair of skew-symmetric bilinear forms in an odd-dimensional vector space* V *corresponds (as above) to the Kronecker pair*

of mappings $K_k^- : W_1 \to W_2^*$, then the subspace $W_1 \subset V$ is canonically defined. It is spanned by 1-dimensional kernels (i.e., by the vectors which are orthogonal to the whole space) of linear combinations $\alpha - \lambda\beta$ of forms α and β. These kernels considered as points in the projectivization PW_1 of the space W_1 form a Veronese curve, i.e., a curve of minimal possible degree (equal to $dimPW_1$) spanning the whole space PW_1.

3.4 The Main Theorem

Let (τ, β_q^a) denote the spectral data for $(T_qV, \omega_0(q), \omega_1(q))$. The existence of ϕ would imply that the spectral data (τ, β_q^0) and $(\tau, \beta_{\phi(q)}^1)$ are equivalent $\forall q \in K \cap \mathcal{U}_0$. Turiel [T] gives a formulation different from Theorem 3.3.16 for equivalence which is more utilitarian for our purpose although it is less general in scope, corresponding only to the Jordan cases.

Theorem 3.4.1 *([T]) Let (α, β) be a pair of bilinear skew-symmetric forms defined on an even-dimensional vector space V, and suppose that α is non-degenerate. Let H be the endomorphism of V defined by the relation $\beta(v,w) = \alpha(Hv, w)$ $\forall v, w \in V$. Then there exists a direct-sum decomposition $V = \oplus_{j=1}^m V_j$ into even-dimensional subspaces and a family of (not necessarily distinct) polynomials $\varphi_1^{l_1}, \ldots, \varphi_m^{l_m}$ (where each φ_j is irreducible and l_j are positive integers) such that*

1. *$\alpha(V_j, V_k) = \beta(V_j, V_k) = 0$ if $j \neq k$. If we put $\alpha_j \stackrel{d}{=} \alpha$ restricted to V_j, then $\alpha = \oplus_{j=1}^m \alpha_j$ and $\beta = \oplus_{j=1}^m \beta_j$.*

2. *If we define H_j by the relation $\beta_j(v,w) = \alpha_j(H_j v, w)$ $\forall v,w \in V_j$, then the elementary divisors of H_j are $\{\varphi_j^{l_j}, \varphi_j^{l_j}\}$.*

3. *The elementary divisors of H are $\{\varphi_j^{l_j}, \varphi_j^{l_j}\}_{j=1}^{m}$.*

Consequently, the algebraic structure of (α, β) is completely determined by the rational canonical form for H.

Remark 3.4.2 If an endomorphism H of V^{2n} has a set of elementary divisors that can be written in the form $\{\varphi_j^{l_j}, \varphi_j^{l_j}\}_{j=1}^{m}$, and if α is a linear symplectic form on H, then the equation

$$\beta(v,w) = \alpha(Hv, w) \;\; \forall v,w \in V.$$

defines a skew-symmetric bilinear form β on V.

The vector subspaces V_j given by Theorem 3.4.1 (corresponding to the spaces $X_1 \oplus X_1^*$ for the Jordan case of Theorem 3.3.16) are indecomposable.

Define endomorphisms H^a of $\mathbf{R}^{2(n-r)}$ by

$$\beta^a(v,w) = \tau(H^a v, w) \;\; \forall v,w \in \mathbf{R}^{2(n-r)}.$$

Turiel's Theorem proves

Theorem 3.4.3 (The Main Theorem) *Any two generic symplectic 2-arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ are equivalent (see Definition 3.1.2) iff their associated endomorphisms H^0 and H^1 have the same rational canonical form.*

3.5 A Moduli Space for the Local Classification of Symplectic Pairs

Let us go back to the following situation (see Problem 3.1.1): (V, Γ) is symplectic where ω_0 and ω_1 are weakly allowable symplectic forms on V, and we are looking for a diffeomorphism $\phi : \mathcal{U}_0 \to \mathcal{U}_1$ between neighborhoods of points of Γ in V such that ϕ

1. preserves Γ,
2. $\phi^*\omega_1 = \omega_0$ on $T_K V$, and
3. $\phi^*\omega_1$ and ω_0 induce equal forms on M and on N.

That is, we want to locally transform a weakly allowable pair into an allowable one.

By Lemma 3.1.10 we may assume that the open sets $\mathcal{U}_a$ are normal, and that in fact K, M, and N are in standard position in $\mathbf{R}^{2n}$ (so $K = (\mathbf{0}, \mathbf{y}, \mathbf{0})$, $M = (\mathbf{x}, \mathbf{y}, \mathbf{0}), N = (\mathbf{0}, \mathbf{y}, \mathbf{z})$). Now that we understand the linearization of Problem 3.1.1 (see the Main Theorem, Theorem 3.4.3) it remains to be seen how this knowledge can be patched together.

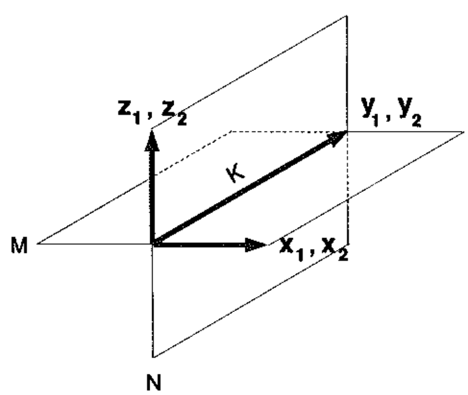
Applying the construction from Section 3.4 to each point q of K gives us the spectral data (τ, β_q^a) of $T_q V$ and so defines a smooth section H^a of End(E), where E is the trivial bundle $E \xrightarrow{\pi} K$ with fibers F_q (given by the linear $\mathbf{z}$-coordinates).

Definition 3.5.1 Let $\mathbf{R}_K[t]$ denote the polynomial algebra of one variable whose coefficents are smooth, real-valued functions defined on K. A polynomial $\varphi \in \mathbf{R}_K[t]$ is said to be **irreducible** iff it is irreducible at each point of K. We similarly define φ to be **prime** iff it is prime at each point.

Definition 3.5.2 *If H is a smooth section of End(E), then the* **algebraic type of** H **is constant** *iff there exist distinct, irreducible, prime polynomials $\varphi_1, \ldots, \varphi_\mu \in \mathbf{R}_K[t]$ and if there exist positive integers $l_{i,j}, 1 \leq i \leq \eta_j$, $1 \leq j \leq \mu$, such that $\{\varphi_j^{l_{i,j}}, \varphi_j^{l_{i,j}}\}_{j=1}^{\mu}$ are the elementary divisors of H. The coefficients $\lambda_j^{\delta_j}$ of φ_j (either $\varphi_j = t + \lambda_j^1$ in which case $\delta_j = 1$, or $\varphi_j = t^2 + \lambda_j^1 t + \lambda_j^2$ in which case $1 \leq \delta_j \leq 2$) are called the* **eigenfunctions** *of H.*

Note that the set of points in K where the algebraic type of H is locally constant is open and dense.

Example 3.5.3 In $\mathbf{R}^6$, let M, N and K be as pictured below.



The symplectic form

$$\omega = \tau_6 + x_1 \sin y_1 \, dy_1 \wedge dz_1 + x_2 \cos y_1 \, dy_1 \wedge dz_2 + \cos y_1 \, dx_1 \wedge dz_1 + \sin y_1 \, dx_2 \wedge dz_2$$

induces the standard sympletic forms on M, N, and K. In fact, the matrix representation of ω with respect to the standard coordinates is

$$\omega = \begin{pmatrix} 0 & 1 & 0 & 0 & cos\, y_1 & 0 \\ -1 & 0 & 0 & 0 & 0 & sin\, y_1 \\ 0 & 0 & 0 & 1 & -x_1\, sin\, y_1 & x_2\, cos\, y_1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

and so $\kappa_C = cos\, y_1 \cdot sin\, y_1 \neq 1$ since

$$C = \begin{pmatrix} cos\, y_1 & 0 \\ 0 & sin\, y_1 \end{pmatrix}.$$

For $q = (\mathbf{0}, \mathbf{y}, \mathbf{0}) \in K$, the spectral data $(\tau, \beta_q) = (dz_1 \wedge dz_2, \kappa_C\, dz_1 \wedge dz_2)$ and

$$H_q = \begin{pmatrix} \kappa_C & 0 \\ 0 & \kappa_C \end{pmatrix}.$$

The elementary divisors of H are then $\{t - \kappa_C, t - \kappa_C\}$ and so the algebraic type of H is constant.

Note that the invariant here, κ_C, is given by $\omega^3 = 3!(1 - \kappa_C)\tau_6$. In general, though, it's not so easy to determine the algebraic type of H. I suspect, however, that the eigenfuctions of H are polynomial functions whose entries are products of the terms $\det C_{I,J}$ which define κ_C (see Section 3.3). ∎

Let $f : \mathcal{O}_0 \to \mathcal{O}_1$ be a symplectomorphism between neighborhoods in K. Solving Problem 3.1.1 amounts to finding necessary and sufficient conditions for the existence of a bundle map $\tilde{f} : E \to E$ lifting f in such a way that $\tilde{f}$ is an isomorphism between $\pi^{-1}(q)$ and $\pi^{-1}(f(q))$ transforming $H^0(q)$ to $H^1(f(q))$.

Definition 3.5.4 If such an f exists we will say that the family of spectral data (τ, β_q^0) is f-**deformable** to $(\tau, \beta_{f(q)}^1)$.

Theorem 3.5.5 *Let $f : \mathcal{O}_0 \to \mathcal{O}_1$ be a symplectomorphism between neighborhoods in K. The spectral data (τ, β_q^0) is equivalent to $(\tau, \beta_{f(q)}^1)$ $\forall q \in \mathcal{O}_0$ iff they are f-deformable to each other iff $H^0(q)$ has the same rational canonical form as $H^1(f(q))$ for each $q \in U_0$.*

Proof. This is a direct result of the Main Theorem, Theorem 3.4.3. ∎

Let

$$\varphi_j^{l_{i,j}}, 1 \le i \le \eta_j, 1 \le j \le \mu$$

and

$$\psi_j^{m_{i,j}}, 1 \leq i \leq \xi_j, 1 \leq j \leq \nu$$

form the elementary divisors of H^0 and H^1 respectively (see Definition 3.5.2). Necessary conditions for f-deformability, independent of the choice of f, would then include

$$(*) \quad \begin{cases} \mu = \nu \\ \eta_j = \xi_j \ \ \forall j \in \{1, \dots, \mu\} \\ l_{i,j} = m_{i,j} \ \ \forall j \in \{1, \dots, \mu\}, \ \forall i \in \{1, \dots, \eta_j\} \end{cases}$$

Additionally there must exist a symplectomorphism $f : \mathcal{O}_0 \to \mathcal{O}_1$ such that

$$(**) \ \ \varphi_j(f(q)) = \psi_j(q) \ \ \forall j \in \{1, \dots, \mu\}.$$

This last condition amounts to finding symplectic normal forms for families of functions, a well-known, and unsolved, problem.

Corollary 3.5.6 Local Classification Theorem for Pairs *Any generic, weakly allowable, symplectic pair (V, Γ) is locally symplectomorphic to an allowable one iff the spectral data are f-deformable for some local symplectomorphism f.*

We thus have a Moduli space as the model for locally classifying generic symplectic pairs.

Example 3.5.7 (2-Dimensional Case) Suppose that K is a codimension 2 submanifold of N. We can write (see Example 3.5.3) $\beta_q = \kappa_C \, dz_1 \wedge dz_2$, where $q = (\mathbf{0}, \mathbf{y}, \mathbf{0})$ and $\kappa_C = \det C$. H_q is simply the 2×2 matrix

$$\begin{pmatrix} \kappa_C & 0 \\ 0 & \kappa_C \end{pmatrix}.$$

Two germs of spectral data $(\tau, \beta_0(p))$ and $(\tau, \beta_1(q))$ are then f-deformable iff there is a diffeomorphism $f : \mathcal{O}_0 \to \mathcal{O}_1$ between neighborhoods of p and q in K such that

- $f^* \tau = \tau$
- $\kappa_C(f(q)) = \kappa_D(q)$

This depends entirely on giving local, symplectic nomal forms for smooth real-valued functions. Assume that $\kappa_C(p) = \kappa_D(q)$. If κ_C and κ_D are regular-valued functions, then f exists. Otherwise the classification becomes much more complicated. If these eigenfunctions have isolated, non-degenerate singularities at the points in question, then what we require is a symplectic Morse theory (see [E]). It is interesting to note that there is no such theory for the case where the co-dimension of K is greater than 2. ■

Definition 3.5.8 Let H be a $(1,1)$ tensor field on E. A point $q \in K$ is said to be a **regular point for** H (or just **regular** for short) if the algebraic type of H is locally constant at q and if the eigenfunctions of H near q are regular. Let $\mathcal{R}$ denote the set of regular points of H.

Let $\mathcal{R}^0$ and $\mathcal{R}^1$ denote the set of regular points for H^0 and H^1. Then $\mathcal{R}^0$ and $\mathcal{R}^1$ are open, dense subsets of K. So if $q \in \mathcal{R}^0$, then the datum of condition $(*)$ are constant near q, and solving for f-deformability would amount to finding some f that satisfied condition $(**)$. Example 4 shows that this case can sometimes be solved.

Appendix A

The Relative Poincaré Lemma

We give here the outlines of a proof for the Relative Poincaré Lemma.

Theorem A.0.9 (Relative Poincaré Lemma) *Let N be a submanifold of the smooth manifold V, and let $\mathcal{U}$ be a tubular neighborhood of N in V. Suppose that λ is a smooth, closed k-form on $\mathcal{U}$ such that the form induced on N by λ vanishes identically, i.e.*

$$i_N^* \lambda = 0$$

Then there exists a smooth $(k-1)$-form γ on $\mathcal{U}$ which vanishes on $T_N V$ such that

$$d\gamma = \lambda.$$

If, in addition, λ vanishes on $T_N V$, we may choose γ such that the first-order partial derivatives of its components with respect to the local coordinates, in any chart, vanish on $T_N V$.

We describe below the tools needed to construct the proof.

Lemma A.0.10 *Let $f_t : V \to V$ be a smooth 1-parameter family of mappings, and let X_t be the time-dependent tangent field of f_t. Let λ_t be a smooth 1-parameter family of k-forms on V. Then the folowing formula holds good*

$$\frac{d}{dt} f_t^* \lambda_t = f_t^* \frac{d\lambda_t}{dt} + i_{X_t} d\lambda_t + d(i_{X_t} \lambda_t)$$

For a proof, see [V] for example.

Lemma A.0.11 *(Homotopy Lemma) Let $\mathcal{U}$ and V be smooth manifolds, and let f be a smooth map from an open neighborhood $[0,1] \times \mathcal{U}$ in $\mathbf{R} \times \mathcal{U}$ into V. For every $t \in [0,1]$, let $f_t : \mathcal{U} \to V$ be the map $p \mapsto f_t(p) = f(t,p)$. For every smooth k-form λ on V, let $H\lambda$ be the smooth $(k-1)$-form on $\mathcal{U}$ defined by the formula*

$$H\lambda(p)(v_1, \ldots, v_{k-1}) = \int_0^1 \lambda(f_t(p))(\frac{d}{d\theta} f(\theta, p) \mid_{\theta=t}, \mathbf{d}_p f_t(v_1), \ldots, \mathbf{d}_p f_t(v_{k-1}))dt$$

where $p \in \mathcal{U}$ and $v_1, \ldots, v_{k-1} \in T_pV$.

The map H so defined is called the **homotopy operator associated with** *f, and is a linear map from the exterior algebra $A(V)$ of the manifold V into the exterior algebra $A(\mathcal{U})$ of the manifold $\mathcal{U}$ which has the following properties:*

1. *For every natural number k, H maps the space $A^k(V)$ of smooth k-forms on V into the space $A^{k-1}(\mathcal{U})$ of smooth $(k-1)$-forms on $\mathcal{U}$, under the convention $A^{-1}(\mathcal{U}) = \{0\}$.*
2. *The map H satisfies*

$$f_1^* - f_0^* = H \circ d + d \circ H$$

For a proof, see [LM] for example. Now let us prove the Relative Poincaré Lemma.

Proof. We may identify the tubular neighborhood $\mathcal{U}$ of N in V with a normal bundle $\nu_N \xrightarrow{\pi} N$ of N in V by the tubular neighborhood theorem ([H]). Let f_t denote the map from $\mathcal{U}$ into itself which is multiplication by $(1-t)$ on the fibers. The map f_0 is the identity map on $\mathcal{U}$, and the map f_1 is the composition $i_N \circ \pi$. Let H be the homotopy operator associated with the homotopy f. Then every smooth k-form λ on $\mathcal{U}$ may be written

$$\lambda = \pi^*(i_N^*\lambda) - H(d\lambda) - d(H\lambda).$$

By hypothesis we obtain

$$\lambda = d(-H\lambda)$$

on $\mathcal{U}$. Putting $\gamma = -H\lambda$, and writing

$$(*)\ \ H\lambda = \int_0^1 f_t^*(i_{X_t}\lambda)dt$$

(f is the reduced flow of the time-dependent vector field X_t) we see that γ vanishes on T_NV (since X_t vanishes on N). We also see that if λ vanishes on T_NV, the components of $i_{X_t}\lambda$, in any chart and for any $t \in [0,1]$, are sums of products of two differentiable functions, both of which are zero on N. The first-order partial derivatives of these components with respect to the local coordinates therefore vanish on N, and formula $(*)$ shows that the same is true for the first-order partial derivatives of the components of $\gamma = -H\lambda$. This finishes the proof of the Relative Poincaré Lemma. ■

Appendix B

The Relative Poincaré Lemma for 2-Arrangements

We will now prove a version of the Relative Poincaré Lemma for an arrangement $\mathcal{A} = (M, N)$ in $\mathbf{R}^{2n}$ (see Definition 3.1.2).

WLOG we may assume that the elements M and N are transversal. We may also suppose that $\mathbf{R}^{2n} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$, $M = (\mathbf{x}, \mathbf{y}, \mathbf{0})$, and $N = (\mathbf{0}, \mathbf{y}, \mathbf{z})$. So $K = (\mathbf{0}, \mathbf{y}, \mathbf{0})$ and $\Gamma = M \cup N$.

Theorem B.0.12 (Local Poincaré Lemma) *([MP2]) Suppose that for the arrangement $\mathcal{A}$ above there is a smooth, closed k-form $\lambda \in H^k(\mathbf{R}^{2n})$ which has the following properties:*

- $\lambda = 0$ *on* $T_K\mathbf{R}^{2n}$
- $i_M^*\lambda = 0,\ i_N^*\lambda = 0$

Then there exists a smooth $(k-1)$-form γ so that $d\gamma = \lambda$ everywhere and

$\gamma = 0$ on $T_\Gamma \mathbf{R}^{2n}$. Furthermore, if there is a smooth family of such k-forms $\lambda_t, 0 \leq t \leq 1$, then there is a smooth family of $(k-1)$-forms γ_t such that $d\gamma_t = \lambda_t$, where $\gamma_t = 0$ on $T_\Gamma \mathbf{R}^{2n}$.

Proof. As a first initial guess for γ the Relative Poincaré Lemma gives a $(k-1)$-form α defined on all of $\mathbf{R}^{2n}$ such that $d\alpha = \lambda$ everywhere, $\alpha = 0$ on $T_K\mathbf{R}^{2n}$, and all first-order partial derivatives at K of the components of α vanish with respect to any chart.

If we could write $\gamma = \alpha + d\eta$ where η is some smooth $(k-2)$-form on $\mathbf{R}^{2n}$ satisfying $d\eta = -\alpha$ on Γ, then we would be done. We are going to build η by using the Relative Poincaré Lemma twice, one application for each plane M and N, and then piece our results together.

Let S^l_m denote the set of all subsets of $\{1, 2, 3, \ldots, l\}$ with length m. If $I = \{I_1, I_2, I_3, \ldots, I_m\} \in S^l_m$, then $d\mathbf{x}_I$ will denote $dx_{I_1} \wedge dx_{I_2} \wedge dx_{I_3} \wedge \cdots \wedge dx_{I_m}$ (where we are assuming that $m \leq l$ and $I_1 < I_2 < I_3 < \cdots < I_m$). Write

$$\alpha = \sum a_{IK}\, d\mathbf{x}_I \wedge d\mathbf{y}_K + \sum b_{JK}\, d\mathbf{z}_J \wedge d\mathbf{y}_K + \sum c_{IJK}\, d\mathbf{x}_I \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K$$

where a_{IJ}, b_{JK} and c_{IJK} denote smooth real-valued functions on $\mathbf{R}^{2n}$.

Let's focus our attention first on the element M. Note that $0 = i^*_M(-\lambda) = i^*_M(-d\alpha) = d\,(-i^*_M\alpha)$ and also that $(-i^*_M\alpha) = 0$ on $T_K\mathbf{R}^{2n}$. Working entirely inside M the Relative Poincaré Lemma may be applied to the smooth $(k-1)$-form $-i^*_M\alpha$ to give us a smooth $(k-2)$-form η_1 such that

- $\eta_1 = 0$ on $T_K M$

- $d\eta_1 = -i_M^*\alpha = -\sum a_{IK}(\mathbf{x},\mathbf{y},\mathbf{0})\,d\mathbf{x}_I \wedge d\mathbf{y}_K$
- $\dfrac{\partial\eta_1}{\partial\mathbf{x}} = \dfrac{\partial\eta_1}{\partial\mathbf{y}} = 0$ on $T_K M$

(where, for example, $\frac{\partial}{\partial\mathbf{x}}$ denotes all first-order partial derivatives with respect to the x-coordinates).

There is, of course, a similar construction for the element N.

Define $S^k(d\mathbf{x}_I)$ (or just $S(d\mathbf{x}_I)$ for short) to be the primitive $\sum_u (-1)^u x_u dx_{I_1} \wedge \cdots \wedge \widehat{dx_{I_u}} \wedge \cdots \wedge dx_{I_k}$, where the hat denotes exclusion.

Put

$$\eta(\mathbf{x},\mathbf{y},\mathbf{z}) = \eta_1(\mathbf{x},\mathbf{y},\mathbf{0}) + \eta_2(\mathbf{0},\mathbf{y},\mathbf{z}) + \sum_{JK} b_{JK}(\mathbf{x},\mathbf{y},\mathbf{0})\,S(d\mathbf{z}_J) \wedge d\mathbf{y}_K$$

$$+ \sum_{IJ} a_{IK}(\mathbf{0},\mathbf{y},\mathbf{z})\,S(d\mathbf{x}_I) \wedge d\mathbf{y}_K + \sum_{IJK} (-1)^i c_{IJK}(\mathbf{x},\mathbf{y},\mathbf{0})\,d\mathbf{x}_I \wedge S(d\mathbf{z}_J) \wedge d\mathbf{y}_K$$

$$+ \sum_{IJK} c_{IJK}(\mathbf{0},\mathbf{y},\mathbf{z})\,S(d\mathbf{x}_I) \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K$$

So

$$d\eta(\mathbf{x},\mathbf{y},\mathbf{z}) = -\sum a_{IK}(\mathbf{x},\mathbf{y},\mathbf{0})\,d\mathbf{x}_I \wedge d\mathbf{y}_K - \sum b_{JK}(\mathbf{0},\mathbf{y},\mathbf{z})\,d\mathbf{z}_J \wedge d\mathbf{y}_K$$

$$- \sum a_{IK}(\mathbf{0},\mathbf{y},\mathbf{z})\,d\mathbf{x}_I \wedge d\mathbf{y}_K - \sum b_{JK}(\mathbf{x},\mathbf{y},\mathbf{0})\,d\mathbf{z}_J \wedge d\mathbf{y}_K$$

$$+ \sum \frac{\partial b_{JK}(\mathbf{x},\mathbf{y},\mathbf{0})}{\partial\mathbf{x}}\,d\mathbf{x} \wedge S(d\mathbf{z}_J) \wedge d\mathbf{y}_K + \sum \frac{\partial b_{JK}(\mathbf{x},\mathbf{y},\mathbf{0})}{\partial\mathbf{y}}\,d\mathbf{y} \wedge S(d\mathbf{z}_J) \wedge d\mathbf{y}_K$$

$$+ \sum \frac{\partial a_{IK}(\mathbf{0},\mathbf{y},\mathbf{z})}{\partial\mathbf{z}}\,d\mathbf{z} \wedge S(d\mathbf{x}_I) \wedge d\mathbf{y}_K + \sum \frac{\partial a_{IK}(\mathbf{0},\mathbf{y},\mathbf{z})}{\partial\mathbf{y}}\,d\mathbf{y} \wedge S(d\mathbf{x}_I) \wedge d\mathbf{y}_K$$

$$+\sum(-1)^i \frac{\partial c_{IJK}(\mathbf{x},\mathbf{y},\mathbf{0})}{\partial \mathbf{x}}\, d\mathbf{x} \wedge d\mathbf{x}_I \wedge S(d\mathbf{z}_J) \wedge d\mathbf{y}_K$$

$$+\sum(-1)^i \frac{\partial c_{IJK}(\mathbf{x},\mathbf{y},\mathbf{0})}{\partial \mathbf{y}}\, d\mathbf{y} \wedge d\mathbf{x}_I \wedge S(d\mathbf{z}_J) \wedge d\mathbf{y}_K$$

$$+\sum \frac{\partial c_{IJK}(\mathbf{0},\mathbf{y},\mathbf{z})}{\partial \mathbf{z}}\, d\mathbf{z} \wedge S(d\mathbf{x}_I) \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K$$

$$+\sum \frac{\partial c_{IJK}(\mathbf{0},\mathbf{y},\mathbf{z})}{\partial \mathbf{y}}\, d\mathbf{y} \wedge S(d\mathbf{x}_I) \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K - \sum c_{IJK}(\mathbf{x},\mathbf{y},\mathbf{0})\, d\mathbf{x}_I \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K$$

$$-\sum c_{IJK}(\mathbf{0},\mathbf{y},\mathbf{z})\, d\mathbf{x}_I \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K$$

So

$$d\eta(\mathbf{x},\mathbf{y},\mathbf{0}) = -\sum a_{IJ}(\mathbf{x},\mathbf{y},\mathbf{0})\, d\mathbf{x}_I \wedge d\mathbf{y}_K$$

$$-\sum b_{JK}(\mathbf{x},\mathbf{y},\mathbf{0})\, d\mathbf{z}_J \wedge d\mathbf{y}_K - \sum c_{IJK}(\mathbf{x},\mathbf{y},\mathbf{0})\, d\mathbf{x}_I \wedge d\mathbf{z}_J \wedge d\mathbf{y}_K = -\alpha(\mathbf{x},\mathbf{y},\mathbf{0})$$

since

$$\frac{\partial a_{IK}(\mathbf{0},\mathbf{y},\mathbf{0})}{\partial \mathbf{y}} = a_{IK}(\mathbf{0},\mathbf{y},\mathbf{0}) = b_{JK}(\mathbf{0},\mathbf{y},\mathbf{0}) = c_{IJK}(\mathbf{0},\mathbf{y},\mathbf{0}) = 0$$

and $S(d\mathbf{z}_J) = 0$ on M.

Similarly,

$$d\eta(\mathbf{0},\mathbf{y},\mathbf{z}) = -\alpha(\mathbf{0},\mathbf{y},\mathbf{z})$$

We have now constructed η, and putting $\gamma = \alpha + d\eta$ finishes the proof, noting that the existence of the path γ_t follows from a similar result for the Relative Poincaré Lemma. ∎

Appendix C

Symplectic Vector Spaces

Definition C.0.13 A **symplectic vector space** (V, Ω) is a vector space V with a non-degenerate, skew-symmetric bilinear form Ω called the **symplectic form.** If $W \subset V$ is a subspace of V then

$$W^s \stackrel{d}{=} \{v \in V \mid \forall w \in W,\ \Omega(v, w) = 0\}$$

is the **skew-orthogonal** (or **symplectically orthogonal**) **subspace of W**. The subspace $W^i \subset W$ defined by

$$W^i \stackrel{d}{=} W \cap W^s$$

is the **radical of W**. Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are **symplectomorphic** iff there is a linear isomorphism $\Phi : V_1 \to V_2$ such that $\Omega_1(v, w) = \Omega_2(\Phi(v), \Phi(w))\ \forall v, w \in V_1$.

We write $V = W_1 \oplus_s W_2$ if $V = W_1 \oplus W_2$ and $W_1^s = W_2$: in this case we say that V is the skew-orthogonal direct sum of W_1 and W_2. Note that Ω is uniquely determined by W_1, W_2, and the forms it induces on them.

Below we give a (non-unique) direct sum decomposion of V into skew-orthogonal terms given a subspace A of V.

Proposition C.0.14 *Let A and B be subspaces of (V, Ω). Then*

1. $A \subset B \Rightarrow B^s \subset A^s$.
2. $A^{ss} = A$.
3. $(A + B)^s = A^s \cap B^s$.
4. $(A \cap B)^s = A^s + B^s$.

Proof. Straightforward (+ denotes union of subspaces). ∎

Definition C.0.15 A^u, the **union subspace of** A, is given as

$$A^u = A + A^s.$$

The **reduced** symplectic space A^r is given as

$$A^r = A/A^i.$$

We may choose a subspace $B \subset A$ symplectomorphic to A^r and write $A = A^i \oplus B$. Similarly we may choose a subspace $C \subset A^s$ symplectomorphic to the symplectic space $A^{sr} = A^s/A^i$ and write $A^s = A^i \oplus C$. Of course, the choice of B and C is not unique. Finally,

Theorem C.0.16 *Given a subspace $A \subset V$, there is a (non-unique) isotropic subspace $Q \subset V$ (isotropic means that $Q \subset Q^s$), supplementary to A^u, such that $A^\sigma \stackrel{d}{=} A^i \oplus Q$ is a symplectic subspace of V and*

$$V = A^\sigma \oplus_s B \oplus_s C.$$

Proof. For a proof, see [V] for example. ∎

Definition C.0.17 *A* **symplectic vector bundle** *is a smooth vector bundle equipped with a smooth field ω of symplectic forms on its fibers.*

If A is a smooth sub-bundle of a symplectic vector bundle E of constant fiberwise symplectic rank, than the above decomposition can be applied fiberwise to give

$$E \equiv A^{\sigma} \oplus_s B(A) \oplus_s C(A)$$

where A^{σ}, $B(A)$, $C(A)$ are all smooth, symplectic sub-bundles of E (see [V] for details).

Finally, we have the linear version of Darboux's Theorem.

Theorem C.0.18 *(***Linear Darboux Theorem***) Let V be an n-dimensional vector space over a subfield of the complex numbers, and let Ω be a skew-symmetric bilinear form on V. Then the rank r of Ω is even, and if $r = 2k$ there is an ordered basis for V in which the matrix of Ω is the direct sum of the $(n-r) \times (n-r)$ zero matrix and k copies of the 2×2 matrix*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Bibliography

[A] V. I. Arnold, "Symplectic Geometry and Topology," Preprint.

[AG] V. I. Arnold and A. B. Givental, "Symplectic Geometry," *Encyclopedia of Mathematical Sciences, Dynamical Systems*, **4** (1990), pp. 4-136.

[B] Banyaga, "Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique," *Comment. Math. Helvetici*, **53** (1978), pp. 174-227.

[BT] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics **82**, Springer Verlag, New York, 1986.

[E] L. H. Eliasson, "Normal Forms for Hamiltonian Systems with Poisson Commuting Integrals–Elliptic Case," *Comment. Math. Helvetici*, **65** (1990), pp. 4-35.

[FR] Michael Falk and Richard Randell, "On the Homotopy Theory of Arrangements," *Advanced Studies in Pure Mathematics*, **8** (1986), pp. 101-124.

[Go] Robert E. Gompf, "A New Construction of Symplectic Manifolds," To be published.

[GP] M. Goresky and R. MacPherson. *Stratified Morse Theory.* Springer-Verlag, Berlin, Heidelberg, 1988.

[GS1] V. Guillemin and S. Sternberg. *Geometric Asymptotics.* American Mathematical Society, Providence, 1977.

[GZ] Israel M. Gelfand and Ilya Zakharevich, "On the Local Geometry of a Bi-hamiltonian Structure," *The Gelfand Mathematical Seninars*, Birkhäuser, Boston, 1993.

[H] Morris W. Hirsch. *Differential Topology.* Graduate Texts in Mathematics **33**, Springer Verlag, New York, 1976.

[HK] Kenneth Hoffman and Ray Kunze. *Linear Algebra.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971

[LM] Paulette Libermann and Ch. M. Marle. *Symplectic Geometry and Analytical Mechanics.* D. Reidel Publishing Co., Dordrecht, 1987.

[M] Ch. M. Marle, "Sous-variétés de rang constant d'une variété symplectique," *Astérisque*, **107-108** (1983), pp. 69 - 86.

[Ma] J. Martinet, "Sur les Singularités des Formes Différentielles," *Ann. Inst. Fourier*, **20**, No. 1 (1970), pp. 95-178.

[Me] R. B. Melrose, "Equivalence of Glancing Hypersurfaces," *Inventiones Mathematicae*, **37** (1976), pp. 165 - 191.