Homology of
Generalized Piecewise Differentiable Currents
on a Combinatorial Manifold

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Abstract of the Dissertation

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In two papers which appeared in the 1991 Journal of Functional Analysis, N. Teleman introduced the complex of generalized piecewise differentiable currents on a smooth manifold. The homology of this complex contains an isomorphic image of the deRham homology, and the Chern-Weil construction applied to any piecewise differentiable metric produces currents which represent the characteristic classes. In this dissertation, the definition of generalized piecewise differentiable currents is extended to combinatorial manifolds and the homology of the resulting complex is computed. The homology of this complex is shown to be isomorphic to the relative homology of a pair of subcomplexes lying in some power of the manifold. The simplicial homology

of the manifold does not inject into the homology of the pair. A modification of the definition produces a new complex whose homology agrees with the homology of the manifold. A multiplication on this complex induces a multiplication on homology which is related to the intersection pairing on simplicial homology.

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INTRODUCTION

In 1989, Nicolae Teleman introduced the concept of generealized piecewise differentiable currents on a triangulated smooth manifold M (compact, orientable, with $\partial M = \emptyset$). They are equivalence classes of piecewise differentiable currents on a k-fold product M_k of M modulo currents which "vanish on the diagonal". The exterior derivative of currents makes the set of generalized piecewise differentiable currents into a complex, denoted $\mathcal{C}^*(M_k)$. In two papers in the 1991 Journal of Functional Analysis (Teleman [T_2], [T_3]), he develops the properties of the complex $\mathcal{C}^*(M_k)$. In [T_2], he computes the homology of the complex $\mathcal{C}^*(M_k)$ in terms of a relative homology group of the pair (T, N) where T is a triangulation of M_k, and N is the subcomplex of T consisting of simplices which are not transverse to the diagonal ∇_M in M_k.

When all simplices of the triangulation which intersect the diagonal are transverse to the diagonal, the homology of the complex $\mathcal{C}^*(M_k)$ agrees with the real simplicial homology of M. This implies that the homological information of M is contained in the complex $\mathcal{C}^*(M_k)$.

By taking a cell structure on M_k consisting of polysimplices (the natural structure arising from a product of the triangulations of the factors of M_k), an operation on the tensor product $\mathcal{C}^*(M_k) \bigotimes \mathcal{C}^*(M_l) \rightarrow \mathcal{C}^*(M_{k+l})$ is shown to induce an operation on the corresponding exterior algebra making it into a differential algebra. From this algebra, a Chern-Weil construction can be made to obtain the characteristic classes from a (discontinuous) piecewise differentiable metric on M.

In this dissertation the corresponding object on combinatorial manifolds will be studied.

Combinatorial manifolds are a natural setting for piecewise differentiable currents. There are natural coordinates (barycentric coordinates), a natural measure on the simplices (from the differentials of the barycentric coordinate functions), and a triangulation. This is exactly the information needed to define piecewise differentiable currents.

Chapter 1 makes precise the the preliminary notions summarized below.

A piecewise differentiable distribution is a distribution T which can be represented as a finite sum of terms of the form

$$T_\sigma(\phi) = \int_\sigma \sum_\alpha \omega_\alpha(x) D_x^\alpha \phi d\sigma$$

where σ is a simplex in M, $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, ω_α is a C^∞-function on σ, and $d\sigma$ is the measure on σ. σ is called the carrier of the distribution T_σ.

The Sullivan complex of piecewise differentiable forms, denoted $\Omega^*(M)$ and defined in Chapter 1, replaces the deRham complex, and a current is defined as a continuous linear functional $U : \Omega^*(M) \to R$.

A simplicial chain c defines a current by

$$[c](\phi) = \int_c \phi$$

for $\phi \in \Omega^*(M)$.

Differential forms also define currents by

$$[\omega](\phi) = \int\limits_M \omega \wedge \phi.$$

A piecewise differentiable current is a piecewise differentiable form with piecewise differentiable distributions as coefficients.

The k-fold product $M_k = M \times \cdots \times M$ inherits the cell structure given by polysimplices $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ from the combinatorial structure on M. The concept of a piecewise differentiable current with carrier Σ generalizes, in a straightforward manner, to M_k.

In Chapter 2 the complex of piecewise differentiable currents $C^*(M_k)$ is defined, an explicit formula for the deRham boundary $b : C^*(M_k) \to C^*(M_k)$ (defined by $bU(\phi) = U(d\phi)$ for $\phi \in \Omega^*(M_k)$) is obtained, and a unique representation for the elements of $C^*(M_k)$ is given. An element U of $C^*(M_k)$ decomposes uniquely into a sum $U = \sum_{\Sigma \in M_k} \sum_{P \supset \Sigma} U_{P,\Sigma}$ for each pair of polysimplices (P, Σ) with P containing Σ and where $U_{P,\Sigma}$ is a piecewise differentiable current on P with carrier Σ. Let $C^*(\Sigma)$ denote the subcomplex of $C^*(M_k)$ consisting of currents with carrier Σ or a face of Σ, and let $C^*(\Sigma, \partial\Sigma) = C^*(\Sigma)/C^*(\partial\Sigma)$ denote the quotient complex. The subcomplex containing the terms corresponding to P will be denoted $C^*(P : \Sigma, \partial\Sigma)$, and $C^*(P; \Sigma, \partial\Sigma)$ will denote the sum of the complexes $C^*(P' : \Sigma, \partial\Sigma)$ for polysimplices P' contained in P. $\nabla^*(P; \Sigma, \partial\Sigma)$ will denote $C^*(P; \Sigma, \partial\Sigma) \cap \nabla^*(M_k)$.

The proof of uniqueness of the decomposition of the complex $C^*(Mk)$

requires the construction of functions in $\Omega^*(M_k)$ whose transverse derivatives to Σ are zero along a given compact subset of Σ.

Chapter 3 introduces the definition of a generalized piecewise differentiable current. For any positive integer k, let $i : M \to M_k$ be the diagonal embedding, where M_k is the k-fold product of M. The diagonal in M_k will be denoted by ∇_M. Let $I^*(M_k)$ be the ideal in $\Omega^*(M_k)$ consisting of forms ω which satisfy $i^*\omega = 0$ and let

$$\nabla^r(M_k) = \bigoplus_{p+q=r} I^p(M_k) \cdot C^q(M_k).$$

The quotient complex

$$\mathcal{C}^*(M_k) = C^*(M_k)/\nabla^*(M_k)$$

is called the complex of generalized piecewise differentiable currents on M (in M_k).

The study of this complex begins with finding an explicit representation of the ideal $I^*(M_k)$ in terms of functions and differentials which generate it.

The next step is to introduce the notion of combinatorial transversality. A pair of polysimplices (P, Σ) are transverse to the diagonal, written $(P, \Sigma) \amalg \nabla_M$, if every linear function on P which is zero on Σ and zero on $P \cap \nabla_M$ is zero on all of P. A polysimplex Σ is transverse to the diagonal if (P, Σ) is transverse to the diagonal for every polysimplex P containing Σ. Proposition 3.3.8 gives a characterization of those polysimplices $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ which are transverse to the diagonal in terms of the simplices σ_j, for $j = 1, \cdots, k$.

It is important to notice that the usual notion of transversality for combinatorial manifolds is not useful in this case.

Let $\nabla^*(\Sigma) = C^*(\Sigma) \cap \nabla^*(M_k)$ and $\mathcal{C}^*(\Sigma)$ denote the quotient complex of $\mathcal{C}^*(\Sigma) = C^*(\Sigma)/\nabla^*(\Sigma$. $\mathcal{C}^*(\Sigma, \partial\Sigma)$ will denote the quotient complex $\mathcal{C}^*(\Sigma)/\mathcal{C}^*(\partial\Sigma)$. Chapter 3 establishes $\mathcal{C}^*(\Sigma, \partial\Sigma) = 0$ if $\Sigma \cap \nabla_M = \emptyset$ or if Σ is not transverse to the diagonal.

To determine the complex $\mathcal{C}^*(\Sigma, \partial\Sigma)$ when Σ is transverse to the diagonal is more involved. Chapter 3 concludes with a complete characterization of the elements of the complex $\nabla^*(P; \Sigma, \partial\Sigma) = C^*(P; \Sigma, \partial\Sigma) \cap \nabla^*(M_k)$. With this result, the case $\Sigma \amalg \nabla_M$ is solved. This is used to determine $\mathcal{C}^*(\Sigma, \partial\Sigma)$ in the remaining cases.

In Chapter 4 the homology of the complex $\mathcal{C}^*(M_k)$ is computed. The first six sections are devoted to the construction of three chain homotopies on the complexes $C^*(P; \Sigma, \partial\Sigma)$ and $\nabla^*(P; \Sigma, \partial\Sigma)$. The first chain homotopy is in the directions along Σ except for the variables which correspond to $\Sigma \cap \nabla_M$. The second chain homotopy acts on the variables which correspond to $\Sigma \cap \nabla_M$. These two homotopies are variations of the homotopy used in the Poincaré lemma.

The third chain homotopy is the transpose of the chain homotopy acting in the directions transverse to Σ.

From the explicit formulas, the effect of the above chain homotopies on the complexes $C^*(P, \Sigma, \partial\Sigma)$ and $\nabla^*(P, \Sigma, \partial\Sigma)$ can be followed. They show

$$H_*(\mathcal{C}^*(\Sigma, \partial\Sigma)) = 0 \text{ , if } \Sigma \not\amalg \nabla_M,$$

$$H_p(\mathcal{C}^*(\Sigma, \partial\Sigma)) = 0 \text{ , if } \Sigma \amalg \nabla_M \text{ and } p \neq 0,$$

$$H_0(\mathcal{C}^*(\Sigma, \partial\Sigma)) = R \cdot [\Sigma] \text{ , if } \Sigma \amalg \nabla_M.$$

The homology of these complexes is related to the homology of $\mathcal{C}^*(M_k)$ by the spectral sequence corresponding to the filtration

$$F^{p,q} = \mathcal{C}^*(K^{p+q}(M_k))$$

where $K^r(M_k)$ is the r-skeleton of M_k, p is the degree of the current, and q is the filtration index. The homology of the complexes $\mathcal{C}^*(\Sigma, \partial\Sigma)$ are exactly the E_1 terms of the spectral sequence determined by the filtration $F^{p,q}$. From this there follows

$$E_1^{p,0} = \bigoplus_{\substack{\text{codim}(\Sigma) = p \\ \Sigma \amalg \nabla_M}} R \cdot [\Sigma],$$

$$E_1^{p,q} = 0 \text{ , if } q \neq 0.$$

As the operator d_1 is shown to correspond to the connecting homomorphism of the pair $(\Sigma, \partial\Sigma)$, it follows that the homology of the complex $\mathcal{C}^*(M_k)$ corresponds (after a flip of indices) to the homology of the pair (M_k, N) where N is the subcomplex of polysimplices of M_k which are not transverse to the diagonal. More precisely, it follows that

$$H_p(\mathcal{C}^*(M_k)) \cong H_{km-p}(M_k, N) \text{ for } 0 \leq p \leq \text{ dim} M,$$

$$H_p(\mathcal{C}^*(M_k)) = 0, \text{ otherwise.}$$

The final section of this chapter gives an example where the the homology of the complex $\mathcal{C}^*(M_k)$ does not agree with the simplicial homology of M. The barycentric subdivision of a compact orientable 2-manifold without boundary produces a combinatorial manifold M with $\mathcal{C}^p(M) = 0$ unless

$p = 2\dim(M)$. This shows the homology of the complex does not agree (after a flip of indices) to the homology of M.

In Chapter 5 the complex $\mathcal{N}^*(M_k)$ is introduced. It is defined as the quotient complex $C^*(M_k)/C^*(N_0(M_k))$ where $N_0(M_k)$ is the subcomplex consisting of those currents whose carriers do not intersect the diagonal. Using the techniques developed in Chapters 3 and 4, it is shown that the homology of the complex $\mathcal{N}^*(M_k)$ agrees with $H_*(M)$, the real simplicial homology of M, (after a flip of indices). There is a multiplication

$$\mu : \mathcal{N}^*(M_k) \bigotimes \mathcal{N}^*(M_l) \to \mathcal{N}^*(M_{k+l})$$

which induces a multiplication on homology. The induced multiplication on homology is related to the intersection pairing on $H_*(M)$ under the identification of $H_*(\mathcal{N}^*(M))$ with $H_*(M)$.

This chapter concludes with an example of a current $U_{\mathcal{E}}$ on $\mathcal{N}^*(M)$ with the property $U_{\mathcal{E}}(\mu) = \chi(M)$ where μ is the canonical volume form on M, and $\chi(M)$ is the Euler characteristic of M.

Chapter 1

PRELIMINARIES

This chapter will give the basic definitions, establish notation, and collect the standard results about the basic objects that will be used. It consists entirely of known material and its straightforward generalization to combinatorial manifolds. The main goal of this chapter is express a piecewise differentiable current in convenient form. In the next chapter, this form will be used to obtain a unique representation. General references for this chapter are deRham [dR], and Teleman [T_1] and [T_2].

Throughout this work M will denote a compact oriented combinatorial manifold of dimension m (a simplicial complex which is locally homeomorphic to R^m) without boundary. For $p = 0, 1, 2, \cdots, m$, $K^p(M)$ will denote the p-skeleton of M, and $\mid M \mid$ will denote the underlying polyhedron of M. The term simplex will mean a closed simplex, and the closed p-simplex σ will also be denoted by $(v^0, \cdots, v^p)$ where $v^a \in K^0(\sigma)$ for $a = 0, 1, 2, \cdots, p$.

Each simplex σ in M has a natural differentiable structure arising from an affine embedding of σ into R^m. The differentiable structure is independent of the choice of the embedding.

1.1 The Sullivan Complex of Piecewise Differentiable Forms

Let p be a nonnegative integer.

Definition 1.1.1: A **piecewise differentiable** p**-form** on M is a function ω which associates to any maximal simplex $\sigma \in M$ a real-valued C^∞ p-form ω_σ on σ such that the collection $\{\omega_\sigma\}$ satisfies the following compatibility condition:

If ρ and σ are two maximal polysimplices of M with $\rho \cap \sigma \neq \emptyset$, and $i_\sigma : \rho \cap \sigma \to \rho$ and $i_\rho : \rho \cap \sigma \to \sigma$ are the inclusions of $\rho \cap \sigma$ into ρ and σ respectively, then

$$(i_\rho)^* \omega_\rho = (i_\sigma)^* \omega_\sigma.$$

Definition 1.1.2: $\Omega^p(M) = \{$ piecewise differentiable p-forms ω on $M\}$.

Definition 1.1.3: If $\omega \in \Omega^r(M)$, $\theta \in \Omega^s(M)$, then $\omega \wedge \theta$ is the form of degree $r + s$ defined by $(\omega \wedge \theta)_\sigma = \omega_\sigma \wedge \theta_\sigma$.

Definition 1.1.4: If $\omega \in \Omega^p(M)$, we define the **differential** of ω to be the form $d\omega$ which assigns to a maximal simplex $\sigma \in M$ the $(p+1)$-form $d(\omega_\sigma)$. Symbolically, $(d\omega)_\sigma = d(\omega_\sigma)$. Note that $d\omega \in \Omega^{p+1}(M)$ as the compatibility condition $d(i_\sigma^* \omega_\sigma) = i_\sigma^* d\omega_\sigma$ is satisfied.

Remark 1.1.5: For $\omega \in \Omega^r(M)$, $\theta \in \Omega^s(M)$,

(a) $\omega \wedge \theta \in \Omega^{r+s}(M)$,

(b) $d\omega \in \Omega^{r+1}(M)$,

(c) $d^2\omega = 0$.

These three properties imply $\Omega^*(M) = \{\Omega^r(M), d\}_{r \in N}$ is a graded differential algebra.

Definition 1.1.6: The graded differential algebra

$$\Omega^*(M) = \{\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0 \rightarrow \cdots\}$$

is called the **Sullivan complex of piecewise differentiable forms** on M.

Let σ be a p-simplex in M with $(y^1, \cdots, y^p)$ a smooth coordinate system on σ. The symbol D_y^α will represent the differential operator

$$D_y^\alpha = (\frac{\partial}{\partial y^1})^{\alpha_1} \cdots (\frac{\partial}{\partial y^p})^{\alpha_p}$$

where $\alpha = (\alpha_1, \cdots, \alpha_p)$ is a multi-index.

$\Omega^*(M)$ is given the Fréchet space topology associated to the family of semi-norms $\| \omega \|_{\alpha,\beta}$, defined on $\omega = \{\omega_\sigma\} \in \Omega^*(M)$ by

$$\omega_\sigma = \sum \omega_{\sigma,\beta}(y) dy^\beta$$

on maximal simplices σ of M, by

$$\| \omega \|_{\alpha,\beta} = \sup_{y,\sigma} \mid D_y^\alpha \omega_{\sigma,\beta}(y) \mid,$$

the supremum is taken with respect to all maximal simplices $\sigma \in M$, and $y \in \mid \sigma \mid$ where

$$\omega_\sigma = \sum \omega_{\sigma,\beta}(y) dy^\beta$$

on maximal simplices σ of M.

1.2 Currents

Definition 1.2.1: A continuous linear functional

$$T : \Omega^{m-p}(M) \to R$$

will be called a **current** of degree p on M.

Remark 1.2.2: A linear functional $T : \Omega^{m-p}(M) \to R$ is continuous in the Fréchet space topology if, and only if, there exist a natural number N (which depends on T), and a positive constant C, such that for all $\omega \in \Omega^{m-p}(M)$,

$$| T\omega | \leq C \ \sup_{\sigma,y} | D_y^{\alpha}\omega_{\sigma,\beta}(y) |,$$

where the supremum is with respect to:

(a) any partial derivative D_y^{α}, $| \alpha | \leq N$,

(b) any maximal simplex σ, and any $y \in | \sigma |$,

(c) any $\omega_{\sigma,\beta}$ where $\omega_{\sigma} = \sum_{\beta} \omega_{\sigma,\beta}(y) dy^{\beta}$.

This characterization of currents is from [T_2], as are the following remarks.

Remark 1.2.3: As in the smooth case, any form $\theta \in \Omega^p(M)$ or any chain γ of dimension $m - p$ defines a current of degree p by

$$\theta(\omega) = \int_M \omega \wedge \theta \ , \ \omega \in \Omega^{m-p}(M),$$

$$[\gamma](\omega) = \int_{\gamma} \omega \ , \ \omega \in \Omega^{m-p}(M).$$

Let T be a current of degree p. If $f \in \Omega^0(M)$, then fT is a current of degree p defined by $(fT)(\omega) = T(f\omega)$. This makes the set of currents of degree p into a module over $\Omega^0(M)$. More generally, if $\theta \in \Omega^q(M)$, then the exterior product $T \wedge \theta$ is a current of degree $p+q$ defined as

$$(T \wedge \theta)(\omega) = T(\theta \wedge \omega) \ , \ \omega \in \Omega^{m-p-q}(M).$$

This makes the set of currents (of all degrees) into a module over $\Omega^*(M)$.

Definition 1.2.4: If T is a current of degree p, then the **deRham boundary** of the current T is the current bT of degree $p+1$ defined by

$$(bT)(\omega) = T(d\omega) \ , \ \omega \in \Omega^{m-p-1}(M).$$

Remark 1.2.5: As $b^2 = 0$, and for currents T of degree p, $\omega \in \Omega^q(M)$,

$$b(T \wedge \omega) = (-1)^q bT \wedge \omega + T \wedge b\omega,$$

it follows that the operator b makes the vector space of currents into a differential graded module over $\Omega^*(M)$.

Definition 1.2.6: If T is a current of degree p, the **exterior derivative** of the current T is the current dT of degree $p+1$ defined by

$$(dT)(\omega) = (-1)^{p+1} T(d\omega) \ , \ \omega \in \Omega^{m-p+1}(M).$$

Remark 1.2.7 : As $d^2 = 0$, and as

$$d(T \wedge \theta) = dT \wedge \theta + (-1)^p T \wedge d\theta$$

for currents T of degree p, the operator d also makes the vector space of currents into a differential graded module over $\Omega^*(M)$.

Remark 1.2.8: As M is compact and orientable, for any $\omega \in \Omega^{m-1}(M)$, it follows by Stokes' Theorem (pairwise cancellation of the forms on the boundaries of the simplices) that $\int_M d\omega = 0$.

1.3 Piecewise Differentiable Distributions

Let (ρ, σ) be a pair of simplices in M with $\sigma \subset \rho$, and let $r = \dim(\rho)$.

Definition 1.3.1: A **piecewise differentiable distribution on ρ with carrier** σ is a linear functional $T : \Omega^0(M) \to R$ which has a representation of the form

$$T(\phi) = \int_\sigma \sum_\alpha \eta_\alpha(y) \cdot D_y^\alpha \phi(y) d\sigma \text{ , for } \phi \in \Omega^0(M)$$

where

1. the sum is over a finite collection of multi-indices α,
2. $y = (y^1, \cdots, y^r)$ are smooth local coordinates for ρ, $(r = \dim(\rho))$,
3. $d\sigma = dy^1 \wedge \cdots \wedge dy^r$,
4. σ is given the orientation such that $\int_\sigma d\sigma > 0$, and
5. $\eta_\alpha \in C^\infty(\sigma)$.

Remark 1.3.2: By the chain rule and the convention that the simplices are closed, it follows that a distribution which has a representation in the above form in some coordinate system has such a representation in any other smooth coordinate system.

Definition 1.3.3: A **piecewise differentiable distribution on M with carrier** σ is a linear functional $T : \Omega^0(M) \to R$ which has a representation of the form

$$T(\phi) = \sum_{\rho \supset \sigma} T_\rho \ , \text{ for } \phi \in \Omega^0(M)$$

where T_ρ represents a piecewise differentiable distribution on ρ with carrier σ, and where the sum is over all simplices ρ containing σ.

Definition 1.3.4: A **piecewise differentiable distribution on** M is a linear functional $T : \Omega^0(M) \to R$ which can be expressed as a sum of piecewise differentiable distributions which have carriers consisting of simplices of M.

Remark 1.3.5: If $\Omega^0(M)$ is topologized as the Fréchet space of currents of degree zero, then the compactness of M and the continuity of η_α imply that piecewise differentiable distributions are continuous linear functionals on the Fréchet space $\Omega^0(M)$.

1.4 Piecewise Differentiable Currents

A piecewise differentiable current on M is a piecewise differential form having piecewise differentiable distributions as coefficients. More precisely, one has the following definition.

Definition 1.4.1: A **piecewise differentiable current** of degree p on M is a continuous linear functional

$$U : \Omega^p(M) \to R$$

which has a representation as a double sum

$$U = \sum_{\sigma \subset M} \sum_{\rho \supset \sigma} U_{\rho,\sigma}$$

where the first sum is over all simplices σ of M, the second sum is over all maximal simplices ρ which contain σ, and where $U_{\rho,\sigma}$ are currents which have a representation as

$$U_{\rho,\sigma}(\phi) = \sum_{\alpha} T_\alpha(\phi_\alpha(y))$$

on a form ϕ such that $j_\rho^* \phi$ has a representation as

$$j_\rho^* \phi = \sum_{\alpha} \phi_\alpha(y) dy^\alpha$$

where $y = (y^1, \cdots, y^m)$ are local coordinates for ρ, $j_\rho : \rho \to M$ is the inclusion, and T_α is a piecewise differentiable distribution on ρ with carrier σ.

Using local coordinates (as in the definition of a piecewise differential distribution), $U_{\rho,\sigma}$ can be expressed as

$$U_{\rho,\sigma}(\phi) = \int_{\sigma} \sum_{\alpha,\beta} \omega_{\alpha,\beta}(y) \cdot D_y^\beta(\phi_\alpha(y)) d\sigma \text{ , for } \phi \in \Omega^p(M)$$

where $j_\rho^*\phi = \sum_\alpha \phi_\alpha(y)dy^\alpha$ as above.

Remark 1.4.2: Any linear functional of the form

$$U = \sum_{\sigma \subset M} \sum_{\rho \supset \sigma} U_{\rho,\sigma}$$

is continuous in the Fréchet space topology as the constant C can be chosen to be the sum (over the finite sets σ, ρ, α, and β) of the maximum of the continuous functions $\mid \omega_{\alpha,\beta}(y) \mid$ on the compact set σ.

Definition 1.4.3: $C^p(M)$ will denote the vector space of piecewise differentiable currents on M of degree p under the operation $+$ given by

$$(U + U')(\omega) = U(\omega) + U'(\omega),$$

and scalar multiplication by real numbers.

Definition 1.4.4: For each vertex $v^a \in M$, let b^a denote the barycentric coordinate function which has value 1 at v^a, and is zero on the other vertices of $\operatorname{star}(v^a)$. Extend b^a to M, by defining b^a to be zero on the complement of $\operatorname{star}(v^a)$. These functions are in the Sullivan complex of piecewise differentiable functions on M.

Definition 1.4.5: Associated to a p-simplex σ, there are the sets of barycentric coordinate functions

$$B(\sigma) = \{b^a \mid \operatorname{star}(v^a) \supset \sigma\},$$

$$T(\sigma) = \{b^a \in B(\sigma) \mid b^a \equiv 0 \text{ on } \sigma\} \text{ , and}$$

$$L(\sigma) = B(\sigma) - T(\sigma) = \{b^a \in B(\sigma) \mid b^a \not\equiv 0 \text{ on } \sigma\}.$$

The elements of T will be called **transverse barycentric coordinates** to σ, and the elements of L will be called **longitudinal barycentric coordinates**. It is convenient to denote the elements b^a which lie in T as t^a, and to denote the elements b^a which lie in L as l^a.

Definition 1.4.6: An **origin** O for a simplex σ is a choice of a vertex v^a of σ.

Remark 1.4.7: Choose an origin O for σ, and let $L' = L'(\sigma)$ denote the set

$$L'(\sigma) = L(\sigma) - \{l^{a_0}\}$$

where l^{a_0} is the longitudinal barycentric coordinate function associated to the vertex corresponding to the origin O for σ. Then the elements x of $\mid \sigma \mid$ have a unique representation as $x = (l_1(x), \cdots, l_p(x))$ in terms of (some ordering of) the elements of L', $p = \dim(\sigma)$.

For any simplex ρ of M containing σ, by ordering the subset $A \subset B(\sigma)$ defined by

$$A = L'(\sigma) \cup \{b^a \in T(\sigma) \mid b^a \not\equiv 0 \text{ on } \rho\},$$

one obtains a coordinate system for ρ.

Remark 1.4.8: The values of the barycentric coordinate functions at a point P in ρ are independent of the choice of origin O for σ.

Remark 1.4.9: The differentials db^a of the barycentric coordinate functions are piecewise differentiable 1-forms, $db^a \in \Omega^1(M)$. Once an origin O

for σ has been chosen, then for any q-simplex ρ containing σ, an element $\phi \in j_\rho^* \Omega^r(M)$ (where $j_\rho : \rho \to M$ is the inclusion), has a unique representation as $\phi = \sum_\alpha \phi_\alpha(b^1, \cdots, b^q) db^\alpha$ where α is a multi-index with $\alpha^a = 0$ if $v^a \notin \rho$ or $v^a = O$.

1.5 The Canonical Volume Form

Remark 1.5.1: For any ordering of the vertices $v^0, \cdots, v^p$ of a p-simplex σ of M, there canonically corresponds the p-form $db^1 \wedge \cdots \wedge db^p$ where db^j is the differential of the barycentric coordinate corresponding to the vertex v^j, for $j = 1, \cdots, p$. This form has the property that given any permutation

$$\pi : (v^0, \cdots, v^p) \to (v^{\pi(0)}, \cdots, v^{\pi(p)})$$

of the vertices, then

$$db^{\pi(1)} \wedge \cdots \wedge db^{\pi(p)} = \operatorname{sgn}(\pi) db^1 \wedge \cdots \wedge db^p.$$

This follows by the skew-symmetry of the exterior product and the relation $\sum_{v^a \in \sigma} b^a = 1$ (which implies $\sum_{v^a \in \sigma} db^a = 0$).

Definition 1.5.2: If σ is an oriented simplex and the orientation of σ can be represented by the ordered $(p+1)$-tuple $(v^0, \cdots, v^p)$, then the form $db^1 \wedge \cdots \wedge db^p$ associated to the oriented p-simplex σ will be denoted $d\sigma$.

As the orientation for M determines an orientation for each m-simplex ρ of M, the above procedure produces a nonvanishing m-form $d\rho$ on each maximal m-simplex ρ of M. As the pull-back of an m-form to an $(m-1)$-face is zero, it follows that these forms combine to give an element $\mu \in \Omega^m(M)$ defined by $\mu \mid_\rho = d\rho$ for maximal simplices $\rho \in K^m(M)$.

Definition 1.5.3: The element $\mu \in \Omega^m(M)$ with $\mu \mid_\rho = d\rho$ for maximal simplices $\rho \in K^m(M)$ will be called the **canonical volume form** on M asssociated to the orientation of M.

1.6 A Representation for Piecewise Differentiable Currents

Let ρ be a maximal simplex of M with $J_\rho : \rho \to M$ the inclusion, let σ be a face of ρ, and $n = \dim(\sigma)$. Choose a coordinate system $(b^1, \cdots, b^m)$ for ρ consisting of barycentric coordinate functions such that $(b^1, \cdots, b^n)$ forms a coordinate system on σ. Set

$$d\rho = db^1 \wedge \cdots \wedge db^m,$$

$$d\sigma = db^1 \wedge \cdots \wedge db^n.$$

Definition 1.6.1: $[\sigma]_\rho : j_\rho^* \Omega^*(M) \to R$ will denote the current defined by

$$[\sigma]_\rho(\phi) = \int_\sigma \phi_\rho(b^1, \cdots, b^m) d\sigma$$

if $\phi = \phi_\rho(b^1, \cdots, b^m) db^1 \wedge \cdots \wedge db^m = \phi_\rho(b^1, \cdots, b^m) d\rho$ is a form of degree m, and

$$[\sigma]_\rho(\phi) = 0$$

if ϕ is a form of lower degree.

Definition 1.6.2: $[\sigma] : \Omega^m(M) \to R$ will denote the current

$$[\sigma](\phi) = \sum_{\rho \supset \sigma, \ \dim(\rho)=m} [\sigma]_\rho(\phi).$$

Let $U_{\rho,\sigma}$ be a piecewise differentiable current on ρ with carrier σ, then for any $\phi \in \Omega^*(M)$ on M, $U_{\rho,\sigma}(\phi) = U_{\rho,\sigma}(j_\rho^*\phi)$. In the coordinate system for ρ, $j_\rho^*\phi$ has a representation as

$$j_\rho^*\phi = \sum_\beta \phi_\beta(b^1, \cdots, b^m)db^\beta.$$

Definition 1.6.3: The partial differential opertators D_b^α operate on forms by

$$D_b^\alpha\phi = \sum_\beta (D_b^\alpha\phi_\beta(b^1, \cdots, b^m))db^\beta \text{ for } \phi = \sum_\beta \phi_\beta(b^1, \cdots, b^m)db^\beta \text{ on } \rho.$$

Definition 1.6.4: The partial differential operators D_b^α operate on currents by the formula

$$(D_b^\alpha U)(\phi) = (-1)^{|\alpha|} \cdot U(D_b^\alpha\phi).$$

If db^β represents a differential form which is not identically zero on ρ, let $db^{\beta'}$ denote the differential form defined by

$$db^\beta \wedge db^{\beta'} = d\rho = db^1 \wedge \cdots \wedge db^m.$$

Remark 1.6.5: Let $\phi \in \Omega^*(M)$, then $j_\rho^*\phi = \sum_\beta \phi_\beta(b^1, \cdots, b^m)db^\beta$ in terms of the local coordinates for ρ. A piecewise differentiable current $U_{\rho,\sigma}$ on ρ with carrier σ can be expressed as

$$U_{\rho,\sigma}(\phi) = \int_\sigma \sum_{\alpha,\beta} (D_b^\alpha \phi_\beta)(b^1, \cdots, b^m)\omega_{\alpha,\beta}(b^1, \cdots, b^m)d\sigma.$$

Define a differential form ω_α on ρ by setting

$$\omega_\alpha = \sum_\beta \omega_{\alpha,\beta}(b^1, \cdots, b^m)db^{\beta'}.$$

Then $U_{\rho,\sigma}$ can be written as

$$U_{\rho,\sigma} = \sum_\alpha D_b^\alpha\{\omega_\alpha \wedge [\sigma]\} \circ j_\rho^*$$

where $\omega_\alpha \in \Omega^*(\rho)$ is a differential form on ρ and σ is a face of ρ.

From this we conclude $U \in C^p(M)$ can be expressed as

$$U = \sum_{\sigma \in M} \sum_{\rho \supset \sigma} \sum_\alpha D_b^\alpha\{\omega_\alpha \wedge [\sigma]\} \circ j_\rho^*$$

where the first sum is over all simplices σ of M, the second sum is over all maximal simplices ρ which contain σ, the third sum is over a finite collection of multi-indices relative to a coordinate system on ρ, $\omega_\alpha \in \Omega^*(\rho)$, and where $j_\rho : \rho \to M$ is the inclusion. This is the basic representation for a piecewise differentiable current.

To introduce a product structure on piecewise differentiable currents as in $[T_2]$, it is necessary to consider piecewise differentiable currents defined on products $M \times \cdots \times M$ of M.

1.7 The Cell Complex M_k

Definition 1.7.1: For a positive integer k, M_k will denote the k-fold product of M,

$$M_k = M \times \cdots \times M \ (k \text{ factors}).$$

Definition 1.7.2: A **polysimplex** Σ in M_k is a closed subspace of $| M_k |$ which is of the form $\Sigma = | \sigma_1 | \times \cdots \times | \sigma_k |$, where $\sigma_1, \cdots, \sigma_k$ are simplices of M. A polysimplex of dimension km ($m = \dim(M)$) in M_k will be called a **maximal polysimplex**.

Remark 1.7.3: M_k together with the collection of all polysimplices contained in M_k, makes M_k into a cell complex, where the cells in M_k are polysimplices.

Let N be a subcomplex of M_k.

Definition 1.7.4: Let $P(N)$ denote the free abelian group with generators the polysimplices P of N, and let $P_j(N)$ denote the the subgroup of $P(N)$ consisting of polysimplices of dimension j. Set $P_j(N) = 0$ for $j < 0$ and for $j > \dim(N)$.

For a polysimplex $P = \rho_1 \times \cdots \times \rho_k$, and for integers $j = 0, \cdots, k$, let $n_j = \dim(\rho_j)$, and let ρ_j^i denote the i-face of ρ_j with respect to some ordering of the vertices of the simplices ρ_j, i.e., $\rho_j^i = \rho_j - v^i$ where v^i is the i^{th} vertex of ρ_j, and where $\rho_j - v^i$ is the simplex spanned by all vertices of ρ_j except v^i. $\rho_j^i = \emptyset$ if ρ_j consists of a single vertex.

Definition 1.7.5:For an integer j, $1 \leq j \leq k$, and integers i, $1 \leq i \leq \dim(\rho_j)$, let

$$P_j^i = \rho_1 \times \cdots \times \rho_{j-1} \times \rho_j^i \times \rho_{j+1} \times \cdots \times \rho_k.$$

Definition 1.7.6: For an integer j, $S_j(N)$ will denote the vector space

$$S_j(N) = P_j(N) \bigotimes_Z R.$$

$$\partial : S_j(N) \to S_{j-1}(N)$$

is the linear map defined on generators P of $S_j(N)$ by

$$\partial(P) = \sum_{j=1}^{k} (-1)^{n_1 + \cdots n_{j-1} + j - 1} \sum_{i=0}^{n_j} (-1)^i \cdot P_j^i.$$

Define $\partial : S_j(N) \to S_{j-1}(N)$ to be the zero homomorphism, if $j < 1$ or $j > \dim(N)$. $S_*(N) = P(N) \bigotimes_Z R$, and let

Definition 1.7.7: The **Real Cellular Chain Complex** of N is the complex

$$S_*(N) = (\bigoplus_{j \in Z} S_j(N), \partial).$$

It is immediate that $\partial^2 = 0$.

Definition 1.7.8: If (N_1, N_2) are a pair of subcomplexes of M_k with $N_1 \supset N_2$, then, for integers j, $S_j(N_1, N_2)$ will denote the quotient space

$$S_j(N_1, N_2) = S_j(N_1)/S_j(N_2)$$

and $S_*(N_1, N_2)$ will denote the quotient complex

$$S_*(N_1, N_2) = (\bigoplus_{j \in Z} S_j(N_1, N_2), \partial).$$

1.8 Piecewise Differential Currents on M_k

This section is devoted to generalizing the material given above from M to M_k. It concludes with a remark that shows a piecewise differentiable current on M_k can be expressed in a certain form.

Definition 1.8.1: A **piecewise differentiable** p**-form** on M_k is a function ω which associates to any maximal polysimplex $\Sigma \in M$ a real-valued C^∞ p-form ω_Σ on Σ such that the collection $\{\omega_\Sigma\}$ satisfies the following compatibility condition: if Σ and P are two maximal polysimplices of M with $\Sigma \cap P \neq \emptyset$, and i_Σ and i_P are the inclusions of $\Sigma \cap P$ into Σ and P respectively, then $(i_\Sigma)^* \omega_\Sigma = (i_P)^* \omega_P$. $\Omega^p(M_k)$ will denote the set

$$\Omega^p(M_k) = \{\omega \mid \omega \text{ is a piecewise differential p-form on } M_k\}.$$

Definition 1.8.2: If $\omega \in \Omega^p(M_k)$, we define the **differential of** ω to be the form $d\omega$ which assigns to a maximal polysimplex $\Sigma \in M_k$ the $(p+1)$-form $d(\omega_\Sigma)$.

Definition 1.8.3: If $\omega \in \Omega^r(M_k)$, $\theta \in \Omega^s(M_k)$, then $\omega \wedge \theta$ is the form of degree $r + s$ defined by $(\omega \wedge \theta)_\Sigma = \omega_\Sigma \wedge \theta_\Sigma$.

Definition 1.8.4: The graded differential algebra

$$\Omega^*(M_k) = \{\Omega^r(M_k), d\}_{r\in N}$$

is called the **Sullivan complex of piecewise differentiable forms** on M_k. It is given the Fréchet space topology from the family of semi-norms $\| \omega \|_{\alpha,\beta}$, defined on $\omega \in \Omega^*(M_k)$, with

$$\omega \mid_\Sigma = \sum \omega_{\Sigma,\beta}(y) dy^\beta$$

on maximal polysimplices Σ of M_k, by

$$\| \omega \|_{\alpha,\beta} = \sup_{y,\Sigma} \mid D_y^\alpha \omega_{\Sigma,\beta}(y) \mid ,$$

where the supremum is taken with respect to all maximal simplices $\Sigma \in M_k$, and $y \in \mid \Sigma \mid$.

Definition 1.8.5: The **canonical volume form** on M_k is

$$\mu^k = \text{proj}_1^* \mu \wedge \cdots \wedge \text{proj}_k^* \mu$$

where $\text{proj}_j : M_k = M \times \cdots \times M \to M$ is projection onto the j^{th} factor.

Definition 1.8.6: A continuous linear functional

$$T : \Omega^{km-p}(M_k) \to R$$

will be called a **current of degree p** on M_k.

Remark 1.8.7: Any $\theta \in \Omega^p(M_k)$ or $\gamma \in S_{km-p}(M_k)$ defines a current of degree p by

$$\theta(\omega) = \int_{M_k} \omega \wedge \theta \ , \ \omega \in \Omega^{km-p}(M_k),$$

$$[\gamma](\omega) = \int_{\gamma} \omega \ , \ \omega \in \Omega^{km-p}(M_k).$$

Remark 1.8.8: If T is a current of degree p and $\theta \in \Omega^q(M)$, then $T \wedge \theta$ is a current of degree $p+q$ defined by

$$(T \wedge \theta)(\omega) = T(\theta \wedge \omega) \ , \ \omega \in \Omega^{km-p-q}(M_k).$$

This makes the set of currents of degree p into a module over $\Omega^0(M_k)$.

Definition 1.8.9: The **deRham boundary** of a current T of degree p is the current bT is of degree $p+1$ defined by

$$(bT)(\omega) = T(d\omega) \ , \ \omega \in \Omega^{km-p+1}(M_k).$$

Remark 1.8.10: It follows that $b^2 = 0$, and for currents T of degree p, $\omega \in \Omega^q(M_k)$,

$$b(T \wedge \omega) = (-1)^q \cdot bT \wedge \omega + T \wedge b\omega.$$

Definition 1.8.11: The **exterior derivative** of a current T of degree p is the current dT is of degree $p+1$ defined by

$$(dT)(\omega) = (-1)^{p+1} T(d\omega) \ , \ \omega \in \Omega^{km-p+1}(M_k).$$

Remark 1.8.12: It follows that $d^2 = 0$, and for currents T of degree p, $\omega \in \Omega^q(M_k)$,

$$d(T \wedge \theta) = dT \wedge \theta + (-1)^p T \wedge d\theta$$

and $bT = (-1)^{p+1} \cdot dT$.

Definition 1.8.13: A **piecewise differentiable current** of degree p on M_k is a continuous linear functional

$$U : \Omega^{km-p}(M_k) \to R$$

which has a representation of the form

$$U = \sum_{\Sigma \subset M_k} \sum_{\Pi \supset \Sigma} U_{P,\Sigma}$$

where the first sum is over all polysimplices Σ of M_k, the second sum is over all maximal polysimplices Π which contain Σ, and where $U_{P,\Sigma}$ denotes a piecewise differentiable currents on P with carrier Σ, i.e. each $U_{P,\Sigma}$ can be represented as

$$U_{P,\Sigma} = \sum_{\alpha} D_x^{\alpha}\{\omega_\alpha \wedge [\Sigma]\} \circ j_\Pi^*$$

where the sum is over a finite collection of multi-indices α, $x = (x^1, \cdots, x^{km})$ are local coordinates for Π, $\omega_\alpha \in \Omega^*(P)$, and $j_\Pi : \Pi \to M_k$ is the inclusion. The polysimplex Σ will be called the **carrier** of the current $U_{P,\Sigma}$.

Remark 1.8.14: Any functional of the above form is continuous in the Fréchet space topology as the constant C can be chosen to be the sum (over the finite sets Σ, Π, and α) of the maximum of the continuous functions $\mid \omega_\alpha(x) \mid$ on Σ.

Definition 1.8.15: For any nonnegative integer r, $C^r(M_k)$ denote the vector space of piecewise differentiable currents on M_k of degree r under the operation $+$, given by $(U+U')(\omega) = U(\omega) + U'(\omega)$, and real multiplication $(r \cdot U)(\omega) = r \cdot U(\omega)$. Define $C^r(M_k) = 0$ for negative integers r.

Definition 1.8.16: $C^*(M_k)$ will denote the direct sum

$$C^*(M_k) = \bigoplus_{r \in Z} C^r(M_k).$$

Definition 1.8.17: If $N \subset M_k$ is a subcomplex of M_k, then $C^r(N)$ will denote the subspace of piecewise differentiable currents in M_k with carrier contained in N. For a pair of subcomplexes (N_1, N_2) of M_k with $N_2 \subset N_1$, $C^r(N_1, N_2)$ will denote the quotient space

$$C^r(N_1, N_2) = C^r(N_1)/C^r(N_2).$$

We also adopt the notation

$$C^*(N) = \bigoplus_{r \in Z} C^r(N),$$

$$C^*(N_1, N_2) = \bigoplus_{r \in Z} C^r(N_1, N_2).$$

As we will need to perform some calculations involving piecewise differentiable currents, it will be useful to introduce the generalization of barycentric coordinates to the cell complex M_k.

Let $\text{proj}_j : M_k = M \times \cdots \times M \to M$ be projection onto the j^{th} factor.

Definition 1.8.18: A **barycentric coordinate function** $b_j^a : M_k \to [0,1]$ on the cell complex M_k is a piecewise differentiable function on M_k of the form

$$b_j^a(X) = b^a(\ \text{proj}_j(X)) \text{ for } X \in M_k.$$

Definition 1.8.19: Associated to a polysimplex $\Sigma = \sigma_1 \times \cdots \times \sigma_k$, are the sets of barycentric coordinates

$$B_j(\Sigma) = \{b_j^a \mid \operatorname{star}(v^a) \supset \sigma_j\} \text{ for } 1 \le j \le k,$$

$$B(\Sigma) = \cup_{j=1}^k B_j(\Sigma),$$

$$T(\Sigma) = \{b_j^a \in B(\Sigma) \mid b_j^a \equiv 0 \text{ on } \Sigma\} \text{ , and}$$

$$L(\Sigma) = B(\Sigma) - T(\Sigma).$$

The elements of $T(\Sigma)$ will be called **transverse coordinates** to Σ, and the elements of $L(\Sigma)$ will be called **longitudinal coordinates**.

Definition 1.8.20: An **origin** O for a polysimplex $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ is a k-tuple of vertices $O = (v_1, \cdots, v_k)$ with $v_j \in \sigma_j$.

Remark 1.8.21: The choice of an origin $O = \{v_1, \cdots, v_k\}$ for a polysimplex Σ of dimension n determines a set $L'(\Sigma)$ defined by

$$L'(\Sigma) = \{l_j^a = b_j^a \mid b_j^a \in L \text{ , } v^a \ne v_j \text{ for } 1 \le j \le k\},$$

then the elements x of $\mid \Sigma \mid$ have a unique representation as $(l^1(x), \cdots, l^n(x))$ in terms of the elements of $L'(\Sigma)$.

For any polysimplex P of M_k containing Σ, the subset $A \subset B$ defined by

$$A = \{b_j^a \in T(\Sigma) \mid b_j^a \ne 0 \text{ on } P\} \cup L'$$

forms a system of coordinates for P.

Definition 1.8.22: If $(l^1, \cdots, l^n)$ is a system of longitudinal barycentic coordinates for an oriented polysimplex Σ in M_k, then $d\Sigma$ will represent the the differential form $d\Sigma = \pm dl^1 \wedge \cdots \wedge dl^n$ where the sign is chosen so that

$$\int_{\Sigma} d\Sigma > 0.$$

Remark 1.8.23: By choosing an origin O for each polysimplex Σ of M_k, elements of $C^r(M_k)$ can be expressed in terms of the elements of L'_{Σ} and $T_{\Pi,\Sigma}$, and it has a representation of the form

$$U = \sum_{\Sigma \subset M_k} \sum_{\Pi \supset \Sigma} U_{\Pi,\Sigma}$$

where

$$U_{\Pi,\Sigma} = \sum_{\alpha,\delta} D_t^{\alpha} D_l^{\delta} \{\omega_{\alpha,\delta} \wedge [\Sigma]\} \circ j_{\Pi}^{*},$$

where the coordinate functions $l^1, \cdots, l^{\dim(\Sigma)}, t^1, \cdots, t^{km-\dim(\Sigma)}$ are the longitudinal and transverse coordinate functions to Σ in Π, and where $j_{\Pi} : \Pi \to M_k$ is the inclusion.

The following remark is a restatement of Proposition 1.1 of [T_2].

Remark 1.8.24: If $\delta \neq 0$, express D_l^{δ} as $D_l^{\delta} = D_{l^j} D_l^{\delta'}$. Then for any

$\phi \in \Omega^{km-r}(M_k)$

$$\begin{aligned}
U_{\Pi,\Sigma}(\phi) &= \sum_{\alpha,\delta} D_t^\alpha D_l^\delta \{\omega_{\alpha,\delta} \wedge [\Sigma]\} \circ j_\Pi^* \phi \\
&= \sum_{\alpha,\delta} (-1)^{|\alpha|+|\delta|} \cdot [\Sigma]\{\omega_{\alpha,\delta} \wedge D_t^\alpha D_l^\delta (j_\Pi^* \phi)\} \\
&= \sum_{\alpha,\delta} [\Sigma](-1)^{|\alpha|+|\delta|} \cdot D_{l^j}\{\omega_{\alpha,\delta} \wedge D_t^\alpha D_l^{\delta'} (j_\Pi^* \phi)\} \\
&- (-1)^{|\alpha|+|\delta|} \cdot \sum_{\alpha,\delta} [\Sigma]\{D_{l^j}\omega_{\alpha,\delta} \wedge D_t^\alpha D_l^{\delta'} j_\Pi^* \phi\} \\
&= \sum_{\alpha,\delta} \nu_i [\Sigma^i](-1)^{|\alpha|+|\delta|} \cdot \{\omega_{\alpha,\delta} \wedge D_t^\alpha D_l^{\delta'} j_\Pi^* \phi\} \\
&- (-1)^{|\alpha|+|\delta|} \cdot \sum_{\alpha,\delta} \nu_0 [\Sigma^0]\{\omega_{\alpha,\delta} \wedge D_t^\alpha D_l^{\delta'} j_\Pi^* \phi\} \\
&- [\Sigma](-1)^{|\alpha|+|\delta|} \cdot \{(D_t^\alpha D_l^{\delta'} j_\Pi^* \phi) \wedge D_l^j \omega_{\alpha,\delta}\} \\
&= -\sum_{\alpha,\delta} D_t^\alpha D_l^{\delta'} \{\nu_i \cdot \omega_{\alpha,\delta} \wedge [\Sigma^i]\}(j_\Pi^* \phi) \\
&+ \sum_{\alpha,\delta} D_t^\alpha D_l^{\delta'} \{\nu_0 \cdot \omega_{\alpha,\delta} \wedge [\Sigma^0]\}(j_\Pi^* \phi) \\
&+ D_t^\alpha D_l^{\delta'} \{D_l^j \omega_{\alpha,\delta} \wedge [\Sigma]\}(j_\Pi^* \phi)
\end{aligned}$$

where Σ^i is the polysimplex obtained from Σ by collapsing the polysimplex with respect to the vertex v_j^a (with $b_j^a = l^i$), Σ^0 is obtained from Σ by collapsing with respect to the vertex v_j (the j^{th} coordinate vertex of the origin O), and where ν_i and ν_0 are ± 1 depending on the orientation given to $[\Sigma^i]$ and $[\Sigma^0]$ respectively (with $\nu_i = -1$ if $d\Sigma = dl^i \wedge d\Sigma^i$)).

This implies $U \in C^*(M_k)$ has a representation of the form

$$U = \sum_{\Sigma \subset M_k} \sum_{\Pi \supset \Sigma} U_{\Pi,\Sigma}$$

with

$$U_{\Pi,\Sigma} = \sum_{\alpha,\gamma} D_t^{\alpha}\{\omega_{\alpha,\gamma} \wedge dt^{\gamma} \wedge [\Sigma]\} \circ j_{\Pi}^{*},$$

with $\omega_{\alpha,\gamma} \in \Omega^*(P)$.

This expression only involves transverse derivatives. In the next chapter a unique representation will be obtained.

Chapter 2

PIECEWISE DIFFERENTIABLE CURRENTS

This chapter is devoted to establishing a unique way of representing piecewise differentiable currents on M_k in a form similar to that given in the previous chapter. An explicit formula for the deRham boundary on the complex of piecewise differentiable currents is calculated. It is shown that the complex $C^*(M_k)$ decomposes into a sum of subcomplexes. These subcomplexes will be used to obtain homology results in subsequent chapters.

2.1 The deRham Boundary on the Complex $C^*(M_k)$

This section will be devoted to calculating an explicit formula for the deRham boundary $b : C^p(M_k) \to C^{p+1}(M_k)$ and the exterior derivative d : $C^p(M_k) \to C^{p+1}(M_k)$. These formulas will express the deRham boundary and exterior derivative of a piecewise differentiable current in the form required by the definition of a piecewise differentiable current. As a consequence, it will follow that

$$C^*(M_k) = (\bigoplus_{p \in Z} C^p(M_k), b)$$

is a chain complex. The formulas obtained in this section will be used in the computation of homology in Chapter 4.

Let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, $n = \dim(\Sigma)$, $n_j = \dim(\sigma_j)$ for $1 \le j \le k$.

Let Π be a maximal polysimplex in M_k containing Σ, and let $U \in C^p(M_k)$ be of the form

$$U = \sum_{\alpha} D_t^{\alpha}\{\omega \wedge [\Sigma]\}.$$

Let $O = (v_1, \cdots, v_k)$ be an origin for Σ, and $n' = km - n$. $(l^1, \cdots, l_n)$ and $(t^1, \cdots, t^{n'})$ will denote the corresponding longitudinal and transverse coordinates to Σ in Π.

Express d as $d_l + d_t$, where

$$d_l = \sum_{i=1}^{n} dl^i \wedge \frac{\partial}{\partial l^i}$$

$$d_t = \sum_{j=1}^{n'} dt^j \wedge \frac{\partial}{\partial t^j}.$$

$$\begin{aligned} dU =& (d_t + d_l)U \\ &= (-1)^p \cdot \sum_{j=1}^{n'} D_t^{\alpha} D_{t^j}\{\omega \wedge dt^j \wedge [\Sigma]\} \\ &+ D_t^{\alpha}\{(d_l\omega) \wedge [\Sigma]\} \\ &+ (-1)^p \cdot D_t^{\alpha}\{\omega \wedge d_l[\Sigma]\}. \end{aligned}$$

To compute the last term, express

$$d_l = \sum_{i=1}^{n} dl^i \wedge \frac{\partial}{\partial l^i} \text{ as } d_l = \sum_{j=1}^{k} dl_j$$

where

$$dl_j = \sum_{h=1}^{n_j} dl_j^h \wedge \frac{\partial}{\partial l_j^h}.$$

dl_j^0 will denote the differential of the longitudinal coordinate corresponding to the vertex of σ_j chosen as the j^{th} vertex in the origin O. For $\phi \in \Omega^{km-p-1}(M_k)$,

$$\begin{aligned}(-1)^p \cdot D_t^\alpha\{\omega \wedge d_l[\Sigma]\}\phi &= (-1)^{|\alpha|} \cdot d_l[\Sigma]\{\omega \wedge D_t^\alpha \phi\} \\ &= (-1)^{|\alpha|} \cdot \sum_{j=1}^{k}\sum_{h_j=1}^{n_j} \frac{\partial}{\partial l^{h_j}}[\Sigma]\{dl^{h_j} \wedge \omega \wedge D_t^\alpha \phi\} \\ &= -(-1)^{|\alpha|} \cdot \sum_{j=1}^{k}\sum_{h_j=1}^{n_j} [\Sigma]\frac{\partial}{\partial l^{h_j}}\{dl^{h_j} \wedge \omega \wedge D_t^\alpha \phi\}.\end{aligned}$$

For $j = 1, \cdots, k$, the identity $\sum_{h=0}^{n_j} l_j^h = 1$ implies $\sum_{h=0}^{n_j} dl_j^h = 0$. This implies $dl_j^0 = -\sum_{h=1}^{n_j} dl_j^h$. Let $[\Sigma_j^h]$ denote the polysimplex obtained from Σ by collapsing to the face opposite the vertex v_j^h with the induced orientation. Then

$$[\Sigma]\partial/\partial l_j^h(\theta) = [\Sigma_j^0] dl_j^0 \wedge \theta - [\Sigma_j^h] dl_j^h \wedge \theta \qquad \text{for} \qquad \theta \in \Omega^{km-1}(M_k)$$

where $[\Sigma_j^0]$ is the face opposite the vertex corresponding to the $v_j \in O$.

Integration along the directions parallel to the j^{th} face yields

$$
\begin{aligned}
-(-1)^{|\alpha|}\sum_{h=1}^{n_j}[\Sigma]\frac{\partial}{\partial l_j^h}\{dl_j^h\wedge\omega\wedge D_t^\alpha\phi\} =& -(-1)^{|\alpha|}\sum_{h=1}^{n_j}[\Sigma_j^0]\{dl_j^h\wedge\omega\wedge D_t^\alpha\phi\} \\
&+(-1)^{|\alpha|}\sum_{h=1}^{n_j}[\Sigma_j^h]\{dl_j^h\wedge\omega\wedge D_t^\alpha\phi\} \\
&=(-1)^{|\alpha|}[\Sigma_j^0]\{dl_j^0\wedge\omega\wedge D_t^\alpha\phi\} \\
&+(-1)^{|\alpha|}\sum_{h=1}^{n_j}[\Sigma_j^h]\{dl_j^h\wedge\omega\wedge D_t^\alpha\phi\} \\
&=(-1)^{|\alpha|}\sum_{h=0}^{n_j}[\Sigma_j^h]\{dl_j^h\wedge\omega\wedge D_t^\alpha\phi\} \\
&=D_t^\alpha\{\sum_{h=0}^{n_j}dt_j^h\wedge\omega\wedge[\Sigma_j^h]\}(\phi) \\
&=(-1)^p\cdot D_t^\alpha\{\omega\wedge\sum_{h=0}^{n_j}dt_j^h\wedge[\Sigma_j^h]\}(\phi)
\end{aligned}
$$

where dl_j^h is changed to dt_j^h as the barycentric coordinate represented by dl_j^h is a transverse barycentic coordinate to the face Σ_j^h.

Definition 2.1.1: Let $\partial[\Sigma]$ denote the current

$$\partial[\Sigma]=\sum_{\Sigma^i}dt^i\wedge[\Sigma^i]$$

where the sum is over all codimension one faces of Σ, and dt^i is the transverse coordinate to the i-face of Σ which is a longitudinal barycentric coordinate to Σ, and $[\Sigma^i]$ denotes the i-face of Σ with the induced orientation. With this notation,

$$(-1)^p\cdot D_t^\alpha\{\omega\wedge d_l[\Sigma]\}=D_t^\alpha\{\omega\wedge\partial[\Sigma]\}.$$

Combining the above results,

$$
\begin{aligned}
dU = & (-1)^p \cdot \sum_{j=1}^{n'} D_t^\alpha D_{t^j}\{\omega \wedge dt^j \wedge [\Sigma]\} \\
& + D_t^\alpha\{(d_l\omega) \wedge [\Sigma]\} \\
& + D_t^\alpha\{\omega \wedge \partial[\Sigma]\}
\end{aligned}
$$

and

$$
\begin{aligned}
bU = & -\sum_{j=1}^{n'} D_t^\alpha D_{t^j}\{\omega \wedge dt^j \wedge [\Sigma]\} \\
& + (-1)^{p+1} \cdot D_t^\alpha\{(d_l\omega) \wedge [\Sigma]\} \\
& + (-1)^{p+1} \cdot D_t^\alpha\{\omega \wedge \partial[\Sigma]\}.
\end{aligned}
$$

These expressions show dU and bU have representations in the form required for a piecewise differential current on M_k. As any element of $C^p(M_k)$ is a linear combination of currents which have the form of the current U chosen above, it follows that the deRham boundary makes $C^*(M_k)$ into a complex.

2.2 Unique Representation of the Elements of $C^*(M_k)$

In the previous chapter it was shown any $U \in C^*(M_k)$ can be expressed as

$$
U = \sum_{\Sigma \in M_k} \sum_{\Pi \supset \Sigma} \sum_{\alpha,\beta} U_{\Pi,\Sigma,\alpha,\beta}
$$

where

$$
U_{\Pi,\Sigma,\alpha,\beta} = D_t^\alpha\{\omega_{\alpha,\beta} \wedge dt^\beta \wedge [\Sigma]\} \circ j_\Pi^*
$$

with $\omega_{\alpha,\beta} = \sum_\gamma \omega_{\alpha,\beta,\gamma}(l,0)dl^\gamma \in \Omega^*(\Pi)$, (l,t) denoting the longitudinal and transverse barycentric coordinates to Σ in Π, and $j_\Pi : \Pi \to M_k$ the inclusion.

Definition 2.2.1: For each polysimplex Σ of M_k, and for each maximal polysimplex $\Pi = \pi_1 \times \cdots \times \pi_k$ containing Σ, the representation for $U_{\Pi,\Sigma,\alpha,\beta}$ determines a unique polysimplex $P = \rho_1 \times \cdots \times \rho_k$ where for $j = 1, \cdots, k$, ρ_j is the simplex whose verices are the union of:

1. the vertices of σ_j,
2. the vertices v^a which correspond to $\alpha_j^a \neq 0$.
3. the vertices v^a which correspond to $\beta_j^a = 0$.

the polysimplex P produced by this procedure will be called the polysimplex **associated** to the pair (α, β). A pair (α, β) of multi-indices will be called **assigned** to the polysimplex P, if P is the polysimplex determined by the pair (α, β) by the above procedure.

Remark 2.2.2: For a pair (Π, Σ) of polysimplices of M_k with $\Pi \supset \Sigma$ and Π maximal, and a current $U_{\alpha,\beta}$ of the form

$$U_{\alpha,\beta} = D_t^\alpha\{\omega_{\alpha,\beta} \wedge dt^\beta \wedge [\Sigma]\} \circ j_\Pi^*,$$

as above, let P be the polysimplex determined by (α, β). Let $n = \dim(\Sigma)$, $n' = \dim(P) - n$, $n'' = km - n - n'$ and let the barycentric coordinates (l, t) of Σ in Π be ordered so that $(t^1, \cdots, t^{n''})$ are the transverse coordinates to P in Π. Define $\theta = \theta_{\Pi,P}$ as $\theta = dt^1 \wedge \cdots \wedge dt^{n''}$.

By the construction of P, $\beta_j^a = 1$ whenever $v^a \in \pi_j - \rho_j$. This implies $dt^\beta = \nu dt^{\beta'} \wedge \theta$ where $\nu = \pm 1$ and $dt^{\beta'}$ contains only those differentials of

dt^β which correspond coordinate functions whose partial derivatives appear in the multi-index α. Express $U_{\alpha,\beta}$ as

$$U_{\alpha,\beta} = D_t^\alpha\{\nu \cdot \omega_{\alpha,\beta} \wedge dt^{\beta'} \wedge \theta \wedge [\Sigma]\}$$

and let $U'_{\alpha,\beta}$ be the current

$$U'_{\alpha,\beta} = D_t^\alpha\{\nu \cdot \omega_{\alpha,\beta} \wedge dt^{\beta'} \wedge \theta \wedge [\Sigma]\} \circ j_P^* = U_{\alpha,\beta} \circ j_P^*$$

where $j_P : P \to M_k$ is the inclusion.

Then, for $\phi \in \Omega^p(M_k)$,

$$\begin{aligned} U_{\alpha,\beta}(\phi) &= (-1)^{|\alpha|} \cdot [\Sigma](\omega_{\alpha,\beta} \wedge dt^\beta \wedge D_t^\alpha \phi) \\ &= (-1)^{|\alpha|} \cdot [\Sigma](\nu \cdot \omega_{\alpha,\beta} \wedge dt^{\beta'} \wedge \theta \wedge D_t^\alpha j_P^* \phi) \\ &= D_t^\alpha\{\nu \cdot \omega_{\alpha,\beta} \wedge dt^{\beta'} \wedge \theta \wedge [\Sigma]\} \circ j_P^*(\phi) \\ &= U'_{\alpha,\beta}(\phi) \end{aligned}$$

as any term in ϕ which contains a differential of a transverse barycentric coordinate to P is annihilated when multiplied by θ.

Hence, $U_{\alpha,\beta} = U_{\alpha,\beta} \circ j_P^*$ where P is the polysimplex determined by the pair (α, β).

Remark 2.2.3: By construction, P is the smallest polysimplex with this property. This will follow from Proposition 2.3.6.

Definition 2.2.4: Given a pair (P, Σ) of polysimplices in M_k with $P \supset \Sigma$ and a choice of orientation for P, let dP denote the volume form on P as in 1.8.20. Let θ_P denote the form defined by

$$dP \wedge \theta_P = \mu^k$$

and define the current $[\Sigma]_{[P]} : \Omega^*(P) \to R$ by

$$[\Sigma]_{[P]}(\phi) = [\Sigma]\{j_P^*\phi \wedge \theta_P\}.$$

Remark 2.2.5: For each polysimplex Σ of M_k, the above procedure shows currents U_Σ which can be represented as

$$U_\Sigma = \sum_{\Pi,\Sigma} \sum_{\alpha,\beta} U_{\Pi,\Sigma,\alpha,\beta}$$

decompose into a sum of currents, each naturally defined on a polysimplex P containing Σ. By defining

$$U_{P,\Sigma} = \sum_{(\alpha,\beta)\subset P} U_{\Pi,\Sigma,\alpha,\beta}$$

where the sum is over all pairs (α, β) assigned to P, it follows that any $U \in C^*(M_k)$ has a representation of the form

$$U = \sum_{\Sigma \in M_k} \sum_{P \supset \Sigma} (\sum_{\alpha,\beta} D_t^\alpha \{\omega_{\alpha,\beta} \wedge dt^\beta \wedge [\Sigma]_{[P]}\} \circ j_P^*),$$

where the first sum is over all polysimplices of M_k, the second sum is over all polysimplices P which contain Σ, the third sum is over all multi-indices which are associated to P, (l, t) are the longitudinal and transverse barycentric coordinates to Σ in P, and $\omega_{\alpha,\beta} = \sum_\gamma \omega_{\alpha,\beta,\gamma}(l, 0) dl^\gamma \in \Omega^*(P)$.

Proposition 2.2.6: Let p be a nonnegative integer. Any element $U \in C^p(M_k)$ has a unique representation of the form

$$U = \sum_{\Sigma \in M_k} \sum_{P \supset \Sigma} (\sum_{(\alpha,\beta)} D_t^\alpha \{\omega_{P,\Sigma,\alpha,\beta} \wedge dt^\beta \wedge [\Sigma]_{[P]}\} j_P^*)$$

where the first sum is over all polysimplices Σ of M_k, the second sum is over all polysimplices P of M_k containing Σ, the third sum is over pairs of multi-indices assigned to P, (l, t) represent logitudinal and transverse barycentric coordinates to Σ in P, and $\omega_{P,\Sigma,\alpha,\beta} = \sum_\gamma \omega^{P,\Sigma}_{\alpha,\beta,\gamma}(l)dl^\gamma \in \Omega^*(P)$.

Existence of this type of representation was proved above. Uniqueness will require the construction of special elements in $\Omega^*(M_k)$. The proof of uniqueness will be supplied immediately following this construction.

Remark 2.2.7: This is a global result. This result should be compared with the corresponding result (Proposition 1.2 of $[T_2]$) on smooth manifolds. In that paper, Teleman gives a uniqueness statement for the complex $C^*(\Sigma, \partial\Sigma)$. In the combinatorial case, the global statement of Proposition 2.2.6 can be proved.

2.3 Functions with Zero Transverse Derivatives

Let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, O an origin for Σ, and let L' be the coordinate system on Σ determined by O. For any polysimplex $P = \rho_1 \times \cdots \times \rho_k$, let T denote the set of transverse coordinates to Σ in P. Let $d = \dim(\Sigma)$, $p = \operatorname{codim}_P(\Sigma)$. Set $S = \operatorname{Int}(\operatorname{star}(O))$, i.e., the interior of the union of all closed polysimplices of M_k containing O.

Define $\pi : P \to \Sigma$ by

$$\pi : (a^0, \cdots, a^d, b^1, \cdots, b^p) \mapsto (a^0, \cdots, a^h)$$

where $a^0, \cdots, a^h$ are the L' coordinates of $p \in P$, and $b^1, \cdots, b^p$ are the T coordinates of p.

For integers j, $1 \le j \le k$, $\text{proj}_j : M_k = M \times \cdots \times M \to M$ will denote projection onto the j th factor, and $\pi_j : P \to \sigma_j$ will be $\text{proj}_j \circ \pi$.

Given the choice of O, these projection mappings are "independent of the choice of P", i.e. if P' is another polysimplex which contains Σ, and $\pi' : P' \to \Sigma$ is the corresponding projection map, then $\pi |_{P \cap P'} = \pi' |_{P \cap P'}$.

Hence, this produces is a projection map $\pi : \cup_{P \supset \Sigma} P \to \Sigma$.

By composition with p_j, the projections π_j are also well-defined on $\cup_{P \supset \Sigma} P$.

Let K be any compact set in the interior of Σ, then $\text{proj}_j(K)$ is a compact subset contained in the interior of σ_j, let U_j be an open set in σ_j containing $\text{proj}_j(K)$ with the closure of U_j contained in the interior of σ_j. Let V_j be $\pi_j^{-1}(U_j)$, and let V be an open set in S. Set $U = V \cap V_1 \cap \cdots \cap V_k$. Let $\Phi : S \to R^{km}$ be a piecewise linear embedding which is linear on each simplex of $S \subset M_k$. Let $\psi : R^{km} \to [0,1]$ be a smooth function with $\psi \equiv 1$ on $\Phi(U)$, and $\psi \equiv 0$ in the complement of the image of Φ. Let $\Psi_K : M_k \to R$ be defined by $\Psi_K = \psi \circ \Phi$ on S, and $\Psi_K = 0$ in the complement of S.

Any smooth function F on Σ with support contained in the interior of Σ, can be extended to function $\tilde{F} \in \Omega^0(M_k)$ with zero transverse derivatives by defining

$$\tilde{F}(P) = \Psi(P) \cdot F(\pi_1(P), \cdots, \pi_k(P)) \text{ , for } P \in \cup_{P \supset \Sigma} P,$$

$$\tilde{F}(P) = 0 \text{ , for } P \in M_k - \cup_{P \supset \Sigma} P.$$

As Ψ is identically zero on any polysimplex P' of $\cup_{P \supset \Sigma} P$ when P' does not contain Σ, this gives a piecewise differentiable function on M_k.

$\tilde{F}$ has the property $D_t \tilde{F} = 0$ along Σ for any transverse barycentric coordinate function t. $\tilde{F}$ also has the property that $\tilde{F} \equiv 0$ in a neighborhood of any polysimplex of M_k which does not contain Σ.

Remark 2.3.1: The projection $\pi : P \to \Sigma$ may be replaced by any other projection $\pi' : P \to \Sigma$ as long as the projections $\pi' : P_1 \to \Sigma$ and $\pi' : P_2 \to \Sigma$ agree on $P_1 \cap P_2$ when $P_1 \cap P_2 \neq \emptyset$.

2.4 Proof of Uniqueness

Any $U \in C^p(M_k)$ can be represented as

$$U = \sum_{\Sigma \subset M_k} \sum_{P \supset \Sigma} U_{P,\Sigma}$$

where each $U_{P,\Sigma}$ can be expressed as

$$U_{P,\Sigma} = \sum_{\alpha,\beta,\gamma} D_t^{\alpha} \{\omega_{\alpha,\beta,\gamma}(l) dl^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]_{[P]}\} j_P^*$$

in terms of coordinates $(l,t) \in L' \cup T$, corresponding to a choice of origin O for each polysimplex Σ of M_k.

Suppose $U \in C^p(M_k)$ has two representations of the above type. If the representations are not identical, there exist polysimplices P and Σ, and multi-indices α, β, γ such that the corresponding functions $\omega_{\alpha,\beta,\gamma}(l)$ and $\tilde{\omega}_{\alpha,\beta,\gamma}(l)$ do not agree. Let

$$\eta_{\alpha,\beta,\gamma}(l) = \omega_{\alpha,\beta,\gamma}(l) - \tilde{\omega}_{\alpha,\beta,\gamma}(l).$$

Let $x \in \Sigma$ be a point in the interior of Σ where $\eta(x) = \eta_{\alpha,\beta,\gamma}(x) \neq 0$, and let V be a neighborhood of x in Σ where η has the same sign. Let U' be the current which is represented by

$$U' = \sum_{\Sigma \subset M_k} \sum_{P \supset \Sigma} U'_{P,\Sigma}$$

where $U'_{P,\Sigma}$ is given by

$$U'_{P,\Sigma} = \sum_{\alpha,\beta,\gamma} D_t^{\alpha}\{(\omega_{\alpha,\beta,\gamma}(l) - \tilde{\omega}_{\alpha,\beta,\gamma}(l) dl^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]_P\} j_P^*.$$

As this current represents $U - U$ it should vanish on all forms $\phi \in \Omega^{km-p}(M_k)$.

Given the pair (P,Σ) and the triple (α,β,γ) as above, let $\theta_{\beta,\gamma} \in \Omega^*(M_k)$ denote a form $\theta = dt^{\zeta}$ defined by $dl^{\gamma} \wedge dt^{\beta} \wedge \theta_{\beta,\gamma} = (-1)^{\nu} j_P^* \mu^k$, where μ^k is the canonical volume form on M_k, $j_P : P \to M_k$ is the inclusion, and $\nu = (-1)^{(km-p)(km-\dim(P))}$.

Let $F : \Sigma \to R$ be a nonnegative smooth function on Σ with support contained in V and which is positive at x. By the previous section, F has an extension $\tilde{F}$ to M_k with the property that $D_t\tilde{F} = 0$ along Σ for all transverse directions.

Define

$$\phi = \frac{1}{\alpha!} \cdot t^{\alpha} \cdot F(l,t)\theta_{\beta,\gamma} \in \Omega^{km-p}(M_k).$$

Then

$$U'(\phi) = \sum_{\Sigma \in M_k} \sum_{P \supset \Sigma} [\sum_{\alpha,\beta,\gamma} D_t^{\alpha}\{[\omega_{\alpha,\beta,\gamma}(l) - \tilde{\omega}_{\alpha,\beta,\gamma}(l)]dl^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]_{[P]}\} j_P^* \phi]$$
$$= \sum_{P \supset \Sigma} [\sum_{\alpha,\beta,\gamma} D_t^{\alpha}\{[\omega_{\alpha,\beta,\gamma}(l) - \tilde{\omega}_{\alpha,\beta,\gamma}(l) dl^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]_{[P]}\}\phi]$$
$$= \sum_{\alpha,\beta,\gamma} D_t^{\alpha}\{[\omega_{\alpha,\beta,\gamma}(l) - \tilde{\omega}_{\alpha,\beta,\gamma}(l)]dl^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]_{[P]}\}(\frac{1}{\alpha!} \cdot t^{\alpha} \cdot \tilde{F}(l,t)\theta_{\beta,\gamma})$$
$$= (-1)^{|\alpha|} \cdot [\Sigma]\{\sum_{\alpha} \eta_{\alpha,\beta,\gamma}(l) \cdot D_t^{\alpha}(\frac{1}{\alpha!} \cdot t^{\alpha} \cdot \tilde{F}(l,t))\mu_k\}$$
$$= (-1)^{|\alpha|} \cdot \int_{\Sigma} \eta_{\alpha,\beta,\gamma}(l,0) \cdot F(l) d\Sigma \neq 0.$$

This contradiction completes the proof of Proposition 2.2.6.

Remark 2.4.1: The same result holds for any choice of transverse coordinates which satisfy the conditions of Remark 2.3.1.

Remark 2.4.2: If $U \in C^*(M_k)$ and (P, Σ) are a pair of polysimplices with $P \supset \Sigma$, then $U_{P,\Sigma}$ can be determined as follows: By induction on the dimension of Σ, define $\tilde{U}_{P,\Sigma}$ by

$$\tilde{U}_{P,\Sigma}(\phi) = U(j_P^*\phi) - \sum_{\Sigma' \subset \Sigma} \tilde{U}_{P,\Sigma'}(j_P^*\phi) \text{ for } \phi \in \Omega^{km-p}(M_k).$$

Then, by induction on dimension of P, define

$$U_{P,\Sigma} = \tilde{U}_{P,\Sigma}(j_P^*\phi) - \sum_{P' \subset P} U_{P',\Sigma}(j_{P'}^*\phi).$$

This shows that $U_{P,\Sigma}$ can be determined without using the coordinate description given in Definition 2.2.1.

Definition 2.4.3: For integers r, let $C^r(P : \Sigma)$ denote the subspace of $C^r(\Sigma)$ consisting of those currents associated to the polysimplex P with carrier Σ by the procedure of definition 2.2.1.

Definition 2.4.4: For integers r, $C^r(P;\Sigma)$ will denote the vector space

$$C^r(P;\Sigma) = \{U \in C^r(\Sigma) \mid U = U \circ j_P^*\}.$$

$C^*(P;\Sigma)$ will denote the complex

$$C^*(P;\Sigma) = \bigoplus_{r \in Z} C^r(P;\Sigma).$$

Definition 2.4.5: For integers r, $C^r(P;\Sigma,\partial\Sigma)$ will denote to be the quotient space

$$C^r(P;\Sigma,\partial\Sigma) = C^r(P;\Sigma)/C^r(P;\partial\Sigma)$$

and $C^*(P;\Sigma,\partial\Sigma)$ will denote the direct sum

$$C^*(P;\Sigma,\partial\Sigma) = \bigoplus_{r \in Z} C^r(P;\Sigma)/C^r(P;\partial\Sigma).$$

Remark 2.4.6: From the explict formula for the deRham boundary b, it follows that each of the groups $C^*(P;\Sigma,\partial\Sigma)$ becomes a complex under the action of the deRham boundary. If $P' \subset P$ is a subpolysimplex of P which contains Σ, then the image of $C^*(P' : \Sigma)$ in $C^*(P;\Sigma,\partial\Sigma)$ is a subcomplex, denoted $C^*(P' : \Sigma, \partial\Sigma)$, and there is a bijection $C^*(P' : \Sigma) \to C^*(P : \Sigma, \partial\Sigma)$. With these definitions,

$$C^*(P; \Sigma, \partial\Sigma) = \bigoplus_{\Sigma \subset P' \subset P} C^*(P' : \Sigma, \partial\Sigma).$$

In the next chapter, the complex of generalized piecewise differentiable currents will be defined, and some of its basic properties will be explored.

Chapter 3

GENERALIZED PIECEWISE DIFFERENTIABLE CURRENTS

A generalized piecewise differential current on M_k is an equivalence class of piecewise differential currents on M_k, where the equivalence relation between currents is that their difference "vanishes" on the diagonal. This chapter will be devoted to characterizing the elements of subcomplex of $C^*(M_k)$ corresponding to the zero element. The chapter closely parallels the ideas introduced by Teleman in [T_2] with the exception of the notion of combinatoral transversality. The notion of combinatorial transversality will be defined below.

3.1 Definitions and Elementary Properties

We begin by making the above ideas more precise with the following definitions:

Let ∇_M denote the diagonal in M_k, i.e.

$$\nabla_M = \{(x, \cdots, x) \in M \times \cdots \times M = M_k \mid x \in M\}.$$

Definition 3.1.1: Let $i : \nabla_M \to M_k$ be the inclusion map. $I^*(M_k)$ will denote the subcomplex of $\Omega^*(M_k)$ defined by

$$I^*(M_k) = \{\omega \in \Omega^*(M_k) \mid i^*\omega = 0\}.$$

Definition 3.1.2: For any integer r, $\nabla^r(M_k)$ will denote the subspace

$$\nabla^r(M_k) = \bigoplus_{p+q=r} I^p(M_k) \cdot C^q(M_k),$$

and $\nabla^*(M_k) \subset C^*(M_k)$ will denote the subcomplex

$$\nabla^*(M_k) = \bigoplus_{r \in Z} \nabla^r(M_k).$$

Definition 3.1.3: For any integer r, $\mathcal{C}^r(M_k)$ will denote the quotient space

$$\mathcal{C}^r(M_k) = C^r(M_k)/\nabla^r(M_k),$$

and $\mathcal{C}^*(M_k)$ will denote the quotient complex

$$\mathcal{C}^*(M_k) = (\bigoplus_{r \in Z} \mathcal{C}^r(M_k), b)$$

where b is the deRham boundary.

An element of $\mathcal{C}^*(M_k)$ will be called a **generalized piecewise differentiable current** on M_k.

Definition 3.1.4: For subcomplexes K of M_k and integers r, the following standard notation is adopted. $C^r(K)$ will denote the subspace of $C^r(M_k)$ consisting of the currents with carriers in the subcomplex K, $\nabla^r(K)$ will denote

$$\nabla^r(K) = C^r(K) \cap \nabla^r(M_k)$$

and $\mathcal{C}^r(K)$ will denote

$$\mathcal{C}^r(K) = C^r(K)/\nabla^r(K).$$

$C^*(K)$, $\nabla^*(K)$, $\mathcal{C}^*(K)$ will denote the subcomplexes of $C^*(M_k)$, $\nabla^*(M_k)$, $\mathcal{C}^*(M_k)$, consisting of those currents with carriers contained in K.

Definition 3.1.5: If (K, L) is a pair of complexes with $K \supset L$, then for integers r,

$$C^r(K, L) = C^r(K)/C^r(L)),$$

$$\nabla^r(K, L) = \nabla^r(K)/\nabla^r(L)),$$

$$\mathcal{C}^r(K, L) = \mathcal{C}^r(K)/\mathcal{C}^r(L))$$

will denote the quotient spaces, and

$$C^*(K, L) = C^*(K)/C^*(L),$$

$$\nabla^*(K, L) = \nabla^*(K)/\nabla^*(L)$$

$$\mathcal{C}^*(K, L) = \mathcal{C}^*(K)/\mathcal{C}^*(L)$$

will denote the quotient complexes.

Remark 3.1.6: The complex $\mathcal{C}^*(M_k)$ is given the structure of a $\Omega^*(M)$-module in the following manner: for any $\omega \in \Omega^*(M)$, for any $U \in C^r(M_k)$, and for any integer $1 \leq i \leq k$, let

$$\omega_i = 1 \otimes \cdots \otimes 1 \otimes \omega \otimes 1 \otimes \cdots \otimes 1 \in \Omega^*(M_k),$$

with ω as the i^{th}-factor, and define

$$\omega \wedge U = \omega_i \wedge U \ \mathrm{mod}\nabla^r(M_k).$$

The class of this current does not depend on the choice of i as $\omega_i - \omega_j \in I^*(M_k)$.

The remainder of this chapter will be to characterize the elements of $\nabla^*(M_k)$. The first step is to characterize the elements of $I^*(M_k)$.

3.2 Characterization of the Elements of $I^*(M_k)$

We determine the elements of $I^*(M_k)$ by determining $j_\Pi^* I^*(M_k)$ for maximal polysimplices $\Pi = \pi_1 \times \cdots \times \pi_k$ in M_k, where $j_\Pi : \Pi \to M_k$ is the inclusion.

Proposition 3.2.1:

(1) If $k = 1$, $I^*(M_k) = 0$.

(2) If $k \geq 2$, $I^*(M_k)$ is the ideal generated by

$$\{w_j^a, dw_j^a \mid w_j^a = b_j^a - b_1^a \ , 2 \leq j \leq k \ , v^a \in K^0(M)\}.$$

Furthermore, for every vertex $v^a \in M$, one is free to choose an index $h(a)$ with $1 \leq h(a) \leq k$, and use $b_{h(a)}^a$ in place of b_1^a in the above characterization of the elements w_j^a, i.e., $I^*(M_k)$ is also the

ideal generated by

$$\{w_j^a, dw_j^a \mid w_j^a = b_j^a - b_{h(a)}^a, v^a \in K^0(M) \ , 1 \leq j \leq k \ , j \neq h(a)\}.$$

Proof: (1) is immediate. Assume $k \geq 2$.

Let $w_j^a : M_k \to [0,1]$ for $2 \leq j \leq k$, $v^a \in K^0(M)$, be defined by

$$w_j^a = b_j^a - b_1^a.$$

As the functions w_j^a vanish on the diagonal ∇_M, it follows that w_j^a and dw_j^a lie in the ideal $I^*(M_k)$ for all (j,a).

Remark: For any a, and for any $1 \leq i,j \leq k$ with $i \neq j$,

$$w_i^a - w_j^a = b_i^a - b_j^a,$$

and from this it follows that the collection

$$\{w_j^a, dw_j^a \mid w_j^a = b_j^a - b_1^a \ , 2 \leq j \leq k\}$$

generates the same ideal as the collection

$$\{w_j^a, dw_j^a \mid w_j^a = b_j^a - b_{h(a)}^a \ , 1 \leq j \leq k \ , j \neq h(a)\}$$

where $1 \leq h(a) \leq k$ is arbitrary.

Let $\Pi = \pi_1 \times \cdots \times \pi_k$ is a maximal polysimplex in M_k, and let $j_\Pi : \Pi \to M_k$ denote the inclusion.

For thoses polysimplices Π with $\Pi \cap \nabla_M \neq \emptyset$, let $\pi = \pi_1 \cap \cdots \cap \pi_k$ be represented by $(v^{a(0)}, \cdots, v^{a(p)})$ where $v^{a(0)}, \cdots, v^{a(p)}$ are the vertices of π. Then

$$\Pi \cap \nabla_M = \{(x, \cdots, x) \mid x \in \pi\}.$$

Let $O = (v^{a(0)}, \cdots, v^{a(0)})$ be an origin for Π, and let L' denote the corresponding set of longitudinal coordinates for Π.

Let

$$B = \{s^{a(h)} = l_1^{a(h)} \mid 1 \leq h \leq p\}.$$

Define $w_j^a = b_j^a - b_1^a$, for $1 \leq j \leq k$, all a except $a(0)$.

Let

$$A = \{w_j^a \mid w_j^a \mid_\Pi \neq 0\}.$$

Then the collection

$$\{w_j^a, s^a \mid w_j^a \in A, s^a \in B\}$$

forms a system of linear coordinates for Π.

Any C^∞-function $f(w, s)$ on Π which vanishes on ∇_M can be expressed as

$$f(w, s) = \sum_{w(j,a) \in A} w_j^a \cdot g_j^a(w, s)$$

for smooth functions $g_j^a(w, s)$ on Π. The functions g_j^a can be computed as follows (Cf. Milnor $[M_1]$, page 5): As $(0, 0) \in \nabla_M$, $f(0, 0) = 0$, and as Π is a convex set, it follows that

$$
\begin{aligned}
f(w,s) &= \int_0^1 \frac{d}{dt} f(tw,ts)\,dt \\
&= \int_0^1 \sum_{(j,a)\in A} \frac{\partial}{\partial w_j^a} f(tw,ts)\cdot w_j^a\,dt + \int_0^1 \sum_{j=1}^{p} \frac{\partial}{\partial s^j} f(tw,ts)\cdot s^j\,dt \\
&= \sum_{(j,a)\in A} w_j^a \int_0^1 \frac{d}{dw_j^a} f(tw,ts)\,dt
\end{aligned}
$$

The last equality follows as $f \equiv 0$ on $\Pi \cap \nabla_M$, which implies $\frac{\partial}{\partial s^j} f = 0$. Take

$$
g_j^a(w,s) = \int_0^1 \frac{d}{dw_j^a} f(tw,ts)\,dt
$$

Conversely, any function which can be expressed as

$$
f(w,s) = \sum_{w(j,a)\in A} w_j^a \cdot g(w,s)
$$

vanishes on the diagonal.

The differentials dw_j^a, for $w_j^a \in A$, vanish on the diagonal, and the differentials ds^a, $1 \le a \le p$ span the cotangent space of $\Pi \cap \nabla_M$. This implies that

$$
\phi = \sum_{\alpha,\beta} f_{\alpha,\beta}(w,s)\,dw^\alpha \wedge ds^\beta
$$

vanishes on the diagonal if, and only if, for each pair (α,β) either $f_{\alpha,\beta} \equiv 0$ on $\Pi \cap \nabla_M$, or $\alpha \neq 0$.

If $\Pi \cap \nabla_M = \emptyset$, then, for each vertex $v^a \in \pi_j$, there is an index i with $v^a \notin \pi_i$. Define w_j^a by $w_j^a = b_j^a - b_j^a$. Then $b_j^a = j_\Pi^* w_j^a$. Hence all differentials forms of degree > 0 belong to $j_\Pi^* I^*(M_k)$. Also, any function f is in the ideal $j_\Pi^* I^*(M_k)$, if, any only if, it is of the form

$$f(w) = \sum_{j,a} w_j^a g_{j,a}(w).$$

It is only necessary to take

$$g_j^a(w) = \int_0^1 \frac{d}{dw_j^a} f(tw)dt.$$

This implies $j_\Pi^* I^*(M_k)$ is the ideal generated by the functions and 1-forms $\{w_j^a, dw_j^a \mid w_j^a \in A\}$.

As the above holds for all maximal polysimplices Π of M_k, it follows that $I^*(M_k)$ is the ideal generated by

$$\{w_j^a, dw_j^a \mid w_j^a = b_j^a - b_1^a \ , 2 \leq j \leq k \ , v^a \in K^0(M)\}.$$

This completes the proof of Proposition 3.2.1.

The next step is to characterize the elements of $\nabla^*(M_k)$. The characterization of the elements will depend on the notion of combinatorial transversality, which will be developed in the next section.

3.3 Combinatorial Transversality

Definition 3.3.1: Let Δ be a linear simplex contained in some polysimplex of M_k, and let Σ, P be a pair of polysimplices in M_k with $\Sigma \subset P$. The pair

(P, Σ) is **combinatorially transverse** to Δ, written

$$(P, \Sigma) \pitchfork \Delta$$

in M_k if, and only if, $\Sigma \cap \Delta \neq \emptyset$, and every linear function $y : P \to R$ with $y(\Sigma \cup \Delta) = 0$ vanishes identically on P.

Definition 3.3.2: A polysimplex Σ is **combinatorially transverse** to Δ, written

$$\Sigma \pitchfork \Delta,$$

if $(P, \Sigma) \pitchfork \Delta$ for every maximal polysimplex P containing Σ.

Remark 3.3.3: If P is a polysimplex containing Σ, then $(P, \Sigma) \pitchfork \nabla_M$ and $P \pitchfork \nabla_M$ does not imply $\Sigma \pitchfork \nabla_M$. Nevertheless, for any polysimplex $P \supset \Sigma$, it follows that $\Sigma \not\pitchfork \nabla_M$ implies that either $P \not\pitchfork \nabla_M$ or $(P, \Sigma) \not\pitchfork \nabla_M$.

Definition 3.3.4: For a polysimplex $P = \rho_1 \times \cdots \times \rho_k$ of M_k, and an integer j with $1 \leq j \leq k$, let $\mathcal{A}(P, j)$ be the set of all vertices v in $\operatorname{star}(\rho_j)$ satisfying

1. $v \notin \rho_j$, and
2. $v \in \rho_1 \cap \cdots \cap \rho_{j-1} \cap \rho_{j+1} \cap \cdots \cap \rho_k$.

$\mathcal{A}(P, j)$ will be called the set of **admissible transverse vertices** associated to the pair (P, j).

Remark 3.3.5: As $\rho_j \cap \mathcal{A}(P, j) = \emptyset$ and $\mathcal{A}(P, j) \subset \rho_i$ for $i \neq j$, it follows that the sets $\mathcal{A}(P, j)$ are disjoint.

Remark 3.3.6: As $\mathcal{A}(P,j) \subset \rho_1 \cap \cdots \cap \rho_{j-1} \cap \rho_{j+1} \cap \cdots \cap \rho_k$, the set $\mathcal{A}(P,j)$ determines a simplex in M.

Definition 3.3.7: Let ρ be a simplex in M. Let $\mathcal{B}(\rho)$ denote

$$\mathcal{B}(\rho) = \{v \in K^0(M) - K^0(\rho) \mid v \in K^0(\pi) \text{ for } \pi \supset \rho \text{ , } \pi \in K^m(M)\}.$$

The next proposition relates the definition of combinatorial transversality (of a polysimplex $P = \rho_1 \times \cdots \times \rho_k$ to the diagonal ∇_M) with the geometric condition that all nearby vertices to the simplices ρ_j are admissible transverse vertices, where nearby means that the vertex lies in $\mathcal{B}(\rho_j)$.

Proposition 3.3.8: A polysimplex $P = \rho_1 \times \cdots \times \rho_k$ in M_k, of positive codimension, is transverse to ∇_M if, and only if,

$$\mathcal{B}(\rho_j) \subset \mathcal{A}(P,j) \text{ , for } 1 \leq j \leq k.$$

Proof: As $P \cap \nabla_M \neq \emptyset$, $P \cap \nabla_M = \{(x, \cdots, x) \mid x \in \rho\}$ where $\rho = \rho_1 \cap \cdots \cap \rho_k$. As P is not a maximal polysimplex, some ρ_j is not maximal. Assume ρ_1 is not maximal. Let v be a vertex of M such that there exists a maximal simplex π_1 of M which contains both ρ_1 and v. Let $\pi_2, \cdots, \pi_k$ be any set of maximal simplices of M with $\pi_j \supset \rho_j$ for $j = 2, \cdots, k$. Let $\Pi = \pi_1 \times \cdots \times \pi_k$.

Let t be the barycentric coordinate function on π_1 with $t(v) = 1$, and extend t to all of Π by $t(X) = t(\text{ proj}_1(X))$, where $X \in \Pi$ and proj_1 is projection onto the first factor. Then $t(P) = 0$, and $(\Pi, P) \amalg \nabla_M$ implies there exists a point $X = (x, \cdots, x) \in \nabla_M$ with $t(X) \neq 0$. This implies $t(X) \neq 0$ where t is a function on π_1, and note $x \in \rho$. As ρ is a simplex which contains a point x with $t(x) \neq 0$, ρ contains v. This implies $(v, \cdots, v) \in \nabla_M$, and hence $v \in \pi_2 \cap \cdots \cap \pi_k$.

Since this holds for all π_j with $\pi_j \supset \rho_j$, $j = 2, \cdots, k$, it follows that $v \in \rho_2 \cap \cdots \cap \rho_k$. Hence v lies in the required set $\mathcal{A}(P, 1)$. As this is true for all v in any maximal simplex π_1 which contains ρ_1, and as the argument can be repeated for the other simplices ρ_j, it follows that any maximal polysimplex Π which contains P is of the required form.

For the converse, suppose $\mathcal{B}(\rho_j) \subset \mathcal{A}(P, j)$. It will be shown that any choice of codim(ρ_j) elements of $\mathcal{A}(P, j)$ which completes ρ_j to a maximal simplex π_j in M, produces a maximal polysimplex $\Pi = \pi_1 \times \cdots \times \pi_k$ such that (Π, P) is transverse to ∇_M.

This follows by induction. Assume (Π', P) is transverse to ∇_M and that Π' has been constructed from P by steps consisting of joining vertices from $\mathcal{A}(P, j)$ to ρ_j. Let Π be the polysimplex obtained from Π' by the addition of a vertex from $\mathcal{A}(P, j)$ to some $\pi_j of \Pi'$. Then $\dim \Pi - \dim P = 1$, and $\dim(\Pi \cap \nabla_M) = \dim(\Pi' \cap \nabla_M) + 1$.

Let L be a linear function on Π with $L(P) = 0$ and $L(\nabla_M) = 0$. Then $L(\Pi') = 0$. Since Π is a polysimplex containing Π' of codimension one, any linear function on Π is determined by its value on Π' and one other point in Π. However, the diagonal in Π is not contained in Π', and the value of L on the diagonal is zero. This implies $L = 0$ on Π. The assumption $\mathcal{B}(\rho_j) \subset \mathcal{A}(P, j)$ guarentees that any maximal polysimplex Π which contains

P can be obtained by the above procedure. This completes the induction step, and the proposition.

Example 3.3.9: Let $M^2 = \partial\Delta^3$, where $\Delta^3 = (v^0, v^1, v^2, v^3)$. Let $P = \rho_1 \times \cdots \times \rho_k$, with $\rho_1 = (v^0, v^3)$, $\rho_2 = \cdots = \rho_k = (v^0, v^1, v^2)$. If $\Pi = \pi_1 \times \cdots \times \pi_k$ is a polysimplex with $\pi_j = \rho_j$ for $2 \leq j \leq k$, and $\pi_1 \supset \rho_1$, then the pair $(P, \Pi) \amalg \nabla_M$ for all choices of maximal simplices π_1 containing ρ_1, i.e. by taking $\pi_1 = (v^0, v^1, v^2)$, or by taking $\pi_1 = (v^0, v^2, v^3)$. This shows $P \amalg \nabla_M$.

Remark 3.3.10: The proof of the above proposition also shows $(P, \Sigma) \amalg \nabla_M$ for any polysimplex P which can be constructed from Σ by taking $P = \rho_1 \times \cdots \times \rho_k$, where ρ_j is any simplex in M of the form $\sigma_j * v^1 * \cdots * v^n$ where $v^i \in \mathcal{A}(\Sigma, j)$ for $1 \leq i \leq m - \dim(\sigma_j)$.

3.4 Transverse Polysimplices and Currents

Let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, and let P be a polysimplex in M_k containing Σ. $C^*(P : \Sigma, \partial\Sigma)$ denotes the subcomplex of $C^*(\Sigma, \partial\Sigma)$ given by

$$\{\sum_{\alpha,\beta} D_t^\alpha\{\omega_{\alpha,\beta} \wedge dt^\beta \wedge [\Sigma]_P\} \circ j_P^* \mid (\alpha, \beta) \text{ is assigned to } P\}.$$

Definition 3.4.1: $\nabla^*(P : \Sigma, \partial\Sigma)$ will denote the subcomplex of $\nabla^*(\Sigma, \partial\Sigma)$ given by

$$\nabla^*(P : \Sigma, \partial\Sigma) = \nabla^*(\Sigma, \partial\Sigma) \cap C^*(P : \Sigma, \partial\Sigma).$$

The next proposition shows that

$$\nabla^*(P : \Sigma, \partial\Sigma) = C^*(P : \Sigma, \partial\Sigma)$$

unless $(P, \Sigma) \amalg \nabla_M$ and $P \amalg \nabla_M$.

Proposition 3.4.2: Let P be a polysimplex in M_k containing Σ.

(1) If $\Sigma \cap \nabla_M = \emptyset$, then $\nabla^*(\Sigma, \partial\Sigma) = C^*(\Sigma, \partial\Sigma)$.

(2) If $P \not\amalg \nabla_M$, then $\nabla^*(P : \Sigma, \partial\Sigma) = C^*(P : \Sigma, \partial\Sigma)$.

(3) If $(P, \Sigma) \not\amalg \nabla_M$, then $\nabla^*(P : \Sigma, \partial\Sigma) = C^*(P : \Sigma, \partial\Sigma)$.

Proof: Let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, and let P be a polysimplex in M_k containing Σ.

We first consider the case $\Sigma \cap \nabla_M = \emptyset$. For $j = 1, \cdots, k$, let U_j be an open containing σ_j such that there exists a piecewise linear embedding $\psi_j : U_j \to R^m$ (linear on $U_j \cap$ (a simplex of M)). Let $U = U_1 \times \cdots \times U_k$, and define $\Psi : U \to R^{km}$ by

$$\Psi(x_1, \cdots, x_k) = (\psi_1(x_1), \cdots, \psi_k(x_k)).$$

Let $V \subset U$ be an open set contains Σ and is disjoint from the diagonal ∇_M. Let $\phi \in C_0^\infty(\Psi(V))$ with $\phi \equiv 1$ on $\Psi(\Sigma)$.

Let $g : M_k \to R^{km}$ be defined by

$$g(x) = \phi(\Psi(x)) \text{ for } x \in V,$$

$$g(x) = 0 \text{ for } x \notin V.$$

Then $g \in I^0(M_k)$, and $g \mid_\Sigma \equiv 1$.

As $U(\phi) = U(g\phi)$ for all $\phi \in \Omega^*(M_k)$ and for all $U \in C^*(\Sigma, \partial\Sigma)$, it follows that

$$C^*(\Sigma, \partial\Sigma) = \nabla^*(\Sigma, \partial\Sigma) \text{ if } \Sigma \cap \nabla_M = \emptyset.$$

This proves (1).

We now consider the cases with $\Sigma \cap \nabla_M \neq \emptyset$.

Let $U_P \in C^*(\Sigma, \partial\Sigma)$ be represented as

$$U_P = \sum_\alpha D_t^\alpha \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^*,$$

where ω_α are smooth forms on P and $j_P : P \to M_k$ is the inclusion.

Proof of (2): If $P \not\amalg \nabla_M$, then there exists an integer j, $1 \leq j \leq k$, and a vertex $v^a \in \mathcal{B}(\rho_j) - \mathcal{A}(P, j)$. Let $\Pi = \pi_1 \times \cdots \times \pi_k$ be a maximal polysimplex in M_k with $\pi_j \supset \rho_j * v^a$. Let $\tau = b_j^a = t_j^a$ as b_j^a is a transverse coordinate to $\Sigma \subset P$. Then $\tau(P) = 0$, and $\tau(P \cap \nabla_M) = 0$. If $(v^a, \cdots, v^a) \notin \Pi \cap \nabla_M$ take $y = \tau$. If $(v^a, \cdots, v^a) \in \Pi \cap \nabla_M$ then $v^a \in \mathcal{B}(\rho_i)$ for all i, and $v^a \notin \rho_i$ for some $i \neq j$, as $v^a \notin \mathcal{A}(P, j)$. In this case, take $y = \tau - b_i^a$. Then $y(P) = 0$, $y(\Pi \cap \nabla_M) = 0$, $y \not\equiv 0$, and $D_\tau y = 1$. Hence for

$$U_P = D_t^\alpha \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \in C^*(\Sigma, \partial\Sigma)$$

this implies

$$\begin{aligned} y \cdot D_\tau U_P &= y \cdot D_\tau D_t^\alpha \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ &= D_\tau D_t^\alpha \{y \cdot \omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* - D_t^\alpha \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ &= -U_P. \end{aligned}$$

The last equality follows as

$$D_\tau D_t^\alpha \{y \cdot \omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^*$$

is the current which results in integrating the zero function on Σ as y vanishes on Σ.

As y vanishes on the diagonal, $y \cdot D_\tau U_P \in \nabla^*(\Sigma, \partial\Sigma)$. This implies $-U_P \in \nabla^*(\Sigma, \partial\Sigma)$. Hence

$$C^*(P : \Sigma, \partial\Sigma) \subset \nabla^*(\Sigma, \partial\Sigma).$$

This completes case (2) and a similar argument can be made for case (3).

Assume $(P, \Sigma) \not\amalg \nabla_M$. By Remark 3.3.10, P is not obtained from Σ by adjoining admissible transverse vertices. Hence there exists a transverse barycentric coordinate $\tau = t_j^a$ to Σ in P, and $\tau(\Sigma) = 0$, $\tau(\nabla_M \cap P) = 0$. For any maximal simplex Π in M_k which contains P, choose y as above. Express U_P as

$$U_P = D_t^{\alpha'} D_\tau^p \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^*,$$

where $p = \alpha_j^a$, and $\alpha_i'^b = \alpha_i^b$, $(i, b) \neq (j, a)$ and $\alpha_j'^a = 0$. This implies

$$\begin{aligned} y \cdot D_\tau U_P = & y \cdot D_t^{\alpha'} D_\tau^{p+1} \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ & = D_t^{\alpha'} D_\tau^{p+1} \{y \cdot \omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ & - (p+1) \cdot D_t^\alpha D_\tau^p \{\omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ & = -(p+1) U_P. \end{aligned}$$

Note $D_t^{\alpha'} D_\tau^{p+1} \{y \cdot \omega_\alpha \wedge [\Sigma]_{[P]}\} \circ j_P^* = 0$ as before.

As y vanishes on the diagonal, y is in $I^0(P)$. This implies the current $-(p+1) \cdot U_P = y \cdot D_\tau U_P$ lies in $\nabla^*(\Sigma, \partial\Sigma)$.

This shows $C^*(P : \Sigma, \partial\Sigma) \subset \nabla^*(\Sigma, \partial\Sigma)$, and completes the proof of Proposition 3.4.2.

Corollary 3.4.3: Let Σ be a polysimplex in M_k. If $\Sigma \not\pitchfork \nabla_M$, then $\nabla^*(\Sigma, \partial\Sigma) = C^*(\Sigma, \partial\Sigma)$.

Proof: As $\Sigma \not\pitchfork \nabla_M$ implies for polysimplex $P \supset \Sigma$ either $P \not\pitchfork \nabla_M$ or $(P, \Sigma) \not\pitchfork \nabla_M$, this is an immediate consequence of Proposition 3.4.2.

3.5 Characterization of the Elements of $\nabla^*(M_k)$

By corollary 3.4.3, if $\Sigma \not\pitchfork \nabla_M$, then $\nabla^*(\Sigma, \partial\Sigma) = C^*(\Sigma, \partial\Sigma)$. We now consider the case $\Sigma \pitchfork \nabla_M$.

Suppose $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ is transverse to the diagonal, and set $\sigma = \sigma_1 \cap \cdots \cap \sigma_k$. By Proposition 3.4.2, any polysimplex $P \supset \Sigma$ is of the form $P = \rho_1 \times \cdots \times \rho_k$ with $\rho_j = \sigma_j$ or $\rho_j = \sigma_j * v^a$ with $v^a \in \mathcal{A}(\Sigma, j)$. As the sets $\mathcal{A}(\Sigma, j)$ are disjoint, the index a determines the index j. Let $j(a)$ denote the index j corresponding to $v^a \in \mathcal{A}(\Sigma, j)$.

To obtain a useful representation of the elements of $\nabla^*(P; \Sigma, \partial\Sigma)$, it is useful to introduce as coordinate system similar to the one used by Teleman in $[T_2]$. Choose an origin $O = (v, \cdots, v) \in \Sigma \cap \nabla_M$ for Σ, and adopt a coordinate system for P consisting of the coordinate functions

$l^a = l_1^a$ for indices a with $v^a \in K^0(\sigma) - \{v\}$,

$t^a = t_j^a$ for indices j, a with $j = j(a)$, and $v^a \in \rho_j - \sigma_j$, and

$w_j^a = l_j^a - b_{j(a)}^a$ for indices j, a with $j \neq j(a)$

where $b_{j(a)}^a = b_1^a = l^a$ if $v^a \in \sigma$, and $b_{j(a)}^a = t^a$ if $v^a \notin \sigma$.

Remark 3.5.1: As $l^a = l_1^a$, $t^a = t_{j(a)}^a$, and $w_j^a = l_j^a - b_{j(a)}^a$, it follows that

$$dl^a = dl_1^a \ , \ dt^a = dt_{j(a)}^a \ , \text{ and } dw_j^a = dl_j^a - db_{j(a)}^a.$$

Using

$$l_1^a = l^a \text{ for } v^a \in \sigma,$$

$$l_j^a = w_j^a + l^a \text{ (for } v^a \in \sigma \text{)},$$

$$l_j^a = w_j^a + t^a \text{ (for } v^a \notin \sigma \text{)},$$

$$t_{j(a)}^a = t^a \text{ for } v^a \in \rho_j,$$

the following formulas for the partial derivatives are obtained.

$$\frac{\partial}{\partial l^a} = \sum_{b \in T} \frac{\partial t^b}{\partial l^a} \frac{\partial}{\partial t^b} + \sum_{(j,b) \in L'} \frac{\partial l_j^b}{\partial l^a} \frac{\partial}{\partial l_j^b} = \frac{\partial}{\partial l_1^a} + \sum_{j=2}^{k} \frac{\partial}{\partial l_j^a} \text{ for } v^a \in \sigma$$

$$\frac{\partial}{\partial t^a} = \sum_{b \in T} \frac{\partial t^b}{\partial t^a} \frac{\partial}{\partial t^b} + \sum_{(j,b) \in L'} \frac{\partial l_j^b}{\partial t^a} \frac{\partial}{\partial l_j^b} = \frac{\partial}{\partial t_{j(a)}^a} + \sum_{j \neq j(a)} \frac{\partial}{\partial l_j^a} \text{ for } v^a \notin \sigma$$

$$\frac{\partial}{\partial w_i^a} = \sum_{b \in T} \frac{\partial t^b}{\partial w_i^a} \frac{\partial}{\partial t^b} + \sum_{(j,b) \in L'} \frac{\partial l_j^b}{\partial w_i^a} \frac{\partial}{\partial l_j^b} = \frac{\partial}{\partial l_i^a} \text{ for all} w_i^a$$

where $\sum_{b \in T}$ means the sum over all barycentric coordinates which are transverse to Σ in P, and where $\sum_{(j,b) \in L'}$ means the sum over all longitudinal barycentric coordinates to Σ in the coordinate system determined by the choice of origin O for Σ.

Remark 3.5.2: As $\partial/\partial l^a$ and $\partial/\partial w_i^a$ are linear combinations of the partial derivatives $\partial/\partial l_j^a$ of the longitudinal coordinates l_j^a, it follows (as in 1.8.24) any current in $C^*(\Sigma)$ can be represented by linear combinations of currents of the form $D_t^\alpha\{\omega \wedge dt^\beta \wedge [\Sigma]\} \circ j_P^*$ where

$$\omega = \sum_{\gamma,\delta} \omega_{\gamma,\delta}(l,w) dw^\delta \wedge dl^\gamma \in \Omega^*(P)$$

and P is a maximal simplex in M_k containing Σ.

Remark 3.5.3: If $\Sigma \subset P \subset P'$ are polysimplices in M_k with $\Sigma \amalg \nabla_M$, then the coordinate system (w,l,t) on P agrees with the coordinate system on $P \subset P'$ obtained by eliminating the transverse coordinates to P in P'. This implies, by remarks 2.3.1 and 2.4.1, currents $U \in C^*(\Sigma)$ can be uniquely expressed as $U = \sum_{P\supset\Sigma} U_{P,\Sigma}$ with

$$U_{P,\Sigma} = \sum_{\alpha,\beta,\gamma,\delta} D_t^\alpha\{\omega_{\alpha\beta\gamma\delta}(w,l) dw^\delta \wedge dl^\gamma \wedge dt^\beta \wedge [\Sigma]_{[P]}\} \circ j_P^*$$

where P is the polysimplex determined by the pair (α,β).

Remark 3.5.4: If P is the polysimplex determined by the pair (α,β), then P is the smallest polysimplex such that $U_{P,\Sigma} = U_{P,\Sigma} \circ j_P^*$. This follows as the smallest polysimplex $P' = \rho_1' \times \cdots \times \rho_k'$ with the property that $U_{P,\Sigma} = U_{P,\Sigma} \circ j_{P'}^*$ must satisfy the following conditions for each vertex $v^a \in \rho_j' - \sigma_j$.

(1) If D_t^α contains $\partial/\partial t_j^a$, then $v^a \in \rho_j'$.

(2) If D_t^α does not contain $\partial/\partial t_j^a$ and dt^β contains dt_j^a, then $v^a \notin \rho'_j$.

(3) If D_t^α does not contain $\partial/\partial t_j^a$ and dt^β does not contain dt_j^a, then $v^a \in \rho'_j$.

These are exactly the conditions defining the polysimplex P.

Hence, $P = P'$.

Remark 3.5.5: Let P be a polysimplex in M_k containing Σ, and let $n = \dim(\sigma)$ where $\sigma = \sigma_1 \cap \cdots \cap \sigma_k$. Let $N = \dim(\Sigma) - n$, and let $n' = \dim(P) - \dim(\Sigma)$.

To simplify notation, let

$$d_w = \sum_{i=1}^{N} dw^i \wedge \partial/\partial w^i$$

$$d_l = \sum_{i=1}^{n} dl^i \wedge \partial/\partial l^i$$

$$d_t = \sum_{i=1}^{n'} dt^i \wedge \partial/\partial t^i,$$

then $d = d_w + d_l + d_t$.

For $U_{P,\Sigma} \in C^p(P;\Sigma,\partial\Sigma)$ of the form $U_{P,\Sigma} = D_t^\alpha\{\omega \wedge dt^\beta \wedge [\Sigma]\}$,

$$\begin{aligned} dU_{P,\Sigma} =& (-1)^p \sum_{j=1}^{n'} D_{tj} D_t^\alpha\{\omega \wedge dt^\beta \wedge dt^j \wedge [\Sigma]\} \\ &+ D_t^\alpha\{d_w\omega \wedge dt^\beta \wedge [\Sigma]\} \\ &+ D_t^\alpha\{d_l\omega \wedge dt^\beta \wedge [\Sigma]\} \end{aligned}$$

in $C^{p+1}(\Sigma, \partial\Sigma)$ as the terms $d_l[\Sigma]$ and $d_w[\Sigma]$ involve partial derivative longitudinal to Σ, and this implies these terms give currents in $C^{p+1}(\partial\Sigma)$. The corresponding formula for bU in $C^{p+1}(\Sigma, \partial\Sigma)$ is

$$\begin{aligned} bU_{P,\Sigma} = &- \sum_{j=1}^{n'} D_{tj} D_t^\alpha\{\omega \wedge dt^\beta \wedge dt^j \wedge [\Sigma]\} \\ &+ (-1)^{p+1} \cdot D_t^\alpha\{d_w\omega \wedge dt^\beta \wedge [\Sigma]\} \\ &+ (-1)^{p+1} \cdot D_t^\alpha\{d_l\omega \wedge dt^\beta \wedge [\Sigma]\}. \end{aligned}$$

Definition 3.5.6: If P, Σ are polysimplices in M_k with $\Sigma \subset P$ and $\Sigma \amalg \nabla_M$, then $\overline{C}^*(P : \Sigma, \partial\Sigma)$ will denote the subcomplex in $C^*(\Sigma, \partial\Sigma)$ generated by currents of the form

$$U = D_t^\alpha\{\omega(w, l) dw^\delta \wedge dl^\gamma \wedge dt^\beta \wedge [\Sigma]\}$$

where P is the polysimplex determined by the pair (α, β). The formula from the above remark guarentees this is a complex.

Also define

$$\overline{\nabla}^*(P : \Sigma, \partial\Sigma) = \overline{C}^*(P : \Sigma, \partial\Sigma) \cap \nabla^*(\Sigma, \partial\Sigma),$$

$$\overline{\mathcal{C}}^*(P : \Sigma, \partial\Sigma) = \overline{C}^*(P : \Sigma, \partial\Sigma) / \overline{\nabla}^*(P : \Sigma, \partial\Sigma).$$

Remark 3.5.7: The ideal $I^*(M_k)$ is generated by $\{w_i^a, dw_i^a\}$ and as

$$w_i^a \cdot D_t^\alpha\{\omega \wedge dt^\beta \wedge [\Sigma]\} = D_t^\alpha\{w_i^a \cdot \omega \wedge dt^\beta \wedge [\Sigma]\}$$

$$dw_i^a \wedge D_t^\alpha\{\omega \wedge dt^\beta \wedge [\Sigma]\} = D_t^\alpha\{dw_i^a \wedge \omega \wedge dt^\beta \wedge [\Sigma]\},$$

it follows that $\overline{\nabla}^*(P:\Sigma,\partial\Sigma)\subset\overline{I}^*(P:\Sigma,\partial\Sigma)$ where $\overline{I}^*(P:\Sigma,\partial\Sigma)$ denotes the subcomplex of $\overline{C}^*(P:\Sigma,\partial\Sigma)$ given by

$$\overline{I}^*(P:\Sigma,\partial\Sigma)=\{\sum_{\alpha,\beta}D_t^\alpha\{\omega_{\alpha,\beta}\wedge dt^\beta\wedge[\Sigma]\}\in C^*(P:\Sigma,\partial\Sigma)\mid\omega\in j_P^*I^*(M_k)\}.$$

By reading the above formulas in the reverse order, one obtains $\overline{\nabla}^*(P;\Sigma,\partial\Sigma)\supset\overline{I}^*(P;\Sigma,\partial\Sigma)$. These results combine to give

Proposition 3.5.8: If a polysimplex Σ is combinatorially transverse to the diagonal ∇_M, then

$$\overline{\nabla}^*(P:\Sigma,\partial\Sigma)=\overline{I}^*(P:\Sigma,\partial\Sigma)$$

where $\overline{I}^*(P:\Sigma,\partial\Sigma)$ denotes the subcomplex of $\overline{C}^*(P:\Sigma,\partial\Sigma)$ given by

$$\{\sum_{\alpha,\beta}D_t^\alpha\{\omega_{\alpha,\beta}\wedge dt^\beta\wedge[\Sigma]\}\in C^*(P:\Sigma,\partial\Sigma)\mid\omega_{\alpha,\beta}\in j_P^*I^*(M_k)\}.$$

Remark 3.5.9: The above results imply

$$C^*(\Sigma,\partial\Sigma)=\nabla^*(\Sigma,\partial\Sigma)\text{ for }\Sigma\not\amalg\nabla_M,$$

$$C^*(\Sigma,\partial\Sigma)=\bigoplus_{\Sigma\subset P'\subset P}\overline{C}^*(P':\Sigma,\partial\Sigma)\text{ for }\Sigma\amalg\nabla_M,$$

$$\nabla^*(\Sigma,\partial\Sigma)=\bigoplus_{\Sigma\subset P'\subset P}\overline{I}^*(P':\Sigma,\partial\Sigma)\text{ for }\Sigma\amalg\nabla_M.$$

In the next chapter, three homotopies will be constructed on the complex $C^*(\Sigma,\partial\Sigma)$. Using the above description of the complex $\nabla^*(\Sigma,\partial\Sigma)$, it will be possible to show that these chain homotopies map $\nabla^*(\Sigma, \partial\Sigma)$ into itself. This will show that the chain homotopies induce chain homotopies on the quotient complex, i.e. on the complex $\mathcal{C}^*(\Sigma, \partial\Sigma)$. A spectral sequence calculation then gives the homology of the complex $\mathcal{C}^*(M_k)$.

Chapter 4

HOMOLOGY OF THE COMPLEX $C^*(Mk)$

This chapter will be devoted to computing the homology of the complex $C^*(M_k)$ by relating it to the real cellular homology of the real relative cellular chain complex $S_*(M_k, N_0)$ introduced in 1.8., where N_0 is the subcomplex of M_k consisting of the polysimplices which are not transverse to the diagonal. For integers p, $0 \leq p \leq m$,

$$H_p(C^*(M_k)) \cong H_{km-p}(M_k, N_0; R).$$

The other homology groups of the complex $C^*(M_k)$ are zero. We begin by using chain homotopies on the complex $C^*(\Sigma, \partial\Sigma)$ to show the homology of this complex agrees with the homology of a real relative cellular complex $S_*(\Sigma, \partial\Sigma)$ if Σ is transverse to the diagonal in M_k. The next section introduces the first chain homotopy.

4.1 The Chain Homotopy $\overline{H}_w$

For each polysimplex $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ in M_k with $\Sigma \amalg \nabla_M$, let $\sigma = \sigma_1 \cap \cdots \cap \sigma_k$, and $n = \dim(\sigma)$. As $\Sigma \cap \nabla_M \neq \emptyset$, choose an origin O for Σ with $O \in \nabla_M$.

For each polysimplex $P = \rho_1 \times \cdots \times \rho_k$ of M_k containing Σ let $\rho = \rho_1 \cap \cdots \cap \rho_k$, $n' = \dim(\rho)$, $N = \dim(P) - n - n'$, give P the coordinate system

$$(w^1, \cdots, w^N, l^1, \cdots, l^n, t^1, \cdots, t^{n'}),$$

introduced in Section 3.5.

To simplify notation, let

$$d_w = \sum_{i=1}^{N} dw^i \wedge \partial/\partial w^i$$

$$d_l = \sum_{i=1}^{n} dl^i \wedge \partial/\partial l^i$$

$$d_t = \sum_{i=1}^{n'} dt^i \wedge \partial/\partial t^i,$$

then $d = d_w + d_l + d_t$.

If $dw^\delta = dw^{\delta_1} \wedge \cdots \wedge dw^{\delta_p}$, then for $i = 1, \cdots, p$, let $dw^{\delta-\delta(i)}$ denote the form

$$dw^{\delta-\delta(i)} = dw^{\delta_1} \wedge \cdots \wedge dw^{\delta_{i-1}} \wedge dw^{\delta_{i+1}} \wedge \cdots \wedge dw^{\delta_p}.$$

Definition 4.1.1: Let $h_w : \Omega^*(P) \to \Omega^*(P)$ be defined on forms

$$\phi = \omega(w, l, t) dw^\delta \wedge dl^\gamma \wedge dt^\beta$$

by

$$h_w \phi = \sum_{i=1}^{|\delta|} (-1)^{i-1} \cdot w^{\delta(i)} \cdot \int_0^1 \lambda^{|\delta|-1} \omega(\lambda w, l, t) dw^{\delta-\delta(i)} \wedge dl^\gamma \wedge dt^\beta$$

on forms $\delta \neq 0$, and define

$$h_w\phi = 0,$$

on forms ϕ with $\delta = 0$, and extend h_w to $\Omega^*(P)$ linearly.

Remark 4.1.2: This is a homotopy operator on $\Omega^*(P)$ which gives a chain homotopy between the identity and the map

$$\phi \mapsto \omega(0,l) \wedge dl^\gamma \wedge dt^\beta$$

for $\delta \neq 0$, and $\phi \mapsto 0$ if $\delta = 0$.

Remark 4.1.3: Note that $d_l h_w + h_w d_l = 0$ and $d_t h_w + h_w d_t = 0$ as on $\phi \in \Omega^*(P)$ with

$$\phi = \omega(w,l,t) dw^\delta \wedge dl^\gamma \wedge dt^\beta$$

then

$$\begin{aligned} d_l h_w \phi &= \sum_{i=1}^{|\delta|} (-1)^{i-1} \cdot w^{\delta(i)} \cdot \int_0^1 \lambda^{|\delta|-1} \omega(\lambda w, l, t) dw^{\delta-\delta(i)} \wedge dl^\gamma \wedge dt^\beta \\ &= (-1)^{|\delta|-1} \sum_{j=1}^{n} \sum_{i=1}^{|\delta|} (-1)^{i-1} w^{\delta(i)} \int_0^1 \lambda^{|\delta|-1} \frac{\partial}{\partial l^i} \omega(\lambda w, l, t) dw^{\delta-\delta(i)} dl^j dl^\gamma dt^\beta \\ &= -(-1)^{|\delta|} \sum_{j=1}^{n} \sum_{i=1}^{|\delta|} (-1)^{i-1} w^{\delta(i)} \int_0^1 \lambda^{|\delta|-1} \frac{\partial}{\partial l^i} \omega(\lambda w, l, t) dw^{\delta-\delta(i)} dl^j dl^\gamma dt^\beta \\ &= -h_w d_l \phi. \end{aligned}$$

An analogous calculation proves the statement $d_t h_w + h_w d_t = 0$.

Definition 4.1.4: Let $H_w : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ be defined on currents of the form

$$U = D_t^\alpha \{\omega \wedge [\Sigma]_{[P]}\} \circ j_P^*$$

by

$$H_w U = D_t^\alpha\{h_w \omega \wedge [\Sigma]_{[P]}\} \circ j_P^*.$$

Let $\overline{H}_w : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ be defined by

$$\overline{H}_w(U) = (-1)^p H_w(U) \text{ for } U \in \overline{C}^*(P : \Sigma, \partial\Sigma).$$

Remark 4.1.5: As $\overline{H}_w$ does not alter the pair (α, β), it defines a map $\overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$.

Remark 4.1.6: If $U \in \overline{\nabla}^*(P : \Sigma, \partial\Sigma) = \overline{I}^*(P : \Sigma, \partial\Sigma)$ with $\delta \neq 0$, then $\overline{H}_w U \in \overline{\nabla}^*(P : \Sigma, \partial\Sigma)$ as ω_i contains a factor of $w^{\delta(i)}$. If $\delta = 0$, then $\overline{H}_w U = 0 \in \overline{\nabla}^*(P : \Sigma, \partial\Sigma)$. This implies $\overline{H}_w : \overline{\nabla}^*(P : \Sigma, \partial\Sigma) \to \overline{\nabla}^*(P : \Sigma, \partial\Sigma)$, and thus induces a well-defined operator $\tilde{H}_w$ on the quotient complex $\overline{C}^*(P : \Sigma, \partial\Sigma)$.

Remark 4.1.7: This is the analogue of the chain homotopy used by Teleman $[T_2]$ on the Koszul complex which has been adapted to act on the complex $\overline{C}^*(P : \Sigma, \partial\Sigma)$.

4.2 The Isomorphism Induced by $\overline{H}_w$

Definition 4.2.1: $\overline{Z}^*(P : \Sigma, \partial\Sigma) \subset \overline{C}^*(P : \Sigma, \partial\Sigma)$ will denote the subcomplex

$$\overline{Z}^*(P : \Sigma, \partial\Sigma) = \{\sum_{\alpha,\beta,\gamma} D_t^\alpha\{\omega_{\alpha,\beta,\gamma}(0,l) dl^\gamma \wedge dt^\beta \wedge [\Sigma]\} \in \overline{C}^*(P : \Sigma, \partial\Sigma)\}.$$

$Z^*(\Sigma;\partial\Sigma)$ will denote the complex

$$Z^*(\Sigma,\partial\Sigma)=\bigoplus_{P\supset\Sigma}\overline{Z}^*(P:\Sigma,\partial\Sigma).$$

Definition 4.2.2: $r:\overline{C}^*(P;\Sigma,\partial\Sigma)\to\overline{Z}^*(P;\Sigma,\partial\Sigma)$ will denote the linear map determined by

$$r: D_t^\alpha\{\omega_{\alpha,\beta,\gamma}(w,l)dl^\gamma\wedge dt^\beta\wedge[\Sigma]_{[P]}\}\circ j_P^*\mapsto D_t^\alpha\{\omega_{\alpha,\beta,\gamma}(0,l)dl^\gamma\wedge dt^\beta\wedge[\Sigma]_{[P]}\}\circ j_P^*$$

on forms whose representation does not contain any differentials of the form dw^δ, and the condition

$$r: D_t^\alpha\{\omega_{\alpha,\beta,\gamma,\delta}(w,l)dw^\delta\wedge dl^\gamma\wedge dt^\beta\wedge[\Sigma]_{[P]}\}\circ j_P^*\mapsto 0$$

on currents with $\delta\neq 0$.

Let $i:\overline{Z}^*(P:\Sigma,\partial\Sigma)\to\overline{Z}^*(P:\Sigma,\partial\Sigma)$ denote the identity homomorphism.

Proposition 4.2.3: Let k be a positive integer, and let Σ be a polysimplex in M_k which is transverse to the diagonal ∇_M, and let P be any polysimplex in M_k containing Σ, then

(1) $\overline{H}_w:\overline{C}^*(P:\Sigma,\partial\Sigma)\to\overline{C}^*(P:\Sigma,\partial\Sigma)$ is a chain homotopy between r and i.

(2) $\overline{H}_w:\overline{\nabla}^*(P:\Sigma,\partial\Sigma)\to\overline{\nabla}^*(P:\Sigma,\partial\Sigma)$ is a chain homotopy between the identity and the zero homomorphism.

(3) $\overline{H}_w$ induces a chain homotopy $\tilde{H}_w:\overline{C}^*(P:\Sigma,\partial\Sigma)\to\overline{Z}^*(P:\Sigma,\partial\Sigma)$ between the induced homomorphisms $\tilde{r}$ and $\tilde{i}$.

(4) $H_*(\mathcal{C}^*(\Sigma, \partial\Sigma)) \cong H_*(Z^*(\Sigma, \partial\Sigma))$.

Proof: Let $U \in \overline{C}^p(P : \Sigma, \partial\Sigma)$ be a current of the form

$$\begin{aligned} U = & D_t^\alpha\{\omega \wedge [\Sigma]_{[P]}\} \circ j_P^*\} \\ = & D_t^\alpha\{\omega(w,l)dw^\delta \wedge dl^\gamma \wedge dt^\beta \wedge [\Sigma]_{[P]}\} \circ j_P^*\}. \end{aligned}$$

It follows immediately from the definition of $\tilde{H}_w$ that $b\tilde{H}_w = dH_w$ and $\tilde{H}_w b = H_w d$. Hence

$$b\tilde{H}_w + \tilde{H}_w b = dH_w + H_w d.$$

Let H denote H_w and we begin with the calculation of $dH_w + H_w d$.

$$\begin{aligned} dHU = & (d_t + d_l + d_w)D_t^\alpha\{h_w\omega \wedge [\Sigma]_{[P]}j_P^*\} \\ = & (-1)^{p-1}\sum_{i=1}^{n'} D_{t^i}D_t^\alpha\{h_w\omega \wedge dt^i \wedge [\Sigma]_{[P]}j_P^*\} \\ & + D_t^\alpha\{d_l h_w\omega \wedge [\Sigma]\} + (-1)^{|\delta|-1}D_t^\alpha\{h_w\omega \wedge d_l[\Sigma]_{[P]}\} \circ j_P^* \\ & + D_t^\alpha\{d_w h_w\omega \wedge [\Sigma]\} + (-1)^{|\delta|-1}D_t^\alpha\{h_w\omega \wedge d_w[\Sigma]_{[P]}\} \circ j_P^* \end{aligned}$$

Also,

$$\begin{aligned} H(dU) = & Hd_tU + Hd_lU + Hd_wU \\ = & H((-1)^p \cdot \sum_{i=1}^{n'} D_{t^i}D_t^\alpha\{\omega \wedge dt^i \wedge [\Sigma]_{[P]}\} \circ j_P^* \\ & + H(D_t^\alpha\{d_l\omega \wedge [\Sigma]_{[P]}\}j_P^*) + (-1)^{|\delta|}H(D_t^\alpha\{\omega \wedge d_l[\Sigma]_{[P]}\}j_P^*) \\ & + H(D_t^\alpha\{d_w\omega \wedge [\Sigma]_{[P]}\}j_P^*) + (-1)^{|\delta|}H(D_t^\alpha\{\omega \wedge d_w[\Sigma]_{[P]}\}j_P^*) \\ = & (-1)^p\sum_{i=1}^{n'} D_{t^i}D_t^\alpha\{h_w\omega \wedge dt^i \wedge [\Sigma]_{[P]}\}j_P^*) \\ & + D_t^\alpha\{h_w d_l\omega \wedge [\Sigma]_{[P]}\}j_P^*) + (-1)^{|\delta|}H(D_t^\alpha\{\omega \wedge d_l[\Sigma]_{[P]}\}j_P^*) \\ & + D_t^\alpha\{h_w d_w\omega \wedge [\Sigma]_{[P]}\}j_P^*) + (-1)^{|\delta|}H(D_t^\alpha\{\omega \wedge d_w[\Sigma]_{[P]}\}j_P^*). \end{aligned}$$

As d_l and d_w involve only derivatives longitudinal to Σ, it follows that the terms involving $d_l[\Sigma]_{[P]}$ and $d_w[\Sigma]_{[P]}$ are zero in $\overline{C}^*(P : \Sigma, \partial\Sigma)$. The terms involving $d_l h_w$ and $h_w d_l$ sum to zero.

Thus, by combining terms, we have

$$dHU + HdU = D_t^\alpha\{(d_w h_w + h_w d_w)\omega \wedge [\Sigma]_{[P]}\}j_P^*$$

From the basic properties of the standard chain homotopy h_w, it follows that $dH_wU + H_wdU = U$ if $\delta \neq 0$, and $dH_wU + H_wdU = U' - U$ if $\delta = 0$ where

$$U' = D_t^\alpha\{\omega(0,l)dl^\gamma \wedge dt^\beta \wedge [\Sigma]_{[P]}\}j_P^*.$$

This calculation proves (1).

The proof of (2) is the observation that $\overline{H}_w$ maps $\overline{\nabla}^*(P : \Sigma, \partial\Sigma)$ to itself, and that an element $U \in \overline{\nabla}^*(P : \Sigma, \partial\Sigma)$ which does not contain a differential of the form dw^j must vanish on the diagonal.

The result follows as any function of the form $\omega(0, l, t)$ only assumes values that it assumes on the diagonal. This implies $\overline{H}_w$ is a chain homotopy between the identity and zero homomorphism on the complex $\overline{\nabla}^*(P : \Sigma, \partial\Sigma)$.

Parts (3) is immediate, and part (4) follows as $\tilde{H}_w$ has the required properties when restricted to any polysimplex P in M_k containing Σ.

4.3 The Chain Homotopy $\overline{H}_l$

The same procedure can be done for the variables along the diagonal in Σ (where Σ is a polysimplex transverse to the diagonal). As the calculations are exactly the same, only the definitions and a statement of the result will be given.

Let P be a polysimplex in M_k containing Σ, and let (w, l, t) be coordinates on P as in sections 4.1 and 4.2. If $dl^\gamma = dl^{\gamma_1} \wedge \cdots \wedge dl^{\gamma_p}$, then for $i = 1, \cdots, p$, let $dl^{\gamma-\gamma(i)}$ denote the form

$$dl^{\gamma-\gamma(i)} = dl^{\gamma_1} \wedge \cdots \wedge dl^{\gamma_{i-1}} \wedge dl^{\gamma_{i+1}} \wedge \cdots \wedge dl^{\gamma_p}.$$

Definition 4.3.1: Let $h_l : \Omega^*(P) \to \Omega^*(P)$ be defined on forms

$$\phi = \omega(w, l, t) dw^\delta \wedge dl^\gamma \wedge dt^\beta$$

by

$$h_l\phi = \sum_{i=1}^{|\gamma|} (-1)^{i-1} \cdot l^{\gamma(i)} \cdot \int_0^1 \lambda^{|\gamma|-1} \omega(w, \lambda l, t) dw^\delta \wedge dl^{\gamma-\gamma(i)} \wedge dt^\beta$$

on forms $\gamma \neq 0$, and define

$$h_l\phi = 0,$$

on forms ϕ with $\gamma = 0$, and extend h_l to $\Omega^*(P)$ linearly.

Remark 4.3.2: This is a homotopy operator on $\Omega^*(P)$ which gives a chain homotopy between the identity and the map

$$\phi \mapsto \omega(w, 0, t) \wedge dl^\gamma \wedge dt^\beta$$

for $\gamma \neq 0$, and $\phi \mapsto 0$ if $\gamma = 0$.

Definition 4.3.3: Let $H_l : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ be defined on currents of the form

$$U = D_t^\alpha\{\omega \wedge [\Sigma]_{[P]}\}j_P^*$$

by

$$H_l U = D_t^\alpha\{h_l\omega \wedge [\Sigma]_{[P]}\}j_P^*.$$

Let $\overline{H}_l : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ be defined by

$$\overline{H}_l(U) = (-1)^p H_l(U) \text{ for } U \in \overline{C}^*(P : \Sigma, \partial\Sigma).$$

Remark 4.3.4: As $\overline{H}_l$ does not alter the pair (α, β), it defines a linear transformation $\overline{H}_l : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ by restriction.

4.4 The Isomorphism Induced by $\overline{H}_l$

Definition 4.4.1: $\overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma) \subset \overline{Z}^*(P : \Sigma, \partial\Sigma)$ will denote the subcomplex

$$\overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma) = \{\sum_{\alpha,\beta} D_t^\alpha\{r dt^\beta \wedge [\Sigma]_{[P]}\}j_P^* \in \overline{Z}^*(P : \Sigma, \partial\Sigma) \mid r \in R\},$$

$\mathcal{D}^*(\Sigma, \partial\Sigma)$ will denote the direct sum

$$\mathcal{D}^*(\Sigma, \partial\Sigma) = \bigoplus_{P \supset \Sigma} \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma).$$

Definition 4.4.2: $s : \overline{Z}^*(P : \Sigma, \partial\Sigma) \to \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$ will denote the linear map determined by

$$r : D_t^\alpha\{\omega(0,l)dl^\gamma \wedge dt^\beta \wedge [\Sigma]_{[P]}\}j_P^* \mapsto D_t^\alpha\{\omega(0,0)dt^\beta \wedge [\Sigma]_{[P]}\}j_P^*$$

on forms whose representation does not contain any differentials of the form dl^γ, and the condition

$$r : D_t^\alpha\{\omega(0,l)dl^\gamma \wedge dt^\beta \wedge [\Sigma]_{[P]}\} \circ j_P^* \mapsto 0$$

on currents with $\gamma \neq 0$.

Let $i : \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma) \to \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$ denote the identity homomorphism.

Proposition 4.4.3: Let k be a positive integer, and let Σ be a polysimplex in M_k which is transverse to the diagonal ∇_M, and let P be any polysimplex in M_k containing Σ, then

(1) $\overline{H}_l : \overline{Z}^*(P;\Sigma,\partial\Sigma) \to \overline{Z}^*(P;\Sigma,\partial\Sigma)$ is a chain homotopy between s and i.

(2) $\overline{H}_l$ induces a chain homotopy $\tilde{H}_l : \overline{Z}^*(P : \Sigma, \partial\Sigma) \to \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$ between the induced homomorphisms $\tilde{s}$ and $\tilde{i}$.

(3) $H_*(Z^*(\Sigma,\partial\Sigma)) \cong H_*(\mathcal{D}^*(\Sigma,\partial\Sigma))$.

Proof: Interchange the roles of l and w in Proposition 4.2.3.

4.5 The Chain Homotopy $\overline{H}_t$

The chain homotopy constructed in this section is the transpose of the homotopy operator h_t defined on $\Omega^*(P)$. More precisely, let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k which is combinatorially transverse to the diagonal, and for polysimplices $P = \rho_1 \times \cdots \times \rho_k$ in M_k containing Σ, let (w, l, t) denote the coordinate system introduced above.

Definition 4.5.1: $h^t_\Sigma : \Omega^*(P) \to \Omega^*(P)$ will denote the linear map determined by

$$h^t_\Sigma \phi = \sum_{\delta,\epsilon} \sum_{i=1}^{|\delta|} (-1)^i t^{\delta(i)} \int_0^1 \lambda^{|\delta|-1} \phi_{\delta,\epsilon}(w, l, \lambda t) d\lambda dw^\zeta \wedge dl^\epsilon \wedge dt^{\delta-\delta(i)},$$

for $\phi = \sum_{\delta,\epsilon} \phi_{\delta,\epsilon}(w, l, t) dw^\zeta \wedge dl^\epsilon \wedge dt^\delta \in \Omega^*(P)$ with $\delta \neq 0$, and set $h^t(\phi) = 0$ if $\delta = 0$.

Definition 4.5.2: For a pair (P, Σ) of polysimplices in M_k with Σ combinatorially transverse to the diagonal, and for nonnegative integers p, define $H^P_t : \overline{C}^p(P : \Sigma, \partial\Sigma) \to \overline{C}^{p+1}(P : \Sigma, \partial\Sigma)$, on currents

$$U = D^\alpha_t\{\omega \wedge dt^\beta \wedge [\Sigma]_{[P]}\} j^*_P,$$

with $\beta \neq 0$, $\omega = \sum_{\gamma,\delta} \omega_{\gamma,\delta}(w, l, t) dw^\delta \wedge dl^\gamma \in \Omega^*(P)$ as

$$H^P_t(U) = \sum_{j=1}^{|\beta|} (-1)^{j-1} \frac{\alpha(\beta(j))}{\tilde{N} + \alpha} D^{\alpha\beta(j)}_t \{\omega_{\alpha,\beta} \wedge dt^{\beta-\beta(j)} \wedge [\Sigma]_{[P]}\} j^*_P,$$

where for $1 \leq j \leq \mid \gamma \mid$, and $\alpha(\beta(j))$ is the value of the multi-index α corresponding to the variable $t^{\beta(j)}$, and $\tilde{N} = \ \mathrm{codim}_P(\Sigma) - \mid \beta \mid$.

If $\beta = 0$, define $H_t^P(U) = 0$.

Let $\overline{H}_t^P : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ be defined by

$$\overline{H}_t^P(U) = (-1)^p H_t^P(U) \text{ for } U \in \overline{C}^*(P : \Sigma, \partial\Sigma).$$

Remark 4.5.3: As H_t only reduces D_t^α to $D_t^{\alpha-\beta(j)}$ in the terms it removes the differential $dt^{\beta(j)}$, it follows that $\overline{H}_t : \overline{C}^*(P : \Sigma, \partial\Sigma) \to \overline{C}^*(P : \Sigma, \partial\Sigma)$ by the defining conditions of $\overline{C}^*(P : \Sigma, \partial\Sigma)$.

Definition 4.5.6: Let $\overline{H}_t : C^*(\Sigma, \partial\Sigma) \to C^*(\Sigma, \partial\Sigma)$ be the linear map determined by $\overline{H}_t \mid_{\overline{C}^*(P;\Sigma,\partial\Sigma)} = \overline{H}_t^P$.

Remark 4.5.7: For each pair (P, Σ) with Σ combinatorially transverse to the diagonal, H_t satisfies

$$H_t U(\phi) = U(h_\Sigma^t \phi) \text{ for } U \in \overline{C}^*(P : \Sigma, \partial\Sigma) \text{ , } \phi \in \Omega^*(M_k).$$

The proof of this remark is a calculation that will be given in the appendix. From this remark it follows that H_t is the chain homotopy claimed in Proposition 4.6.3. This remark will not be used for any results. $\overline{H}_t$ will be applied to the complex $\overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$ and it will be directly verified that $\overline{H}_t$ has the required properties.

4.6 The Isomorphism Induced by $\overline{H}_t$

Definition 4.6.1: For a polysimplex Σ in M_k which is transverse to the diagonal, $\overline{S}^*(\Sigma, \partial\Sigma)$ will denote the 1-dimensional subcomplex of $\overline{\mathcal{D}}^*(\Sigma, \partial\Sigma)$ generated by the current $\{[\Sigma]_{[\Sigma]}\}j^*_\Sigma$.

Definition 4.6.2: $u : \mathcal{D}^*(\Sigma, \partial\Sigma) \to \overline{S}^*(\Sigma, \partial\Sigma)$ will denote the linear map

$u : D^\alpha_t\{r \cdot dt_\beta \wedge [\Sigma]_{[P]}\} \circ j^*_P \mapsto 0$, if α or β is not zero, and

$u : \{r \cdot [\Sigma]_\Sigma\} \circ j^*_\Sigma \mapsto \{r \cdot [\Sigma]_{[\Sigma]}\}j^*_\Sigma$, if $\alpha = 0$ and $\beta = 0$.

Let $i : \mathcal{D}^*(\Sigma, \partial\Sigma) \to \mathcal{D}^*(\Sigma, \partial\Sigma)$ denote the identity homomorphism,

Proposition 4.6.3: For polysimplices Σ in M_k which are transverse to the diagonal

(1) $\overline{H}_t$ is a chain homotopy between the identity on $\mathcal{D}^*(\Sigma, \partial\Sigma)$, and the chain map $u : \mathcal{D}^*(\Sigma, \partial\Sigma) \to S^*(\Sigma, \partial\Sigma)$

(2) If $P \neq \Sigma$, then the complex $\overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$ is acyclic.

Proof: As $b\overline{H}_t = dH_t$ and $\overline{H}_t b = H_t d$, it suffices to show

$$dH_tU + H_tdU = U$$

for $U \in \overline{\mathcal{D}}(P : \Sigma, \partial\Sigma)$ with $P \neq \Sigma$, and

$$dHU + HtdU = U - U = 0$$

for $U \in \overline{\mathcal{D}}(P : \Sigma, \partial\Sigma)$ with $P = \Sigma$.

Let H denote H_t. For

$$U = D^\alpha_t\{r \cdot dt^\beta \wedge [\Sigma]_{[P]}\}j^*_P \in \overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma),$$

with $\beta \neq 0$, $H(U)$ is given by

$$H(U) = \sum_{j=1}^{|\beta|}(-1)^{j-1}\frac{\alpha(\beta(j))}{\tilde{N}+\mid\alpha\mid}D_t^{\alpha-\beta(j)}\{r\cdot dt^{\beta-\beta(j)}\wedge[\Sigma]_{[P]}\}j_P^*,$$

where for $1 \leq j \leq\mid \beta \mid$, and $\alpha(\beta(j))$ is the value of the multi-index α corresponding to the variable $t^{\beta(j)}$, and $\tilde{N} =$ codim$_P(\Sigma)-\mid \beta \mid$. Let $N =$ dim(Σ).

Then the class of dHU in $\overline{\mathcal{D}}^*(P:\Sigma,\partial\Sigma)$ is given by

$$\begin{aligned}dHU &= d\Big(\sum_{j=1}^{|\beta|}(-1)^{j-1}\frac{\alpha(\beta(j))}{\tilde{N}+\mid\alpha\mid}D_t^{\alpha-\beta(j)}\{r\cdot dt^{\beta-\beta(j)}\wedge[\Sigma]_{[P]}\}\circ j_P^*\Big)\\ &= \sum_{j=1}^{|\beta|}\sum_{i=1,i\neq j}^{N'}(-1)^{j-1}\frac{\alpha(\beta(j))}{\tilde{N}+\mid\alpha\mid}D_{t^i}D_t^{\alpha-\beta(j)}\{r\cdot dt^i\wedge dt^{\beta-\beta(j)}\wedge[\Sigma]_{[P]}\}j_P^*\end{aligned}$$

as $d_l(r)$ and $d_w(r)$ are zero.

Let $\alpha(\beta'(j))$ denote the value of α of the j^{th} variable in $(t^1,\cdots,t^{n'})$ with the property that its differential does not appear in dt^β.

Then, in $\overline{\mathcal{D}}^*(P:\Sigma,\partial\Sigma)$

$$
\begin{aligned}
H(dU) = & H\Big(\sum_{i=1}^{N'} D_{t^i} D_t^{\alpha}\{r \cdot dt^i \wedge dt^{\beta} \wedge [\Sigma]_{[P]}\} \circ j_P^*\Big) \\
= & \sum_{j=1}^{\tilde{N}} \frac{1+\alpha(\beta'(j))}{\tilde{N}+\mid \alpha \mid} D_t^{\alpha}\{r dt^{\beta} \wedge [\Sigma]_{[P]}\} \circ j_P^* \\
& + \sum_{i=1}^{N'} \sum_{j=1}^{|\beta|} \frac{(-1)^{j-1} \cdot \alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_{t^i} D_t^{\alpha-\beta(j)}\{r \cdot dt^i \wedge dt^{\beta-\beta(j)} \wedge [\Sigma]_{[P]}\} \circ j_P^* \\
= & \sum_{j=1} \tilde{N} \frac{1+\alpha(\beta'(j))}{\tilde{N}+\mid \alpha \mid} D_t^{\alpha}\{r \cdot dt^{\beta} \wedge [\Sigma]_{[P]}\} j_P^* \\
& + \sum_{i=1}^{N'} \sum_{j=1}^{|\beta|} \frac{(-1)^{j-1} \cdot \alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_{t^i} D_t^{\alpha-\beta(j)}\{r \cdot dt^i \wedge dt^{\beta-\beta(j)} \wedge [\Sigma]_{[P]}\} j_P^*.
\end{aligned}
$$

Hence

$$
\begin{aligned}
dHU + HdU = & \sum_{j=1} \tilde{N} \frac{1+\alpha(\beta'(j))}{\tilde{N}+\mid \alpha \mid} U \\
& + \sum_{i=1}^{N'} \sum_{j=1}^{|\beta|} \frac{(-1)^{j-1} \alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_{t^i} D_t^{\alpha-\beta(j)}\{r dt^i \wedge dt^{\beta-\beta(j)} [\Sigma]_{[P]}\} j_P^* \\
& - \sum_{j=1}^{|\beta|} \sum_{i=1, i \neq j}^{N'} \frac{(-1)^{j-1} \alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_{t^i} D_t^{\alpha-\beta(j)}\{r dt^i \wedge dt^{\beta-\beta(j)} [\Sigma]_{[P]}\} j_P^* \\
= & \sum_{j=1} \tilde{N} \frac{1+\alpha(\beta'(j))}{\tilde{N}+\mid \alpha \mid} U \\
& + \sum_{j=1}^{|\beta|} \frac{(-1)^{j-1} \alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_{t^{\beta(j)}} D_t^{\alpha-\beta(j)}\{r dt^{\beta(j)} \wedge dt^{\beta-\beta(j)} [\Sigma]_{[P]}\} j_P^*
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1} \tilde{N} \frac{1+\alpha(\beta'(j))}{\tilde{N}+\mid \alpha \mid} U \\
&+ \sum_{j=1}^{|\beta|} \frac{\alpha(\beta(j))}{\tilde{N}+\mid \alpha \mid} D_t^{\alpha}\{r \cdot dt^{\beta} \wedge [\Sigma]_{[P]}\} \circ j_P^* \\
&= \frac{\tilde{N}}{\tilde{N}+\mid \alpha \mid} U + \frac{\mid \alpha \mid}{\tilde{N}+\mid \alpha \mid} U \\
&= U.
\end{aligned}
$$

This shows $\beta \neq 0$ implies $(dH + Hd)(U) = U$.

If $\beta = 0$, then $H(U) = 0$. Hence $H(dU) + d(HU) = H(dU)$, and if either $\alpha \neq 0$ or $N' \neq 0$, then, in $\overline{\mathcal{D}}^*(P : \Sigma, \partial\Sigma)$,

$$
\begin{aligned}
H(dU) &= H(\sum_{i=1}^{N'} D_{t^i} D_t^{\alpha}\{r \cdot dt^i \wedge [\Sigma]_{[P]}\} \circ j_P^*) \\
&= \frac{N'+\mid \alpha \mid}{N'+\mid \alpha \mid} D_t^{\alpha}\{r dt^{\beta} \wedge [\Sigma]_{[P]}\} \circ j_P^* \\
&= U.
\end{aligned}
$$

If $\alpha = 0$, $\beta = 0$, and $N' = 0$, then $U = \{r \cdot [\Sigma]_{[\Sigma]}\} \circ j_{\Sigma}^*$. Then

$$
H(dU) = 0 = U - \{r \cdot [\Sigma]_{[\Sigma]}\} \circ j_{\Sigma}^*.
$$

This establishes the required properties of H, and completes the proof of Proposition 4.6.3.

4.7 Homology of the Complex $\mathcal{C}^*(\Sigma, \partial\Sigma)$

Combining the above results gives

Theorem 4.7.1: Let M be a compact orientable manifold of dimension m without boundary, k a positive integer. Let Σ denote a polysimplex in M_k.

(1) If Σ is combinatorially transverse to the diagonal in M_k, then

$$H_*(\mathcal{C}^*(\Sigma, \partial\Sigma)) \cong H_*(\overline{S}^*(\Sigma, \partial\Sigma)).$$

(2) If Σ is not combinatorially transverse to the diagonal in M_k, then

$$H_*(\mathcal{C}^*(\Sigma, \partial\Sigma) = 0.$$

Proof: In the propositions it has been established that

$$H_*(\mathcal{C}^*(\Sigma, \partial\Sigma)) \cong H_*(\mathcal{Z}^*(\Sigma, \partial\Sigma)) \cong H_*(\mathcal{D}^*(\Sigma, \partial\Sigma)) \cong H_*(\overline{S}^*(\Sigma, \partial\Sigma; R))$$

if Σ is combinatorially transverse to the diagonal in M_k.

If Σ is not combinatorially transverse to the diagonal, Corollary 3.4.3 implies $\mathcal{C}^*(\Sigma, \partial\Sigma) = 0$. This implies $H_*(\mathcal{C}^*(\Sigma, \partial\Sigma) = 0$.

4.8 The spectral sequence

In this section, the spectral sequence argument is completed and the homology of the complex of piecewise differentiable currents is calculated in terms of the combinatoral structure of the manifold M.

To compute the homology of the complex $\mathcal{C}^*(M_k)$, introduce the filtration

$$F^{p,q}(\mathcal{C}^*(M_k)) = \mathcal{C}^{p+q}(K^{km-p}(M_k))$$

where $K^{km-p}(M_k)$ is the $(km-p)$-skeleton of M_k. p is the filtration index and $p+q$ is the degree of the currents.

$\{E_r^{p,q}, d_r\}$ will denote the corresponding spectral sequence.

Let $\Sigma = \sigma^1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, and let P be a polysimplex in M_k containing Σ.

Adopt the following notation: $P = \rho_1 \times \cdots \times \rho_k$, $\sigma = \sigma_1 \cap \cdots \cap \sigma_k$, $\rho = \rho_1 \cap \cdots \cap \rho_k$, $d = \dim(\sigma)$, and $p = \dim(\rho)$.

For a positive integer h, let $J_0^\infty R^h$ denote the ∞-jets of smooth functions at the origin in R^h.

Definition 4.8.1: $\mathcal{K}^*(R^h)$ will denote the **Koszul complex** of all ∞-jets of smooth differential forms at $0 \in R^h$.

Definition 4.8.2: $\mathcal{K}^*(P;\Sigma)$ will denote the Koszul complex of the pair (P,Σ), defined by,

$$\mathcal{K}^*(P;\Sigma) = \sum_{\epsilon} J_0^\infty R^h dx^\epsilon$$

where $h = p - d = \dim(\rho) - \dim(\sigma)$.

Definiton 4.8.3: The **Dirac complex** for the pair (P, Σ) is

$$\mathcal{D}^*(P;\Sigma) = \{U = \sum_{\alpha,\gamma} A_{\alpha,\gamma} D_t^{\alpha}\{dt^{\gamma} \wedge [\delta_0]\} \mid U \in C^*(P;\Sigma)\}$$

and $[\delta_0]$ represents the Dirac distribution at the origin in R^h where h is the codimension of Σ in P.

Remark 4.8.4: From Proposition 3.4.2, following Teleman in $[T_2]$, one has: For $k \geq 1$,

$$\mathcal{C}^*(\Sigma, \partial\Sigma) = \sum_P (\Omega^*(\Sigma \cap \nabla_M) \bigotimes_R \mathcal{K}^*(P;\Sigma) \bigotimes_R \mathcal{D}^*(P;\Sigma))$$

where the sum is over all polysimplices P containing Σ, and satisfying $(P, \Sigma)\amalg \nabla_M$ and $P \amalg \nabla_M$.

Remark 4.8.5: By the above remark, it is possible to give the following characterization of the $E_0^{p,q}$ terms of the spectral sequence: for $k \geq 1$,

$$E_0^{p,q} = \bigoplus_{\substack{\mathrm{codim}(\Sigma) = p \\ \Sigma \amalg \nabla_M}} \sum_{P \supset \Sigma} (\Omega^*(\Sigma \cap \nabla_M) \bigotimes_R \mathcal{K}^*(P;\Sigma) \bigotimes_R \mathcal{D}^*(P;\Sigma)).$$

Remark 4.8.6: In [T₁], Teleman obtains a similar characterization. In his decomposition, there is an isomorphism

$$\mathcal{C}^*(\Sigma, \partial\Sigma) \cong \Omega^*(\Sigma \cap \nabla_M) \bigotimes_R \mathcal{K}^*(\Sigma) \bigotimes_R \mathcal{D}^*(\Sigma),$$

and he constructs chain homotopies on each of these three complexes.

In the approach given in this work, homotopies have been constructed directly on each of the complexes $\overline{C}^*(P : \Sigma, \partial\Sigma)$ (which restrict to chain homotopies on $\overline{\nabla}^*(P : \Sigma, \partial\Sigma)$). From the explicit formulas for the chain homotopies, it follows that they combine to give a chain homotopies on the complexes $C^*(\Sigma, \partial\Sigma)$, and $\nabla^*(\Sigma, \partial\Sigma)$.

This approach avoids using the explicit representation of the E_0 terms of the spectral sequence. The E_1 terms were calculated directly.

From

$$E_0^{p+q} = \bigoplus_{\text{codim}(\Sigma) = p} C^{p+q}(\Sigma, \partial\Sigma),$$

it follows that

$$E_1^{p,q} = H^q(E_0^{p,*}) = \bigoplus_{\text{codim}(\Sigma) = p} H_q(C^{p+q}(\Sigma, \partial\Sigma)).$$

Theorem 4.7.1 implies

$$E_1^{p,q} = \bigoplus_{\text{codim}(\Sigma) = p} H_q(C^*(\Sigma, \partial\Sigma))$$

satisfies

$$E_1^{p,q} = 0 \text{ , if } q \neq 0 \text{ , and}$$

$$E_1^{p,0} = \bigoplus_{\substack{\text{codim}(\Sigma) = p \\ \Sigma \amalg \nabla_M}} R \cdot \{[\Sigma]_{[\Sigma]}\} \circ j_\Sigma^*$$

Let N denote the subcomplex of M_k consisting of the polysimplices of M_k which are not transverse to the diagonal ∇_M. Let $S_r(M_k, N; R)$ denote the relative cellular chain complex

$$S_r(M_k, N; R) = S_r(M_k; R)/S_r(N; R)$$

as defined in 1.8.17.

The isomorphisms between the vector space $S_{km-p}(M_k, N; R)$ and the vector space $\overline{S}^p(M_k)$ given by $[\Sigma] \to \{[\Sigma]_{[\Sigma]}\} \circ j^*_\Sigma$ induce an isomorphism between chain complexes

$$\overline{S}^*(M_k) \cong S_{km-*}(M_k, N)$$

(after a flip of indices). This follows by showing the homomorphism

$$d_1 : H^*(C^*(\Sigma, \partial\Sigma)) \to H^*(C^*(\Sigma, \partial\Sigma))$$

agrees with the boundary homomorphism on $S_*(M_k, N)$.

On an element $\{[\Sigma]_{[\Sigma]}\} \circ j^*_\Sigma$ of $E_1^{p,0}$, the homomorphism d_1 is the deRham boundary b. It follows that

$$\begin{aligned} d_1\{[\Sigma]_{[\Sigma]}\} \circ j^*_\Sigma \phi &= b(\{[\Sigma]_{[\Sigma]}\} \circ j^*_\Sigma)(\phi) \\ &= [\Sigma]_{[\Sigma]}\{j^*_\Sigma(d\phi)\} \\ &= [\Sigma]_{[\Sigma]}\{d(j^*_\Sigma\phi)\} \\ &= [\Sigma]\{d(j^*_\Sigma\phi) \wedge \theta_\Sigma\} \\ &= \int_{[\Sigma]} d(j^*_\Sigma\phi) \\ &= \int_{\partial[\Sigma]} i^*_{\partial\Sigma} j^*_\Sigma \phi \end{aligned}$$

where θ_Σ is defined by $d\Sigma \wedge \theta_\Sigma = \mu^k$ (as in Definition 2.2.4) and $i_{\partial\Sigma} : \partial\Sigma \to \Sigma$ is the inclusion.

As

$$\{[\partial\Sigma]_{[\partial\Sigma]}\} \circ j^*_{\partial\Sigma}(\phi) = [\partial\Sigma] j^*_{\partial\Sigma}\phi \wedge \theta_{\partial\Sigma} = \int_{[\partial\Sigma]} j^*_{\partial\Sigma}\phi,$$

it follows that

$$d_1\{[\Sigma]_{[\Sigma]}\} \circ j^*_\Sigma = \{[\partial\Sigma]_{[\partial\Sigma]}\} \circ j^*_{\partial\Sigma}$$

corresponds to the connecting homomorphism of the long exact homology sequence of the pair $(\Sigma, \partial\Sigma)$ (which is induced by the boundary homomorphism of the complex $S_*(M_k, N)$).

Hence, under the isomorphism

$$E_1^{p,0} \cong S_{km-p}(M_k, N; R),$$

d_1 corresponds to the boundary homomorphism on $S_*(M_k, N; R)$.

$$E_2^{p,q} = H^q(E_1^{p,*}).$$

As $E_1^{p,q} = 0$, if $q \neq 0$, and $E_1^{p,0} \cong S_{km-p}(M_k, N; R)$, it follows that

$$E_2^{p,0} \cong H_{km-p}(M_k, N; R)$$

$$E_2^{p,q} = 0 \text{ if } q \neq 0.$$

Theorem 4.8.7: For any positive integer k, the homology of $(\mathcal{C}^*(M_k), d)$, the complex of generalized piecewise differentiable currents associated to the k-fold product M_k of a compact oriented combinatorial manifold M with $\partial M = \emptyset$, is given by

$H_p(\mathcal{C}^*(M_k)) \cong H_{km-p}(M_k, N; R)$, $0 \leq p \leq \dim M$,

$H_p(\mathcal{C}^*(M_k)) = 0$, $p > \dim M$,

where $N = \{\Sigma \in M_k \mid \Sigma \not\pitchfork \nabla_M\}$.

Proof: As $H_p(\mathcal{C}^*(M_k)) \cong \bigoplus_{q \in Z} E_2^{p,q}$ it follows that

$$H_p(\mathcal{C}^*(M_k)) \cong H_{km-p}(M_k, N; R).$$

To prove $H_{km-p}(M_k, N; R) = 0$ if $p > m$, it suffices to show N contains the $m(k-1)$-skeleton of M_k.

Assume $k \geq 2$, and let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k transverse to the diagonal and with $\dim(\Sigma) \leq km - 1$. Immediately from the definition of combinatorial transversality, it follows that any polysimplex $\Sigma' = \sigma_1' \times \cdots \times \sigma_k'$ which contains Σ is transverse to the diagonal in M_k. Choose Σ' with $\dim(\Sigma') = km - 1$. As each σ_j', $1 \leq j \leq k$, is obtained from σj by taking the join of σ_j with elements of $\mathcal{B}(\sigma_j)$, and $\mathcal{B}(\sigma_j) \subset \mathcal{A}(\Sigma, j)$, it follows that

$$\dim(\sigma_1' \cap \cdots \cap \sigma_k') = \dim(\sigma_1 \cap \cdots \cap \sigma_k) + \dim(\Sigma') - \dim(\Sigma).$$

Hence

$\dim(\Sigma') - \dim(\Sigma) \leq \dim(\sigma_1' \cap \cdots \cap \sigma_k') \leq m - 1$ as Σ' is not maximal. This implies $\dim(\Sigma) \geq \dim(\Sigma') - m + 1$, and shows any polysimplex which is transverse to the diagonal has dimension greater than $m(k-1)$.

Hence, N contains the $m(k-1)$-skeleton of M_k. This implies

$$H_{km-p}(N) \cong H_{km-p}(M_k) \text{ if } p > m.$$

As this isomorphism is induced by the inclusion, it follows directly from the long exact sequence of the pair (M_k, N), that $H_{km-p}(M_k, N; R) = 0$ if $p > m$. This completes the proof of Theorem 4.8.7.

Remark 4.8.8: This theorem is the analogue of Theorem 2.6 from [T_1]. It expresses the homology of the complex $\mathcal{C}^*(M_k)$ as the relative homology of the pair (M_k, N) which is obtained from M using only the combinatorial structure of M. In [T_1], N. Teleman proceeds to show, in the case of smooth manifolds, that these relative homology groups depend only on the underlying topological structure of M, if the simplices of the triangualtion are transverse to the diagonal.

In the final section of this chapter, it will shown that the corresponding result does not hold in the combinatorial case.

4.9 Dependence on the Combinatorial Structure

In this section, it is shown that the relative homology groups obtained in Theorem 4.8.7 are not invariant under subdivision.

Hence, the results of this work are strongly dependent on the combinatorial structure, and cannot be expressed in terms of the underlying piecewise-linear structure. It will be shown that the derived complex M' of any compact orientable 2-manifold without boundary is a complex which does not have any transverse polysimplices of positive codimension in any k-fold product of M'_k if $k > 1$. Hence, a single barycentric subdivision of a combinatorial 2-manifold distroys all of the homological data. We begin this section by proving this fact.

Remark 4.9.1: If M is a combinatorial 2-manifold, and M' is the derived complex of M, then the star of any vertex contains at least four 2-simplices. This follows by considering the barycentric subdivision of two neighboring 2-simplices in M.

With this observation, the following proposition is easily proved.

Proposition 4.9.2: Let M be a compact, orientable, combinatorial 2-manifold without boundary. Let M' be the derived complex. If $k > 1$, then M'_k contains no polysimplices Σ of positive codimension which satisfy $\Sigma \amalg \nabla_M$.

Proof: Suppose Σ is a polysimplex in M'_k with $\Sigma \amalg \nabla_M$. Then $\Sigma \cap \nabla_M \neq \emptyset$, and let σ be the simplex given by

$$\Sigma \cap \nabla_M = \{(x, \cdots, x) \in M'_k \mid x \in \sigma\}.$$

Express Σ as $\Sigma = \sigma_1 \times \cdots \times \sigma_k$.

First assume $\operatorname{codim}(\Sigma) = 1$ and σ is a 0-simplex, $\sigma = (v^0)$. Then some σ_i is 1-dimensional, and σ_i is of the form $\sigma_i = (v^0, v^1)$. As M' is a 2-manifold, σ_i is a face of two 2-simplices of the form (v^0, v^1, v^2) and (v^0, v^1, v^3). As $\Sigma \amalg \nabla_M$, it follows that $v^2, v^3 \in \mathcal{A}(\Sigma, i)$.

Hence, the set of vertices $\{v^0, v^2, v^3\}$ is contained in σ_j for all $j \neq i$. This implies $\sigma_j = (v^0, v^2, v^3)$ for $j \neq i$.

From this it follows that the star of v^0 in M' contains only three 2-simplices. This contradicts the above remark.

If σ is a 1-simplex, $\sigma = (v^0, v^1)$, then $\sigma_i = \sigma$. As σ is contained in two 2-simplices (v^0, v^1, v^2) and (v^0, v^1, v^2), transversality of Σ to the diagonal would imply that σ_j must contain the four vertices v^0, v^1, v^2, v^3. This is impossible.

This implies M_k' has no polysimplices Σ with $\Sigma \amalg \nabla_M$ and $\operatorname{codim}(\Sigma) = 1$.

Suppose Σ is any polysimplex in M_k' with $\Sigma \amalg \nabla_M$ and which has positive codimension. Then, by adjoining $\operatorname{codim}(\Sigma) - 1$ admissible transverse vertices to Σ, one obtains a polysimplex P with $P \amalg \nabla_M$ and $\operatorname{codim}(P) = 1$. As there do not exist polysimplices with these properties, there are no polysimplices Σ of positive codimension with $\Sigma \amalg \nabla_M$.

This establishes the proposition.

The next chapter will study a version of the concept of a generalized piecewise differentiable current which does carry the homological information of M.

Chapter 5

HOMOLOGY OF THE COMPLEX $\mathcal{N}^*(M_k)$

In this chapter, a modification of the notion of a generalized piecewise differentiable current is introduced. Using the Thom Isomorphism theorem, it will be shown the homology of this new complex $\mathcal{N}^*(M_k)$ (defined in 5.1.2 below) agrees with the (real) simplicial homology of M. Moreover, there is a natural multiplication

$$\mu_{k,l} : \mathcal{N}^r(M_k) \bigotimes_R \mathcal{N}^s(M_l) \to \mathcal{N}^{r+s}(M_{k+l})$$

which induces an operation

$$\mu_{k,l*} : H_r(\mathcal{N}^*(M_k)) \bigotimes_R H_s(\mathcal{N}^*(M_l)) \to H_{r+s}(\mathcal{N}^*(M_{k+l}))$$

for all integers r and s. The multiplication on homology is related to the intersection pairing of the real simplicial homology classes of M by the Thom isomorphism.

5.1 Definitions

Let M be a compact orientable combinatorial manifold of dimension m without boundary. For any positive integer k, M_k will denote the k-fold product of M, and M_k will be given the cell structure consisting of polysimplices $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ with σ_j a simplex of M, for $1 \leq j \leq k$.

Definition 5.1.1: For any integer $k \geq 1$, let $N_0(M_k)$ denote the subcomplex of M_k consisting of those simplices with do not intersect the diagonal ∇_M in M_k.

Definition 5.1.2: $\mathcal{N}^*(M_k)$ will denote the quotient complex

$$\begin{aligned}\mathcal{N}^*(M_k) &= C^*(M_k)/C^*(N_0(M_k)) \\ &= C^*(M_k, N_0(M_k)).\end{aligned}$$

The elements of $\mathcal{N}^*(M_k)$ are piecewise differentiable currents on M_k with carriers contained in the complement of $N_0(M_k)$ in M_k.

5.2 Multiplicative Structure

Definition 5.2.1: For a pair of positive integers k, l,

$$\mu_{k,l} : C^r(M_k) \bigotimes_R C^s(M_l) \to C^{r+s}(M_{k+l}), \text{ for } r, s \in Z,$$

be defined on elements

$$U_1 = D_t^{\alpha_1}\{\omega_1 \wedge [\Sigma_1]\} \circ j_P^* \in C^r(M_k))$$

$$U_2 = D_t^{\alpha_2}\{\omega_2 \wedge [\Sigma_2]\} \circ j_{P'}^* \in C^*(M_l))$$

where P is a maximal polysimplex in M_k, and P' is a maximal polysimplex in M_l, by

$$\mu_{k,l}(U_1, U_2) = D_t^{\alpha_1+\alpha_2}\{(p^k)^*\omega_1 \wedge (p_l)^*\omega_2 \wedge [\Sigma_1 \times \Sigma_2]\}.$$

Extend $\mu_{k,l}$ to $C^*(M_k) \bigotimes_R C^*(M_l)$ linearly.

For a positive integers k, l, let

$$p^k : M_{k+l} = M_k \times M_l \rightarrow M_k$$

be projection onto the first factor, and let

$$p_l : M_{k+1} = M_k \times M_l \rightarrow M_l$$

be projection onto the second factor.

Proposition 5.2.2: For U_1 and U_2 as above,

$$d\mu_{k,l}(U_1, U_2) = \mu_{k,l}(dU_1, U_2) + (-1)^r \mu_{k,l}(U_1, dU_2).$$

Proof: Let $(l, t) = (l^1, \cdots, l^{N_1+N_2}, t^1, \cdots, t^{n_1+n_2})$ denote the longitudinal and transverse barycentric coordinates to $\Sigma_1 \times \Sigma_2$ in $P \times P'$ chosen so that $(l', t') = (l^1, \cdots, l^{N_1}, t^1, \cdots, t^{n_1})$ give the longitudinal and transverse barycentric coordinates to Σ_1 in P under the identification of P as the first factor in $P \times P'$. This implies $(l'', t'') = (l^{N_1+1}, \cdots, l^{N_1+N_2}, t^{n_1+1}, \cdots, t^{n_1+n_2})$ are the longitudinal and transverse barycentric coordinates to Σ_2 in P' under the inclusion of P' as the second factor of $P \times P'$.

To simplify notation, let μ denote $\mu_{k,l}$,

$$d_{l'} = \sum_{i=1}^{N_1} dl^i \wedge \partial/\partial l^i$$

$$d_{l''} = \sum_{i=1+N_1}^{N_1+N_2} dl^i \wedge \partial/\partial l^i$$

Also set

$$\tilde{\omega}_1 = (p^k)^* \omega_1$$

$$\tilde{\omega}_2 = (p_l)^* \omega_2$$

$$d_l = d_{l'} + d^*_{l''}.$$

Then

$$
\begin{aligned}
d\mu(U_1,U_2) =& d(D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\}) \\
=& \sum_{i=1}^{n_1+n_2} D_{t^i} D_t^{\alpha_1+\alpha_2}\{dt^i \wedge \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ D_t^{\alpha_1+\alpha_2}\{d_l(\tilde{\omega}_1 \wedge \tilde{\omega}_2) \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ (-1)^{r+s} D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge d_l[\Sigma_1 \times \Sigma_2]\} \\
=& \sum_{i=1}^{n_1} D_{t^i} D_t^{\alpha_1+\alpha_2}\{dt^i \wedge \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ (-1)^r \sum_{i=n_1+1}^{n_1+n_2} D_{t^i} D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge dt^i \wedge \tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ D_t^{\alpha_1+\alpha_2}\{d_{l'}\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ (-1)^r D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge d_{l''}\tilde{\omega}_2 \wedge [\Sigma_1 \times \Sigma_2]\} \\
&+ (-1)^{r+s} D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge d_{l'}[\Sigma_1] \wedge [\Sigma_2]\} \\
&+ (-1)^{r+s} D_t^{\alpha_1+\alpha_2}\{\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge [\Sigma_1] \wedge d_{l''}[\Sigma_2]\} \\
=& \mu(dU_1,U_2) + (-1)^r \mu(U_1,dU_2).
\end{aligned}
$$

This implies

$$d\mu_{k,l}(U_1,U_2) = \mu_{k,l}(dU_1,U_2) + (-1)^r \mu_{k,l}(U_1,dU_2)$$

and establishes the proposition.

As these elements generate $C^r(M_k)$ and $C^s(M_l)$, the following corollary is immediate.

Corollary 5.2.3: For all $U_1 \in C^r(M_k)$, and for all $U_2 \in C^s(M_l)$

$$d\mu_{k,l}(U_1,U_2) = \mu_{k,l}(dU_1,U_2) + (-1)^r \mu_{k,l}(U_1,dU_2).$$

$$b\mu_{k,l}(U_1, U_2) = (-1)^s \mu_{k,l}(bU_1, U_2) + \mu_{k,l}(U_1, dU_2).$$

5.3 Induced Multiplication on Homology

The next proposition establishes the induced multiplication on Homology is well-defined.

Proposition 5.3.1: For any pair of positive integers k, l,

$$\mu = \mu_{k,l} : C^*(M_k) \bigotimes_R C^*(M_l) \to C^*(M_{k+l})$$

induces linear transformations for integers r and s

$$\mu_* : H_r(\mathcal{N}^*(M_k)) \bigotimes_R H_s(\mathcal{N}^*(M_l)) \to H_{r+s}(\mathcal{N}^*(M_{k+l}))$$

given by

$$\mu_*([U_1], [U_2]) = [\mu(U_1, U_2)]$$

for $[U_1] \in H_r(\mathcal{N}^*(M_k))$, and $[U_2] \in H_s(\mathcal{N}^*(M_l))$.

Proof: Let $U_1 \in Z^r(M_k, N_0(M_k))$ represent $[U_1] \in H_r(\mathcal{N}^*(M_k))$, and let $U_2 \in Z^s(M_l, N_0(M_l))$ represent $[U_2] \in H_s(\mathcal{N}^*(M_l))$. Then

$$bU_1 \in C^{r+1}(N_0(M_k)), \text{ and } bU_2 \in C^{s+1}(N_0(M_l)).$$

By the above corollary,

$$b\mu(U_1, U_2) = (-1)^s \mu(bU_1, U_2) + \mu(U_1, bU_2).$$

As $bU_1 \in C^{r+1}(N_0(M_k))$,

$$bU_1 = \sum_{\Sigma' \subset N_0(M_k)} U_{\Sigma'} \text{ with } U_{\Sigma'} \in C^{r+1}(\Sigma') \subset C^{r+1}(N_0(M_k)).$$

However, as $\Sigma' \cap \nabla_M = \emptyset$ (in M_k), it follows that

$$(\Sigma' \times M_l) \cap \nabla_M = \emptyset \text{ in } M_{k+l}.$$

This implies

$$\mu(bU_1, U_2) \subset C^{p+q+1}(N_0(M_{k+l})),$$

i.e., $[\mu(bU_1, U_2)] = 0$ in $\mathcal{N}^{r+s+1}(M_{k+l})$.

Similarly,

$$[\mu(U_1, bU_2)] = 0 \text{ in } \mathcal{N}^{r+s+1}(M_{k+l}).$$

This implies

$$\mu : Z^r(M_k, N_0(M_k)) \bigotimes_R Z^s(M_l, N_0(M_l)) \to Z^{r+s}(M_{k+l}, N_0(M_{k+l})).$$

To show μ is well-defined, suppose

$$U_1' = U_1 + bU' + U'' \text{ with } U' \in C^{r-1}(M_k) \text{ and } U'' \in C^r(N_0(M_k)).$$

Then

$$\mu(U_1', U_2) = \mu(U_1 + bU' + U'', U_2) = \mu(U_1, U_2) + \mu(bU', U_2) + \mu(U'', U_2).$$

However,

$$b\mu(U', U_2) = (-1)^s \mu(bU', U_2) + \mu(U', bU_2)$$
$$= (-1)^s \mu(bU', U_2) + \mu(U', U_2'') \text{ with } U_2'' \in C^s(N_0(M_l)).$$

As $U_2'' \in C^s(N_0(M_l))$ implies $\mu(U', U_2'') \in C^{r+s+1}(N_0(M_{k+l}))$, it follows that

$$\mu(bU', U_2) = (-1)^s b\mu(U', U_2) - (-1)^s \mu(U', U_2'').$$

Also $\mu(U'', U_2) \in C^{r+s+1}(N_0(M_{k+l}))$.

Hence

$$\mu(U_1', U_2) = \mu(U_1, U_2) + (-1)^s b\mu(U', U_2) + (\mu(U'', U_2) - (-1)^s \mu(U', U_2'')).$$

This implies

$$[\mu(U_1', U_2)] = [\mu(U_1, U_2)] \text{ in } H_{r+s}(\mathcal{N}^*(M_{k+l})).$$

A similar argument holds for the second factor.

This completes the proof of Proposition 5.3.1.

5.4 Intersection Pairing and Homology of $\mathcal{N}^*(M_k)$

The next theorem relates the homology of $\mathcal{N}^*(M_k)$ to the homology of M with real coefficients.

Theorem 5.4.1: Let M be a compact oriented combinatorial manifold of dimension m without boundary. Let k be a positive integer. Then

$$H_p(\mathcal{N}^*(M_k)) \cong H_{m-p}(M) \text{ for } p \in Z.$$

Proof: To compute the homology of the complex $\mathcal{N}^*(M_k)$ introduce the filtration $F^{p,q}$ on $C^*(M_k)$ used in Chapter 4. Given a pair of polysimplices (P, Σ) in M_k with $\Sigma \subset P$, let (l, t) denote the longitudinal and transverse barycentric coordinates of Σ in P. Let H_l and H_t denote the homotopy operators associated to the longitudinal and transverse coordinates, i.e.,

$$H_l(D_t^\alpha\{\omega \wedge [\Sigma]_P\} \circ j_P^*) = (-1)^p D_t^\alpha\{h_l\omega \wedge [\Sigma]_P\} \circ j_P^*,$$

where h_l is the homotopy operator on $\Omega^*(P)$ formed by using all longitudinal coordinates,

$$h_l\omega = \sum_{i=1}^{|\gamma|}(-1)^{i-1} l^{\gamma(i)} \cdot \int_0^1 \lambda^{|\gamma|-1}\omega(\lambda l, t)d\lambda dl^{\gamma-\gamma(i)} \wedge dt^\beta$$

on forms ω with $j_P^*\omega = \omega(l,t)dl^\gamma \wedge dt^\beta$.

$$H_t(D_t^\alpha\{rdt^\beta \wedge [\Sigma]_P\} \circ j_P^*) = \sum_{j=1}^{|\beta|} \frac{(-1)^{p+j-1}\alpha(\beta(j))}{\tilde{N}+ \mid \alpha \mid} D_t^{\alpha-\beta(j)}\{rdt^{\beta-\beta(j)} \wedge [\Sigma]_P\} j_P^*$$

$\tilde{N} = \dim(P) - \dim(\Sigma) + \mid \beta \mid$, and $\alpha(\beta(j))$ is the value of the multi-index α at the variable corresponding to $\beta(j) = j^{th}$ nonzero component of the index β.

From these homotopy operators and the spectral sequence, there follows

$$H_p(\mathcal{N}^*(M_k)) \cong H_{km-p}(S^*(M_k, N_0(M_k)) \text{ for } 0 \leq p \leq m,$$

$$H_p(\mathcal{N}^*(M_k)) = 0 \text{ for } p > m.$$

The next step is to construct a retraction $r : M_k - \nabla_M \to N_0(M_k)$ and to show it gives rise to an isomorphism

$$H_q(M_k, M_k - \nabla_M) \cong H_q(M_k, N_0(M_k))$$

for each integer q.

For a polysimplex $P = \rho_1 \times \cdots \times \rho_k$ with $P \cap \nabla_M \neq \emptyset$,

$$P \cap \nabla_M = \{(x, \cdots, x) \in M \times \cdots \times M = M_k \mid x \in \rho_1 \cap \cdots \cap \rho_k\}.$$

Let $\rho = \rho_1 \cap \cdots \cap \rho_k$, and $\nabla_\rho = P \cap \nabla_M$. Let $\hat{b}$ denote the barycenter of ρ. Then $(\hat{b}, \cdots, \hat{b})$ will be called the barycenter of ∇_ρ.

A retraction $r : M_k - \nabla_M \to N_0(M_k)$ can be constructed in the following manner: For $j = 1, \cdots, km$, let n_j be the number of polysimplices of dimension j which intersect the diagonal ∇_M in M_k and let $N = \sum_{j=1}^{km} n_j$.

For $j = 1, \cdots, km$, $i = 1, \cdots, n_j$, let P_i^j be an ordering of the set of simplices of dimension j with $P_i^j \cap \nabla_M \neq \emptyset$.

For $j = 1, \cdots, km$, $i = 1, \cdots, n_j$, let

$$s_i^j : P_i^j - \nabla_M \to \partial P_i^j - \nabla_M$$

by radial projection from the barycenter $\hat{b}$ of $P_i^j \cap \nabla_M$, i.e.

$$s(x) = (1 - t_x)\hat{b} + t_x x \text{ for } x \in P_i^j - \nabla_M$$

where $t_x \geq 0$ is chosen so that $s(x) \in \partial P_i^j$. This depends continuously on x.

For $j = 1, \cdots, km$, $i = 1, \cdots, n_j$, define

$$r_i^j : M_k - \nabla_M \to M_k - (\nabla_M \cup \ \mathrm{Int}(P_i^j))$$

by

$$r_i^j(x) = x \text{ for } x \in M_k - (\nabla_M \cup \ \mathrm{Int}(P_i^j))$$

$$r_i^j(x) = s_i^j(x) \text{ for } x \in P_i^j - (P_i^j \cap \nabla_M)$$

As s_i^j is the identity on $\partial P_i^j - (\partial P_i^j - \nabla_M)$, r_i^j is a continuous mapping.

A homotopy

$$h_i^j : (M_k - \nabla_M) \times [0,1] \to M_k - \nabla_M$$

between r_i^j and the identity on $M_k - \nabla_M$ is given by

$$h_i^j(x,t) = (1-t)x + t \cdot r_i^j(x) \text{ for } x \in M_k - \nabla_M \ , t \in [0,1].$$

For $j = 1, \cdots, km$, let $r^j = r_{n_j}^j \circ \cdots \circ r_1^j$. Then

$$r_{km} : M_k - \nabla_M \to M_k - (\nabla_M \cup \cup_{i=1}^{n_{km}} \mathrm{Int}(P_i^{km})),$$

$$r^{km-1} \circ r_{km} : M_k - \nabla_M \to M_k - (\nabla_M \cup \cup_{i=1}^{n_{km}} \mathrm{Int}(P_i^{km}) \cup \cup_{i=1}^{n_{km}} \mathrm{Int}(P_i^{km})).$$

Continuing in this manner, one has

$$r^1 \circ r^2 \circ \cdots \circ r_{km} : M_k - \nabla_M \to M_k - (\nabla_M \cup \cup_{l=1}^{km} \cup_{i=1}^{n_l} \mathrm{Int}(P_i^l)).$$

As

$$M_k - (\nabla_M \cup \cup_{l=1}^{km} \cup_{i=1}^{n_l} \mathrm{Int}(P_i^l)) = N_0(M_k),$$

$r = r^1 \circ \cdots \circ r^{km} : M_k - \nabla_M \to N_0(M_k)$ is continuous.

r is a retraction as $r \mid_{N_0(M_k)}=$ identity, and a homotopy

$$h : (M_k - \nabla_M) \times [0,1] \to M_k - \nabla_M$$

between r and the identity on $M_k - \nabla_M$ can be constructed as follows: Order the set $\{r_i^j\}$ such that $r_i^j < r_s^t$ if $j > t$, or $j = t$ and $i < s$. Order the set $\{h_i^j\}$ in the same manner, i.e. $h_i^j < h_s^t$ if $j > t$, or $j = t$ and $i < s$.

Let $\overline{r}_1 = r_1^{km}$ and define $\overline{r}_i = r_l^j \circ \overline{r}_{i-1}$ where r_l^j represents the i^{th} element of $\{r_i^j\}$ with respect to the ordering. Define

$$h : (M_k - \nabla_M) \times [0,1] \to M_k - \nabla_M$$

by

$$h(x,t) = h_i(\overline{r}_{i-1}(x), \frac{Nt - i + 1}{N}) \text{ for } t \in [\frac{i-1}{N}, \frac{i}{N}]$$

for $i = 1, \cdots, N$.

Then h is continuous, and $h(x,0) = x$, $h(x,1) = r(x)$, and $h \mid_{N_0(M_k)}=$ identity.

This implies $N_0(M_k)$ is a strong deformation retact of $M_k - \nabla_M$, and

$$r_* : H_q(M_k - \nabla_M) \to H_q(N_0(M_k))$$

is an isomorphism for all integers q. The long exact sequence of the pair $(M_k - \nabla_M, N_0(M_k))$ implies $H_q(M_k - \nabla_M, N_0(M_k)) = 0$ for all integers q. By the long exact sequence of the triple $(M_k, M_k - \nabla_M, N_0(M_k))$, it follows that

$$H_q(M_k, M_k - \nabla_M) \cong H_q(M_k, N_0(M_k)) \text{ for } q \in Z.$$

Remark: If one takes a triangulation of the cell structure on M_k which gives a triangulation of the diagonal, then the second barycentric subdivision of this triangulation produces a regular neighborhood $W^k \subset M_k$ of the diagonal (in the sense of Definition 6.3.1 of Stallings $[S_2]$), and by Proposition 6.6.8 of $[S_2]$, it follows that W^k collapses to the diagonal.

As ∇_M has an open neighborhood M' which retracts to the diagonal, the Thom Isomorphism Theorem (Corollary 11.20 of Dold [D] with $X = \emptyset$) implies

$$H_{q+(k-1)m}(M_k, M_k - \nabla_M) \cong H_q(M) \text{ for } q \in Z.$$

This establishes the isomorphism $H_q(\mathcal{N}^*(M_k)) \cong H_{m-q}(M)$ for $q \in Z$, and completes the proof of theorem 5.4.1.

Remark 5.4.2: This implies the homology of the complex $\mathcal{N}^*(M_k)$ does not depend on the combinatorial structure of M and is determined by the underlying topological manifold.

Remark 5.4.3: The chain homotopies of Theorem 5.4.1 show any chain $c \in C^*(M_k)$ representing an element of $H^*(\mathcal{N}^*(M_k))$ of the form

$$c = \sum_{(i,\alpha)\in A} c_{i,\alpha} D_t^{\alpha}\{\omega_{i,\alpha} \wedge [\Sigma_{i,\alpha}]\} \in C^*(M_k)$$

lies in the same homology class as a current of the form

$$\overline{c} = \sum_{j\in J} \overline{c}_j \{[\Sigma_j]_{[\Sigma_j]}\} j^*_{\Sigma_j} \in \overline{S}^*(M_k)$$

where the sets $\{[\Sigma_{i,\alpha}]\}_{(i,\alpha)\in A}$ and $\{[\Sigma_j]\}_{j\in J}$ contain the same elements, $\overline{S}^*(M_k)$ as defined in 4.6.1.

To simplify notation, let $\tilde{\Sigma}$ denote the current $\tilde{\Sigma} = \{[\Sigma]_{[\Sigma]}\} \circ j^*_{\Sigma}$ for an oriented polysimplex Σ in M_k. Also let $(\Sigma \times \Sigma')\tilde{}$ denote the current $(\Sigma \times \Sigma')\tilde{} = \{[\Sigma \times \Sigma']_{[\Sigma\times\Sigma']}\} \circ j^*_{\Sigma\times\Sigma'}$. (These is the same correspondence of $\overline{S}^*(M_k)$ with $S_*(M_k)$ as in 4.6).

If Σ is an oriented polysimplex in M_k, and Σ' is an oriented polysimplex in M_l, then the definition of μ implies

$$\mu(\tilde{\Sigma}, \tilde{\Sigma'}) = (\Sigma \times \Sigma')\tilde{}.$$

If

$$c_1 = \sum_{i=1}^{n_1} c_i \Sigma_i \in H_{km-r}(M_k),$$

$$c_2 = \sum_{j=1}^{n_2} c'_j \Sigma'_j \in H_{lm-s}(M_l),$$

let

$$\tilde{c}_1 = \sum_{i=1}^{n_1} c_i \tilde{\Sigma}_i \in C^r(M_k) \text{ , and}$$

$$\tilde{c}_2 = \sum_{j=1}^{n_2} c'_j \tilde{\Sigma}'_j \in C^s(M_l).$$

With this notation

$$\begin{aligned}\mu(\tilde{c}_1, \tilde{c}_2) &= \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} c_i c'_j (\Sigma_i \times \Sigma'_j)\tilde{} \\ &= (c_1 \times c_2)\tilde{}.\end{aligned}$$

This shows the identification of $\overline{S}^*(M_k, N_0(M_k))$ with $S_*(M_k, N_0(M_k))$ (denoted by $\tilde{}$) gives an identification between $\mu(\tilde{c}1, \tilde{c}2)$ and $c_1 \times c_2$. This implies $\mu_*([c_1], [c_2])$ corresponds to $[c_1 \times c_2]$ in homology.

Remark 5.4.4: Let $\overline{r}_k : M_k - \nabla_M^k \to N_0(M_k)$ denote the retraction of Theorem 5.4.1. In $M_{k+1} = M_k \times M_l$, let ∇_{M_2} denote $\nabla_M^k \times \nabla_M^l \subset M_k \times M_l$, and let $N_0(M_k, M_l)$ denote the subcomplex of $M_k \times M_l$ consisting of those polysimplices which do not intersect $\nabla_M^k \times \nabla_M^l$. By defining the barycenter of the polysimplex $\nabla_\sigma \times \nabla_\tau$ in $\nabla_M^k \times \nabla_M^l$ as the pair $(\hat{b}_\sigma, \hat{b}_\tau)$, the procedure of the theorem constructs a retraction

$$\overline{r}_2^{k,l} : M_k \times M_l - (\nabla_M^k \times \nabla_M^l) \to N_0(M_k, M_l).$$

This implies, as in the theorem,

$$H_*(M_{k+l}, M_{k+l} - \nabla_{M_2}) \cong H_*(N_0(M_k, M_l)).$$

It also follows that ∇_{M_2} has a regular neighborhood $W^{k,l}$ which retracts to the submanifold ∇_{M_2} ($W^{k,l}$ can be taken as the interior of the subcomplex consisting of simplices in the second barycentric subdivision of M_{k+l} which intersect the diagonal). Let $r_{k,l} : W^{k,l} \to \nabla_M^k \times \nabla_M^l$ denote this retraction, and let $r_k : W_k \to \nabla_M$ be the retraction associated to the collapse of W^k to ∇_M as in the remark of Theorem 5.4.1.

Remark 5.4.5: These observations imply (by 11.20 of Chapter 8 in Dold [D])

$$H_q(M_{k+l}, M_{k+l} - (\nabla_M^k \times \nabla_M^l) \cong H_{q-2m}(M_2)$$

by an isomorphism

$$\overline{c} \mapsto (r_{k,l})_*(\tau \cap \overline{c}) \text{ , for } \overline{c} \in H_q(M_{k+l}, M_{k+l} - (\nabla_M^k \times \nabla_M^l))$$

where $\tau = \tau_{1,1}^{k,l} = \tau_{M \times M}^{M_k \times M_l} : H^*(M_{k+l}, M_{k+l} - \nabla_{M_2})$ is the Thom class of the embedding of M_2 into M_{k+l} as $\nabla_M^k \times \nabla_M^l$.

Remark 5.4.6: The results of this section give the following sequence of isomorphisms

$$H_q(\mathcal{N}^*(M_k)) \to H_q(\overline{S}^*(M_k, N_0(M_k)) \to H_q(M_k, N_0(M_k)) \xleftarrow{\overline{r}_*^k} H_q(M_k, M_k - \nabla_M)$$

$$\xrightarrow{\tau\cap} H_{q-(k-1)m}(W^k) \xrightarrow{r_{k*}} H_{q-(k-1)m}(\nabla_M) \xleftarrow{d_*} H_{q-(k-1)m}(M)$$

where the first two isomorphisms are from the chain homotopies and spectral sequence of Theorem 5.4.1, $\overline{r}_*^k$, r_{k*} are induced by the retractions, τ is the Thom class of the diagonal, and $d : M \to M_k$ is the diagonal embedding.

Remark 5.4.7: Let $d : M \to M \times M$ denote the diagonal embedding, and let $j : (M \times M, \emptyset) \to (M \times M, M \times M - \nabla_M)$ be the inclusion. If $c_1 \in H_r(M)$, $c_2 \in H_s(M)$, then the intersection product $c_1 \bullet c_2 \in H_{r+s-m}(M)$ satisfies, (and is determined by)

$$d_*(c_1 \bullet c_2) = (-1)^{m(m-s)} r_{2*} \circ \tau \cap j_*(c_1 \times c_2)$$

where τ is the Thom class of d. This follows immediately from the definitions of τ and the intersection pairing (as given in Dold [D]).

Remark 5.4.8: Let $c_1 \in S_r(M)$, $c_2 \in S_s(M)$ represent homology classes $[c_1]$, $[c_2] \in H_*(M)$. Let $\tilde{c}_1$, $\tilde{c}_2$ denote the corresponding chains in $\overline{S}^*(M)$, and let

$$h_* : \overline{S}^{m-r}(M) \to S_r(M)$$

denote the inverse to $\tilde{\ }: S_r(M) \to \overline{S}^{m-r}(M)$.

Then

$$([c_1] \times [c_2])\tilde{}=[(c_1 \times c_2)\tilde{}]$$
$$= [\tilde{c}_1 \times \tilde{c}_2]$$
$$= [\mu(\tilde{c}_1, \tilde{c}_2)].$$

Hence, (with the notation of 5.4.7)

$$d_*([c_1] \bullet [c_2])=(-1)^{m(m-s)} r_{2*} \circ \tau \cap j_*([\tilde{c}_1] \times [\tilde{c}_2])$$
$$= (-1)^{m(m-s)} r_{2*} \circ \tau \cap j_* h_*(([c_1] \times [c_2])\tilde{})$$
$$= (-1)^{m(m-s)} r_{2*} \circ \tau \cap j_* h_*[\mu(\tilde{c}_1, \tilde{c}_2)].$$

This establishes the relation between the intersection pairing on $H_*(M)$ and

$$\mu_{1,1} : \mathcal{N}^*(M) \times \mathcal{N}^*(M) \to \mathcal{N}^*(M_2).$$

Remark 5.4.9: For the general case, let

$$[U_1] \in H_{r+(k-1)m}(\mathcal{N}^*(M_k))$$

$$[U_2] \in H_{s+(l-1)m}(\mathcal{N}^*(M_l)).$$

Let $c_1 \in S_r(M)$, $c_2 \in S_s(M)$ represent homology classes in $H_*(M)$ corresponding to $[U_1]$, $[U_2]$ under the isomorphism of Theorem 5.4.1, i.e.

$$[c_1] = r_{k*} \circ \tau_1^k \cap [U_1]$$

$$[c_2] = r_{l*} \circ \tau_1^l \cap [U_2].$$

Then

$$[c_1] \times [c_2]=(r_{k*} \circ \tau_1^k \cap [U_1]) \times (r_{l*} \circ \tau_1^l \cap [U_2])$$
$$= (r_{k+l})_* \circ \tau_{1,1}^{k,l} \cap ([U_1] \times [U_2])$$
$$= (r_{k+l})_* \circ \tau_{1,1}^{k,l} \cap [\mu(U_1 \times U_2)].$$

Hence

$$\begin{aligned}
d_*([c_1] \bullet [c_2]) &= (-1)^{m(m-s)}(r_{k+l})_* \circ \tau_1^{1,1} \cap ([c_1] \times [c_2]) \\
&= (-1)^{m(m-s)}(r_{k+l})_* \circ \tau_1^{1,1} \cap (\tau_{1,1}^{k,l} \cap [\mu(U_1 \times U_2)] \\
&= (-1)^{m(m-s)}(r_{k+l})_* \circ (\tau_1^{1,1} \cup \tau_{1,1}^{k,l}) \cap [\mu(U_1 \times U_2)] \\
&= (-1)^{m(m-s)}(r_{k+l})_* \circ \tau_1^{k,l} \cap [\mu(U_1, U_2)]
\end{aligned}$$

i.e.

$$d_*(\tau_1^k \cap [U_1]) \bullet (\tau_1^l \cap [U_2]) = (-1)^{m(m-s)}(r_{k+l})_* \circ \tau_1^{k,l} \cap [\mu(U_1, U_2)].$$

The equivalence of $\tau_1^{1,1} \cup \tau_{1,1}^{k,l}$ with $\tau_1^{k,l}$ is by Proposition 11.26, Chapter 8 of Dold [D].

This expresses the relation between μ, the intersection pairing, and the Thom class.

Example 5.4.10: Let M be a compact oriented combinatorial manifold without boundary. Define a current $U_\mathcal{E} \in C^*(M)$ by the equation

$$U_\mathcal{E} = \sum_{p=0}^{m} \sum_{\dim(\sigma^p)=p} (-1)^p \cdot \frac{p!}{N(\sigma^p)} \{[\sigma^p]\}$$

where the second sum is over all p-simplices σ^p of M, and $N(\sigma^p)$ is the number of maximal simplices of M which contain σ. When this current is evaluated on the Sullivan form μ, it gives the Euler characteristic of M. This example shows how a current can contain information about the entire combinatorial structure of the combinatorial manifold.

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Appendix

The Chain Homotopy H_t.

This appendix is devoted to the proof of the claim made in Remark 4.5.7. The chain homotopy constructed in this appendix is the transpose of the homotopy operator h_t defined on $\Omega^*(P)$ where P is a polysimplex containing Σ. More precisely, let $\Sigma = \sigma_1 \times \cdots \times \sigma_k$ be a polysimplex in M_k, and for polysimplices $P = \rho_1 \times \cdots \times \rho_k$ in M_k containing Σ, let (x, t) denote a coordinate system with the x-coordinates longitudinal to Σ, and the t-coordinates transverse to Σ. Let $N = \dim(\Sigma)$.

Define $h^t_\Sigma : \Omega^*(P) \to \Omega^*(P)$ by

$$h^t_\Sigma \phi = \sum_{\delta,\epsilon} \sum_{i=1}^{|\delta|} (-1)^i t^{\delta(i)} \int_0^1 \lambda^{|\delta|-1} \phi_{\delta,\epsilon}(x, \lambda t) d\lambda dt^{\delta-\delta(i)} \wedge dx^\epsilon,$$

for $\phi = \sum_{\delta,\epsilon} \phi_{\delta,\epsilon}(x,t) dt^\delta \wedge dx^\epsilon \in \Omega^*(P)$ with $\delta \neq 0$, and set $h^t(\phi) = 0$ if $\delta = 0$.

For each pair (P, Σ) define $H_t : C^*(M_k) \to C^*(M_k)$ by

$$H_t U(\phi) = U(h^t_\Sigma \phi) \text{ for } U \in C^*(P, \Sigma, \partial\Sigma)\ ,\ \phi \in \Omega^*(M_k)$$

Proposition A.1: For any nonnegative integer p,

$$H : C^p(M_k) \to C^{p+1}(M_k),$$

and for

$$U = D_t^{\alpha}\{\omega_{\alpha,\beta} \wedge dt^{\beta} \wedge [\Sigma]\} \circ j_P^*,$$

with $\beta \neq 0$, $H(U)$ is given by

$$H(U) = \sum_{j=1}^{|\beta|}(-1)^{j-1}\frac{\alpha(\beta(j))}{\tilde{N}+\alpha}D_t^{\alpha-\beta(j)}\{\omega_{\alpha,\beta} \wedge dt^{\beta-\beta(j)} \wedge [\Sigma]\} \circ j_P^*,$$

where for $1 \leq j \leq\mid \gamma \mid$, and $\alpha(\beta(j))$ is the value of the multi-index α corresponding to the variable $t^{\beta(j)}$, and $\tilde{N} =$ $\text{codim}_P(\Sigma)-\mid \beta \mid$.

If $\beta = 0$, then $H(U) = 0$.

Proof: To show H maps $C^p(M_k) \to C^{p-1}(M_k)$, it is necessary to show that H has a representation of a certain form. Hence, it is necessary to calculate an explicit formula for $H(U)$ for $U \in C^*(M_k)$. Let (x,t) be coordinates for the pair (P,Σ) as above.

Let

$$U = D_t^{\alpha}\{\omega(x,t)dx^{\gamma} \wedge dt^{\beta} \wedge [\Sigma]\} \circ j_P^* \in C^*(P,\Sigma,\partial\Sigma),$$

$$\phi = \sum_{\epsilon,\delta}\phi_{\epsilon,\delta}(x,t)dt^{\delta} \wedge dx^{\epsilon} \in \Omega^*(P).$$

Remark: $U(\phi_{\epsilon,\delta}dt^{\delta} \wedge dx^{\epsilon}) = 0$ unless $dt^{\delta} \wedge dx^{\epsilon} \wedge dx^{\gamma} \wedge dt^{\beta} = \pm\mu^k$. Let γ', β' denote the multi-indices with $dt^{\beta'} \wedge dx^{\gamma'} \wedge dx^{\gamma} \wedge dt^{\beta} = \mu^k$. Hence

$$U(\phi) = U(\phi_{\gamma',\beta'}dt^{\beta'} \wedge dx^{\gamma'})$$

(after combining all terms of ϕ which can be expressed as a function times $dx^{\beta'} \wedge dx^{\gamma'}$).

Remark: If $dt^{\delta-\delta(i)} = dt^{\beta'}$, then the differential $dt^{\delta(i)}$ does not appear in $dt^{\beta'}$. This implies the differential $dt^{\delta(i)}$ appears in dt^{β}. As $dt^{\delta-\delta(i)} = dt^{\beta'}$ does not contain any differentials in dt^{β}, it follows that $dt^{\delta} = \pm dt^{\delta(i)} \wedge dt^{\beta'}$. Let j be such that $dt^{\delta(i)} = dt^{\beta(j)}$, and define $\nu(j) = \pm 1$ and $\nu_{\beta,\gamma} = \pm 1$ by the conditions

$$\begin{aligned} dt^{\delta} \wedge dx^{\gamma'} \wedge dx^{\gamma} \wedge dt^{\beta} &= \nu(j) dt^{\delta-\delta(i)} \wedge dx^{\gamma'} \wedge dt^{\delta(i)} \wedge dx^{\gamma} \wedge dt^{\beta} \\ &= (-1)^{j-1} \cdot \nu_{\beta,\gamma}(j) \cdot \mu^{k}. \end{aligned}$$

Set $A = \{j \mid t^{\beta(j)} = t_i^a \text{ with } \alpha_i^a \neq 0\}$, and set $\tilde{N} = \operatorname{codim}_P(\Sigma) - \mid \beta \mid$. For an index $j \in A$, $\partial/\partial t^{\beta(j)}$ appears in the expression D_t^{α}, and therefore corresponds to some index $\alpha(h)$ occuring in D_t^{α}. Denote the corresponding h by $h = \beta(j)$, and the corresponding value of α by $\alpha(\beta(j))$. With these notational conventions,

$$\begin{aligned} HU(\phi) = U(h_t\phi) &= U(\sum_{\epsilon,\delta} \sum_{i=1}^{|\delta|} (-1)^i t^{\delta(i)} \int_0^1 \lambda^{|\delta|-1} \phi_{\epsilon,\delta}(x, \lambda t) d\lambda dt^{\delta-\delta(i)} \wedge dx^{\epsilon}) \\ &= U(\sum_{i=1}^{|\beta|} \nu(j) \cdot t^{\beta(j)} \int_0^1 \lambda^{\tilde{N}} \phi_{\gamma',(\beta-\beta(j))'}(x, \lambda t) d\lambda dt^{\beta'} \wedge dx^{\gamma'}) \\ &= \sum_{j=1}^{|\beta|} \nu(j) [\Sigma] \{\omega(x,t) D_t^{\alpha}(t^{\beta(j)} \int_0^1 \lambda^{\tilde{N}} \phi d\lambda)(x, 0) dt^{\beta'} \wedge dx^{\gamma'} \wedge dx^{\gamma} \wedge dt^{\beta}\} \\ &= \sum_{j=1}^{|\beta|} \nu_{\beta,\gamma}(j) \cdot [\Sigma] \{\omega(x,t) D_t^{\alpha}(t^{\beta(j)} \int_0^1 \lambda^{\tilde{N}} \phi d\lambda)(x, 0) \mu^{k}\} \end{aligned}$$

$$= \sum_{j=1}^{|\beta|} \nu_{\beta,\gamma}(j) \cdot [\Sigma]\{\omega(x,t) t^{\beta(j)} D_t^{\alpha}(\int_0^1 \lambda^{\tilde{N}} \phi(x, \lambda t) d\lambda)(x,0)\mu^k\}$$

$$+ \sum_{A} \nu_{\beta,\gamma}(j) \cdot [\Sigma]\{\alpha(\beta(j)) \cdot \omega(x,t) D_t^{\alpha-\beta(j)}(\int_0^1 \lambda^{\tilde{N}} \phi(x, \lambda t) d\lambda)(x,0)\mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \cdot \alpha(\beta(j)) \cdot [\Sigma]\{\omega(x,t) D_t^{\alpha-\beta(j)}(\int_0^1 \lambda^{\tilde{N}} \phi(x, \lambda t) d\lambda)(x,0)\mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \alpha(\beta(j)) [\Sigma]\{\omega(x,t)(\int_0^1 \lambda^{\tilde{N}} D_t^{\alpha-\beta(j)}(\phi(x, \lambda t)) d\lambda)(x,0)\mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \alpha(\beta(j)) [\Sigma]\{\omega(x,t)(\int_0^1 \lambda^{\tilde{N}+\alpha-1} (D_t^{\alpha-\beta(j)}\phi)(x, \lambda t) d\lambda)(x,0)\mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \cdot \alpha(\beta(j)) \cdot [\Sigma]\{\omega(x,t) \cdot (\int_0^1 \lambda^{\tilde{N}+\alpha-1} (D_t^{\alpha-\beta(j)}\phi)(x, 0) d\lambda) \cdot \mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \cdot \alpha(\beta(j)) \cdot [\Sigma]\{\omega(x,t) \cdot (\int_0^1 \lambda^{\tilde{N}+\alpha-1} d\lambda)(D_t^{\alpha-\beta(j)}\phi)(x, 0) \cdot \mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \cdot \frac{\alpha(\beta(j))}{\tilde{N}+\alpha} \cdot [\Sigma]\{\omega(x,t) \cdot (D_t^{\alpha-\beta(j)}\phi)(x, 0) \cdot \mu^k\}$$

$$= \sum_{A} \nu_{\beta,\gamma}(j) \cdot \frac{\alpha(\beta(j))}{\tilde{N}+\alpha} \cdot [\Sigma]\{\omega(x,t)(D_t^{\alpha-\beta(j)}\phi_{\gamma',(\beta+\beta(j))'})(x, 0)\mu^k\}$$

$$= \sum_{j=1}^{|\beta|} (-1)^{j-1} \frac{\alpha(\beta(j))}{\tilde{N}+\alpha} D_t^{\alpha-\beta(j)}\{\omega_{\alpha,\beta} \wedge dt^{\beta-\beta(j)} \wedge [\Sigma]\} \circ j_P^* \phi.$$

Remark: $\sum_A$ can be replaced by $\sum_{j=1}^{|\beta|}$ as the terms with transverse coordinates $t_i^a = t^{\beta(j)}$ corresponding to the additional indices j are such that the $\alpha(\beta(j))$ corresponding to these terms are zero.

For the last claim of the proposition, if

$$U = D_t^\alpha\{\omega \wedge [\Sigma]\} \circ j_P^* \in C^*(M_k),$$

and $\phi \in \Omega^*(M_k)$ with $j_P^*\phi = \sum_{\delta,\epsilon} \phi_{\delta,\epsilon}(x,t)dt^\epsilon \wedge dx^\delta$, then $H(U)(\phi) = U(h^t\phi)$ which is zero, as $h^t j_P^*(\phi)$ does not contain the differential $dt^P = dt^1 \wedge \cdots \wedge dt^{N'}$. As ϕ was an arbitrary element of $\Omega^*(M_k)$, it follows that $H(U) = 0$ if $\beta = 0$. This completes the proof of the proposition.