On the geodesic flow of Zoll manifolds

A Dissertation Presented

by

Carlos Eduardo Durán Fernández

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

December 1994

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation
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1994

We examine in detail the geodesic flow of *Zoll manifolds*, that is, Riemannian or Finsler manifolds such that all of its geodesics are closed with the same period.

Our main results are:

1) We give several geometric conditions which indicate 'closeness' of Riemannian manifolds under which two Zoll metrics geometrically close to each other have symplectically conjugate geodesic flows.

2) The set of Finsler Zoll metrics is locally path connected: if g_0 and g_1 are two Finsler Zoll metrics on a manifold M which are

close to each other, then there is a one parameter family of Finsler Zoll metrics, all with mutually conjugate geodesic flows, joining g_0 and g_1.

3) The geodesic flow of a Zoll manifold is completely integrable in the Liouville sense. Applications to the canonical examples are given.

A mis padres

Hay dos modos de conciencia:
una es luz, y otra, paciencia.
Una estriba en alumbrar
un poquito en el hondo mar;
otra, en hacer penitencia
con caña o red, y esperar
el pez, como pescador.
Dime tú: ¿Cual es mejor?
¿ Conciencia de visionario
que mira en el hondo acuario
peces vivos,
fugitivos,
que no se pueden pescar,
o esa maldita faena
de ir arrojando a la arena,
muertos, los peces del mar?

Antonio Machado

Contents

Acknowledgements

I would like to acknowledge special mentors and friends through the years:

Detlef Gromoll, my thesis advisor, who in these years has always patiently tried to share his enormous wealth of geometric insight with me. For him a geodesic is a line, more than the solution of a second order ODE; the curvature tensor is a measure of how a manifold and its geodesics bend, more than a tensor which is given by a second order non-linear PDE on the metric. Each time I went to our meetings depressed and tired of wrestling with the Jacobi equation, he would lift my spirits with his congenial ways and a new arsenal of geometric ideas.

Lázaro Recht, who first introduced me to the wonders of Differential and Riemannian geometry. It all began with a rather shy undergraduate student asking for a project, and him coming back with the assignment of reading Milnor's 'Morse Theory'.

Gerardo Mendoza, who made me appreciate the strange beauty that lies in the harshness of advanced, hard mathematics.

My alter ego Juan Carlos Alvarez, 'la otra Urraca'; for his friendship and collaboration since our undergraduate years. I hope it continues flourishing in the future.

I would have never gotten past the undergraduate level were it not for the unselfish help of Neidha Nadal. Thanks, Neidha.

All my friends here, specially Robert Stingley, Lori Stone, LeRoy Wenstrom, Jane Hurley and Bob Donley, who did their best to make me feel at home in this very strange but interesting country, and Pablo Ares and Jan Kiwi, who oddly enough represented normalcy for me, a stranger in a strange land.

To all my colleagues and friends back home, for making my vacations wonderful, specially Julio César Alvarez, Jose Vicente Mandé, and the researchers and staff of the mathematics department at IVIC.

I would like to give special thanks to Santiago Simanca; he is always willing and able to help; both in mathematics and in any other way. His constant mentoring is one of the main reasons I survived this five years with some degree of sanity left. Now he is bracing himself to help a new generation. Keep up the good work, Santiago.

And, last but not least, thanks to all the wonderful folks at the Suffolk Institute dojo.

Chapter 1

Introduction

A *Zoll manifold* (M, g) is, roughly speaking, a Riemannian manifold 'all of whose geodesics are closed'. This concept can be made precise in numerous (a priori non-equivalent) ways:

1. The less restrictive definition requires that, for all geodesics γ emanating from a point $p \in M$, γ eventually returns to p, and this happens for all $p \in M$.

2. The most restrictive definition requires that the geodesic flow is periodic, with the same minimal period L for all geodesics, and all geodesics are required to be simple.

For definitions in between and pointwise conditions, see [Bes78, chapter 7]. Although *a priori* the second definition is more restrictive, all known examples of smooth metrics satisfying (1) also satisfy (2), but it is a mostly open problem to determine if that happens in general. It is know that any metric on S^2 satisfying (1) also satisfies (2) ([GG81]).

We will work with the following definition:

Definition. A Riemannian manifold (M, g) is a *Zoll manifold* if the cogeodesic flow of g on the unit cotangent bundle U^*M is periodic, with all orbits of the same minimal period 2π.

Remark. The definition makes perfect sense also for a *Finsler* manifold. Most of our constructions, and theorems A, D and E, will work in this more general category.

Remark. We assume that the manifold M is simply connected.

A natural problem is then to characterize Zoll manifolds. In 1903, O. Zoll ([Zol03]) constructed examples of non-standard Zoll metrics on the sphere S^2. After a long period of dormancy, the subject was revitalized in the seventies by works of Guillemin ([Gui76]), Weinstein ([Wei77]) and, specially, the beautiful book of Besse ([Bes78]).

The compact rank one symmetric spaces (from here on referred to as CROSSes) with their canonical metrics are Zoll manifolds. The natural question to be asked is: is the Zoll condition geometrically rigid? Are there any other Zoll manifolds not isometric to CROSSes? It is a very interesting problem to topologically or differentiably characterize which manifolds admit a Zoll metric, although all known examples are metrics on manifolds diffeomorphic to CROSSes. Also, topologically, Zoll manifolds are very similar to CROSSes, by the work of Bott and Samelson (see chapter 5 for the statement of the Bott-Samelson theorem; see also [Bes78, chapter 7]), [Bot54], [Sam63])

Therefore most of the research on Zoll manifolds concentrates on the following question: given a metric g on a manifold M diffeomorphic to a CROSS, must (M, g) be isometric to (M, can)? The state of the question is as follows:

- There are non-trivial Zoll deformations of the canonical metric on the sphere S^n for any n. ([Zol03], [Bes78, chapter 4])
- The real projective spaces are rigid, i.e. any Zoll metric on $\mathbb{R}P^n$ is isometric to the canonical metric on $\mathbb{R}P^n$ ([Bes78, Appendix D])
- The projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbf{Ca}P^2$ are *infinitesimally* rigid, i.e. for any 1-parameter family of Zoll metrics g_t with $g_0 =$ the canonical metric, the derivative at zero $h = \frac{d}{dt}g_t|_{t=o}$ is tangent to the action of the diffeomorphism group. ([Tsu81])
- The projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbf{Ca}P^2$ are *locally* rigid, i.e. any Zoll metric g which is sufficiently close (in a C^k norm, with k depending on the dimension) to the canonical metric is isometric to the canonical metric. ([Kiy87])

The techniques used in dealing with Zoll manifolds seem to indicate that, intuitively, the Zoll property is a *dynamical* property of the geodesic flow of the manifold, rather than a *geometrical* property on M. That is, the natural place to study Zoll manifolds is not M, but the tangent bundle TM or the unit tangent bundle UM (or their duals). Therefore a basic problem is to understand dynamically the geodesic flow of Zoll manifolds, in the sense of problem 10 in the list of problems in the introduction of [Bes78]. We will see (proposition 1.1.1) that such an understanding can lead to a solution of the isometry problem.

In what follows we describe our results concerning the dynamics of Zoll manifolds:

1.1 Symplectic Conjugacy of Geodesic Flows of Zoll manifolds

One necessary condition for geometric rigidity is *symplectic rigidity*, which we define below:

Let (M, g_0), (M, g_1) be two Riemannian metrics on the same manifold M, H_0, H_1 the associated energy functions on T^*M, $H_i(\xi) = g_i(\xi^\flat, \xi^\flat)$ where $\flat$ is the musical isomorphism between T^*M and TM.

Definition. The metrics g_0 and g_1 are said to be *symplectically equivalent* (or *dynamically equivalent*) if there exists an exact symplectomorphism $\phi : T^*M\backslash 0 \to T^*M\backslash 0$ such that $H_1 = H_0 \circ \phi$.

Observe that if two metrics are symplectically equivalent, their cogeodesic flows (the geodesic flow translated to the cotangent bundle via the canonical 'musical' isomorphism between TM and T^*M) are conjugated by ϕ; thus the name dynamically equivalent. Also we need to remove the zero section from the domain of ϕ to avoid trivialities, in view of the following result (see [AM78, page 186, 3.2F]):

Proposition 1.1.1 *Let* (M, g_0) *and* (M, g_1) *be two symplectically equivalent metrics. Then the symplectomorphism* ϕ *can be extended smoothly to the zero section if and only if* (M, g_0) *and* (M, g_1) *are isometric.*

This proves that 'symplectic rigidity' is a necessary condition for 'geometric rigidity'. Therefore a possible program to attack the isometry problem for Zoll manifolds is: first, show that given two Zoll metrics $(M, g_0), (M, g_1)$ are symplectically conjugate. Next, show that this conjugacy extends smoothly to the zero section, thus solving the isometry problem in view of proposition 1.1.1. We have given partial solutions to the first part of this program, for general Zoll metrics (Theorem I) or versions adapted to the case in which (M, g_0) is a CROSS (Theorems II and III).

All known examples of Zoll metrics are symplectically equivalent to the canonical metric on their model manifolds (including the non-isometric examples on S^n); thus a natural problem is to determine if this always happens. The following results are known:

- It is true for Zoll metrics on S^2 ([Gui76, appendix B].
- It is true for conformal Zoll deformations of an arbitrary Zoll manifold ([Bes78, chapter 4]).
- It is true for any smooth one parameter family of Zoll metrics ([Wei76]).

We prove the following discrete versions of symplectic equivalence:

Theorem I *Let (M, g_0), (M, g_1) be two Zoll metrics which are C^2 close. Then g_0 and g_1 are symplectically equivalent.*

Remark. This theorem remains true also for Zoll Finsler metrics.

We improve theorem A by stating geometric conditions (weaker than the metrics being C^2-close) that indicate that (M, g) is 'close' to the canonical metrics on a CROSS; specifically, we want the curvature tensor to behave as it does in a CROSS:

A Riemannian metric is said to be ϵ-almost symmetric if $|\nabla R| < \epsilon$. The *pinching* of a compact Riemannian manifold of nonnegative curvature is defined as $\delta_M = K_{min}/K_{max}$, where K_{min} (resp. K_{max}) is the minimum (resp. maximum) of the sectional curvatures over all 2-planes of TM. A Riemannian manifold is said to be ϵ-quarter pinched if the pinching $\delta_M > 1/4 - \epsilon$, ϵ-one pinched if the pinching satisfies $\delta_M > 1 - \epsilon$.

Then we have

Theorem II *There is $\epsilon > 0$ such that if (M, g) is an ϵ-almost symmetric, Zoll metric, then M is diffeomorphic to a CROSS and (M, g) is symplectically equivalent to (M, can).*

Theorem III *There is $\epsilon > 0$ such that*

- *If (M, g) is an ϵ-one pinched Zoll metric then M is diffeomorphic to S^n and (M, g) is symplectically equivalent to (M, can)*

- *If (M, g) is an ϵ-quarter pinched Zoll metric and M is not diffeomorphic to S^n, then M is diffeomorphic to a projective space and (M, g) is symplectically equivalent to (M, can).*

In the realm of Finsler Zoll metrics, we have the following local path connectedness result:

Theorem IV *Let (M, g_0), (M, g_1) be two Zoll metrics which are symplectically equivalent and C^2-close. Then there is a 1-parameter family g_t of Finsler Zoll metrics, all symplectically equivalent to each other, joining g_0 and g_1.*

It is probably true that, if g_0 and g_1 are Riemannian metrics, then the deformation g_t can be chosen to be through *Riemannian* metrics.

1.2 Integrability of the Geodesic Flow of Zoll manifolds

There has been a renewed interest in recent years on the study of completely integrable Hamiltonian flows, with many new examples discovered relatively recently. ([Fom88]). However, for geodesic flows, the classical origin of the theory, progress has been slower. The condition of having completely integrable geodesic flow on a manifold seems to be rather restrictive on both the topology and the metric on (M, g); see for example [Pat91], and there are very few examples of Riemannian manifolds with completely integrable geodesic flows. For a recent review of complete integrability of geodesic flows, see [Spa90].

We show that if (M, g) is any Zoll manifold, its geodesic flow is completely integrable, giving us a potential new source of examples.

Theorem V *The geodesic flow on any Zoll manifold (M^n, g) is completely integrable.*

As an immediate consequence, we have

Corollary *The geodesic flow on a compact symmetric space of rank one is completely integrable.*

This has been proved relatively recently, using some rather involved Lie-algebraic calculations. The integrating functions we find are not as explicit as in ([Thi81] or [GS83]), but the construction of the functions using the Zoll property and the space of geodesics is a lot less involved.

Chapter 2

Mathematical Preliminaries

In this chapter we review the basic mathematical concepts used in this work. Most of the material is covered in detail in [AM78]; another reference is [Arn89]. For specific topics, basic references can be: [MS95] (symplectic geometry); [BR95] (contact geometry) and [KN63] (principal fiber bundles). For a very nice discussion of the geodesic flow on Riemannian manifolds, see chapter 3 of [Kli82].

2.1 Symplectic and Contact Geometry

2.1.1 Symplectic geometry

A *symplectic form* on a manifold X is 2-form ω satisfying:

1) ω is closed, i.e. $d\omega = 0$.

2) ω is non-degenerate, i.e the transformation $T_\omega : T_xX \to T_x^*X$ given by $v \to \omega_x(v, \cdot)$ is an isomorphism for all $x \in X$.

A pair (X,ω) of a manifold and symplectic form ω on X is called a *symplectic manifold.*

Example.

Let $X = \mathbb{C}^n$, with coordinates $(z_1,\ldots,z_n)$, $z_k = x_k + iy_k$. Set $\omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$. Then ω is a symplectic structure on $\mathbb{C}^n$.

A crucial property of symplectic manifolds is that they are locally isomorphic to previous example:

Theorem 2.1.1 (Darboux) *Let (X,ω) be a symplectic manifold, $x \in X$. Then there is an open neighborhood U_x of x and a chart $\phi : U_x \to \mathbb{C}^n$ such that $\phi^*\omega_0 = \omega$.*

In particular, *symplectic manifolds have no local invariants.* The only 'local invariant' of a symplectic manifold is its dimension, which has to be even. Since ω is closed, there is an obvious global invariant attached to a symplectic manifold, namely the cohomology class $[\omega] \in H^2(M,\mathbb{R})$.

Another aspect of symplectic manifolds is that symplectic structures cannot be deformed non-trivially in a given cohomology class, in view of the following theorem of Moser:

Theorem 2.1.2 *Let ω_t, $t \in [0,1]$, be a one parameter family of cohomologous symplectic forms on a compact manifold M. Then there exists a one parameter family of diffeomorphisms $\phi_t : M \to M$ such that $\phi_t^*\omega_0 = \omega_t$.*

A consequence of Moser's technique is the following:

Theorem 2.1.3 *Let ω_0 and ω_1 be two cohomologous symplectic form on a compact manifold M, which are close in the supremum norm. Then there exists a diffeomorphism $\phi : M \to M$, close to the identity in the C^1 norm, such that $\phi^*\omega_0 = \omega_1$.*

An important special class of symplectic manifolds are those for which ω is *integral*, that is $[\omega]$ lies in the image of $H^2(M, \mathbb{Z})$ under the coefficient homomorphism $c : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. Next we give the 'universal' example of such a situation:

Example.

Let $X = \mathbb{C}P^n$. Define Ω as follows: let $p \in \mathbb{C}P^n$. The tangent space $T_p\mathbb{C}P^n$ can be thought as the space $\mathcal{L}(p, p^\perp)$ of linear transformations from the one-dimensional subspace p into its orthogonal complement. Given to such transformations X_p, Y_p, define $h(X, Y) = trace(X^*Y)$. Writing h in its real and imaginary parts, we have $h(X, Y) = g(X, Y) + i\omega(X, Y)$. The real part g is the canonical Fubini-Study Riemannian metric on $\mathbb{C}P^n$, and ω is the canonical symplectic form, which is also the Kaehler form of the metric.

This example is 'universal' in the following way: if (X, ω) and (Y, σ) are symplectic manifolds, a *symplectic embedding* $F : (X, \omega) \to (Y, \sigma)$ is an embedding $F : X \to Y$ satisfying $F^*\sigma = \omega$. Then we have [D'A87]:

Theorem 2.1.4 *Let (X, ω) be an integral symplectic manifold. Then there is a symplectic embedding $F : (X, \omega) \to (\mathbb{C}P^N, \Omega)$.*

We will use a delicate version of theorem 2.1.4 adapted to our particular case.

2.1.2 Contact geometry

Roughly speaking, contact geometry is to odd-dimensional manifolds what symplectic geometry is to even dimensional manifolds.

A *contact structure* on a manifold M^{2n+1} is a $2n$ dimensional distribution Δ which is non-degenerate in the following sense:

Locally, the distribution Δ is given by $\Delta_p = \ker \alpha_p$, for some 1-form $\alpha \in \Lambda^1(M)$; Δ is *non-degenerate* if $\alpha \wedge (d\alpha)^n$ is locally a volume form. A 1-form α satisfying the non-degeneracy condition is called a *contact form.* Observe that $\alpha \wedge d\alpha^n$ being a volume form is equivalent to $d\alpha|_\Delta$ being non-degenerate. Two contact forms defining the same distribution are proportional, and the non-degeneracy is independent of the choice of the contact forms.

Let us remark that the contact structure is given by the *distribution*, not the contact form. Given a contact manifold (M, Δ), a *contact transformation* is a diffeomorphism $f : M \to M$ that preserves the distribution Δ; i.e $f_*\Delta = \Delta$. Given a contact form α adapted to the distribution Δ, this is equivalent to the condition $f^*\alpha = \lambda\alpha$, where $\lambda : M \to \mathbb{R}$ is a smooth, never vanishing function.

Given a contact manifold (M, Δ) with contact form α, the *characteristic line field* E_Δ is the one dimensional distribution given by $E_\Delta(x) = \{v \in T_xM \,/\, d\alpha(v, \cdot) \equiv 0\}$. The *characteristic vector field* ξ is the unique vector field defined by $\xi_x \in E_\Delta(x)$, $\alpha(\xi) = 1$. The characteristic distribution is independent of α, but α is necessary for the normalization required to define the characteristic vector field.

Example.

Let $M = \mathbb{C}^n \times \mathbb{R}$ with coordinates $(\vec{z}, t)$. Let α_0 be given by $\alpha_0 = dt + \sum_{i=1}^{n} x_i dy_i$. Then α_0 is a contact form on M, and the distribution is given by $\Delta_{z,t} = \{a_i \partial_{x_i} + b_i \partial_{y_i}, a_i, b_i \in \mathbb{R}\}$

As in the symplectic case, all contact manifolds locally look like their model spaces:

Theorem 2.1.5 *Let (M, Δ) be a contact manifold, with contact form α. Then given any point $p \in M$, there is a neighborhood U_p of p and a local chart $\phi : U_p \to \mathbb{C}^n \times \mathbb{R}$ such that $\phi^* \alpha_0 = \alpha$.*

The following example will be useful in our work. It is the contact analogue to the canonical symplectic structure on $\mathbb{C}P^n$.

Example.

Let $M = S^{2n+1} \subset \mathbb{C}^{n+1}$. Define the one-form α on M by $\alpha_p(x) = \mathrm{Re}\langle ip, x \rangle$. Then α is a contact form on S^{2n+1}.

The Hopf fibration $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ relates this contact structure to the canonical symplectic structure on $\mathbb{C}P^n$. In fact, $d\alpha_{S^{2n+1}} = \pi^* \omega_{\mathbb{C}P^n}$. We will exploit that relationship from several viewpoints in this work.

2.1.3 Geometry of Cotangent Bundles

There is a canonical 1-form α_{can} on the cotangent bundle T^*M of any manifold, defined as follows: if $\pi : T^*M \to M$ is the natural projection, define

$$\alpha_{can_v}(w) = v(\pi_* w)$$

Then it can be shown that $\omega_{can} = -d\alpha_{can}$ is a symplectic form on T^*M.

In local coordinates $(q_1, \ldots q_n, p_1, \ldots p_n)$, where q_i are coordinates in M and p_i are the fiber coordinates, α_{can} and ω_{can} are given by

$$\begin{aligned} \alpha_{can} &= \sum_{i=1}^{n} p_i dq_i \\ \omega_{can} &= \sum_{i=1}^{n} dq_i \wedge dp_i \end{aligned}$$

Now let F be a Finsler metric on T^*M; that is $F : T^*M \to \mathbb{R}$ is a smooth function (off the zero section), homogeneous of degree. Let U^*M be the unit tangent bundle $U^*M = F^{-1}(1)$, $i : U^*M \to T^*M$ the inclusion.

Define the vector field X_F by $\omega_{can}(X_F, \cdot) = dF$. (This is the *Hamiltonian vector field of* F; we will study Hamiltonian vector fields in more detail in the next section). In local coordinates,

$$X_F = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \partial_{q_i} - \frac{\partial F}{\partial p_i} \partial_{p_i}$$

We claim that U^*M is a contact manifold, with $i^*\alpha_{can}$ as contact from. Since $di^*\alpha_{can} = i^* d\alpha_{can} = i^*\omega_{can}$, $di^*\alpha_{can}$ has maximal rank $2n-2$.

Note that

$$\alpha_{can}(X_F) = \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i} = F(q,p)$$

since F is homogeneous of degree one. Therefore $i^*\alpha_{can}(X_H) = 1$. Also, $i^*\omega_{can}(X_F, v) = dF(i_* v) = 0$ since $i_* v$ is tangent to the level surface $F^{-1}(1)$. Therefore, $T_x U^*M$ is spanned by X_F and $\ker\alpha$, from which it follows that α is a contact form and X_F is the characteristic vector field of the contact structure.

Homogeneous symplectic maps $\phi : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ and contact maps $f : U^*M \to U^*M$ are intimately related:

First, note how α_{can} behaves with respect to 'homothety' maps. If $\lambda : T^*M\backslash 0_{section} \to \mathbb{R}$ is a function, we define $H_\lambda : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ by $H_\lambda(v) = \lambda(v)v$. In coordinates, $H_\lambda(q,p) = (Q,P) = (q, \lambda(q,p)p)$.

Then we have

$$H_\lambda^*\alpha = \sum_i^n P_i dQ_i = \lambda p_i dq_i = \lambda\alpha$$

Now let $\phi : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ be an homogeneous of degree one symplectic map, and let $F_0 : \phi : T^*M\backslash 0_{section} \to \mathbb{R}$ be a Finsler metric. Define a new metric $F_1 = F_0 \circ \phi$. Let U_0 and U_1 denote their respective unit cotangent bundles. Define $\lambda : \phi : T^*M\backslash 0_{section} \to \mathbb{R}$ by $\lambda(v) = F_1(v)/F_0(v)$.

Then H_λ takes U_0 onto U_1, and $H_\lambda^*\alpha|_{U_1} = \lambda\alpha|_{U_0}$. Then the map $\psi : U_0 \to U_0$ given by $\psi(v) = H_\lambda^{-1} \circ \phi$ is a contact map of $(U_0, \alpha|_{U_0})$ into itself.

Conversely, if we have a contact transformation $\psi : U_0 \to U_0$, then $\psi^*\alpha = \lambda\alpha$ for some $\lambda : U_0 \to \mathbb{R}$.

Extend λ to a function on $T^*M\backslash 0_{section}$ by degree one homogeneity. Then the map $\phi : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ given by $\phi(v) = \lambda(v)\psi(F(v)^{-1}v)$ is a homogeneous symplectomorphism.

2.1.4 Symplectic Reduction

Let M be a manifold and Ω be a closed 2-form of constant rank on M. Define the characteristic distribution E_Ω by $E_\Omega(x) = \{v \in T_xM \,/\, \Omega(v, \cdot) \equiv 0\}$. The fact that Ω is closed implies that E_Ω is integrable; thus defining a foliation $\mathcal{F}$ on M. Assume that the quotient space $N = M/\mathcal{F}$ obtained by identifying the leaves of the foliation has a smooth manifold structure. Since $T_{[x]}N \cong T_xM/E_\Omega(x)$, Ω descends to a *non-degenerate*, closed two form ω on N; thus N is a symplectic manifold. If $\pi : M \to N$ is the projection, Ω and ω are related by $\Omega = \pi^*\omega$.

The process of obtaining (N, ω) from (M, Ω) is called *symplectic reduction.*

An important case happens when there is a group G acting on a symplectic manifold (X, ω) by symplectomorphisms. We will describe the procedure for $G = S^1$, which is the case of immediate interest to us; that way we will avoid the complications of non-commutativity. For general Lie groups G, see [AM78, chapter 4].

Assume we have an free symplectic action $\Theta : S^1 \times X \to X$ on (X, ω). Let the infinitesimal generator of the action be denoted by ξ; ξ is the vector field on X given by

$$\xi_x = \frac{d}{ds}|_{s=0}\Theta(s, x)$$

Since the action is symplectic, $\mathcal{L}_\xi\omega = 0$.

Observe that the 1-form $\beta = i_\xi\omega = \omega(\xi, \cdot)$ is closed, since

$$d\beta = i_\xi(d\omega) + \mathcal{L}_\xi\omega = 0$$

Assume β is exact, that is, there is a function J such that $dJ = \beta$. We call J a *moment map* for the action.

Then we have

Proposition 2.1.1 *The map J is invariant under the action.*

Proof.

Let $x \in M$, and let $\gamma_x(s)$ be the orbit through p. Then

$$\frac{d}{ds}J(\gamma_x(s)) = dJ_{\gamma_x(s)}(\gamma_x'(s)) = dJ_{\gamma_x(s)}(\xi_{\gamma_p(s)}) = \beta(\xi) = \omega(\xi, \xi) = 0$$

Thus J is constant along the orbits.

Q.E.D.

Let c be a regular value of J. Since J is invariant under the action, S^1 acts on $M = J^{-1}(c)$. We apply the symplectic reduction procedure to $(M, \omega|_M)$. Observe that the characteristic distribution is generated by ξ, since ξ is tangent to M and $\omega(\xi, v) = dJ(v) = 0$ for any v tangent to $M = J^{-1}(c)$. Therefore the characteristic foliation is given by the orbits of the action, and the quotient space is a symplectic manifold (N, ω).

This construction will be the cornerstone of our study of Zoll manifolds, the S^1 action being given by the geodesic flow of the metric.

2.2 Hamiltonian Dynamics

A *Hamiltonian system* is a triple (X, ω, H), where (X, ω) is a symplectic manifold and $H : X \to \mathbb{R}$ is a smooth function, called the Hamiltonian of the system.

Given such data, we can construct a vector field X_H on X, the Hamiltonian vector field of H, defined by the equation

$$\omega(X_H, \cdot) = dH$$

The flows of such vector fields are called Hamiltonian flows.

The theory of Hamiltonian systems has its origin in classical mechanics: given a mechanical system, its dynamics can be described by a Hamiltonian system (T^*X, ω_{can}, H), where T^*X is the *position-momentum* space, and $H = T - V$, where T is the kinetic energy and V is the potential energy. See [Arn89] for a very good introduction to classical mechanics from the Newtonian, Lagrangian and Hamiltonian points of view.

An crucial property of Hamiltonian flows is that they preserve the symplectic form, i.e. for each $t \in \mathbb{R}$ the flow $\phi_t : M \to M$ is a symplectomorphism.

To see that, note that, infinitesimally,

$$\begin{aligned} \mathcal{L}_{X_H}\omega &= d(i_{X_H}\omega) - i_{X_H}(d\omega) \\ &= d^2 H - i_{X_H} 0 \\ &= 0 \end{aligned}$$

Integrating, we get the invariance of ω under the Hamiltonian flow.

In this section we study some fundamental properties of Hamiltonian flows, and specifically, the geodesic flow of a Riemannian manifold.

2.2.1 Geodesic Flows

Let (M, g) be a Riemannian manifold. We can go -via g- from TM to T^*M and vice versa using the *musical isomorphisms.* For example, if $v \in TM$ we define $v^\sharp$ by $v^\sharp(w) = g(v, w)$. The inverse isomorphism is denoted by $\flat$.

Any tensor can be pulled back and forth between TM and T^*M via the musical isomorphisms (classically, 'raising and lowering the indices'). In particular, there is an induced Riemannian metric on T^*M, $g^\sharp(v, w) = g(v^\flat, v^\flat)$.

Let $H : T^*M \to \mathbb{R}$ be given by $H(v) = \frac{1}{2}g^\sharp(v, v)$.Then we have

Definition.

The *cogeodesic flow* of M is the Hamiltonian system (T^*M, ω_{can}, H)

Remark.

We can use the musical isomorphism to pull back the canonical symplectic form of T^*M to TM, and define the *geodesic flow* on TM as the Hamiltonian system $(TM, (\omega_{can})^\flat, H)$, where $H(v) = \frac{1}{2}g(v, v)$. In this work it is convenient to have a fixed symplectic form when the metrics vary, so we use the cogeodesic flow all around. The musical isomorphisms intertwine all the relevant constructions.

Remark.

Sometimes the geodesic and cogeodesic flow are defined by the length function $L(v) = \sqrt{g}(v, v)$. Since $X_H = L^{-1}X_L$, these two flows are proportional, with the proportionality factor being constant in each energy level. The main advantages of H over L is that it is defined in the zero section, and that the quadratic character of the Hamiltonian is easily recognized.

2.2.2 Complete Integrability

Given a symplectic manifold (X, ω), we can define the *Poisson bracket* $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ as follows:

$$\{f, g\} = \omega(X_f, X_g),$$

where X_f, X_g are the Hamiltonian vector fields corresponding to f and g respectively.

If $\{f, g\} = 0$, the functions f and g are said to be *in involution.*

Now let (X^{2n}, ω, H) be a Hamiltonian system. It is said to be *completely integrable* if there are n functions $H = f_1, \ldots, f_n$ in involution, which are linearly independent almost everywhere. In formulas, we have

- $\{f_i, f_j\} = 0$
- The set
$$\{x : df_1(x), \ldots df_n(x) \text{ is linearly independent}\}$$
is open and dense.

Remark.

In the definition of complete integrability, it is sometimes required that the set on which the functions $f_1, \ldots f_n$ are independent is also of full measure (with respect to the volume form ω^n).

The importance of completely integrable Hamiltonian systems is that they can, in principle, be integrated by quadratures; that is, if we know explicitly integrating functions $f_2, \ldots, f_n$, we can actually more or less explicitly solve for the flow of X_H. For good examples of complete integrability in action, see [Bes78, chapter 4], where the geodesic flow of Zoll metrics of revolution in S^2 is integrated with the aid of Claireaut's first integral, and [Kli82, chapter 3], where the classical Jacobi integration of the geodesic flow of the 2-dimensional ellipsoid is done.

Recently, it has been shown that the complete integrability of the geodesic flow of a manifold (with some non-degeneracy conditions on the integrals) imposes rather severe restrictions on the topology of the manifold: if M admits a metric with completely integrable geodesic flow through non-degenerate integrals then M must be *rationally elliptic* ([Pat91]).

2.3 Connections on Principal Fiber Bundles

A *principal fiber bundle* (PFB for short) with *base space* B, *structure group* G and *total space* P is given by a right action of G (the *total space*) on P satisfying:

1. G acts freely on P, that is, if $p \cdot g = p$ for some $p \in P$, then $g = e \in G$.

2. M is the quotient of P by the action of G; $M = P/G$ and the projection map π is differentiable.

3. P is locally trivial, that is for every $x \in M$ there is a neighborhood U of x such that $\pi^{-1}(U)$ is isomorphic to $U \times G$, the isomorphism being equivariant with respect to the action of G in P and the trivial action of G on $U \times G$.

We denote a PFB by $\xi : \{G \to P \xrightarrow{\pi} B\}$. Observe that the first condition implies that the fiber over a given point b is isomorphic to the group G; fixing $p \in \pi^{-1}(b)$, the map $g \mapsto p \cdot g$ is an isomorphism. However, we need to fix p in the fiber for such a construction, and is therefore not canonical. Note that if a PFB has a section $s : B \to P$, then is globally trivial, the isomorphism $\Phi : B \times G \to P$ being given by $\Phi(b, g) = s(b) \cdot g$.

Observe that the tangent space to a given fiber at p is canonically isomorphic to the lie algebra $\mathfrak{g}$ of G; given $\eta \in \mathfrak{g}$, set $f(\eta) = d/dt|_{t=0} p \cdot \exp(t\eta)$. We call the tangent space to the fiber at a point p the *vertical* subspace V_p. Let $q : V_p \to \mathfrak{g}$ denote the inverse of f.

However, we have no canonical way of obtaining a 'horizontal' subspace H_p complementary to V_p. A (consistent) choice of such a complement is called a *connection* in the PFB.

Definition.

A *connection* in a PFB consists of a smooth distribution H_p on P satisfying:

1. $H_p \oplus V_p = T_p P$

2. H_p is equivariant; that is $H_{p \cdot g} = R_{g_*} H_p$

Observe that, restricted to H_p, the projection $\pi^* : H_p M \to T_{\pi(p)} B$ is an isomorphism. Given $X \in T_{\pi(p)} B$, the *horizontal lift* $\tilde{X}$ of X is the unique vector in $H_p P$ such that $\pi_*(\tilde{X}) = X$.

Then we have

Proposition 2.3.1 *Let $\gamma : [0,1] \to B$ be a curve, $\gamma(0) = b \in B$. Then, given $p \in \pi^{-1}(b)$ there exists a unique curve $\tilde{\gamma} : [0,1] \to P$ satisfying $\tilde{\gamma}(0) = p, \pi_* \tilde{\gamma}'(t) = \gamma'(t)$.*

Such a curve is called the *horizontal lift* of γ with initial value p.

2.3.1 Connection Forms, Curvature

Observe that choosing a splitting of $T_p P = V_p \oplus H_p$ is equivalent to choosing a projection $\rho : T_p P \to T_p P$ with image V_p and kernel H_p. since V_p is canonically isomorphic to $\mathfrak{g}$, we are led to the following definition:

Definition. A *connection form* on a PFB $\xi : \{G \to P \xrightarrow{\pi} B\}$ is a $\mathfrak{g}$-valued 1-form α satisfying:

1. α is G-equivariant; that is, $R_g^* \alpha = Ad_{g^{-1}} \circ \alpha$.

2. α restricted to the fiber is the identity; that is, if $v \in V_p$, then $\alpha(v) = q(v) \in \mathfrak{g}$.

Setting $H_p = \ker\alpha$ gives us a connection if α is a connection form. Reciprocally, if we have a connection, we define α by $\alpha_p(X) = q(\rho_V(X))$, where $\rho_V(X)$ is the vertical part of X in the decomposition $T_pP = H_p \oplus V_p$.

Now denote by ρ_H the projection of T_pP onto H_p with kernel V_p. The *covariant derivative* D of a k-form β is defined by

$$D\beta(X_1, \ldots, X_{k+1}) = d\beta(\rho_H(X_1), \ldots, \rho_H(X_{k+1})$$

Given a connection form α, we define the *curvature form* Ω by $\Omega = D\alpha$. Observe that since Ω is a 'horizontal' form, there is a $\mathfrak{g}$-valued 2-form $\omega \in \Lambda^2(B, \mathfrak{g})$ such that $\Omega = \pi^*\omega$.

Then we have

Theorem 2.3.1 (Structural equation) *Let α be a connection form with curvature form Ω. Then*

$$d\alpha(X, Y) = -\frac{1}{2}[\alpha(X), \alpha(Y)] + \Omega(X, Y)$$

Observe that in particular, if G is Abelian, $d\alpha = \Omega$.

The following lifting result will be used in an essential way in the next chapter: Having a principal bundle with a connection allows us to lift curvature preserving maps of the base to connection-preserving maps of the total spaces:

Proposition 2.3.2 *Let $G \to P_0 \xrightarrow{\pi_0} B_0$, $G \to P_1 \xrightarrow{\pi_1} B_1$ be principal bundles with connection forms α_0, α_1 respectively. Assume that the base spaces B_0, B_1*

are simply connected. Let $f : B_0 \to B_1$ *be a curvature preserving map, i.e.* $f^*\omega_1 = \omega_0$.

Then there is G*-equivariant map* $F : P_0 \to P_1$ *such that* $F^*\alpha_0 = \alpha_1$.

2.3.2 Classifying Maps

It is well known that there are *universal classifying spaces* for principal G-bundles [Ste40] , that is, given a compact group G there is a principal G bundle $\xi = \{G \to EG \to BG\}$ that is universal in the following sense:

1) Given any principal G-bundle, $\eta = \{G \to P \to B\}$ there is a bundle map

$$\begin{array}{ccc} G & & G \\ \downarrow & & \downarrow \\ P & \xrightarrow{F} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{array}$$

Such that $f^*\xi$ is isomorphic to η.

2) The isomorphism classes of principal G-bundles over a compact manifold B are classified by the homotopy classes of maps $F : B \to BG$.

The classifying spaces are directs limits of a sequence of finite dimensional smooth principal G-bundles $\xi_N = \{G \to EG_N \to BG_N\}$, and given a principal G-bundle over a finite dimensional compact smooth manifold B, we can choose the map f to be an embedding into one of the spaces BG_N for N big enough.

There is a similar situation in the category of *principal bundles-with-connection.* The bundles $\xi_N = \{G \to EG_N \to BG_N\}$ have a canonical connection 1-form, and the universality conditions (1) and (2) are satisfied, but now in the bundle-with-connection category. This was discovered by [NR61]; see also [D'A87], [Sch80] and [PR86]. More precisely, we have:

Theorem 2.3.2 *[NR61] There are connection forms $A \in \Lambda^1(EG_N, \mathfrak{g})$ in the universal bundles $\{G \to EG_N \to BG_N\}$, such that for any principal bundle $\eta = \{G \to P \to B\}$ with a connection 1-form α on P, there is (for big enough N) a bundle map*

$$\begin{array}{ccc} G & & G \\ \downarrow & & \downarrow \\ P & \xrightarrow{F} & EG_N \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG_N \end{array}$$

Such that $F^*\xi$ is isomorphic to η, and $F^*A = \alpha$.

There is also a statement corresponding to the second universality condition, but with a restricted concept of homotopy, essentially 'connection-preserving' [Sch80] or 'straight' [PR86].

In the case of S^1 bundles, the constructions and the proof of the classification theorem in [NR61] can be greatly simplified. In this case, the classification spaces are the Hopf bundles $S^1 \to S^{2N+1} \xrightarrow{q} \mathbb{C}P^N$, with their canonical connection, which we analyze more carefully below.

Consider the unit sphere $S^{2N+1} \subset \mathbb{C}^{N+1}$. The unit circle in $\mathbb{C}$, $S^1 = \{\theta \in \mathbb{C}/|\theta|^2 = 1\}$ acts on S^{2N+1} by multiplication: $p \cdot \theta = \theta p$. This is a free S^1

action whose quotient is $\mathbb{C}P^N$. Observe that the infinitesimal generator T for this action is given by $T_p = ip$.

Define a $i\mathbb{R}$-valued 1-form A on S^{2N+1} by $A_p(X) = i\langle ip, X\rangle$, where the inner product is the standard inner product on $\mathbb{C}^{N+1}$, $\langle X, Y\rangle = Re(\sum \bar{X}_k Y_k)$

The form A is clearly invariant under the S^1 action, and $A_p(T_p) = i\langle ip, ip\rangle = i$. Therefore A is a *connection form* on the Hopf bundle $\xi_N = \{S^1 \to S^{2N+1} \to \mathbb{C}P^N\}$.

Proposition 2.3.3 *The form A can be written as $A_p = \sum \bar{p}_i dp_i$.*

Proof.

Let $X_p = (x_1, \ldots x_{N+1}) \in \mathbb{C}^{N+1}$ be a tangent vector to S^{2N+1} at p, that is $\langle p, X\rangle = 0$.

Then $A_p(X) = i\langle ip, X\rangle = i\sum \bar{ip}_k x_k = -i^2 \sum \bar{p}_k x_k = \sum \bar{p}_k dp_k(X)$.

Q.E.D.

Now let $\Phi : M \to S^{2N+1}$ be a map (it will be a classifying map in the applications). Consider $\Phi = (\phi_1, \ldots, \phi_{N+1})$ as a map into $\mathbb{C}^{N+1}$ satisfying $\sum_{k=1}^{N+1} \phi_k \bar{\phi}_k = 1$. Then we have

Corollary 2.3.1 *The pullback of A by Φ is given by $\Phi^* A = \sum_{k=1}^{N+1} \bar{\phi}_k d\phi_k$.*

Therefore, if we have a bundle-with connection $\eta = (\{S^1 \to P \to B\}, \alpha)$ with α the connection 1-form, in order to find a classifying map all we need to do is to construct $2N+1$ S^1-invariant functions $\phi_1, \ldots \phi_n : P \to \mathbb{C}$ satisfying the equations:

$$\begin{aligned} \sum_{k=1}^{2N+1} \phi_k \bar{\phi_k} &= 1 \\ \sum_{k=1}^{2N+1} \bar{\phi_k} d\phi_k &= \alpha \end{aligned}$$

Chapter 3

The Space of Geodesics

Let (M, g) be a *Zoll manifold.* Recall that according to our definition, the cogeodesic flow on U^*M is periodic with minimal period the same for all orbits and we normalized the period to be 2π.

This situation gives rise to the fibration $S^1 \to U^*M \xrightarrow{p} \mathbf{Geod}(M)$, where S^1 acts on U^*M via the cogeodesic flow (which is an $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ action because of the periodicity of the flow); two vectors in U^*M are identified iff they are velocity vectors of the same geodesic on M. Therefore the quotient (which is a differentiable manifold under our hypothesis on the action) $\mathbf{Geod}(M)$ can be considered the 'space of geodesics' on M. One of the most important consequences of the Zoll property is that the space of geodesics has the structure of a differentiable manifold $\mathbf{Geod}(M)$, on which many structures (Riemannian, symplectic, almost complex) naturally arise, with a rich interplay between the structures on M, U^*M and $\mathbf{Geod}(M)$.

In this chapter we will describe the main features of the principal S^1-bundle $\mathcal{G} = \{S^1 \to U^*M \xrightarrow{p} \mathbf{Geod}(M)\}$. For additional details, see [Bes78, chapter 2]. However, we warn the reader that in [Bes78] all constructions are done in UM, instead of U^*M. The musical isomorphisms intertwines all the relevant constructions.

3.1 Symplectic geometry and geometry of connections on $\mathcal{G}$

The first extra structure we will consider on $\mathbf{Geod}(M)$ is its *symplectic* structure:

Since the cogeodesic flow is an S^1 action on T^*M *by symplectomorphisms*, (because it is a Hamiltonian flow on T^*M), the most direct way of realizing the space of geodesics as a symplectic manifold is the following:

Definition. The space of geodesics $(\mathbf{Geod}(M), \omega)$ is the symplectic reduction of (T^*M, Ω_{can}) under the S^1 action given by the geodesic flow.

Observe that the moment map of the action can be taken to be H itself. Having said that, let us apply the symplectic reduction process to this particular case: we first restrict to $H^{-1}(1) = U^*M$ by definition. Let $\alpha = \alpha_{can}|_{U^*M}, \Omega = \Omega_{can}|_{U^*M} = d\alpha$. Then Ω is *horizontal*, that is, there is a 2-form $\omega \in \Lambda^2\mathbf{Geod}(M)$ such that $\Omega = p^*\omega$. The symplectic reduction of (T^*M, Ω_{can}, H) is then $(\mathbf{Geod}(M), \omega)$.

The link between the symplectic structure and the theory of connections in a principal bundle is given by the following proposition:

Proposition 3.1.1 *The form $i\alpha$ is a connection form on the principal $U(1)$ bundle $S^1 \to U^*M \to \mathbf{Geod}(M)$.*

Proof.

All we have to prove are conditions (1) and (2) from the definition of a connection form:

The invariance of $i\alpha$ under the action of the group follows from the invariance of α_{can} and of H under the cogeodesic flow.

On the other hand, the infinitesimal generator of the cogeodesic flow is just the cogeodesic spray Z_g on U^*M. Thus $\alpha_p(Z_g(p)) = H^2(p) = 1$ on U^*M.

Q.E.D.

Observe that, from the bundles-with-connections point of view, a symplectomorphism $f : \mathbf{Geod}(M)_0 \to \mathbf{Geod}(M)_1$ is a 'curvature preserving map'. Therefore, applying proposition 2.3.2, we have

Proposition 3.1.2 *Let (M, g_0) and (M, g_1) be Zoll metrics on a manifold M. Assume that there is a symplectomorphism $f : \mathbf{Geod}(M)_0 \to \mathbf{Geod}(M)_1$ between the manifold of geodesics. Then there is a S^1 equivariant map $F : U_0^*M \to U_1^*M$ such that $F^*\alpha_1 = \alpha_0$.*

As in section 1.3 of chapter 2, it follows that extending F by degree 1 homogeneity gives us a symplectomorphism $\tilde{F} : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ such that $\tilde{F}(U_0^*M) = U_1^*M$. Therefore, since $\tilde{F}$ is homogeneous of degree 1, we have:

Proposition 3.1.3 *Let (M, g_0) and (M, g_1) be two Zoll manifolds. Then the following are equivalent:*

1) The spaces of geodesics $\mathbf{Geod}(M)_0$ *and* $\mathbf{Geod}(M)_1$ *are symplectomorphic.*

2) There exists a symplectomorphism $\tilde{F} : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ such that $H_0 \circ \tilde{F} = H_1$.

Therefore, we have the following

Principle 3.1.1 *Finding symplectic conjugacies of the geodesic flows, is the same as finding symplectomorphisms between the respective space of geodesics.*

When we have two Zoll metrics and their corresponding fibrations with connections $S^1 \to U_0^*M \to \mathbf{Geod}(M)_0$, $S^1 \to U_1^*M \to \mathbf{Geod}(M)_1$, it will sometimes be convenient to have a common total space. To achieve that we use the homothety map given by $T(v) = \lambda(v)v$, where $\lambda : T^*M - 0_{section} \to \mathbb{R}$ is given by $\lambda(v) = H_0(v)/H_1(v)$, $H_i(v) = \sqrt{g_i(v, v)}$.

Then T restricted to U_0^*M gives us a function $T : U_0^*M \to U_1^*M$, satisfying $T^*\alpha_1 = \lambda(v)\alpha_0$.

We pull back all the information of the system on U_1^*M to U_0^*M via T. If $\phi : U_0^*M \to: U_0^*M$ is a contact transformation satisfying $\phi^*\lambda\alpha_0 = \alpha_0$, the homogeneous of degree one extension of $T \circ \phi : U_0^*M$ is a symplectomorphism $F : T^*M\backslash 0_{section} \to T^*M\backslash 0_{section}$ (cf. sec 1.3 of chapter 1), which carries U_0^*M onto U_1^*M. Then we have the next

Principle 3.1.2 *Finding symplectic conjugacies of the geodesic flows, is the same as finding contact transformations* $\phi : U_0^*M \to U_0^*M$ *satisfying* $\phi^*\lambda\alpha_0 = \alpha_0$.

Going back to connections in a PFB , we have the translation of the preceding principle to this viewpoint:

Principle 3.1.3 *Finding symplectic conjugacies of the geodesic flows, is the same as finding a connection preserving map* $\phi : (U_0^*M, \Theta_0, \alpha_0) \to (U_0^*M, \Theta_1, \lambda\alpha_0)$, *where* Θ_0, Θ_1 *are the* S^1 *actions on* U_0^*M *given by the respective contact flows.*

3.2 Symplectic geometry and Riemannian geometry on $\mathcal{G}$

Several naturally defined Riemannian metrics can be constructed on the manifold of geodesics $\mathbf{Geod}(M)$. With the exception of the computation of the geodesics and the curvature of the g_0 metric (Besse's terminology) on [Bes78], they have barely been studied.

Here we study the metric g_1 on $\mathbf{Geod}(M)$; from the Riemannian and symplectic viewpoint, this metric behaves rather nicely with respect to the submersion $S^1 \to U^*M \xrightarrow{p} \mathbf{Geod}(M)$ and the 'averaged' metric $\bar{g}$ on U^*M. In this section, the relations between the Riemannian invariants of the submersion $S^1 \to U^*M \xrightarrow{p} \mathbf{Geod}(M)$ and the symplectic invariants of $\mathbf{Geod}(M)$ are studied; roughly speaking, they determine each other.

The notation everywhere is as follows:

- Geometric objects of the connection metric on U^*M will be denoted by a 'c' superscript: g^c, ∇^c, R^c.

- Geometric objects of the averaged metric on U^*M will be denoted by bars: $\bar{g}, \bar{\nabla}, \bar{R}$.

- Geometric objects of the quotient metric on $\mathbf{Geod}(M)$ will be denoted by 1's subscripts: g_1, ∇_1, R_1.

- geometric objects on M itself will not have any markings: g, ∇, R.

3.2.1 Construction of g_1

We follow [Bes78]. In general, the connection metric on U^*M is *not* invariant under the cogeodesic flow. In fact, it is easy to prove that g^c is invariant under the geodesic flow if and only if M is a round sphere. ([Bes78]). So what we do is to force g^c to be invariant by averaging it over the orbits of the flow. More precisely, let

$$\bar{g}(X,Y) = \frac{1}{2\pi}\int_{s\in S^1}(L_s^*)g^c(X,Y)ds = \frac{1}{2\pi}\int_{s\in S^1} g^c(L_{s_*}X, L_{s_*}Y)ds,$$

where L_s means left translation under the action of the cogeodesic flow. Now this metric is S^1 invariant and therefore defines a metric on the quotient $\mathbf{Geod}(M)$. this quotient metric is the metric g_1 on $\mathbf{Geod}(M)$.

The tangent space to $\mathbf{Geod}(M)$ at a geodesic γ can be identified with the space of Jacobi fields Y that are orthogonal to γ (i.e. 'true variations' of γ). With this identification, we have that given two Jacobi fields Y_1, Y_2 on

$T_\gamma \mathbf{Geod}(M)$,

$$g_1(Y_1, Y_2) = \int_0^{2\pi} g(Y_1, Y_2) + g(\nabla_t Y_1, \nabla_t Y_2)$$

Or, since Y_1, Y_2 are periodic Jacobi fields, integrating by parts we have

$$g_1(Y_1, Y_2) = \int_0^{2\pi} g(Y_1, Y_2) + g(R(Y_1, \dot{\gamma})\dot{\gamma}, Y_2)$$

3.2.2 Submersion Invariants

Following [O'N66], given a Riemannian submersion $F \to E \to B$, we define the tensors T and A as follows:

$$T_F G = \mathcal{H}\nabla_{\mathcal{V}F}\mathcal{V}G + \mathcal{V}\nabla_{\mathcal{V}F}\mathcal{H}G$$

$$A_F G = \mathcal{V}\nabla_{\mathcal{H}F}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}F}\mathcal{V}G$$

Where $\mathcal{H}$ and $\mathcal{V}$ are respectively the horizontal and vertical projections of a vector.

Now we prove the promised relation between symplectic and submersion invariants:

Theorem 3.2.1 *Let X, Y be horizontal vectors. Then $A_X Y = \omega(X, Y)$.*

Proof. Observe that for the canonical 1-form α, we have

$$\alpha(V) = g^c(Z_g, V)$$

Since $L_{Z_g}\alpha = 0$, α is invariant under the cogeodesic flow. Therefore, integrating both sides of the previous equation along an orbit we get

$$\alpha(V) = (2\pi)^{-1} \int_{S^1} \alpha({\zeta^s}_* V) = (2\pi)^{-1} \int_{S^1} g^c(Z_g, \zeta^* V) = \bar{g}(Z_g, V)$$

Taking the exterior derivative of α and evaluating at two horizontal vectors X, Y, we get

$$\Omega(X,Y) = d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) = \bar{g}(Z_g, [X,Y])$$

Q.E.D.

Corollary 3.2.1 *Let $N \in$ **Geod**(M) be a Lagrangian submanifold. Then the restricted fibration $S^1 \to p^{-1}(N) \to N$ is locally isometric to a product $N \times S^1$.*

Proof of Corollary. The submersion $S^1 \to p^{-1}(N) \to N$ with the restricted metric is a Riemannian submersion with totally geodesic fibers and zero O'Neill tensor. Therefore it is locally isometric to a product. Q.E.D.

3.3 Volumes of Zoll manifolds: Weinstein's Theorem

One of the most striking applications of the interplay between the different structures on the bundle $\mathcal{G}$ is the following theorem, due to A. Weinstein: (see [Bes78, chapter 2], [Wei77]):

Theorem 3.3.1 (Weinstein) *Let (M^n, g) be a Zoll manifold, with minimal period of the geodesics normalized to ℓ. Then the ratio*

$$\frac{2\pi}{\ell} \frac{Vol(M^n, g)}{Vol(S^n, can)}$$

is an integer $i(M,g)$ *called the* Weinstein's *integer.*

In particular, if $\ell = 2\pi$ *then the volume of* (M,g) *is an integral multiple of the volume of* (S^n, can).

There are two major ingredients in the proof: first, since the symplectic form ω is the curvature form of a connection, then ω/ℓ represents the Euler class of the bundle $\mathcal{G} = \{S^1 \to U^*M \to \mathbf{Geod}(M)\}$. Thus ω/ℓ is an *integral* form, i.e. it lies in the image of the coefficient homomorphism $c : H^2(\mathbf{Geod}(M), \mathbb{Z}) \to H^2(M, \mathbb{R})$. Therefore the number

$$j(M,g) = \int_{\mathbf{Geod}(M)} (\frac{\omega}{\ell})^{n-1} = \frac{1}{\ell^{n-1}} \int_{\mathbf{Geod}(M)} \omega^{n-1}$$

is an integer.

The right hand side of the last equation is just $\ell^{1-n} Vol(\mathbf{Geod}(M), \omega)$. Then the relationship between the different metrics on $\mathbf{Geod}(M), U^*M$ and M itself, plus a basic topological lemma, finishes the proof.

Observe that, being a discrete quantity and a continuous function of the metric, Weinstein's integer is constant under deformations. Thus the natural question: is $i(M,g)$ independent of g (g in the set of Zoll metrics)? This is a mostly topological question, which consist basically in studying the possible Euler classes of a bundle $\mathcal{G} = \{S^1 \to U^*M \to \mathbf{Geod}(M)\}$. It has been slowly answered, in the positive sense, and it is completely settled if M is diffeomorphic to a CROSS:

Theorem 3.3.2 *Let* (M,g) *be a Zoll manifold,* M *diffeomorphic to a CROSS. Then* $i(M,g) = i(M, can)$ *for:*

- $M = S^{2n}$ *([Wei77])*
- $M = S^{2n+1}$ *([Yan80])*
- $M = \mathbb{C}P^n$ *([Yan91])*
- $M = \mathbb{H}P^n$, *or* $M = \mathbf{Ca}P^2$ *([Rez85])*

3.4 Examples of Spaces of Geodesics

The first example we present is $\mathbf{Geod}(S^n, can)$, the space of geodesics of the canonical round sphere of constant curvature 1. This example already illustrates the richness of the structure of spaces of geodesics.

Since an image of a geodesic in S^n is given by the intersection of a 2-plane trough the origin in $\mathbb{R}^{n+1}$, the space of unparametrized geodesics of S^n is given by the Grassmann manifold $Gr(2, n+1)$. However, one has to remember the orientation of the geodesics (since the orbit through v is not the same as the orbit through $-v$); therefore, we have

$$\mathbf{Geod}(S^{n+1}, can) = Gr^+(2, n+1)$$

It is a classical fact that this space is isomorphic to the quadric $Q_n \subset \mathbb{C}P^n$ (see [GH78]). In fact, as a symplectic manifold, $\mathbf{Geod}(S^n, can)$ is isomorphic to Q_n endowed with the symplectic structure inherited from $\mathbb{C}P^n$.

Another example in which the space of geodesics can be readily described in a very geometrical fashion is $(\mathbb{C}P^n, can)$. We have [1]

[1]The author is indebted to J.C. Alvarez who pointed out this description of $\mathbf{Geod}(\mathbb{C}P^n, can)$, see [Alv94].

Proposition 3.4.1 *The space* $\mathbf{Geod}(\mathbb{C}P^n, can)$ *can be identified with the flag manifold* $F_{1,2}(\mathbb{C}^{n+1})$

Proof.

Let γ be an oriented geodesic in $\mathbb{C}P^n$. The geodesic γ is contained in a unique totally geodesic holomorphic 2-sphere $\mathbb{C}P^1 \subset \mathbb{C}P^n$. This sphere is given by the set of lines contained in a given 2-plane P_γ in $\mathbb{C}^{n+1}$.

Now given an oriented geodesic γ in S^2, using the 'right hand rule' it determines, and is determined by, a point $p \in S^2$, which corresponds to a line $\ell \subset P_\gamma$. This completes the identification.

Q.E.D.

Chapter 4

Symplectomorphisms Between Spaces of Geodesics

In this chapter, given two Zoll metrics which are 'close' in the appropriate sense, we construct an equivariant map $\Delta : (U^*M, \Theta_1) \to (U^*M, \Theta_2)$ which naturally descends to a diffeomorphism $\delta : \mathbf{Geod}(M)_1 \to \mathbf{Geod}(M)_2$ satisfying that $\delta^*\omega_2$ is close enough to ω_1 to apply Moser's theorem; therefore, by the principles stated in chapter 3, this suffices to give us a symplectic conjugacy of the geodesic flows.

We will use the continuity of classifying maps from appendix A plus a 'geometric isotopy' construction explained in the next section.

4.1 Joining Nearby Embeddings

In this section we 'join' two nearby embeddings in a geometrical fashion which will be quite useful for our purposes:

Proposition 4.1.1 *Let $F_1 : M \to Q$ be an embedding, M compact and simply connected. Then there is $\epsilon > 0$ such that for any other embedding $F_2 : M \to Q$ which ϵ-close to F_1 in the C^1 norm, the following property holds: given any point $q \in F_2(M)$, there is a unique point $p = D_0(q) \in F_1(M)$ realizing the distance from q to $F_1(M)$. The function $\Delta_0 = F_1^{-1} \circ D_0 \circ F_2$ is a diffeomorphism.*

Proof.

Let $M_i = F_i(M)$, ν_i the normal bundle of M_i in Q, and $\nu_i^a = \{v \in \nu_i : |v| < a\}$ let $\delta > 0$ be such that $\exp_1 : \nu_1^\delta \to Q$ is a diffeomorphism onto its image, so that the tubular neighborhood of M_1 of radius δ is parametrized by the normal disc bundle ν_1^δ.

Choose $\epsilon < \frac{1}{2}\delta$, so that all of M_2 is contained in $\exp_1(\nu_1^\delta)$.

We use Fermi coordinates around M_1 and do everything in the bundle $\nu_1 \xrightarrow{\pi} M_1$:

Let $\tilde{F} = \exp_1^{-1} \circ F_2 : M \to \nu_1$. $\tilde{F}$ makes sense by the conditions above. In this coordinates, the map D_0 is just the projection onto M_1. More precisely,

$$\pi \circ \tilde{F} = D_0 \circ F_2$$

The map F_1 in this coordinates is just the canonical inclusion of M_1 in ν_1 as the zero section. Then we have reduced our problem to the following lemma:

Lemma 4.1.1 *Let $E \xrightarrow{\pi} B$ be a vector bundle over a compact, simply connected manifold B, $z : B \to E$ the zero section, and let $z_0 : B \to E$ be an embedding C^1-close to z. Then the projection $\pi \circ z_0$ is a diffeomorphism.*

Proof of Lemma.

If z_1 is C^1-close enough to the zero section, then $z_{1*}(T_bB) \cap T_{z_1(b)}Fiber = \{0\}$. Thus $z_1(B)$ is always transversal to the fibers. Therefore the derivative $(\pi \circ z_1)_{*b}$ is an isomorphism. Thus $\pi \circ z_1$ is a local diffeomorphism. Since B is compact, it is also a covering map. Since B is simply connected, any covering map must be a homeomorphism. q.e.d.

Unraveling the identifications, lemma 3.1.1 implies that D_0 is a diffeomorphism.

Q.E.D.

For notational convenience, we will work with the inverses $D = D_0^{-1}, \Delta = \Delta_0^{-1}$. Then observe that there is a section N of ν_1 such that $D(p) = \exp_p(N_p)$. We can extend N to a vector field $\tilde{N}$ on Q such that $\tilde{N}|_{M_1} = N$. Then we have an extension $\tilde{D} : Q \to Q$ given by $\tilde{D}(q) = \exp_q(\tilde{N}_q)$ such that the following diagram commutes (i denotes the inclusion in Q):

$$\begin{array}{ccccc} M & \xrightarrow{F_1} & M_1 & \xrightarrow{i} & Q \\ \Delta\downarrow & & D\downarrow & & \tilde{D}\downarrow \\ M & \xrightarrow{F_2} & M_2 & \xrightarrow{i} & Q \end{array}$$

Let us compute the derivative of the map $\tilde{D}$ (or, for that matter, any map of the form $\tilde{D}(q) = \exp(\tilde{N}_q)$, where $\tilde{N}$ is a vector field on any manifold Q).

Proposition 4.1.2 *Let $\tilde{N}$ be a vector field on a complete Riemannian manifold Q, and define $\tilde{D} : Q \to Q$ by $\tilde{D}(q) = \exp(\tilde{N}_q)$. Let $X \in T_qQ$. Then:*

1) If $N_q \neq 0$, $\tilde{D}_*X = Y(1)$ where Y is the unique Jacobi field along the geodesic $\gamma(t) = \exp_q(t\tilde{N}_q)$ satisfying $Y(0) = X, (\nabla_t Y)(0) = \nabla_X N$.

2) If $N_q = 0$, then $\tilde{D}_* X = X + \nabla_X N$.

Proof.

First assume $N_q \neq 0$.

Let $\sigma(s)$ be a curve adapted to X. Then $\sigma(0) = q, \frac{d\sigma}{ds}|_{s=0} = X$. Let $\Gamma(t,s) = \exp_{\sigma(s)}(t\tilde{N}_{\sigma(s)})$. Observe that $\Gamma(t,0) = \gamma$ and $\Gamma(0,s) = \sigma$.

Then $Y(t) = \frac{\partial}{\partial s}\Gamma(s,t)|_{s=0}$ is a Jacobi field, being a variation of γ through geodesics. Clearly, $Y(1) = \tilde{D}_* X$. All we need to show is that the initial values of Y are as claimed. Since $\Gamma(0,s) = \sigma$, $Y(0) = X$. On the other hand,

$$\begin{aligned} \nabla_t \frac{\partial}{\partial s}\Gamma(t,s)|_{t=0,s=0} &= \nabla_s \frac{\partial}{\partial t}\Gamma(t,s)|_{t=0,s=0} \\ &= \nabla_s \frac{\partial}{\partial t}\exp(t\tilde{N}_{\sigma(s)}|_{t=0})|_{s=0} \\ &= \nabla_s \tilde{N}_{\sigma(s)}|_{s=0} \\ &= \nabla_X \tilde{N}. \end{aligned}$$

The case $N_q = 0$ follows from the previous one by a limiting argument.

Q.E.D.

Observe that proposition 4.1.2 in particular implies the following translation: 'F_1 and F_2 are close in the C^1 norm' translates to 'N is close to the zero vector field in the C^1 norm'.

For our applications, we will need an equivariant version of proposition 3.1.1:

Proposition 4.1.3 *Let $F_i : M \to Q$ be as in proposition 4.1.1. Assume there are G-actions Θ_1, Θ_2 on M and a G-action '$\cdot$' on X by isometries. If the maps F_i are equivariant, then the map Δ is equivariant.*

Proof.

It suffices to prove that, for any $g \in G$, if $p = D(q)$, then $D(g \cdot q) = g \cdot D(q)$. This follows from the following facts:

1) If the map F_1 is G-equivariant and $Q \in T_pN$ is normal to $F_1(M)$ at p, then so are the G-translates $g_* Q$ for all $g \in G$.

2) If $\gamma(t)$ is a minimal geodesic with $\gamma'(0)$ normal to $F_1(M)$, then so is $g \cdot \gamma(t)$ for any $g \in G$.

3) Given $p = D(q)$, the geodesic $\gamma \in \nu_1$ such that $\gamma(0) = p, \gamma(t_0) = q$ is *unique*.

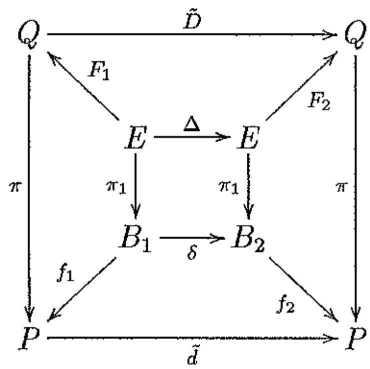
Q.E.D.

Observe that in this case, the corresponding vector field N is G-equivariant. Moreover, since N must be orthogonal to the image M_1 and $p \in M_1 \Rightarrow$ the whole fiber over $p \in M_1$, we can take N to be horizontal, that is, orthogonal to the fiber. Then we have arrived at the version of proposition 3.1.1 for principal bundles:

Proposition 4.1.4 *Let $G \to E \xrightarrow{\pi_1} B_1$, $G \to E \xrightarrow{\pi_2} B_2$ be principal G-bundles (The two G-actions on E are different, say Θ_1 and Θ_2). Let $G \to Q \xrightarrow{\pi} P$ a principal G bundle with metrics on P and Q that makes the fibration a Riemannian submersion. Assume there equivariant maps $F_1, F_2 : E \to Q$ which are ϵ-close in the C^1 metric as in proposition 4.1.1. Then:*

1) There is an equivariant diffeomorphism $\Delta : (E, \Theta_1) \to (E, \Theta_2)$ which is C^1-close to the identity.

2) There are vector fields $\tilde{N}$ on Q, $\tilde{n}$ on P, with $\tilde{N}$ horizontal and basic, $\pi_* N = n$ such that, if $\tilde{D}(q) = \exp_q(N_q)$, $\tilde{d}(p) = \exp_p(\tilde{n}_p)$, we have the following equivariant commutative diagram:



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where the maps $\tilde{D}, \Delta, \tilde{d}, \delta$ are diffeomorphisms.

Observe that, in particular, we have proven that B_1 and B_2 are diffeomorphic.

4.2 Joining Nearby Classifying Maps

In this section we use propositions 3.1.1 and 3.1.4 in our context:

Let g_0, g_1 be two Zoll metrics on a manifold M, with the same period. We assume:

1) g_0 and g_1 are C^1-close

2) The geodesic flows of g_0 and g_1 are C^1-close.

Condition (2) is, in particular, satisfied if g_0 and g_1 are C^2-close, but in the next chapter we will give some more natural curvature conditions which (possibly after diffeomorphisms) assure conditions (1) and (2).

Then we have the principal fibre bundles with connection forms

$$(S^1 \to U_0^*M \to \mathbf{Geod}(M)_0, \alpha_0), \qquad (S^1 \to U_0^*M \to \mathbf{Geod}(M)_0, \alpha_0)$$

As in chapter 3, we pull back the information on U_1^*M to U_0^*M via the homothety $T(v) = (H_0(v)/H_1(v))v$. Then we have the total spaces of the new principal fibre bundles with connections

$$(U_0^*M, \Theta_0, \alpha_0), \qquad (U_0^*M, \Theta_1, \alpha_1),$$

where $\alpha_1 = (H_0(v)/H_1(v))\alpha_0$, and Θ_1 is the pullback by T of the flow on U_1^*M, or , equivalently, the flow of the contact vector field of α_1.

Then we have two principal fiber bundles with connections, with the same total space (but with different S^1 actions and connections) and such that

1) The actions of the group are C^1 close (this is guaranteed by the geodesic flows being C^1 close)

2) The connection forms are C^1 close (this is guaranteed by the metrics being C^1 close)

In appendix A, we prove the following continuity theorem for classifying maps of bundles with connections: let Θ_1, Θ_2 be free S^1-actions on a manifold P, such that the quotients $B_1 = P/\Theta_1$, $B_2 = P/\Theta_2$ are manifolds. Let α_1 (resp. α_2) be 'connection forms' for the actions, that is $i\mathbb{R}$-valued 1-forms satisfying, for $k = 1, 2$,

$$\begin{aligned} \alpha_k(T_k) &= i \\ \mathcal{L}_{T_k}\alpha_k &= 0, \end{aligned}$$

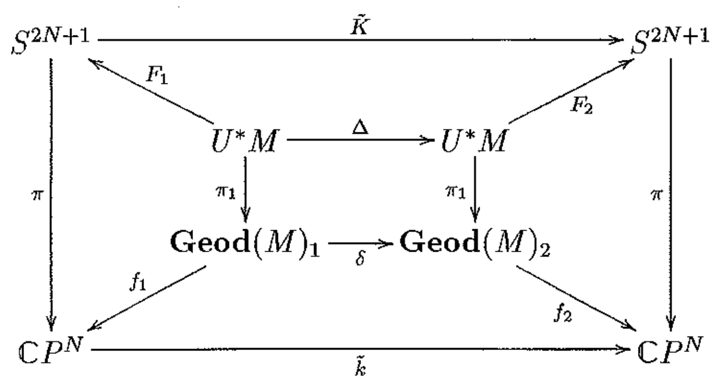
where T_k is the infinitesimal generator for the group action Θ_k.

This gives rise to two bundles-with-connection $S^1 \to P \xrightarrow{p_1} B_1$, $S^1 \to P \xrightarrow{p_2} B_2$. As such, they admit classifying maps into the universal *bundles-with-connection* $S^1 \to S^{2N+1} \xrightarrow{q} \mathbb{C}P^N$, that is, Θ_k-equivariant maps from P into S^{2N+1} such that $\Phi_k^* A = \alpha_k$.

Then we prove that the classifying map construction is *continuous* in the following sense:

Theorem A.1 *Assume the actions Θ_1, Θ_2 and the connection 1-forms α_1, α_2 are C^1-close. Then the maps $\Phi_1, \Phi_2 : P \to S^{2N+1}$ can be chosen to be C^1-close.*

Applying theorem A.1 to our particular situation, we find that we can find classifying maps $F_1, F_2 : U^*M \to S^{2N+1}$ that are C^1 close and equivariant with respect to the adequate S^1 actions. Then we have, from proposition 4.1.4,



,

With the additional property that $F_1^* A = \alpha_1, F_2^* A = \alpha_2$ and therefore $f_1^* \Omega = \omega_1, f_2^* \Omega = \omega_2$.

Next we prove that $\mathbf{Geod}(M)_1$ and $\mathbf{Geod}(M)_2$ are not only diffeomorphic, but symplectomorphic. The next theorem will allow us to apply Moser's method to this problem:

Theorem 4.2.1 *If the maps F_1, F_2 are C^1 close, then ω_1 and $\delta^*\omega_2$ are C^0 close.*

Proof.

The vector field n on $\mathbb{C}P^N$ such that $\tilde{k}(p) = \exp_p(n_p)$ is close to the zero field in the C^1 norm, being just the projection of a basic vector field N which is C^1 close to the zero field. Therefore the map $\tilde{k}$ is C^1 close to the identity.

We also have

$$\delta^*\omega_2 = \delta^* f_2^*\Omega = f_1^*\tilde{k}^*\Omega$$

Since $\tilde{k}$ is C^1 close to the identity, $\tilde{k}^*\Omega$ is C^0-close to Ω. Thus

$$|\omega_1 - \delta^*\omega_2|_0 = |f_1^*\Omega - f_1^*\tilde{k}^*\Omega|_0 = |f_1^*(\Omega - \tilde{k}^*\Omega)|_0 \leq C|(\Omega - \tilde{k}^*\Omega)|_0 \leq C\epsilon$$

Where C is a constant that depends on the original embedding F_1.

Q.E.D.

On the other hand, ω_1 and $\delta^*\omega_0$ represent the same cohomology class; since they both represent the Euler class of their respective S^1 bundles ($S^1 \to U_0^*M \to \mathbf{Geod}(M)_0, \alpha_0$), ($S^1 \to U_1^*M \to \mathbf{Geod}(M)_1, \alpha_1$), and δ is induced from the bundle map Δ. Then using Moser's stability theorem, we have

Corollary 4.2.1 *There is a map $h : \mathbf{Geod}(M)_1 \to \mathbf{Geod}(M)_1$ such that $h^*\delta^*\omega_2 = \omega_1$. The map h is C^1-close to the identity.*

And then taking $\phi = \delta \circ h$, we have

Corollary 4.2.2 *There exists a symplectomorphism $\phi : \mathbf{Geod}(M)_1 \to \mathbf{Geod}(M)_2$.*

Then applying principles 3.1.1, 3.1.2, 3.1.3 of chapter 3, we have

Theorem 4.2.2 *Let g_0, g_1 be Zoll metrics on a given manifold M satisfying:*

1) g_0 and g_1 are C^1-close

2) The geodesic flows of g_0 and g_1 are C^1-close.

Then the geodesic flows of g_0 and g_1 are symplectically conjugate, by a conjugacy ϕ which is C^1-close to the identity.

This implies in particular our first main theorem:

Theorem I *Let (M, g_0), (M, g_1) be two Zoll metrics which are C^2 close. Then g_0 and g_1 are symplectically equivalent.*

Translating this result in terms of principle 3.1.2, we have that given to C^2-close Zoll metrics g_0, g_1, there is a contact transformation $\psi : U^*M \to U^*M$ such that $\psi^*\alpha = \lambda\alpha$, where ψ is C^1 close to the identity and $\lambda = 1/H_1(v)$.

On the other hand, the group of contact transformations is locally path connected (see [BR95]):

Theorem 4.2.3 *Let (X, Δ) be a compact contact manifold, $\psi : X \to X$ be a contact transformation which is C^1-close to the identity. Then there exists a smooth family ψ_t of contact transformations with $\psi_0 = Id, \psi_1 = \psi$.*

Then we can prove

Theorem IV *Let (M, g_0), (M, g_1) be two Zoll metrics which are C^2-close. Then there is a 1-parameter family g_t of Finsler Zoll metrics, all symplectically equivalent to each other, joining g_0 and g_1.*

Proof.

Let ϕ_t be as in theorem 4.2.3, and define $\tilde{H}_t$ by $\phi_t^*\alpha = \tilde{H}_t^{-1}\alpha$. Extend $\tilde{H}_t^{-1}$ by homogeneity to a function H_t on T^*M. Then by construction all H_t are a one parameter family of symplectically conjugate Hamiltonians joining H_0 and H_1.

Q.E.D.

It should be remarked that if it is possible to find such a family with the additional property of H_t being a *quadratic* Hamiltonian, then the 'local' rigidity problem for Zoll manifolds is reduced to the 'infinitesimal' problem, since we could construct a smooth path of Riemannian Zoll metrics joining g_0 and g_1.

Chapter 5

The curvature tensor and the geodesic flow

In the previous chapter, we proved that two Zoll metrics on a given manifold have symplectically conjugate geodesic flows if the metrics are C^1-close and their geodesic flows are C^1-close. The easiest way to assure the fulfillment of the hypothesis is to assume that the given metrics are C^2-close.

In this chapter we exploit the intimate relationship between the geodesic flow and the curvature tensor to give more geometric 'closeness' conditions under which two Zoll metrics are symplectically conjugate.

In the first section, we review the relationship between the geodesic flow and the curvature, allowing us to improve theorem I: instead of requiring the metrics to be C^2-close, we only require C^0-closeness of the respective curvature tensors.

In the second section, we specialize to the case of g_0 being a rank one symmetric space. In this case, we can give simple conditions on the curvature tensor of g_1 which imply hypothesis (1) and (2) of theorem 4.2.2, thus allowing us to prove theorems II and III of chapter 1.

Some technical details concerning the Jacobi differential equation and Sturm-Liouville systems will be delayed until appendix B.

5.1 The curvature tensor and the geodesic flow

We want to compute, for each $t \in \mathbb{R}$, the derivative of the time t geodesic flow ϕ^t. Let $\sigma(s) : (-\epsilon, \epsilon) \to TM$ be a smooth curve, $\sigma(0) = \xi \in T_pM, \sigma'(0) = \Xi \in T_\xi TM$. Then

$$D\phi^t_\xi(\Xi) = \frac{d}{ds}|_{s=0}\phi_t(\sigma(s)).$$

Observe that, for each fixed s_0, $t \to \pi\phi^t(s_0)$ is a geodesic in M with initial velocity $\sigma(s_0) \in TM$. This motivates the following construction:

Let $\gamma(t) : [0,1] \to M$ be a geodesic, $\gamma'(0) = \xi \in TM$. Let $\Gamma(t,s)$: $[0,1] \times (-\epsilon, \epsilon)$ be a variation of γ through geodesics, i.e. for each $s_0 \in (-\epsilon, \epsilon)$, $\Gamma(t, s_0)$ is a geodesic in t, and $\Gamma(t, 0) = \gamma(t)$.

Define a vector field $Y(t)$ along γ by

$$Y(t) = \frac{\partial}{\partial s}|_{s=0}\Gamma(t,s).$$

Such a vector field is called a *Jacobi field* along γ. Then we have the following principle:

Principle 5.1.1 *There exists a natural correspondence*

$$\textit{Jacobi fields along } \gamma \leftrightarrow T_\xi TM,$$

given by

$$\Xi \in T_\xi TM \to Y(t) = \frac{d}{ds}|_{s=0}\pi\phi^t(\sigma(s)),$$

where $\sigma(s)$ is a curve adapted to Ξ.

Recall that the Levi-Civita connection induces a decomposition

$$T_\xi TM = H_\xi \oplus V_\xi$$

Where V_ξ is the tangent space to the fiber and H_ξ is the 'horizontal' subspace, generated by curves of the form $\tilde{\rho}(s), \tilde{\rho}(0) = \xi, \tilde{\rho}(s)$ is the parallel transport of ξ along some curve $\rho : (-\epsilon, \epsilon) \to M$. Observe that $\pi_* : H_\xi \to T_{\pi(\xi)}M$ is an isomorphism, given by $\tilde{\rho}'(0) \to \rho'(0)$. The vertical subspace V_ξ is also canonically isomorphic to $T_{\pi(\xi)}M$, being the tangent space of a vector space at a point.

Using these identifications, we have the following lemma: (see [Kli82, Lemma 3.1.17])

Lemma 5.1.1 *The Jacobi fields $Y(t)$ along a geodesic $\gamma(t)$ on M are in 1:1 correspondence with the flow invariant vector fields $\eta(t)$ along the corresponding flow line $\phi^t(\gamma'(0)) = \gamma'(t) = T$. The correspondence is given by*

$$Y(t) \in T_{\gamma(t)}M \leftrightarrow (Y(t), \nabla_T Y(t)) \in H_T \oplus V_T$$

Using freely the correspondences of principle 5.1.1 and lemma 5.1.1, we have

Theorem 5.1.1 *Let* $\Xi \in T_\xi TM$. *Decompose* $\Xi = \Xi_H \oplus \Xi_V$ *in its horizontal and vertical components. Then*

$$D\phi_\xi^t(\Xi) = Y(t),$$

where $Y(t)$ *is the unique Jacobi field along the geodesic* $\gamma(t) = exp(t\xi)$ *satisfying* $Y(0) = \Xi_H, \nabla_t Y(0) = \Xi_V$.

The Jacobi fields along a geodesic γ are characterized be the *Jacobi equation*, which links the curvature tensor and the geodesic flow:

$$\nabla_T^2 Y + R(Y,T)T = 0,$$

where T is the velocity vector field of γ, and R is the curvature tensor. (see [Mil70]). Since the Jacobi equation is a second order linear differential equation, a Jacobi field Y is completely determined by the initial conditions $Y(0), \nabla_t Y(0)$ (thus the word 'unique' in the statement of the previous theorem).

Putting principle 5.1.1, lemma 5.1.1 and theorem 5.1.1 together, plus the constructions from appendix B, we have the following lemma:

Lemma 5.1.2 *Let* g_0, g_1 *be Riemannian metrics on a compact manifold* M. *Assume that:*

1) The metrics g_0, g_1 *are* C^1*-close.*

2) The respective curvature tensors R_0 *and* R_1 *are* C^0 *close.*

Then the geodesic flows of the metrics are C^1 *close.*

Then applying theorem 4.2.2 of chapter 4, we get

Theorem 5.1.2 *Let g_0, g_1 be Zoll metrics on a manifold M satisfying:*

1) g_0 and g_1 are C^1 close.

2) The respective curvature tensors R_0 and R_1 are C^0-close.

Then the geodesic flows of g_0 and g_1 are symplectically conjugate, by a conjugacy ϕ which is C^1 close to the identity.

On the other hand, Jacobi fields gives us via Morse theory insights into the topology of manifolds with special geodesic flows (see [Mil70]). Let us recall a few basic facts:

Let $p, q \in (M, g)$, and let $\gamma : [0, a] \to M$ be a geodesic joining p and q, with velocity vector T. The *index form* is the symmetric, bilinear form

$$I(X,Y) = -\sum_t g(\Delta_t \nabla_T X, Y) + \int_0^a g(\nabla_t^2 X, Y) + g(R(X,T)T, Y)dt$$

acting on the space $T_\gamma \Omega_{pq} = \{$ piecewise smooth vector fields X along γ such that $X(0) = X(a) = 0\}$ and, for a piecewise continuous vector field Z, $\Delta_t Z = \lim_{s \to t^+} Z - \lim_{s \to t^-} Z$.

The *index* of γ is the maximal dimension of a subspace of $T_\gamma \Omega_{pq}$ in which I is negative definite.

Then we have (see [Mil70, §15])

Theorem 5.1.3 (Morse Index Lemma) *The index of a geodesic γ is always finite, and it coincides with the number of zeros on $(0, a)$ (counted with multiplicity) of Jacobi fields Y along γ satisfying $Y(0) = Y(a) = 0$.*

One of the most beautiful applications of Morse theory is the following theorem (see [Bes78, chapter 7]), [Bot54], [Sam63]):

Definition. A Riemannian manifold (M, g) is said to be a *L_l^p-manifold* if there is a point $p \in M$ such that all geodesics emanating from p are simple loops returning to p at length l.

Theorem 5.1.4 (Bott-Samelson theorem) *Let M be an d-dimensional L_l^p-manifold for some $p \in M$. Let α be the index of one of the closed geodesic loops emanating from p. Then α is the same for all the loops, and only the following possibilities can occur:*

- $\alpha = 0$, *and M has the homotopy type of* $\mathbb{R}P^2$.
- $\alpha = 1, d = 2n$, *and M has the homotopy type of* $\mathbb{C}P^n$.
- $\alpha = 3, d = 4n$, *and M has the integral cohomology ring of* $\mathbb{H}P^n$.
- $\alpha = 7, d = 8$ *and M has the integral cohomology ring of* $\mathbf{Ca}P^2$.
- $\alpha = d - 1$ *and M has the homotopy type of* S^n.

We will use the Bott-Samelson theorem to limit the possibilities of possible curvature tensors in a Zoll manifold. Given a Zoll manifold (M, g), let the *Bott-Samelson number* $\beta(M)$ be defined as the common index of its geodesic loops. The Bott-Samelson theorem in particular asserts that $\beta(M)$ is well defined, depending only on the cohomology ring of M and not on the metric.

5.2 Curvature tensors similar to curvatures of a CROSS

A simply connected symmetric space can be characterized locally by the condition $|\nabla R| = 0$. Recall that the pinching δ_g of a Riemannian manifold of positive curvature is given by $\delta_g = K_{min}/K_{max}$ where K_{min} and K_{max} are respectively the minimum and the maximum of the sectional curvature over the set of all two planes in TM.

By the classification of symmetric spaces and the sphere theorem and its rigidity version ([CE75]), the CROSSes can be characterized by their curvature tensors as follows:

- A simply connected manifold (M, g) is isometric to a CROSS if and only if g has positive curvature and $|\nabla R| = 0$.
- A simply connected manifold (M, g) is isometric to a round sphere (S^n, can) if and only if $\delta_g = 1$.
- A simply connected manifold (M, g), not diffeomorphic to S^n, is isometric to a CROSS if and only if $\delta_g = 1/4$.

The following conditions therefore express the similarity of a Riemannian manifold and a CROSS:

A Riemannian manifold (M, g) of positive curvature is said to be:

- ϵ-almost symmetric if $|\nabla R| < \epsilon$.

- ϵ-one pinched if $\delta_g > 1 - \epsilon$.

- ϵ-quarter pinched if $\delta_g > 1/4 - \epsilon$.

Observe that the first condition does not mean anything without a normalization; but in our case the metrics are normalized by the length of the closed geodesics being fixed. Throughout this section, ϵ denotes either ϵ itself or a function of ϵ that goes to zero as $\epsilon \longrightarrow 0$.

5.2.1 Almost symmetric Zoll manifolds

In this section, we assume that (M, g) is an ϵ-almost symmetric Zoll manifold. Recall that for us, 'Zoll manifold' also means that the common length of the closed geodesics is normalized to 2π. Observe that, if γ is a geodesic with velocity vector T and P is a parallel field along γ, then

$$\frac{d}{dt} g(R(P,T)T, P) = g(\nabla R(P,T)T, P) < \epsilon$$

Thus the curvature transformation $K_T = R(\cdot, T)T$ is 'almost constant' in a parallel basis. We use that fact repeatedly to prove

Theorem II *There is $\epsilon > 0$ such that if (M, g) is an ϵ-almost symmetric, Zoll metric, then M is diffeomorphic to a CROSS and (M, g) is symplectically equivalent to (M, can).*

Proof.

The strategy of the proof is to use the fact that any Jacobi field satisfying $Y(0) = 0$ also satisfies $Y(2\pi) = 0$, plus the Bott-Samelson theorem, to give a rather detailed description of the structure of the curvature tensor; in fact, it looks very much like the curvature tensor of a CROSS. Then theorem 5.1.2 completes the proof.

Let $\gamma : [0, 2\pi] \to M$ be a geodesic, with velocity vector T and curvature transformation $K_T = R(\cdot)T, T$.

First, given 'big' eigenvalues of the curvature transformation K_T, we construct vector fields along γ with $I(X, X) < 0$:

Lemma 5.2.1 *Let λ be an eigenvalue K_T with corresponding eigenvector P (which we extend by parallel transport to a vector field P_t along γ). Assume $\lambda \geq 1$ so that $\pi/\sqrt{\lambda} \leq \pi$. Then there are vector fields $X = f(t)P_t$ along γ satisfying $I(X, X) = 0$.*

Proof of Lemma.

Define the vector field Y_λ^α by

$$Y_\lambda^\alpha(t) = \begin{cases} 0, & t \in [0, \alpha) \\ \frac{1}{\lambda}\sin(t\sqrt{\lambda})P_t, & t \in [\alpha, \alpha + \pi/\sqrt{\lambda}) \\ 0, & t \in (\alpha + \pi/\sqrt{\lambda}, 2\pi] \end{cases}$$

Observe that in the interval $[\alpha, \alpha + \pi/\sqrt{\lambda}]$, Y_λ^α is an 'almost Jacobi field', that is,

$$
\begin{aligned}
I(Y^\alpha_\lambda, Y^\alpha_\lambda) &= g(\nabla^2_T Y^\alpha_\lambda + R(Y^\alpha_\lambda, T), T), Y^\alpha_\lambda \\
&= \frac{1}{\lambda}\int_\alpha^{\alpha+\pi/\sqrt{\lambda}} -\lambda \sin^2(t\sqrt{\lambda}) g(P_t, P_t) + \sin^2(t\sqrt{\lambda}) g(R_t(P,T)T, P) \\
&= \frac{1}{\lambda}\int_\alpha^{\alpha+\pi/\sqrt{\lambda}} -\lambda \sin^2(t\sqrt{\lambda}) + \sin^2(t\sqrt{\lambda})\{g((R_t - R_0)(P,T)T, P) + \lambda\} \\
&\leq \frac{1}{\lambda}\int_\alpha^{\alpha+\pi/\sqrt{\lambda}} \sin^2(t\sqrt{\lambda})\epsilon \\
&= \frac{1}{\lambda^{3/2}}\epsilon
\end{aligned}
$$

Observe also that $\Delta_\alpha \nabla_t Y^\alpha_\lambda = P_\alpha, \quad \Delta_{\alpha+\pi/\sqrt{\lambda}} \nabla_t Y^\alpha_\lambda = -P_{\alpha+\pi/\sqrt{\lambda}}$.

Now we can estimate the index of γ by using the almost-Jacobi fields and mimicking the 'index producing vector fields' of [Mil70, §15]: given λ an eigenvalue of the curvature vector, and α such that $\alpha + 3\pi/2\sqrt{\lambda} < 2\pi$, let $X^\alpha_\lambda = Y^\alpha_\lambda + Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda$. Note that $g(\Delta_t \nabla_t Y^\alpha_\lambda, Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda) = \sqrt{\lambda}^{-1}$.

Then, on $[\alpha, \alpha + 3\pi/2\sqrt{\lambda}]$, we have

$$
\begin{aligned}
|I(Y^\alpha_\lambda, Y^\alpha_\lambda)| &< \frac{1}{\lambda^{3/2}}\epsilon \\
|I(Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda, Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda)| &< \frac{1}{\lambda^{3/2}}\epsilon \\
|I(X^\alpha_\lambda, Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda) + 1/\sqrt{\lambda}| &< \frac{1}{\lambda^{3/2}}\epsilon
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I(X^\alpha_\lambda, X^\alpha_\lambda) &= I(Y^\alpha_\lambda, Y^\alpha_\lambda) + I(Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda, Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda) + 2I(X^\alpha_\lambda, Y^{\alpha+\pi/2\sqrt{\lambda}}_\lambda) \\
&\leq -\frac{1}{\sqrt{\lambda}} + 3\frac{1}{\lambda^{3/2}}\epsilon < 0
\end{aligned}
$$

Thus $I(X^\alpha_\lambda, X^\alpha_\lambda) < 0$.

q.e.d.

Observe that if λ is *too* big, there are many disjoint intervals of the form $[\alpha, \alpha + 3\pi/2\sqrt{\lambda}]$, producing linearly independent vector fields X_k satisfying $I(X_k, X_l) = -\delta_{kl}$. Thus the index of γ exceeds the limits imposed by the Bott-Samelson theorem. These reasoning implies that we have an a priori upper bound on the sectional curvature.

The next lemma shows that for any $T \in TM$, the eigenvalues of K_T are clustered around $\frac{1}{4} \times n^2$:

Lemma 5.2.2 *Let (M, g) be an ϵ-almost symmetric, Zoll manifold. Let λ be an eigenvalue of the curvature transformation K_T. Then there is $n \in \mathbb{Z}^+$ such that $|4\lambda - n^2| < \epsilon$.*

Proof of Lemma.

Let λ be an eigenvalue of K_T with corresponding unit eigenvector P; let Y the Jacobi field with initial conditions $Y(0) = 0, \nabla_t Y(0) = P$.

By lemma B.1.4 of appendix B,

$$|Y(t) - \lambda^{-1} \sin(t\sqrt{\lambda})| < tK\epsilon$$

Evaluating at $t = 2\pi$, we get

$$|\lambda^{-1} \sin(2\pi\sqrt{\lambda})| < K\pi\epsilon.$$

Since λ is a priori bounded by lemma 5.2.1, we have

$$\sin(2\pi\sqrt{\lambda})| < \epsilon$$

from which it follows that $|2\pi\sqrt{\lambda} - k\pi| < \epsilon$. q.e.d.

In particular, the curvature satisfies $K_M \geq 1/4 - \epsilon$. Denote by $E_T(n)$ the subspace of $T^\perp$ spanned by the eigenvalues of K_T clustered around $n^2/4$. Next we show

Lemma 5.2.3 *The dimension of $E_T(n)$ is zero for $n > 2$.*

Proof of Lemma.

Let $\lambda \in (n^2/4 - \epsilon, n^2/4 + \epsilon)$. Then we can find index producing vector fields X_λ^α with $X_\lambda^\alpha = 0$ for $t \in [\pi/2 - \pi/10, 2\pi]$.

Therefore, if $E_T(n)$ is non-trivial for $n > 2$ then the index of $\gamma|_{[0,\pi]} > 0$. But by continuity and connectedness, if $E_T(n)$ is non-trivial for some $T \in U_pM$ then $E_T(n)$ is non trivial for *all* $T \in U_pM$.

Thus any geodesic from p of length π is *not* minimizing, which contradicts the fact that $diam(M) = \pi$ for any Zoll manifold.

q.e.d.

Therefore, we only have two possibilities for the cluster of eigenvalues: Only $E_T(1)$ (corresponding to eigenvalues close to $1/4$) and $E_T(2)$ (corresponding to eigenvalues close to 1) are non-trivial.

Morse theory, via the Bott-Samelson theorem, restricts drastically the the possible dimension of $E_T(2)$, i.e. the multiplicity of the space of 'big' eigenvalues of the curvature transformation. Roughly speaking, given the topology of M we know -independently of the metric- which approximate eigenvalues and multiplicities can occur:

Lemma 5.2.4 *Let (M,g) be a Zoll manifold. Then the Bott-Samelson number of M equals the dimension of $E_T(2)$.*

Proof of Lemma.

Let $d = \dim E_T(2)$, and let $P_i, i = 1, \ldots, d$ be a basis of eigenvectors corresponding to eigenvalues $\lambda_i \sim 1$. Let $X^0_\lambda(i)$ be the index producing vector fields corresponding to the direction P_i. Then as in lemma 5.2.1, $I(X_i, X_j) = -\delta_{ij} + \epsilon$. Thus $d \leq \beta(M)$.

On the other hand, $n - d = \dim E_T(1) < n - \beta(M)$, since by the comparison lemma B.1.4 on appendixB, any Jacobi field Y with $Y(0) = 0$, $\nabla_T Y(0) = P \in E_T(1)$ *cannot* have a zero on $(0, 2\pi)$, since its norm is close to the function $\sin(\frac{1}{2}t)$.

q.e.d.

Lemmas 5.2.1, 5.2.2 and 5.2.3 complete the proof that the curvature tensor of M is C_0 close to the curvature tensor of its model CROSS, since on both of them the space $E_T(1)$ corresponds to Jacobi fields which are zero at 2π. Since the corresponding variations are C_0-close, the subspaces are also close to each other.

Then theorem 5.1.2 finishes the proof of theorem II.

Q.E.D.

5.2.2 Pinched Zoll manifolds

Let us prove theorem III:

Theorem III *There is $\epsilon > 0$ such that*

- *If (M, g) is an ϵ-one pinched Zoll metric then M is diffeomorphic to S^n and (M, g) is symplectically equivalent to (M, can)*
- *If (M, g) is an ϵ-quarter pinched Zoll metric and M is not diffeomorphic to S^n, then M is diffeomorphic to a projective space and (M, g) is symplectically equivalent to (M, can).*

Proof.

We normalize the common length of the closed geodesics to be 2π, and for each case find the 'right' normalization of the curvature tensor.

First we do the almost 1-pinched case. This condition implies that there is $\Lambda, \epsilon > 0$ such that

$$\Lambda - \epsilon < K_M < \Lambda + \epsilon$$

Let $i(M)$ be the injectivity radius of M. We have the inequalities

$$\min\{\frac{\pi}{\sqrt{\Lambda + \epsilon}}, \pi\} \leq i(M) \leq \frac{\pi}{\sqrt{\Lambda - \epsilon}}$$

Where the first inequality comes from comparison theory and the second inequality is Klingenberg's lemma ([CE75]).

It then follows that $1 - \epsilon' \leq K_M \leq 1 + \epsilon'$, which is the right normalization for the curvature.

On the one hand, a manifold satisfying $1 - \epsilon \leq K_M \leq 1 + \epsilon$ is diffeomorphic to the standard sphere S^n ([Gro66], [CE75, chapter 7]). Moreover, in the appropriate coordinates, (namely, exponential coordinates from the 'north' and 'south' poles), the metric is C^1 close to the canonical metric of constant curvature 1 on S^n.

On the other hand, having the pinching condition plus the normalization implied by the length of the closed geodesics means that the curvature transformation $K_T = R(\cdot, T)T$ is C^0 close to the identity transformation. Thus the curvature tensors of g and the round sphere of constant curvature one are C_0 close.

Then both conditions of theorem 5.1.2 are satisfied. This proves theorem III for the almost 1-pinched case.

The almost $\frac{1}{4}$-pinched case is somewhat more delicate; we have to probe a little deeper in the structure of the curvature tensor.

Let λ (resp. Λ) denote the absolute minimum (resp. maximum) of the sectional curvature. Then $0 < \lambda/\Lambda - 1/4 < \epsilon$ (where the first inequality is by the sphere theorem; we are assuming M is *not* diffeomorphic to S^n)

On one hand, since

$$\frac{\pi}{\sqrt{\Lambda}} \leq i(M) \leq \min\{\pi/\sqrt{\lambda}, \pi\},$$

it follows that $\Lambda \geq 1$. On the other hand, by the Bott-Samelson theorem and the pigeonhole principle, given any geodesic γ there exists Jacobi fields Y along γ such that $Y(0) = Y(\pi) = 0$ such that Y *has no zeros* on $(0, \pi)$. Therefore, by comparison theory, it follows that $\lambda \leq 1/4$. Putting the estimates for Λ and λ together plus the pinching hipothesis gives

$$1/4 - \epsilon < K_M < 1 + \epsilon$$

This condition implies, by Gromov's compactness technique ([MGP81]) as applied by Berger ([Ber83]), that (M, g) is diffeomorphic to a projective space and (possibly after diffeomorphism) the metric is C^1 close to the symmetric metric ([Kas89]). Also, for a study of the structure of Jacobi fields on almost-quarter pinched manifolds, see [AM94].

On the other hand, as in the previous theorem, the 'index producing Jacobi fields' lie close to eigenvalues of the curvature tensor which are close to one. The number of them is the same as in their model CROSSes, by the Bott-Samelson theorem. Therefore the curvature transformation of (M, g) is C_0 close to the curvature transformation of its model CROSS. Thus by theorem 5.1.2, the geodesic flows of (M, g) and (M, can) are symplectically conjugate.

Q.E.D.

Chapter 6

Integrability of the Geodesic flow on Zoll manifolds

In this chapter we show that if (M, g) is any Zoll manifold, its geodesic flow is completely integrable, giving us a potential source of new examples of manifolds with completely integrable geodesic flows:

Theorem V *The geodesic flow on any Zoll manifold* (M^n, g) *is completely integrable.*

As an immediate consequence, we have

Corollary *The geodesic flow on a compact symmetric space of rank one is completely integrable.*

The only previous proofs of this fact use some rather involved Lie-algebraic techniques ([Thi81], [GS83]).

6.1 Proof of Theorem V

The key lemma is the following, see for example [Fom88, page 145]:

Lemma 6.1.1 *Let (X^{2n}, Ω) be a compact symplectic manifold. Then there exists functions $h_1, \ldots, h_n$ which are in involution and linearly independent almost everywhere.*

Applying the lemma to $(X, \Omega) = (\mathbf{Geod}(\mathrm{M}), \omega_G)$ we get $n-1$ functions $h_1, \ldots, h_{n-1}$ in involution, which are linearly independent almost everywhere in $\mathbf{Geod}(M)$. Now let $\tilde{h}_i : UM \to \mathbb{R}$ be given by $\tilde{h}_i = h_i \circ \pi$. Extend each $\tilde{h}_i$ to functions $F_i : TM - \{\text{ 0 section}\} \to \mathbb{R}$ by degree zero homogeneity, i.e. $F_i(x) = h_i(x/|x|)$. Then we have

Lemma 6.1.2 *The set $F_1 \ldots, F_{n-1}$ satisfies*

1) $\{H, F_i\} = \{F_i, F_j\} = 0$

2) The set $\{dH_x, dF_1(x), \cdots dF_{n-1}(x)\}$ is linearly independent for almost all $x \in TM$.

Proof.

Proof of (1):

Clearly $\{H, F_i\} = 0$ since each F_i is constant on the orbits of X_H by construction.

For the rest of the proof of lemma 3.2, it is convenient to write the symplectic form ω in 'polar coordinates': let $\Theta : \mathbb{R}^+ \times UM \to TM - \{\text{ 0 section}\}$ be the diffeomorphism given by $\Theta(r, x) = rx$. Then a computation shows that

$$\Theta^* \omega = dr \wedge \alpha_1 + r\omega_1$$

Observe that in this coordinates, $F(r,\theta) = \tilde{h}_i(\theta)$; therefore $dF_i(\partial_r) = 0$. Since each F_i is invariant under the flow, $dF_i(X_H) = 0$ also. On the other hand, $span\{\partial_r, X_H\}$ is symplectically orthogonal to the horizontal distribution.

Therefore, we have that $X_{F_i}(r,\theta)$ is tangent to $\{r\} \times UM$ and horizontal for each (r,θ). Thus

$$\omega(X_{F_i}, X_{F_j}) = \omega_1(X_{F_i}, X_{F_j}) = \omega_1(X_{h_i}, X_{h_j}) = \omega_G(\pi_* X_{\tilde{h}_i}, \pi_* X_{\tilde{h}_j}) = \omega_G(X_{\tilde{h}_i}, X_{\tilde{h}_j}) = 0,$$

which ends the proof of part 1.

Proof of (2):

Let $c_1, \ldots, c_n$ be real numbers such that $c_0 dH + \sum c_i dF_i = 0$. Observe that $dF_i(\partial_r) = 0$, $dH(\partial_r)(x) = H(x) \neq 0$ if $x \neq 0$. Therefore evaluating $c_0 dH = \sum c_i dF_i$ at ∂_r we get $c_0 = 0$.

The sum is then reduced to $\sum c_i dF_i(x) = 0$. But since $dF_i = \pi^* dh_i$, we have $\sum c_i dh_i(\pi x) = 0$ since π is a submersion. Thus the dependent set is given by

$$\{x : \{dh_1(\pi(x)), \ldots dh_{n-1}(\pi(x)\} \text{ is linearly dependent }\}$$

which is a measure zero set.

Q.E.D.

Therefore the set $F_1, \ldots F_{n-1}$ constitutes a complete set of first integrals for the geodesic flow of H on $TM\backslash 0_{section}$. We can extend these functions to the zero section using the following lemma:

Lemma 6.1.3 *Let $H : TM \to \mathbb{R}$ be the Hamiltonian corresponding to a Riemannian metric on M, and assume there are functions $F_1 \ldots F_{n-1} : TM\backslash 0_{section} \to$*

$\mathbb{R}$ *such that* $\{H, F_i\} = \{F_i, F_j\} = 0, \{dH_x, dF_1(x), \ldots, dF_{n-1}(x)\}$ *is linearly independent for almost all* x. *Then there are* C^∞ *functions* $f_1, \ldots f_{n-1} : TM \to \mathbb{R}$ *satisfying the same properties as* $F_1, \ldots F_{n-1}$.

Proof.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function vanishing to all orders at zero. Define

$$f_i(x) = \phi(H(x))F_i(x), \quad x \neq 0, \qquad , f_i(0) = 0$$

Then each f_i is smooth on TM. To prove independence a.e., compute:

$$df_i(x) = \phi'(H(x))dH_x + \phi(H(x))dF_i(x)$$

Thus if $c_0 dH_x + \sum c_i df_i(x) = 0$, we have

$$(c_0 + \phi'(H(x)) \sum c_i)dH_x + \phi(H(x)) \sum c_i dF_i(x) = 0$$

For all $x \in TM - \{$ 0 section$\}$, this means

$$\begin{aligned} (c_0 + \phi'(x) \sum c_i) &= 0 \\ \phi c_i &= 0 \quad 1 \leq i \leq n-1 \end{aligned}$$

Since both $\phi(H(x)), \phi'(H(x))$ are positive off the zero section, the complement of the set where the previous conditions are satisfied is an open dense set.

On the other hand,

$$X_{f_i} = \phi'(H(x))X_H + \phi(H(x))X_{F_i}$$

Then it follows that $\omega(X_H, X_{f_i}) = \omega(X_{f_i}, X_{f_j}) = 0$.

Q.E.D.

Applying lemmas 6.1.1, 6.1.2 and 6.1.3, we complete the proof of theorem E.

6.2 An Example

We saw in chapter 3 that the space of geodesics of the round sphere is given by $\mathbf{Geod}(S^n, can) = Gr^+(2, n+1) = Q_n$, where $Q_n \subset \mathbb{C}P^n$ is the n-dimensional quadric given in homogeneous coordinates by $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$.

In view of the constructions of this chapter, it suffices to give n functions $f_i : Q_n \to \mathbb{R}$ which are in involution with respect to the symplectic form of Q_n.

We proceed to do that in the 3 dimensional case; a deeper understanding of the geometry of the general quadric Q_n gives natural geometrically defined integrals of the n-dimensional case.

It is a classical fact that $Q_3 \cong S^2 \times S^2$, with the product symplectic form $\omega \ominus \omega$ (see [GH78], [Bes78, chapter 4]). Put coordinates $(x, y) \in S^2 \times S^2$; and let $f_0(x, y) = h(x), f_1(x, y) = h(y)$, where $h : S^2 \to \mathbb{R}$ is any function having critical points only at the north and south poles. Then (the lifts to T^*M) of f_0, f_1 is a set of integrals in involution for the geodesic flow of S^3.

Appendix A

Classifying Principal S^1 Bundles with a connection

In this appendix we prove the continuity of classifying maps for S^1 bundles-with-connections.

Let Θ_1, Θ_2 be free S^1-actions on a manifold P, such that the quotients $B_1 = P/\Theta_1$, $B_2 = P/\Theta_2$ are manifolds. Let α_1 (resp. α_2) be 'connection forms' for the actions, that is $i\mathbb{R}$-valued 1-forms satisfying, for $k = 1, 2$,

$$\begin{aligned} \alpha_k(T_k) &= i \\ \mathcal{L}_{T_k}\alpha_k &= 0, \end{aligned}$$

where T_k is the infinitesimal generator for the group action Θ_k.

This gives rise to two bundles-with-connection $S^1 \to P \xrightarrow{p_1} B_1$, $S^1 \to P \xrightarrow{p_2} B_2$. As such, they admit classifying maps into the universal *bundles-with-connection* $S^1 \to S^{2N+1} \xrightarrow{q} \mathbb{C}P^N$, that is, Θ_k-equivariant maps from P into S^{2N+1} such that $\Phi_k^* A = \alpha_k$.

Here we will prove that the classifying map construction is *continuous* in the following sense:

Theorem A.1 *Assume the actions Θ_1, Θ_2 and the connection 1-forms α_1, α_2 are C^1-close. Then the maps $\Phi_1, \Phi_2 : P \to S^{2N+1}$ can be chosen to be C^1-close.*

Such a result is certainly true for arbitrary structure group G, but we do only the S^1 case since is technically simpler and we are mainly concerned with the applications to the geometry of Zoll manifolds, which involves the principal S^1 bundle $S^1 \to U^*M \to \mathbf{Geod}(M)$ where U^*M is the unit cotangent bundle of the manifold and $\mathbf{Geod}(M)$ is the 'manifold of geodesics'.

A.1 Construction of the Classifying map

In this section we construct a classifying map for an arbitrary connection on a principal S^1 bundle. We follow closely the constructions in [NR61], modifying it slightly so that the classifying map can be chosen to be an embedding. Recall that, in order to classify an S^1 bundle with connection form α, we need to find N functions $\Phi_i : E \to \mathbb{C}$ such that $\sum \Phi_i \bar{\Phi}_i = 1$ and $\sum \bar{\Phi}_i d\Phi_i = \alpha$. We will show that the choice of such functions is a continuous function of the data (i.e. the action and the connection form), and that $\vec{\Phi} = (\Phi_1, \ldots, \Phi_N) : E \to S^{2N-1}$ can be chosen to be an embedding.

We will first construct the classifying map locally. Then a partition of unity argument will give us the global classification result. The next theorem is the key local construction:

Theorem A.1.1 *Let $U \subset \mathbb{R}^n$ be an open set, $\alpha = \Sigma_{s=1}^n \alpha_s dx_s$ be a $u(1)$-valued 1-form on U. (That is, $\alpha = i\alpha^0$, α^0 a real 1-form) Assume $\sup_U |\alpha_s(x)| < \frac{1}{2n}$. Then there exists functions $\phi_1, \ldots, \phi_{4n} : U \to \mathbb{C}$ satisfying*

$$\Sigma_{j=1}^{4n} \bar{\phi}_j \phi_j = 1$$

$$\Sigma_{j=1}^{4n} \bar{\phi}_j d\phi_j = \alpha.$$

The functions ϕ_j can be chosen so that the map $x \mapsto (\phi_1, \ldots, \phi_{4n}) \in \mathbb{C}^{4n}$ is an embedding.

Proof.

Define functions μ_s, ν_s by

$$\mu_s = \frac{1}{2}(\frac{1}{2n} + \alpha_s^0)$$
$$\nu_s = \frac{1}{2}(\frac{1}{2n} - \alpha_s^0).$$

Then for all s, m_s and ν_s are strictly positive functions, and they satisfy $\mu_s - \nu_s = \alpha_s^0$, $\Sigma_{s=1}^n \mu_s + \nu_s = \frac{1}{2}$. Let p_s and q_s be the square roots of μ_s and ν_s respectively.

Now define the functions ϕ_j by

$$\phi_j = \begin{cases} p_s e^{ix_j}, & 1 \leq j \leq n \\ q_s e^{-ix_{j-n}}, & n+1 \leq j \leq 2n \\ \frac{1}{\sqrt{2n}} e^{ix_{j-2n}}, & 2n+1 \leq j \leq 3n \\ \frac{1}{\sqrt{2n}} e^{-ix_{j-3n}} & 3n+1 \leq j \leq 4n \end{cases}$$

Then, on the one hand,

$$\begin{aligned}\Sigma_{j=1}^{4n}\bar{\phi}_j\phi_j &= \Sigma_{j=2n+1}^{4n}\bar{\phi}_j\phi_j + \Sigma_{j=2n+1}^{4n}\bar{\phi}_j\phi_j \\ &= \Sigma_{s=1}^{n}p_s^2 + q_s^2 + \Sigma_{2n+1}^{4n}\frac{1}{2n} \\ &= \Sigma_{s=1}^{n}\mu_s + \nu_s + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

On the other hand,

$$\Sigma_{j=1}^{4n}\bar{\phi}_j d\phi_j = \Sigma_{j=1}^{2n}\bar{\phi}_j d\phi_j + \Sigma_{j=2n+1}^{4n}\bar{\phi}_j d\phi_j$$

The first part gives us

$$\begin{aligned}\Sigma_{j=1}^{2n}\bar{\phi}_j d\phi_j &= \Sigma_{s=1}^{n}p_s e^{-ix_s}\{ip_s e^{ix_s}dx_s + dp_s e^{ix_s}\} + \Sigma_{s=1}^{n}q_s e^{ix_s}\{-iq_s e^{-ix_s}dx_s + dq_s e^{ix_s}\} \\ &= \Sigma_{s=1}^{n}i(p_s^2 - q_s^2)dx_s + p_s dp_s + q_s dq_s \\ &= (\Sigma_{s=1}^{n}i(\mu_s - \nu_s)dx_s) + d(\Sigma_{s=1}^{n}\mu_s + \nu_s) \\ &= \Sigma_{s=1}^{n}i\alpha_s^0 dx_s \\ &= \alpha\end{aligned}$$

The second part does not add anything to α:

$$\Sigma_{j=2n+1}^{4n}\bar{\phi}_j d\phi_j = \Sigma_{s=1}^{n}ie^{-ix_s}e^{ix_s}dx_s - ie^{-ix_s}e^{ix_s}dx_s = 0$$

The functions $\phi_j = e^{ix_{j-2n}}, 2n+1 \leq j \leq 3n$ are clearly independent, so the map $x \mapsto (\phi_1, \ldots, \phi_{4n})$ has maximal rank.

Q.E.D.

Remark. The construction $\alpha \to \vec{\phi}$ in A.1.1 is *continuous*, in the sense that if two forms α_0, α_1 are C^k close then the corresponding maps $\vec{\phi}_1, \vec{\phi}_2$ will be C^k close.

We use theorem A.1.1 to locally classify connections on a principal S^1-bundle:

Theorem A.1.2 *Let $S^1 \to P \xrightarrow{p} B$ be a principal S^1-bundle, dim $P = n+1$ with a connection 1-form α_0. Let $\mathcal{U} \subset B$ be a trivializing open set and $\sigma : \mathcal{U} \to P$ be a local section. Then there is an S^1-equivariant map $\Phi_{\mathcal{U}} : p^{-1}\mathcal{U} \to S^{4n-1}$ such that $\Phi_{\mathcal{U}}^* A = \alpha_0$.*

Proof.

Choose a local coordinate map $\tau : U \to \mathcal{U}$ where $U \subset \mathbb{R}^n$.

Define a function $T : U \times S^1 \to p^{-1}U$ by $T(u, \theta) = \sigma(\tau(u)) \cdot \theta$. Then the following diagram commutes:

$$\begin{array}{ccc} U \times S^1 & \xrightarrow{T} & p^{-1}\mathcal{U} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\tau} & \mathcal{U} \end{array}$$

Let $\tilde{\alpha} = T^*\alpha_0$, $\alpha = \tau^*\sigma^*\alpha_0$. Changing coordinates if necessary, we can assume that $-i\alpha$ satisfies the conditions of Theorem A.1.1, which then gives us functions $\phi_1, \ldots, \phi_{2n} : U \to \mathbb{C}$ satisfying equations (1) and (2). Write the functions ϕ_i as a vector in $\mathbb{C}^n$, $\vec{\phi} = (\phi_1, \ldots, \phi_{2n})$. Observe that $|\vec{\phi}| = 1$.

Define then $\vec{\Phi} : U \times S^1 \to S^{4n-1}$ by $\vec{\Phi}(u, \theta) = \theta\vec{\phi}(u)$. Then by construction $\vec{\Phi}$ is S^1-equivariant (with respect to the tautological action of S^1 on $U \times S^1$), and the fact that the functions $(\phi_1, \ldots, \phi_{2n})$ are constructed as in theorem A.1.1 translates to $\vec{\Phi}^* A = \tilde{\alpha}$.

Finally, define $\Phi_{\mathcal{U}} : p^{-1}\mathcal{U} \to S^{4n-1}$ by $\Phi_{\mathcal{U}} = \vec{\Phi} \circ T^{-1}$. Then clearly $\Phi_{\mathcal{U}} : p^{-1}\mathcal{U} \to S^{4n-1}$ is an S^1-equivariant map satisfying $\Phi_{\mathcal{U}}^* A = \alpha_0$.

Q.E.D.

In order to construct a global classifying map, we use a specialized partition of unity: let $\{\kappa_i\}$ be a 'quadratic' partition of unity (that is, $\sum_i \kappa_i^2 = 1$) subordinated to a covering by trivializing open sets $\mathcal{U}_i$. Let $k_i = p \circ \kappa_i$. Then $\{k_i\}$ is a S^1 invariant quadratic partition of unity subordinated to the covering $p^{-1}\mathcal{U}_i$.

Define $F : P \to \mathbb{C}^N$ by

$F(x) = (k_1 \Phi_{\mathcal{U}_1}, \ldots k_n \Phi_{\mathcal{U}_n})$ where $\Phi_{\mathcal{U}_i}$ is as constructed in the last theorem. Then we have

Theorem A.1.3 *The map F satisfies*

1) $|F(x)| = 1$

2) $F^* A = \alpha_0$ *Therefore,* $F : P \to S^{2N-1}$ *is a classifying map for the bundle-with-connection* $(S^1 \to P \to B, \alpha_0)$.

Proof.

On the one hand,

$$|F(x)|^2 = \sum_i k_i^2 |\Phi_{\mathcal{U}_i}|^2 = \sum_i k_i^2 = 1$$

Therefore F is actually a map from P into S^{2N-1}.

On the other hand,

$$\begin{aligned} F^*A &= \sum_{i,j} k_i\overline{(\Phi_{\mathcal{U}_i})}_j d(k_i(\Phi_{\mathcal{U}_i})_j) \\ &= \sum_{i,j} k_i\overline{(\Phi_{\mathcal{U}_i})}_j (\Phi_{\mathcal{U}_i})_j dk_i + k_i d(\Phi_{\mathcal{U}_i})_j \\ &= \sum_i (\sum_j k_i dk_i \overline{(\Phi_{\mathcal{U}_i})}_j (\Phi_{\mathcal{U}_i})_j + sum_j \sum_i k_i^2 \overline{(\Phi_{\mathcal{U}_i})}_j d(\Phi_{\mathcal{U}_i})_j \end{aligned}$$

But, for each fixed i, $\sum_j \overline{(\Phi_{\mathcal{U}_i})}_j (\Phi_{\mathcal{U}_i})_j = |\Phi_{\mathcal{U}_i}|^2 = 1$, and $\overline{(\Phi_{\mathcal{U}_i})}_j d(\Phi_{\mathcal{U}_i})_j = \alpha_0$. Therefore we have

$$F^*A = \sum_i k_i dk_i + k_i^2 \alpha_0 = \alpha_0$$

Since $\sum_i k_i^2 = 1, \sum_i k_i dk_i = \frac{1}{2} d(\sum_i k_i^2) = 0$.

Q.E.D.

A.2 The Continuity Theorem

Here we prove Theorem A.1. Recall that the situation is: there are principal S^1-bundles with connections $(S^1 \to P \to B_1, \alpha_1)$, $(S^1 \to P \to B_2, \alpha_2)$ where we denote the respective S^1 actions by Θ_1, Θ_2. Then we have

Theorem A.1 *Assume the actions Θ_1, Θ_2 and the connection 1-forms α_1, α_2 are C^1-close. Then the maps $\Phi_i : P \to S^{2N+1}$ can be chosen to be C^1-close.*

Proof.

The proof is given in several steps:

1) Finding appropriate invariant coverings and trivializations for each action that are 'close' to each other. This will be expressed precisely in lemma A.2.1 below.

2) Do the local construction of the last section with respect to the coverings $\tilde{\mathcal{U}}_i, \tilde{\mathcal{V}}_i$ and trivializations T_i, R_i. Since the connection 1-forms are 'close', then the local constructions are 'close' to each other.

3) Paste the local constructions together using partitions of unity. We shall construct partitions of unity $\{f_i\}$ (resp $\{\hat{f}_i\}$), subordinated to the covering $\{\tilde{\mathcal{U}}_i\}$ (resp $\{\tilde{\mathcal{V}}_i\}$) that are Θ_1 (resp Θ_2) invariant, and such that f_i and $\hat{f}_i$ are C^1 close; that is done in lemma A.2.2 below.

Lemma A.2.1 *There exists coverings $\tilde{\mathcal{U}}_i = p_1^{-1}\mathcal{U}_i$ (resp. $\tilde{\mathcal{V}}_i = p_2^{-1}\mathcal{V}_i$) satisfying: there are local equivariant diffeomorphisms $T_i : U \times S^1 \to \tilde{\mathcal{U}}_i$, $R_i : U \times S^1 \to \tilde{\mathcal{V}}_i$, where $U \subset \mathbb{R}^n$, giving local trivializations*

$$\begin{array}{ccccc}
\tilde{\mathcal{U}}_i & \xleftarrow{T_i} & U \times S^1 & \xrightarrow{R_i} & \tilde{\mathcal{V}}_i \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle pr_1}\downarrow & & {\scriptstyle p_2}\downarrow \\
\mathcal{U}_i & \xleftarrow{\tau_i} & U & \xrightarrow{\rho_i} & \mathcal{V}_i
\end{array}$$

And such that R_i and T_i are C^1-close, and their inverses (where both are defined) are also C^1 close.

Proof of Lemma.

First we construct a Riemannian metric g_1 on P such that $S^1 \to P \xrightarrow{p_1} B_1$ becomes a Riemannian submersion with totally geodesic fibers, and α_1 is given by $\alpha_1(X) = g_1(T_1, X)$.

Remark. The construction of g_1 and the trivializing coverings in step (1) can be done choosing an arbitrary connection on the bundle $S^1 \to P \to B_1$; the existence of such coverings only depends on the actions Θ_1, Θ_2 being C^1 close. But for our porposes it is better to use the connection form given by α_1.

Define the horizontal distribution H on P by $H_p = \ker(\alpha_{1_p})$. Given any Riemannian metric g on B_1, define a metric g_1 on P by declaring $T_1(p) \perp H_p$, $|T_1| = 1$, and for $X, Y \in H_p$, $g_1(X,Y) = g(p_{1*}X, p_{1*}Y)$. Clearly, under the metric g_1, $S^1 \to P \overset{p_1}{\to} B_1$ is a Riemannian submersion, and $\alpha_1(X) = g_1(T_1, X)$ (since both sides are equal in a basis consisting on $T_1, v_1, \ldots v_{n-1}$ with each v_i horizontal). Under the metric g_1, the fibers are totally geodesic. Denote by E_1 (resp E) the exponential map of the metric g_1 (resp g).

Cover B_1 with a finite number of geodesic balls $\mathcal{U}_i = B(b_i, r) = E(B(0_{b_i}, r))$, where $r > 0$ will be chosen so that the pertinent constructions work. Choosing a point $e_i \in p_1^{-1}(b_i)$, let $\mathcal{H}_i$ be the 'geodesic lift' of $\mathcal{U}_i$, i.e. $\mathcal{H}_i = E_1(B(0_{e_i}, r) \cap H_{e_i})$. We adopt the following notation: given $\lambda > 0$, $\lambda\mathcal{H}_i = E_1(B(0_{e_i}, \lambda r) \cap H_{e_i})$.

Let $\tilde{\mathcal{U}}_i = p_1^{-1}(B_i)$. The sets $\tilde{\mathcal{U}}_i$ are of the form

$$\tilde{\mathcal{U}}_i = \cup_{z \in S^1} \{\Theta_1(z, p) / p \in \mathcal{H}_i\}$$

We identify $T_{b_i}B_1 \cong H_{e_i} \cong \mathbb{R}^n$ by isometries, and we speak of a set $U \in \mathbb{R}^n$ as living in H_{e_i} or $T_{b_i}B_1$ without mentioning the identification. Let $U = B(0_{\mathbb{R}^n}, r)$, $\tau_i : U \to \mathcal{U}_i$ be given the exponential coordinates from $T_{b_i}B$, forming a set of local coordinates for B; the first condition on r is that $0 < r <$

$inj(B,g)$. We also have local sections $\sigma_i : \mathcal{U}_i \to P$ given by $\sigma_i(b) = E_1\overline{E^{-1}b}$, where $\overline{v}$ is the horizontal lift of a vector $v \in T_bB$. Observe that $\tau_i^*\sigma_i^*\alpha_1 = E_1^*\alpha_1$.

Define $T_i : U \times S^1 \to \tilde{\mathcal{U}}_i$ by $T_i(x,z) = \Theta_1(E_1(x),z)$. Since Θ_1 acts by isometries of (P,g_1), this map is Θ_1- equivariant. Thus we have the following trivializing diagram:

$$\begin{array}{ccc} U \times S^1 & \xrightarrow{T_i} & \tilde{\mathcal{U}}_i \\ \downarrow & & \downarrow \\ U & \xrightarrow{\tau_i} & \mathcal{U}_i \end{array}$$

Note that at the point e_i, the Θ_1-orbit is orthogonal to $\mathcal{H}$, and, consequently, $\tau^*\sigma_i^*\alpha_1|_0 = E_1^*\alpha_1|_0 = 0$. Also the derivative $T_*|_0$ is an isometry. Therefore, we can choose r small enough so that:

- The angle between $2\mathcal{H}_i$ and any Θ_1 orbit intersecting it is close to $\pi/2$
- $|\tau_i^*\sigma_i^*\alpha_1| = |E_1^*\alpha_1| < \frac{1}{4n}$
- $0 < 1/2 \leq \det T_* \leq 3/2$

Since we are assuming that both actions and connection forms are C^1 close to each other, note that we can assume that both conditions also hold if we replace 'Θ_1' and 'α_1' with 'Θ_2' and 'α_2'. The first condition guarantees that an appropriate open subset of H_{e_i} will be a trivializing coordinate set for the actions; the second condition is needed for the construction of the local classifying maps, and the third one guarantees that any map that is C^1 close to T_i will also be a diffeomorphism, with inverse C^1 close to the inverse of T_i.

Now define the sets $\tilde{\mathcal{V}}_i^0$ as the translates under Θ_2 of the sets $\tilde{\mathcal{U}}_i$:

$$\tilde{\mathcal{V}}_i^0 = \cup_{z\in S^1}\{\Theta_2(z,p)/p \in \tilde{\mathcal{U}}_i\}.$$

The final trivializations for the Θ_2 action will be slight modifications of the covering $\{\mathcal{V}_i^0\}$.

Denote by $D(p)$ the maximum distance between the Θ_1-orbit through p and the Θ_2 orbit through p, $D(p) = \max_{z\in S^1} dist(\Theta_1(p,z), \Theta_2(p,z))$. We are assuming that the actions are C^1 close, so in particular $D(p) < \epsilon$ which we assume to be less than r.

Let $\delta_i = max_{u\in\tilde{\mathcal{V}}_i^0\cap E_1(H_{e_i})} dist(e_i, u)$. We have $\delta < \epsilon + r < 2r$, so in particular, $\tilde{\mathcal{V}}_i^0 \cap E_1(H_{e_i}) \subset 2\mathcal{H}_i$.

Define $\tilde{\mathcal{V}}_i = \{\Theta_2(u,z), u \in \delta\mathcal{H}_i, z \in S^1\}$. Then we have $\tilde{\mathcal{U}}_i \subset \tilde{\mathcal{V}}_i^0 \subset \tilde{\mathcal{V}}_i$; therefore the sets $\{\tilde{\mathcal{V}}_i\}$ form a Θ_2-invariant covering of P.

Define $R_i : U \times S^1 \to \tilde{\mathcal{V}}_i$ by $R_i(x,z) = \Theta_2(E_1(\delta^{-1}x, z))$.

The trivializations $\{\mathcal{U}_i, T_i\}, \{\mathcal{V}_i, \mathcal{U}_)\}$ then satisfy the conditions in lemma 1. q.e.d.

Now for each i, let α_1^i (resp(α_2^i) be the induced forms on U by pulling back α_1 (resp α_2) by the maps $x \mapsto T_i(x,1)$ (resp. $x \mapsto R_i(x,1)$). These forms are C^1 close, being the pullback by C^1 close maps of C^1 close 1-forms. Therefore, the local classification maps of last section are C^1 close. Diagrammatically, the situation is as follows:

$$\begin{array}{ccccc} \tilde{\mathcal{U}}_i & \xleftarrow{T_i} & U \times S^1 & \xrightarrow{\vec{\Phi}_i} & S^{2N+1} \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle pr_1}\downarrow & & \downarrow \\ \mathcal{U}_i & \xleftarrow{\tau_i} & U & \xrightarrow{\vec{\phi}_i} & \mathbb{C}P^N \end{array}$$

$$\begin{array}{ccccc}
\tilde{\mathcal{V}}_i & \xleftarrow{R_i} & U \times S^1 & \xrightarrow{\vec{\Psi}_i} & S^{2N+1} \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle pr_1}\downarrow & & \downarrow \\
\mathcal{V}_i & \xleftarrow{\rho_i} & U & \xrightarrow{\vec{\psi}_i} & \mathbb{C}P^N
\end{array}$$

Where (restricted to $\mathcal{U}_i$, where both are defined) the maps $\Phi_{\mathcal{U}_i} = \vec{\Phi}_i \circ T_i^{-1}$ and $\Psi_{\mathcal{V}_i} = \vec{\Psi}_i \circ R_i^{-1}$ are C^1 close.

Therefore we can find *local* classifications that are C^1 close. To globalize the construction, all we need to do is to find C^1-close partitions of unity, which we do now:

Lemma A.2.2 *Let $\{k_i\}$ be a quadratic Θ_1-invariant partition of unity subordinated to the covering $\{\mathcal{U}_i\}$. Then there is a Θ_2-invariant quadratic partition of unity $\{\hat{k}_i\}$ such that for each i, k_i and $\hat{k}_i$ are C^1 close.*

Proof of Lemma.

Let $\hat{k}_i^0$ be the average of k_i over the Θ_2 orbits,

$$\hat{k}_i^0(p) = \frac{1}{\pi} \int_{z \in S^1} k_i(\Theta_2(p,z))dz.$$

Since $k_i(p) = k_i(\Theta_1(p,z))$ for any $z \in S^1$, and the actions Θ_1, Θ_2 are C^1 close, it is clear that k_i and $\hat{k}_i^0$ are C^1 close for each i. Then the functions

$$\hat{k}_i = (\sum_i (\hat{k}_i^0)^2)^{-1})\hat{k}_i^0$$

give us the desired partition of unity. q.e.d.

Lemmas A.2.1 and A.2.2 plus the continuity of the local construction of last section with respect to α prove Theorem A.1. Q.E.D.

Appendix B

The Jacobi equation and Sturm-Liouville systems

We can translate the Jacobi equation to a second order ordinary differential equation in $\mathbb{R}^n$ in the following way:

Let $\gamma : \mathbb{R} \to M$ be a geodesic, $\gamma(0) = p$. Denote by T the velocity vector $\gamma'(t)$, $T_0 = \gamma'(0)$

Denote by $K_T : T^\perp \to T^\perp$ the transformation $X \mapsto R(X,T)T$. The map K_T is a symmetric linear map at each point $\gamma(t)$.

Choose an orthonormal basis $\{P_1, \ldots, P_k\}$ of $T_0^\perp$ of eigenvectors of K_{T_0}. Thus $R(P_i, T_0)T_0 = \lambda_i P_i$. The condition $K_M \geq \kappa$ translates to each $\lambda_i \geq \kappa$.

Extend the basis $\{P_1, \ldots, P_k\}$ along γ by parallel transport. Any vector field V along γ which is orthogonal to γ can be expressed as

$$V(t) = x_1(t)P_1 + \ldots + x_k(t)P_k$$

The Jacobi equation

$$\nabla_T^2 V + R(V,T)T = 0$$

Expressed in terms of the parallel fields $\{P_1, \ldots, P_k\}$ is given by

$$x''(t) + R(t)x(t) = 0 \tag{B.1}$$

Where $x(t) = (x_1(t), \ldots, x_k(t)) \in \mathbb{R}^k$, and $R(t)$ is the matrix defined by $R_{ij}(t) = \langle R(P_i, T)T, P_j \rangle$. In the basis we have chosen, we have $R_{ij}(0) = \text{diag}(\lambda_1, \ldots, \lambda_k)$. Let Λ (resp. λ) denote the supremum (resp. infimum) of the eigenvalues of $R(t)$. We assume $\lambda > 0$.

In this trivialization, we have the following translations:

- (M, g) is symmetric $\Leftrightarrow \nabla R = 0 \Leftrightarrow \frac{d}{dt}R(t) = 0$
- (M, g) is ϵ-almost symmetric $\Leftrightarrow |\nabla R| < \epsilon \Leftrightarrow |dR/dt| < \epsilon$.
- (M, g) is ϵ almost 1-pinched $\Leftrightarrow |K_M - 1| < \epsilon \Leftrightarrow |R - \mathbb{I}| < \epsilon$.
- (M, g) is ϵ almost $1/4$ -pinched $\Leftrightarrow 0 < \lambda$ and $|\lambda/\Lambda - 1/4| < \epsilon$.

Therefore we must study second order differential equations of the form B.1. We review such systems in what follows; we follow [CH89]. Since in our case all the solutions z will satisfy $z(0) = z(2\pi) = 0$; thence the name 'Sturm-Liouville' in the title.

B.1 Review of vector-valued Sturm systems

B.1.1 The constant coefficient case:

We look at the constant coefficient second order ordinary differential equation

$$x''(t) + Rx(t) = 0 \tag{B.2}$$

Where x(t) is a vector in $\mathbb{R}^k$ and R is a constant $k \times k$ matrix. In our applications, R will also be symmetric and positive definite, so we assume that as well.

The unique solution of equation (1) satisfying $x(0) = q_0, x'(0) = v_0$ is given by

$$x(t) = \mathrm{cn}(\mathrm{t}\sqrt{\mathrm{R}})\mathrm{q}_0 + \mathrm{sn}(\mathrm{t}\sqrt{\mathrm{R}})\mathrm{v}_0,$$

where the functions $\mathrm{cn}(\mathrm{t}\sqrt{\mathrm{R}})$ and $\mathrm{sn}(\mathrm{t}\sqrt{\mathrm{R}})$ are defined as follows: choose an orthonormal basis $P_1, ..., P_k$ of eigenvalues R, i.e. $R(P_i) = \lambda_i P_i$. Then the solution $x(t)$ satisfying $x(0) = q_1 P_1 + \ldots + q_k P_k, x'(0) = v_1 P_1 + \ldots v_k P_k$ is given by

$$x(t) = \sum q_i \mathrm{cn}_{\lambda_i}(\mathrm{t})\mathrm{P_i} + \mathrm{v_i}\mathrm{sn}_{\lambda_i}(\mathrm{t})\mathrm{P_i},$$

where the functions cn_λ and sn_λ are defined by

$$\begin{aligned} \mathrm{cn}_\lambda &= \cos(t\sqrt{\lambda}) \\ \mathrm{sn}_\lambda &= \sqrt{\lambda}^{-1}\sin(t\sqrt{\lambda}). \end{aligned}$$

B.1.2 The variable coefficient case:

We study the variable coefficient system

$$x''(t) + R(t)x(t) \tag{B.3}$$

Where $R(t)$ is not assumed to be constant anymore but we still assume is symmetric.

Let $R = R(0)$. We want to compare the solutions of the constant coefficient system B.2 with the solutions of the system B.3.

Let $\varepsilon(t) = R(t) - R$. Then (B.3) can be rewritten as

$$x''(t) + Rx(t) + \varepsilon(t)x(t) = 0 \tag{B.4}$$

Let $z(t)$ be the solution of (4) satisfying $z(0) = q_0, z'(0) = v_0$. Let $u(t)$ be the solution of (B.2) satisfying the same initial conditions: $u(0) = q_0, u'(0) = v_0$. Let $w(t) = z(t) - u(t)$.

Then w satisfies the differential equation with initial conditions

$$w''(t) + Rw(t) = -\varepsilon(t)z(t) \qquad w(0) = w'(0) = 0$$

The solution to such an equation is given by

$$w(t) = \int_0^t \mathrm{sn}((\mathrm{t} - \mathrm{s})\sqrt{\mathrm{R}})\varepsilon(\mathrm{s})\mathrm{z}(\mathrm{s})\mathrm{ds}$$

Since $w = z - u$ and u is a solution of B.2, we arrive at the following recursive relation:

Lemma B.1.1 *Let $z(t)$ be the solution of the system (3) with initial conditions $z(0) = q_0, z'(0) = v_0$. Then z satisfies the following recursive relation:*

$$z(t) = \mathrm{cn}(\mathrm{t}\sqrt{\mathrm{R}})\mathrm{q}_0 + \mathrm{sn}(\mathrm{t}\sqrt{\mathrm{R}})\mathrm{v}_0 + \int_0^t \mathrm{sn}((\mathrm{t}-\mathrm{s})\sqrt{\mathrm{R}})\varepsilon(\mathrm{s})\mathrm{z}(\mathrm{s})\mathrm{ds}$$

Thus, if u is the solution of B.2 satisfying $u(0) = q_0, u'(0) = v_0$, we have

$$z(t) - u(t) = \int_0^t \mathrm{sn}((\mathrm{t}-\mathrm{s})\sqrt{\mathrm{R}})\varepsilon(\mathrm{s})\mathrm{z}(\mathrm{s})\mathrm{ds}$$

Differentiating, we also get a relation for the derivatives:

$$z'(t) - u'(t) = \sqrt{R}\int_0^t \mathrm{cn}((\mathrm{t}-\mathrm{s})\sqrt{\mathrm{R}})\varepsilon(\mathrm{s})\mathrm{z}(\mathrm{s})\mathrm{ds}$$

B.1.3 The almost constant coefficient case

Now assume that the matrix R is *almost constant*, i.e. there is $\epsilon > 0$ such that $|dR/dt| < \epsilon$ for all t. Thus if $\varepsilon(t)$ is as in (4), we have $|\varepsilon(t)| < |t|\epsilon \leq 2\pi\epsilon$ since we are only concerned with $t \in [0, 2\pi]$. Let z and u be as in the last section. Let λ, Λ be respectively the smallest and greatest eigenvalues of $R(t)$. By comparison theory, a solution $z(t)$ of (4) with $z(0) = 0, |z'(0)| = 1$ satisfies $|z(t)| \leq \mathrm{sn}_\lambda(\mathrm{t})$ on [0,T], where T is the first zero of z.

Then using lemma B.1.1 and just bounding each term, we get

Lemma B.1.2 *Under the conditions of the previous paragraph, we have*

$$|z(t) - u(t)| \;<\; \epsilon \textstyle\int_0^t \mathrm{sn}_\lambda^2(t-s) < Kt\epsilon$$

$$|z'(t) - u'(t)| \;<\; \Lambda K t\epsilon.$$

Notice that all we used is the closeness of R and R_t. Thus we also have

Lemma B.1.3 *Let $R_0(t)$, $R_1(t)$ be bounded symmetric, positive definite matrix functions satisfying $|R_0 - R_1| < \epsilon$, and let $z(t)$ (resp. $(u(t))$ be solutions of $z'' + R_0(t)z = 0$ (resp. $u'' + R_1(t)u = 0$). Then there is K such that*

$$|z(t) - u(t)| \;<\; \epsilon \textstyle\int_0^t \mathrm{sn}_\lambda^2(t-s) < Kt\epsilon$$

$$|z'(t) - u'(t)| \;<\; \Lambda K t\epsilon .$$

B.1.4 The Jacobi Equation in an almost-symmetric space

Let M be a compact Riemannian manifold which is ϵ-*almost symmetric*, i.e. $|\nabla R| < \epsilon$. Assume M has positive curvature, $K_M \geq \kappa > 0$. We want to compare the Jacobi equation in M with the Jacobi equation of an actual symmetric space.

We want to compare the solutions of (B.1) with the solutions of what would be the Jacobi equation if M were symmetric:

$$x''(t) + R(0)x(t) = 0 \tag{B.5}$$

Thus from lemma B.1.2 we get the following application to the Jacobi fields in an almost symmetric space:

Lemma B.1.4 *Let M be an ϵ-almost symmetric space, γ a geodesic in M, and $\{P_1, \ldots, P_k\}$ as before. Let J be the unique Jacobi field along γ satisfying*

$$J(0) \;=\; a_1 P_1 + \ldots + a_k P_k$$

$$\nabla_t J(0) \;=\; b_1 P_1 + \ldots + b_k P_k$$

Let V be the vector field $V(t) = x_1(t)P_1(t) + \ldots + x_k(t)P_k(t)$, where$x_i(t)$ is given by $x_i(t) = a_i \mathrm{cn}_{\lambda_i}(\mathrm{t}) + \lambda_{\mathrm{i}}^{-1}\mathrm{sn}_{\lambda_i}(\mathrm{t})$. Then

$$|J(t) - V(t)| \;<\; t^2 K\epsilon$$

$$|\nabla_t J(t) - \nabla_t V(t)| \;<\; t^2 K\epsilon.$$

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