Transition Functions for Monopole-Free Abelian Lattice Gauge Fields on the Torus

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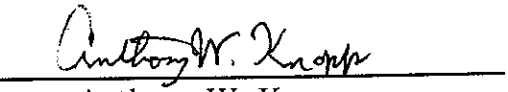

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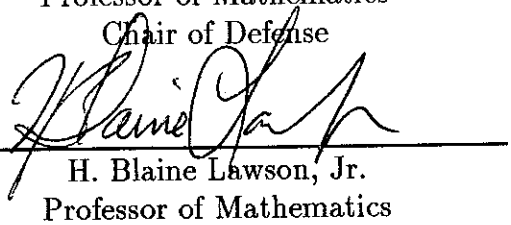

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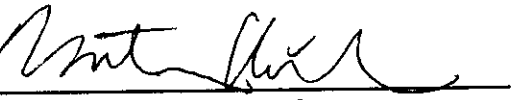

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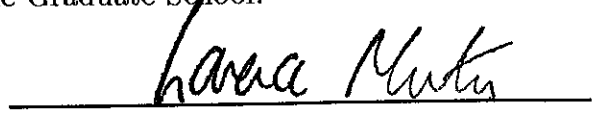

Graduate School

Abstract of the Dissertation

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A lattice gauge field (under certain conditions) represents a principal bundle with a connection. In this dissertation we show how such a bundle may be constructed by producing transition functions for a coordinate bundle based on the lattice gauge field data. We focus on the case of a continuous, monopole-free lattice gauge field defined on a cubical lattice on the n-torus for arbitrary n.

We formulate an Ansatz for the form of the transition functions as they depend on the lattice data: each one is defined in terms of a linear combination of nearby plaquette products, where

the coefficients only depend on n and on the remoteness of the plaquette from the site in question. Applying the relevant cocycle conditions produces a system of linear equations in these unknown coefficients. The monopole-free condition is shown to be necessary and sufficient for the existence of a solution to this system. Furthermore we show, using the theory of continuants, that the coefficients satisfy a certain recursion relation which allows them to be generated independently of the linear algebra.

To my family

Contents

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Chapter 1

Introduction

In all contemporary theories, the forces between elementary particles are mediated by quantum gauge fields. The classical example of a gauge theory is electromagnetism, which is the theory of interactions of charged particles with the electromagnetic field. Here the gauge group is U_1, corresponding to the internal symmetry (the phase) of a charged particle. In geometric language, an electromagnetic field in a region R of space-time is a connection in the bundle of phases over R, a principal U_1-bundle. In fact, the connection form may be exactly identified with the vector 4-potential A_μ. As a particle moves along a path in R, its phase undergoes parallel transport by A_μ in proportion to its charge. The primacy of A_μ over the usual electomagnetic field (whose six components are the six components of dA_μ) [5] is illustrated by the Aharnov-Bohm effect. Yang-Mills fields generalize this interpretation of electromagnetism to connections in principal G-bundles, where G is the group of internal symmetries of particles in the theory in question.

Quantizing the theory means that we no long consider a single path joining

two points. If we want to know what the change of phase will be when a charged particle moves from point X to point Y in space-time, we average the parallel transport along all possible paths from X to Y, weighting the contribution of each path by the action along that path. This is the path-integral procedure [6]. An analogous procedure is used to determine expected values of quantities associated to gauge fields, when (for example) the fields are required to satisfy certain boundary conditions. Now the "path integral" is taken over the space of all connections satisfying the boundary conditions. Note that in general this space is non-compact and infinite-dimensional.

For electromagnetism the results of the path-integral procedure can be calculated by perturbative methods. This has led to a highly successful theory. But for general gauge fields, and in particular for the gauge theory of the strong force, these methods do not work. A non-perturbative approach became possible with Kenneth Wilson's invention of lattice gauge fields in 1974 [17].

The lattice approach substitutes a finite-dimensional, compact model for the infinite-dimensional and non-compact spaces mentioned above. For example, instead of integrating over the space of all connections in principal G-bundles over a space X, we integrate over the space of all G-valued LGFs on a cell-decomposition Λ of X. When G and X are compact, this space is compact (although in general so large that refined Monte-Carlo methods must be used to evaluate the integrals). By definition, a G-valued LGF on a simplicial complex is a collection of group elements, one $u_{\alpha\beta}$ for each oriented 1–simplex $< \alpha\beta >$ in the complex, with $u_{\beta\alpha} = u_{\alpha\beta}^{-1}$. A G-valued LGF represents a (continuum) gauge field, i.e. a connection in a principal G-bundle over

a domain in space-time, in the following sense: Suppose the relevant domain has been triangulated as a simplicial complex Λ, and that a trivialization has been chosen at each vertex of Λ; this identifies the fiber over that vertex with G. Let α and β be adjacent vertices in Λ. If we start at the identity element in the fiber over α and use the connection to parallel-transport this element to the fiber over β, we should arrive at the group element $u_{\alpha\beta}$ in that fiber. It turns out [14] that if X is a compact manifold, certain conditions on the LGF (essentially, bounds on the "plaquette angles", the products of group elements around the edges of 2-simplexes) lead to the possibility of reconstructing the bundle and an approximation to the original connection. Then the bundle and connection allow us to compute characteristic numbers, or charges, which can be used in the lattice computation of the expected value of topological invariants of the system in question. Recovering the bundle from the LGF allows such numbers to be computed from discrete information; for example, the "topological charge", an invariant of $SU(n)$-gauge fields on 4-manifolds, admits a lattice computation [10],[14].

In this work we address the problem of reconstructing abelian (U_1) gauge fields from lattice data on rectangular complexes of arbitrary dimension. The expected values of quantities associated with abelian fields have recently become of renewed interest in topological quantum field theories [2], [18]. For U_1-valued LGFs it is not only possible to show that the corresponding bundles exist; it is possible to repesent them explicitly as coordinate bundles, whose transition functions are given by universal formulas in terms of the lattice data. This procedure works for all LGFs off a set of measure zero, except that

in dimensions 3 and above the "monopole-free" condition must be satisfied. This condition is necessary for the existence of a bundle.

Our procedure for reconstructing the bundle follows an Ansatz which may be summarized as follows (see Chapter 3). We construct the bundle via a representative coordinate bundle, given by a set of transition functions on overlaps of local trivializations. For the case of U_1-valued LGFs on a cubical lattice on an n-dimensional torus, we have verified a pattern which was known for $n = 2, 3, 4$ [13]; this work gives an explicit algorithm for the transition functions for all $n \geq 2$. The sticking point in this problem is determining the value of transition functions on multiple intersections, where they must satisfy certain cocycle conditions. Extension to the rest of the domain may be achieved through linear interpolation. The exponent of the transition function consists of a certain linear combination whose coefficients are determined by solving an $n \times n$ system of equations. The results are stated below:

Theorem 1 *Let* $\mathbf{u}$ *be a continuous LGF on a (periodic) cubical lattice* Λ *on the torus* T^{n+1}*, and* α *a vertex of the lattice. If* $\mathbf{u}$ *is monopole-free, then there is a set of coefficients* $\{a_\ell^{n+1}\}$ *such that all relevant cocycle conditions at the given point are satisfied by the transition function given by:*

$$v_{\alpha;j_0}(\alpha + \frac{1}{2}(j_1 + j_2 + \ldots + j_n)) = u_{\alpha;j_0} \cdot exp(i\Theta(\alpha; j_0)), \tag{1.1}$$

where

$$\Theta(\alpha; j_0) = \sum_{\ell=1}^{n} a_{n+1}^{(\ell)} \sum_{m \neq 0} K_{0,m}^{(\ell)}(\alpha). \tag{1.2}$$

Theorem 2 *There is a recursive relationship among the coefficients for* T^{n+1}*:*

$$-(n-\ell)a_{n+1}^{(\ell+1)} + (n+3)a_{n+1}^{(\ell)} - (\ell-1)a_{n+1}^{(\ell-1)} = 0$$

for $1 \leq \ell \leq n$.

Theorem 3 *The following relationship holds between the coefficients for* T^n *and* T^{n+1}*:*

$$a_n^{(\ell)} = a_{n+1}^{(\ell)} + a_{n+1}^{(\ell+1)}.$$

Chapter 2

Preliminaries

2.1 Fiber Bundles

The following definitions are adapted from Steenrod [S].

A **principal coordinate bundle** $(X, G, \mathcal{U} = \{U_\alpha\}, \{g_{\alpha\beta}: U_\alpha \cap U_\beta \to G\})$ consists of a topological space X, a Lie group G, a covering $\mathcal{U}$ of X by open sets U_α (so $X = \cup_\alpha U_\alpha$) and a continuous map $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ whenever $U_\alpha \cap U_\beta \neq \emptyset$, satisfying the following *cocycle condition*: if $x \in U_\alpha \cap U_\beta \cap U_\gamma$, then

$$g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) g_{\beta\gamma}(x).$$

The *total space* E of the bundle is defined as

$$E = \bigcup_\alpha U_\alpha \times G / \sim$$

where $(x, g_\alpha) \in U_\alpha \times G$ is identified with $(x, g_\beta) \in U_\beta \times G$ iff $g_\alpha = g_{\alpha\beta}(x) g_\beta$. The *projection* $p: E \to X$ takes (x, g_α) to x. The *global right action* of G on E is defined on $U_\alpha \times G$ by $(x, g_\alpha) \cdot g = (x, g_\alpha g)$; since it commutes with the

identifications, it extends to E. Clearly, $X = E/G$. We can interpret the obvious map $h_\alpha: U_\alpha \times G \to p^{-1}U_\alpha$ as a local coordinate on E; over $U_\alpha \cap U_\beta$ the two coordinates are related by $h_\alpha(x, g_\alpha) = h_\beta(x, g_\beta)$ if and only if $g_\alpha = g_{\alpha\beta}(x)g_\beta$; hence the name *transition functions* for the $g_{\alpha\beta}$.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$ with say $V_\rho \subset U_{\eta(\rho)}$ for each ρ, then the coordinate bundle with covering $\mathcal{V}$ and transition functions defined by $g'_{\rho\sigma} = g_{\eta(\rho)\eta(\sigma)}|_{V_\rho \cap V_\sigma}$ is considered *equivalent* to $(X, G, \mathcal{U}, \{g_{\alpha\beta}\})$. So to compare two principal coordinate bundles $(X, G, \mathcal{U} = \{U_\alpha\}, \{g_{\alpha\beta}\})$ and $(X, G, \mathcal{U}' = \{U'_\lambda\}, \{g'_{\lambda_\mu}\})$ with same base X and group G we may suppose that $\mathcal{U}' = \mathcal{U}$ by passing to a common refinement if necessary. Then the two bundles are considered *equivalent* if there exists a family of continuous functions $\lambda_\alpha: U_\alpha \to G$ such that $g_{\alpha\beta}\lambda_\alpha = \lambda_\beta g'_{\alpha\beta}$.

A **principal G-bundle** is an equivalence class of principal coordinate G-bundles under this relation.

Remark The dual cells, and their pairwise intersections, are closed; usually open sets are used for local trivializations. This was justified in [P]; for a construction which corresponds to this one but uses open sets, see [GKSW].

2.2 Connections

There are several equivalent ways of defining a connection in a principal G-bundle $\xi = (\pi: E \to X)$. For our purposes the most convenient is the following one defining connection in terms of parallel transport. A **connection** associates to each piecewise smooth curve $c: [0, 1] \to X$ and to each element

$e \in p^{-1}c(0)$ an element $e' \in p^{-1}c(1)$ in such a way that 1) e' varies continuously with c, and 2) considering the right action of $g \in G$ on E, $(e \cdot g)' = e' \cdot g$. We say that e' is obtained from parallel transport of e along c by the connection.

2.3 Lattice Gauge Fields

In the case of U_1-valued LGFs on tori, it is natural to let Λ be a periodic, cubical lattice. Working on T^{n+1}, let us denote by $\{j_0, j_1, \ldots, j_n\}$ unit vectors parallel to the coordinate axes (and by $-j_0$ the unit vector in the direction opposite j_0, etc.), so that given a vertex $\alpha \in \Lambda$, $\alpha + j_k$ is the adjacent vertex in the lattice in the kth direction. A U_1-valued **lattice gauge field u** is the assignment of an element of U_1 to each oriented 1-cell in the lattice. It assigns $u_{\alpha;j_i}$ to the 1-cell going from α to $\alpha + j_i$. The definition requires $u_{\alpha+j_i;-j_i} = u^{-1}_{\alpha;j_i}$. We call a square (2-cell) P in the lattice a **plaquette**, and we will use the notation $P_{i,k}(\alpha)$ for the plaquette with vertices listed counterclockwise as $\alpha, \alpha + j_i, \alpha + j_i + j_k, \alpha + j_k$. The **plaquette product** is then the product $u_{\alpha;j_i}\, u_{\alpha+j_i;j_k}\, u^{-1}_{\alpha+j_k;j_i}\, u^{-1}_{\alpha;j_k}$ of the transporters assigned to each edge of the square, and this element of U_1 is denoted $u_{P_{i,k}(\alpha)}$. Its argument (using the branch of the logarithm between $-\pi$ and π) is called the **plaquette angle** and denoted $K_{P_{i,k}(\alpha)}$ or $K_{i,k}(\alpha)$. The LGF **u** is said to be **continuous** if no $u_P = -1$; then K_P is unambiguously defined and never takes the values π or $-\pi$. Some useful plaquette product identities are

$$K_{i,k}(\alpha) = -K_{k,i}(\alpha) = K_{k,-i}(\alpha + j_i). \tag{2.1}$$

Having chosen α and two directions j_i and j_k, it will turn out to be convenient to have a concise notation for sums of angles of plaquettes "parallel" to $P_{i,k}(\alpha)$ (i.e. also involving the same two directions). We will group them by their remoteness from α as follows:

$$K_{i,k}^{(1)}(\alpha) = K_{i,k}(\alpha)$$

$$K_{i,k}^{(2)}(\alpha) = \sum_{m\neq i,k} K_{i,k}(\alpha + j_m)$$

$$K_{i,k}^{(3)}(\alpha) = \sum_{m\neq i,k} \sum_{\ell\neq i,k,m} K_{i,k}(\alpha + j_m + j_\ell)$$

$$K_{i,k}^{(4)}(\alpha) = \sum_{m\neq i,k} \sum_{\ell\neq i,k,m} \sum_{p\neq i,k,\ell,m} K_{i,k}(\alpha + j_m + j_\ell + j_p)$$

$$\vdots$$

$$K_{i,k}^{(n-2)}(\alpha) = \sum_{m\neq i,k} \sum_{\ell\neq i,k,m} K_{i,k}(\alpha + \sum_{p\neq i,k,\ell,m} j_p)$$

$$K_{i,k}^{(n-1)}(\alpha) = \sum_{m\neq i,k} K_{i,k}(\alpha + \sum_{\ell\neq i,k,m} j_\ell)$$

$$K_{i,k}^{(n)}(\alpha) = K_{i,k}(\alpha + \sum_{m\neq i,k} j_m)$$

We also set

$$K_i^{(\ell)}(\alpha) = \sum_{k\neq i} K_{i,k}^{(\ell)}(\alpha), \quad \ell = 1, \ldots, n.$$

In the higher-dimensional cases, there is another important property which we must require in addition to the continuity condition. It is convenient to

describe a cube (3-cell) by a starting vertex and the three directions one travels to trace the cube; for example, $(\beta; j_i, j_k, j_\ell)$. The **monopole-free condition** states that the following relationship holds for the faces (plaquettes or 2-cells) of that cube:

$$K_{i,k}(\beta) - K_{i,k}(\beta + j_\ell) = K_{k,\ell}(\beta + j_i) - K_{k,\ell}(\beta) + K_{i,\ell}(\beta) - K_{i,\ell}(\beta + j_k) \quad (2.2)$$

The monopole-free condition is necessary for the existence of the bundle, as it guarantees that we can extend our construction from the 2-skeleton to the 3-skeleton of the lattice. In fact, according to [13], the sum

$$K_{i,k}(\beta) - K_{i,k}(\beta + j_\ell) - K_{k,\ell}(\beta + j_i) + K_{k,\ell}(\beta) - K_{i,\ell}(\beta) + K_{i,\ell}(\beta + j_k)$$

is 2π times the first Chern number C_1 of the bundle defined by the lattice gauge field $\mathbf{u}$ on the boundary of the cube (topologically a 2-sphere). This bundle can be extended to the interior of the cube if and only if it is trivial, which is equivalent to $C_1 = 0$ [8].

This topological fact will have an algebraic manifestaton when we carry out the implementation of our Ansatz in Chapter 3. The Ansatz leads to a certain system of linear equations which is overdetermined and admits no solution. At a corner x of a dual cube on the $(n+1)$-torus, enforcing the cocycle condition around a plaquette P leads to an inhomogeneous system of $(2n-1)$ linear equations in the n weights assigned to $K_i^{(1)}(\alpha), \ldots, K_i^{(n)}(\alpha)$. Applying the monopole-free condition to each of the $(n-1)$ 3-cubes in the link of x which have P as a face will reduce the system to one with a unique solution.

2.4 Continuants

We now include some material from linear algebra which will be useful in the proof of the main theorems. We refer the reader to Muir [M] for further details.

A **continuant** is a determinant all of whose elements are zero except those on the main diagonal and in the two adjacent minor diagonals. We will adopt the notation $D(1,n)$ for the $n \times n$ determinant with a_1,b_1 and c_1 in the upper left corner and a_n,b_{n-1} and c_{n-1} in the lower right corner:

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & 0 & \dots & 0 \\ & & \vdots & & & & \\ 0 & \dots & 0 & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \dots & 0 & 0 & 0 & c_{n-1} & a_n \end{pmatrix}$$

or,

$$D\begin{pmatrix} & b_1 & & b_2 & \dots & b_{n-1} & \\ a_1 & & a_2 & \dots & n-1 & & a_n \\ & c_1 & & c_2 & \dots & c_{n-1} & \end{pmatrix}$$

Theorem 1 *Let Δ_n be the following $n \times n$ continuant:*

$$\begin{pmatrix} b & & 2b & \dots & (n-1)b \\ a & a-(b+c) & & \dots & a-(n-1)(b+c) \\ -(n-1)c & & -(n-2)c & \dots & -c \end{pmatrix}$$

Then $\Delta_n = [a-(n-1)c][a-(n-2)c-b][a-(n-3)c-2b]\cdots[a-(n-1)b]$.

Chapter 3

Ansatz

In this chapter we explain the Ansatz we employ for constructing a coordinate bundle (i.e. a set of transition functions satisfying the cocycle condition) from a U_1-valued lattice gauge field on a cubical complex. This is a generalization and systematization of a procedure used in [13] for dimensions 2,3 and 4. As we proceed, we will illustrate the Ansatz by carrying it out completely in dimensions 2 and 3.

3.1 Beginning of the Ansatz

To construct the coordinate bundle, we first form the dual complex to Λ (we denote the n-cell dual to the vertex $\alpha \in \Lambda$ by c_α). The top-dimensional dual cells will be our "trivializing sets", and we will define the transition functions on pairwise intersections (codimension-1 faces of dual cubes), e.g. $v_{\alpha;j_i} : c_\alpha \cap c_{\alpha+j_i} \to U_1$. At triple intersections $c_\alpha \cap c_{\alpha+j_i} \cap c_{\alpha+j_k}$ (codimension-2 cells in the dual complex) we must ensure that the product of the four

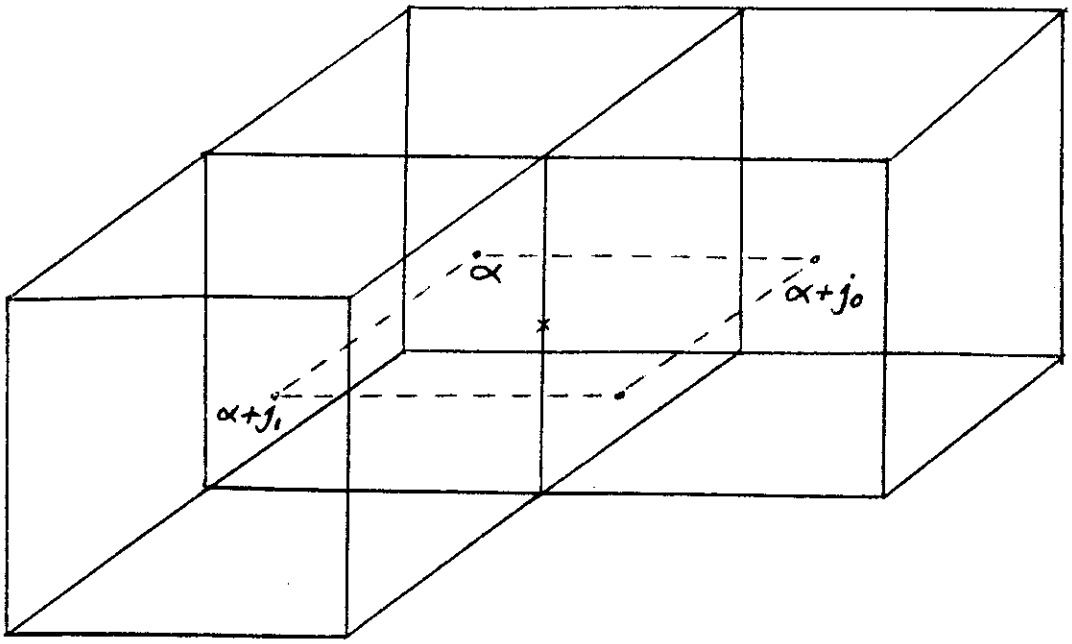

Figure 3.1: $c_\alpha \cap c_{\alpha+j_0} \cap c_{\alpha+j_1}$ is dual to the plaquette $P_{0,1}(\alpha)$ (dim 3)

transition functions (corresponding to the four incident codimension-1 cells) is the identity. This is a **cocycle condition**, and there is one for each dual codimension-2 cell; i.e. for each plaquette in Λ, since the triple intersection $c_\alpha \cap c_{\alpha+j_i} \cap c_{\alpha+j_k}$ is dual to the plaquette $P_{i,k}(\alpha)$ (see Figure 3.1).

We will define the transition functions inductively on dimension. At each stage v is already defined on intersections of the dual face with lower-dimensional lattices. We will define v at the far corners of the dual intersection, and interpolate to match the definitions established at the center and from the previous stages. The general procedure will be to define v from the center value and from the plaquette angles of certain 2-faces related to the point in question. To take advantage of lattice symmetries, all plaquettes equally distant from this point will be weighted equally in our formula.

Note: It is important to remark that the Ansatz uses only the geometric relations between its various ingredients. We will use specific coordinate directions to illustrate its operation and to calculate the values of certain a priori unknown coefficients, but the calculation is completely invariant under symmetries of the lattice. In particular it will only be necessary to define the transition function at one new corner of one particular face in each dimension, and in each dimension it will only be necessary to check one cocycle condition.

We will construct the transition functions to agree with $\mathbf{u}$ in the following obvious sense: since the 1-cell (in Λ) labelled $\alpha; j_i$ runs through the center $\alpha + \frac{1}{2}j_i$ of $c_\alpha \cap c_{\alpha+j_i}$, we set $\nu_{\alpha;j_i}(\alpha+\frac{1}{2}j_i) = u_{\alpha;j_i}$.

3.2 In dimension 2

We will construct $v_{\alpha;j_0} : c_\alpha \cap c_{\alpha+j_0} \to U_1$. The domain of this transition function is one-dimensional. We have defined it at the midpoint of the intersection; now to define it at either endpoint (the "corners" $\alpha + \frac{1}{2}(j_0 \pm j_1)$) is sufficient, for we can use linear interpolation to extend $v_{\alpha;j_0}$ to the rest of the intersection (see Figure 3.2). We will define

$$v_{\alpha;j_0}(\alpha + \frac{1}{2}j_0 + \frac{1}{2}j_1) = u_{\alpha;j_0} \exp(ia_2^{(1)} K_{0,1}(\alpha)) \tag{3.1}$$

where $a_2^{(1)}$ is universal (the subscript indicates the dimension of the lattice, and will be omitted when there is no confusion; the superscript indexing ensures compatibility with the higher levels of the algorithm).

To determine $a_2^{(1)}$, we apply the only relevant cocycle condition at the

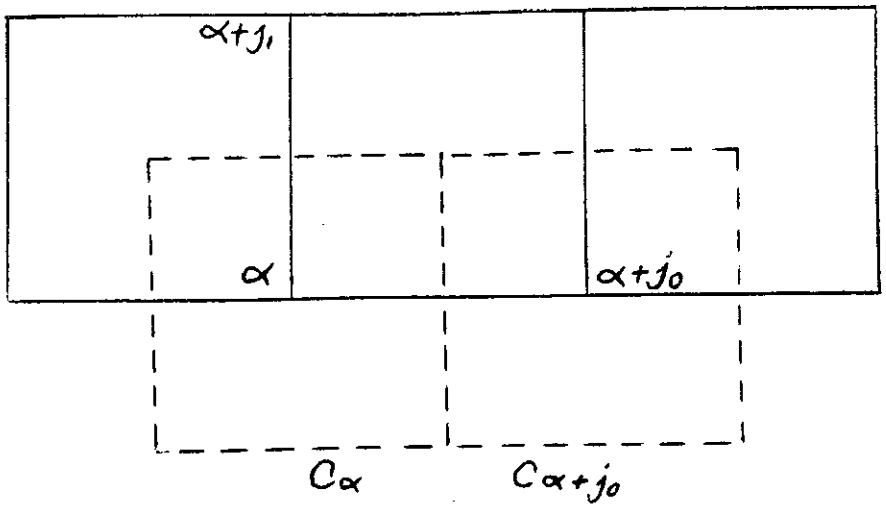

Figure 3.2: $dim(\Lambda) = 2$

chosen point, which is the one corresponding to the plaquette $(\alpha; j_0, j_1)$. If

$$A = v_{\alpha;j_0}(\alpha + \frac{1}{2}j_0 + \frac{1}{2}j_1)$$

$$B = v_{\alpha+j_0;j_1}(\alpha + \frac{1}{2}j_0 + \frac{1}{2}j_1) = v_{\alpha+j_0;j_1}(\alpha + j_0 - \frac{1}{2}j_0 + \frac{1}{2}j_1)$$

$$C = v_{\alpha+j_0+j_1;-j_0}(\alpha + \frac{1}{2}j_0 + \frac{1}{2}j_1) = v_{\alpha+j_0+j_1;-j_0}(\alpha + j_0 + j_1 - \frac{1}{2}j_0 - \frac{1}{2}j_1)$$

$$D = v_{\alpha+j_1;-j_1}(\alpha + \frac{1}{2}j_0 + \frac{1}{2}j_1) = v_{\alpha+j_1;-j_1}(\alpha + j_1 + \frac{1}{2}j_0 - \frac{1}{2}j_1)$$

(using the plaquette identities 2.1) then the cocycle condition is $ABCD = 1$.

By the equation 3.1,

$$B = u_{\alpha+j_0;j_1} \exp(ia_2^{(1)} K_{1,-0}(\alpha + j_0))$$

$$C = u_{\alpha+j_0+j_1;-j_0} \exp(ia_2^{(1)} K_{-0,-1}(\alpha + j_0 + j_1))$$

$$D = u_{\alpha+j_1;-j_1} \exp(ia_2^{(1)} K_{-1,0}(\alpha + j_1)).$$

Now we set $ABCD = 1$ and regroup the u's and K's:

$$1 = u_{\alpha;j_0} u_{\alpha+j_0;j_1} u_{\alpha+j_0+j_1;-j_0} u_{\alpha+j_1;-j_1} \times$$

$$\exp(ia_2^{(1)})[K_{0,1}(\alpha) + K_{1,-0}(\alpha + j_0) + K_{-0,-1}(\alpha + j_0 + j_1) + K_{-1,0}(\alpha + j_1)].$$

But the product of the u's is $\exp(iK_{0,1}(\alpha))$; and the plaquette identities 2.1 reveal that each of the terms in the exponent is also this plaquette angle. Therefore, we have

$$1 = \exp(iK_{0,1}(\alpha))(1 + 4a_2^{(1)}),$$

which determines the coefficient: $a_2^{(1)} = -1/4$.

3.3 In dimension 3; monopole-free condition

We now move to dimension 3, where the monopole-free condition first appears. We will show that this condition is necessary and sufficient for determining the coefficients used in the transition functions. Here, the domain of the transition function is 2-dimensional. We will begin with the same construction developed for dimension 2 – *i.e.*

$$v_{\alpha;j_0}(\alpha + \frac{1}{2}j_0) = u_{\alpha;j_0}$$

$$v_{\alpha;j_0}(\alpha + \frac{1}{2}(j_0 \pm j_k)) = u_{\alpha;j_0} \exp(-i/4K_{0,\pm k}(\alpha; j_0)), k = 1, 2.$$

Next we calculate the value of $v_{\alpha;j_0}$ at the corners of $c_\alpha \cap c_{\alpha+j_0}$ (which are $\alpha + \frac{1}{2}(j_0 \pm j_1 \pm j_2)$) and interpolate. Because of the symmetries of the lattice and the expression, it will suffice to provide the construction for only one of these;

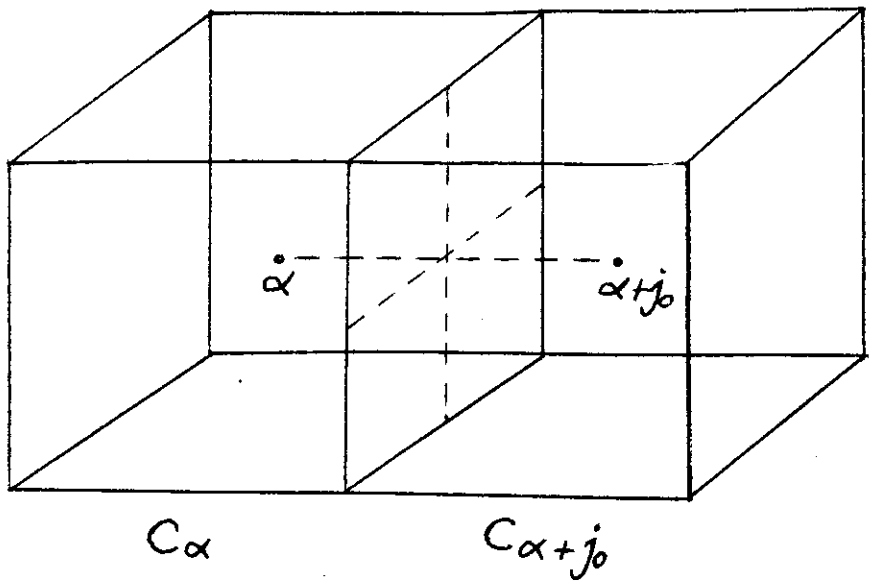

Figure 3.3: $dim(\Lambda) = 3$

it is convenient to use the one with all positive directions, which is the center of the 3-cube $(\alpha; j_0, j_1, j_2)$. It is convenient to restrict our choices of plaquettes to this cube; and we will further restrict ourselves to those plaquettes involving the direction j_0, since it is the determining direction for the transition function in question. In that cube there are the "nearer" plaquettes (which contain the edge $\alpha; j_0$), to which we give the weight $a_3^{(1)}$, and we give the weight $a_3^{(2)}$ to the two "farther" plaquettes, so that

$$v_{\alpha;j_0}(x_0) = u_{\alpha;j_0} \exp(i)\{a_3^{(1)}[K_{0,1}(\alpha)+K_{0,2}(\alpha)]+a_3^{(2)}[K_{0,1}(\alpha+j_2)+K_{0,2}(\alpha+j_1)]\}$$

At this corner, there are two cocycle conditions to be satisfied, corresponding to the two codimension 2 faces of the intersection which meet there.

The cocycle condition $(\alpha; j_0, j_k), k = 1, 2$ requires that

$$v_{\alpha;j_0}(x_0)v_{\alpha+j_0;j_k}(x_0)v^{-1}_{\alpha+j_1;j_0}(x_0)v^{-1}_{\alpha;j_k}(x_0) = 1,$$

i.e. (taking k=1, for example)

$$\begin{aligned}
1 \;=\; & (u_{\alpha;j_0}u_{\alpha+j_0;j_1}u^{-1}_{\alpha+j_1;j_0}u^{-1}_{\alpha;j_0})\times \\
& \exp\{i\left(a_3^{(1)}[K_{0,1}(\alpha) + K_{0,2}(\alpha) + K_{1,-0}(\alpha+j_0) + K_{1,2}(\alpha+j_0)\right. \\
& -K_{0,-1}(\alpha+j_1) - K_{0,2}(\alpha+j_1) - K_{1,0}(\alpha) - K_{1,2}(\alpha)] \\
& +a_3^{(2)}[K_{0,1}(\alpha+j_2) + K_{0,2}(\alpha+j_1) + K_{1,-0}(\alpha+j_0+j_2) + K_{1,2}(\alpha+j_0-j_0) \\
& \left.-K_{0,-1}(\alpha+j_1+j_2) - K_{0,2}(\alpha+j_1-j_1) - K_{1,0}(\alpha+j_2) - K_{1,2}(\alpha+j_0)]\right)\}
\end{aligned}$$

Now we apply the plaquette identities 2.1 and combine similar terms:

$$\begin{aligned}
1 \;=\; & \exp(iK_{0,1}(\alpha))exp(i)\left(a_3^{(1)}[4K_{0,1}(\alpha)\right. \\
& +K_{0,2}(\alpha) + K_{1,2}(\alpha+j_0) - K_{0,2}(\alpha+j_1) - K_{1,2}(\alpha)] \\
& \left.+a_3^{(2)}[4K_{0,1}(\alpha+j_2) + K_{0,2}(\alpha+j_1) + K_{1,2}(\alpha) - K_{0,2}(\alpha) - K_{1,2}(\alpha+j_0)]\right)
\end{aligned}$$

$$\begin{aligned}
1 \;=\; & \exp(i)\left((1+4a_3^{(1)})K_{0,1}(\alpha) + a_3^{(2)}K_{0,1}(\alpha+j_2)\right. \\
& \left.+(a_3^{(1)} - a_3^{(2)})[K_{0,2}(\alpha) - K_{0,2}(\alpha+j_1) + K_{1,2}(\alpha+j_0) - K_{1,2}(\alpha)]\right)
\end{aligned}$$

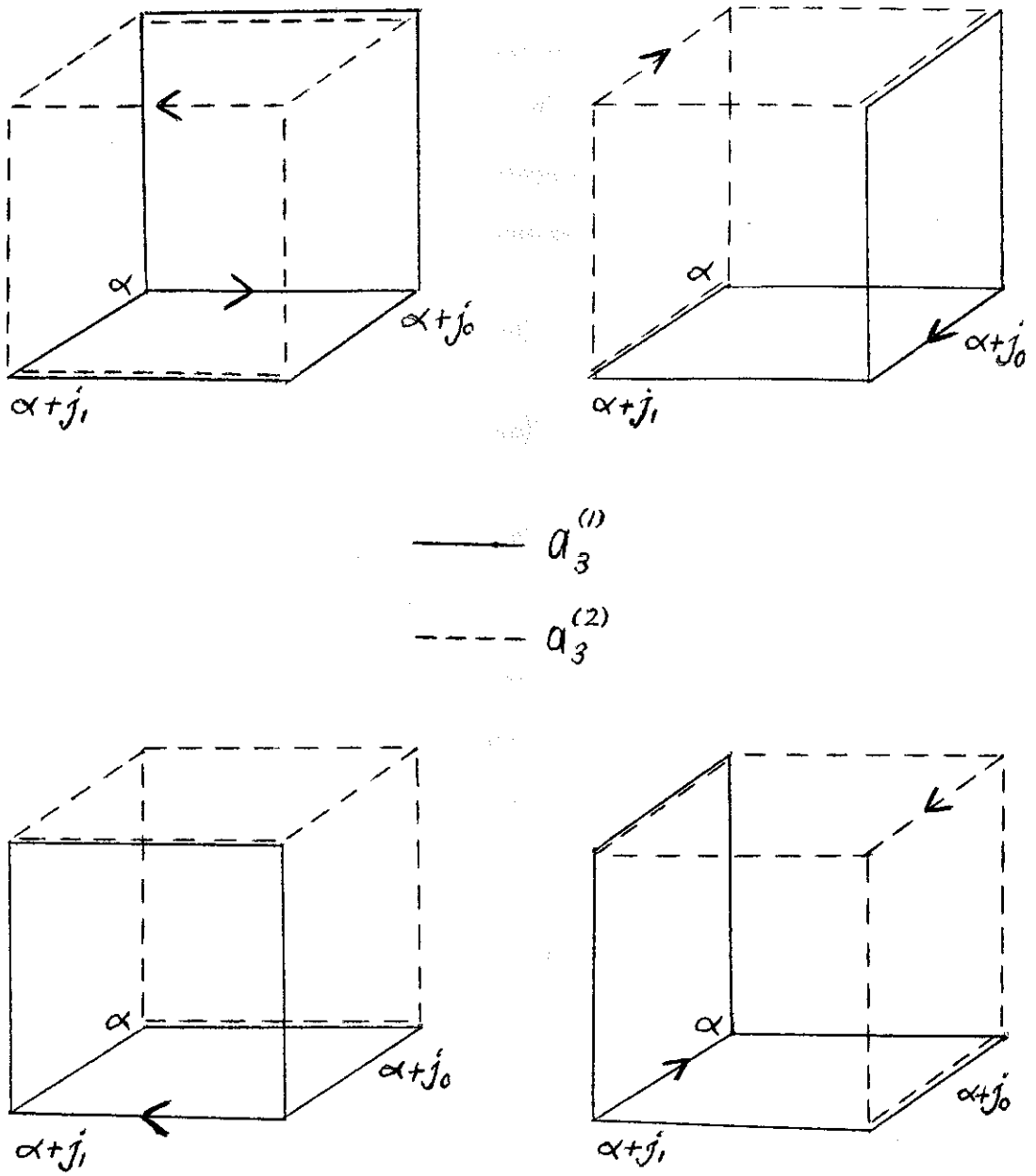

Figure 3.4: Plaquettes used in the four terms of the cocycle condition $(\alpha; j_0, j_1)$

Thus we see (either from this expression or by examining Figure 3.3) that to each face of the cube $(\alpha; j_0, j_1, j_2)$ that links our corner is associated a coefficient of $1 + 4a_3^{(1)}, 4a_3^{(2)}$, or $a_3^{(1)} - a_3^{(2)}$. Since the plaquette angles in the equation are arbitrary, these coefficients must be simulaneously zero for the cocycle condition to balance. Thus our Ansatz leads to the system

$$\begin{aligned} 1 + 4a_3^{(1)} &= 0 \\ 4a_3^{(2)} &= 0 \\ a_3^{(1)} - a_3^{(2)} &= 0 \end{aligned}$$

which clearly admits no solutions.

The only way to continue with the Ansatz is to reduce the dimension of the space of equations. Let us suppose that there is some linear relationship among the plaquette angles of this cube: *i.e.*

$$\sum_{i=1}^{6} \lambda_i K_i = 0, \lambda_i \text{ not all } 0 \tag{3.2}$$

(where we have abbreviated $K_1 = K_{0,1}(\alpha), K_2 = -K_{0,1}(\alpha + j_2), K_3 = K_{1,2}(\alpha)$, $K_4 = -K_{1,2}(\alpha + j_0), K_5 = -K_{0,2}(\alpha), K_6 = K_{0,2}(\alpha + j_1)$)

The cocycle condition $(\alpha; j_0, j_1)$ shows that, because four of the plaquette angles (namely, K_3, K_4, K_5, and K_6) have the same coefficient, that $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$; in exactly the same manner, applying the cocycle condition $(\alpha; j_0, j_2)$ gives us $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Therefore, all the λ's are equal; now equation 3.2 becomes

$$K_1 + K_2 + K_3 + K_4 + K_5 + K_6 = 0,$$

or

$$K_{0,1}(\alpha) - K_{0,1}(\alpha + j_2) + K_{1,2}(\alpha) - K_{1,2}(\alpha + j_0) - K_{0,2}(\alpha) + K_{0,2}(\alpha + j_1) = 0,$$

which is precisely the monopole-free condition (2.2). Thus, the existence of a solution for the coefficients (given this Ansatz) is equivalent to the imposition of the monopole-free constraint.

Since the coefficients λ_i are all equal, we now have $1 + 4a_3^{(1)} = 4a_3^{(2)} = a_3^{(1)} - a_3^{(2)}$, yielding $a_3^{(1)} = -5/24$ and $a_3^{(2)} = -1/24$ in the 3-dimensional case.

3.4 In dimension $n+1$

We now move to higher dimensions. On the interface between two top-dimensional dual cells in an $(n+1)$-dimensional cubical lattice Λ we will we will define $v_{\alpha;j_0}$ as above at the points $\alpha+\frac{1}{2}j_0$ and $\alpha+\frac{1}{2}j_0+\frac{1}{2}j_k$, and $\alpha+\frac{1}{2}(j_0\pm j_k\pm j_\ell)$, for $1 \leq k, \ell \leq n$ and we assume by induction that this pattern has been carried out through dimension n. This defines $v_{\alpha;j_0}$ at all the intersections of the interface between c_α and $c_{\alpha+j_0}$ with lower-dimensional sublattices of Λ.

At the furthest corners of $c_\alpha \cap c_{\alpha+j_0}$, which have the form $\alpha + \frac{1}{2}(\pm j_0 \pm \ldots \pm j_n)$, we define

$$v_{\alpha;j_0} = u_{\alpha;j_0} \exp(i)(\textit{linear combination of nearby plaquette angles}),$$

where, as before, "nearby" will mean that a plaquette is a face of the top-dimensional cube in Λ which has as its center the corner in question; those

plaquettes at the same relative distance will carry the same weight:

$$v_{\alpha;j_0}(x) = u_{\alpha;j_0} \exp(i) \sum_{\ell=1}^{n} \sum_{m \neq 0} a_{n+1}^{(\ell)} K_{0,m}^{(\ell)}(\alpha),$$

using the notation from Section 2.3.

This completes the Ansatz. We will show in the next section that, using the monopole-free condition, it can indeed be carried out.

Chapter 4

Main Theorem and Proof

Theorem 4 *Let* $\mathbf{u}$ *be a continuous lattice gauge field on a periodic, cubical lattice* Λ *on the torus* T^{n+1}*, and* α *a vertex of the lattice. If* $\mathbf{u}$ *is monopole-free, then there is a set of coefficients* $\{a_\ell^{n+1}\}$ *such that all relevant cocycle conditions at the given point are satisfied by the transition function given by:*

$$v_{\alpha;j_0}(\alpha + \frac{1}{2}(j_1 + j_2 + \ldots + j_n)) = u_{\alpha;j_0} \cdot exp(i\Theta(\alpha; j_0)), \tag{4.1}$$

where

$$\Theta(\alpha; j_0) = \sum_{\ell=1}^{n} a_{n+1}^{(\ell)} \sum_{m \neq 0} K_{0,m}^{(\ell)}(\alpha). \tag{4.2}$$

Remark Defining $v_{\alpha;j_0}$ at $\alpha + \frac{1}{2}\sum_{\ell=0}^{n} j_\ell$ is one of several possible choices, given a cocycle; any one of the "corners" of the triple intersection could be used, as previously stated. If we choose the corner $x = \alpha + \frac{1}{2}(\sum \epsilon_i j_i)$, $(\epsilon_i = \pm 1)$, we can proceed with the stated formula, substituting $\epsilon_i j_i$ for j_i, and applying the plaquette identities (2.1). The same coefficients will result.

Proof: Assume that $v_{\alpha;j_0}(x)$ has the form 4.1 where $x = \alpha + \frac{1}{2}\sum_{\ell=0}^{n} j_\ell$. As remarked in Section 3.3, satisfying the cocycle condition coming from one

plaquette is equivalent, because of symmetry, to satisfying all such conditions, so consider the cocycle condition from the plaquette $(\alpha; j_0, j_k)$:

$$v_{\alpha;j_0}(x)v_{\alpha+j_0;j_k}(x)v^{-1}_{\alpha+j_k;j_0}(x)v^{-1}_{\alpha;j_k}(x) = 1.$$

Subsituting the values from 4.1 and rearranging factors gives

$$1 = u_{\alpha;j_0}u_{\alpha+j_0;j_k}u^{-1}_{\alpha+j_k;j_0}u^{-1}_{\alpha;j_k} \cdot exp(i\Theta_k) = \exp(iK_{0,k}(\alpha)) \cdot \exp(i\Theta_k),$$

where

$$\Theta_k = \Theta(\alpha; j_0) + \Theta(\alpha + j_0; j_k) - \Theta(\alpha + j_k; j_0) - \Theta(\alpha; j_k).$$

Substituting the expressions 4.2 leads to an equation in the coefficients $a^{(\ell)} = a^{(\ell)}_{n+1}$ and the various $K^{(\ell)}$'s. Let us write it as

$$1 = \exp[i(K_{0,k}(\alpha) + \sum_{\ell} a^{(\ell)}C_\ell)]. \tag{4.3}$$

Here

$$C_\ell = K^{(\ell)}_0(\alpha) + K^{(\ell)}_k(\alpha + j_0) - K^{(\ell)}_0(\alpha + j_k) - K^{(\ell)}_k(\alpha).$$

We will analyze these four terms separately.

$$
\begin{aligned}
K_0^{(\ell)}(\alpha) &= \sum_{m\neq 0} K_{0,m}^{(\ell)}(\alpha) \\
&= K_{0,k}^{(\ell)}(\alpha) + \sum_{m\neq 0,k} K_{0,m}^{(\ell)}(\alpha) \\
&= K_{0,k}^{(\ell)}(\alpha) + \sum_{m\neq 0,k} \sum_{\substack{P\not\ni 0,m \\ |P|=\ell-1}} K_{0,m}(\alpha + j_P) \\
&= K_{0,k}^{(\ell)}(\alpha) + \sum_{m\neq 0,k} [\sum_{\substack{P\not\ni 0,m,k \\ |P|=\ell-1}} K_{0,m}(\alpha + j_P) \\
&\qquad + \sum_{\substack{\widehat{P}\not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{0,m}(\alpha + j_k + j_{\widehat{P}})].
\end{aligned}
$$

$$
\begin{aligned}
K_k^{(\ell)}(\alpha + j_0) &= \sum_{m\neq k} K_{k,m}^{(\ell)}(\alpha + j_0) \\
&= K_{k,-0}^{(\ell)}(\alpha + j_0) + \sum_{m\neq 0,k} K_{k,m}^{(\ell)}(\alpha + j_0) \\
&= K_{k,-0}^{(\ell)}(\alpha + j_0) + \sum_{m\neq 0,k} \sum_{\substack{P\not\ni k,m \\ |P|=\ell-1}} K_{k,m}(\alpha + j_0 + j_P) \\
&= K_{k,-0}^{(\ell)}(\alpha + j_0) + \sum_{m\neq 0,k} [\sum_{\substack{P\not\ni 0,m,k \\ |P|=\ell-1}} K_{k,m}(\alpha + j_0 + j_P) \\
&\qquad + \sum_{\substack{\widehat{P}\not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{k,m}(\alpha + j_0 - j_0 + j_{\widehat{P}})].
\end{aligned}
$$

And similarly

$$K_0^{(\ell)}(\alpha + j_k) = K_{0,-k}^{(\ell)}(\alpha + j_k) + \sum_{m \neq 0,k} [\sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{0,m}(\alpha + j_k + j_P) + \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{0,m}(\alpha + j_k - j_k + j_{\widehat{P}})].$$

$$K_0^{(\ell)}(\alpha) = K_{k,0}^{(\ell)}(\alpha) + \sum_{m \neq 0,k} [\sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{k,m}(\alpha + j_P) + \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{k,m}(\alpha + j_0 + j_{\widehat{P}})].$$

Now substituting the last line of each of these computations in the expression for C_ℓ, and applying the identities 2.1 yields

$$4K_{0,k}^{(\ell)}(\alpha) + \sum_{m \neq 0,k} \left\{ \begin{array}{l} \sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{0,m}(\alpha + j_P) + \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{0,m}(\alpha + j_k + j_{\widehat{P}}) \\ + \sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{k,m}(\alpha + j_0 + j_P) + \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{k,m}(\alpha + j_{\widehat{P}}) \\ - \sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{0,m}(\alpha + j_k + j_P) - \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{0,m}(\alpha + j_{\widehat{P}}) \\ - \sum_{\substack{P \not\ni 0,m,k \\ |P|=\ell-1}} K_{k,m}(\alpha + j_P) - \sum_{\substack{\widehat{P} \not\ni 0,m,k \\ |\widehat{P}|=\ell-2}} K_{k,m}(\alpha + j_0 + j_{\widehat{P}}) \end{array} \right\}$$

For each P, the left-hand column of this array is the sum of the plaquette angles of the four "lateral" faces of the cube based at $\alpha + j_P$ with edges

along j_0, j_k, j_m and oriented by that 3-frame. If this cube is not to contain a monopole, this sum must be equal to minus the sum of the plaquette angles of the top and bottom faces, i.e.

$$K_{0,k}(\alpha + j_P) - K_{0,k}(\alpha + j_P + j_m).$$

Similarly for the right-hand column, substituting $\widehat{P}$ for P and noting that the signs are reversed; the monopole-free condition requires the sum to equal

$$-K_{0,k}(\alpha + j_{\widehat{P}}) + K_{0,k}(\alpha + j_{\widehat{P}} + j_m).$$

Replacing the column sums by these new values gives

$$\begin{aligned} C_\ell &= 4K_{0,k}^{(\ell)} \\ &\quad + \sum_{m \neq 0,k} \sum_{P \not\ni 0,k,m} [K_{0,k}(\alpha + j_P) - K_{0,k}(\alpha + j_m + j_P)] \\ &\quad - \sum_{m \neq 0,k} \sum_{\widehat{P} \not\ni 0,k,m} [K_{0,k}(\alpha + j_{\widehat{P}}) - K_{0,k}(\alpha + j_m + j_{\widehat{P}})] \end{aligned}$$

Now compare $A = \sum_{m \neq 0,k} \sum_{P \not\ni 0,k,m} K_{0,k}(\alpha + j_P)$ with $K_{0,k}^{(\ell)} = \sum_{P \not\ni 0,k} K_{0,k}(\alpha + j_P)$. Each term in $K_{0,k}^{(\ell)}$ appears once in A for each possible choice of $m \notin P \cup \{0,k\}$. Since there are $n+1-(\ell-1)-2 = (n-\ell)$ such choices, we have $A = (n-\ell)K_{0,k}^{(\ell)}$. Similarly, $\sum_{m \neq 0,k} \sum_{\widehat{P} \not\ni 0,k,m} K_{0,k}(\alpha + j_{\widehat{P}}) = (n-\ell-1)K_{0,k}^{(\ell-1)}$. On the other hand, $\sum_{m \neq 0,k} \sum_{P \not\ni 0,k,m} K_{0,k}(\alpha + j_m + j_P)$ is clearly the same as $\ell \sum_{m \neq 0,k} \sum_{\bar{P} \not\ni 0,k} K_{0,k}(\alpha + j_{\bar{P}}) = \ell K_{0,k}^{(\ell+1)}$ since there are ℓ different places to insert the "m", and similarly $\sum_{m \neq 0,k} \sum_{\widehat{P} \not\ni 0,k,m} K_{0,k}(\alpha + j_m + j_{\widehat{P}}) = (\ell-1)K_{0,k}^{(\ell)}$. With these simplifications, the expression for C_ℓ becomes:

$$\begin{aligned} C_\ell &= 4K^{(\ell)} + [(n-\ell)K^{(\ell)} - \ell K^{(\ell+1)}] - [(n-(\ell-1))K^{(\ell-1)} - (\ell-1)K^{(\ell)}] \\ &= K^{(\ell)}[4+n-\ell+\ell-1] - \ell K^{(\ell+1)} - (n-\ell+1)K^{(\ell-1)} \\ &= -(n-\ell+1)K^{(\ell-1)} + (n+3)K^{(\ell)} - \ell K^{(\ell+1)}. \end{aligned}$$

Now substituting into equation 4.3 gives

$$1 = \exp\{i(K^{(1)}(\alpha) + \sum_\ell a^{(\ell)}[-(n-\ell+1)K^{(\ell-1)} + (n+3)K^{(\ell)} - \ell K^{(\ell+1)}]\}$$

and regrouping in terms of the K's gives

$$1 = \exp\{i(K^{(1)}(\alpha) + \sum_{\ell=1}^{n-1}[-(\ell-1)a^{(\ell-1)} + (n+3)a^{(\ell)} - (n-\ell)a^{(\ell+1)}]K^{(\ell)})\}.$$

A priori this means that the argument of the exponential is a multiple of $2\pi i$; but since the K's are arbitrary the only way for a linear combination to be constant is for all the coefficients to be zero, and then in fact the argument must be zero:

$$0 = K^{(1)} + \sum_{\ell=1}^{n-1}[-(\ell-1)a^{(\ell-1)}] + (n+3)a^{(\ell)} - (n-\ell)a^{(\ell+1)}]K^{(\ell)}.$$

We can interpret this as an $n \times n$ system ($1 \leq k \leq n$) of the form

$$\mathbf{0} \;=\; \mathbf{K}\,(\mathbf{B} \cdot \mathbf{A} \;+\; \vec{e}_1),$$

where $\vec{e}_1$ is the first standard basis vector for $\mathbf{R}^n$, $\mathbf{A}$ is the column vector of the unknowns a_ℓ, $\mathbf{K}$ is the $n \times n$ matrix of plaquette product terms ($K_{s,t} = K_{0,s}^{(t)}$), and $\mathbf{B}$ is the $n \times n$ matrix

$$\begin{pmatrix} n+3 & -(n-1) & 0 & 0 & \dots & 0 \\ -1 & n+3 & -(n-2) & 0 & \dots & 0 \\ 0 & -2 & n+3 & -(n-3) & 0 & \dots \\ & & \vdots & & & \\ 0 & \dots & 0 & -(n-2) & n+3 & -1 \\ 0 & \dots & 0 & 0 & -(n-1) & n+3 \end{pmatrix}$$

Since $\mathbf{K}$ is arbitrary, the solution requires $\mathbf{0} = \mathbf{B} \cdot \mathbf{A} + \vec{e}_1$, or

$$\mathbf{B} \cdot \mathbf{A} \;=\; -\vec{e}_1. \tag{4.4}$$

Now the matrix $\mathbf{B}$ has the special form of a centrosymmetric continuant; taking $a = n+3, b = -1, c = 1$ in Theorem 1 yields immediately

$$\det \mathbf{B} = 2^n (n+1)!,$$

so the system has a unique solution. ◇

Theorem 5 *The last two coefficients are*

$$a_{n+1}^{(n)} = \frac{-1}{2^n (n+1) n}$$

and

$$a_{n+1}^{(n-1)} = \frac{-(n+3)}{2^n(n+1)n(n-1)}.$$

Furthermore the coefficients $a_{n+1}^{(\ell)}$ *satisfy the recursion relation*

$$(n-\ell)a_{n+1}^{(\ell+1)} = (n+3)a_{n+1}^{(\ell)} - (\ell-1)a_{n+1}^{(\ell-1)}.$$

Proof: This is clear from equation 4.4 and the form of $\mathbf{B}$. ◇

Theorem 6 *The coefficients for dimensions* n *and* $n+1$ *are related as follows:*

$$a_n^{(\ell)} = a_{n+1}^{(\ell)} + a_{n+1}^{(\ell+1)}.$$

Proof: For simplicity we relabel them as b_ℓ and a_ℓ, respectively. The proof goes by induction, using the recursive relationship among the coefficients on each level. We know that

$$-(\ell-1)a_{\ell-1} + (n+3)a_\ell - (n-\ell)a_{\ell+1} = 0 \text{ for } 1 \leq \ell \leq n+1$$

and

$$-(\ell-1)b_{\ell-1} + (n+2)b_\ell - (n-1-\ell)b_{\ell+1} = 0 \text{ for } 1 \leq \ell \leq n.$$

Assume that we know $a_{\ell+1} + a_{\ell+2} = b_{\ell+1}$; we wish to show that $a_\ell + a_{\ell+1} = b_\ell$. Consider

$$Z = \ell[b_\ell - (a_\ell + a_{\ell+1})].$$

Then

$$
\begin{aligned}
Z &= \ell b_\ell - \ell a_\ell - \ell a_{\ell+1} \\
&= (n+2)b_{\ell+1} - (n-\ell-1-1)b_{\ell+2} - \ell a_\ell - \ell a_{\ell+1} \\
&= (n+2)a_{\ell+1} + (n+2)a_{\ell+2} - (n-\ell-1-1)a_{\ell+2} - (n-\ell-1-1)a_{\ell+3} \\
&\quad -\ell a_\ell - \ell a_{\ell+1} + a_{\ell+1} - a_{\ell+1} \\
&= -\ell a_\ell + (n+3)a_{\ell+1} - (n-(\ell+1))a_{\ell+2} \\
&\quad -(\ell+1)a_{\ell+1} + (n+3)a_{\ell+2} - (n-(\ell+2))a_{\ell+3}
\end{aligned}
$$

Therefore, $Z = 0$. ◇

This allows the entire array of coefficients to be reconstructed from $a^{(n)}_{n+1} = (-1)/2^n(n+1)n$. The beginning of the table looks like this:

T^2 : $-\frac{1}{4}$

T^3 : $-\frac{5}{24}$ $-\frac{1}{24}$

T^4 : $-\frac{17}{96}$ $-\frac{1}{32}$ $-\frac{1}{96}$

T^5 : $-\frac{49}{320}$ $-\frac{23}{960}$ $-\frac{7}{960}$ $-\frac{1}{320}$

T^6 : $-\frac{43}{320}$ $-\frac{3}{160}$ $-\frac{1}{192}$ $-\frac{1}{480}$ $-\frac{1}{960}$

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