Critical metrics for the L^2-norm of the curvature tensor

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François Lamontagne

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François Lamontagne

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Michael Anderson
Professor of Mathematics
Dissertation Director

Christophe Golé
Assistant Professor of Mathematics
Chairman of Defense

Claude LeBrun
Associate Professor of Mathematics

Folkert Tangerman
Research Scientist, Department of Applied Mathematics and Statistics
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation

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Given a compact, differentiable manifold M^n, the L^2-norm of the curvature tensor,

$$\mathcal{R}(g) = \int_M |R|^2 \, dvol_g$$

defines a Riemannian functional on the space of metrics of fixed volume on M. In dimension four Einstein metrics are critical for $\mathcal{R}$. Our first result is a partial converse to this statement, namely given a $\mathcal{R}$-critical metric g of non-positive sectional curvature on a four dimensional manifold then g is Einstein. Next we offer a partial classification of three and four dimensional homogeneous spaces that are $\mathcal{R}$-critical. Essentially if the isotropy group

is non-trivial or the dimension is equal to three, the classification is complete. It remains to classify $\mathcal{R}$-critical, left invariant metrics on four dimensional Lie groups. Under the assumption that the group is unimodular and has a non-trivial center we can complete the classification. This dissertation was written under Professor Michael Anderson.

To my mother.

Contents

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Anyone in this department who has dealt with the administration, remembers how crucial the secretaries are to guide us through the "Kafkian" maze of forms and deadlines. I wish to thank them all, with special thanks to Lucille and Joann.

Also I have had the pleasure of interacting with a number of faculty members, it is in order to mention some of them. I wish to thank Emili Bifet, Christophe Golé, Detlef Gromoll, Blaine Lawson, Claude LeBrun, Ralf Spatzier and Nicolai Teleman.

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to mind, I have been fortunate to be surrounded with Jongsu Kim, Sunil Nair, Janet Woodland, Pablo Ares, Ed Taylor, Jennifer Shaw, Ana-Maria Baeza and Thalia Jeffres. I must stop here this list, it could easily go on. After all I've been here for seven years, it's only too tempting to plunge into nostalgia and to spend hours remembering faces. That, I will do in my spare time as many other graduates have done.

Chapter 1

Motivation and Results

This thesis outgrew from a rather vague question posed by René Thom, "Given a differentiable manifold M^n is there a canonical Riemannian metric living on it?".

The notion of "canonical" should be loose enough not to overdetermine the topological type of M, while it should be restrictive enough that the moduli space of such metrics be finite dimensional. Metrics of constant sectional curvature are definitely "canonical" but they impose too many restrictions on the topology to satisfy the first requirement. On the other hand, metrics of constant scalar curvature on a given manifold tend to form infinite dimensional families, thereby breaking the second requirement. As a middle ground one could look for metrics of constant ricci curvature, the so-called Einstein metrics. These do satisfy the above requirements.

So far we have defined "canonical" in terms of the local geometry. As Hilbert and Einstein pointed out Einstein metrics also occur as the critical points of the scalar curvature functional. Here a metric is canonical provided it is critical for some energy functional, namely the integral of the scalar curvature. One could define other Riemannian functionals, thereby obtaining new meanings to the word canonical. It is this perspective we shall adopt. The present work is a very partial study of the L^2-norm of the curvature tensor

$$\mathcal{R}(g) = \int_M |R|^2 \, dvol_g.$$

Metrics that are critical for this functional, are called $\mathcal{R}$-critical. We should mention at least one other functional, the L^2-norm of the Weyl tensor, which has been studied by both mathematicians and physicists . In chapter one we'll see how, in dimension four, the above functionals relate to one another. These are well known facts but definitely worth mentioning; in particular Einstein metrics are $\mathcal{R}$-critical. The converse is not true. Even in dimension four there are metrics which are $\mathcal{R}$-critical but are not Einstein. On the other hand, if the metric is $\mathcal{R}$-critical and has non- positive sectional curvature we have the following

Theorem 1 *Let (M^4, g) be a four dimensional Riemannian manifold. Suppose that g is $\mathcal{R}$-critical and has non-positive sectional curvature. Then g is Einstein.*

The proof of this statement will conclude our first chapter.

The second chapter consists of a partial classification of three and four dimensional $\mathcal{R}$-critical, homogeneous spaces.

In dimension three, starting with the Cartan's classification of homogeneous Riemannian spaces, we obtain the following classification

Theorem 2 *Let (M,g) be a three-dimensional, simply connected, $\mathcal{R}$-critical homogeneous space. Then (M,g) is one of the following: $C_k(3)$, or a three dimensional Lie group with an $\mathcal{R}$-critical, left invariant metric.*

Theorem 3 *Let (M,g) be a three-dimensional, simply connected unimodular Lie group with a left invariant, $\mathcal{R}$-critical metric. Then (M,g) is one of the following: $\mathbf{R}^3$ or $E(2)$ with a flat metric, or $SU(2)$ with the round metric or $SU(2)$ with a Berger-type metric (one rescales the length of the fibers in the Hopf fibration to have length $\frac{4\pi}{\sqrt{11}}$).*

In dimension four, Gary Jensen in [Jen69] has already provided us with a complete classification of four dimensional homogeneous Einstein spaces. His classification is based on the earlier work of Ishihara [Ish55] and as we'll see this work is more than enough to handle all homogeneous spaces with non-trivial isotropy group. We are left with the problem of classifying $\mathcal{R}$-critical left invariant metrics on four dimensional Lie groups. As in [Jen69] the analysis required to handle the group manifold situation is reduced to solving a number of algebraic equations. Unfortunately these are of degree four and are far more complicated than the Einstein equations for left invariant metrics. Modulo the assumption that the group is unimodular and has a non-trivial center we have obtained the following

Theorem 4 *Let (M,g) be a four dimensional, simply connected, $\mathcal{R}$-critical homogeneous space. Then (M,g) is one of the following: $C_k(4)$, $C_k(2) \times C_{\pm k}(2)$, $\mathbf{CP}_2$, $\mathbf{CH}_2$ or a four dimensional Lie group with a $\mathcal{R}$-critical left-invariant metric.*

Theorem 5 *Let (M, g) be a four dimensional, unimodular, simply connected Lie group with a non-trivial center and g a left invariant, $\mathcal{R}$-critical metric. Then (M, g) is one of the following:* $\mathbf{R}^4$ *or* $E(2) \times \mathbf{R}$ *with a flat metric.*

Here $C_k(n)$ is the simply connected space form of curvature k and dimension n, $\mathbf{CP}_2$ is the complex projective plane and $\mathbf{CH}_2$ is the complex hyperbolic space. Finally $E(2)$ is the group of Euclidean motions on $\mathbf{R}^2$.

We have used the Mathematica package for most of the symbolic manipulations.

Chapter 2

A theorem of rigidity

2.1 Review of differential geometry

Given a Riemannian manifold of dimension n, (M^n, g), one defines the Levi-Civita connection ∇ and from it the Riemann curvature tensor and its contractions.

Definition 1 *Let X, Y, Z, W be vector fields on M, $\{e_i\}_{i=1}^n$ an orthonormal frame at a given point; define*

1. *The Riemann curvature tensor*

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)$$

2. *The Ricci tensor*

$$r(X,Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i)$$

3. The scalar curvature

$$s = \sum_{i=1}^{n} r(e_i, e_i)$$

4. The traceless Ricci tensor

$$z = r - \frac{s}{n} g$$

2.1.1 Tensors of curvature type

The Riemann curvature tensor satisfies three fundamental identities.

1. $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$

2. $R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0$

3. $R(X,Y,Z,W) = R(Z,W,X,Y)$

Let us point out that the third identity is a formal consequence of the first two and the second identity is called *the first Bianchi identity.*

Definition 2 *A four-tensor satisfying the above identities is said to be of* curvature type.

Observation 1 *Let T be of curvature type, a,b $\in T_pM$ and $\{e_i\}$ be an orthonormal basis for the tangent space at p. Then T yields*

1. an endomorphism of the space of symmetric two-tensors

$$\dot{T} : S^2_p(M) \longrightarrow S^2_p(M)$$

$$\dot{T}s\,(a,b) = \sum_{i,j} T(a, e_i, b, e_j)\, s(a,b)$$

2. an endomorphism of the space of two forms

$$\hat{T} : \Lambda_p^2(M) \longrightarrow \Lambda_p^2(M)$$

$$\hat{T}\omega\,(a,b) = \sum_{i,j} T(a,b,e_i,e_j)\,\omega(e_i,e_j)$$

3. a bilinear form

$$\check{T}(a,b) = \sum_{i,j,k} T(a,e_i,e_j,e_k)\,T(b,e_i,e_j,e_k)$$

Finally we recall the Kulkarni-Nomizu product $\odot$ on two-tensors.

Definition 3 *Define* $\odot : T^2(M) \times T^2(M) \longrightarrow T^4(M)$ *by*

$$s \odot t\,(a,b,c,d) = s(a,c)\,t(b,d) + s(b,d)\,t(a,c) - s(a,d)\,t(b,c) - s(b,c)\,t(a,d)$$

The remarkable fact about the Kulkarni-Nomizu product is that: if s, t are symmetric then $s \odot t$ is of curvature type.

This product allows us to write with ease the well-known orthogonal decomposition of the Riemann curvature tensor.

$$R = W + Z + U$$

where $Z = (z \odot g)/(n-2), U = s\,(g \odot g)/2n(n-1)$ and W is the Weyl tensor.

2.1.2 Riemannian geometry in dimension four

In dimension four the Hodge star operator yields an endomorphism of the space of two forms. Moreover it has two eigenvalues $\{+1, -1\}$ which decompose the space of two forms as a direct sum of the corresponding eigenspaces. These are of equal dimensions.

1. $\wedge^2(M) = \wedge^2_+(M) \oplus \wedge^2_-(M)$

2. $* \mid_{\wedge^2_+(M)} = +Id$

3. $* \mid_{\wedge^2_-(M)} = -Id$

Under this decomposition the curvature operator has the following matrix decomposition.

$$\hat{R} = \begin{pmatrix} W_+ + \frac{s}{12}Id & Z \\ & \\ Z^t & W_- + \frac{s}{12}Id \end{pmatrix}$$

where

$$\hat{W} = \begin{pmatrix} W_+ & 0 \\ & \\ 0 & W_- \end{pmatrix}$$

This decomposition has numerous consequences. From our perspective the most saliant are the Chern-Weil integrands for the signature and the Euler characteristic.Precisely,

1. $\chi(M) = \frac{1}{8\pi^2}\int_M (\mid W_+ \mid^2 + \mid W_- \mid^2 + \frac{s^2}{24} - \frac{1}{6} \mid Z \mid^2) dvol_g$

2. $\tau(M) = \frac{1}{12\pi^2}\int_M (\mid W_+ \mid^2 - \mid W_- \mid^2) dvol_g$

2.2 Riemannian functionals

Given a compact differentiable manifold M^n we consider the space $\mathcal{M}$ of metrics on M^n . $\mathcal{M}$ is an open cone in the space of bilinear forms on M^n.

For technical reasons $\mathcal{M}$ is often endowed with a Sobolev type topology. That is, one fixes a metric on M and defines a $L_{2,K}$ norm on the space of bilinear forms with $K > \frac{n}{2} + 1$. Since K is large enough, this topology is finer than the compact-open topology. We will interchangeably use $\mathcal{M}$ for the space of smooth metrics and its $L_{2,K}$ completion. For a thorough treatment of the space of metrics we refer to [Bes87] and [Ebi68].

Definition 4 *A Riemannian functional F is a real valued function over the space of metrics invariant under the diffeomorphism group.*

$$F : \mathcal{M} \longrightarrow \mathbb{R}$$

$$F(\psi^* g) = F(g)$$

where $g \in \mathcal{M}$ and $\psi \in diffeo(M)$.

Definition 5 *F is said to be differentiable at g provided there exists a linear functional $F'(g)$ on $S^2(M)$, such that*

$$\lim_{\|h\|_{2,K} \to 0} \frac{F(g+h) - F(g) - F'(g)h}{\| h \|_{2,K}} = 0$$

Definition 6 *We say that F has a* gradient *at g if $F'(g)$ can be represented by a smooth bilinear form a such that*

$$F'(g)h = \int_M g(a,h) dvol_g$$

a is denoted by $grad_g F$.

We shall consider Riemannian functionals that are given as polynomial expressions of the curvature tensor.

$$F(g) = \int_M P(R) dvol_g$$

All functionals of this type are differentiable and admit a gradient at all points.

Note 1 *The gradient is often called the Euler-Lagrange operator and the equation $grad_g F = 0$ the Euler-Lagrange equation.*

2.2.1 Examples of Riemannian functionals

We will restrict ourselves to list three functionals, the integral of the scalar curvature, the L^2-norm of the Weyl tensor and the L^2-norm of the curvature tensor. The first one has been extensively studied in the context of Lorentzian geometry, for its Euler-Lagrange equation yields the Einstein equations of General Relativity. It is now deeply studied in Riemannian geometry as a mean to induce "canonical geometries" on differentiable manifolds. The second one also takes its origin in physics and serves a similar purpose in Riemannian geometry.

The L^2-norm of the curvature tensor is a newcomer in Riemannian geometry but we will see that in dimension four these functionals are deeply interrelated.

Let us now list the functionals with their Euler-Lagrange equation (since these are matrix equations, perhaps we should say "Euler-Lagrange

equations ". We will use singular and plural interchangeably). We refer to [Bes87] [1] for a thorough treatment of these functionals.

1. The $\mathcal{S}$-functional,

$$\mathcal{S}(g) = \int_M s\, dvol_g$$

$$grad_g\mathcal{S} = \frac{s}{2}g - r$$

2. The $\mathcal{W}$-functional

$$\mathcal{W}(g) = \int_M |W|^2\, dvol_g$$

$$grad_g W = 4\delta^D D^* W - 2\check{W}(r)$$

3. The $\mathcal{R}$-functional

$$\mathcal{R}(g) = \int_M |R|^2\, dvol_g$$

$$grad_g\mathcal{R} = 4\delta^D d^D r - 2\check{R} - \frac{|R|^2}{2}g$$

Here D is the Levi-Civita connection and D^* is its formal adjoint. The definitions of d^D and δ^D are more involved. In general, given a connection D on a vector bundle $E \to M$, we can define the following operators on E-valued k-forms.

---

[1]In [Bes87] p.131 the inner product induced on the space of symmetric 2-tensors is twice the definition we use, see also [Bou81].

- $d^D : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$

$$d^D\omega_{(X_0,\ldots,X_k)} = \sum_{i=0}^{k}(-1)^i(D_{X_i}\omega)(X_0,\ldots,\hat{X}_i,\ldots,X_k)$$

- $\delta^D : \Omega^{k+1}(M,E) \longrightarrow \Omega^k(M,E)$

$$\delta^D\omega_{(X_1,\ldots,X_k)} = -\sum_{i=1}^{k}(D_{e_i}\omega)(e_i,X_1,\ldots,X_k)$$

$\{e_i\}$ is an orthonormal frame for the tangent space.

In general these functionals are not invariant under scaling of the metric. To remedy this problem one usually considers their restrictions to metrics of fixed volume. But in dimension four the $\mathcal{W}$-functional and the $\mathcal{R}$-functional are invariant under scaling. Hence their Euler-Lagrange equations remain unchanged when restricted to volume one metrics. On the other hand, the $\mathcal{S}$-functional is sensitive to scaling. Its normalised Euler-Lagrange equation in dimension n is

$$r - \frac{s}{n}g = 0 \tag{2.1}$$

These are the famous Einstein's equations. If g satisfies (2.1) then it is called an Einstein metric.

2.2.2 Interplay between the $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{R}$ functionals in dimension four

Due to Pontryagin numbers

1. $\chi(M) = \frac{1}{8\pi^2}\int_M |W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{1}{6}|Z|^2\, dvol_g$
2. $\tau(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2\, dvol_g,$

one may rewrite the expression for the $\mathcal{R}$-functional in the following ways (see [Bes87], [Leb93]).

1. $\mathcal{R}(g) = \frac{1}{12\pi^2}\int_M |Z|^2\, dvol_g + 2\chi(M)$
2. $\mathcal{R}(g) = \frac{1}{\pi^2}\int_M |W_\pm|^2 + \frac{s^2}{48} dvol_g - (2\chi(M) \pm 3\tau(M)).$

Metrics for which W_+ or W_- vanishes are called "half-conformally flat". These correspond to absolute minimum of the $\mathcal{W}$-functional.

Similarly metrics that are either Einstein or half-conformally flat with zero scalar curvature correspond to absolute minimum of the $\mathcal{R}$-functional.

From the above formulas one easily derives topological obstructions to the existence of Einstein metrics or half-conformally flat metrics. Despite these examples there is no known obstructions to the existence of $\mathcal{R}$-critical metrics.

As mentioned above in dimension four Einstein metrics are $\mathcal{R}$-critical, the following theorem is a partial converse to this statement.

Theorem 1 *Let (M,g) be a four dimensional Riemannian manifold such that g is $\mathcal{R}$-critical. If g has non-positive sectional curvature then g is Einstein.*

Proof

In dimension four the Euler-Lagrange equation for the $\mathcal{R}$-functional is given by

$$grad_g R = 4\delta^D d^D r - 4\dot{W}(z) - \frac{2s}{3}z \qquad (2.2)$$

A critical metric g is therefore a solution for the following equation:

$$2\delta^D d^D r = 2\dot{W}(z) + \frac{s}{3} z \tag{2.3}$$

By taking the trace of Equation (2.3) we see that the scalar curvature is constant, see [Bes87]. Therefore

$$\int_M < \delta dr, z > dvol_g = \int_M < dz, dz > dvol_g$$

Hence

$$\int_M < dz, dz > dvol_g = \int_M < 2\,\dot{W}(z) + \frac{s}{3} z, z > dvol_g$$

We will show that non-positive sectional curvature implies that

$$< 2\dot{W}(z) + \frac{s}{3} z, z >$$

is non-positive. A local computation will then yield the theorem.

Let $\{e_i\}$ be an orthonormal frame of eigenvectors for the ricci tensor r as well as for z.

Let $\{\mu_i\}$ and $\{\lambda_i\}$ be the eigenvalues (respectively) of r and z, and $\sigma_{ij} = R(e_i, e_j, e_i, e_j)$.

$$r = \sum_{i=1}^{4} \mu_i e_i \otimes e_i$$

$$z = \sum_{i=1}^{4} \lambda_i e_i \otimes e_i.$$

We first show that,

$$2\dot{W}(z) + \frac{s}{3} z = \sum_{i,j}^{4} (\lambda_i + \lambda_j)^2 \sigma_{ij}.$$

Let's compute

$$
\begin{aligned}
2 < \dot{W}(z), z > &= 2 < \dot{W}(\sum_{i=1}^{4} \lambda_i e_i \otimes e_i), \sum_{j=1}^{4} \lambda_j e_j \otimes e_j > \\
&= 2 \sum_{i,j=1}^{4} \lambda_i \lambda_j < \dot{W}(e_i \otimes e_i), e_j \otimes e_j > .
\end{aligned}
$$

To compute

$$< \dot{W}(e_i \otimes e_i), e_j \otimes e_j >$$

we recall that

$$W = R - \frac{s}{24} g \odot g - \frac{1}{2} z \odot g.$$

Where $\odot$ is the Kulkarni-Nomizu product on symmetric two tensors.

Thus,

$$
\begin{aligned}
< \dot{W}(e_i \otimes e_i), e_j \otimes e_j > &= W(e_i, e_j, e_i, e_j) \\
&= R(e_i, e_j, e_i, e_j) - \frac{s}{24} g \odot g(e_i, e_j, e_i, e_j) - \\
&\quad \frac{1}{2} z \odot g(e_i, e_j, e_i, e_j) \\
&= \sigma_{ij} - \frac{s}{24} 2(1 - \delta_{ij}) - \frac{1}{2}(\lambda_i + \lambda_j)(1 - \delta_{ij})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
< 2\dot{W}(z) + \frac{s}{3} z, z > &= 2 \sum_{i,j=1}^{4} \lambda_i \lambda_j (\sigma_{ij} - \frac{s}{12}(1 - \delta_{ij}) - \\
&\quad \frac{1}{2}(\lambda_i + \lambda_j)(1 - \delta_{ij})) + \frac{s}{3} \sum_{i=1}^{4} \lambda_i^2
\end{aligned}
$$

$$= 2\sum_{i,j=1}^{4}\lambda_i\lambda_j\sigma_{ij} - \frac{s}{6}\sum_{i,j=1}^{4}\lambda_i\lambda_j(1-\delta_{ij}) -$$
$$\sum_{i,j=1}^{4}(\lambda_i+\lambda_j)\lambda_i\lambda_j(1-\delta_{ij}) + \frac{s}{3}\sum_{i=1}^{4}\lambda_i^2.$$

Now, since $\sum_{i=1}^{4}\lambda_i = 0$,

$$\begin{aligned}
< 2\dot{W}(z) + \frac{s}{3}z, z > &= 2\ \ (\sum_{i,j=1}^{4}\lambda_i\lambda_j\sigma_{ij} + \frac{s}{4}\sum_{i=1}^{4}\lambda_i^2 + \sum_{i=1}^{4}\lambda_i^3) \\
&= 2(\sum_{i,j=1}^{4}\lambda_i\lambda_j\sigma_{ij} + \sum_{i=1}^{4}\lambda_i^2(\lambda_i + \frac{s}{4})) \\
&= 2(\sum_{i,j=1}^{4}\lambda_i\lambda_j\sigma_{ij} + \sum_{i=1}^{4}\lambda_i^2\mu_i) \\
&= 2(\sum_{i<j}2\lambda_i\lambda_j\sigma_{ij} + \sum_{i=1}^{4}\lambda_i^2(\sum_{i\neq j}\sigma_{ij})) \\
&= 2(\sum_{i<j}2\lambda_i\lambda_j\sigma_{ij} + \sum_{i<j}(\lambda_i^2+\lambda_j^2)\sigma_{ij}) \\
&= 2\sum_{i<j}(\lambda_i+\lambda_j)^2\sigma_{ij} \\
&= \sum_{i,j=1}^{4}(\lambda_i+\lambda_j)^2\sigma_{ij}
\end{aligned}$$

A quick calculations shows that

$$\begin{aligned}
\sum_{i,j}^{4}(\lambda_i+\lambda_j)^2\sigma_{ij} &= (\sigma_{12}-\sigma_{34})^2(\sigma_{12}+\sigma_{34}) + (\sigma_{13}-\sigma_{24})^2(\sigma_{13}+\sigma_{24}) + \\
&\quad (\sigma_{14}-\sigma_{23})^2(\sigma_{14}+\sigma_{23}) \qquad (2.4)
\end{aligned}$$

We recall from the Euler-Lagrange equation that

$$\int_M < dz, dz > dvol_g = \int_M < 2\dot{W}(z) + \frac{s}{3}z, z > dvol_g \qquad (2.5)$$

If the sectional curvature is non-positive, Equation (2.4) implies

$$< 2\,\dot{W}(z) + \frac{s}{3}z, z > \leq 0 \tag{2.6}$$

but Equation (2.5) makes it impossible for $< 2\,\dot{W}(z) - \frac{s}{3}z, z >$ to take any negative values. Hence $< 2\,\dot{W}(z) - \frac{s}{3}z, z >$ vanishes identically, from Equation (2.4) we conclude that g is Einstein.

Chapter 3

Homogeneous $\mathcal{R}$-critical metrics

3.1 Introduction

The purpose of this chapter is to understand the geometry of homogeneous, $\mathcal{R}$-critical metrics in dimension three and four. Our work follows quite closely the work of Jensen [Jen69] on homogeneous Einstein metrics. We start with the classifications of Cartan [Car46] and Ishihara [Ish55] (respectively) of three dimensional and four dimensional Riemannian homogeneous spaces (see also [Ber81]). This essentially reduces the problem of classifying homogeneous, $\mathcal{R}$-critical metrics to the problem of classifying left invariant, $\mathcal{R}$-critical metrics on a three or four dimensional Lie group.

To simplify the exposition we will allow M to be non-compact and take the Euler-Lagrange equation for $\mathcal{R}$-critical metrics as derived in the compact case to be the definition of $\mathcal{R}$-criticality (since the examples under consideration all admit a co-compact discrete group of isometries this is a minor twist to the definition of $\mathcal{R}$-criticality). We will denote by $C_k(n)$ the simply connected space form of dimension n and of curvature k. Finally all the products involved are Riemannian products. We have obtained the following results.

Theorem 2 *Let (M, g) be a three dimensional, simply connected, $\mathcal{R}$-critical homogeneous space. Then (M, g) is one of the following: $C_k(3)$, or a three dimensional Lie group with an $\mathcal{R}$-critical, left invariant metric.*

Theorem 3 *Let (M, g) be a three dimensional, simply connected unimodular Lie group with a left invariant, $\mathcal{R}$-critical metric. Then (M, g) is one of the following: $\mathbb{R}^3$ or $E(2)$ with a flat metric, or $SU(2)$ with the round metric or $SU(2)$ with a Berger type metric (one rescales the length of the fibers in the Hopf fibration to have lenght $\frac{4\pi}{\sqrt{11}}$).*

Theorem 4 *Let (M, g) be a four dimensional, simply connected, $\mathcal{R}$-critical homogeneous space. Then (M, g) is one of the following: $C_k(4)$, $C_k(2) \times C_{\pm k}(2)$, $\mathbb{CP}_2$, $\mathbb{CH}_2$ or a four dimensional Lie group with a $\mathcal{R}$-critical left-invariant metric.*

Theorem 5 *Let (M, g) be a four dimensional, unimodular, simply connected Lie group with a non-trivial center and g a left invariant, $\mathcal{R}$-critical metric. Then (M, g) is one of the following: $\mathbb{R}^4$ or $E(2) \times \mathbb{R}$ with a flat metric.*

The chapter is organised as follows. First we recall the Euler-Lagrange equation for the $\mathcal{R}$- functional. Second we use Cartan's and Ishihara's classifications to obtain theorem 2 and theorem 4. Third we tackle the classification of Lie groups with $\mathcal{R}$-critical left-invariant metrics in dimension three and four.

Most of the symbolic manipulations needed in this article were executed by the computer program Mathematica. Nevertheless we have checked the computations by hand in the three dimensional case.

3.2 The Euler-Lagrange equation for $\mathcal{R}$-critical metrics

In this chapter we are concerned with three dimensional as well as four dimensional Riemannian compact manifolds. It is therefore natural to consider

$$\mathcal{R}_n(g) = \frac{1}{Vol(M)^{(n-4)/n}} \int_M |R|^2 \, dvol_g$$

The gradient of $\mathcal{R}_n$ simply amounts to the gradient of $\mathcal{R}$ restricted to metrics of fixed volume. Recall from chapter 2,

$$Grad_g R = 4\delta^D d^D r - 2\check{R} + \frac{|R|^2}{2} g$$

Hence

$$\begin{aligned} Grad_g R_n &= \frac{2}{Vol(M)^{(n-4)/n}}(2\delta^D d^D r - \check{R} + \frac{|R|^2}{4} g) - \\ &\quad \frac{n-4}{2nVol(M)^{(2n-4)/n}}(\int_M |R|^2 \, dvol_g) g \end{aligned}$$

If $|R|^2$ is a constant function on M then the equation $Grad_g R_n \equiv 0$ is equivalent to:

$$0 \equiv \frac{1}{Vol(M)^{(n-4)/n}}(2\delta^D d^D r - \check{R} + \frac{|R|^2}{4} g) - \frac{(n-4)Vol(M)\,|R|^2}{4nVol(M)^{(2n-4)/n}} g$$

i.e.

$$0 \equiv 2\delta^D d^D r - \check{R} - \frac{|R|^2}{n} g \tag{3.1}$$

Since we will be dealing with homogeneous Riemannian manifolds having compact quotient, we may take Equation (3.1) as the definition of $\mathcal{R}$-criticality independently of M being compact or not.

Note 2 *If M is a Riemannian product of two manifolds, $M = M_1 \times M_2$ then*

$$\begin{aligned} \delta^D d^D r &= (\delta^D d^D r)_{M_1} \oplus (\delta^D d^D r)_{M_2} \\ \check{R} &= \check{R}_{M_1} \oplus \check{R}_{M_2} \end{aligned}$$

This has the obvious consequence that a product of space forms $C_k(m) \times C_l(n)$ are $\mathcal{R}$-critical if and only if $k = \pm l$. Also if $M = N \times F$ and F is flat then N must be flat, assuming the metric is $\mathcal{R}$-critical.

3.3 Homogeneous metrics in low dimensions

Cartan has classified all the simply connected, three dimensional, homogeneous Riemannian manifolds. These are $C_k(3)$, $C_k(2) \times \mathbf{R}$ and three dimensional Lie groups endowed of a left-invariant metric. As noted above it follows almost immediately from Equation (3.1) that $C_k(3)$ are $\mathcal{R}$-critical spaces while

$C_k(2) \times \mathbf{R}$ is not (unless $k = 0$). In the next section we will classify $\mathcal{R}$-critical metrics on three dimensional Lie groups.

In dimension four, Ishihara has furthered the work of Cartan. He obtained the following classification of simply connected, four dimensional homogeneous Riemannian manifolds, these are: $C_k(4)$, $C_k(3) \times \mathbf{R}$, $C_k(2) \times C_l(2)$, $\mathbf{CP}_2$, $\mathbf{CH}_2$ and four dimensional Lie groups with left invariant metrics. Since $C_k(4)$, $\mathbf{CP}_2$, $\mathbf{CH}_2$ are Einstein spaces, they are $\mathcal{R}$-critical. On the other hand $C_k(2) \times C_l(2)$ is $\mathcal{R}$-critical iff $k = \pm l$ and $C_k(3) \times \mathbf{R}$ is $\mathcal{R}$-critical only if $k = 0$. Following this section we will classify $\mathcal{R}$-critical metrics on four dimensional unimodular Lie groups having a non-trivial center.

Note 3 *In the following sections, we have written our equations up to a constant multiple e.g.*

$$0 = 5(a + b)(a - b^2)$$

is replaced by

$$0 = (a + b)(a - b^2)$$

without any warning.

3.4 Background on left invariant metrics

Let M be a Lie group and g a left invariant metric on M. Let $\{X_i\}$ be a left invariant orthonormal frame for g. Define C_{ij}^k to be the set of structure constants corresponding to this frame.

If we are to compute the Christoffel symbols in this frame,

$$\begin{aligned}\Gamma_{ij}^k &= \langle \nabla_{X_i} X_j, X_k \rangle \\ &= \frac{1}{2}\{X_i\langle X_j, X_k\rangle + X_j\langle X_k, X_i\rangle - X_k\langle X_i, X_j\rangle \\ &\quad -\langle X_i, [X_j, X_k]\rangle + \langle X_j, [X_k, X_i]\rangle + \langle X_k, [X_i, X_j]\rangle\end{aligned}$$

i.e.

$$\Gamma_{ij}^k = \frac{1}{2}\{-C_{jk}^i + C_{ki}^j + C_{ij}^k\}$$

We see that the Christoffel symbols are constant, linear expressions of the structure constants. It follows that the Riemann curvature tensor, its contractions (ricci and scalar curvature tensors) as well as any differential operator defined in terms of the connection (e.g d^∇ or δ^∇) can solely be expressed in terms of the structure constants.

For instance,

$$\begin{aligned}R(X_i, X_j)X_k &= \nabla_{[X_i, X_j]}X_k - [\nabla_{X_i}, \nabla_{X_j}]X_k \\ &= C_{ij}^l \Gamma_{lk}^m X_m - \Gamma_{ik}^l \Gamma_{jl}^m X_m + \Gamma_{ik}^l \Gamma_{jl}^m X_m\end{aligned}$$

It is clear as how to proceed to compute expressions like δdr, $\check{R} - \frac{|R|^2}{n} g$, etc... It should also be clear that one may get lost in a sea of symbols doing so. In the next sections we have fully avoided the actual computations involved in translating the Euler-Lagrange equation into a system of polynomial equations on the structure constants. The interested reader is urged to get hold of a

$$\begin{aligned}\Gamma_{ij}^{k} &= \langle \nabla_{X_i} X_j, X_k \rangle \\ &= \frac{1}{2}\{X_i\langle X_j, X_k\rangle + X_j\langle X_k, X_i\rangle - X_k\langle X_i, X_j\rangle \\ &\quad -\langle X_i, [X_j, X_k]\rangle + \langle X_j, [X_k, X_i]\rangle + \langle X_k, [X_i, X_j]\rangle\end{aligned}$$

i.e.

$$\Gamma_{ij}^{k} = \frac{1}{2}\{-C_{jk}^{i} + C_{ki}^{j} + C_{ij}^{k}\}$$

We see that the Christoffel symbols are constant, linear expressions of the structure constants. It follows that the Riemann curvature tensor, its contractions (ricci and scalar curvature tensors) as well as any differential operator defined in terms of the connection (e.g d^{∇} or δ^{∇}) can solely be expressed in terms of the structure constants.

For instance,

$$\begin{aligned}R(X_i, X_j)X_k &= \nabla_{[X_i, X_j]} X_k - [\nabla_{X_i}, \nabla_{X_j}]X_k \\ &= C_{ij}^{l}\Gamma_{lk}^{m}X_m - \Gamma_{ik}^{l}\Gamma_{jl}^{m}X_m + \Gamma_{ik}^{l}\Gamma_{jl}^{m}X_m\end{aligned}$$

It is clear as how to proceed to compute expressions like δdr, $\check{R} - \frac{|R|^2}{n} g$, etc... It should also be clear that one may get lost in a sea of symbols doing so. In the next sections we have fully avoided the actual computations involved in translating the Euler-Lagrange equation into a system of polynomial equations on the structure constants. The interested reader is urged to get hold of a computer and the appropriate software to derive in an instant the necessary formulas.

3.5 Left invariant metrics on three dimensional, unimodular Lie groups

Let (M, g) be a three dimensional, simply connected, unimodular Lie group with a left invariant metric g. It can be shown (see [Mil76]) that the Lie algebra of M admits an orthonormal basis e_1, e_2, e_3 satisfying the following commutator relations

$$[e_1, e_2] = \lambda_3 e_3$$
$$[e_2, e_3] = \lambda_1 e_1$$
$$[e_3, e_1] = \lambda_2 e_2$$

In this frame the Euler-Lagrange equation for $\mathcal{R}$-critical metrics becomes a system of three polynomial equations in λ_i. That is $0 = 2\delta dr - \check{R} - \frac{|R|^2}{n}$ translates as:

$$\begin{aligned} 0 &= -22\lambda_1^4 + 15\lambda_1^3\lambda_2 - \lambda_1^2\lambda_2^2 - 3\lambda_1\lambda_2^3 + 11\lambda_2^4 + \\ &\quad 15\lambda_1^3\lambda_3 - 6\lambda_1^2\lambda_2\lambda_3 + 3\lambda_1\lambda_2^2\lambda_3 - 12\lambda_2^3\lambda_3 - \lambda_1^2\lambda_3^2 + \\ &\quad 3\lambda_1\lambda_2\lambda_3^2 + 2\lambda_2^2\lambda_3^2 - 3\lambda_1\lambda_3^3 - 12\lambda_2\lambda_3^3 + 11\lambda_3^4 \qquad (3.2) \\ 0 &= 11\lambda_1^4 - 3\lambda_1^3\lambda_2 - \lambda_1^2\lambda_2^2 + 15\lambda_1\lambda_2^3 - 22\lambda_2^4 - \\ &\quad 12\lambda_1^3\lambda_3 + 3\lambda_1^2\lambda_2\lambda_3 - 6\lambda_1\lambda_2^2\lambda_3 + 15\lambda_2^3\lambda_3 + 2\lambda_1^2\lambda_3^2 + \end{aligned}$$

$$3\lambda_1\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3^2 - 12\lambda_1\lambda_3^3 - 3\lambda_2\lambda_3^3 + 11\lambda_3^4 \tag{3.3}$$

$$\begin{aligned} 0 &= 11\lambda_1^4 - 12\lambda_1^3\lambda_2 + 2\lambda_1^2\lambda_2^2 - 12\lambda_1\lambda_2^3 + 11\lambda_2^4 - 3\lambda_1^3\lambda_3 + \\ &\quad 3\lambda_1^2\lambda_2\lambda_3 + 3\lambda_1\lambda_2^2\lambda_3 - 3\lambda_2^3\lambda_3 - \lambda_1^2\lambda_3^2 - 6\lambda_1\lambda_2\lambda_3^2 - \\ &\quad \lambda_2^2\lambda_3^2 + 15\lambda_1\lambda_3^3 + 15\lambda_2\lambda_3^3 - 22\lambda_3^4 \end{aligned} \tag{3.4}$$

We will make the following change of variables : $\lambda_1 = \mu_2+\mu_3$, $\lambda_2 = \mu_1+\mu_3$, $\lambda_3 = \mu_1 + \mu_2$. We obtain the following system of equations,

$$\begin{aligned} 0 &= 3\mu_1^2\mu_2^2 + 6\mu_1\mu_2^3 - 10\mu_1^2\mu_2\mu_3 + 5\mu_1\mu_2^2\mu_3 - 6\mu_2^3\mu_3 + 3\mu_1^2\mu_3^2 + \\ &\quad 5\mu_1\mu_2\mu_3^2 - 6\mu_2^2\mu_3^2 + 6\mu_1\mu_3^3 - 6\mu_2\mu_3^3 \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 &= 6\mu_1^3\mu_2 + 3\mu_1^2\mu_2^2 - 6\mu_1^3\mu_3 + 5\mu_1^2\mu_2\mu_3 - 10\mu_1\mu_2^2\mu_3 - 6\mu_1^2\mu_3^2 + \\ &\quad 5\mu_1\mu_2\mu_3^2 + 3\mu_2^2\mu_3^2 - 6\mu_1\mu_3^3 + 6\mu_2\mu_3^3 \end{aligned} \tag{3.6}$$

$$\begin{aligned} 0 &= -6\mu_1^3\mu_2 - 6\mu_1^2\mu_2^2 - 6\mu_1\mu_2^3 + 6\mu_1^3\mu_3 + 5\mu_1^2\mu_2\mu_3 + 5\mu_1\mu_2^2\mu_3 + \\ &\quad 6\mu_2^3\mu_3 + 3\mu_1^2\mu_3^2 - 10\mu_1\mu_2\mu_3^2 + 3\mu_2^2\mu_3^2 \end{aligned} \tag{3.7}$$

We form a new system of equations, namely Equations (3.5) - (3.6), (3.6) - (3.7), (3.7) - (3.5)

$$\begin{aligned} 0 &= (-\mu_1 + \mu_2)(2\mu_1^2\mu_2 + 2\mu_1\mu_2^2 - 2\mu_1^2\mu_3 + 3\mu_1\mu_2\mu_3 - 2\mu_2^2\mu_3 - 3\mu_1\mu_3^2 - \\ &\quad 3\mu_2\mu_3^2 - 4\mu_3^3) \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 &= (-\mu_2 + \mu_3)(-4\mu_1^3 - 3\mu_1^2\mu_2 - 2\mu_1\mu_2^2 - 3\mu_1^2\mu_3 + 3\mu_1\mu_2\mu_3 + 2\mu_2^2\mu_3 - \\ &\quad 2\mu_1\mu_3^2 + 2\mu_2\mu_3^2) \end{aligned} \tag{3.9}$$

$$0 = (-\mu_1 + \mu_3)(2\mu_1^2\mu_2 + 3\mu_1\mu_2^2 + 4\mu_2^3 - 2\mu_1^2\mu_3 - 3\mu_1\mu_2\mu_3 + 3\mu_2^2\mu_3 -$$

$$2\mu_1\mu_3^2 + 2\mu_2\mu_3^2) \tag{3.10}$$

Let us assume for the time being that none of the variables are pairwise equal. Then one has to solve the following set of equations:

$$\begin{aligned} 0 &= 2\mu_1^2\mu_2 + 2\mu_1\mu_2^2 - 2\mu_1^2\mu_3 + 3\mu_1\mu_2\mu_3 - 2\mu_2^2\mu_3 - \\ &\quad 3\mu_1\mu_3^2 - 3\mu_2\mu_3^2 - 4\mu_3^3 \qquad (3.11)\\ 0 &= -4\mu_1^3 - 3\mu_1^2\mu_2 - 2\mu_1\mu_2^2 - 3\mu_1^2\mu_3 + 3\mu_1\mu_2\mu_3 + \\ &\quad 2\mu_2^2\mu_3 - 2\mu_1\mu_3^2 + 2\mu_2\mu_3^2 \qquad (3.12)\\ 0 &= 2\mu_1^2\mu_2 + 3\mu_1\mu_2^2 + 4\mu_2^3 - 2\mu_1^2\mu_3 - 3\mu_1\mu_2\mu_3 + \\ &\quad 3\mu_2^2\mu_3 - 2\mu_1\mu_3^2 + 2\mu_2\mu_3^2 \qquad (3.13) \end{aligned}$$

If we take the difference between the first two equations (3.11), (3.12), we obtain:

$$0 = (\mu_1 - \mu_3)(4\mu_1^2 + 5\mu_1\mu_2 + 4\mu_2^2 + 5\mu_1\mu_3 + 5\mu_2\mu_3 + 4\mu_3^2), \tag{3.14}$$

but modulo the initial hypothesis that none of the variables are pairwise equal. We obtain from Equation (3.14), the following equation:

$$0 = 4\mu_1^2 + 5\mu_1\mu_2 + 4\mu_2^2 + 5\mu_1\mu_3 + 5\mu_2\mu_3 + 4\mu_3^2 \tag{3.15}$$

equivalently

$$0 = \frac{5}{2}(\mu_1 + \mu_2 + \mu_3)^2 + \frac{3}{2}(\mu_1^2 + \mu_2^2 + \mu_3^2). \quad (3.16)$$

This last equation equals zero iff all the variables are equal to zero.

Now consider the case where two of the variables are pairwise equal, say $\mu_2 = \mu_3$. Since the equation is homogeneous, we can assume that for a non-trivial solution $\mu_1 = 1$. Substituting in (3.5) $\mu_1 = 1, \mu_2 = t, \mu_3 = t$ we obtain:

$$0 = 3t^2(-1+t)(-2+9t) \quad (3.17)$$

One can easily check that, up to permutation, the following triples

$$(\mu_1, \mu_2, \mu_3) = \{(1,1,1), (1, 2/9, 2/9), (1,0,0), (0,0,0)\}$$

are solutions for the Euler-Lagrange equation and as we've shown above, these are all the solutions. We can restate these solutions in terms of the structure constants (λ_i). We obtain $(1,1,1), (11/9, 11/9, 4/9), (1,1,0), (0,0,0)$.

The Lie group and metrics correponding to these structure constant are given in [Mil76]. It follows that the triple $(1,1,1)$ corresponds to $SU(2)$ with the round metric, (11/9,11/9,4/9) corresponds to $SU(2)$ with a Berger type metric, (1,1,0) corresponds to $E(2)$ with a flat metric and (0,0,0) is simply $\mathbf{R}^3$ with its flat metric.

3.6 Remarks on the geometry of a Berger sphere

Recall that a Berger sphere is obtained from the Hopf fibration $S^3 \longrightarrow S^2$ by rescaling the length of the S^1-fibers by a constant.

In order to see how the $SU(2)$-left invariant metric corresponding to the triple $(\lambda_1, \lambda_2, \lambda_3) = (11/9, 11/9, 4/9)$ is actually realised as Berger metric we need a clear correspondance between $SU(2)$ and S^3. This is achieved by the identifying S^3 with the unit quaternions, the isomorphism between $SU(2)$ and S^3 writes as follows (here $(z, w) \in \mathbf{C}^2$. In real coordinates the pair (z, w) will be written as (z_1, z_2, w_1, w_2)):

$$SU(2) \longrightarrow S^3$$

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \longrightarrow z + \mathbf{j}w.$$

It follows that $SU(2)$ acts on $\mathbf{C}^2$ the same way S^3 acts on $\mathbf{R}^4$ by quaternionic multiplication, i.e.

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (z + \mathbf{j}w)(a + \mathbf{j}b)$$

This isomorphism makes it easy to find a left invariant orthonormal frame on S^3 with its canonical metric.

Take

$$\frac{\partial}{\partial z_2}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2} \,\epsilon\, T_{(1,0,0,0)}S^3$$

define their left translates by

$$e_1 = dL_{(z,w)}\,(\frac{\partial}{\partial w_2})$$

$$e_2 = dL_{(z,w)}\,(\frac{\partial}{\partial w_1})$$

$$e_3 = dL_{(z,w)}\,(\frac{\partial}{\partial z_2}).$$

Here $dL_{(z,w)}$ has the following real form

$$dL_{(z,w)} = \begin{pmatrix} z_1 & -z_2 & -w_1 & -w_2 \\ z_2 & z_1 & w_2 & -w_1 \\ w_1 & -w_2 & z_1 & z_2 \\ w_2 & w_1 & -z_2 & z_1 \end{pmatrix}$$

Since $\frac{\partial}{\partial z_2}, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}$ form an orthonormal basis for $T_{(1,0,0,0)}S^3$ and $dL_{(z,w)} \,\epsilon\, SO(4)$. It follows that e_1, e_2, e_3 is a left invariant orthonormal frame for (S^3, g_{can}).

The Hopf fibration may be obtained by a right S^1 action on S^3.

$$S^3 \times S^1 \longrightarrow S^3$$

$$(z,w) \times e^{i\theta} \longmapsto (z\,e^{i\theta}, w\,e^{i\theta})$$

The vector field e_3 is tangent to this action. Indeed

$$\begin{aligned}
\frac{\partial}{\partial\theta}(z+\mathbf{j}w)\,e^{i\theta}\,|_{\theta=0} &= \frac{\partial}{\partial\theta}L_{(z,w)}((1,0)\,e^{i\theta})\,|_{\theta=0}\\
&= dL_{(z,w)}(\frac{\partial}{\partial\theta}(sin\theta, cos\theta, 0, 0)\,|_{\theta=0})\\
&= dL_{(z,w)}(\frac{\partial}{\partial z_2}).
\end{aligned}$$

Now consider the Berger metric:

$$\tilde{g} = e_1 \otimes e_1 + e_2 \otimes e_2 + \frac{4}{11} e_3 \otimes e_3$$

An orthonormal frame for $\tilde{g}$ would be $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = \frac{\sqrt{11}}{2} e_3$.

Since the structure constants of the frame (e_1, e_2, e_3) are $(2, 2, 2)$. We obtain

$$\begin{aligned}
[\tilde{e}_1, \tilde{e}_2] &= \frac{4}{\sqrt{11}} \tilde{e}_3\\
[\tilde{e}_2, \tilde{e}_3] &= \sqrt{11} \tilde{e}_1\\
[\tilde{e}_3, \tilde{e}_1] &= \sqrt{11} \tilde{e}_2
\end{aligned}$$

These structure constants $(\sqrt{11}, \sqrt{11}, 4/\sqrt{11})$ are a multiple of $(11/9, 11/9, 4/9)$. Hence they define the same metric up to dilation. Therefore $\tilde{g}$ is the required (up to dilation) $\mathcal{R}$-critical metric.

We conclude by giving some idea of the geometry of $\tilde{g}$.

In [Mil76], one has a formula to compute the sectional curvatures σ_{ij} from the triple $(\lambda_1, \lambda_2, \lambda_3)$.

Define

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$$

then

$$\begin{aligned}\sigma_{12} &= -\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_{13} &= \mu_1\mu_2 - \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_{23} &= \mu_1\mu_2 + \mu_1\mu_3 - \mu_2\mu_3\end{aligned}$$

For the metric $\tilde{g}$ in the basis $\{\tilde{e}_i\}$, we obtain

$$\begin{aligned}(\mu_1, \mu_2, \mu_3) &= (\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{9}{\sqrt{11}}) \\ (\sigma_{12}, \sigma_{13}, \sigma_{23}) &= (\frac{32}{11}, \frac{4}{11}, \frac{4}{11})\end{aligned}$$

Therefore,

$$\begin{aligned}Vol(\tilde{g}) &= \frac{2}{\sqrt{11}} Vol(S^3) \\ Pinching(\tilde{g}) &= \frac{1}{8} \\ \mathcal{R}_n(\tilde{g}) &= (\frac{4}{11})^{2/3} \frac{1056}{121} Vol(S^3)^{4/3} \\ &= 4.44621\ldots Vol(S^3)^{4/3}\end{aligned}$$

This metric comes a bit of a surprised, for it doesn't have constant sectional curvature (hence it's not Einstein). It has positive sectional curvature. Finally it has more"energy" than the round metric $\mathcal{R}_n(g_{S^3}) = 3\, Vol(S^3)^{4/3}$. The metric is obviously of Berger type, it is obtained by shrinking the S^1-fibers by a factor of $2/\sqrt{11}$.

3.7 Left invariant, $\mathcal{R}$-critical metrics in dimension four

In this section we classify $\mathcal{R}$-critical left invariant metrics g on four dimensional, unimodular Lie groups M having non-trivial center.

Let $\mathcal{A}$ be the Lie algebra of M and take X_4 to be in the center of $\mathcal{A}$. Let $\mathcal{I}$ be the orthogonal complement of X_4. By restricting g to $\mathcal{I}$ we induce an inner product $\hat{g}$ on the quotient algebra $\hat{\mathcal{I}} = \mathcal{A}/\langle X_4\rangle$. $\hat{\mathcal{I}}$ being unimodular, the pair $(\hat{\mathcal{I}}, \hat{g})$ admits a canonical frame $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$. Each $\hat{X}_i$ comes from a unique vector X_i lying in $\mathcal{I}$. The commutator relations of $\mathcal{A}$ written in the basis X_1, X_2, X_3, X_4 become:

$$\begin{aligned}
[X_i, X_4] &= 0 \\
[X_1, X_2] &= \lambda_3 X_3 + \xi_3 X_4 \\
[X_2, X_3] &= \lambda_1 X_1 + \xi_1 X_4 \\
[X_3, X_1] &= \lambda_2 X_2 + \xi_2 X_4
\end{aligned}$$

Note 4 *Observe that the Jacobi identities put no restrictions on the variables λ_1, λ_2, λ_3, ξ_1, ξ_2 and ξ_3. Also the relations admit a cyclic permutation of the indices and if we are ready to let a variable absorb a minus sign, we can interchange two indices. This comment is behind the more or less obvious "w.l.o.g." that will be used below. As in the previous section all the equations are derived up to a constant multiple.*

3.7.1 The Euler-Lagrange equations

The Euler-Lagrange equation $0 = 2\delta dr - \check{R} - \frac{|R|^2}{n}$ when written in a left invariant orthonormal frame yields a matrix equation whose entries are polynomial expressions of the structure constants.

Since $grad\mathcal{R}_n(g)$ is a symmetric two-tensor there are twelve equations to solve, namely:

$$\begin{aligned}
0 &= -11\xi_1^4 + 22\xi_1^2\xi_2^2 + 33\xi_2^4 + 22\xi_1^2\xi_3^2 + 66\xi_2^2\xi_3^2 + 33\xi_3^4 - \\
&\quad 66\xi_1^2\lambda_1^2 + 2\xi_2^2\lambda_1^2 + 2\xi_3^2\lambda_1^2 - 55\lambda_1^4 + 12\xi_1^2\lambda_1\lambda_2 - \\
&\quad 12\xi_2^2\lambda_1\lambda_2 + 4\xi_3^2\lambda_1\lambda_2 + 36\lambda_1^3\lambda_2 - 2\xi_1^2\lambda_2^2 + 66\xi_2^2\lambda_2^2 \\
&\quad -6\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 - 12\lambda_1\lambda_2^3 + 33\lambda_2^4 + 12\xi_1^2\lambda_1\lambda_3 + \\
&\quad 4\xi_2^2\lambda_1\lambda_3 - 12\xi_3^2\lambda_1\lambda_3 + 36\lambda_1^3\lambda_3 + 4\xi_1^2\lambda_2\lambda_3 - 36\xi_2^2\lambda_2\lambda_3 - \\
&\quad 36\xi_3^2\lambda_2\lambda_3 - 12\lambda_1^2\lambda_2\lambda_3 + 12\lambda_1\lambda_2^2\lambda_3 - 36\lambda_2^3\lambda_3 - 2\xi_1^2\lambda_3^2 - \\
&\quad 6\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + 12\lambda_1\lambda_2\lambda_3^2 + 6\lambda_2^2\lambda_3^2 - \\
&\quad 12\lambda_1\lambda_3^3 - 36\lambda_2\lambda_3^3 + 33\lambda_3^4 \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
0 &= \xi_1\xi_2(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_1^2 - 6\lambda_1\lambda_2 - 11\lambda_2^2 + 6\lambda_1\lambda_3 + \\
&\quad 6\lambda_2\lambda_3 + \lambda_3^2) \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
0 &= \xi_1\xi_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 6\lambda_1\lambda_3 + \\
&\quad 6\lambda_2\lambda_3 - 11\lambda_3^2) \qquad (3.20)
\end{aligned}$$

$$\begin{aligned}
0 &= \xi_1\lambda_1(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 + 6\lambda_1\lambda_3 - \\
&\quad 2\lambda_2\lambda_3 + \lambda_3^2) \qquad (3.21)
\end{aligned}$$

$$
\begin{aligned}
0 &= 33\xi_1^4 + 22\xi_1^2\xi_2^2 - 11\xi_2^4 + 66\xi_1^2\xi_3^2 + 22\xi_2^2\xi_3^2 + \\
&\quad 33\xi_3^4 + 66\xi_1^2\lambda_1^2 - 2\xi_2^2\lambda_1^2 - 6\xi_3^2\lambda_1^2 + 33\lambda_1^4 - \\
&\quad 12\xi_1^2\lambda_1\lambda_2 + 12\xi_2^2\lambda_1\lambda_2 + 4\xi_3^2\lambda_1\lambda_2 - 12\lambda_1^3\lambda_2 + 2\xi_1^2\lambda_2^2 - \\
&\quad 66\xi_2^2\lambda_2^2 + 2\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 + 36\lambda_1\lambda_2^3 - 55\lambda_2^4 - \\
&\quad 36\xi_1^2\lambda_1\lambda_3 + 4\xi_2^2\lambda_1\lambda_3 - 36\xi_3^2\lambda_1\lambda_3 - 36\lambda_1^3\lambda_3 + 4\xi_1^2\lambda_2\lambda_3 + \\
&\quad 12\xi_2^2\lambda_2\lambda_3 - 12\xi_3^2\lambda_2\lambda_3 + 12\lambda_1^2\lambda_2\lambda_3 - 12\lambda_1\lambda_2^2\lambda_3 + 36\lambda_2^3\lambda_3 - \\
&\quad 6\xi_1^2\lambda_3^2 - 2\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 + 6\lambda_1^2\lambda_3^2 + 12\lambda_1\lambda_2\lambda_3^2 - \\
&\quad 2\lambda_2^2\lambda_3^2 - 36\lambda_1\lambda_3^3 - 12\lambda_2\lambda_3^3 + 33\lambda_3^4 \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \xi_2\xi_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_2 - 11\lambda_2^2 + 6\lambda_1\lambda_3 - \\
&\quad 6\lambda_2\lambda_3 - 11\lambda_3^2) \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \xi_2\lambda_2(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_2 - 11\lambda_2^2 - 2\lambda_1\lambda_3 + \\
&\quad 6\lambda_2\lambda_3 + \lambda_3^2) \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
0 &= 33\xi_1^4 + 66\xi_1^2\xi_2^2 + 33\xi_2^4 + 22\xi_1^2\xi_3^2 + 22\xi_2^2\xi_3^2 - \\
&\quad 11\xi_3^4 + 66\xi_1^2\lambda_1^2 - 6\xi_2^2\lambda_1^2 - 2\xi_3^2\lambda_1^2 + 33\lambda_1^4 - \\
&\quad 36\xi_1^2\lambda_1\lambda_2 - 36\xi_2^2\lambda_1\lambda_2 + 4\xi_3^2\lambda_1\lambda_2 - 36\lambda_1^3\lambda_2 - 6\xi_1^2\lambda_2^2 + \\
&\quad 66\xi_2^2\lambda_2^2 - 2\xi_3^2\lambda_2^2 + 6\lambda_1^2\lambda_2^2 - 36\lambda_1\lambda_2^3 + 33\lambda_2^4 - \\
&\quad 12\xi_1^2\lambda_1\lambda_3 + 4\xi_2^2\lambda_1\lambda_3 + 12\xi_3^2\lambda_1\lambda_3 - 12\lambda_1^3\lambda_3 + 4\xi_1^2\lambda_2\lambda_3 - \\
&\quad 12\xi_2^2\lambda_2\lambda_3 + 12\xi_3^2\lambda_2\lambda_3 + 12\lambda_1^2\lambda_2\lambda_3 + 12\lambda_1\lambda_2^2\lambda_3 - 12\lambda_2^3\lambda_3 + \\
&\quad 2\xi_1^2\lambda_3^2 + 2\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - 12\lambda_1\lambda_2\lambda_3^2 - \\
&\quad 2\lambda_2^2\lambda_3^2 + 36\lambda_1\lambda_3^3 + 36\lambda_2\lambda_3^3 - 55\lambda_3^4 \qquad (3.25)
\end{aligned}
$$

$$
0 = \xi_3\lambda_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 6\lambda_1\lambda_3 +
$$

$$6\lambda_2\lambda_3 - 11\lambda_3^2) \tag{3.26}$$

$$\begin{aligned}
0 \; = \; & -55\xi_1^4 - 110\xi_1^2\xi_2^2 - 55\xi_2^4 - 110\xi_1^2\xi_3^2 - 110\xi_2^2\xi_3^2 - \\
& 55\xi_3^4 - 66\xi_1^2\lambda_1^2 + 6\xi_2^2\lambda_1^2 + 6\xi_3^2\lambda_1^2 - 11\lambda_1^4 + \\
& 36\xi_1^2\lambda_1\lambda_2 + 36\xi_2^2\lambda_1\lambda_2 - 12\xi_3^2\lambda_1\lambda_2 + 12\lambda_1^3\lambda_2 + 6\xi_1^2\lambda_2^2 - \\
& 66\xi_2^2\lambda_2^2 + 6\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 + 12\lambda_1\lambda_2^3 - 11\lambda_2^4 + \\
& 36\xi_1^2\lambda_1\lambda_3 - 12\xi_2^2\lambda_1\lambda_3 + 36\xi_3^2\lambda_1\lambda_3 + 12\lambda_1^3\lambda_3 - 12\xi_1^2\lambda_2\lambda_3 + \\
& 36\xi_2^2\lambda_2\lambda_3 + 36\xi_3^2\lambda_2\lambda_3 - 12\lambda_1^2\lambda_2\lambda_3 - 12\lambda_1\lambda_2^2\lambda_3 + 12\lambda_2^3\lambda_3 + \\
& 6\xi_1^2\lambda_3^2 + 6\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - 12\lambda_1\lambda_2\lambda_3^2 - \\
& 2\lambda_2^2\lambda_3^2 + 12\lambda_1\lambda_3^3 + 12\lambda_2\lambda_3^3 - 11\lambda_3^4
\end{aligned} \tag{3.27}$$

Our goal is to show that the only family of solutions to this system of equations is, up to permutation, $\xi_1 = \xi_2 = \xi_3 = \lambda_3 = 0, \lambda_2 = \lambda_3$. The vanishing of ξ_1, ξ_2, ξ_3 implies that M is a Riemannian product of a three dimensional Lie group and the real line, i.e.

$$M = G \times \mathbf{R}.$$

While $\lambda_3 = 0$, $\lambda_1 = \lambda_2$ implies if $\lambda_1 = \lambda_2 = 0$:

$$G = \mathbf{R}^3.$$

If $\lambda_1 = \lambda_2$ are non-zero, we get:

$$G = E(2)$$

with a flat metric.

The strategy is simple; one essentially assumes the contrary and derives a contradiction. We divide our analysis into three cases, first we will assume that none of the λ_i's vanish and show that no solutions are possible in that case. Second we assume that only one of the λ_i's is zero and derive the above solution. Third we assume that two or more λ_i's are zero and derive that all other variables must vanish. We will also establish, that in any case one of the ξ_i's has to vanish; w.l.o.g. we will assume $\xi_1 = 0$ through out all the computations.

Assume none of the ξ_i's are zero. It follows from equations (3.19), (3.20), (3.23) that

$$\begin{aligned} 0 &= -11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_1^2 - 6\lambda_1\lambda_2 - 11\lambda_2^2 + 6\lambda_1\lambda_3 + \\ &\quad 6\lambda_2\lambda_3 + \lambda_3^2 \\ 0 &= -11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_1^2 + 6\lambda_1\lambda_2 + \lambda_2^2 - 6\lambda_1\lambda_3 + \\ &\quad 6\lambda_2\lambda_3 - 11\lambda_3^2 \\ 0 &= -11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_2 - 11\lambda_2^2 + 6\lambda_1\lambda_3 - \\ &\quad 6\lambda_2\lambda_3 - 11\lambda_3^2 \end{aligned}$$

Taking the sum we obtain:

$$\begin{aligned} 0 &= -33\xi_1^2 - 33\xi_2^2 - 33\xi_3^2 - 15(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 3(\lambda_1 - \lambda_2)^2 - \\ &\quad 3(\lambda_2 - \lambda_3)^2 - 3(\lambda_3 - \lambda_1)^2 \end{aligned}$$

which vanishes only if all variables vanishes, consequently we may assume that w.l.o.g. $\xi_1 = 0$.

3.7.2 Case 1: $\lambda_i \neq 0$

Our goal is to show that all ξ_i's vanish. W.l.o.g. we may assume that $\xi_1 = 0$. Suppose that ξ_2,ξ_3 are both non-zero. Then Equation (3.23) and Equation (3.24) imply:

$$0 = -11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_2 - 11\lambda_2^2 + 6\lambda_1\lambda_3 - 6\lambda_2\lambda_3 - 11\lambda_3^2 \quad (3.28)$$

$$0 = -11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_2 - 11\lambda_2^2 - 2\lambda_1\lambda_3 + 6\lambda_2\lambda_3 + \lambda_3^2 \quad (3.29)$$

Adding (3.28) to (3.29) we obtain:

$$0 = \lambda_3(-2\lambda_1 + 3\lambda_2 + 3\lambda_3) \quad (3.30)$$

Hence

$$\lambda_1 = 3(\lambda_2 + \lambda_3)/2$$

We make the following substitution in the Euler-Lagrange equations

$$\lambda_1 \to 3/2(\lambda_2 + \lambda_3)$$

Under that substitution the Euler-Lagrange equations become:

$$\begin{aligned} 0 = {} & 176\xi_2^4 + 352\xi_2^2\xi_3^2 + 176\xi_3^4 + 280\xi_2^2\lambda_2^2 + 24\xi_3^2\lambda_2^2 - 781\lambda_2^4 \\ & -208\xi_2^2\lambda_2\lambda_3 - 208\xi_3^2\lambda_2\lambda_3 - 3732\lambda_2^3\lambda_3 + 24\xi_2^2\lambda_3^2 + 280\xi_3^2\lambda_3^2 - \\ & 5134\lambda_2^2\lambda_3^2 - 3732\lambda_2\lambda_3^3 - 781\lambda_3^4 \end{aligned} \quad (3.31)$$

$$\begin{aligned} 0 = {} & -176\xi_2^4 + 352\xi_2^2\xi_3^2 + 528\xi_3^4 - 840\xi_2^2\lambda_2^2 - 88\xi_3^2\lambda_2^2 + 1937\lambda_2^4 + \\ & 432\xi_2^2\lambda_2\lambda_3 - 1392\xi_3^2\lambda_2\lambda_3 + 8244\lambda_2^3\lambda_3 - 8\xi_2^2\lambda_3^2 - 24\xi_3^2\lambda_3^2 + \end{aligned}$$

$$9238\lambda_2^2\lambda_3^2 + 4308\lambda_2\lambda_3^3 + 609\lambda_3^4 \quad (3.32)$$

$$0 = \xi_2\xi_3(-44\xi_2^2 - 44\xi_3^2 + \lambda_2^2 + 66\lambda_2\lambda_3 + \lambda_3^2) \quad (3.33)$$

$$0 = \xi_2\lambda_2(-44\xi_2^2 - 44\xi_3^2 + \lambda_2^2 + 66\lambda_2\lambda_3 + \lambda_3^2) \quad (3.34)$$

$$\begin{aligned} 0 = {} & 528\xi_2^4 + 352\xi_2^2\xi_3^2 - 176\xi_3^4 - 24\xi_2^2\lambda_2^2 - 8\xi_3^2\lambda_2^2 + 609\lambda_2^4 - \\ & 1392\xi_2^2\lambda_2\lambda_3 + 432\xi_3^2\lambda_2\lambda_3 + 4308\lambda_2^3\lambda_3 - 88\xi_2^2\lambda_3^2 - 840\xi_3^2\lambda_3^2 + \\ & 9238\lambda_2^2\lambda_3^2 + 8244\lambda_2\lambda_3^3 + 1937\lambda_3^4 \end{aligned} \quad (3.35)$$

$$\begin{aligned} 0 = {} & -880\xi_2^4 - 1760\xi_2^2\xi_3^2 - 880\xi_3^4 + 24\xi_2^2\lambda_2^2 + 24\xi_3^2\lambda_2^2 - 203\lambda_2^4 + \\ & 1584\xi_2^2\lambda_2\lambda_3 + 1584\xi_3^2\lambda_2\lambda_3 - 1356\lambda_2^3\lambda_3 + 24\xi_2^2\lambda_3^2 + 24\xi_3^2\lambda_3^2 - \\ & 3074\lambda_2^2\lambda_3^2 - 1356\lambda_2\lambda_3^3 - 203\lambda_3^4 \end{aligned} \quad (3.36)$$

Modulo the assumption that ξ_2,ξ_3 are both non-zero, Equation (3.33) implies

$$0 = -44\xi_2^2 - 44\xi_3^2 + \lambda_2^2 + 66\lambda_2\lambda_3 + \lambda_3^2 \quad (3.37)$$

This allows us to make the substitution

$$\xi_2^2 + \xi_3^2 \to (\lambda_2^2 + 66\lambda_2\lambda_3 + \lambda_3^2)/44$$

into the Equation (3.36). We obtain after simplifications:

$$0 = 279\lambda_2^4 + 1848\lambda_2^3\lambda_3 + 3682\lambda_2^2\lambda_3^2 + 1848\lambda_2\lambda_3^3 + 279\lambda_3^4 \quad (3.38)$$

One can see easily that the only solution to this equation is the trivial one. Therefore another ξ_i must vanish, say $\xi_2 = 0$

$\xi_1, \xi_2 = 0$

Assume that $\xi_1, \xi_2 = 0$ then the Euler-Lagrange equations become:

$$\begin{aligned}
0 &= 33\xi_3^4 + 2\xi_3^2\lambda_1^2 - 55\lambda_1^4 + 4\xi_3^2\lambda_1\lambda_2 + 36\lambda_1^3\lambda_2 - 6\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 - \\
&\quad 12\lambda_1\lambda_2^3 + 33\lambda_2^4 - 12\xi_3^2\lambda_1\lambda_3 + 36\lambda_1^3\lambda_3 - 36\xi_3^2\lambda_2\lambda_3 - 12\lambda_1^2\lambda_2\lambda_3 + \\
&\quad 12\lambda_1\lambda_2^2\lambda_3 - 36\lambda_2^3\lambda_3 + 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + 12\lambda_1\lambda_2\lambda_3^2 + 6\lambda_2^2\lambda_3^2 - \\
&\quad 12\lambda_1\lambda_3^3 - 36\lambda_2\lambda_3^3 + 33\lambda_3^4 \qquad (3.39)\\
0 &= 33\xi_3^4 - 6\xi_3^2\lambda_1^2 + 33\lambda_1^4 + 4\xi_3^2\lambda_1\lambda_2 - 12\lambda_1^3\lambda_2 + 2\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 + \\
&\quad 36\lambda_1\lambda_2^3 - 55\lambda_2^4 - 36\xi_3^2\lambda_1\lambda_3 - 36\lambda_1^3\lambda_3 - 12\xi_3^2\lambda_2\lambda_3 + 12\lambda_1^2\lambda_2\lambda_3 - \\
&\quad 12\lambda_1\lambda_2^2\lambda_3 + 36\lambda_2^3\lambda_3 + 66\xi_3^2\lambda_3^2 + 6\lambda_1^2\lambda_3^2 + 12\lambda_1\lambda_2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 - \\
&\quad 36\lambda_1\lambda_3^3 - 12\lambda_2\lambda_3^3 + 33\lambda_3^4 \qquad (3.40)\\
0 &= -11\xi_3^4 - 2\xi_3^2\lambda_1^2 + 33\lambda_1^4 + 4\xi_3^2\lambda_1\lambda_2 - 36\lambda_1^3\lambda_2 - 2\xi_3^2\lambda_2^2 + 6\lambda_1^2\lambda_2^2 - \\
&\quad 36\lambda_1\lambda_2^3 + 33\lambda_2^4 + 12\xi_3^2\lambda_1\lambda_3 - 12\lambda_1^3\lambda_3 + 12\xi_3^2\lambda_2\lambda_3 + 12\lambda_1^2\lambda_2\lambda_3 + \\
&\quad 12\lambda_1\lambda_2^2\lambda_3 - 12\lambda_2^3\lambda_3 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - 12\lambda_1\lambda_2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 + \\
&\quad 36\lambda_1\lambda_3^3 + 36\lambda_2\lambda_3^3 - 55\lambda_3^4 \qquad (3.41)\\
0 &= \xi_3\lambda_3(-11\xi_3^2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 6\lambda_1\lambda_3 + 6\lambda_2\lambda_3 - 11\lambda_3^2) \qquad (3.42)\\
0 &= -55\xi_3^4 + 6\xi_3^2\lambda_1^2 - 11\lambda_1^4 - 12\xi_3^2\lambda_1\lambda_2 + 12\lambda_1^3\lambda_2 + 6\xi_3^2\lambda_2^2 - 2\lambda_1^2\lambda_2^2 + \\
&\quad 12\lambda_1\lambda_2^3 - 11\lambda_2^4 + 36\xi_3^2\lambda_1\lambda_3 + 12\lambda_1^3\lambda_3 + 36\xi_3^2\lambda_2\lambda_3 - 12\lambda_1^2\lambda_2\lambda_3 - \\
&\quad 12\lambda_1\lambda_2^2\lambda_3 + 12\lambda_2^3\lambda_3 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - 12\lambda_1\lambda_2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 + \\
&\quad 12\lambda_1\lambda_3^3 + 12\lambda_2\lambda_3^3 - 11\lambda_3^4 \qquad (3.43)
\end{aligned}$$

Equation (3.42) implies

$$\xi_3^2 = (\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 6\lambda_1\lambda_3 + 6\lambda_2\lambda_3 - 11\lambda_3^2)/11 \tag{3.44}$$

We can use the substitution

$$\xi_3^2 \to (\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 6\lambda_1\lambda_3 + 6\lambda_2\lambda_3 - 11\lambda_3^2)/11$$

to obtain a new set of Euler-Lagrange equations. Namely,

$$\begin{aligned}
0 &= -75\lambda_1^4 + 48\lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 - 16\lambda_1\lambda_2^3 + 45\lambda_2^4 + 54\lambda_1^3\lambda_3 - \\
&\quad 18\lambda_1^2\lambda_2\lambda_3 + 18\lambda_1\lambda_2^2\lambda_3 - 54\lambda_2^3\lambda_3 - \lambda_1^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3^2 + 3\lambda_2^2\lambda_3^2 \quad (3.45) \\
0 &= 45\lambda_1^4 - 16\lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 + 48\lambda_1\lambda_2^3 - 75\lambda_2^4 - 54\lambda_1^3\lambda_3 \\
&\quad +18\lambda_1^2\lambda_2\lambda_3 - 18\lambda_1\lambda_2^2\lambda_3 + 54\lambda_2^3\lambda_3 + 3\lambda_1^2\lambda_3^2 + 2\lambda_1\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3^2 \quad (3.46) \\
0 &= 45\lambda_1^4 - 48\lambda_1^3\lambda_2 + 6\lambda_1^2\lambda_2^2 - 48\lambda_1\lambda_2^3 + 45\lambda_2^4 - 18\lambda_1^3\lambda_3 + \\
&\quad 18\lambda_1^2\lambda_2\lambda_3 + 18\lambda_1\lambda_2^2\lambda_3 - 18\lambda_2^3\lambda_3 - \lambda_1^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3^2 \quad (3.47) \\
0 &= -15\lambda_1^4 + 16\lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 + 16\lambda_1\lambda_2^3 - 15\lambda_2^4 + 18\lambda_1^3\lambda_3 - \\
&\quad 18\lambda_1^2\lambda_2\lambda_3 - 18\lambda_1\lambda_2^2\lambda_3 + 18\lambda_2^3\lambda_3 - \lambda_1^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3^2 \quad (3.48)
\end{aligned}$$

We form the equation (3.47)+3×(3.48):

$$0 = (\lambda_1 + \lambda_2)\lambda_3(9\lambda_1^2 - 18\lambda_1\lambda_2 + 9\lambda_2^2 - \lambda_1\lambda_3 - \lambda_2\lambda_3) \tag{3.49}$$

This equation strongly suggests the following change of variables:

$$\lambda_1 \to (\mu_1 + \mu_2), \lambda_2 \to (\mu_1 - \mu_2).$$

Moreover let us take advantage of the homogeneity of the Euler-Lagrange equations by setting $\lambda_3 \to 1$. The above system of equations then becomes:

$$0 = \mu_1^2 - 2\mu_1\mu_2 + 72\mu_1^2\mu_2 - 88\mu_1^3\mu_2 - 44\mu_1^2\mu_2^2 + 36\mu_2^3 - 152\mu_1\mu_2^3 - 16\mu_2^4 \quad (3.50)$$

$$0 = \mu_1^2 + 2\mu_1\mu_2 - 72\mu_1^2\mu_2 + 88\mu_1^3\mu_2 - 44\mu_1^2\mu_2^2 - 36\mu_2^3 + 152\mu_1\mu_2^3 - 16\mu_2^4 \quad (3.51)$$

$$0 = -\mu_1^2 - 36\mu_1\mu_2^2 + 132\mu_1^2\mu_2^2 + 48\mu_2^4 \quad (3.52)$$

$$0 = -\mu_1^2 + 36\mu_1\mu_2^2 - 44\mu_1^2\mu_2^2 - 16\mu_2^4 \quad (3.53)$$

Equation (3.49) becomes:

$$0 = \mu_1(18\mu_2^2 - \mu_1) \quad (3.54)$$

To finish "case 1", we will treat two different alternatives. First let us assume μ_1 is non-zero then we can make the substitution

$$\mu_1 \to 18\mu_2^2$$

in equations (3.50), (3.51), (3.52), (3.53). We obtain:

$$0 = \mu_2^2(72 + 5841\mu_2 - 132556\mu_2^2) \quad (3.55)$$

$$0 = \mu_2^2(90 - 5841\mu_2 + 125420\mu_2^2) \quad (3.56)$$

$$0 = \mu_2^2(-27 - 54\mu_2 + 3568\mu_2^2) \quad (3.57)$$

$$0 = \mu_2^2(-81 + 162\mu_2 - 3568\mu_2^2) \quad (3.58)$$

A simple computation shows that $90 - 5841\mu_2 + 125420\mu_2^2$ admits no real roots. Therefore $\mu_2 = 0$ hence $\mu_1 = 0$ which contradicts the hypothesis that none of the λ_i's vanish. The other possibility is that μ_1 vanishes in which case (3.50), (3.51), (3.52), (3.53) yield:

$$0 = 36\mu_2^3 - 16\mu_2^4 \tag{3.59}$$

$$0 = -36\mu_2^3 - 16\mu_2^4 \tag{3.60}$$

$$0 = 48\mu_2^4 \tag{3.61}$$

$$0 = -16\mu_2^4 \tag{3.62}$$

Which again contradicts the assumption that none of the λ_i's vanish.

Conclusion of case 1 Therefore if none of the λ_i's vanish, all the ξ_i's must vanish. But this last possibility yields a Riemannian product between a three manifold and a line. This product is $\mathcal{R}$-critical iff the three manifold is flat. This in turn is not possible if all the λ_i's are non-zero (see [Mil76]).

3.7.3 Case 2: only one λ_i vanishes

We have established that one of the λ_i has to vanish, as well as one of the ξ_i has to vanish. In what follows we distinguish two cases: First $\lambda_1 = 0$ and $\xi_1 = 0$, second $\lambda_2 = 0$ and $\xi_1 = 0$. In both cases we show that solving the Euler-Lagrange equations leads to the vanishing of the other ξ_i's and the conclusion that $\lambda_2 = \lambda_3$ in the first case, $\lambda_1 = \lambda_3$ in the second.

Throughout this section we will assume that only one λ_i vanishes; the situation where two of them vanish, is treated at the end.

$\lambda_1 = 0,\ \xi_1 = 0$

In this case the Euler-Lagrange equations take the following form:

$$0 = 11\xi_2^4 + 22\xi_2^2\xi_3^2 + 11\xi_3^4 + 22\xi_2^2\lambda_2^2 - 2\xi_3^2\lambda_2^2 + 11\lambda_2^4 - 12\xi_2^2\lambda_2\lambda_3 - 12\xi_3^2\lambda_2\lambda_3 - 12\lambda_2^3\lambda_3 - 2\xi_2^2\lambda_3^2 + 22\xi_3^2\lambda_3^2 + 2\lambda_2^2\lambda_3^2 - 12\lambda_2\lambda_3^3 + 11\lambda_3^4 \tag{3.63}$$

$$0 = -11\xi_2^4 + 22\xi_2^2\xi_3^2 + 33\xi_3^4 - 66\xi_2^2\lambda_2^2 + 2\xi_3^2\lambda_2^2 - 55\lambda_2^4 + 12\xi_2^2\lambda_2\lambda_3 - 12\xi_3^2\lambda_2\lambda_3 + 36\lambda_2^3\lambda_3 - 2\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 - 12\lambda_2\lambda_3^3 + 33\lambda_3^4 \tag{3.64}$$

$$0 = \xi_2\xi_3(11\xi_2^2 + 11\xi_3^2 + 11\lambda_2^2 + 6\lambda_2\lambda_3 + 11\lambda_3^2) \tag{3.65}$$

$$0 = \xi_2\lambda_2(-11\xi_2^2 - 11\xi_3^2 - 11\lambda_2^2 + 6\lambda_2\lambda_3 + \lambda_3^2)/4 \tag{3.66}$$

$$0 = 33\xi_2^4 + 22\xi_2^2\xi_3^2 - 11\xi_3^4 + 66\xi_2^2\lambda_2^2 - 2\xi_3^2\lambda_2^2 + 33\lambda_2^4 - 12\xi_2^2\lambda_2\lambda_3 + 12\xi_3^2\lambda_2\lambda_3 - 12\lambda_2^3\lambda_3 + 2\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 + 36\lambda_2\lambda_3^3 - 55\lambda_3^4 \tag{3.67}$$

$$0 = \xi_3\lambda_3(-11\xi_2^2 - 11\xi_3^2 + \lambda_2^2 + 6\lambda_2\lambda_3 - 11\lambda_3^2)/4 \tag{3.68}$$

$$0 = -55\xi_2^4 - 110\xi_2^2\xi_3^2 - 55\xi_3^4 - 66\xi_2^2\lambda_2^2 + 6\xi_3^2\lambda_2^2 - 11\lambda_2^4 + 36\xi_2^2\lambda_2\lambda_3 + 36\xi_3^2\lambda_2\lambda_3 + 12\lambda_2^3\lambda_3 + 6\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_2^2\lambda_3^2 + 12\lambda_2\lambda_3^3 - 11\lambda_3^4 \tag{3.69}$$

Under the assumption that no other variables vanish, equations (3.65) and (3.66) imply that:

$$0 = 11\xi_2^2 + 11\xi_3^2 + 11\lambda_2^2 + 6\lambda_2\lambda_3 + 11\lambda_3^2 \tag{3.70}$$

$$0 = -11\xi_2^2 - 11\xi_3^2 - 11\lambda_2^2 + 6\lambda_2\lambda_3 + \lambda_3^2 \tag{3.71}$$

If one considers the equation obtained by (3.70) −(3.71):

$$0 = 22\xi_2^2 + 22\xi_3^2 + 22\lambda_2^2 + 10\lambda_3^2 \tag{3.72}$$

Hence one more variable should vanish, let us assume that it is one of the ξ_i's, we will treat in the next section the situation where two of λ_i's vanish. W.l.o.g. let us assume that $\xi_2 = 0$.

$\lambda_1 = 0, \xi_1 = 0, \xi_2 = 0$

Under the assumption $\lambda_1 = 0, \xi_1 = 0, \xi_2 = 0$, we show that one more variable has to vanish. For we have the following Euler-Lagrange system of equations:

$$\begin{aligned}
0 &= 11\xi_3^4 - 2\xi_3^2\lambda_2^2 + 11\lambda_2^4 - 12\xi_3^2\lambda_2\lambda_3 - 12\lambda_2^3\lambda_3 + 22\xi_3^2\lambda_3^2 + \\
&\quad 2\lambda_2^2\lambda_3^2 - 12\lambda_2\lambda_3^3 + 11\lambda_3^4 && (3.73)\\
0 &= 33\xi_3^4 + 2\xi_3^2\lambda_2^2 - 55\lambda_2^4 - 12\xi_3^2\lambda_2\lambda_3 + 36\lambda_2^3\lambda_3 + 66\xi_3^2\lambda_3^2 - \\
&\quad 2\lambda_2^2\lambda_3^2 - 12\lambda_2\lambda_3^3 + 33\lambda_3^4 && (3.74)\\
0 &= -11\xi_3^4 - 2\xi_3^2\lambda_2^2 + 33\lambda_2^4 + 12\xi_3^2\lambda_2\lambda_3 - 12\lambda_2^3\lambda_3 - 66\xi_3^2\lambda_3^2 - \\
&\quad 2\lambda_2^2\lambda_3^2 + 36\lambda_2\lambda_3^3 - 55\lambda_3^4 && (3.75)\\
0 &= \xi_3\lambda_3(-11\xi_3^2 + \lambda_2^2 + 6\lambda_2\lambda_3 - 11\lambda_3^2) && (3.76)\\
0 &= \xi_3\lambda_3(-11\xi_3^2 + \lambda_2^2 + 6\lambda_2\lambda_3 - 11\lambda_3^2) && (3.77)\\
0 &= -55\xi_3^4 + 6\xi_3^2\lambda_2^2 - 11\lambda_2^4 + 36\xi_3^2\lambda_2\lambda_3 + 12\lambda_2^3\lambda_3 - 66\xi_3^2\lambda_3^2 - \\
&\quad 2\lambda_2^2\lambda_3^2 + 12\lambda_2\lambda_3^3 - 11\lambda_3^4 && (3.78)
\end{aligned}$$

Equation (3.76) allows us to make the substitution:

$$\xi_3^2 \to (\lambda_2^2 + 6\lambda_2\lambda_3 - 11\lambda_3^2)/11.$$

This yield the following system of equations:

$$0 = \lambda_2^2(15\lambda_2^2 - 18\lambda_2\lambda_3 + \lambda_3^2) \quad (3.79)$$

$$0 = \lambda_2^2(-75\lambda_2^2 + 54\lambda_2\lambda_3 - \lambda_3^2) \quad (3.80)$$

$$0 = \lambda_2^2(45\lambda_2^2 - 18\lambda_2\lambda_3 - \lambda_3^2) \quad (3.81)$$

$$0 = \lambda_2^2(-15\lambda_2^2 + 18\lambda_2\lambda_3 - \lambda_3^2) \quad (3.82)$$

Let us point out that λ_2 must vanish. If not we could set it to be one in which case the resulting system doesn't yield any solutions. Therefore if ξ_3 is non-zero then λ_2 must vanish. This will be treated in the next section. Let us assume that ξ_3 vanishes as well as ξ_2, ξ_1 and λ_1. We then obtain the following Euler-Lagrange equations

$$0 = (-\lambda_2 + \lambda_3)^2(11\lambda_2^2 + 10\lambda_2\lambda_3 + 11\lambda_3^2) \quad (3.83)$$

$$0 = (-\lambda_2 + \lambda_3)(55\lambda_2^3 + 19\lambda_2^2\lambda_3 + 21\lambda_2\lambda_3^2 + 33\lambda_3^3) \quad (3.84)$$

$$0 = (\lambda_2 - \lambda_3)(33\lambda_2^3 + 21\lambda_2^2\lambda_3 + 19\lambda_2\lambda_3^2 + 55\lambda_3^3) \quad (3.85)$$

$$0 = (-\lambda_2 + \lambda_3)^2(11\lambda_2^2 + 10\lambda_2\lambda_3 + 11\lambda_3^2) \quad (3.86)$$

It follows from (3.86) that the only solution is $\lambda_2 = \lambda_3$.

$\lambda_2 = 0$ **and** $\xi_1 = 0$

Under this assumption we obtain the following set of equations

$$\begin{aligned}
0 &= 33\xi_2^4 + 66\xi_2^2\xi_3^2 + 33\xi_3^4 + 2\xi_2^2\lambda_1^2 + 2\xi_3^2\lambda_1^2 - 55\lambda_1^4 + \\
&\quad 4\xi_2^2\lambda_1\lambda_3 - 12\xi_3^2\lambda_1\lambda_3 + 36\lambda_1^3\lambda_3 - 6\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - \\
&\quad 12\lambda_1\lambda_3^3 + 33\lambda_3^4 && (3.87)\\
0 &= -11\xi_2^4 + 22\xi_2^2\xi_3^2 + 33\xi_3^4 - 2\xi_2^2\lambda_1^2 - 6\xi_3^2\lambda_1^2 + 33\lambda_1^4 + \\
&\quad 4\xi_2^2\lambda_1\lambda_3 - 36\xi_3^2\lambda_1\lambda_3 - 36\lambda_1^3\lambda_3 - 2\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 + 6\lambda_1^2\lambda_3^2 - \\
&\quad 36\lambda_1\lambda_3^3 + 33\lambda_3^4 && (3.88)\\
0 &= \xi_2\xi_3(-11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_3 - 11\lambda_3^2) && (3.89)\\
0 &= 33\xi_2^4 + 22\xi_2^2\xi_3^2 - 11\xi_3^4 - 6\xi_2^2\lambda_1^2 - 2\xi_3^2\lambda_1^2 + 33\lambda_1^4 + \\
&\quad 4\xi_2^2\lambda_1\lambda_3 + 12\xi_3^2\lambda_1\lambda_3 - 12\lambda_1^3\lambda_3 + 2\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + \\
&\quad 36\lambda_1\lambda_3^3 - 55\lambda_3^4 && (3.90)\\
0 &= \xi_3\lambda_3(-11\xi_2^2 - 11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_3 - 11\lambda_3^2) && (3.91)\\
0 &= -55\xi_2^4 - 110\xi_2^2\xi_3^2 - 55\xi_3^4 + 6\xi_2^2\lambda_1^2 + 6\xi_3^2\lambda_1^2 - 11\lambda_1^4 - \\
&\quad 12\xi_2^2\lambda_1\lambda_3 + 36\xi_3^2\lambda_1\lambda_3 + 12\lambda_1^3\lambda_3 + 6\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + \\
&\quad 12\lambda_1\lambda_3^3 - 11\lambda_3^4 && (3.92)
\end{aligned}$$

We will show that ξ_3 must vanish. For Equation (3.91) allows us to make the following substitution

$$\xi_2^2 \to (-\xi_3^2 + \lambda_1^2/11 + 6\lambda_1\lambda_3/11 - \lambda_3^2)$$

back into the equations. We obtain,

$$0 = -75\lambda_1^4 - 22\xi_3^2\lambda_1\lambda_3 + 56\lambda_1^3\lambda_3 + 99\xi_3^2\lambda_3^2 + 2\lambda_1^2\lambda_3^2 - 76\lambda_1\lambda_3^3 + 99\lambda_3^4 \quad (3.93)$$

$$0 = 45\lambda_1^4 - 22\xi_3^2\lambda_1\lambda_3 - 52\lambda_1^3\lambda_3 + 33\xi_3^2\lambda_3^2 + 12\lambda_1^2\lambda_3^2 - 40\lambda_1\lambda_3^3 + 33\lambda_3^4 \quad (3.94)$$

$$0 = 45\lambda_1^4 - 22\xi_3^2\lambda_1\lambda_3 - 16\lambda_1^3\lambda_3 - 33\xi_3^2\lambda_3^2 + 14\lambda_1^2\lambda_3^2 - 4\lambda_1\lambda_3^3 - 33\lambda_3^4 \quad (3.95)$$

$$0 = -15\lambda_1^4 + 66\xi_3^2\lambda_1\lambda_3 + 12\lambda_1^3\lambda_3 - 99\xi_3^2\lambda_3^2 - 28\lambda_1^2\lambda_3^2 + 120\lambda_1\lambda_3^3 - 99\lambda_3^4 \quad (3.96)$$

By considering (3.94)+3×(3.96) we obtain:

$$0 = \lambda_3(-2\lambda_1 + 3\lambda_3)(-11\xi_3^2 + \lambda_1^2 + 6\lambda_1\lambda_3 - 11\lambda_3^2) \quad (3.97)$$

Let us plug back in (3.93), (3.94), (3.95), (3.96) successively

$$\lambda_1 = (3/2)\lambda_3,$$

$$\xi_3^2 = \lambda_1^2/11 + 6\lambda_1\lambda_3/11 - \lambda_3^2$$

The situation $\lambda_3 = 0$ and $\lambda_2 = 0$ is treated in next section.

$\lambda_1 = (3/2)\lambda_3$ yields:

$$0 = \lambda_3^2(352\xi_3^2 - 1073\lambda_3^2) \quad (3.98)$$

$$0 = \lambda_3^4 \quad (3.99)$$

$$0 = \lambda_3^2(-352\xi_3^2 + 887\lambda_3^2) \quad (3.100)$$

$$0 = \lambda_3^4 \quad (3.101)$$

This implies the vanishing of all the variables, similarly

$$\xi_3^2 = \lambda_1^2/11 + 6\lambda_1\lambda_3/11 - \lambda_3^2$$

yields:

$$0 = \lambda_1^2(-75\lambda_1^2 + 54\lambda_1\lambda_3 - \lambda_3^2) \quad (3.102)$$

$$0 = \lambda_1^2(15\lambda_1^2 - 18\lambda_1\lambda_3 + \lambda_3^2) \quad (3.103)$$

$$0 = \lambda_1^2(45\lambda_1^2 - 18\lambda_1\lambda_3 - \lambda_3^2) \quad (3.104)$$

$$0 = \lambda_1^2(-15\lambda_1^2 + 18\lambda_1\lambda_3 - \lambda_3^2) \quad (3.105)$$

These equations have only one common solution, namely $\lambda_1 = 0$. Therefore let us assume that $\xi_3 = 0$.

Therefore if $\xi_1 = 0, \lambda_2 = 0$ then $\xi_3 = 0$

Now in this situation the Euler-Lagrange system becomes:

$$0 = 33\xi_2^4 + 2\xi_2^2\lambda_1^2 - 55\lambda_1^4 + 4\xi_2^2\lambda_1\lambda_3 + 36\lambda_1^3\lambda_3 - 6\xi_2^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 - 12\lambda_1\lambda_3^3 + 33\lambda_3^4 \quad (3.106)$$

$$0 = -11\xi_2^4 - 2\xi_2^2\lambda_1^2 + 33\lambda_1^4 + 4\xi_2^2\lambda_1\lambda_3 - 36\lambda_1^3\lambda_3 - 2\xi_2^2\lambda_3^2 + 6\lambda_1^2\lambda_3^2 - 36\lambda_1\lambda_3^3 + 33\lambda_3^4 \quad (3.107)$$

$$0 = (33\xi_2^4 - 6\xi_2^2\lambda_1^2 + 33\lambda_1^4 + 4\xi_2^2\lambda_1\lambda_3 - 12\lambda_1^3\lambda_3 + 2\xi_2^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + 36\lambda_1\lambda_3^3 - 55\lambda_3^4 \quad (3.108)$$

$$0 = -55\xi_2^4 + 6\xi_2^2\lambda_1^2 - 11\lambda_1^4 - 12\xi_2^2\lambda_1\lambda_3 + 12\lambda_1^3\lambda_3 + 6\xi_2^2\lambda_3^2 - 2\lambda_1^2\lambda_3^2 + 12\lambda_1\lambda_3^3 - 11\lambda_3^4 \quad (3.109)$$

Let us make the following change of variables:

$$\lambda_1 \to (a+b), \lambda_2 \to (a-b)$$

we obtain:

$$
\begin{aligned}
0 &= -256a^3b - 128a^2b^2 - 448ab^3 - 48b^4 + 16ab\xi_2^2 - \\
&\quad 8b^2\xi_2^2 + 33\xi_2^4 && (3.110)\\
0 &= 384a^2b^2 + 144b^4 - 8b^2\xi_2^2 - 11\xi_2^4 && (3.111)\\
0 &= 256a^3b - 128a^2b^2 + 448ab^3 - 48b^4 - 16ab\xi_2^2 - \\
&\quad 8b^2\xi_2^2 + 33\xi_2^4 && (3.112)\\
0 &= -128a^2b^2 - 48b^4 + 24b^2\xi_2^2 - 55\xi_2^4 && (3.113)
\end{aligned}
$$

It is clear from the last equation that the only possible solution is $b = 0$ and $\xi_2 = 0$. $b = 0$ is on the other hand equivalent to $\lambda_1 = \lambda_3$.

This closes the case $\xi_1 = 0, \lambda_2 = 0$. We have exhausted all the possible cases, up to symmetry, and they all lead to $\xi_i = 0$ for all i and $\lambda_1 = \lambda_3$.

3.7.4 Case 3: $\lambda_1 = 0$ and $\lambda_2 = 0$

Under the assumption that only $\lambda_1 = 0$ and $\lambda_2 = 0$ the Euler-Lagrange equation becomes:

$$
\begin{aligned}
0 &= -11\xi_1^4 + 22\xi_1^2\xi_2^2 + 33\xi_2^4 + 22\xi_1^2\xi_3^2 + 66\xi_2^2\xi_3^2 + 33\xi_3^4 - \\
&\quad 2\xi_1^2\lambda_3^2 - 6\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 + 33\lambda_3^4 && (3.114)
\end{aligned}
$$

$$0 = \xi_1\xi_2(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 + \lambda_3^2) \quad (3.115)$$

$$0 = \xi_1\xi_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_3^2) \quad (3.116)$$

$$\begin{aligned} 0 = {} & 33\xi_1^4 + 22\xi_1^2\xi_2^2 - 11\xi_2^4 + 66\xi_1^2\xi_3^2 + 22\xi_2^2\xi_3^2 + 33\xi_3^4 - \\ & 6\xi_1^2\lambda_3^2 - 2\xi_2^2\lambda_3^2 + 66\xi_3^2\lambda_3^2 + 33\lambda_3^4 \end{aligned} \quad (3.117)$$

$$0 = \xi_2\xi_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_3^2) \quad (3.118)$$

$$\begin{aligned} 0 = {} & 33\xi_1^4 + 66\xi_1^2\xi_2^2 + 33\xi_2^4 + 22\xi_1^2\xi_3^2 + 22\xi_2^2\xi_3^2 - 11\xi_3^4 + \\ & 2\xi_1^2\lambda_3^2 + 2\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 55\lambda_3^4 \end{aligned} \quad (3.119)$$

$$0 = \xi_3\lambda_3(-11\xi_1^2 - 11\xi_2^2 - 11\xi_3^2 - 11\lambda_3^2) \quad (3.120)$$

$$\begin{aligned} 0 = {} & -55\xi_1^4 - 110\xi_1^2\xi_2^2 - 55\xi_2^4 - 110\xi_1^2\xi_3^2 - 110\xi_2^2\xi_3^2 - 55\xi_3^4 + \\ & 6\xi_1^2\lambda_3^2 + 6\xi_2^2\lambda_3^2 - 66\xi_3^2\lambda_3^2 - 11\lambda_3^4 \end{aligned} \quad (3.121)$$

It follows from Equation (3.121) that all the variables must vanish.

This concludes the analysis of the Euler-Lagrange equations for a left invariant metric on a four dimensional, unimodular Lie group with non-trivial center. We have shown that the only solutions, in terms of the structure constants are (up to permutation) $\xi_i = 0$ for all i's while $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$. These correspond to $\mathbf{R}^4$ if all variables vanish, to $E(2) \times \mathbf{R}$ if λ_1 is non-zero, moreover the metric on $E(2)$ is flat.

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