Structure Jumping in Holomorphic Families

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Abstract of the Dissertation

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In this thesis we establish criteria under which complex structure-jumping may occur in certain holomorphic families of compact manifolds. Further results are obtained for the converse problem of extension of complex-analytic automorphisms on a given sub-family, in the distinct cases of compact and strictly pseudoconvex manifolds. A central role is played by the theory of deformation of complex structures, developed by Kodaira and Spencer, in conjunction with the direct image theorem for coherent analytic sheaves, and the theory of deformation of complex spaces, due to Grauert.

For Jennie Shaw, and for Ruth and Richard Harris.

Contents

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Chapter 1

Introduction - Complex Deformations and Structure-Jumping

1.1 A historical synopsis of the moduli problem

The possibility of defining a real, two-dimensional surface as the domain of holomorphy of a multivalued, or "algebraic" function was first realised by Riemann around 1850. Such surfaces were commonly found to coincide with the loci of algebraic curves, X, determined by the vanishing of a homogeneous polynomial (ie., $P(x_0, x_1, x_2) = 0$) on $\mathbf{P}_2(\mathbf{C})$. By fixing a point, $p_0 \in X$, and projecting stereographically from X onto any line embedded in the complex projective plane (which, in modern language, would correspond to defining a meromorphic function, $f : X \to \mathbf{P}_1$), it was observed that the set of "branch points" of the projection is associated with an important invariant. If d denotes the degree of the homogeneous polynomial, P, then, in the absence of singular points, the "genus", p, was found to be independent of the choice of $p_0 \in X$, and is simply expressed by the formula, $\frac{1}{2}(d-1)(d-2)$. The corresponding real surface was seen, via "cutting and pasting", to be equivalent to a "sphere with p handles", thus introducing the modern topological conception of the "Riemann Surface".

The underlying motive of these discoveries, however, came from the question of complex-analytic equivalence of curves. In particular, when X is a conic in $\mathbf{P}_2$ (ie., $d = 2, p = 0$), Riemann easily demonstrated that all such curves are conformally equivalent to $\mathbf{P}_1$. When X is a cubic (or "elliptic") curve (ie., $d = 3, p = 1$), Abel had already proved that there exists some number, $\tau \in \mathbf{H}_+$ (the complex upper half-plane), such that $X \cong \mathbf{C}/G_\tau$, where $G_\tau = \{a\tau + b \mid a, b \in \mathbf{Z}\}$ corresponds to a discrete group of affine transformations of $\mathbf{C}$. In fact, there are as many conformal equivalence classes of elliptic X as there are points in the space, $\mathbf{H}_+/PSL(2,\mathbf{Z})$. The general classification of such "moduli spaces" for algebraic curves led Riemann to conclude that all non-singular curves of genus, $p \geq 2$, are parametrised, up to conformal equivalence, by spaces of complex dimension, $\mu = 3p - 3$. We may pre-empt modern terminology by remarking that $3p - 3 = 2p - 2 + p - 1$, where $2p - 2 = c(K)$ (the Chern class of the canonical line bundle), and $p = dim_{\mathbf{C}} H^0(X, \Omega^1_X)$ (the space of global holomorphic 1-forms). Note, for X of complex dimension one, that $K = \Omega^1_X$.

An analogous result for algebraic surfaces, $\Sigma \hookrightarrow \mathbf{P}_3$, was partially found by M. Noether about 1870. The "postulation formula", $\mu = 10(p_a + 1) - 2c_1^2$, where p_a denotes the "arithmetic genus", corresponding to $dim_{\mathbf{C}}H^0(\Sigma, K) - dim_{\mathbf{C}}H^0(\Sigma, \Omega^1_\Sigma)$, while c_1^2 is the second "cup-power" of the first Chern class, was shown to hold only for surfaces of degree at most four, however, and was later corrected through the work of Enriques and the Italian school. While no further advances were made on moduli of higher-dimensional manifolds for over half a century, the study of Riemann surfaces underwent further development with Klein, who extended Riemann's formula to moduli of non-compact (Riemann) surfaces with boundary. The approach introduced by Klein has led ultimately to the theory of "quasi-conformal" mappings, and the construction of metrics on the moduli space, initiated by Teichmüller.

1.2 Infinitesimal deformation

The modern notion of deformation of complex structures for compact manifolds of higher dimension first appeared in a paper of Frölicher and Nijenhuis, about 1957. The germ implicit in this work, however, was taken up and developed fully as a theory by Kodaira and Spencer, who had already been considering problems related to the variation of almost complex structures. The idea of "complex-analytic (or holomorphic) family" provides the foundation of this theory, and signals its radical departure from the approach to the classical moduli problem. Consider a differentiable map, $\mathcal{M} \xrightarrow{f} D$, where $D \subseteq \mathbf{R}^m$ is a domain containing the origin, and $\mathcal{M}$ is a differentiable manifold, such that for all $t \in D$, the preimage, $f^{-1}(t) = M_t$, is a compact, complex manifold of constant dimension. $\mathcal{M}$ is therefore a fiber space over D, having complex structure in the direction of each fiber, and differentiable structure in the normal direction, and is termed a "differentiable family" of compact, complex manifolds. $\mathcal{M} \xrightarrow{f} D$ is said to be a "holomorphic family" when the stronger requirements that $D \subseteq \mathbf{C}^n, \mathcal{M}$ is a complex manifold, and f holomorphic, are satisfied. Under these circumstances, the constancy of dimension for M_t is guaranteed by the regularity condition, $rank_{\mathbf{C}}(J(f))_q = dim_{\mathbf{C}}(D)$, for all $q \in \mathcal{M}$, where $J(f)$ denotes the Jacobian matrix. For the moment, however, the distinction between differentiable and holomorphic families need not be stressed. Let $U_i, U_j \in \mathcal{U}$ be charts on $\mathcal{M}$, and consider complex coordinate functions, $z_k^i(t), z_k^j(t), 1 \leq k \leq dim_{\mathbf{C}} M_t$, such that the restriction to $U_i \cap M_t$ (resp. $U_j \cap M_t$) is holomorphic. The holomorphic transition map between U_i and U_j is assumed to preserve the complex structure of the fibers, hence

$$z_k^j(t) = f_{i,j}^k(z_k^i(t), t), \quad 1 \leq k \leq dim_{\mathbf{C}} M_t.$$

For simplicity of notation, let $\frac{\partial}{\partial t}$ denote a differentiable vector field on D (the parameter, t, may be either real or complex). Denote by $(\frac{\partial}{\partial t})_i$, the natural pullback, $f^*(\frac{\partial}{\partial t}) \mid_{U_i}$, and similarly, $(\frac{\partial}{\partial t})_j = f^*(\frac{\partial}{\partial t}) \mid_{U_j}$. From the chain rule, it follows that

$$(\frac{\partial}{\partial t})_j = \Sigma_k \frac{\partial f_{i,j}^k}{\partial t} \frac{\partial}{\partial z_k^i} + (\frac{\partial}{\partial t})_i.$$

Hence, for each $t \in D$,

$$(\frac{\partial}{\partial t})_i - (\frac{\partial}{\partial t})_j \in C^1(U_i \cap U_j \cap M_t, \Theta_t),$$

where Θ_t is the notation used by Kodaira to represent the sheaf of holomorphic sections of TM_t (cf. [10]). Note that the sets, $U_i \cap M_t$, define an open cover of M_t. Passing to the direct limit, if necessary, we obtain a map,

$$\rho_t : T_t D \to H^1(M_t, \Theta_t),$$

such that

$$\rho_t(\frac{\partial}{\partial t}) = [\{(\frac{\partial}{\partial t})_i - (\frac{\partial}{\partial t})_j\}_{i,j}].$$

Though introduced by Frölicher and Nijenhuis, ρ_t is commonly known as the "Kodaira-Spencer map", which measures the "rate of variation" of complex structure in the parameter, t. In particular, $\rho_t(\frac{\partial}{\partial t}) = 0 \Rightarrow$

$$(\frac{\partial}{\partial t})_i - (\frac{\partial}{\partial t})_j = \sigma_i(z^i(t), t) - \sigma_j(z^j(t), t)$$

for some sections, $\sigma_i \in \Gamma(U_i \cap M_t, \Theta_t), \sigma_j \in \Gamma(U_j \cap M_t, \Theta_t)$. The equations,

$$\sigma_i - (\frac{\partial}{\partial t})_i = \sigma_j - (\frac{\partial}{\partial t})_j,$$

define a global, holomorphic vector field, ς, along M_t. It is by no means clear, however, that ς is holomorphic, or even continuous, in t, ie., $\rho_t(\frac{\partial}{\partial t}) = 0$ for all $t \in U \subseteq D$ does not automatically imply that $\frac{\partial}{\partial t}$ lifts to a holomorphic (resp. differentiable) vector field on $f^{-1}(U) \subseteq \mathcal{M}$. For Kodaira and Spencer, the solution of this problem required elliptic methods of partial differential equations, applied to the variation of almost complex structures, along with the assumption that $dim_{\mathbf{C}} H^i(M_t, \Theta_t), i = 0, 1$, is constant for all $t \in U$ (cf. [10], [12]).

A somewhat different approach, modelled on work of Grothendieck in the algebraic category, was adopted by Grauert, who considered the "direct image sheaves", $R^i f_*(\Theta)$, where Θ denotes that subsheaf of $\mathcal{TM}$ whose sections lie parallel to the fibers, M_t, over D. Now, if $\mathcal{M}$ is a holomorphic family of complex *spaces*, admitting arbitrary singularities, and $\mathcal{M} \xrightarrow{f} D$ is a *proper* holomorphic map (hence the fibers, $f^{-1}(t)$, are all compact), Grauert's "direct image theorem" states that $R^i f_*(\mathcal{F}), i \geq 0$, is coherent, for any coherent analytic sheaf, $\mathcal{F} \to \mathcal{M}$ (cf. [7]). In particular, when the infinitesimal Kodaira-Spencer maps, ρ_t, are replaced by a sheaf homomorphism,

$$\rho : \mathcal{T}D \to R^1 f_*(\Theta),$$

and $\frac{\partial}{\partial t}$ represents a holomorphic section of $\mathcal{T}D$, then from the coherence of $R^1 f_*(\Theta), \rho(\frac{\partial}{\partial t}|_U) \equiv 0$ implies that $\frac{\partial}{\partial t}$ lifts holomorphically to $f^{-1}(U)$, when $\mathcal{M}$ is once more assumed to be a regular family of compact, complex manifolds. Moreover, $\rho|_U \equiv 0$ if and only if there exists a biholomorphic map,

$$\Phi_{t_0} : f^{-1}(U) \to U \times M_{t_0},$$

for each $t_0 \in U$, thereby defining a "local trivialization" of the family (cf. [12]). Under these circumstances, integration of $f^*(\frac{\partial}{\partial t})$ induces Φ_{t_0}. It should be remarked that holomorphic triviality is a much stronger condition than the requirement that the C^∞-structure of the fibers be locally invariant. In fact, by means of a partition of unity, it can be shown relatively directly, that any smooth vector field on D may be lifted globally to $\mathcal{M}$, hence inducing a diffeomorphic trivialization of the family (cf. [12]).

Given a compact manifold, M, each embedding of M as a fiber, $M_{t_0} = f^{-1}(t_0)$, of some holomorphic family, $\mathcal{M}$, determines a linear subspace, or "space of infinitesimal deformations", $\mathcal{D}_{\mathcal{M}} \subseteq H^1(M_{t_0}, \Theta_{t_0})$, corresponding to the image of ρ_{t_0}. $\mathcal{M}$ is then referred to as a "deformation" of M. Conversely, given $\theta \in H^1(M, \Theta)$, does $\mathcal{M} \xrightarrow{f} D$, such that $\theta = \rho_0(\frac{\partial}{\partial t})$, necessarily exist ? If $\theta = [\{\theta_{i,j}\}]$, define

$$\zeta_{i,j,k} = [\theta_{i,j}, \theta_{j,k}],$$

the Lie bracket on $U_i \cap U_j \cap U_k$. It can be shown that

$$\zeta = [\{\zeta_{i,j,k}\}] \in H^2(M, \Theta)$$

defines an obstruction to the variation of complex structure in the parameter, t, hence a necessary condition that $\theta = \rho_0(\frac{\partial}{\partial t})$, for some $\mathcal{M} \xrightarrow{f} D, f^{-1}(0) = M$, is given by $\zeta = 0$ (for a discussion of the general problem of existence, cf. [12]).

Definition 1: A holomorphic family, $\mathcal{M}$ is said to be "effectively parametrised" at $t_0 \in D$ if ρ_{t_0} is injective.

Definition 2: $\mathcal{M}$ is said to be "versal" at $t_0 \in D$, if any deformation, $\mathcal{N} \xrightarrow{g} B$ of $M = g^{-1}(s_0)$ is such that there exists $\phi : B \to D$ holomorphic, with $\phi(s_0) = t_0$, and the diagram,

$$\begin{array}{ccccc} M & \hookrightarrow & \mathcal{N} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow g & & \downarrow f \\ s_0 & \hookrightarrow & B & \xrightarrow{\phi} & D \end{array}$$

commutes in a sufficiently small neighbourhood of t_0.

(*Remark:* The term "complete" was originally used in this context by Kodaira and Spencer, who assumed that all parameter spaces are complex manifolds. The modern setting of definition 2 allows B and D to be "non-reduced" complex spaces, to which a more general notion of infinitesimal deformation applies (cf. chapter four). Under these conditions, "versality" entails that ρ is surjective.)

Now if $\mathcal{M} \xrightarrow{f} D$ is complete and effective at each $t \in D$, define the "number of moduli", $\mu = dim_{\mathbf{C}}(D)$.

Much subsequent work in the development of Kodaira-Spencer deformation theory has centered on the "postulation formula",

$$\mu = dim_{\mathbf{C}} H^1(M_{t_0}, \Theta_{t_0}).$$

Agreement is found in many instances (cf. [10], [12]), perhaps the simplest being the case where M is a Riemann surface of genus, p. Note that $\Theta_{t_0} = \mathcal{T}M$ implies $c(\Theta_{t_0}) = c(\mathcal{T}M) = 2 - 2p < 0$, when $p \geq 2$, hence $H^0(M_{t_0}, \Theta_{t_0}) = 0$. It follows from Riemann-Roch that

$$dim_{\mathbf{C}} H^1(M_{t_0}, \Theta_{t_0}) = 3 - 3p,$$

which coincides with the number of moduli determined classically. For deformations of higher dimension, however, there are serious obstacles to the construction of a metric on the space of moduli. In fact, as a topological space, the set of biholomorphic equivalence classes need not even be Hausdorff, as will be seen in the next section.

1.3 Structure-jumping

Prior to the appearance of their work on deformation theory in 1958, Kodaira and Spencer had already demonstrated that $dim_{\mathbf{C}} H^i(M_t, \Theta_t)$ is an upper semicontinuous function of t (cf. chapter three). The following examples illustrate holomorphic families, $\mathcal{M} \xrightarrow{f} D$, in which the restriction over $D \setminus \{0\}$ is locally trivial, ie., all fibers $f^{-1}(t)$, are biholomorphically equivalent, $t \neq 0$. However, $f^{-1}(0) \not\cong f^{-1}(t)$, giving rise to the phenomenon of "structure-jumping".

Example 1(cf. [10]): Consider the one-parameter family of "Hopf Surfaces", defined as follows. Let $W = \mathbf{C}^2 \setminus \{0\}$, and let g_t be an automorphism of W, given by

$$(z_1, z_2) \mapsto (\alpha z_1 + t z_2, \alpha z_2),$$

where $0 < |\alpha| < 1, t \in \mathbf{C}$. Let

$$G_t = \{g_t^m \mid m \in \mathbf{Z}\}$$

be an infinite cyclic group, which acts on W in a properly discontinuous manner, without fixed point, hence $M_t = W/G_t$ is a compact, complex surface. Now the automorphism,

$$g : (z_1, z_2, t) \mapsto (\alpha z_1 + t z_2, \alpha z_2, t),$$

generates an infinite cyclic group, G, which acts on $W \times \mathbf{C}$ in a similar manner, hence $\mathcal{M} = (W \times \mathbf{C})/G$ is a complex manifold. Since g commutes with the projection: $W \times \mathbf{C} \to \mathbf{C}$, the induced map: $\mathcal{M} \to \mathbf{C}$ is holomorphic, and has a Jacobian of rank one at each point of $\mathcal{M}$.

Biholomorphic equivalence classes amongst fibers of $\mathcal{M}$ correspond to orbits of the matrices, $\begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix}$, under conjugation by elements in $PSL(2, \mathbf{C})$ (cf. [10], [12]). In particular,

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

indicates that $M_t \cong M_1$ for all $t \neq 0$. It may be seen by explicit computation, however, that $dim_{\mathbf{C}} H^0(M_t, \Theta_t) = 2, t \neq 0$, while $dim_{\mathbf{C}} H^0(M_0, \Theta_0) = 4$, hence $M_0 \not\cong M_t$ (cf. [10], [12]). Further discussion of deformation and structure-jumping for families of Hopf surfaces will appear in the next chapter. The role played by the Lie group of complex analytic automorphisms of M_t, and its Lie algebra, $H^0(M_t, \Theta_t)$, will be a recurrent theme in the following chapters.

Example 2(cf. [10]): Let $\mathcal{L} \to \mathbf{P}_1$ be a line bundle, with base coordinates, z_1, z_2 on $U_1, U_2 \subseteq \mathbf{P}_1$, and corresponding fiber coordinates, ζ_1, ζ_2, such that $\zeta_1 = z_2^3 \zeta_2$, hence $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}_1}(-3)$. Compactification of fibers yields a $\mathbf{P}_1$-bundle over $\mathbf{P}_1$, commonly known as the Hirzebruch surface, Σ_3 (cf. [10], [4]). It can be shown explicitly that $H^1(\Sigma_3, \mathcal{T}\Sigma_3)$ is generated by $z_2^i \frac{\partial}{\partial \zeta_1}, i = 1, 2$, hence the deformation,

$$\zeta_1 = z_2^3 \zeta_2 + t_2 z_2^2 + t_1 z_2,$$

detrmines a two-parameter family, $\mathcal{M} \xrightarrow{f} D \subseteq \mathbf{C}^2$, such that

$$\rho_0(\frac{\partial}{\partial t_i}) = z_2^i \frac{\partial}{\partial \zeta_1}.$$

Since ρ_0 is clearly surjective, it follows that $\mathcal{M} \xrightarrow{f} D$ is versal at $(t_1, t_2) = (0,0)$. Now for $t_1 t_2 \neq 0$, the holomorphic change of coordinates defined by

$$\zeta_1' = \frac{(-z_1 + \frac{t_2}{t_1})\zeta_1 + t_1}{-z_1\zeta_1 + t_1}, \qquad \zeta_2' = \frac{(-\frac{t_2}{t_1} z_2 + 1)\zeta_2 - t_2^2/t_1}{z_2\zeta_2 + t_2},$$

implies $\zeta_1' = z_2 \zeta_2'$, ie., $f^{-1}(t) \cong \Sigma_1, t = (t_1, t_2) \in \mathbf{C}^2 \setminus \{t_1 t_2 = 0\}$. For $t_2 = 0, t_1 \neq 0$, or vice versa, the transformations,

$$\zeta_1' = \frac{z_1\zeta_1 - t_i}{t_i \zeta_1}, \qquad \zeta_2' = \frac{\zeta_2}{t_i z_2^2 \zeta_2 + t_i^2}, \qquad i = 1, 2,$$

will achieve the same result. Note that the above is an explicit instance of local trivialization over $D \setminus \{0\}$. In fact no one change of coordinates suffices to trivialise the entire family over $D \setminus \{0\}$. We remark, in particular, that $dim_{\mathbf{C}} H^0(\Sigma_3, \mathcal{T}\Sigma_3) = 8$, while $dim_{\mathbf{C}} H^0(\Sigma_1, \mathcal{T}\Sigma_1) = 6$, and it is clear that $f^{-1}(0) \ncong f^{-1}(t)$ (cf. chapter two).

1.4 Complex spaces and structure-jumping

The concept of deformation theory introduced by Grothendieck extends to complex spaces with arbitrary singularities, both as objects for deformation, and as parameter spaces. In this approach, an important role is played by a notion of infinitesimal deformation in which the parameter space may consist of a single point, but with a more complicated structure algebra than before. The relationship between the theory of Kodaira and Spencer, and that of Grothendieck, is partly analogous to the relationship between an analytic function, and finite segments of its Taylor expansion at a given point. Further details of this theory, including the fundamental definition of tangent cohomology for complex spaces, will be discussed in chapter four. By way of a simple example, consider the following deformation, $X \xrightarrow{f} Y$, for which the structure sheaf, $\mathcal{O}_{X_t}$, on each fiber, X_t, is equipped with *nilpotent* elements. Let f correspond to the branched, two-fold covering of $\mathbf{C}$ given by $w = z^2$. Fibers corresponding to $w \neq 0$ consist of two points, hence the algebra of "holomorphic functions" on these fibers is a space of two complex dimensions. $X_0 = f^{-1}(0)$ is a one-point space, however, corresponding to $z^2 = 0$. In order to preserve the dimension of the structure sheaf on this fiber, it must therefore be viewed as a "fusion" of two simple points. $\mathcal{O}_{X_0}$ is thus identified with $H/(z^2)$, where H is the algebra of convergent power series in z, hence

$$\mathcal{O}_{X_0} = \{a + b\varepsilon \mid a, b \in \mathbf{C}, \varepsilon^2 = 0\}$$

(cf. [18]).

Versality of deformations of isolated singular points was treated by G. N. Tjurina, [24]. In particular, the versal deformation of the smooth compact surface, Σ_3, examined in the previous section, has its singular analogue in the following construction. Recall $\mathcal{L} \cong \mathcal{O}_{\mathbf{P}_1}(-3)$, and consider the holomorphic functions,

$$w_i = z_2^i \zeta_2 = z_1^{3-i} \zeta_1, \qquad \text{such that} \qquad w_i \in \mathcal{O}_{\mathcal{L}}, 0 \le i \le 3.$$

Here $\mathcal{L}$ is considered as the non-compact manifold corresponding to the total space. The induced holomorphic map,

$$\sigma : \mathcal{L} \to \mathbf{C}^4, \qquad (z, \zeta) \mapsto (w_0, ..., w_3),$$

is proper, such that $\sigma^{-1}(0) = E$, the exceptional curve in Σ_3, corresponding to the zero section of $\mathcal{L}$. Hence σ is identified with a "blowing-down" of Σ_3, for which the image is a singular surface in $\mathbf{C}^4$, defined by the relations,

$$\frac{w_0}{w_1} = \frac{w_1}{w_2} = \frac{w_2}{w_3}.$$

The versal deformation of this surface (cf. [24]) corresponds to the relations,

$$\frac{w_0}{w_1} = \frac{w_1 + t_1}{w_2} = \frac{w_2 + t_2}{w_3},$$

such that, for $t_1 \neq 0$, each fiber is a smooth surface, biholomorphically equivalent to $\mathcal{O}_{\mathbf{P}_1}(-1)$, while $t_1 = 0, t_2 \neq 0$ yields a one-parameter family of smooth surfaces, isomorphic to $\mathcal{O}_{\mathbf{P}_1}(+1)$.

1.5 Summary of Results

The following thesis is divided into four parts :

(1) (cf. chapter two) Given a holomorphic family, $\mathcal{M} \xrightarrow{f} D$, where D is a domain containing the origin in $\mathbf{C}^n$, such that the restriction over the punctured domain $D \setminus \{0\}$ is locally trivial, under what circumstances does f define a trivial deformation over D ? (An example of this type of problem

is a conjecture of Kodaira and Spencer [12], recently proved by Siu [22]: if $D \subset \mathbf{C}$, and $f^{-1}(t) \cong \mathbf{P}_m$, the complex projective space, for all $t \in D \setminus \{0\}$, then $f^{-1}(0) \cong \mathbf{P}_m$.) The results of this chapter may be summarised in the following

Theorem: *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact, complex surfaces, locally trivial over $D \setminus \{0\}$, such that $f^{-1}(t)$ is isomorphic to a fiber bundle over $\mathbf{P}_1$, for all $t \in D \setminus \{0\}$, then $f^{-1}(t) \not\cong f^{-1}(0) \Rightarrow dim(D) \leq 3$.*

The above result applies moreover to locally trivial families of certain quadratic transforms (ie., "blow-ups") of fiber bundles over $\mathbf{P}_1$. Note that the term "fiber" refers here to a Riemann surface of genus g, and is not to be confused with the two-dimensional fibers, $f^{-1}(t)$, of the family, $\mathcal{M} \xrightarrow{f} D$.

(2) (cf. chapter three) For a holomorphic family, $\mathcal{M} \xrightarrow{f} D, D \subseteq \mathbf{C}^n$, let Θ denote the subsheaf of the tangent sheaf, $\mathcal{TM}$, having sections which lie parallel to the fibers, $f^{-1}(t)$. $R^i f_*(\Theta)$ will then denote the i-th direct image of Θ over D. It was originally remarked by Mumford, in the case $D \subseteq \mathbf{C}$, having a single structure jump at the origin, that the vector field, $\frac{\partial}{\partial t}$, is mapped by the Kodaira-Spencer homomorphism,

$$\rho : \mathcal{T}D \rightarrow R^1 f_*(\Theta),$$

to a non-trivial section which vanishes on $D \setminus \{0\}$, with a pole of finite order at $t = 0$. Multiplication by a suitable power of t provides a lifting of $\frac{\partial}{\partial t}$ to $\mathcal{TM}$, which is tangent to $M_0 = f^{-1}(0)$, hence inducing an exceptional automorphism on that fiber. Via the theory of "jump-cocycles", [8], Griffiths translated Mumford's idea from the algebraic category to the holomorphic.

The results of this chapter refer to structure-jumping over an analytic subvariety, $A \subseteq D$, such that the restricted family, $\mathcal{M} \setminus f^{-1}(A) \xrightarrow{f} D \setminus A$, is locally trivial. After establishing an appropriate generalisation of [8], theorem 1.1 (cf. chapter three - main theorem), there are two main applications:

Theorem: *Let $H \subseteq D$ be a smooth analytic hypersurface containing A, then each $\sigma \in \Gamma(H, R^0 f_*(\Theta_{\mathcal{M}}))$ extends to a vector field $\hat{\sigma} \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that if $\nu(\hat{\sigma})_t$ denotes the projection of $\hat{\sigma} \mid_{M_t}$ onto the normal bundle, $\mathcal{N}_{M_t}$ in $\mathcal{M}$, then $\nu(\hat{\sigma})$ vanishes to order $k(\sigma)$on $f^{-1}(H) \subseteq \mathcal{M}$.*

(For the definition of $k(\sigma)$, see chapter three). Note that σ corresponds to a family of automorphisms on fibers above H.) The second application involves a regularity criterion for structure jumping, when $A = \{0\} \subset D$, under different assumptions from those of part (1):

Theorem: *Suppose*

(i) $H^1(M_t, \Theta_t) = 0$ *for all* $t \neq 0$,

(ii) $dim_{\mathbf{C}} H^2(M_t, \Theta_t)$ *is constant for all* $t \in D$, *and*

(iii) $\mathcal{M}$ *is effectively parametrised at* 0, *ie., the infinitesimal Kodaira-Spencer map* $\rho_0 : T_0 D \to H^1(M_0, \Theta_0)$ *is injective, then* $M_0 \not\cong M_t \Rightarrow$

$$\text{either} \quad dim(D) \leq 2, \quad \text{or} \quad 3 \leq dim(D) \leq r.$$

(Here r denotes the constant rank of $R^0 f_*(\Theta)\mid_{D\setminus\{0\}}$, corresponding to the dimension of the Lie algebra, $H^0(M_t, \Theta_t)$, for all $t \neq 0$.)

(3)(cf. chapter four) We now turn to the case of a family, X, of germs of deformations of a singular curve, C, defined in $\mathbf{C}^2$ by the zero locus of a weighted homogeneous polynomial,

$$\phi(x, y) = \prod_{i=1}^{k} (y^p - \lambda_i x^q).$$

Here, the $\mathbf{C}^*$-action, defined by

$$(x, y) \mapsto (t^p x, t^q y),$$

is the unique automorphism of C, and does not extend to neighbouring fibers. Methods applied to the family of simultaneous resolutions of fibers of X, which are pseudoconvex rather than compact, differ in certain respects from the smooth case (cf. [15], [16]). In particular, we explicitly construct the space of "infinitesimal deformations", $\mathcal{D}$, of the embedded resolution, $(M, \tilde{C})$, of C, such that

$$\mathcal{D} = \bigoplus_{1 \leq k \leq n} H^1(A_1, \mathcal{L}_k),$$

where A_1 denotes that irreducible component of the exceptional set in M for which the self-intersection is minus one. The line bundles, $\mathcal{L}_k, 1 \leq k \leq n$, will be defined in chapter four. The relationship between structure-jumping in the resultant family, $\mathcal{M}$, of resolutions, and the $\mathbf{C}^*$-action of $\tilde{C}$ is also examined, though a conclusive answer is not reached.

(4) (cf. Appendix). The cohomology group, $H^1(D \setminus \{0\}, R^0 f_*(\Theta))$, plays

a central role in the results of chapters two and three. Since $R^0 f_*(\Theta)$ is a coherent sheaf, it follows from work of Andreotti and Grauert, [1], that the above cohomology, is finite-dimensional over $\mathbf{C}$. We are led to consider a relatively compact domain, D, inside a complex manifold, $\mathcal{M}$, for which the boundary, ∂D, consists of multiple components, C_i, of varying pseudoconvexity, q_i, as well as components, C_j, of varying pseudoconcavity, q_j. The natural extension of [1], theorem 11, is then

Theorem: *If $\mathcal{F} \to \mathcal{M}$ is a coherent analytic sheaf, $q = sup_{1\leq i\leq r} q_i, \hat{q} = sup_{r+1\leq j\leq s} q_j$, then*

$$dim_{\mathbf{C}} H^k(D, \mathcal{F}) < +\infty, \qquad q \leq k \leq dh(\mathcal{F}) - \hat{q} - 1,$$

where $dh(\mathcal{F}) = inf_{p\in\mathcal{M}} dh_p(\mathcal{F})$, denotes the homological dimension of $\mathcal{F}$.

A straightforward extension of Andreotti-Tomassini, [2], theorem 4, on the vanishing of certain cohomology groups, is also obtained for $D \subset\subset \mathcal{M}$ above:

Theorem: *If $D = \cap_{1\leq\mu\leq s} D_\mu, \mathcal{M} = \cup_{1\leq\mu\leq s} D_\mu$, where D_i is a "strictly q_i-complete space", $1 \leq i \leq r$, and D_j is a "strictly q_j-pseudoconcave space", $r+1 \leq j \leq s$, then for any metrically pseudoconvex line bundle, $L \to \mathcal{M}$, and any coherent sheaf, $\mathcal{F} \to \mathcal{M}$, there exists $\hat{\nu} \in \mathbf{Z}_+$ such that*

$$H^k(D, \mathcal{F} \otimes \mathcal{O}(L^{\hat{\nu}})) = 0, \qquad q \leq k \leq dh(\mathcal{F}) - \hat{q} - 2.$$

(For the definition of all terms above, see the appendix).

Chapter 2

A Regularity Theorem for Deformations of Compact Surfaces

2.1 Locally trivial families on a punctured domain

In questions related to the theory of deformation of complex structures on a complex manifold, it is usual to start with a fixed manifold M, and consider a family $\mathcal{M} \xrightarrow{f} \mathcal{P}$, with fibers $f^{-1}(p) = M_p$, corresponding to structures which neighbour $M_q = M$. If M is a compact Riemann Surface, small, nontrivial deformations yield a continuum of different Riemann Surfaces. Kodaira and Spencer [12] observed, in their paper of 1958, that for compact, complex manifolds of higher dimension. however, small deformations might yield only one biholomorphically distinct structure, a phenomenon known as 'structure jumping'. The main emphasis of this chapter is on a question of the converse type: given a holomorphic family $\mathcal{M} \xrightarrow{f} D$, where D is a domain containing the origin in $\mathbf{C}^m$, such that the restriction over the punctured domain $D \setminus \{0\}$ is locally trivial, under what circumstances does f define a trivial deformation over D ? An example of this type of problem is a conjecture of Kodaira and Spencer [12], recently proved by Siu [22]: if $D \subset \mathbf{C}$, and $f^{-1}(t) \cong \mathbf{P}_n$, (the complex projective space), for all $t \in D \setminus \{0\}$, then $f^{-1}(0) \cong \mathbf{P}_n$. Local triviality implies that over sufficiently small neighbourhoods, $U \subseteq D \setminus \{0\}$, holomorphic vector fields in D may be lifted (non-uniquely) to the total space $\mathcal{M}$, inducing a local equivalence of structures along the fibers. If the parameter space D has dimension at least two, 'patching together' these local liftings gives rise to obstructions in $H^1(D \setminus \{0\}, R^0 f_*(\Theta))$, where $R^0 f_*(\Theta)$ denotes the direct image of the sheaf of germs of holomorphic vector fields lying parallel to the fibers of $\mathcal{M}$. When the obstruction vanishes, the lifting extends globally from $D \setminus \{0\}$ to $\mathcal{M} \setminus M_0$. But the holomorphic vector field so defined on the total space must now extend across codimension at least two, by Hartogs' Theorem, hence $\mathcal{M} \xrightarrow{f} D$ is trivial. In the following sections, a class of compact, complex surfaces will be examined, for which an important subsheaf of $R^0 f_*(\Theta) \mid_{D \setminus \{0\}}$ is shown to have the structure of a Whitney sum of holomorphic line bundles when $dim_{\mathbf{C}}(D) \geq 4$. The class considered is that of holomorphic fiber bundles over $\mathbf{P}_1$. The essential idea is to show that on each fiber of $f \mid_{\mathcal{M} \setminus M_0}$, an element of the appropriate Lie Algebra (ie., a holomorphic vector field), is determined by its 'values' at a finite number of uniformly marked points. This condition implies vanishing of the obstructions, via Scheja's work on extension of cohomology groups [20]. It will then follow that locally trivial deformations of these surfaces over $D \setminus \{0\}$ extend trivially to D when $dim_{\mathbf{C}}(D) \geq 4$. In section two, this result will be established for bundles in which the fibers have genus zero, namely the Hirzebruch Surfaces, while section three will cover the case of Cartesian Products. Section four will be devoted to non-trivial bundles with elliptic and hyperelliptic fibers. The treatment of the elliptic case will in fact be applied to the family of (primary) Hopf Surfaces, in which many of the former are contained. In section five, the preceding results will be extended to deformations in which the generic fiber is a surface of one of the above types, blown up at one or more distinct points, under certain restrictions. Finally, section six treats an example of a locally trivial family of compact, complex surfaces over $D \setminus \{0\}$, with non-trivial extension to D, where $dim_{\mathbf{C}}(D) = 2$. No example of this behaviour when $dim_{\mathbf{C}}(D) = 3$ is known to the author.

2.2 Bundles with projective fibers

Let Σ_n denote the n-th Hirzebruch Surface, and $\mathcal{G}$ the Lie Algebra of $Aut(\Sigma_n)$, ie $\mathcal{G} = H^0(\Sigma_n, \mathcal{T}\Sigma_n)$, where $\mathcal{T}\Sigma_n$ represents the tangent sheaf of Σ_n. Given that $\pi : \Sigma_n \rightarrow E$ corresponds to the fiberwise compactification of $\mathcal{O}_{\mathbf{P}_1}(-n)$, where E denotes the base space (isomorphic to $\mathbf{P}_1$), let $\mathcal{G}_F$ represent the Lie subalgebra corresponding to vector fields parallel to the fibers of Σ_n.

(*Remark:* $\mathcal{G}_F$ is in fact canonical, for suppose $F \subset \Sigma_n$ is a holomorphic curve corresponding to a fiber, hence $F \cdot F = 0, F \cdot E = 1, E \cdot E = -n$. Let F' denote the image of F under some biholomorphism of Σ_n. Now F, E generate $H_2(\Sigma_n, \mathbf{Z}) \Rightarrow F' = lF + mE$, for some $l, m \in \mathbf{Z}$. Since intersection numbers are preserved under biholomorphism, it follows that $F' \cdot F' = 0, F' \cdot E = 1$ (note E is canonical, since it is the unique holomorphic curve of self-intersection, $-n$ in Σ_n). From these follow the relations,

$$2lm - nm^2 = 0, \qquad l - nm = 1.$$

Now $n = 0$ implies either $l = 0$ or $m = 0$. $n \geq 1 \Rightarrow m = 0$, or $m = -\frac{2}{n}, l = -1$. The latter situation, however, is impossible, since it follows that $F \cdot F' = -\frac{2}{n} < 0$. Hence F' must represent the same homology class, and is therefore a fiber.)

Now the sequence

$$0 \longrightarrow \mathcal{G}_F \longrightarrow \mathcal{G} \xrightarrow{\pi_*} \mathcal{G}_E \longrightarrow 0$$

is exact [10], and for all $\sigma \in \mathcal{G}, \sigma = \sigma_F + \sigma_E$, where $\sigma_F \in \mathcal{G}_F, \pi_*(\sigma_E) \in \mathcal{G}_E$. Explicitly,

$$\sigma_F = (\Sigma_{j=0}^{n} \alpha_j z^j)\zeta^2 \frac{\partial}{\partial \zeta} + \beta \zeta \frac{\partial}{\partial \zeta}, \qquad \alpha_i, \beta \in \mathbf{C}, \text{ and}$$

$$\sigma_E = (az^2 + bz + c)\frac{\partial}{\partial z} - az\zeta \frac{\partial}{\partial \zeta}, \qquad a, b, c \in \mathbf{C},$$

with respect to a coordinate system (z, ζ) on Σ_n. In the following, "E" will also be understood to refer to the rational curve lying in Σ_n which corresponds to E above. For all $p \in \Sigma_n, \mathcal{F}_p$ will denote the subspace of $T_p\Sigma_n$ lying parallel to the fiber through p.

Proposition 1 *There exist points,* $p_i \in \Sigma_n \setminus E, 1 \leq i \leq n+2$, *such that for all* $\sigma \in \mathcal{G}_F$, *the vector field is uniquely determined by* $\{\sigma(p_i) \in \mathcal{F}_{p_i} \mid 1 \leq i \leq n+2\}$. *Conversely, given* $s_i \in \mathcal{F}_{p_i}, 1 \leq i \leq n+2$, *there exists* $\sigma \in \mathcal{G}_F$ *such that* $\sigma(p_i) = s_i$.

Proof: Assume for the moment that it is sufficient to choose the first $n+1$ points on distinct fibers of $\Sigma_n \setminus E$, ie., $i \neq j \Rightarrow \pi(p_i) \neq \pi(p_j), 1 \leq i, j \leq n+1$. Choose $\sigma \in \mathcal{G}_F$, and define

$$\Psi_\sigma : \mathcal{G}_F \times \Sigma_n \to \mathbf{C}.$$

such that

$$\Psi_\sigma(\tau, p) = \tau(p) - \sigma(p).$$

Let $\Psi_{\sigma,i} = \Psi_\sigma(*, p_i), 1 \leq i \leq n+1$, and let $\mathcal{G}_i \subset \mathcal{G}_F$ denote the Lie subalgebra corresponding to the kernel of $\Psi_{\sigma,i}$. Note that $dim_{\mathbf{C}}(\mathcal{G}_F) = n+2$, hence $dim_{\mathbf{C}}(\mathcal{G}_i) = n+1, 1 \leq i \leq n+1$. Moreover, suppose that the initial assumption implies

$$dim_{\mathbf{C}}(\cap_{1 \leq i \leq n+1} \mathcal{G}_i) = 1.$$

Now choose ξ, a generator of $\cap_{1 \leq i \leq n+1} \mathcal{G}_i$, and let Δ denote the zero locus of ξ. Note that $\Delta \subset \Sigma_n$ is a divisor, hence $p_{n+2} \in \Sigma_n \setminus \Delta, \tau \in \cap_{1 \leq i \leq n+1} \mathcal{G}_i \cap ker(\Psi_{\sigma,n+2}) \Rightarrow \tau - \sigma \equiv 0$, ie., $\tau = \sigma$, from which the uniqueness of σ follows.

The sufficiency assumption for the first $n+1$ points will now be justified, and at the same time, it will be observed that $\Delta \cap \Sigma_n \setminus E$ is a smooth hypersurface, a fact which will be required presently. To show this, let $N \subset \Sigma_n$ be a coordinate neighbourhood, with $p_i = (z_i, \zeta_i) \in N \setminus E, 1 \leq i \leq n+1, p_{n+2} = (z, \zeta)$

(note $\zeta_i \neq 0, i \neq j \Rightarrow z_i \neq z_j$). Let ξ, as defined above, be given by

$$(\alpha(z)\zeta^2 + \beta\zeta)\frac{\partial}{\partial\zeta},$$

where $\alpha(z) = \Sigma_{j=0}^{n}\alpha_j z^j$ in the above notation. Then $a_i = \alpha(z_i) \Rightarrow a_i\zeta_i^2 + \beta\zeta_i = 0, 1 \leq i \leq n+1$. Note that $\alpha(z)$ is uniquely determined by the a_i, via the interpolation formula

$$\alpha(z) = \Sigma_{j=1}^{n+1} a_i \lambda_i(z), \qquad \lambda_i(z) = \frac{P(z)}{P'(z_i)(z - z_i)},$$

and $P(z) = (z - z_1)...(z - z_{n+1})$. Clearly $p_i \in \Delta, 1 \leq i \leq n+1$. Hence, if $p_{n+2} \in \Delta$, we may write the values $\xi(p_i) = 0, 1 \leq i \leq n+2$ in N, in terms of a linear system of equations

$$\mathbf{A}\underline{X} = \underline{0},$$

where $\underline{X} = (a_1, ..., a_{n+1}, \beta), \underline{0} = (0, ..., 0, 0)$, and $\mathbf{A}$ is given by

$$\begin{pmatrix} \zeta_1^2 & 0 & \ldots & 0 & \zeta_1 \\ 0 & \zeta_2^2 & \ldots & 0 & \zeta_2 \\ . & . & . & . & . \\ 0 & 0 & \ldots & \zeta_{n+1}^2 & \zeta_{n+1} \\ \lambda_1(z)\zeta^2 & \lambda_2(z)\zeta^2 & \ldots & \lambda_{n+1}(z)\zeta^2 & \zeta \end{pmatrix}.$$

Note that $dim_{\mathbf{C}} ker(\mathbf{A}) \leq 1$. If p_{n+2} is chosen so that $det(\mathbf{A}) \neq 0$, then $\underline{X} = \underline{0}$. Conversely, p_{n+2} satisfies $det(\mathbf{A}) = 0 \Rightarrow ker(\mathbf{A})$ contains non-trivial solutions of the form $(a_1, ..., a_{n+1}, \beta)$, ie., $\xi \not\equiv 0$. Hence $det(\mathbf{A}) = 0$ corresponds precisely to the defining equation of $\Delta \cap N$.

Lemma 1

$$det(\mathbf{A}) = \kappa\zeta(\zeta\Lambda(z) + \kappa),$$

where

$$\kappa = \Pi_{i=1}^{n+1}\zeta_i, \qquad \Lambda(z) = \Sigma_{i=1}^{n+1}(\frac{\kappa}{\zeta_i})\lambda_i(z).$$

Proof: Compute $det(\mathbf{A})$ above.

Let $\psi(z, \zeta) = \zeta\Lambda(z) + \kappa$, then $\frac{\partial\psi}{\partial z} = \zeta\Lambda'(z), \frac{\partial\psi}{\partial\zeta} = \Lambda(z), \kappa \neq 0 \Rightarrow \Delta \cap (N \setminus E)$ is smooth. Since the above argument applies to any neighbourhood, N, such that $p_i \in N \setminus E, 1 \leq i \leq n+2$, it follows that $\Delta \cap \Sigma_n \setminus E$ is a smooth hypersurface. Moreover, the existence of a σ such that $\sigma(p_i) = s_i, 1 \leq i \leq n+2$, follows from the invertibility of $\mathbf{A}$. The proof of proposition 1 is now complete.

Now consider a holomorphic family, $\mathcal{M} \xrightarrow{f} D, 0 \in D \subseteq \mathbf{C}^m$, with fibers $M_t = f^{-1}(t) \cong \Sigma_n$ such that $f \mid_{\mathcal{M} \setminus M_0}$ is locally trivial, ie., there exists a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $D \setminus \{0\}$, such that for each $i \in I$, there is a biholomorphism

$$\Phi_i : U_i \times \Sigma_n \longrightarrow f^{-1}(U_i),$$

with $\Phi_i(t, \Sigma_n) = M_t$ for all $t \in U_i$. Let $\vec{v} \in \Gamma(U_i, \mathcal{T}D)$ be a holomorphic vector field, then the vanishing of the Kodaira-Spencer map,

$$\rho : \Gamma(U_i, \mathcal{T}D) \to R^1 f_*(\Theta) \mid_{U_i},$$

implies the existence of a local lifting, $f^*(\vec{v}) \in \Gamma(f^{-1}(U_i), \mathcal{T}\mathcal{M})$, which induces Φ_i.

Let $E_t = \Phi_i(t, E)$ for all $t \in U_i, i \in I, \mathcal{E} = \cup_{t \in D \setminus \{0\}} E_t$. Since $E_t \subset M_t$ is the unique curve such that the self-intersection, $E_t \cdot E_t = -n$, it follows that E_t is stable under biholomorphism, hence $\mathcal{E} \subseteq \mathcal{M} \setminus M_0$ is a smooth subvariety of codimension one.

Lemma 2 *If $m = dim(D) \geq 4$, then the closure $\bar{\mathcal{E}}$ of $\mathcal{E}$ in $\mathcal{M}$ is an analytic subvariety. Moreover $E_0 = \bar{\mathcal{E}} \cap M_0$ is a proper subvariety of M_0.*

Proof: That $\bar{\mathcal{E}}$ is analytic follows immediately from the Remmert-Stein Extension Lemma [9], since $\mathcal{E}$ is a hypersurface, and $m \geq 2$. Note that $\bar{\mathcal{E}}$ is also a hypersurface. Suppose $M_0 \subseteq \bar{\mathcal{E}}$, ie the ideal

$$\mathcal{I}_{\bar{\mathcal{E}}} \subseteq \mathcal{I}_{M_0} = (t_1, ..., t_m).$$

For any element $g \in \mathcal{I}_{\bar{\mathcal{E}}}$, it follows that $g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \in (t_1, ..., t_m)$, where x, y represent any local system of coordinate functions along the fibers of $\mathcal{M}$. Now let g be a generator for $\mathcal{I}_{\bar{\mathcal{E}}}$. If $\bar{\mathcal{E}}_1$ denotes the locus of (g, g_x, g_y), then it follows that $M_0 \subseteq \bar{\mathcal{E}}_1$. But E_t is smooth for all $t \neq 0$, hence $\bar{\mathcal{E}}_1 \cap \mathcal{M} \setminus M_0 = \emptyset$, ie $\bar{\mathcal{E}}_1 = M_0$. Now $codim_{\mathcal{M}}(\bar{\mathcal{E}}_1) \leq 3$, while $codim_{\mathcal{M}}(M_0) = m \geq 4$, which is a contradiction. This completes the proof of lemma 2.

Our aim now is to apply proposition 1 uniformly to the fibers of $\mathcal{M} \setminus M_0$, for which the following "marking" procedure is needed. Choose $p \in M_0 \setminus E_0$, and a neighbourhood V of p which is relatively compact in $\mathcal{M} \setminus \bar{\mathcal{E}}$, moreover assume that $dim(D) = m \geq 4$.

Step one: Consider $U_0 = f(V)$, and define a holomorphic map $\phi_1 : U_0 \rightarrow V$ such that $f \circ \phi_1 = id_{U_0}$. Recall that each M_t is a $\mathbf{P}_1$-bundle over E_t for

$t \neq 0$, hence let $F_1(t)$ denote the unique fiber of M_t containing $\phi_1(t)$. Now $F_1 = \cup_{t \in U_0 \setminus \{0\}} F_1(t)$ is another smooth hypersurface in $V \setminus M_0$, to which the argument of lemma 2 applies, hence the closure $\bar{F}_1 \cap M_0$ is a proper subvariety.

Step 2: Define $\phi_2 : U_0 \to V \setminus \bar{F}_1$ such that $f \circ \phi_2 = id$, and for all $t \neq 0$ let $F_2(t)$ denote the unique fiber containing $\phi_2(t)$. The corresponding subvariety F_2 is once again smooth in $V \setminus M_0$, hence it extends appropriately across the central fiber of $\mathcal{M}$....

Step $n+1$: Define $\phi_{n+1} : U_0 \to V \setminus \cup_{1 \leq j \leq n} \bar{F}_j$ such that $f \circ \phi_{n+1} = id$, and $\bar{F}_{n+1}$ accordingly.

Step $n+2$: Recall $\Delta \subseteq \Sigma_n$ from the proof of proposition 1. Define $\Delta_i(t)$ analogously in terms of the points $p_j(t) = \Phi_i^{-1}(\phi_j(t)), 1 \leq j \leq n+1$ for all $t \in U_i \cap U_0, i \in I$, and let $\Delta_i^{'}(t) = \Phi_i(t, \Delta_i(t))$, and $\Delta^{'} = (\cup_{t \in U_0 \setminus \{0\}} \Delta_i^{'}(t)) \cap V \setminus M_0$. $\Delta^{'}$ so defined is a hypersurface in $V \setminus M_0$, hence smoothness of $\Delta_i(t) \Rightarrow \Delta^{'}$ is also smooth, and consequently lemma 2 applies once more to its closure in V. Finally, consider $\phi_{n+2} : U_0 \to V \setminus \bar{\Delta}^{'} \cup_{1 \leq j \leq n+1} \bar{F}_j$, which completes the marking of each fiber in $\mathcal{M} \setminus M_0$.

The above procedure is preliminary to a discussion of the group of obstructions to a global trivialization of the deformation $\mathcal{M} \setminus M_0$. Recall that for each $i \in I$, the biholomorphism Φ_i is induced by a lifting $f_i^*(\vec{v})$ for some $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$. Now, for $t \in U_i \cap U_0$, consider

$$\delta_k(t) = (\phi_k)_*(\vec{v}(t)) - f_i^*(\vec{v})_{\phi_k(t)} \in T_{\phi_k(t)} M_t, \qquad 1 \leq k \leq n+2.$$

Recall the Lie Algebra, $\mathcal{G}$, of Σ_n and the decomposition, $\sigma = \sigma_E + \sigma_F$ for all

$\sigma \in \mathcal{G}$. Choose any three of the points, $\phi_k(t)$ (eg., let $k = 1, 2, 3$), and for all $t \in U_i \cap U_0, i \in I$, let σ_i denote a holomorphic t-parameter family of vector fields in $\mathcal{G}$ such that

$$\sigma_i(p_k(t)) = (\Phi_i)_*^{-1}(\delta_k(t)), \qquad k = 1, 2, 3.$$

Now $\hat{f}_i^*(\vec{v}) = f_i^*(\vec{v}) + (\Phi_i)_*(\sigma_i)$ has the property that

$$(\phi_k)_*(\vec{v}(t)) - \hat{f}_i^*(\vec{v})_{\phi_k(t)} = \hat{\delta}_k(t), \qquad 1 \leq k \leq n+2,$$

where $\hat{\delta}_k(t) = 0, k = 1, 2, 3$. Similarly, let $\hat{f}_j^*(\vec{v}) = f_j^*(\vec{v}) + (\Phi_j)_*(\sigma_j)$ for $j \in I$ such that $U_j \cap U_i \neq \emptyset$, then for all $t \in U_0 \cap U_i \cap U_j$,

$$(\hat{f}_i^*(\vec{v}) - \hat{f}_j^*(\vec{v}))_{\phi_k(t)} = [\hat{f}_i^*(\vec{v})_{\phi_k(t)} - (\phi_k)_*(\vec{v}(t))] - [\hat{f}_j^*(\vec{v})_{\phi_k(t)} - (\phi_k)_*(\vec{v}(t))] = 0,$$

k= 1,2,3. Consider $W = (\cup_{1 \leq k \leq 3} \phi_k(U_0)) \cap V \setminus M_0$. Then if $R^0 f_*(\Theta)$ denotes the zeroth direct image of the subsheaf of $T\mathcal{M}$ corresponding to germs of vector fields parallel to the fibers of $\mathcal{M}$, let $\mathcal{W}$ represent the subsheaf of $R^0 f_*(\Theta) \mid_{U_0 \setminus \{0\}}$ having sections τ such that $\tau \mid_W \equiv 0$. It follows that the cocycle determined by

$$\hat{f}_i^*(\vec{v}) - \hat{f}_j^*(\vec{v}) \in \Gamma(U_i \cap U_j, \mathcal{W}),$$

for all $i, j \in I$, determines an obstruction in $H^1(U_0 \setminus \{0\}, \mathcal{W})$.

Proposition 2 *$\mathcal{W}$ is free.*

Proof: Recall the Lie subalgebra, $\mathcal{G}_F$, of proposition 1, and let $\mathcal{G}_{F(t)}$ denote the t-parameter family of Lie algebras determined by $(\Phi_i)_*(\mathcal{G}_F), t \in U_i \cap U_0, i \in I$.

Consequently, each stalk $\mathcal{W}_t = \{\sigma \in \mathcal{G}_{F(t)} \mid \sigma(\phi_k(t)) = 0, k = 1,2,3\}$. Now let $\mathcal{F}_k(t) = T_{\phi_k(t)}F_k(t), 4 \leq k \leq n+2$, then $\mathcal{F}_k$ defines a locally free sheaf of rank one on $U_0 \setminus \{0\}$. By proposition 1, there exists an isomorphism

$$\mathcal{W}_t \cong \bigoplus_{4\leq k\leq n+2} \mathcal{F}_k(t)$$

for all $t \in U_0 \setminus \{0\}$, hence $\mathcal{W} \cong \bigoplus_{4\leq k\leq n+2} \mathcal{F}_k$, considered as the Whitney sum of line bundles $\mathcal{F}_k$. If $U_0 \subseteq \mathbf{C}^m, m \geq 4$, is assumed explicitly to be a ball, then a well known result, due to Scheja [20], indicates that $H^1(U_0 \setminus \{0\}, \mathcal{O}) = 0$. Moreover, since $U_0 \setminus \{0\}$ has the homotopy type of a sphere, $\mathbf{S}^{2m-1}$, it follows that $H^2(U_0 \setminus \{0\}, \mathbf{Z}) = 0$, and hence exactness of the sequence

$$H^1(U_0 \setminus \{0\}, \mathcal{O}) \longrightarrow H^1(U_0 \setminus \{0\}, \mathcal{O}^*) \longrightarrow H^2(U_0 \setminus \{0\}, \mathbf{Z})$$

implies that $H^1(U_0 \setminus \{0\}, \mathcal{O}^*) = 0$, ie., every line bundle over $U_0 \setminus \{0\}$ is trivial, and extends freely, via Hartogs' theorem, to U_0. It follows at once that $\mathcal{F} = \bigoplus_{4\leq k\leq n+2} \mathcal{F}_k$ also extends freely across the origin, but now $\mathcal{F} \mid_{U_0\setminus\{0\}} \cong \mathcal{W}$, and the result is proven.

An immediate corollary of proposition 2, and the results of Scheja cited above, is that $H^1(U_0 \setminus \{0\}, \mathcal{W}) = 0$. It now remains simply to state the main result of this note.

Theorem 1 *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, such that $f^{-1}(t) \cong \Sigma_n, n \geq 1$, for all $t \in D \setminus \{0\}$, then $f^{-1}(t) \ncong f^{-1}(0) \Rightarrow dim(D) \leq 3$.*

Proof: Suppose $dim(D) \geq 4$, then since $H^1(U_0 \setminus \{0\}, \mathcal{W}) = 0$, it follows that the obstruction to a global lifting, $f^*(\vec{v})$, of $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$ to $\mathcal{M} \setminus M_0$ vanishes. But then Hartogs' theorem implies that the holomorphic vector field $f^*(\vec{v})$ must extend uniquely across M_0, thereby negating structure jumping, which is a contradiction.

2.3 Cartesian products

We turn now to the case of holomorphic families, the generic fibers of which all have the structure of a Cartesian product, $X \times \mathbf{P}_1$, where X denotes a Riemann Surface of genus g. The following simplified marking of $\mathcal{M} \setminus M_0$ is required. As before, consider $p \in M_0$, a relatively compact neighbourhood $V \subseteq \mathcal{M}$, and $U_0 = f(V)$. Given a holomorphic map $\phi_1 : U_0 \to V, f \circ \phi_1 = 1$, then for each $t \in U_0 \setminus \{0\}$, let $\chi_1(t)$ denote the unique pair of intersecting lines in $f^{-1}(t)$ containing $\phi_1(t)$. If $V' = V \setminus \phi_1(U_0)$, then the hypersurface, $\chi_1 = \cup_{t \in U_0 \setminus \{0\}} \chi_1(t) \subset \mathcal{M} \setminus M_0$, is such that $\chi_1 \cap (V' \setminus M_0)$ is smooth. By the argument of lemma 2, it follows that the topological closure of χ_1 in V' extends properly, as a hypersurface, across $M_0 \cap V'$ when $dim(D) \geq 4$. Now let $\bar{\chi}_1$ denote the closure in V, and define $\phi_2 : U_0 \to V \setminus \bar{\chi}_1$, and $\phi_3 : U_0 \to V \setminus (\bar{\chi}_1 \cup \bar{\chi}_2)$ accordingly.

Theorem 2 *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, such that $f^{-1}(t) \cong X \times \mathbf{P}_1$ for all $t \neq 0$, then $f^{-1}(t) \ncong f^{-1}(0) \Rightarrow dim(D) \leq 3$.*

Proof: Consider the following cases.

(i) $g = 0$. Note that for $\mathbf{P}_1 \times \mathbf{P}_1$, the Lie Algebra $\mathcal{G} \cong \mathcal{G}_{\mathbf{P}_1} \oplus \mathcal{G}_{\mathbf{P}_1}$, hence each $\sigma \in \mathcal{G}$ is uniquely determined by its "values" at three points (z_i, ζ_i) in a coordinate neighbourhood of $\mathbf{P}_1 \times \mathbf{P}_1$, such that $i \neq j \Rightarrow z_i \neq z_j, \zeta_i \neq \zeta_j$. The sheaf $\mathcal{W}$, analogous to the one defined in section two, is consequently trivial (ie., $\mathcal{W} \equiv 0$), and the obstruction to a global lifting $f^*(\vec{v})$ vanishes automatically.

(ii) $g = 1$. Here X represents an elliptic curve, hence $\mathcal{G} \cong \mathcal{G}_{\mathbf{P}_1} \oplus \mathbf{C}$. With respect to the marking procedure above, $\mathcal{W}_t \cong T_{p_2(t)}\mathbf{P}_1 \oplus T_{p_3(t)}\mathbf{P}_1$, where $p_k(t) = \Phi_i^{-1}(\phi_k(t)), i \in I, t \in U_i \cap U_0$, and hence $\mathcal{W}$ is shown to be free by the argument of proposition 2, section two.

(iii) $g \geq 2$. For the case where X is hyperelliptic, $\mathcal{G} \cong \mathcal{G}_{\mathbf{P}_1}$,

$$R^0 f_*(\Theta)_t = \mathcal{W}_t \cong \bigoplus_{1 \leq k \leq 3} T_{p_k(t)}\mathbf{P}_1,$$

for $t \in U_i \cap U_0, i \in I$, and $\mathcal{W}$ is once more free by the argument of proposition 2, section two.

It will be remarked in the following section that in fact all hyperelliptic bundles over $\mathbf{P}_1$ are of this type.

2.4 Hopf surfaces and bundles with hyperelliptic fibers

The case of elliptic fiber bundles, other than Cartesian products, is partially treated in the following discussion of holomorphic families, the generic fibers of which are (primary) Hopf Surfaces (ie., homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^3$). In general, all non-trivial elliptic bundles over $\mathbf{P}_1$ are Hopf Surfaces, though these may only admit a finite, unramified cover which is primary. Kodaira and Spencer treat the family of Hopf Surfaces, [12], as a fundamental example of structure jumping. Each fiber is determined as the quotient of $W = \mathbf{C}^2 \setminus \{0\}$ modulo the action of an infinite cyclic group, G_μ, generated by a matrix

$$\mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

such that $|\alpha + \delta| > 3, |(\alpha - \delta)^2 + 4\beta\gamma| < 1$, (these inequalities ensure that the eigenvalues of μ have norm greater than one). The parameter space $P \subset \mathbf{C}^4$, so defined, is then fibered over $Q \subset \mathbf{C}^2$ by the map $\psi(\mu) = (\epsilon, \Delta)$, where $\epsilon = \frac{1}{2}(\alpha + \delta), \Delta = \frac{1}{4}(\alpha - \delta)^2 + \beta\gamma$, hence Q is defined by $|\epsilon| > \frac{3}{2}, |\Delta| < \frac{1}{4}$. Modulo conjugation with elements in $GL(2, \mathbf{C})$, the elements of P fall into three equivalence classes:

$\Delta \neq 0$. For fixed $\epsilon \in Q, \psi^{-1}(\epsilon, \Delta)$ consists of matrices μ equivalent to

$$\begin{pmatrix} \epsilon + \sqrt{\Delta} & 0 \\ 0 & \epsilon - \sqrt{\Delta} \end{pmatrix}.$$

This constitutes the largest equivalence class.

$\Delta = 0$. For fixed $(\epsilon, 0) \in Q, \psi^{-1}(\epsilon, 0)$ corresponds to a quadratic surface $S_\epsilon \subset P$, defined by the equations

$$\alpha + \delta - 2\epsilon = 0, \qquad (\alpha - \delta)^2 + 4\beta\gamma = 0,$$

with a singular point corresponding to $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$, which lies in its own equivalence class. Each smooth point, $\mu \in S_\epsilon$, however, is equivalent to $\begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}$.

It is proven, moreover, in [12], theorem 15.1, that the three equivalence classes on P correspond precisely to biholomorphic equivalence classes on the fibers of the family, ie.,if $M_t \cong W/G_{\mu(t)}$, and $M_s \cong W/G_{\mu(s)}$, then $M_t \cong M_s$ if and only if there exists $g \in GL(2, \mathbf{C})$ such that $\mu(t) = g^{-1}\mu(s)g$. There are consequently three cases to be considered in the discussion of a family $\mathcal{M} \xrightarrow{f} D$, locally trivial over $D \setminus \{0\}$, for which the generic fiber $M_t = f^{-1}(t)$ is a Hopf Surface. It should be remarked, however, that the above family does not correspond to a complete classification. Within the versal family (cf. Wehler,

[26]), the following cases correspond to Hopf Surfaces of types IV, II_b, and II_c.

(a) $M_t \cong H_\epsilon = W/G_\mu$, where $\mu = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$, for some fixed ϵ. Here μ belongs to the center of $GL(2,\mathbf{C}) \Rightarrow M_t \cong W/G_\mu$ for all $t \neq 0$. The complex structure is strictly constant, ie., $\mathcal{M} \setminus M_0 = M_{t_0} \times D \setminus \{0\}$, for each $t_0 \neq 0$. Hence, for $\vec{v} \in \Gamma(D, \mathcal{T}D)$, the natural lifting $f^*(\vec{v})$ to $\mathcal{M} \setminus M_0$ is both holomorphic and bounded, and extends to $\mathcal{M}$ when $dim(D) \geq 1$, via a simple application of the Riemann Extension Theorem.

(b) $M_t \cong W/G_\mu$, where $\mu = \begin{pmatrix} \epsilon & 1 \\ 0 & \epsilon \end{pmatrix}$. Holomorphic vector fields, σ, on W/G_μ are determined as vector fields on W, the universal covering space, which are invariant under the action of G_μ (cf. [10], [12]). Hence

$$\sigma = (c_1 z_1 + c_2 z_2)\frac{\partial}{\partial z_1} + c_1 z_2 \frac{\partial}{\partial z_2}, \qquad c_1, c_2 \in \mathbf{C}.$$

Let $p = (u_1, u_2) \in W$, then $\sigma(p) = \eta\frac{\partial}{\partial z_1} + \zeta\frac{\partial}{\partial z_2} \Rightarrow c_1, c_2$ are uniquely determined if $u_2 \neq 0$. Note that under the covering map $\pi : W \to W/G_\mu$, the image of the line $z_2 = 0$ is an elliptic curve embedded in this surface (cf [4], [12]). Now in each fiber, $M_t = W/G_{\mu(t)}$, of $\mathcal{M} \setminus M_0$, the elliptic curve, C_t, is canonically determined as the image of the corresponding eigenspace, in W, of $\mu(t) = g_t^{-1}\mu g_t$. Hence $\cup_{t \in D\setminus\{0\}} C_t = \mathcal{C}$ determines a smooth analytic subvariety of $\mathcal{M} \setminus M_0$, which extends properly to $\bar{\mathcal{C}} \subset \mathcal{M}$ by the argument of proposition 2, when $dim(D) \geq 4$.

Now let $p \in M_0 \setminus C_0, V \subset \mathcal{M} \setminus \bar{\mathcal{C}}$ be a relatively compact neighbourhood of p, and $U_0 = f(V)$. Consider a holomorphic map $\phi : U_0 \to V$, and for $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$, define

$$\delta(t) = \phi_*(\vec{v}(t)) - f_i^*(\vec{v})_{\phi(t)} \in T_{\phi(t)} M_t$$

for $t \in U_i \cap U_0, i \in I$, by analogy with the argument of section two. Similarly, if

$$\Phi_i : U_i \times W/G_\mu \to f^{-1}(U_i)$$

is the biholomorphism induced by the lifting $f_i^*(\vec{v})$, and $p(t) = \Phi_i^{-1}(\phi(t)) \in W/G_\mu$, let σ_i denote a holomorphic t-parameter family of vector fields on W/G_μ such that $\sigma_i(p(t)) = (\Phi_i)_*^{-1}(\delta(t))$, for all $t \in U_i \cap U_0, i \in I$. Hence

$$\hat{f}_i^*(\vec{v}) = f_i^*(\vec{v}) + (\Phi_i)_*(\sigma_i)$$

has the property that

$$\phi_*(\vec{v}) - \hat{f}_i^*(\vec{v})_{\phi(t)} = 0.$$

Once again, by analogy with the argument of section two, for $t \in U_0 \cap U_i \cap U_j \neq \emptyset$, it follows that

$$(\hat{f}_i^*(\vec{v}) - \hat{f}_j^*(\vec{v}))_{\phi(t)} = 0.$$

If $W = \phi(U_0) \cap V \setminus M_0$, and $\mathcal{W} \subseteq R^0 f_*(\Theta) \mid_{U_0 \setminus \{0\}}$ is the subsheaf of sections, σ, such that $\sigma \mid_W \equiv 0$, then $\{\hat{f}_i^*(\vec{v}) - \hat{f}_j^*(\vec{v})\}_{i,j \in I}$ determines an obstruction in $H^1(U_0 \setminus \{0\}, \mathcal{W})$. But since σ is uniquely determined by its values at $\phi(t) \in W$, it follows that $\mathcal{W} \equiv 0$, and so all obstructions in $H^1(U_0 \setminus \{0\}, \mathcal{W})$ vanish automatically. Thus there exists a global lifting, $f^*(\vec{v})$, to $\mathcal{M} \setminus M_0$, which extends to $\mathcal{M}$ when $dim(D) \geq 4$.

(c) $M_t \cong W/G_\mu$, where $\mu = \begin{pmatrix} \epsilon + \sqrt{\Delta} & 0 \\ 0 & \epsilon - \sqrt{\Delta} \end{pmatrix}$, for (ϵ, Δ) fixed. Let $\kappa_1 = \epsilon + \sqrt{\Delta}, \kappa_2 = \epsilon - \sqrt{\Delta}$, then it is proven, for example, in [4], that W/G_μ is an elliptic fiber bundle over $\mathbf{P}_1$, if and only if $\kappa_1^k = \kappa_2^l$, for some $k, l \in \mathbf{Z}$. Otherwise W/G_μ contains precisely two irreducible curves, namely the images of $z_1 = 0, z_2 = 0$ under the covering map, $\pi : W \to W/G_\mu$. A simple calculation shows that vector fields on W, which are invariant under G_μ, are of the form

$$\sigma = c_1 z_1 \frac{\partial}{\partial z_1} + c_2 z_2 \frac{\partial}{\partial z_2}, \qquad c_1, c_2 \in \mathbf{C}.$$

If $p = (u_1, u_2) \in W$, then clearly $\sigma(p)$ determines c_1, c_2 uniquely when $u_1, u_2 \neq 0$. Let C_t, C_t' denote the canonically determined elliptic curves in M_t corresponding to the eigenspaces of $\mu(t) = g_t^{-1} \mu g_t$ in W, and let $\mathcal{C} = \cup_{t \neq 0} C_t, \mathcal{C}' = \cup_{t \neq 0} C_t'$. Note that $C_t \cap C_t' = \emptyset$, for all $t \neq 0$, hence $\mathcal{C} \cap \mathcal{C}' = \emptyset$, for if $\mathcal{C} \cup \mathcal{C}'$ were connected, the fibers, $C_t \cup C_t'$ over $D \setminus \{0\}$ would determine topologically a double covering. But $D \setminus \{0\}$ is simply connected when $dim(D) \geq 2$, hence the covering must be trivial, which is a contradiction. Under the assumption that $dim(D) \geq 4$, the smooth subvarieties, $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{M} \setminus M_0$ extend properly to $\mathcal{M}$ by the argument of proposition 1, section two. Now consider $V \subset \mathcal{M} \setminus \bar{\mathcal{C}} \cup \bar{\mathcal{C}}', \phi : U_0 \to V$, and $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$, as before. From here, the argument of case (b) may be followed verbatim. We conclude

Theorem 3 *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, such that $f^{-1}(t) \cong W/G_\mu$, where μ is of type (b) or (c), for all $t \neq 0$, then $f^{-1}(t) \not\cong f^{-1}(0) \Rightarrow dim(D) \leq 3$. If μ is of type (a), then $f^{-1}(t) \cong f^{-1}(0)$.*

We turn, finally, to bundles for which the fiber has genus at least two. It is known that for hyperelliptic bundles, E, with base space, B, there exists an unramified cover, B', of B, such that the pullback of E over B' is a Cartesian product (cf., for example, [4]). But $B = \mathbf{P}_1$ is simply connected, hence the corresponding bundle is automatically of this type, ie., trivial, and has been dealt with in section three.

2.5 Quadratic transforms

The essential idea of the preceding sections was to characterise elements of the (finite dimensional) Lie Algebra of M_t in terms of some finite configuration of points, $\phi_k(t)$, uniformly marked on the fibers of $\mathcal{M} \xrightarrow{f} D$. Note that if $\sigma \in H^0(M_t, \Theta_t)$, such that $\sigma(\phi_k(t)) = 0$, then, replacing $\phi_k(t)$ by an exceptional curve , $A(t)$, of the first kind, we find that σ lifts to a vector field, $\hat{\sigma}$, on $M_t \# \overline{\mathbf{P}_2}$. Conversely, if $\hat{\mathcal{M}} \xrightarrow{\hat{f}} D$ is a holomorphic family, locally trivial over $D \setminus \{0\}$, such that $\hat{f}^{-1}(t) \cong M \# \overline{\mathbf{P}_2}, t \neq 0$, with M a compact surface of one of the above types, the classification of vector fields on $\hat{f}^{-1}(t)$ would reduce to those on M which vanish at the blow-down, q, of the exceptional curve, A. Let

$\mathcal{N} \cong \mathcal{O}_{\mathbf{P}_1}(-1)$ denote the normal bundle of A in $M\#\overline{\mathbf{P}_2}$. It follows from [11], theorem 1, that since $H^1(A, \mathcal{N}) = 0$, A is a "stable submanifold" of $M\#\overline{\mathbf{P}_2}$. In particular, this implies that the family $A(t) \subset \hat{f}^{-1}(t), t \in U_i, i \in I$, is preserved by the biholomorphisms, Φ_i, of the local trivialization. Suppose now that $\hat{f}^{-1}(t)$ contains r disjoint exceptional curves, $A_k(t)$, of the first kind. Let $A_k = \cup_{t \neq 0} A_k(t), A = \cup_{1 \leq k \leq r} A_k$. If $dim(D) \geq 2$. then each A_k is a connected component of A, for if not, then the fibers, $\cup_{1 \leq k \leq r} A_k(t)$, would determine an r-fold topological cover of $D \setminus \{0\}$ with fewer than r connected components, which is impossible, since the latter is simply connected. Hence the A_k are smooth analytic subvarieties of $\hat{\mathcal{M}} \setminus \hat{f}^{-1}(0)$. Now, when M_t is a non-trivial fiber bundle over $\mathbf{P}_1$, for each $A_k(t), t \neq 0, 1 \leq k \leq r$, there is a unique fiber, $F_k(t) \subset M_t$, such that the strict transform, $\hat{F}_k(t)$, intersects $A_k(t)$, and so a smooth subvariety, $\hat{F}_k \subset \hat{\mathcal{M}} \setminus \hat{f}^{-1}(0)$, is similarly determined. We will refer to this situation by saying that the correspondence, $A_k \leftrightarrow \hat{F}_k$ is one to one.

Theorem 4 *If $\hat{\mathcal{M}} \xrightarrow{\hat{f}} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, with $\hat{f}^{-1}(t) \cong \Sigma_n\#\overline{\mathbf{P}_2}\#...\#\overline{\mathbf{P}_2}, n \geq 1, t \neq 0$, and the correspondence, $A_k \leftrightarrow \hat{F}_k$ is one to one, $1 \leq k \leq r \leq n+1$, then $\hat{f}^{-1}(t) \ncong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 3$. If $r \geq n+2$, then $dim(D) \leq 1$.*

Proof: Suppose $dim(D) \geq 4$. The Lie Algebra of $\hat{f}^{-1}(t)$ corresponds to vector fields on $M_t \cong \Sigma_n$ which vanish at $q_k(t)$, the blow- down of $A_k(t), 1 \leq k \leq r$. If $1 \leq r \leq 3$, then the sheaf $\hat{\mathcal{W}}$, the analogue of $\mathcal{W}$ in section two, is similarly isomorphic to a direct sum of $n-1$ holomorphic line bundles. If $4 \leq r \leq n+1$, then $\hat{\mathcal{W}} = R^0\hat{f}_*(\Theta) \mid_{U_0 \setminus \{0\}}$, and is isomorphic to a direct sum of $n+2-r$

line bundles. Note that $r \geq n+2 \Rightarrow R^0\hat{f}_*(\Theta) \mid_{D\setminus\{0\}} \equiv 0$, in which case, for $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$, there exists a global lifting, $\hat{f}^*(\vec{v}) \in \Gamma(\hat{f}^{-1}(U_0), \mathcal{T}\mathcal{M})$, which extends across codimension two by Hartogs' Theorem. Hence $\hat{f}^{-1}(t) \not\cong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 1$.

Now, when M_t is a Cartesian product, $t \neq 0$, each $A_k(t)$ intersects the strict transform, $\hat{\chi}_k(t)$, of a unique pair of fibers, according to the notation of theorem 2. Given $M_t \cong X \times \mathbf{P}_1$, where X is a Riemann Surface of genus g, let ν denote the rank of $\mathcal{W}$, the associated direct sum of line bundles. Recall that (i) $\nu = 0$ if $g = 0$; (ii) $\nu = 2$ if $g = 1$; (iii) $\nu = 3$ if $g \geq 2$.

Theorem 5 *If $\hat{\mathcal{M}} \xrightarrow{\hat{f}} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, with $\hat{f}^{-1}(t) \cong (X \times \mathbf{P}_1)\#\overline{\mathbf{P}_2}\#...\#\overline{\mathbf{P}_2}, t \neq 0$, and the correspondence, $A_k \leftrightarrow \hat{\chi}_k$, is one to one, $1 \leq k \leq r \leq 2$, where $\chi_1 \cap \chi_2 = \emptyset$, then $\hat{f}^{-1}(t) \not\cong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 3$. If $r \geq 3$, then $dim(D) \leq 1$.*

Proof: Suppose $dim(D) \geq 4$. (i) If $g = 0$, then $\hat{\nu}$, the rank of $\hat{\mathcal{W}}$, is zero, ie., $\hat{\mathcal{W}} \equiv 0$. (ii) If $g = 1$, then $\hat{\nu} = 3 - r$, and $\hat{\mathcal{W}} = R^0\hat{f}_*(\Theta) \mid_{D\setminus\{0\}}$. (iii) If $g \geq 2$, then $\hat{\nu} = 3 - r$, and $\hat{\mathcal{W}} = R^0\hat{f}_*(\Theta) \mid_{D\setminus\{0\}}$. Note that $r \geq 3 \Rightarrow R^0\hat{f}_*(\Theta) \mid_{D\setminus\{0\}} \equiv 0$, in which case, for $\vec{v} \in \Gamma(U_0, \mathcal{T}D)$, there exists a global lifting, $\hat{f}^*(\vec{v}) \in \Gamma(\hat{f}^{-1}(U_0), \mathcal{T}\mathcal{M})$, which extends across codimension two by Hartogs' Theorem. Hence $\hat{f}^{-1}(t) \not\cong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 1$.

Finally, recall that when $M_t \cong W/G_\mu$, a Hopf Surface of type (b) or (c) (cf. section four), the families, $\mathcal{C}, \mathcal{C}'$, of elliptic curves, $C_t, C_t' \subset M_t$ form smooth subvarieties in the corresponding total space. Now let $A = \cup_{t\neq 0} A(t)$ be the subvariety of $\hat{\mathcal{M}} \setminus \hat{f}^{-1}(0)$ generated by a single exceptional curve, $A(t)$, on $\hat{f}^{-1}(t) \cong W/G_\mu \# \overline{\mathbf{P}_2}$, and let $\hat{\mathcal{C}}, \hat{\mathcal{C}}'$ denote the smooth families generated by the strict transforms of C_t, C_t' above.

Theorem 6 *If $\hat{\mathcal{M}} \xrightarrow{\hat{f}} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus \{0\}$, with $\hat{f}^{-1}(t) \cong W/G_\mu \# \overline{\mathbf{P}_2}$, where μ is of type (b) or (c), $t \neq 0$, and $A \cap \hat{\mathcal{C}} = \emptyset, A \cap \hat{\mathcal{C}}' = \emptyset$, then $\hat{f}^{-1}(t) \not\cong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 1$.*

Proof: Let $q(t)$ denote the blow-down of $A(t), t \neq 0$, then if σ is an element of the Lie Algebra of M_t such that $\sigma(q(t)) = 0$, it follows from theorem 3 that $\sigma \equiv 0$. Hence $\hat{f}^{-1}(t) \cong W/G_\mu \# \overline{\mathbf{P}_2}, A \cap \hat{\mathcal{C}}, A \cap \hat{\mathcal{C}}' = \emptyset \Rightarrow R^0 \hat{f}_*(\Theta) \mid_{D \setminus \{0\}} \equiv 0$, therefore $\hat{f}^{-1}(t) \not\cong \hat{f}^{-1}(0) \Rightarrow dim(D) \leq 1$.

(Remark: Suppose that instead of a single exceptional curve, $A(t)$, $\hat{f}^{-1}(t)$ contains several disjoint curves, $A_k(t)$, of the first kind. Then, provided the corresponding subvarieties, A_k, are such that $A_k \cap \hat{\mathcal{C}}, A_k \cap \hat{\mathcal{C}}' = \emptyset, 1 \leq k \leq r$, the above theorem still holds.)

2.6 An example of structure jumping in codimension two

Let $M_0 = \Sigma_3$, the third Hirzebruch Surface (ie., the fiberwise compactification of $\mathcal{O}_{\mathbf{P}_1}(-3)$), with coordinate systems $(z_1, \zeta_1), (z_2, \zeta_2)$, such that $z_1 =$

$1/z_2, \zeta_1 = z_2^3\zeta_2$. Now define a holomorphic family, $\mathcal{M} \xrightarrow{f} D \subseteq \mathbf{C}^2$, such that on $f^{-1}(t)$, the coordinate systems are related by

$$z_1 = 1/z_2, \qquad \zeta_1 = z_2^3\zeta_2 + t_2 z_2^2 + t_1 z_2,$$

where $t = (t_1, t_2) \in D$. $\mathcal{M} \xrightarrow{f} D$ is in fact the versal deformation of Σ_3 (cf. [10], [11]), where, for all $t \neq (0,0), f^{-1}(t) \cong \Sigma_1 = \mathbf{P}_2 \# \overline{\mathbf{P}_2}$. Moreover, let $A \subset \mathcal{M} \setminus M_0$ denote the subvariety generated by the exceptional curves $A(t) \subset f^{-1}(t), t \neq 0$. If $\bar{A}$ is the closure of A in $\mathcal{M}$, it can be shown that $M_0 \subset \bar{A}$, ie., $A(0)$ is not a proper subvariety. Recall the discussion of the versal deformation of Σ_3 in chapter one. In particular, A may be represented locally by the defining equation,

$$-z_1\zeta_1 + \frac{t_2}{t_1}\zeta_1 + t_1 = 0, \qquad t_1 t_2 \neq 0.$$

Now consider any one-parameter subfamily corresponding to the relation, $t_2 = \lambda t_1, \lambda \in \mathbf{C}^*$, and the associated subvariety, A_λ, defined by

$$z_1\zeta_1 + \lambda\zeta_1 + t_1 = 0.$$

The extension, $\bar{A}_\lambda \cap M_0$ is therefore determined by

$$\lim_{t_1 \to 0}(z_1\zeta_1 + \lambda\zeta_1 + t_1) = \zeta_1(z_1 + \lambda) = 0.$$

Now

$$A(0) \supset \cup_{\lambda \in \mathbf{C}^*} \bar{A}_\lambda \cap M_0,$$

hence it cannot be a proper subvariety of M_0. The marking argument of section two would consequently fail in this case.

Chapter 3

Structure-Jumping across Analytic Subsets

3.1 The extension problem for holomorphic families

At this point, it will be appropriate to place the phenomena of deformation and structure-jumping in a broader context. Given a domain, $D \subseteq \mathbf{C}^n$, and $A \subset D$, an analytic subset, it may be asked under what conditions certain analytic data, such as functions, sheaves or cohomology classes, can be extended from $D \setminus A$ to D. A classical instance of this problem is the "Riemann Extension Theorem", which states that a holomorphic function extends uniquely when A has codimension at least two. Another example is the Remmert-Stein lemma for extension of analytic subsets, already encountered in chapter two. In the case of a sheaf, $\mathcal{F}$, of abelian groups, the exact cohomology sequence,

$$... \to H^k(D, \mathcal{F}) \to H^k(D \setminus A, \mathcal{F}) \to H_A^{k+1}(D, \mathcal{F}) \to ...$$

indicates that obstructions to extension of k-cohomology classes of $\mathcal{F}$ are determined by $(k+1)$-classes "with supports in A" (cf. [3], [23]). In particular, when $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_D$-modules, Cartan's "Theorem B" implies that $H^k(D,\mathcal{F})=0, k\geq 1$, hence $H^k(D\setminus A,\mathcal{F})\cong H^{k+1}_A(D,\mathcal{F})$.

Another type of extension problem is the following. Let $\mathcal{M}\xrightarrow{f} D$ represent a holomorphic family of compact, complex manifolds, and consider the restriction, $\mathcal{H}=f^{-1}(H)$, where $H\subset D$ is a smooth hypersurface containing A. Given analytic data on $\mathcal{H}$, what are the obstructions to defining an extension of these data to $\mathcal{M}$, or at least to a "formal neighbourood" of $\mathcal{H}$ in $\mathcal{M}$? More precisely, let $\mathcal{I}_{\mathcal{H}}\subset\mathcal{O}_{\mathcal{M}}$ be the sheaf of ideals having $\mathcal{H}$ as locus. The sheaf of "jets of order k", in parameters normal to $\mathcal{H}$, will then correspond to $\mathcal{O}_{\mathcal{M}}/\mathcal{I}^k_{\mathcal{H}}, k=1,2,3....$ The pairs, $(\mathcal{H},(\mathcal{O}_{\mathcal{M}}/\mathcal{I}^k_{\mathcal{H}})\mid_{\mathcal{H}})$, form a sequence of ringed spaces, $\mathcal{H}(k)$, which comprise the formal, or "infinitesimal" neighbourhood of $\mathcal{H}$ (cf. [8], and chapter 4, in which "non-reduced" complex spaces are discussed). For example, if $\mathcal{V}$ denotes the sheaf of sections of a holomorphic vector bundle on $\mathcal{H}$, then extension of $\mathcal{V}$ to $\mathcal{H}(k)$ is obstructed by $H^2(\mathcal{H},\mathcal{E}nd(\mathcal{V})\otimes\mathcal{N}^{-l})=Ext^2_{\mathcal{O}_{\mathcal{H}}}(\mathcal{V},\mathcal{V}\otimes\mathcal{N}^{-l}), 1\leq l\leq k$, where $\mathcal{N}\cong(\mathcal{I}_{\mathcal{H}}/\mathcal{I}^2_{\mathcal{H}})^*$ is the invertible normal sheaf of $\mathcal{H}$ in $\mathcal{M}$ (cf. [8]). If $\sigma\in H^\gamma(\mathcal{H}(k),\mathcal{V})$, then extension of σ to $\mathcal{H}(k+1)$, assuming $\mathcal{V}$ itself extends, is obstructed by $H^{\gamma+1}(\mathcal{H}(k),\mathcal{V}\mid_{\mathcal{H}}\otimes\mathcal{N}^{-(k+1)})$.

The significance of this last example will be examined more carefully. Let $0\in A\subset D$, and suppose the ideal sheaf of H to be generated locally by $\mu\in\mathcal{O}_{D,0}$. If $f:\mathcal{M}\to D$ is assumed to be proper, and everywhere regular, ie., $rank_{\mathbb{C}}(J(f))_q=dim_{\mathbb{C}}(D)$, for all $q\in\mathcal{M}$, then $\mathcal{M}$ has the structure of a holomorphic fiber space on D, ie., each $M_p = f^{-1}(p)$ is a compact, complex manifold, with trivial normal bundle in $\mathcal{M}$. Consequently, the subfamily, $\mathcal{H} = f^{-1}(H)$ is a complex submanifold with trivial normal bundle, since $\tilde{\mu} = \mu \circ f$ provides a global generator for $\mathcal{I}_{\mathcal{H}}$. Under the assumption that $\mathcal{V}$ is a holomorphic vector bundle on $\mathcal{M}$, let $U \subset \mathcal{M}$ be an arbitrary open set. The short exact sequence of $\mathcal{O}_{\mathcal{M}}$-modules,

$$0 \to \Gamma(U, \mathcal{V}) \xrightarrow{\cdot\tilde{\mu}} \Gamma(U, \mathcal{V}) \xrightarrow{\varepsilon} \Gamma(U \cap \mathcal{H}, \mathcal{V} \mid_{\mathcal{H}}) \to 0,$$

gives rise, in the direct limit, to a short exact sequence,

$$0 \to \mathcal{V}_q \xrightarrow{\cdot\tilde{\mu}} \mathcal{V}_q \xrightarrow{\varepsilon} \mathcal{V} \mid_{\mathcal{H},q} \to 0.$$

of sheaves, for all $q \in \mathcal{H} \cap U$. Let $R^\gamma f_*(\mathcal{V})$ denote the γ-th direct image sheaf. Then there is a corresponding pair of long exact sequences,

$$\ldots \to R^\gamma f_*(\mathcal{V}) \xrightarrow{\varepsilon_*} R^\gamma f_*(\mathcal{V} \mid_{\mathcal{H}}) \xrightarrow{\bar{\delta}} R^{\gamma+1} f_*(\mathcal{V}) \to \ldots$$

$$\ldots \to \Gamma(D \cap H, R^\gamma f_*(\mathcal{V}) \mid_H) \xrightarrow{\varepsilon_*} \Gamma(D \cap H, R^\gamma f_*(\mathcal{V} \mid_{\mathcal{H}})) \xrightarrow{\bar{\delta}} \Gamma(D \cap H, R^{\gamma+1} f_*(\mathcal{V}) \mid_H) \to \ldots$$

on H. The following theorems are fundamental to the results of this chapter:

Theorem-α (Upper Semi-Continuity of Cohomology):

(cf. [7]) *For every $d \geq 0$, the set, $A_{\gamma,d} = \{p \in D \mid dim_{\mathbf{C}} H^\gamma(M_p, \mathcal{V}_p) \geq d\}$, where $\mathcal{V}_p = \mathcal{V} \mid_{M_p}$, is an analytic subset of D.*

Theorem-β (cf. [7]) *If $dim_{\mathbf{C}} H^\gamma(M_p, \mathcal{V}_p)$ is independent of p, then $R^\gamma f_*(\mathcal{V})$ is locally free, and $rank_{\mathcal{O}_{D,p}}(R^\gamma f_*(\mathcal{V})) = dim_{\mathbf{C}} H^\gamma(M_p, \mathcal{V}_p)$.*

(By the *rank* of a coherent sheaf, $\mathcal{F}$, at a point, p, is understood the $\mathbf{C}$-rank of the module, $\mathcal{F}/\mathbf{m}_p \cdot \mathcal{F}$, where $\mathbf{m}_p$ denotes the maximal ideal in $\mathcal{O}_D$ corresponding to p.)

Now let $d(\gamma) = inf_{p\in D} dim_{\mathbf{C}} H^\gamma(M_p, V_p)$, and suppose $A \supseteq A_{\gamma,d(\gamma)+1}$, for all integers, $\gamma \geq 0$, ie., the sheaves, $R^\gamma f_*(\mathcal{V})$ are locally free on $D \setminus A$. If $\sigma \in \Gamma(D \cap H, R^{\gamma+1} f_*(\mathcal{V} \mid_{\mathcal{H}}))$, then the image of $\bar{\delta}$, above, will be shown to correspond to a torsion module, obstructing extension of σ to D, namely, $\Gamma_A(D \cap H, R^{\gamma+1} f_*(\mathcal{V}))$ (cf. theorem 7 of the following section). In this way, the second type of extension problem discussed here is seen also to be connected with cohomology supported on A. Theorem 7, which constitutes the main result of this chapter, is in fact a generalisation, via different techniques, of a result due to Griffiths, [8], when $dim_{\mathbf{C}} D = 1$.

Now let $\mathcal{V} = \Theta$ denote the subsheaf of the tangent sheaf, $\mathcal{TM}$, having sections which lie parallel to the fibers, M_t. $R^\gamma f_*(\Theta)$ will then denote the γ-th direct image of Θ over D. It was originally remarked by Mumford, in the case of a parameter space, $D \subseteq \mathbf{C}$, having a single structure-jump at the origin, that the vector field, $\frac{\partial}{\partial t}$, is mapped by the Kodaira-Spencer homomorphism,

$$\rho : \mathcal{T}D \to R^1 f_*(\Theta),$$

to a non-trivial section which vanishes on $D \setminus \{0\}$, with a pole of finite order at $t = 0$. Multiplication by a suitable power of t provides a lifting of $\frac{\partial}{\partial t}$ to $\mathcal{TM}$, which is tangent to M_0, hence inducing an exceptional automorphism on that fiber. Via the theory of "jump cocycles", [8], Griffiths translated Mumford's idea from the algebraic category to the holomorphic.

The emphasis of this chapter is on the extension of Griffiths' work to parameter spaces of higher dimension. Structure jumping is considered to occur with respect to an analytic subvariety, $A \subseteq D$, such that the restricted family, $\mathcal{M} \setminus f^{-1}(A) \xrightarrow{f} D \setminus A$, is locally trivial. The following section will be devoted to a generalisation of [8], theorem 1.1, using sheaf theoretic techniques drawn from [21], [23]. This result has two distinct applications in section three : first, to the problem of extending exceptional automorphisms from fibers above a hypersurface containing A, to the ambient deformation space, $\mathcal{M}$, and secondly, to the special case of $A = \{0\} \subseteq D$, with the additional requirement that $R^1 f_*(\Theta) \mid_{D \setminus \{0\}} \equiv 0$. Under these conditions, a regularity criterion is derived for $\mathcal{M} \xrightarrow{f} D$, relating $dim_{\mathbb{C}}(D)$ to the dimension of the Lie algebra, $\mathcal{G} = H^0(M_p, \Theta_p)$, for all $p \neq 0$.

3.2 Jump-cocycles and extension of automorphisms

Consider a holomorphic vector bundle V on $\mathcal{M}$, of rank r, and let V_p denote the restriction of V to M_p. If $\mathcal{V}$ represents the $\mathcal{O}_{\mathcal{M}}$-module corresponding to the sheaf of holomorphic sections, then the $\gamma - th$ direct image sheaf will be a coherent $\mathcal{O}_D$-module, denoted by $R^\gamma f_*(\mathcal{V})$.

Suppose that $A \subseteq D$ is an analytic subvariety such that $R^\gamma f_*(\mathcal{V})$ restricted to $D \setminus A$ is locally free for all γ, and $H \subseteq D$ any hypersurface containing A.

For any relatively compact subdomain, D' of D, let $\mathcal{H} = f^{-1}(H \cap D')$ be the corresponding hypersurface in $\mathcal{M}$, and

$$G_\gamma = \Gamma(H \cap D', R^\gamma f_*(\mathcal{V} \mid_{\mathcal{H}})),$$

the $\mathcal{O}_{D'}$-module of global sections of $R^\gamma f_*(\mathcal{V} \mid_{\mathcal{H}})$.

$$E_\gamma = \Gamma(H \cap D', R^\gamma f_*(\mathcal{V}) \mid_{H \cap D'}) \subset G_\gamma$$

will then correspond to those sections of $R^\gamma f_*(\mathcal{V} \mid_{\mathcal{H}})$ which extend to a neighbourhood of H in D'. Finally, let J_γ represent $\Gamma_A(D', R^\gamma f_*(\mathcal{V}))$, the module of global sections of $R^\gamma f_*(\mathcal{V})$ with support contained in A.

Theorem 7 *$G_\gamma / E_\gamma \cong J_{\gamma+1}$ as graded $\mathcal{O}_{D'}$- modules.*

Proof: Choose $\mu \in \mathcal{O}_{D'}$ such that $\mathcal{I}_{H \cap D'} = (\mu)$ is the ideal sheaf of $H \cap D'$. The assumption that $\mathcal{N}_{M_p}$ is trivial in $\mathcal{M}$ for all $p \in D \Rightarrow \mathcal{I}_{\mathcal{H}} = (\mu \circ f)$. Let $\mathcal{U}$ be a Leray cover for $\mathcal{M}$, and for all $\sigma \in G_\gamma$ let $\overline{\sigma}$ be an extension to a γ-cochain on the nerve of $\mathcal{U}$. If δ denotes the coboundary map, then $\delta(\overline{\sigma}) \mid_{\mathcal{H}} = 0 \Rightarrow (\mu \circ f)^k$ divides $\delta(\overline{\sigma})$ for some maximal $k < \infty$ (unless $\sigma \in E_\gamma$). hence $\frac{1}{(\mu \circ f)^k} \delta(\overline{\sigma})$ will represent a section of cohomology classes in $J_{\gamma+1}$, since $R^{\gamma+1} f_*(\mathcal{V})$ is assumed to be locally free on $D \setminus A$. For convenience, let $\mathcal{F}$ denote $R^{\gamma+1} f_*(\mathcal{V})$, and define a sheaf homomorphism

$$\phi_k : \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}_{D'}}(\mathcal{I}^k_{H \cap D'}, \mathcal{F})$$

as follows. At the presheaf level, for all $U \subseteq D'$, and for all

$$\xi \in \Gamma(U, \mathcal{F}) = H^{\gamma+1}(f^{-1}(U), \mathcal{V}),$$

let $\phi_k(\xi)$ be an $\mathcal{O}_{D'}$-module homomorphism sending g to $(g \circ f) \cdot \xi$, where $g \in \Gamma(U, \mathcal{I}^k_{H \cap D'})$, and then pass to the direct limit. If $\mathcal{F}_k$ is taken to denote $ker(\phi_k)$, then the ascending chain of coherent sheaves :

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_\infty$$

is locally stationary (cf. [21]). Moreover, that $\mathcal{F}_k \neq \emptyset$ for some k, is guaranteed by the Nullstellensatz for coherent analytic sheaves (cf. [7], [23]). Hence if D' is a relatively compact subdomain, there exists k_0 such that $\mathcal{F}_k \mid_{D'} = \mathcal{F}_{k_0} \mid_{D'}$ for all $k \geq k_0$. Moreover $J_{\gamma+1} = \Gamma_A(D', \mathcal{F}) = \Gamma(D', \mathcal{F}_\infty)$ (cf. [23]), hence

$$\mu^{k_0} \cdot \Gamma_A(D', R^{\gamma+1} f_*(\mathcal{V})) \equiv 0.$$

Consequently, if $\overline{\sigma}'$ is a cochain extension of σ to $\mathcal{U} \cap f^{-1}(D')$, then $(\mu \circ f)^{k_0+1}$ divides $\delta(\overline{\sigma}') \Rightarrow \sigma \in E_\gamma$.

Now, since $\mathcal{H}$ is a hypersurface in $f^{-1}(D')$, for which the normal line bundle $\mathcal{N}$ corresponds to $(\mathcal{I}_\mathcal{H}/\mathcal{I}^2_\mathcal{H})^*$, the cohomology of $\mathcal{V}$ may be expanded in a power series about $\mathcal{H}$, i.e. consider

$$\bigoplus_{k=0}^{\infty} H^{\gamma+1}(\mathcal{H}, \mathcal{V} \mid_\mathcal{H} \otimes \mathcal{N}^{-k}),$$

in which each direct summand will henceforth be abbreviated to $H_k^{\gamma+1}$. In particular there is a well defined, $\mathcal{O}_{D'}$-linear map

$$\omega : G_\gamma \longrightarrow \bigoplus_{k=0}^{\infty} H_k^{\gamma+1}$$

such that $\omega(\sigma) = [\delta(\overline{\sigma})]$. Moreover, if

$$P_k : \bigoplus_{k=0}^{\infty} H_k^{\gamma+1} \longrightarrow H_k^{\gamma+1}$$

is the k-th canonical projection, then define $k(\sigma) = inf\{k \mid P_k(\omega(\sigma)) \neq 0\}$, where $k(\sigma) \leq k_0$ for all $\sigma \in G_\gamma/E_\gamma$, and is defined to be $k_0 + 1$ for $\sigma \in E_\gamma$. Hence, for all $\sigma \in G_\gamma$,

$$\frac{1}{(\mu \circ f)^k}\omega(\sigma) \in J_{\gamma+1} \subseteq H^{\gamma+1}(f^{-1}(D'), \mathcal{V}).$$

Now let $\Sigma_\gamma(k) = \{\sigma \in G_\gamma \mid k(\sigma) \geq k\}$, so that

$$G_\gamma = \Sigma_\gamma(1) \supseteq \Sigma_\gamma(2) \supseteq \ldots \supseteq \Sigma_\gamma(k_0+1),$$

and $\Pi_\gamma(k) = \{\xi \in J_{\gamma+1} \mid \mu^k \cdot \xi \equiv 0\}$, so that

$$\Pi_\gamma(1) \subseteq \Pi_\gamma(2) \subseteq \ldots \subseteq \Pi_\gamma(k_0) = J_{\gamma+1}.$$

The following is now a straightforward consequence:

Lemma: $\Sigma_\gamma(k)/\Sigma_\gamma(k+1) \cong \Pi_\gamma(k)/\Pi_\gamma(k-1)$.

Proof: Consider

$$\psi_k : \Sigma_\gamma(k) \longrightarrow \Pi_\gamma(k)/\Pi_\gamma(k-1)$$

such that $\psi_k(\sigma) = \frac{1}{(\mu \circ f)^k}\omega(\sigma)$. ψ_k is linear, and $ker(\psi_k) = \Sigma_\gamma(k+1)$. Suppose $\xi \in \Pi_\gamma(k)/\Pi_\gamma(k-1)$, then $\mu^k \cdot \xi \equiv 0, \mu^{k-1} \cdot \xi \not\equiv 0 \Rightarrow \xi$ can be represented by cocycles of the form $\frac{1}{(\mu \circ f)^k}\delta(\overline{\sigma})$, for some γ-cochain $\overline{\sigma}$ on $\mathcal{M}$. Since $\delta(\overline{\sigma}) \mid_{\mathcal{H}} = 0$, it follows that $\overline{\sigma} \mid_{\mathcal{H}} = \sigma \in \Sigma_\gamma(k) \subseteq G_\gamma$ such that $\psi_k(\sigma) = \xi$, hence ψ_k is surjective. Q.E.D.

Finally, let $\Sigma_\gamma^k = \Sigma_\gamma(k)/\Sigma_\gamma(k+1), \Pi_\gamma^k = \Pi_\gamma(k)/\Pi_\gamma(k-1)$. Then, as $\mathcal{O}_{D'}$-modules,

$$G_\gamma = \bigoplus_{k=1}^{k_0} \Sigma_k^\gamma \bigoplus \Sigma_\gamma(k_0+1),$$

the last direct summand being equal to E_γ, and

$$J_{\gamma+1} = \bigoplus_{k=1}^{k_0} \Pi_\gamma^k \Rightarrow \psi : G_\gamma / E_\gamma \cong J_{\gamma+1},$$

where ψ is the linear extension of the homomorphisms ψ_k. This completes the proof of Theorem 7.

3.3 Application to complex deformations

Consider $\mathcal{V} = \Theta_{\mathcal{M}}$, corresponding to the subsheaf of $\mathcal{TM}$ having sections which lie parallel to the fibers M_p. Triviality of $\mathcal{N}_{M_p}$ in $\mathcal{M}$, combined with the compactness of M_p for all $p \in D$ implies that the canonical exact sequence

$$0 \longrightarrow \Theta_{\mathcal{M}} \longrightarrow \mathcal{TM} \longrightarrow \mathcal{TM}/\Theta_{\mathcal{M}} \longrightarrow 0$$

induces a long exact sequence of the form :

$$0 \longrightarrow R^0 f_*(\Theta_{\mathcal{M}}) \longrightarrow R^0 f_*(\mathcal{TM}) \xrightarrow{f_*} \mathcal{T}D \xrightarrow{\rho} R^1 f_*(\Theta_{\mathcal{M}}) \longrightarrow \ldots$$

where ρ is the Kodaira-Spencer map. $\mathcal{M}$ now corresponds to the deformation space of holomorphic structures on a compact, complex manifold M_0. It will be assumed here that $\mathcal{M}$ is a versal deformation space, and that $H^2(M_0, \Theta_0) = 0$ (hence, by upper semicontinuity of cohomology, $R^2 f_*(\Theta_{\mathcal{M}}) \equiv 0$ on D sufficiently small), which together imply that ρ is surjective (cf. [7], [10], and Nakayama's lemma). Moreover, as regards the minimal number of generators for the first direct image,

$$rank_{\mathcal{O}_{D,p}}(R^1 f_*(\Theta_{\mathcal{M}})) = dim_{\mathbb{C}} H^1(M_p, \Theta_p)$$

for all $p \in D$. This fact is due to $R^2 f_*(\Theta_{\mathcal{M}})$ being, a fortiori, torsion-free, and hence follows from the proof of Theorem-β, even though $R^1 f_*(\Theta_{\mathcal{M}})$ is not locally free (cf. [7]).

Corollary 1 *Each* $\sigma \in G_0 = \Gamma(H \cap D, R^0 f_*(\Theta_{\mathcal{M}}))$ *extends to a vector field* $\hat{\sigma} \in \Gamma(f^{-1}(D), \mathcal{TM})$ *such that if* $\nu(\hat{\sigma})_p$ *denotes the projection of* $\hat{\sigma} |_{M_p}$ *onto* $\mathcal{N}_{M_p}$, *then* $\nu(\hat{\sigma})$ *vanishes to order* $k(\sigma)$ *on* $\mathcal{H}$.

Proof: Let $\mathcal{K} \subseteq \mathcal{T}D$ be the subsheaf of germs of holomorphic vector fields $\vec{v}$ such that $\rho(\vec{v}) \equiv 0$. Since $R^0 f_*(\mathcal{TM})$ is coherent, there exists a free resolution

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \xrightarrow{\alpha} R^0 f_*(\mathcal{TM}) \longrightarrow 0.$$

Moreover, $\mathcal{T}D$ is free, therefore isomorphic to $\mathcal{O}^n, D \subseteq \mathbf{C}^\mathbf{n}$, from which follows the resolution

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \xrightarrow{f_* \circ \alpha} \mathcal{O}^n \xrightarrow{\rho} R^1 f_*(\Theta_{\mathcal{M}}) \longrightarrow 0.$$

But $im(f_* \circ \alpha) = im(f_*) = ker \rho = \mathcal{K}$, therefore

$$\mathcal{O}^p \longrightarrow \mathcal{O}^q \xrightarrow{f_* \circ \alpha} \mathcal{K} \longrightarrow 0$$

implies $\mathcal{K}$ is coherent. Now consider $\xi \in J_1 = \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}}))$. Since ρ is surjective, for all $p \in D$ there exists a neighbourhood U, and $\vec{v} \in \Gamma(U, \mathcal{T}D)$ such that $\rho(\vec{v}) = \xi |_U$. Given $\vec{v}_i, \vec{v}_j$ such that $\xi |_{U_i} = \rho(\vec{v}_i), \xi |_{U_j} = \rho(\vec{v}_j)$, then

$$U_i \cap U_j \neq \emptyset \Rightarrow \vec{v}_i - \vec{v}_j \in \Gamma(U_i \cap U_j, \mathcal{K}).$$

Passing to a refinement, if necessary, assume $U_i, U_j \in \mathcal{U}$ to be a Stein cover of D. Then $[\{\vec{v}_i - \vec{v}_j\}_{i,j}] \in H^1(D, \mathcal{K})$, which vanishes by Cartan's Theorem B, hence there exists $\vec{v} \in \Gamma(D, \mathcal{T}D)$ such that $\xi = \rho(\vec{v})$. Now $A \subseteq H$ implies $\rho_p(\vec{v}) \equiv 0$ for all $p \in D \setminus H$, hence there exists a neighbourhood U_i of p in $D \setminus H$, and a local lifting $f_i^*(\vec{v})$ to $\Gamma(f^{-1}(U), \mathcal{TM})$. Moreover, $U_i \cap U_j \neq \emptyset \Rightarrow \{f_i^*(\vec{v}) - f_j^*(\vec{v})\}_{i,j}$ determines an obstruction class in $H^1(D \setminus H, R^0 f_*(\Theta_{\mathcal{M}}))$. But $D \setminus H$ is Stein, and $R^0 f_*(\Theta_{\mathcal{M}})$ is coherent, so once again the cohomology vanishes by the Theorem B, and there is a global lifting $f^*(\vec{v}) \in \Gamma(f^{-1}(D \setminus H), \mathcal{TM})$. Now, given $\sigma \in G_0/E_0$, let $\sigma = \Sigma_{k=1}^{k_0} \lambda_k \sigma_k$ be the direct sum decomposition, where $\sigma_k \in \Sigma_0^k$, and $\lambda_k \in \mathcal{O}_D$. By theorem 7, there exists unique $\xi = \Sigma_{k=1}^{k_0} \lambda_k \xi_k \in J_1$ such that $\xi = \psi(\sigma)$. Moreover, by the above argument, there exist vector fields $\vec{v}_k$ such that $\xi_k = \rho(\vec{v}_k), 1 \leq k \leq k_0$, and hence holomorphic liftings $f^*(\vec{v}_k)$ to $\mathcal{M} \setminus \mathcal{H}$. But $\mu^{k(\sigma_k)} \cdot \xi_k \equiv 0 \Rightarrow \mu^{k(\sigma_k)} f^*(\vec{v}_k)$ extends holomorphically across $\mathcal{H}$, hence

$$\hat{\sigma} = \Sigma_{k=1}^{k_0} \mu^{k(\sigma_k)} f^*(\vec{v}_k).$$

By construction, $\hat{\sigma} |_{\mathcal{H}} = \Sigma_{k=1}^{k_0} \lambda_k \sigma_k = \sigma$. Moreover, the normal component, ν_k, of $\mu^{k(\sigma_k)} f^*(\vec{v}_k)$ vanishes to order $k(\sigma_k)$ on $\mathcal{H}$, hence the order of vanishing of $\nu(\hat{\sigma})$ corresponds to $k(\sigma) = inf\{k(\sigma_k) | \lambda_k \neq 0\}$. This completes the proof of corollary 1.

It may be asked whether some analogue of corollary 1 may be extended to any subvariety S, of codimension greater than or equal to two in D, such that $A \subseteq S$. In particular, if $\xi \in \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}}))$, then there exists $\vec{v} \in \Gamma(D, \mathcal{T}D)$ such that $\xi = \rho(\vec{v})$, and for all $p \in D \setminus S$, there exists a neighbourhood U_i, and a local lifting $f_i^*(\vec{v})$ to $\Gamma(f^{-1}(U), \mathcal{TM})$. The obstruction to global lifting lies in $H^1(D \setminus S, R^0 f_*(\Theta_{\mathcal{M}}))$, but $D \setminus S$ is no longer Stein, and in fact for the case $dim(A) \geq 1$:

Corollary 2 *If* $\Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) \neq 0$, *then*

$$dim_{\mathbf{C}} H^1(D \setminus S, R^0 f_*(\Theta_{\mathcal{M}})) = +\infty$$

Proof: Given

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}D \xrightarrow{\rho} R^1 f_*(\Theta_{\mathcal{M}}) \longrightarrow 0,$$

let $\mathcal{K}[S]$ denote the "relative gap-sheaf" in $\mathcal{T}D$ with respect to S, ie., coming from the presheaf, $U \mapsto \{\vec{v} \in \Gamma(U, \mathcal{T}D) | \ \vec{v}\ |_{U \setminus S} \in \Gamma(U \setminus S, \mathcal{K})\}$, for all $U \subseteq D$ (cf. [23]). Consider

$$\hat{\rho} : \Gamma(D, \mathcal{K}[S]) \longrightarrow \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) \longrightarrow 0,$$

where $ker(\hat{\rho}) = \Gamma(D, \mathcal{K})$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a Leray cover of $D \setminus S$, and for $\vec{v} \in \Gamma(D \setminus S, \mathcal{K})$, let $f^*(\vec{v}) = \{f_i^*(\vec{v})\}_{i \in I} \in C^0(\mathcal{U}, R^0 f_*(\mathcal{T}\mathcal{M}))$ represent a lifting. Finally, consider the coboundary operator,

$$\delta : C^0(\mathcal{U}, R^0 f_*(\mathcal{T}\mathcal{M})) \longrightarrow C^1(\mathcal{U}, R^0 f_*(\Theta_{\mathcal{M}})).$$

The composition $\delta \circ f^*$ determines a well defined map

$$\Delta_f : \Gamma(D \setminus S, \mathcal{K}) \longrightarrow H^1(D \setminus S, R^0 f_*(\Theta_{\mathcal{M}})),$$

for which $\vec{v} \in ker \Delta_f$ implies there exists a global lifting $f^*(\vec{v})$ to $f^{-1}(D \setminus S)$. Moreover, $codim_D(S) \geq 2$ implies $f^*(\vec{v})$ extends holomorphically across $f^{-1}(S)$, by Hartogs' theorem, and hence there exists an extension $\overline{\vec{v}} \in \Gamma(D, \mathcal{K}[S])$

such that $\overline{\vec{v}} \in ker\hat{\rho}$, therefore $ker\Delta_f \subseteq ker\hat{\rho}$. Let $\mathcal{Q} = \Gamma(D, \mathcal{K})/ker\Delta_f$, then from the exact sequence

$$0 \longrightarrow \Gamma(D, \mathcal{K}) \longrightarrow \Gamma(D, \mathcal{K}[S]) \xrightarrow{\hat{\rho}} \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) \longrightarrow 0$$

is induced the sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow \Gamma(D, \mathcal{K}[S])/ker\Delta_f \longrightarrow \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) \longrightarrow 0.$$

Now $\Gamma(D, \mathcal{K}[S])/ker\Delta_f \cong im(\Delta_f) \subseteq H^1(D \setminus S, R^0 f_*(\Theta_{\mathcal{M}}))$. But A is a Stein space of positive dimension, hence $\Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}}))$ is a Fréchet space of infinite dimension over $\mathbf{C}$. Moreover

$$im(\Delta_f) \longrightarrow \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) \longrightarrow 0$$

implies $dim_{\mathbf{C}}(im(\Delta_f)) = +\infty$, hence $dim_{\mathbf{C}} H^1(D \setminus S, R^0 f_*(\Theta_{\mathcal{M}})) = +\infty$. This completes the proof of corollary 2.

For the case $dim(A) = 0$, let $\mathcal{T}$ be the torsion subsheaf of $R^1 f_*(\Theta_{\mathcal{M}})$; then $supp(\mathcal{T}) = A = \{0\} \Rightarrow dim_{\mathbf{C}} \Gamma_A(D, R^1 f_*(\Theta_{\mathcal{M}})) = dim_{\mathbf{C}} \Gamma(D, \mathcal{T}) \leq rank_0(\mathcal{T})$, which is finite, since $\mathcal{T}$ is coherent. In addition, if $A = S = \{0\}$, then $dim_{\mathbf{C}} H^r(D \setminus \{0\}, \mathcal{F})$ is finite, for any coherent sheaf $\mathcal{F}, 0 < r < dh(\mathcal{F}) - 1$, under the assumption that the boundary, ∂D, is strictly pseudoconvex (cf. [1], and chapter five). If $\mathcal{F}$ is locally free on $D \setminus \{0\}$, then $dh(\mathcal{F}) = n = dim(D)$; moreover, when $\mathcal{F}$ corresponds to a vector bundle, V, over $D \setminus \{0\}$, equipped with a holomorphic connection,

$$\nabla : \mathcal{V} \longrightarrow \Omega^1(\mathcal{V}),$$

then $H^r(D \setminus \{0\}, \mathcal{V}) = 0$ for $0 < r < n-1$ (cf. [5]). The existence of a free extension of $\mathcal{V}$ across the origin in $\mathbf{C}^n$ then becomes a relevant question, since the group of obstructions to global existence of a connection on $\mathcal{V}$, ie.

$$H^1(D \setminus \{0\}, \Omega^1(End(\mathcal{V}))) = Ext^1_{\mathcal{O}_{D\setminus\{0\}}}(\mathcal{V}, \Omega^1(\mathcal{V})),$$

is similarly finite dimensional.

A theorem of Grauert implies that any holomorphic vector bundle on a contractible Stein space is trivial, hence

$$0 \neq [V] \in H^1(\mathbf{C}^n \setminus \{0\}, GL_r(\mathbf{C})), \qquad rank(V) = r,$$

implies $\mathcal{V}$ does not extend freely across $\{0\}$. For example, consider L corresponding to the total space of the tautological line bundle on $\mathbf{P}_{n-1}$, for which the sheaf of holomorphic sections is denoted $\mathcal{O}_{\mathbf{P}_{n-1}}(-1)$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a Stein cover of $\mathbf{P}_{n-1}$. For all $p \in \mathbf{P}_{n-1}$, the complex dimension of the fiber $\pi^{-1}(p)$ (where π is the canonical projection of the line bundle to its base) is one, hence $\pi^{-1}(p) \setminus \{p\}$ is Stein, and so is the Cartesian product $U_i \times (\pi^{-1}(p) \setminus \{p\})$, from which a Stein cover of $L \setminus \mathbf{P}_{n-1}$ is generated. Now the cohomology of a vector bundle $\tilde{V}$ over $L \setminus \mathbf{P}_{n-1}$ may be expanded in a Laurent series about $\mathbf{P}_{n-1}$, ie.

$$H^r(L \setminus \mathbf{P}_{n-1}, \tilde{\mathcal{V}}) \cong \bigoplus_{k=-\infty}^{\infty} H^r(\mathbf{P}_{n-1}, \tilde{\mathcal{V}} \mid_{\mathbf{P}_{n-1}} \otimes \mathcal{N}^{-k}),$$

where $\mathcal{N}$, the normal bundle of $\mathbf{P}_{n-1}$ in L, is essentially the same as $\mathcal{O}_{\mathbf{P}_{n-1}}(-1)$. In particular, if $\tilde{\mathcal{V}} = \pi^*(\Omega^1_{\mathbf{P}_{n-1}})$, then for $r = 1$, the Bott formula (cf. [17]) indicates that $dim_{\mathbf{C}} H^1(\mathbf{P}_{n-1}, \Omega^1_{\mathbf{P}_{n-1}}(k)) = 1$ if $k = 0$, and zero otherwise, hence $dim_{\mathbf{C}} H^1(L \setminus \mathbf{P}_{n-1}, \tilde{\mathcal{V}}) = 1$. But now, under blowing down, $L \setminus \mathbf{P}_{n-1} \cong$

$\mathbf{C}^n \setminus \{0\}$, and $\tilde{V}\mid_{L\setminus\mathbf{P}_{n-1}}$ is mapped biholomorphically to a vector bundle V on $\mathbf{C}^n \setminus \{0\}$, such that $dim_{\mathbf{C}} H^1(\mathbf{C}^n \setminus \{0\}, \mathcal{V}) = 1$. This means, in particular, that $\mathcal{V}$ is non-free when $n \geq 3$, since $H^1(\mathbf{C}^n \setminus \{0\}, \mathcal{O}^r) \cong H^1(\mathbf{C}^n, \mathcal{O}^r) = 0$ (cf. [20]).

Consider finally the case of a holomorphic family of compact manifolds, $\mathcal{M} \xrightarrow{f} D$, where once more $A = \{0\}, \mathcal{V} = R^0 f_*(\Theta)\mid_{D\setminus\{0\}}$. Under the following conditions, it is possible to determine an upper bound for $dim_{\mathbf{C}}(D)$ in terms of the (constant) rank, r, of $R^0 f_*(\Theta)\mid_{D\setminus\{0\}}$, corresponding to the dimension of the Lie algebra, $H^0(M_p, \Theta_p), p \neq 0$.

Corollary 3 *Suppose*

(i) $H^1(M_p, \Theta_p) = 0$ *for all* $p \neq 0$,

(ii) $dim_{\mathbf{C}} H^2(M_p, \Theta_p)$ *is constant for all* $p \in D$, *and*

(iii) $\mathcal{M}$ *is effectively parametrised at* 0, *ie., the infinitesimal Kodaira-Spencer map* $\rho_0 : T_0 D \to H^1(M_0, \Theta_0)$ *is injective, then* $M_0 \not\cong M_p \Rightarrow$

$$either \quad dim(D) \leq 2, \qquad or \quad 3 \leq dim(D) \leq r.$$

Proof: $H^1(M_p, \Theta_p) = 0, p \neq 0, \Rightarrow R^1 f_*(\Theta)\mid_{D\setminus\{0\}} \equiv 0$, therefore

$$J_1 = \Gamma_{\{0\}}(D, R^1 f_*(\Theta)) = \Gamma(D, R^1 f_*(\Theta)).$$

If $dim_{\mathbf{C}} H^2(M_p, \Theta_p)$ is constant for all $p \in D$, then $R^2 f_*(\Theta)$ is locally free, hence

$$rank(J_1) = rank_0(R^1 f_*(\Theta)) = dim_{\mathbf{C}} H^1(M_0, \Theta_0).$$

Now ρ_0 injective $\Rightarrow dim(D) \leq dim_{\mathbf{C}} H^1(M_0, \Theta_0) = rank(J_1)$, hence if $H \subset D$ is a hypersurface such that $0 \in H, \mathcal{H} = f^{-1}(H)$, and $G_0 = \Gamma(H, R^0 f_*(\Theta\mid_{\mathcal{H}}))$, then theorem 7 ($J_1 \cong G_0/E_0$) implies that $rank(J_1) \leq rank(G_0)$. If it is assumed that $dim(D) \geq 3$, then $codim_H(\{0\}) \geq 2$, hence since $R^0 f_*(\Theta)$ is a normal sheaf, it follows that

$$\Gamma(H \setminus \{0\}, R^0 f_*(\Theta \mid_{\mathcal{H}})) \cong \Gamma(H, R^0 f_*(\Theta \mid_{\mathcal{H}})).$$

But $R^0 f_*(\Theta) \mid_{H\setminus\{0\}}$ is locally free, hence for all $p \in H \setminus \{0\}$,

$$rank(\Gamma(H \setminus \{0\}, R^0 f_*(\Theta \mid_{\mathcal{H}}) \mid_{H\setminus\{0\}})) \leq rank_p(R^0 f_*(\Theta \mid_{\mathcal{H}})),$$

and $R^1 f_*(\Theta) \mid_{H\setminus\{0\}}$ locally free implies

$$rank_p(R^0 f_*(\Theta \mid_{\mathcal{H}})) = dim_{\mathbf{C}} H^0(M_p, \Theta_p) = r,$$

therefore $dim(D) \leq r$.

Example: Consider $\mathcal{M} \xrightarrow{f} D$ for which $M_p \cong \Sigma_1 \# \overline{\mathbf{P}}_2 \# \overline{\mathbf{P}}_2$. It can be shown (cf. [10]) that $H^k(M_p, \Theta_p) = 0, k = 1, 2$, while $r = dim_{\mathbf{C}} H^0(M_p, \Theta_p) = 2$. Hence $M_0 \not\cong M_p \Rightarrow dim(D) \leq 2$, when $H^2(M_0, \Theta_0) = 0$. Otherwise, theorem 4 of chapter two implies $dim(D) \leq 3$.

In the next chapter, an extension problem of a similar type will be discussed, with certain important differences. In particular, "A" will be seen to signify a compact hypersurface, having non-trivial normal bundle within a (non-compact) complex manifold, M. Infinitesimal neighbourhoods of A will play a far more prominent role in the extension process, for which the "analytic data" will correspond to certain sheaves of automorphisms supported on A. Roughly speaking, these sheaves will stand in relation to the tangent sheaf of A as a Lie group stands in relation to its Lie algebra.

Chapter 4

Deformation and Resolution of Reducible Plane Curves

4.1 Deformation and resolution of complex spaces

The theory of deformation of complex spaces generalises the fundamental concepts and definitions of the Kodaira-Spencer theory for compact complex manifolds. In essence, the holomorphic map, $f : \mathcal{M} \to D$, between complex manifolds (D is often assumed to be a domain in $\mathbf{C}^n$) is replaced by a map, $\pi : X \to S$, of complex spaces. The regularity criterion for f, which ensures that fibers, $M_p = f^{-1}(p)$, are manifolds of constant dimension for all $p \in D$, is correspondingly replaced by the requirement that π be "flat" (cf. [18]). If $s_0 \in S$ is a distinguished point, and $X_0 \cong \pi^{-1}(s_0)$ is a given complex space, then the family, (π, X, S) is said to define a "deformation" of X_0. Moreover, if $x_0 \in X_0$, and U is an open neighbourhood of x_0, the induced deformation of U is said to represent a "germ of the deformation of X_0 at x_0".

Further sophistication is required in generalising the notion of "tangent sheaf" to complex spaces with singularities. Let $U \subseteq X_0$ correspond to a reduced analytic subset, V, of a manifold, M, which is the locus of an ideal sheaf, $\mathbf{m} \subset \mathcal{O}_M$, with Ω^1_M denoting the sheaf of holomorphic 1-forms on M. Let $\Omega' \subset \Omega^1_M$ denote the subsheaf generated by $\mathbf{m} \cdot \Omega^1_M$ and df for $f \in \mathbf{m}$, ie., $\omega_x \in \Omega'_x$ if

$$\omega_x = \Sigma_i h_i \theta_i + \Sigma_j g_j df_j,$$

where $\theta_i \in \Omega^1_{M,x}, g_j \in \mathcal{O}_{M,x}$, and $h_j, f_j \in \mathbf{m}_x$ (cf. [14]). The sheaf, Ω^1_V, of germs of holomorphic 1-forms on V, is therefore defined as Ω^1_M/Ω'. Moreover, if $\mathcal{O}_V = (\mathcal{O}_M/\mathbf{m})\ |_V$, then the tangent sheaf, $\mathcal{T}_V$, will correspond to $\mathcal{H}om(\Omega^1_V, \mathcal{O}_V)$. Higher cohomologies of the tangent sheaf will therefore correspond to the groups, $Ext^i_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V) \cong \mathcal{E}xt^i_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})_{x_0}$. Neighbourhoods of non-singular points of X_0 are rigid, in the sense of admitting only trivial deformations, hence $\mathcal{E}xt^i_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})$ is supported on the singular locus of X_0.

In the case of x_0 an isolated singular point, G.N. Tjurina constructed the minimal, locally versal deformation, $\pi : X \to S$, of the germ of X_0 at x_0, where S is a complex manifold with distinguished point, x_0. In addition, it was shown that the canonical homomorphism, $\rho_0 : T_{s_0}S \to \mathcal{E}xt^1_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})_{x_0}$, the analogue of the Kodaira-Spencer map for deformations of compact manifolds, is in fact an isomorphism when $\mathcal{E}xt^2_{\mathcal{O}_{X_0}}(\Omega^1_{X_0}, \mathcal{O}_{X_0})_{x_0} = 0$ (cf. [24]).

By a simultaneous resolution of the deformation, $\pi : X \to S$, is meant a commutative diagram of mappings of complex spaces,

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ T & \xrightarrow{\phi} & S \end{array}$$

in which T is also a manifold with distinguished point, and ϕ is a proper mapping, such that $\phi^{-1}(s)$ is a finite, non-empty set, for all $s \in S$. Moreover, $\tilde{\pi}$ is regular, in the sense that the $\mathbf{C}$-rank of $\tilde{\pi}_*$ is equal to $dim_{\mathbf{C}}(T)$ at each point of $\tilde{X}$, and the fiber, $\tilde{X}_t = \tilde{\pi}^{-1}(t)$, is a resolution of $\pi^{-1}(\phi(t)) = X_{\phi(t)}$, for all $t \in T$.

When X_0 is a complex space of dimension two, the topological type of the singularity is determined by its minimal resolution. Hence, if $A \hookrightarrow \tilde{X}_t$ is the exceptional set, and $A = \cup_i A_i$ its decomposition into irreducible components, for each $t \in T$, then the fibers of $\pi : X \to S$ are all of the same topological type if and only if the intersection matrix, $(A_i \cdot A_j)_{i,j}$, is independent of t. Note that the resolution, $\tilde{X}_0$, of X_0 corresponds to a pseudoconvex neighbourhood of the compact analytic subspace which is the preimage of x_0. Deformation of $\tilde{X}_0$ consequently involves a machinary which is in some respects intermediate to the theory of local deformations of complex spaces, and the Kodaira-Spencer theory for compact manifolds. In particular, the question of existence of a versal deformation of $\tilde{X}_0$ (or, alternatively, the existence of a simultaneous resolution of the fibers of X in the above sense) is quite separate from that addressed in [24], and is treated, in the case of equitopological families of two-dimensional normal singularities, in [15].

The approach taken is to choose a Riemann surface, A_i, belonging to A, deform the normal bundle, N_i, in $\tilde{X}_0$, and hence the embedding of A_i onto the zero section of the total space (taking automorphisms of A into account). Note that the Chern class of N_i, corresponding to $A_i \cdot A_i$, should remain invariant. Deformations, $\tilde{X}_t$, of the resolution are then recovered via a plumbing procedure on coordinate charts of the deformation of each N_i (cf. [15]). Each fiber, $\tilde{X}_t$, determines a sequence of "non-reduced" spaces, for which formal equivalence provides a sufficient condition of biholomorphic equivalence between fibers ([14], theorem 6.13). Recall that a non-reduced space, also called an "analytic space with nilpotents", consists of a pair, $(V, \mathcal{O}_V)$, where V is a Hausdorff topological space, and the structure sheaf of C-algebras, $\mathcal{O}_V$, has the following property. For each $p \in V$, there exist a neighbourhood, U, a subset, Y of D (a polydisc in $\mathbf{C}^n$), corresponding to the locus of a sheaf of ideals, $\mathcal{I}$, and a C-algebra isomorphism, $\Psi : (\mathcal{O}/\mathcal{I}) \mid_Y \rightarrow \psi^*(\mathcal{O}_V \mid_U)$, where $\psi : Y \rightarrow U$ is the underlying homeomorphism. More specifically, suppose that each component, A_i, of the exceptional set in $\tilde{X}_t$ is the locus of an ideal, $\mathcal{I}_i$, in $\mathcal{O}_{\tilde{X}_t}$. Any product of higher powers of the $\mathcal{I}_i$ will then determine an ideal, $\mathcal{I}$, and a corresponding non-reduced structure on A, denoted $A(\mathcal{I})$. Such spaces play a crucial role in this branch of deformation theory, which generalises Kodaira's treatment, [11], of deformations of compact, complex manifolds with analytically "stable" submanifolds. Note by comparison that the spaces, X_t, being deformed here are non-compact, and strongly pseudoconvex, containing a compact subspace with singular points, corresponding to the transverse intersections of the A_i (cf. [6], [14], [15]). Modulo these discrepancies, however, the idea of deformation is the same. On coordinate charts of A, the analytic structure of any non-reduced space may be extended isomorphically. On intersections of these charts, however, the difference between adjacent local extensions generates 1-cocycles associated with the following sheaves. Again consider products of the form, $\Pi_i \mathcal{I}_i^{r_i}, \Pi_i \mathcal{I}_i^{s_i}$, denoted by $\mathbf{m}, \mathbf{n}$ respectively, where $r_i = s_i$ for all r_i different from some chosen i_0, while $s_{i_0} = r_{i_0} + 1$. Let $\mathcal{A}ut(\mathbf{n} : \mathbf{m})$ denote the sheaf of automorphisms, with stalkwise multiplication given by composition, determined by its presheaf of sections. Namely, if $U \subseteq A$ is open, then $\alpha \in \Gamma(U, \mathcal{A}ut(\mathbf{n} : \mathbf{m})) \Rightarrow \alpha$ is an isomorphism on $A(\mathbf{n}) |_U$ (the underlying homeomorphism being the identity), which preserves $\mathbf{m}/\mathbf{n}$ stalkwise, and induces the identity on $A(\mathbf{m}) |_U \subseteq A(\mathbf{n}) |_U$. $\mathcal{A}ut(\mathbf{n} : \mathbf{m})$ is a sheaf of non-abelian groups, hence the cohomology, $H^k(A, \mathcal{A}ut(\mathbf{n} : \mathbf{m}))$, will have the structure of a group when $k = 0$, but will simply be a set with distinguished element, or "pointed set", when $k = 1$. Passing reference has been made to deformation of $\tilde{X}_0$ as a two-fold process, involving both the normal bundle, N_i, of each A_i, and automorphisms of $A(\mathbf{n})$ which fix the points of $A \cap A_i$. More precisely, $\mathcal{A}ut(\mathbf{n} : \mathbf{m})$ may be decomposed via the following short exact sequence:

$$1 \to \mathcal{A}ut(\mathrm{n}, \mathrm{m}) \to \mathcal{A}ut(\mathbf{n} : \mathbf{m}) \to \mathcal{A}n(\mathrm{n}, \mathrm{m}) \to 1,$$

which induces an exact sequence,

$$\ldots \rightarrow H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n},\mathbf{m})) \rightarrow H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n}:\mathbf{m})) \xrightarrow{\varsigma} H^1(A_{i_0}, \mathcal{A}n(\mathbf{n},\mathbf{m})),$$

of pointed sets of cohomology. $\mathcal{A}n(\mathbf{n},\mathbf{m})$ may be identified with a subsheaf, $\mathcal{O}_1^* \subset \mathcal{O}_{A_{i_0}}^*$, whose sections near $A_i \cap A_{i_0}$ are of the form $exp(\mathcal{I}_i^{r_i})$. Moreover, there is a short exact sequence,

$$0 \rightarrow \mathbf{Z}' \stackrel{2\pi i}{\rightarrow} \mathcal{O}_1 \stackrel{exp}{\rightarrow} \mathcal{O}_1^* \rightarrow 1,$$

where $\mathbf{Z}'$ is that subsheaf of the locally constant sheaf of integers having zero stalks at $A_i \cap A_{i_0}$, which induces the corresponding exact sequence,

$$\ldots \rightarrow H^1(A_{i_0}, \mathcal{O}_1) \rightarrow H^1(A_{i_0}, \mathcal{O}_1^*) \xrightarrow{\delta} H^2(A_{i_0}, \mathbf{Z}')$$

(cf. [14], [15]). In particular, if A_{i_0} has genus zero, then $H^1(\mathcal{O}_1) = 0, H^1(\mathcal{O}_1^*) \cong H^1(\mathcal{A}n(\mathbf{n},\mathbf{m}))$, implies that the map, $\delta : H^1(A_{i_0}, \mathcal{A}n(\mathbf{n},\mathbf{m})) \rightarrow H^2(A_{i_0}, \mathbf{Z}')$ is injective. Now let $[\xi] \in H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n}:\mathbf{m}))$ denote an obstruction class, generated by local extensions of the analytic structure, between neighbouring non-reduced fibers. $\varsigma([\xi])$ then corresponds to the difference, $N_{i_0} \otimes (N_{i_0}')^{-1}$, of the two normal bundles over an intersection of charts on A_{i_0}. But the Chern classes, $c(N_{i_0}), c(N_{i_0}')$ are assumed to be the same, hence $\delta \circ \varsigma([\xi]) = 0$. Therefore $\varsigma([\xi])$ must be zero, since δ is injective. We conclude that when $A_{i_0} \cong \mathbf{P}_1$, every $[\xi] \in H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n}:\mathbf{m}))$ may be pulled back to $H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n},\mathbf{m}))$. The sheaf, $\mathcal{A}ut(\mathbf{n},\mathbf{m})$, is in fact isomorphic to $\mathcal{T}_{A_{i_0}} \otimes \mathbf{m}/\mathbf{n}$, where $\mathcal{T}_{A_{i_0}}$ is the tangent sheaf, and $\mathbf{m}/\mathbf{n}$ corresponds to $N_{i_0}^{-r_{i_0}} \ni [-E], E$ being a divisor, associated with the points of $A \cap A_{i_0}$, on which sections of $\mathbf{m}/\mathbf{n}$ must vanish to a certain order (cf. [14]). The crucial role of the set, $H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n} : \mathbf{m}))$, is indicated by the following (cf. [6], [15])

lemma:

Let $\Phi : A(\mathbf{m}) \rightarrow A(\mathbf{m}'), \Psi : A(\mathbf{m}) \rightarrow A(\tilde{\mathbf{m}})$ be distinct isomorphisms of non-reduced spaces, such that local extensions of these isomorphisms to $A(\mathbf{n})$ give rise to obstruction classes, $[\xi'], [\tilde{\xi}] \in H^1(A_{i_0}, \mathcal{A}ut(\mathbf{n} : \mathbf{m}))$. Then there exists an isomorphism, $\Theta : A(\mathbf{n}') \rightarrow A(\tilde{\mathbf{n}})$, which extends $\Psi \circ \Phi^{-1}$ if and only if $[\xi'] = [\tilde{\xi}]$. In particular, Φ(resp.Ψ) will extend globally to $A(\mathbf{n})$ if and only if $[\xi']$(resp.$[\tilde{\xi}]$) $= *$, the distinguished element.

Hence, equivalence of non-reduced spaces, $A(\mathbf{m})$, is a vanishing condition, which must therefore be satisfied at each stage of the extension, $A(\mathbf{m}_1) \subseteq A(\mathbf{m}_2) \subseteq \ldots$, corresponding to stepwise increments in the powers, r_i, of the ideal of each A_i. In fact, if vanishing occurs for r_i sufficiently large, (ie., an isomorphism on $A(\mathbf{m}_k)$ extends globally to $A(\mathbf{m}_{k+1})$, for all $k \leq N$), then global extension follows automatically for all non-reduced $A(\mathbf{m}_k), 1 \leq k \leq \infty$. In this case, it follows that the fibers, say $\tilde{X}_0, \tilde{X}_t$, are actually biholomorphic (cf. [14]. theorem 6.13).

In the following sections, the deformation theory of resolutions will be applied, not to normal surface singularities, but to reducible plane curves, and in particular to those for which the germ of the defining equation is weighted homogeneous. Equitopological deformation of the resolution will give rise to a family of non-reduced spaces, for which the non-trivial analytic structure is concentrated on that vertex of Γ, the weighted dual graph, which has weight,

-1 (ie., $A_{i_0} \cdot A_{i_0} = -1$). If the (weighted homogeneous) singularity, $(C, 0) \subset \mathbf{C}^2$. and its deformations are of type, (p, q), with $\gamma = gcd(p, q), \beta = lcm(p, q)$, then the $\mathbf{C}^*$-action, $(X, Y) \mapsto (t^{\frac{p}{\gamma}} X, t^{\frac{q}{\gamma}} Y)$ on C will induce 1-parameter families of mutually isomorphic deformations. In particular, let $W(X, Y)$ represent the germ of $(C, 0)$, and $(W + \Delta)(X, Y)$ that of an arbitrary deformation, $(C_\varepsilon, 0)$ (for reasons which will be made explicit in the next section, assume

$$\Delta(X, Y) = \Sigma_{i,j} \varepsilon_{i,j} X^i Y^j,$$

such that $\frac{p}{\gamma} i + \frac{q}{\gamma} j \geq \beta + 1$). Then

$$(W + \Delta)(t^{\frac{p}{\gamma}} X, t^{\frac{q}{\gamma}} Y) = t^\beta (W + t\Delta_t).$$

Moreover, if the deformation space of $(C, 0)$ is assumed to be minimal and versal, in the sense of Tjurina, [24], then structure-jumping will occur within each of these 1-parameter families when $t = 0$. After constructing the space of infinitesimal deformations, an attempt is made to examine the parallel phenomenon at the level of the resolution. In particular, it may be asked whether those 1-parameter families induced by the $\mathbf{C}^*$-action give a complete picture of structure-jumping in a neighbourhood of the central fiber.

4.2 The Versal Family

Let $C \subset \mathbf{C}^2$ be an algebraic curve, with a singular point at the origin, 0. Let the ideal sheaf, $\mathcal{I}_C$, be generated at 0 by a holomorphic function germ, such that for any representative, $f(X,Y)$, $f(0,Y)$ is divisible by Y^p, and $f(X,0)$ is divisible by $X^q, p, q \geq 1$. Applying a theorem of G.N. Tjurina, [24], the locally versal space of germs of complex analytic deformations, $[f_\varepsilon]$, of $[f]$ is parametrised by a domain, $D \subseteq \mathbf{C}^\mu$, where

$$\mu = dim_{\mathbf{C}}(\mathcal{O}_{\mathbf{C}^2}/(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}))_0$$

is minimal. If it is assumed that the Weierstrass Polynomial, $W(X,Y)$, associated with $[f]$ is in fact weighted homogeneous, ie

$$W(X,Y) = \Sigma_{i,j} a_{i,j} X^i Y^j,$$

such that $(\frac{p}{\gamma})i + (\frac{q}{\gamma})j = \beta, \gamma = gcd(p,q), \beta = lcm(p,q)$, then from Euler's formula, it follows that $W \in (\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y})_0$. Moreover, the Fundamental Theorem of Algebra implies that

$$W(X,Y) = \Pi_{k=1}^{\gamma}(Y^{\frac{p}{\gamma}} - \lambda_k X^{\frac{q}{\gamma}}), \qquad \lambda_k \in \mathbf{C}^*, 1 \leq k \leq \gamma.$$

Consequently, the intersection of C with the unit sphere, $\mathbf{S}^3$, is homeomorphic to a union of simple torus knots, of type $(\frac{p}{\gamma}, \frac{q}{\gamma})$. If it is further assumed that $\lambda_{k_1} \neq \lambda_{k_2}, 1 \leq k_1 \neq k_2 \leq \gamma$, then Hensel's Lemma implies that deformation germs, $[f_\varepsilon]$, whose locus preserves the topological type of the singularity in C, may be represented by polynomials of the form, $W + \Delta$, where

$$\Delta(X,Y) = \Sigma_{i,j}\varepsilon_{ij} X^i Y^j, \qquad (\frac{p}{\gamma})i + (\frac{q}{\gamma})j \geq \beta + 1,$$

and $i \leq q-2, j \leq p-2$. We will now consider the family of resolutions of those germs parametrised by the domain, $D_1 = \{\varepsilon = (\varepsilon_{i,j}) \in \mathbf{C}^{\mu'}\}$, defined above.

Let $U \subsetneq \mathbf{C}^2$ be a neighbourhood of 0, and let $\sigma : M \to U$ be the canonical sequence of quadratic transformations which simultaneously resolves the loci, C_ε, of $[f_\varepsilon]$, for all $\varepsilon \in D_1$. Denote by $(M_\varepsilon, \tilde{C}_\varepsilon)$ the neighbourhood of the exceptional divisor, $A = \sigma^{-1}(0)$, which is specifically ambient to the strict transform, $\tilde{C}_\varepsilon$, of C_ε. Hence there is a family, $\mathcal{M}$, and a surjective, holomorphic map, $\pi : \mathcal{M} \to D_1$, such that $\pi^{-1}(\varepsilon) = (M_\varepsilon, \tilde{C}_\varepsilon)$, for all ε, ie., given $\mathcal{M} \stackrel{\sigma \times 1_{D_1}}{\to} U \times D_1 \stackrel{P_{D_1}}{\to} D_1$, then $\pi = P_{D_1} \circ (\sigma \times 1_{D_1})$. If C_ε^* denotes the total transform, $\sigma^{-1}(C_\varepsilon)$, then the subvariety,

$$\mathcal{C}^* = \cup_{\varepsilon \in D_1} C_\varepsilon^* \subset \mathcal{M},$$

induces a non-trivial family of embeddings, as will be seen in the next section.The first step will be to return to the resolution of the initial fiber, and construct its space of infinitesimal deformations.

4.3 Infinitesimal Deformations of the Resolution

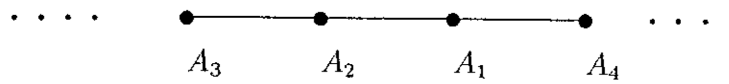
Let $A = \cup_i A_i$ be the decomposition of the exceptional set into embedded rational curves (ie., $\mathbf{P}_1$'s), with intersection matrix determined by relations $A_i \cdot A_j = 1, i \neq j, A_i \cdot A_i = -c_i$ (the Chern class of the normal bundle of A_i in M). If $\mathcal{I}_i$ is the ideal sheaf of A_i, let $\mathrm{m} = \Pi_i \mathcal{I}_i^{r_i}, r_i \geq 1$. By $A(\mathrm{m})$ will be understood the non-reduced space with underlying topological space equal to

A, and structure sheaf, $\mathcal{O}_{A(\mathrm{m})} = (\mathcal{O}_M/\mathrm{m}) \mid_A$. If $\mathcal{V} = (\sigma \times 1_{D_1})^{-1}(0,0), \mathcal{W} = (\sigma \times 1_{D_1})^{-1}(0,\varepsilon)$, then let $\mathcal{O}_\mathcal{V}, \mathcal{O}_\mathcal{W} \cong \mathcal{O}_{A(\mathrm{m})}$ determine the corresponding non-reduced spaces, $(\mathcal{V}, \mathcal{O}_\mathcal{V}), (\mathcal{W}, \mathcal{O}_\mathcal{W})$. Recall that two such non-reduced spaces are isomorphic if there is a homeomorphism, $\phi : \mathcal{V} \to \mathcal{W}$, and an isomorphism, $\Phi : \mathcal{O}_\mathcal{V} \to \phi^*(\mathcal{O}_\mathcal{W})$, of sheaves of C-algebras over $\mathcal{V}$. Let A_1 be the unique component of A such that $A_1 \cdot A_1 = -1$, hence $\tilde{C}_\varepsilon$ intersects A_1 transversely at the points $\lambda_k, 1 \leq k \leq \gamma$. Note that for $r_1 = 1$, each restriction, $\Phi \mid_{A_i}$, is simply a Möbius transformation, varying continuously with $\varepsilon \in D_1 \setminus \{0\}$. The additional requirement that $\{\lambda_k\}$ be fixed, however, forces $\Phi \mid_{A_1}$ to be the identity. Extensions of Φ must successively map formal neighbourhoods of C^* to neighbourhoods of $C^*_\varepsilon, \varepsilon \neq 0$. If $\mathcal{U}$ denotes an open cover of $\mathcal{V}$, then for $U_i, U_j \in \mathcal{U}$, let Φ_i, Φ_j be the appropriate local extensions to $\mathcal{O}'_\mathcal{V} \cong \mathcal{O}_{A(\mathrm{n})}$, where $\mathrm{n} = \Pi_i \mathcal{I}_i^{s_i}, s_i = r_i, i > 1, s_1 = r_1 + 1$. Hence

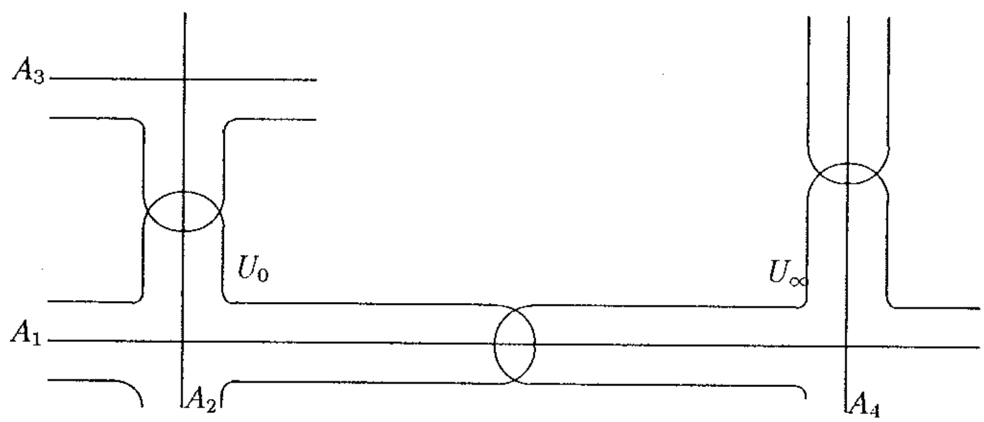
$$\Phi_i : \mathcal{O}'_\mathcal{V} \mid_{U_i} \cong \phi^*(\mathcal{O}'_\mathcal{W}) \mid_{U_i}, \quad \Phi_j : \mathcal{O}'_\mathcal{V} \mid_{U_j} \cong \phi^*(\mathcal{O}'_\mathcal{W}) \mid_{U_j}$$

implies $\xi = \{\Phi_j^{-1} \circ \Phi_i\}_{i,j}$ determines an element of $C^1(\mathcal{U}, \mathcal{A}ut(\mathrm{m} : \mathrm{n}))$. More precisely, let $\mathcal{I}_{\tilde{C},\mathrm{n}}$ denote the ideal sheaf of $\tilde{C}$ in $\mathcal{O}'_\mathcal{V}$. Then the 1-cocycle, ξ, is derived from a subsheaf, $\mathcal{F} \subset \mathcal{A}ut(\mathrm{m} : \mathrm{n})$, consisting of automorphisms which preserve the locus of $\mathcal{I}_{\tilde{C},\mathrm{n}}$. The space of such cocycles will determine sets of obstructions, $\mathcal{D}(r_1) \subseteq H^1(\mathcal{U}, \mathcal{F})$, which may be computed explicitly as follows.

Let Γ denote the weighted dual graph of A, which in the case of a weighted homogeneous curve, C, and its equitopological deformations, has an unbranched, linear sequence of vertices, as illustrated below.



The natural Leray cover, $\mathcal{U}$, of A is comprised of a union of Leray covers on the A_i, and may be represented in the following way.



Alternatively, if $\mathcal{U}_1 = \{U_0, U_\infty\}$, is the standard open cover of A_1, define

$$U_0' = U_0 \cup_j A_j, \qquad U_\infty' = U_\infty \cup_k A_k,$$

where the subindex, j, runs through all vertices of Γ lying to the left of A_1, and k runs through all vertices lying to the right. U_0', U_∞', so defined, are open, connected sets which form a cover, $\mathcal{U}'$ of A, and though it is not Stein, it will be seen presently that $\mathcal{U}'$ is sufficient for computing infinitesimal deformations of the resolution.

Once again, consider an isomorphism,

$$(\Phi, \phi) : (\mathcal{V}, \mathcal{O}_\mathcal{V}) \to (\mathcal{W}, \mathcal{O}_\mathcal{W}),$$

and its local extensions, Φ_0, Φ_∞, relative to $\mathcal{U}'$, which map the formal neighbourhood of C^* to that of $C^*_\varepsilon, \varepsilon \neq 0$. Let A_1, A_2 intersect transversely in the neighbourhood U_0, and choose x, y to be functions in $\mathcal{O}'_{\mathcal{V}}$ which generate $\mathcal{I}_{A_1}, \mathcal{I}_{A_2}$, respectively. $\mathbf{m}$ and $\mathbf{n}$ will then be represented locally by $(x^{r_2}y^{r_1})$ and $(x^{r_2}y^{r_1+1})$, and $\mathcal{I}_{\tilde{C},\mathbf{n}}$ will be generated by $\omega(x) = \Pi_{k=1}^{\gamma}(x - \lambda_k)$. Now $\alpha_0 \in \Gamma(U'_0, \mathcal{F})$ may be expressed locally by the relations,

$$\alpha_0(x) = x + x^{r_2}y^{r_1}\omega(x)g(x), \qquad \alpha_0(y) = y + x^{r_2}y^{r_1}h(x).$$

Set $r_j = 1, j \neq 1, 2$, and suppose A_2, A_3 intersect transversely in a neighbourhood of coordinate functions, $w, z \in \mathcal{O}'_{\mathcal{V}}$, such that in the intersection with $U_0, y = z^{-1}$, and $x = wz^{c_2}$, where $c_2 = -A_2 \cdot A_2$. Now $\alpha_0 \in \Gamma(U'_0, \mathcal{F}) \Rightarrow$

$$\alpha_0(z) = \alpha_0(y)^{-1} = y^{-1}(1 + x^{r_2}y^{r_1-1}h(x))^{-1}$$

$$\equiv z - w^{r_2}z^{c_2r_2-r_1+1}h(0), \qquad \mathrm{mod}(\mathbf{n}),$$

since $\mathbf{m} = \mathbf{n} = (w^{r_2}z)$ locally, and $c_2r_2 - r_1 + 1 \geq 0 \Leftrightarrow r_2 \geq \frac{r_1-1}{c_2}$. Moreover,

$$\alpha_0(w) = \alpha_0(x)\alpha_0(y)^{c_2} = (x + x^{r_2}y^{r_1}\omega(x)g(x)) \cdot y^{c_2}(1 + x^{r_2}y^{r_1-1}h(x))^{c_2}$$

$$\equiv w + w^{r_2}z^{c_2r_2-r_1-c_2}\omega(0)g(0), \qquad \mathrm{mod}(\mathbf{n}),$$

and $c_2(r_2 - 1) - r_1 \geq 0 \Leftrightarrow r_2 \geq 1 + \frac{r_1}{c_2}$.

Similarly, consider A_1, A_4 intersecting transversely in the neighbourhood, U_∞, where $\mathbf{m} = (v^{r_4}u^{r_1}), \mathbf{n} = (v^{r_4}u^{r_1+1}), u = xy, v = x^{-1}$. If $c_4 = -A_4 \cdot A_4, r_k = 1, k \neq 1, 4$, then $\Gamma(U'_\infty, \mathcal{F})$ is non-trivial implies $r_4 \geq \frac{r_1-1}{c_4}$. Conversely, $r_4 \geq 1 + \frac{r_1}{c_4} \Rightarrow \Gamma(U'_\infty, \mathcal{F})$ is non-trivial. We therefore conclude that $\Gamma(U'_0, \mathcal{F}), \Gamma(U'_\infty, \mathcal{F})$ are non-trivial if $r_2 \geq 2 + [\frac{r_1}{c_2}]$, and $r_4 \geq 2 + [\frac{r_1}{c_4}]$, where

"[ ]" denotes the integer part of the given fraction. Note that the above computation assumes at least one vertex on either side of A_1 in Γ. Clearly, if no vertices lie either to the right or left side, then $r_{2,4} \geq 0$ accordingly.

Now without loss of generality, assume $\lambda_k \in A_1 \setminus U_\infty, 1 \leq k \leq \gamma$, and choose extensions, $\Phi_0(x) = x + x^{r_2} y^{r_1} \psi(x), \Phi_0(y) = \tau(y) + x^{r_2} y^{r_1} \zeta(x), \tau = \Phi \mid_{A_2}$ being a Möbius transformation which fixes $A_1 \cap A_2$, while $\Phi_\infty \equiv \Phi$. Then $\xi' = \Phi_\infty^{-1} \circ \Phi_0 \in C^1(\mathcal{U}', \mathcal{F})$, such that

$$\xi'(x) = x + x^{r_2} y^{r_1} \psi(x), \quad \xi'(y) = y + x^{r_2} y^{r_1} \zeta(x) = y(1 + \varphi(x, y)).$$

Denote by $R(\psi)$ the Weierstrass remainder of $\psi(x)$ divided by $\omega(x)$ in $\mathcal{O}_\mathcal{V}$. Now let $\alpha_0 \in \Gamma(U_0', \mathcal{F})$ such that

$$\alpha_0(x) = x + x^{r_2} y^{r_1} (1 + \varphi(x, y))^{-r_1} (R(\psi) - \psi), \quad \alpha_0(y) = y(1 + \varphi(x, y))^{-1}.$$

It follows that

$$\xi' \circ \alpha_0(x) = x + x^{r_2} y^{r_1} R(\psi), \quad \xi' \circ \alpha_0(y) = y.$$

Finally, consider $\alpha_\infty \in \Gamma(U_\infty', \mathcal{F})$, such that $\alpha_\infty^{-1}(v) = v + v^{r_4} u^{r_1} \vartheta(v), \alpha_\infty^{-1}(u) = u$, hence

$$\alpha_\infty^{-1}(x) = x - x^{r_2} y^{r_1} (x^\nu \vartheta(\frac{1}{x})), \quad \alpha_\infty^{-1}(y) = y,$$

where $\nu = r_1 + 2 - r_2 - r_4$. Set $\vartheta = 0$ if $\nu < 0$: otherwise, let

$$\vartheta(\frac{1}{x}) = \frac{1}{x^\nu} \Sigma_{k=0}^{\nu} \frac{1}{k!} \frac{d^k R(\psi)}{dx^k}(0) x^k.$$

It now follows that

$$\alpha_\infty^{-1} \circ \xi' \circ \alpha_0(x) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi), \quad \alpha_\infty^{-1} \circ \xi' \circ \alpha_0(y) = y,$$

where

$$R_{\nu+1}(\psi) = \Sigma_{k=\nu+1}^{\gamma-1} \frac{1}{k!} \frac{d^k R(\psi)}{dx^k}(0) x^k.$$

Thus $\xi' \in C^1(\mathcal{U}', \mathcal{F})$ determines a cohomology class, $[\xi'] \in H^1(\mathcal{U}', \mathcal{F})$, represented uniquely by the polynomial, $R_{\nu+1}(\psi)$. For suppose $[\xi_1'], [\xi_2']$ are cohomology classes, represented by

$$\xi_1'(x) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_1), \qquad \xi_2'(x) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_2).$$

Then $[\xi_1'] = [\xi_2'] \Leftrightarrow \xi_1' = \alpha_\infty^{-1} \circ \xi_2' \circ \alpha_0$, for some $\alpha_0 \in \Gamma(U_0', \mathcal{F}), \alpha_\infty \in \Gamma(U_\infty', \mathcal{F})$. Note that ξ_1', ξ_2', α_0 all extend to U_0' implies α_∞ must also extend, hence

$$\alpha_\infty^{-1}(x) = x - x^{r_2} y^{r_1} \theta(x), \qquad deg(\theta) \leq \nu.$$

Moreover, $\alpha_0(x) = x + x^{r_2} y^{r_1} \omega(x) g(x) \Rightarrow \xi_1'(x) = x + x^{r_2} y^{r_1}(\theta(x) + R_{\nu+1}(\psi_2) + \omega(x) g(x)) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_1)$. But $deg(R_{\nu+1}(\psi_1)) \leq \gamma - 1 \Rightarrow g \equiv 0$, and hence the order, $o(R_{\nu+1}(\psi_1)) \geq \nu + 1 \Rightarrow \theta \equiv 0$, therefore $R_{\nu+1}(\psi_1) = R_{\nu+1}(\psi_2)$.

Let $\mathcal{D}'(r_1) \subseteq H^1(\mathcal{U}', \mathcal{F})$ be the set of all such $[\xi']$.

Proposition 3 *If r_2, r_4 are minimal with respect to r_1, ie., $r_{2,4} = 2 + [\frac{r_1}{c_{2,4}}]$, then $\mathcal{D}(r_1)$ and $\mathcal{D}'(r_1)$ are bijectively equivalent.*

Proof: Define a map, $\chi : \mathcal{D}(r_1) \to \mathcal{D}'(r_1)$ as follows. Note that the support of $\mathcal{F}$ is contained in A_1, since for all $p \in A \setminus A_1, \mathbf{m} \cong \mathbf{n} \Rightarrow \mathcal{A}ut(\mathbf{m} : \mathbf{n})_p$ is trivial, ie., consists only of the identity. Hence if $\xi = \{\Phi_j^{-1} \circ \Phi_i\}_{i,j}$ is a cocycle determined by the set of local extensions, Φ_i, on $\mathcal{U}$, it follows that

$[\xi]\ |_{A_i}, i \neq 1$, is a coboundary, and the isomorphism, Φ, may be extended to Φ_0, Φ_∞, on U'_0, U'_∞ respectively. Hence $\xi' = \Phi_\infty^{-1} \circ \Phi_0$ will determine a class, $[\xi'] = \chi([\xi]) \in \mathcal{D}'(r_1)$. To check that χ is well defined, take $\xi_1 = \{\Phi_j^{-1} \circ \Phi_i\}_{i,j}, \xi_2 = \{\Psi_j^{-1} \circ \Psi_i\}_{i,j}$ to be distinct cocycles, such that $[\xi_1] = [\xi_2]$, ie., for all i, j, there exist $\alpha_i \in \Gamma(U_i, \mathcal{F}), \alpha_j \in \Gamma(U_j, \mathcal{F})$, such that $\Phi_j^{-1} \circ \Phi_i = \alpha_j^{-1} \circ \Psi_j^{-1} \circ \Psi_i \circ \alpha_i$. Let $\Phi_0, \Psi_0, \Phi_\infty, \Psi_\infty$ be the corresponding extensions on $\mathcal{U}'$, then if r_2, r_4 are minimal, it follows that $\Psi_0^{-1} \circ \Phi_0 = \alpha_0 \in \Gamma(U'_0, \mathcal{F}), \Psi_\infty^{-1} \circ \Phi_\infty = \alpha_\infty \in \Gamma(U'_\infty, \mathcal{F}) \Rightarrow \Phi_\infty^{-1} \circ \Phi_0 = \alpha_\infty^{-1} \circ \Psi_\infty^{-1} \circ \Psi_0 \circ \alpha_0$, ie., $[\xi'_1] = [\xi'_2]$.

Conversely, given $[\xi'] \in \mathcal{D}'(r_1)$, any representative cocycle, $\Phi_\infty^{-1} \circ \Phi_0$, may be pulled back to a cocycle over $\mathcal{U}$ by taking restrictions, $\Phi_k = \Phi_0\ |_{U_k}, \Phi_l = \Phi_0\ |_{U_l}$, for all $U_k, U_l \in \mathcal{U}$ such that $U_{k,l} \subset U'_0$, and similarly for Φ_∞ restricted to $U_{k,l} \subset U'_\infty$. Hence $\{\Phi_l^{-1} \circ \Phi_k\}_{k,l} = \xi$, such that $\chi([\xi]) = [\xi']$. If now $\Phi_\infty^{-1} \circ \Phi_0, \Psi_\infty^{-1} \circ \Psi_0$ are distinct representatives of $[\xi']$, then $\Phi_\infty^{-1} \circ \Phi_0 = \alpha_\infty^{-1} \circ \Psi_\infty^{-1} \circ \Psi_0 \circ \alpha_0 \Rightarrow \Phi_l^{-1} \circ \Phi_k = \alpha_l^{-1} \circ \Psi_l^{-1} \circ \Psi_k \circ \alpha_k$, where $\alpha_k = \alpha_0\ |_{U_k}, \alpha_l = \alpha_\infty\ |_{U_l}$, are the appropriate restrictions. Hence there is a well defined inverse map, and χ is a bijection.

From this point on, it will be assumed that $r_{2,4}$ are minimal with respect to r_1, hence the distinction between $\mathcal{D}(r_1)$ and $\mathcal{D}'(r_1)$ will be dropped. Let Λ correspond to the divisor, $\Sigma_{i=0}^{\gamma+1} n_i P_i$ on A_1, where $P_0 = 0, P_i = \lambda_i, 1 \leq i \leq \gamma, P_{\gamma+1} = \infty, n_0 = r_2, n_i = 1, n_{\gamma+1} = r_4$.

Proposition 4 $\mathcal{D}'(r_1) \cong H^1(A_1, \mathcal{T}_{A_1} \otimes [-\Lambda] \otimes \mathcal{N}^{-r_1})$, *where* $\mathcal{T}_{A_1}$ *is the tangent sheaf to* A_1, *and* $\mathcal{N}$ *is the normal sheaf, ie.,* $\mathcal{N} \cong (\mathcal{I}_{A_1}/\mathcal{I}_{A_1}^2)^*$.

Proof: Consider the sheaf homomorphism, $\eta : \mathcal{L} \to \mathcal{F}, \mathcal{L} = \mathcal{T}_{A_1} \otimes [-\Lambda] \otimes \mathcal{N}^{-r_1}$, defined for all $p \in A_1$ by

$$\eta(\varsigma)(f)_p = f_p + \varsigma(df)_p,$$

where ς denotes a given section-germ of $\mathcal{L}$, and f_p is any germ in $\mathcal{O}'_{V,p}$. Let

$$\eta_* : H^1(A_1, \mathcal{L}) \to H^1(\mathcal{U}', \mathcal{F})$$

be the induced map on cohomology. Note that $U_0 \cap U_\infty = U_0' \cap U_\infty'$, and that for each $[\xi'] \in \mathcal{D}'(r_1)$, there exists a unique cocycle of the form, $x^{r_2} y^{r_1} R_{\nu+1}(\psi) \frac{\partial}{\partial x} \in C^1(\mathcal{U}_1, \mathcal{L})$, representing $\eta_*^{-1}([\xi'])$. Hence $\mathcal{D}'(r_1) \hookrightarrow H^1(A_1, \mathcal{L})$, and inherits the structure of a linear subspace.

Now count dimensions. Clearly, $dim_{\mathbf{C}}(\mathcal{D}(r_1)) = deg(R(\psi)) - \nu = \gamma - 1 - \nu$. Conversely, note that the Chern classes, $c(\mathcal{T}_{A_1}) = 2, c(\mathcal{N}^{-r_1}) = r_1$, and $c([-\Lambda]) = -\gamma - r_2 - r_4$, imply $c(\mathcal{L}) = 2 + r_1 - \gamma - r_2 - r_4 = \nu - \gamma$. Now $\nu \geq \gamma - 1 \Rightarrow [\xi] \equiv 0$, therefore $\nu < \gamma - 1 \Rightarrow H^0(A_1, \mathcal{L}) = 0 \Rightarrow -dim_{\mathbf{C}}(H^1(A_1, \mathcal{L})) = c(\mathcal{L}) + 1 - g_{A_1} = \nu - \gamma + 1$, via Riemann-Roch, hence

$$dim_{\mathbf{C}}(H^1(A_1, \mathcal{L})) = dim_{\mathbf{C}}(\mathcal{D}(r_1)) = \gamma - \nu - 1.$$

Note that for r_1 sufficiently large, say $r_1 \geq n + 1$, we have $\nu(r_1) = r_1 - 2 - [\frac{r_1}{c_2}] - [\frac{r_1}{c_4}] \geq \gamma - 1$, hence $\mathcal{D}(r_1) = 0$. As r_1 takes all positive integer values from 1 to n, therefore, let $\mathcal{L}_k$ denote the line bundle $\mathcal{L}$ for the case $r_1 = k, 1 \leq k \leq n$. The space of infinitesimal deformations of $(M, \tilde{C})$ may now be defined as

$$\mathcal{D} = \bigoplus_{1 \leq k \leq n} \mathcal{D}(k) = \bigoplus_{1 \leq k \leq n} H^1(A_1, \mathcal{L}_k).$$

4.4 The Infinitesimal Deformation Map

From the machinary of the previous section, a canonical linear homomorphism from $D_1 \cong T_0 D_1$ to $\mathcal{D}$ has been all but determined. For each vector, $\vec{\varepsilon} \in T_0 D_1$, there is a deformation germ, $W + \Delta$, with strict transform, $\omega + \delta$, ie., $W + \Delta = \sigma_*(\omega + \delta)$, a corresponding locus, $\tilde{C}_\varepsilon$, and unique cohomology class, $[\xi]_1 \in H^1(A_1, \mathcal{L}_1)$. Hence define $[\xi]_1 = \rho_1(\vec{\varepsilon})$. Now let $D_2 = ker(\rho_1)$, and define $\rho_2 : D_2 \to H^1(A_1, \mathcal{L}_2)$, such that $\rho_2(\vec{\varepsilon}) = [\xi]_2$, etc. If $\bar{\rho}_k : D_k / D_{k+1} \to H^1(A_1, \mathcal{L}_k)$ are the corresponding induced isomorphisms, $1 \leq k \leq n$, then $T_0 D_1 \cong \bigoplus_{1 \leq k \leq n} D_k / D_{k+1}$ allows us to define

$$\rho_0 : T_0 D_1 \to \mathcal{D}, \qquad \text{where} \qquad \rho_0 = \Sigma_{k=1}^n \bar{\rho}_k.$$

Note that $\mathcal{D}$ contains the image of ρ_0, but need not coincide with it. ρ_0 is injective, however, for suppose $\vec{\varepsilon} \in D_{n+1}$, ie., $\rho_k(\vec{\varepsilon}) \equiv 0, 1 \leq k \leq n$. Then there exists an isomorphism, $(\Phi, \phi) : (\mathcal{V}, \mathcal{O}_\mathcal{V}) \to (\mathcal{W}, \mathcal{O}_\mathcal{W})$, which extends formally to $\Phi' : \mathcal{O}'_\mathcal{V} \to \phi^*(\mathcal{O}'_\mathcal{W})$ $(r_1 = n + 1, r_i$ sufficiently large$, i \neq 1)$, and hence between all formal neighbourhoods of $\mathcal{V}$ and $\mathcal{W}$. From theorem 6.13 of [14], it follows that there exists a biholomorphism, $B : (M, \tilde{C}) \to (M_\varepsilon, \tilde{C}_\varepsilon)$. But now there is an induced biholomorphism, $\tilde{B} : (U, C) \to (U, C_\varepsilon)$, such that $\tilde{B} \mid_{U \setminus \{0\}} = \sigma \circ B \circ \sigma^{-1} \mid_{U \setminus \{0\}}, \tilde{B}(0) = 0$, hence from the minimality of μ, the number of parameters of the local versal space, it follows that $\vec{\varepsilon} = 0$, ie., $D_{n+1} = \{0\}$.

We will conclude with a partial converse to the observation that the $\mathbf{C}^*$-action: $(X, Y) \mapsto (t^{\frac{p}{\gamma}} X, t^{\frac{q}{\gamma}} Y)$, induces one-parameter biholomorphic equivalence classes of fibers in $\mathcal{M}$, having a structure jump at M_0. Suppose $B : (M_\varepsilon, \tilde{C}_\varepsilon) \to (M_{\varepsilon'}, \tilde{C}_{\varepsilon'})$ is a biholomorphism, $\mathcal{W} = (\sigma \times 1_{D_1})^{-1}(0,\varepsilon), \mathcal{X} = (\sigma \times 1_{D_1})^{-1}(0,\varepsilon')$, and let $\hat{B} : (\mathcal{W}, \mathcal{O}_\mathcal{W}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be the induced isomorphism. Moreover, suppose $\Phi : (\mathcal{V}, \mathcal{O}_\mathcal{V}) \to (\mathcal{W}, \mathcal{O}_\mathcal{W}), \Psi : (\mathcal{V}, \mathcal{O}_\mathcal{V}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X})$ are isomorphisms which map the formal neighbourhood, $A(\mathrm{m})$, corresponding to C^*, to those of $C^*_\varepsilon, C^*_{\varepsilon'}$ respectively. As usual, let $\Phi_0, \Psi_0, \Phi_\infty, \Psi_\infty$ be the local extensions to $\mathcal{O}'_\mathcal{V}$, corresponding to $A(\mathrm{n})$, and define $\varpi_0 = \Psi_0^{-1} \circ \hat{B} \circ \Phi_0, \varpi_\infty = \Psi_\infty^{-1} \circ \hat{B} \circ \Phi_\infty$, such that

$$\varpi_\infty^{-1} \circ \Psi_\infty^{-1} \circ \Psi_0 \circ \varpi_0 = \Phi_\infty^{-1} \circ \Phi_0 \qquad (*).$$

Though automorphisms of $\mathcal{O}'_\mathcal{V}(U'_0), \mathcal{O}'_\mathcal{V}(U'_\infty)$ which restrict to the identity on $\mathcal{O}_\mathcal{V}$,

ϖ_0, ϖ_∞ need not belong to $\mathcal{A}ut(\mathrm{m} : \mathrm{n})$. Now consider

$$\Phi_\infty^{-1} \circ \Phi_0(x) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_1) \qquad \Psi_\infty^{-1} \circ \Psi_0(x) = x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_2),$$

etc.. Once again, these expressions, along with $\varpi_0(x)$, extend to U'_0, hence, in terms of local coordinates on U'_0, U'_∞, the relation $(*)$ implies

$$\varpi_0(x) = x + x^{r_2} y^{r_1} \omega(x) g(x), \qquad \varpi_0(y) = \tau(y) + x^{r_2} y^{r_1} h(x),$$

$$\varpi_\infty^{-1}(x) = x + x^{r_2} y^{r_1} \theta(x), \qquad \varpi_\infty^{-1}(y) = \tau^{-1}(y),$$

where $deg(\theta) \leq \nu$, and $\tau \in PSL(2, \mathbf{C}), \tau(0) = 0$. We therefore have

$$\Phi_\infty^{-1} \circ \Phi_0(x) = x + x^{r_2} y^{r_1} \theta(x) + x^{r_2} \tau^{-1}(y)^{r_1} (R_{\nu+1}(\psi_2) + \omega(x) g(x))$$

$$= x + x^{r_2} y^{r_1} R_{\nu+1}(\psi_1).$$

Note $x^{r_2}\tau^{-1}(y)^{r_1} \equiv x^{r_2}t^{r_1}y^{r_1}$, mod(n), where $\frac{d\tau^{-1}}{dy}(0) = t \neq 0$. Now

$$deg(R_{\nu+1}(\psi_1)) \leq \gamma - 1 \Rightarrow g \equiv 0, o(R_{\nu+1}(\psi_1)) \geq \nu + 1 \Rightarrow \theta \equiv 0,$$

hence $R_{\nu+1}(\psi_1) = t^{r_1}R_{\nu+1}(\psi_2)$. Finally, if $\vec{\varepsilon_1}, \vec{\varepsilon_2} \in D_k \subset D_1$, such that $k = r_1, \rho_0(\vec{\varepsilon_1}) \sim R_{\nu+1}(\psi_1), \rho_0(\vec{\varepsilon_2}) \sim R_{\nu+1}(\psi_2)$, it follows that $\vec{\varepsilon_1} - t^k\vec{\varepsilon_2} \in D_{k+1}$.

It should be remarked that the construction of $\mathcal{D}$ above yields a space of deformations of M_0 which only approximates the simultaneous resolution of the minimal versal family of deformations of $0 \in C$. The precise relationship between $\mathcal{D}$ and $\mathcal{E}xt^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)_0$ is not yet understood.

Chapter 5

Appendix: Finite-Dimensional Cohomology on a Complex Domain

5.1 Pseudoconvexity, pseudoconcavity, and the finiteness theorems of Andreotti-Grauert

Let X initially be taken to be a complex manifold, $dim_{\mathbf{C}}(X) = n$, and consider a relatively compact subdomain, $D \subset\subset X$, with smooth C^2-boundary, ie., for all $p \in \partial D$, there exists an open neighbourhood, U, of p, and an "exhaustion function", $\phi \in C^2(U)$, such that

(i) $\partial D \cap U = \{x \in U \mid \phi(x) = 0\}$, and

(ii) $D \cap U = \{x \in U \mid \phi(x) < 0\}$.

Note that U is diffeomorphic to an open subset of $\mathbf{R}^{2n}$, and $\partial D \cap U$ is a C^2-submanifold, M, such that $dim_{\mathbf{R}}(M) = 2n - 1$. Moreover, suppose $d\phi_p = \partial\phi_p + \bar{\partial}\phi_p \neq 0$, and consider the holomorphic tangent space to M at p, ie.,

$$\mathcal{H}_p(M) = \{w \in \mathbf{C}^n \mid \Sigma_{k=1}^n \frac{\partial \phi}{\partial z_k}(p) w_k = 0\} = ker(\partial\phi)_p.$$

Note $dim_{\mathbf{C}}(\mathcal{H}_p(M)) = n - 1$. The "Hessian Form" corresponds to

$$\bar{\partial}\partial\phi = \Sigma_{l,k} \frac{\partial^2 \phi}{\partial z_l \partial \bar{z}_k} dz_l \wedge d\bar{z}_k \in \Omega^2(U),$$

and is invariant under holomorphic changes of coordinates in U (cf. eg., [9]). The "Levi Form" of ϕ then corresponds to

$$L(\phi) = \bar{\partial}\partial\phi \mid_{\mathcal{H}(M)} .$$

Note that $L(\phi)_p$, with respect to any system of coordinates, $(z_1, ..., z_n)$, on U, is represented by an $(n-1) \times (n-1)$ Hermitian matrix. D is therefore said to be "strictly q-pseudoconvex" at p if $L(\phi)_p$ has at least $n - q$ positive eigenvalues. Conversely, D is "strictly q-pseudoconcave" at p if $L(\phi)_p$ has at least $n - q$ negative eigenvalues. Suppose $L(\phi)_p$ has precisely $q(p)$ positive (resp. negative) eigenvalues, and let C be a connected boundary component of D. Then C is said to be strictly Q-pseudoconvex (resp. Q-pseudoconcave) if it is so at each $p \in C$, and $Q = sup_{p\in C} q(p)$. Note that D is relatively compact implies C is compact, hence consider a finite open cover, $\mathcal{U} = \{U_\alpha\}_{1\leq\alpha\leq N}$, of C by domains of holomorphy, $U_\alpha \subset X$. Let $\phi_\alpha \in C^2(U_\alpha)$ be an exhaustion function such that $d\phi_\alpha \neq 0$, $1 \leq \alpha \leq N$. If $\{\rho_\alpha\}_{1\leq\alpha\leq N}$ denotes a partition of unity subordinate to $\mathcal{U}$, then there exists a function, $\Phi \in C^2(W), W = \cup_{1\leq\alpha\leq N} U_\alpha$, given by

$$\Phi = exp(c\Sigma_{\alpha=1}^N \rho_\alpha \phi_\alpha) - 1,$$

(cf. [1], proposition 15). Φ has the following properties:

(i) $C = \{x \in W \mid \Phi(x) = 0\}$,

(ii) $D \cap W = \{x \in W \mid \Phi(x) < 0\}$,

(iii) $d\Phi_p \neq 0$, for all $p \in C$, and

(iv) For all $p \in W$, $\bar{\partial}\partial\Phi_p$ corresponds to an $n \times n$ Hermitian matrix, having at least $n - Q + 1$ positive (resp. negative) eigenvalues, for c sufficiently large.

In their paper, [1], Andreotti and Grauert introduced the notions of q-pseudoconvexity and q-pseudoconcavity for a complex analytic space, X, and proved that certain cohomology groups, $H^k(X, \mathcal{F})$, where $\mathcal{F}$ is an arbitrary coherent sheaf, are finite dimensional. A key step in the argument developed by these authors is to prove the result first for a relatively compact subdomain, $D \subset\subset X$, having boundary, ∂D, which is either q-pseudoconvex or q-pseudoconcave. The main question addressed in this chapter is whether a finiteness theorem holds for a domain, D, having boundary components, C_i, of varying convexity, as well as components, C_j, of varying concavity. In section two, it will be shown that the argument employed in [1] may be extended to this case, when X is a manifold, or in so far as the intersection of ∂D with the singular locus of X is empty . It will be assumed, for convenience, that each C_i and C_j is a C^2-submanifold of X, in order that the amount of convexity or concavity may be defined intrinsically via a Levi form on the holomorphic tangent bundle of the component (the result is not altered, however, if the C_i and C_j are allowed to be singular). The third section will then contain an extension of the vanishing theorem 4, of [2], with respect to a metrically pseudoconvex line bundle on X.

5.2 Finiteness theorem

Consider $\partial D = \partial' \amalg \partial''$, where $\partial' = \amalg_{1 \leq i \leq r} C_i$, such that C_i is strictly Q_i-pseudoconvex, and $\partial'' = \amalg_{r+1 \leq j \leq s} C_j$, such that C_j is strictly Q_j-pseudoconcave. Let $Q = sup_{1 \leq i \leq r} q_i, \quad \hat{Q} = sup_{r+1 \leq j \leq s} q_j$.

Theorem 8 *If $\mathcal{F} \to X$ is a coherent analytic sheaf, then*

$$dim_{\mathbf{C}} H^k(D, \mathcal{F}) < +\infty, \qquad Q \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1,$$

where $dh(\mathcal{F}) = inf_{x \in X} dh_x(\mathcal{F})$, denotes the homological dimension of $\mathcal{F}$.

Proof: The argument is divided into three lemmas, adapted from [1].

Lemma 3 *(a) There exists an ascending sequence,*

$$D = B_0 \subset B_1 \subset ... \subset B_N,$$

of open sets, relatively compact in X, and a finite open cover, $\mathcal{U} = \{U_\alpha\}_{1 \leq \alpha \leq N}$ of ∂ by domains of holomorphy, such that

(i) $B_\alpha - B_{\alpha-1} \subset\subset U_\alpha, \qquad 1 \leq \alpha \leq N$, and

(ii) $H^k(U_\alpha \cap B_\beta, \mathcal{F}) = 0, \qquad k \geq Q, 1 \leq \alpha, \beta \leq N$.

(b) There exists an ascending sequence,

$$B_N = A_0 \subset A_1 \subset ... \subset A_{\hat{N}},$$

of open sets, relatively compact in X, and a finite open cover, $\hat{\mathcal{U}} = \{\hat{U}_\gamma\}_{1\leq\gamma\leq\hat{N}}$ *of* $\hat{\partial}$*, such that*

(i) $A_\gamma - A_{\gamma-1} \subset\subset \hat{U}_\gamma, \qquad 1 \leq \gamma \leq \hat{N}$*, and*

(ii) $H^k(\hat{U}_\gamma \cap A_\delta, \mathcal{F}) = 0, \qquad 1 \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1, 1 \leq \gamma, \delta \leq \hat{N}$.

Proof: (a) Construct exhaustion functions, Φ_i, with respect to finite open covers, $\mathcal{U}_i = \{U_\alpha^i\}_{1\leq\alpha\leq N_i}$, of $C_i, 1 \leq i \leq r$, such that $supp(\Phi_{i_1}) \cap supp(\Phi_{i_2}) = \emptyset$. Now let

$$\Phi = \Sigma_{1\leq i\leq r}\Phi_i,$$

and apply the following "bumping argument" to ∂ by means of Φ. If $N = \Sigma_{1\leq i\leq r}N_i$, then simply relabel,

$$\mathcal{U} = \cup_{1\leq i\leq r}\mathcal{U}_i = \{U_\alpha^{**}\}_{1\leq\alpha\leq N},$$

so that Φ will satisfy properties (i) - (iv) of the previous section. Now choose

$$U_\alpha \subset\subset U_\alpha^* \subset\subset W = \cup_{1\leq\alpha\leq N}U_\alpha^{**},$$

such that $\cup_{1\leq\alpha\leq N}U_\alpha$ also covers ∂, and there is an isomorphism,

$$\psi_\alpha : U_\alpha^* \to B(\delta_\alpha),$$

where $B(\delta_\alpha) \subset \mathbf{C}^n$ is a ball of radius δ_α, such that

$$\Phi \mid_{U_\alpha^*} = \varphi_\alpha \circ \psi_\alpha,$$

for some $\varphi_\alpha \in C^\infty(B(\delta_\alpha), 1 \leq \alpha \leq N$. Moreover,

$$\psi_\alpha(U_\alpha) \subset\subset \psi_\alpha(U_\alpha^*) \Rightarrow H^k(\psi_\alpha(U_\alpha \cap D), \psi_\alpha^*(\mathcal{F})) = 0, \quad k \geq Q,$$

(cf. [1], theorem 5), hence $H^k(U_\alpha \cap D, \mathcal{F}) = 0$. Now take a partition of unity, $\{\rho_\alpha\}_{1\le\alpha\le N}$, on $\cup_{1\le\alpha\le N} U^*_\alpha$, such that $supp(\rho_\alpha) \subset U_\alpha$, and $\Sigma_{1\le\alpha\le N}\rho_\alpha(\xi) > 0$, for all $\xi \in \partial$. Define $\rho_{\alpha,\beta} \in C_0^\infty(B(\delta_\alpha))$, with

$$\rho_\alpha \mid_{U^*_\beta} = \rho_{\alpha,\beta} \circ \psi_\alpha.$$

Now, for ε_β sufficiently small, let

$$\Phi_m = \Phi - \Sigma_{1\le\beta\le m}\varepsilon_\beta\rho_\beta, \quad 1 \le \beta \le N,$$

such that Φ_m remains strictly Q-pseudoconvex. Similarly, define

$$\varphi_{m,\alpha} = \varphi_\alpha - \Sigma_{1\le\beta\le m}\varepsilon_\beta\rho_{\alpha,\beta},$$

so that if

$$\tilde{B}_m = \{x \in W \mid \Phi_m(x) < 0\},$$

it follows that

$$H^k(\psi_\alpha(\tilde{B}_m \cap U_\alpha), \psi^*_\alpha(\mathcal{F})) = 0 \Rightarrow H^k(\tilde{B}_m \cap U_\alpha, \mathcal{F}) = 0,$$

$k \ge Q$. Note that $\tilde{B}_m - \tilde{B}_{m-1} \subset\subset U_m, \quad 1 \le m \le N$, hence define

$$B_\alpha = D \cup \check{B}_\alpha,$$

and the proof of part (a) is complete.

(b) Similarly, construct exhaustion functions, $\hat{\Phi}_j$ with respect to each $C_j, r+1 \le j \le s$, and apply the bumping argument to $\hat{\partial}$ by means of

$$\hat{\Phi} = \Sigma_{r+1\le j\le s}\hat{\Phi}_j.$$

In particular, if $\hat{U}_\gamma \subset\subset \hat{U}^*_\gamma \subset\subset \hat{U}^{**}_\gamma, \quad 1 \leq \gamma \leq \hat{N}$, is the corresponding open cover of $\hat{\partial}$, with isomorphisms, $\hat{\psi}_\gamma : \hat{U}^*_\gamma \to B(\delta_\gamma)$, such that

$$\hat{\Phi} \mid_{\hat{U}^*_\gamma} = \hat{\varphi}_\gamma \circ \hat{\psi}_\gamma,$$

for some $\hat{\varphi}_\gamma \in C^\infty(B(\delta_\gamma))$, then [1], theorem 9, implies

$$H^k(\hat{\psi}_\gamma(\hat{U}_\gamma \cap B_N), \hat{\psi}^*_\gamma(\mathcal{F})) = H^k(\hat{U}_\gamma \cap B_N, \mathcal{F}) = 0, \quad 1 \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1.$$

Now choose a partition of unity, $\{\hat{\rho}_\gamma\}_{1 \leq \gamma \leq \hat{N}}$ and $\hat{\rho}_{\gamma,\eta} \in C^\infty_0(B(\delta_\gamma))$, such that

$$\hat{\Phi}_n = \hat{\Phi} + \Sigma_{1 \leq \gamma \leq n} \varepsilon_\gamma \hat{\rho}_\gamma, \quad \hat{\phi}_{n,\gamma} = \hat{\varphi}_\gamma + \Sigma_{1 \leq \eta \leq n} \varepsilon_\eta \hat{\rho}_{\gamma,\eta},$$

for ε_γ sufficiently small, remain strictly $\hat{Q}$-pseudoconcave, $1 \leq n \leq \hat{N}$. Then

$$\tilde{A}_n = \{x \in \hat{W} \mid \hat{\Phi}_n(x) > 0\} \Rightarrow H^k(\hat{U}_\gamma \cap \tilde{A}_n, \mathcal{F}) = 0, \quad 1 \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1.$$

Now let $A_\gamma = \tilde{A}_\gamma \cup B_N, 1 \leq \gamma \leq \hat{N}$, and the result for part (b) follows.

Lemma 4 *(a) The homomorphism,*

$$\eta : H^k(B_N, \mathcal{F}) \to H^k(D, \mathcal{F}),$$

induced by restriction, is surjective for $k \geq Q$. Similarly,

(b) The homomorphism,

$$\hat{\eta} : H^k(A_{\hat{N}}, \mathcal{F}) \to H^k(B_N, \mathcal{F}),$$

is surjective, for $1 \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1$.

Proof: Apply inductively the Mayer-Vietoris argument of [1], propositions 16 and 17. Note in particular that for part (a), $B_{\gamma+1} = B_\gamma \cup (B_{\gamma+1} \cap U_{\gamma+1}), B_\gamma \cap (B_{\gamma+1} \cap U_{\gamma+1}) = B_\gamma \cap U_{\gamma+1}$, and $B_{\gamma+1} \cap U_{\gamma+1}, B_\gamma \cap U_{\gamma+1}$ are both acyclic. Hence

$$... \to H^k(B_{\gamma+1}, \mathcal{F}) \to H^k(B_\gamma, \mathcal{F}) \oplus H^k(B_{\gamma+1} \cap U_{\gamma+1}, \mathcal{F}) \to H^k(B_\gamma \cap U_{\gamma+1}, \mathcal{F}) \to ...$$

implies that $H^k(B_{\gamma+1}, \mathcal{F}) \to H^k(B_\gamma, \mathcal{F})$ is surjective, and the result follows. Similarly for part (b).

We conclude that $D \subset\subset A_{\hat{N}}$, and

$$\eta \circ \hat{\eta} : H^k(A_{\hat{N}}, \mathcal{F}) \to H^k(D, \mathcal{F})$$

is surjective, for $Q \leq k \leq dh(\mathcal{F}) - \hat{Q} - 1$.

Lemma 5 $dim_{\mathbf{C}} H^k(D, \mathcal{F}) < +\infty, \qquad q \leq k \leq dh(\mathcal{F}) - \hat{q} - 1.$

Proof: The result follows automatically from [1], theorem 11, which draws on a fundamental theorem of L. Schwartz, concerning continuous linear mappings of Fréchet spaces. Hence the proof of theorem 8 is now complete.

Theorem 8 may now be formally extended to the case of X a (reduced) complex analytic space, $D \subset\subset X$. Consider $p \in \partial D$, and a neighbourhood, U, of p, biholomorphically equivalent, under a map, ψ, to a subvariety, $V = \psi(U)$, of a domain, $U^* \subseteq \mathbf{C}^m$. Let ϕ be defined on U such that $\phi \circ \psi^{-1}$ is the restriction to V of an exhaustion function, $\tilde{\phi} \in C^2(U^*)$, and suppose

$$\psi(U \cap D) = \{x \in V \mid \tilde{\phi}(x) < 0\}.$$

D is then said to be strictly q-pseudoconvex (resp. q-pseudoconcave), if $L(\tilde{\phi})_p = \bar{\partial}\partial\tilde{\phi} \mid_{\mathcal{H}_p(M)}$ has $n - q$ positive (resp. negative) eigenvalues, where $\psi(D \cap U)$ is assumed to be of the form, $M \cap V$, for some C^2-submanifold, $M \subseteq U^*$, of real dimension $2n - 1$. If C is a connected component of ∂D, let $\mathcal{U} = \{U_\alpha\}_{1 \leq \alpha \leq N}$ once more be a finite open cover of domains of holomorphy, and $V_\alpha = \psi_\alpha(U_\alpha) \subset U_\alpha^*, 1 \leq \alpha \leq N$. Suppose there exists a global exhaustion function, Φ, on $\cup_{1 \leq \alpha \leq N} U_\alpha$, such that $\Phi \circ \psi_\alpha^{-1}$ is the restriction to V_α of an exhaustion function, $\tilde{\phi}_\alpha \in C^2(U_\alpha^*)$, having $n - q + 1$ positive (resp. negative) eigenvalues . With the help of [1], theorems 5 and 9, the three lemmas of theorem 8 may then be applied to the case of a complex analytic space, X, without further modification.

Example: Let $X = \mathbf{C}^5 \times \mathbf{P}_1 \times \mathbf{P}_1$, and

$$B_{\epsilon_1} = \{z \in \mathbf{C}^5 \mid |z| < \epsilon_1\}. B_{\epsilon_2} = \{(z, \zeta) \in \mathbf{C}^5 \times \mathbf{P}_1 \mid |(z, \zeta)| < \epsilon_2\},$$

$$B_{\epsilon_3} = \{(z, \zeta, \xi) \in \mathbf{C}^5 \times \mathbf{P}_1 \times \mathbf{P}_1 \mid |(z, \zeta, \xi)| < \epsilon_3\},$$

with $0 < \epsilon_3 < \epsilon_2 < \frac{1}{2}\epsilon_1$. Let $\hat{B}_{\epsilon_3}$ be a translate of B_{ϵ_3} such that

$$\hat{B}_{\epsilon_3} \subset\subset ((B_{\epsilon_1} \times \mathbf{P})_1 - B_{\epsilon_2}) \times \mathbf{P}_1.$$

Now define $D = (((B_{\epsilon_1} \times \mathbf{P}_1) - B_{\epsilon_2}) \times \mathbf{P}_1) - \hat{B}_{\epsilon_3}$. It follows that for any locally free sheaf, $\mathcal{F}$ (therefore $dh(\mathcal{F}) = 7$), over X,

$$dim_{\mathbf{C}} H^k(D, \mathcal{F}) < +\infty, \qquad k = 3, 4.$$

(Remark: When $X = \mathbf{C}^n$ it follows from [1]. theorem 15, that the boundary components, $C_j, r + 1 \leq j \leq s$, may even be isolated points.)

5.3 Vanishing Theorem

Consider a holomorphic line bundle, $L \xrightarrow{\pi} X$, over the (reduced) analytic space, X, and a Hermitian metric, χ, on the fibers of L, given locally by C^∞-functions, $h_i : U_i \to \mathbf{R}_+$, such that $h_i(z^i) = |g_{i,j}(z^j)|^2 h_j(z^j)$, where $g_{i,j} : U_i \cap U_j \to \mathbf{C}^*$ are the transition functions of L with respect to the cover, $\mathcal{U} = \{U_i\}_{i\in I}$, of X. If $\vec{v} = (z^i, \xi^i) \in \pi^{-1}(U_i)$, where $z^i = \pi(\vec{v}), \xi^i \in \mathbf{C}$, then $\|\vec{v}\|^2 = h_i(z^i)|\xi^i|^2 = \chi(z^i, \xi^i)$, represents in local coordinates the length of $\vec{v}$ with respect to χ. Following the terminology of [2], L is said to be "metrically pseudoconvex" if χ exists such that $\chi(z^i, \xi^i) = \|\vec{v}\|^2$ is a strictly pseudoconvex function on $\{(z^i, \xi^i) \mid \xi^i \neq 0\}$, for each $i \in I$. Correspondingly, the dual bundle, L^*, is then said to be "metrically pseudoconcave".

Now consider X compact, and the domain, $D \subset\subset X$, with $\partial D = \partial' \amalg \partial''$, defined as in the previous section. Let $D = \cap_{1\leq\mu\leq s} D_\mu, X = \cup_{1\leq\mu\leq s} D_\mu$, where D_i is a "strictly q_i-pseudoconvex space", $1 \leq i \leq r$, and D_j is a "strictly q_j-pseudoconcave space", $r+1 \leq j \leq s$. By these terms is meant that for each i, j there exist compact $K_i \subset D_i, K_j \subset D_j$, such that the exhaustion functions, $\Phi_i \mid_{D_i - K_i}, \Phi_j \mid_{D_j - K_j}$, are strictly q_i-pseudoconvex and q_j-pseudoconcave respectively. By [2], theorem 2, given a coherent sheaf, $\mathcal{F} \to X$, and a metrically pseudoconvex line bundle, $L \to X$, there exists an integer, $\nu_j = \nu_j(\mathcal{F}, L)$, such that

$$H^k(D_j, \mathcal{F} \otimes \mathcal{O}(L^\nu)) = 0, \qquad \nu \geq \nu_j, 1 \leq k \leq dh(\mathcal{F}) - q_j - 1.$$

Moreover, since X is compact, there exists an integer, $\nu_0 = \nu_0(\mathcal{F}, L)$, such that

$$H^k(X, \mathcal{F} \otimes \mathcal{O}(L^\nu)) = 0, \qquad \nu \geq \nu_0, 1 \leq k \leq dh(\mathcal{F}) - 1,$$

(cf. [1], [2]). Now let $\hat{\nu} = sup_{r+1 \leq j \leq s}\{\nu_0, \nu_j\}$, and $\mathcal{F}_1 = \mathcal{F} \otimes \mathcal{O}(L^{\hat{\nu}})$. If it is assumed that $X = D_M \cup (\cap_{\mu \geq M+1} D_\mu), 1 \leq M \leq s-1$, then from the successive Mayer-Vietoris sequences,

$$H^k(X, \mathcal{F}_1) \rightarrow H^k(D_M, \mathcal{F}_1) \oplus H^k(\cap_{\mu \geq M+1} D_\mu, \mathcal{F}_1) \rightarrow H^k(\cap_{\mu \geq M} D_\mu, \mathcal{F}_1) \rightarrow H^{k+1}(X, \mathcal{F}_1)$$

$1 \leq M \leq s-1$, it follows that

$$H^k(D, \mathcal{F}_1) \cong \bigoplus_{1 \leq \mu \leq s} H^k(D_\mu, \mathcal{F}_1) \cong \bigoplus_{1 \leq i \leq r} H^k(D_i, \mathcal{F}_1),$$

for $1 \leq k \leq dh(\mathcal{F}) - \hat{q} - 2$. Now suppose that D_i is "q_i-complete", ie., the compact set, K_i, defined above, is empty, $1 \leq i \leq r$. By the corollary to [1], theorem 14, it follows that $H^k(D_i, \mathcal{F}_1) = 0, k \geq q_i$, hence, as a straightforward extension of [2], theorem 4, we have

Theorem 9 *If $D \subset\subset X$, with $\partial D = \partial' \amalg \partial''$, as above, and each $D_i \subset X$ is a q_i-complete analytic space, $1 \leq i \leq r$, then for any metrically pseudoconvex line bundle, $L \rightarrow X$, and any coherent sheaf, $\mathcal{F} \rightarrow X$, there exists $\hat{\nu} \in \mathbf{Z}_+$ such that*

$$H^k(D, \mathcal{F} \otimes \mathcal{O}(L^{\hat{\nu}})) = 0, \qquad q \leq k \leq dh(\mathcal{F}) - \hat{q} - 2.$$

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