Singular Integral Operators, Contraction Operators and Principal Currents

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Abstract of dissertation

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In this dissertation we study the relation between unitary invariants of the generator of certain almost commuting C^*-algebras and the metric geometry of the associated principal currents to the generators.

We explicitly construct a principal current for an operator triple $\{P, W, H_2\}$ or equivalently for the unitary-normal operator pair $\{W, P + iH_2\}$, where W is a unitary operator, H_2 is a self-adjoint operator and P is the projection to the absolutely continuous spectral subspace of H_2.

The determinant of the characteristic operator function of the contraction

operator $T = PWP$ is calculated in term of the intersection geometry of the principal current of $\{P, W, H_2\}$. It turns out that extracting the unitary invariant information of T depends on the metric geometry of the principal current, not merely the topology.

As application, we study singular integral operators with unimodular symbols. A necessary and sufficient condition is given for such operators to be unitary operators. And an index theorem for contractive singular integral operators is derived.

Shicong and Tracey

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Chapter 1

Introduction

Principal current theory was invented by J. Pincus in order to study certain almost commuting C^*-algebras with a finite number of generators. In this work, we explore the relation between the unitary invariants of the generators and metric and intersection geometry of the associated principal currents, which is beyond the scope of K-theory.

The following singular integral operator in $L_2(E)$ was studied in [24], [26], [28], [29]:

$$Lf(\lambda) = A(\lambda)f(\lambda) + \frac{1}{\pi i}k^*(\lambda)\int_E \frac{k(t)f(t)}{t-\lambda}dt. \tag{1.1}$$

where E is a bounded measurable set on the real line, $A(\lambda)$ is a real measurable function and $k(\lambda) \in L_2(E)$. L is a bounded self-adjoint operator if the coefficients $A(\lambda), k(\lambda)$ are bounded. It was the diagonalization of this operator which lead to the discovery of the principal function theory. However, L is in general (cf. [28]) an unbounded operator if the coefficients are unbounded. Necessary and sufficient conditions were given for L to be an unbounded self-adjoint operator in [33]. In [24], it was shown that L defines a symmetric operator on a dense domain in $L_2(E)$ when $k(\lambda)$ is square integrable, and the deficiency indices of the symmetric operator in terms of the singularities of the symbols $A(\lambda) \pm |k(\lambda)|^2$.

Let $g(x, \lambda)$ be the characteristic function of the set $D = \{(x, \lambda) \mid A(\lambda) - |k(\lambda)|^2 < x < A(\lambda) + |k(\lambda)|^2\}$. Then $g(x, \lambda)$ is the principal function of a unbounded pair of self-adjoint operators $\{H_1, H_2\}$, see [24]. Let $G(\tau, \lambda)$ be the characteristic function of the image of D on the cylinder $R \times S^1$ under the Cayley transform in the x variable, i.e. $G(\tau, \lambda) = g(x, \lambda)$ for $\tau = (x + i)(x - i)^{-1}$. By the general canonical model construction (see [4], [6]), there is, up to unitary equivalence, a unique pair of unitary and self-adjoint operators $\{W, H_2\}$ having rank one commutator and $G(\tau, \lambda)$ as its principal function. In fact, W is the Cayley transform of H_1 (see [24]). And the symmetric operator defined by (1.1) then is the inverse Cayley transform of the compression of W to a certain canonical subspace found by J. Pincus.

It is an early result of J. Pincus that the von Neumann multiplicity functions of W and H_2 are given by the intersection geometry of the principal function $G(\tau, \lambda)$. That is, the multiplicity of W at τ is computed by the number of intervals formed when a level line ar τ intersects the support of $G(\tau, \lambda)$ while the multiplicity of H_2 at λ is computed by the number of intervals formed when a horizontal circle at λ intersects the support of $G(\tau, \lambda)$. See [26]. Therefore H_2 has almost everywhere multiplicity one.

Let P be the projection to the absolutely continuous spectral subspace of

H_2, $T = PWP$ and

$$h(\lambda) = \frac{1}{2\pi i} \int G(\tau, \lambda) \frac{d\tau}{\tau} \tag{1.2}$$

be the average of $G(\tau, \lambda)$. Then by a result of Verblunsky [38], there are positive Borel measures $\mu^{\pm}$ defined on the real line so that for $\Im z \neq 0$

$$\exp \int \frac{h(\lambda)}{\lambda - z} d\lambda = 1 + \frac{1}{\pi} \int \frac{1}{\lambda - z} d\mu^{+}(\lambda). \tag{1.3}$$

and

$$\exp - \int \frac{h(\lambda)}{\lambda - z} d\lambda = 1 - \frac{1}{\pi} \int \frac{1}{\lambda - z} d\mu^{-}(\lambda). \tag{1.4}$$

Let $\mu_s^{\pm}$ be the singular parts of $\mu^{\pm}$ in the Lebesgue decomposition. It was shown in [24] that the dimension of $L_2(\mu_s^+), L_2(\mu_s^-)$ give the the deficiency indices of the contraction operator T as well as of the symmetric operator L. In this work, we are mainly interested in the case when $\mu_s^{\pm}$ have only a finite number of atoms, i.e. T has finite deficiency indices. Therefore we assume that μ_s^+ has atoms at $\{\lambda_1^+, \lambda_2^+, \ldots, \lambda_n^+\}$ and μ_s^- has atoms at $\{\lambda_1^-, \lambda_2^-, \ldots, \lambda_m^-\}$.

As a projection to the absolutely continuous subspace of H_2, P is a function of H_2, but not a smooth function. The C^*-algebra $C^*(P, W, H_2)$, generated by $\{P, W, H_2\}$, is contained in the von Neuman algebra $W^*(W, H_2)$ generated by W and H_2. In this respect, we are studying a proper C^*-subalgebra of $W^*(W, H_2)$, which contains the generators.

Principal currents have been constructed previously by Carey and Pincus for pairs of operators consisting of a normal operator N and a bounded self adjoint operator X with commutator $[N, X]$ in trace class. The main ideas of that unpublished work are recounted in [35]. A special case was used in the discussion of Toeplitz operators on multiply connected domains in [35]. The present work relates to a normal operator $P + iH_2$ and a unitary operator W. The interest of the present theorem is both that P is not a smooth function of H_2, and that the spectrum of W is the full unit circle.

In chapter 3, we construct a principal current $\Upsilon_{\{P,H_2,W\}}$ supported on the product spectrum $\sigma(P) \times \sigma(W) \times \sigma(H_2)$ for the operator triple $\{P, W, H_2\}$ or the normal and unitary operator pair $\{P + iH_2, W\}$. And the projections of the current to each coordinate plane give the principal currents of the corresponding operator pairs. The support of the current is a certain minimal span of the joint essential spectrum of the generators and certain circles erected over the point spectrum of H_2, the singular numbers $\{\lambda_j^\mp\}$.

The fact that the principal current $\Upsilon_{\{P,H_2,W\}}$ for the operator triple $\{P, W, H_2\}$ has the form indicated below was stated by Pincus (without proof) in [25]. But the proof given here, which has benefited from the advice of Pincus, provides a more detailed understanding of the structure of the current that was previously available.

Theorem 1.1 *There is a rectifiable current $\Upsilon_{\{P,H_2,W\}}$ with support in $[0,1] \times \sigma(W) \times \sigma(H_2)$ such that*

$$\Upsilon_{\{P,H_2,W\}}(df \wedge dh) = \mathrm{tr}[f(P,W,H_2), h(P,W,H_2)]$$

$$= \frac{1}{2\pi} \iint_\Sigma (df \wedge dh, \eta) G(x,\tau,\lambda) d\mathcal{H}^2, \tag{1.5}$$

where Σ is a Hausdorff two measure rectifiable subset of $R \times S^1 \times R$; namely,

$$\Sigma = \{1\} \times \{\text{support of} G(\tau,\lambda)\} \cup (\cup_{r=1}^n [0,1] \times S^1 \times \{\lambda_r^+\}) \cup (\cup_{j=1}^m [0,1] \times S^1 \times \{\lambda_j^-\});$$

with $G(1,\tau,\lambda) = G(\tau,\lambda)$, $G(x,\tau,\lambda_r^+) = G(x,\tau,\lambda_j^-) = 1, 0 \le x \le 1$. The orienting vector, η, of Σ is the outward unit normal on $\{1\} \times \{\text{support of} G(\tau,\lambda)\}$, it *is the inward unit normal on* $[0,1] \times S^1 \times \{\lambda_r^+\}$, *that is it points to the negative* x*-axis and on* $[0,1] \times S^1 \times \{\lambda_j^-\}$ *it points to the positive* x*-axis.*

The Lifshitz characteristic operator function of a contraction is defined as

$$\Theta_T(z) = (-T + D_{T^*}(1 - zT^*)^{-1} D_T)|\mathcal{D}_T \tag{1.6}$$

where $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$, and $\mathcal{D}_T$ is the closure of the range of D_T. $\Theta_T(z)$ is a completely unitary invariant for T when T is completely non-unitary. For convenience, we suppose now that W has almost everywhere finite multiplicity. this is not essential, as we will see later. We know that the multiplicity of W at τ can be read off from the principal function $G(\tau, \lambda)$ by counting the number of intervals formed when a level line at τ intersects the support of $G(\tau, \lambda)$, see [26]. Now we denote these intervals by $\{(\lambda_i^+, \lambda_i^-), i = 1, 2, \ldots, m(\tau)\}$ where $m(\tau)$ is the multiplicity function of W. A fundamental lemma in [24] shows that the set of atoms $\{\lambda_r^+ : r = 1, 2, \ldots, n\} \subset \{\lambda_i^+(\tau), i = 1, 2, \ldots, m(\tau)\}$ while $\{\lambda_j^- : j = 1, 2, \ldots, m\} \subset \{\lambda_i^-(\tau), i = 1, 2, \ldots, m(\tau)\}$. We form $\{a_j(\tau)\} = \{\lambda_i^+(\tau)\} - \{\lambda_r^+\}$ and $\{b_j(\tau)\} = \{\lambda_i^-(\tau)\} - \{\lambda_r^-\}$ by removing the smaller sets from the larger ones. In the case $m = n$, we arrange these points in such a way that the intervals $\{(a_j(\tau), b_j(\tau))\}$ are the intersection of a level line at height τ on the cylinder with the support of $G_T(\tau, \lambda)$, where $G_T(\tau, \lambda) = G(\tau, \lambda) + \{\# \textit{ of } \lambda_j^- : \lambda_j^- < \lambda\} - \{\# \textit{ of } \lambda_j^+ : \lambda_j^+ < \lambda\}$. That is, we let $F_\tau^k = \{: G_T(\tau, \lambda) \geq k\}$. Then F_τ^k is a union of intervals, say $F_\tau^k = \bigcup_{j=1}^{j_k} I_j^k(\tau)$. We arrange $a_j(\tau)$ and $b_j(\tau)$ so that $\{I_j^k(\tau)\} = \{(a_j(\tau), b_j(\tau))\}$.

Now we can define the following "Riemann-Hilbert barrier" :

$$S(\tau) = \left[\prod_{k=1}^{m(\tau)} \prod_{r=1}^{m} \frac{|b_k(\tau) - \lambda_r^+|}{|a_k(\tau) - \lambda_r^+|}\right] \cdot \left[\prod_{k=1}^{m(\tau)} \prod_{j=1}^{m} \frac{|b_k(\tau) - \lambda_j^-|}{|a_k(\tau) - \lambda_j^-|}\right]^{-1} . \tag{1.7}$$

Theorem 1.2 *T with deficiency indices (m, m) is a weak contraction $\sigma(T) = S^1$, the unit circle, and we have* det $\Theta_T^*(\tau)\Theta_T(\tau) = S(\tau)$, for $|\tau| = 1$.

In [34] Pincus and Daoxing Xia showed that a different class of contraction operators had the property that $\det\Theta_T(\tau)$ is determined purely from the intersection geometry of an associated principal current.

One of the main goals of this thesis was to establish that such results hold also for the singular integral contractions by completing the analysis of [24], [25]. This was a conjecture of Pincus, and the analysis has benefited at many stages from his suggestions.

The above result together with a result of Sz. Nagy and C. Foias show that the class of contractions of our form consists of operators quasi-similar but not similar to unitary operators. The proof of Theorem 1.2 will be presented in chapter 4.

Motivated by [25] for Wiener-hopf operators and Toeplitz operators, a wider class of contraction operators as well as contraction operator pair are considered in a joint paper (preprint [37]) with J. Pincus. There, the contraction operators could be similar to unitary operators. These contraction operators will include the Cayley transform of Wiener-hopf operators as well as Toeplitz operators with unimodular symbols.

It is interesting that such unitary invariants of T are determined *entirely by the metric geometry* of the intersections (slices) of "the horizontal line currents at height τ" with the principal current of Theorem 1.1. In this respect the results achieved here are exactly parallel to those in [34] where intersections with a family of lines through the origin played the role of the horizontal lines here. Again it is the metric geometry of the principal current (rather than merely the topology of the spectrum) that enters once we pass to the analysis of structure beyond the index theory. Recall that the principal current is invariant under trace class perturbations of the generating operators, and is not generally invariant under compact perturbations. Note also that in [7] it was discovered that the necessary and sufficient conditions for two self-adjoint operators to be unitarily equivalent modulo the trace class involves *metric* conditions on their spectra.

It is shown in [24] that the contraction T has the following singular integral representation:

$$Tf(\lambda) = \frac{A(\lambda) + |k(\lambda)|^2 + i}{A(\lambda) + |k(\lambda)|^2 - i} f(\lambda) - \frac{1}{2\pi i} \frac{\bar{H}(-i,\lambda)}{k(\lambda)} \int \frac{H(i,t)}{\bar{k}(t)} \frac{f(t)dt}{t - (\lambda - i0)}, \tag{1.8}$$

where $H(-i,\lambda)$ and $H(i,\lambda)$ are certain functions such that $\frac{|H(\pm i,\lambda)|}{|k(\lambda)|^2}$ are square integrable. T has unimodular symbols. Let V be the unitary operator on $L_2(E)$ obtained by multiplying by the unimodular function $(A(\lambda)+|k(\lambda)|^2+ i)(A(\lambda) + |k(\lambda)|^2 - i)^{-1}$. Then the operator V^*T has the following form:

$$Uf(\lambda) = f(\lambda) + \frac{1}{\pi}\bar{\alpha}(\lambda) \int \beta(t)f(t)\frac{dt}{t - (\lambda - i0)} \tag{1.9}$$

for some square integrable functions $\alpha(\lambda), \beta(\lambda)$.

In chapter 5, we study general singular integral operators of form (1.9) with unimodular symbols. That is $1 - 2i\bar{\alpha}(\lambda)\beta(\lambda)$ are of absolute value one. For such an operator, we give necessary and sufficient condition for it to have a unitary operator closure. Therefore we have obtained an inversion formula for such a singular integral operator. And we also give conditions for U to be the compression of the wave operator coming from certain one dimensional self-adjoint operator perturbation problem.

In the classical index theory of singular integral operators with discontinuous coefficients, for example when the coefficients have a finite number of jumps, the essential spectrum of the operators is calculated from the range of the symbols together with the straight line segments which fill in the jumps. The index is then the winding number of the quotient of the modified symbols, see [19], [21]. Here, the singular integral operators we study are of a different type. First, the symbols of (1.9) have the product form $\bar{\alpha}(\lambda)\beta(\lambda)$. Therefore, unlike the classical case, the symbols do not determine the operator uniquely. The operator U in (1.9) is unitary or the compression of a wave operator and corresponds to a kind of symmetric decomposition of the product $\bar{\alpha}(\lambda)\beta(\lambda)$. Secondly, we have no smoothness assumptions on the symbols. But the essential spectrum of the operator is still the range of the symbols on the unit circle together with arcs on the unit circle, which connect the jumps of the symbols. And the index is the winding number computed with a certain weighting of the modified symbols. The proof of these results is an application of results in [24].

Chapter 2

Preliminaries

In this chapter, we first recall some basic properties of the principal function theory and prove some preliminary results we need later. We mainly adopt the notations from [5] and [24].

2.1 Symbol Homomorphism

We recall that the principal function for a pair of unitary and self-adjoint operators can be constructed from the symbols. The abstract symbols were first introduced in [2]. The symbols of a bounded operator A relative to a self-adjoint operator H are defined by

$$S_{\pm}(H,A) = s - \lim_{t \to \pm\infty} e^{iHt} A e^{-iHt} P_a(H). \tag{2.1}$$

whenever the limits exist and $P_a(H)$ represents the projection to the absolutely continuous subspace of H. It is shown in [2] that the symbols of A exist if A is in C^*-algebra $\mathcal{M}(H)$ generated by the collection of operators having commutator in trace class with H. The symbol is an algebraic homomorphism from $\mathcal{M}(H)$ to the algebra of all bounded operators and the symbols commute with H. The kernel of the symbol homomorphism contains the ideal of compact operators, cf [2].

In particular, the symbols of W relative to H_2 exist since the commutator $[W, H_2]$ is in trace class. Since the symbols of W commute with H_2, in the direct integral space diagonalizing H_2, they are decomposable. So we can write

$$S_{\pm}(H_2, W) = \int \oplus S_{\pm}(H_2, W)(\lambda) d\lambda. \tag{2.2}$$

Furthermore, $S_{\pm}(H_2, W)(\lambda)$ are unimodular complex numbers since H_2 has multiplicity one and W is unitary. The principal function $G(\tau, \lambda)$ then are determined by the symbols of W in the following way: For a fixed λ, $G(\tau, \lambda)$ was defined by Pincus to be the characteristic function of the positive arc on the unit circle with endpoints $S_{\pm}(H_2, W)(\lambda)$. So we have the relation:

$$\frac{S_{-}(H_2, A)(\lambda) - \omega}{S_{+}(H_2, A)(\lambda) - \omega} = \exp \int G(\tau, \lambda) \frac{d\tau}{\tau - \omega} \tag{2.3}$$

for $\omega \neq 0$.

Since we also have the relation $G((x + i)(x - i)^{-1}, \lambda) = g(x, \lambda)$, we see that the symbols $S_{\pm}(H_2, W)(\lambda)$ are the Cayley transforms of the symbols $A(\lambda) \pm |k(\lambda)|^2$ of the singular integral (1.1).

Instead of the symbols relative to a self-adjoint operator, we also have the symbols relative to a unitary operator, the so-called polar symbols. The polar symbols will have the same properties which the regular symbols satisfy. For example, the symbols $S_{\pm}(H_2, W) = \lim_{n \to \pm\infty} W^{-n} H_2 W^n P_a(W)$ exist when the commutator $[W, H_2]$ is in trace class and is decomposable in the direct integral space diagonalizing W.

2.2 One Dimensional Perturbation Problem

For a one-dimensional self-adjoint operator perturbation problem $H \to H + \frac{1}{\pi} d \otimes d$, there is a scalar function $h(\lambda)$ taking values between 0 and 1, the so called phase shift function [17] , such that

$$\det(H + \frac{1}{\pi} d \otimes d - z)(H - z)^{-1} = 1 + \frac{1}{\pi} \int \frac{d\mu(\lambda)}{\lambda - z} = \exp \int \frac{h(\lambda)}{\lambda - z} d\lambda \quad (2.4)$$

where $d\mu(\lambda) = d(E_\lambda d, d)$ and E_λ is the spectral resolution of H. For a differentiable function F, the following displacement formula of Lifshitz and M. G. krein (cf. [17]) is valid:

$$tr(f(H + \frac{1}{\pi} d \otimes d) - f(H)) = \int f'(\lambda) H(\lambda) d\lambda, \quad (2.5)$$

The spectral properties of H are completely determined by the phase shift function. In particular, a point λ_0 is in the continuous spectrum of H if and only if ([1], [33]):

$$\int_{\lambda > \lambda_0} \frac{h(\lambda)}{\lambda - \lambda_0} d\lambda + \int_{\lambda < \lambda_0} \frac{1 - h(\lambda)}{\lambda_0 - \lambda} d\lambda = \infty. \quad (2.6)$$

The wave operators of the perturbation problem are defined by (see [15])

$$W_\pm = \lim_{t \to \pm\infty} e^{-it(H + \frac{1}{\pi} d \otimes d)} e^{itH} P_a(H). \quad (2.7)$$

The wave operators $W_\pm$ exist and are complete. That is, $W_\pm$ are partial isometries with initial space $\mathcal{H}_a(H)$-the absolutely continuous subspace of H, and final space $\mathcal{H}_a(H + \frac{1}{\pi} d \otimes d)$- the absolutely continuous subspace of $H + \frac{1}{\pi} d \otimes d$.

These operators $W_\pm$ intertwine the operator pair H and $H + \frac{1}{\pi} d \otimes d$:

$$W_\pm H = (H + \frac{1}{\pi} d \otimes d) W_\pm. \quad (2.8)$$

In the case that H has multiplicity one, it was shown in [5] that any isometry which intertwines the operator pair H and $H + \frac{1}{\pi} d \otimes d$ is of the form $W_- f(\lambda)$, where $f(\lambda)$ is a unimodular function in the absolutely continuous part of the spectral representation space of H. Furthermore, it was also shown that $W_\pm$ has the following singular integral representation:

$$W_- g(\lambda) = g(\lambda) + \frac{1}{\pi} \bar{d}_a(\lambda) \int \frac{d_a(t)}{\det(t - i0)} \frac{g(t)dt}{\lambda - (t - i0)}. \tag{2.9}$$

where $\det(t - i0)$ is the boundary value of the perturbation determinant and $d_a(\lambda)$ is the representative of d_a-the projection of d to the absolutely continuous subspace of H-in the spectral representation space of H.

2.3 Smooth Functional Calculus

In this section, we introduce a certain smooth functional calculus for the non-commutative operator triplet $\{P, W, H_2\}$. In order to define it, we need the fact that all the commutators of the triplet $\{P, W, H_2\}$ are in trace class. In fact, the commutator $[W, H_2]$ is a rank one operator and so is in trace class. P and H_2 commute. And we will see (Lemma 2.1 below) that $[P, M]$ is also in trace class.

Let $\mathcal{M}(R \times S^1 \times R)$ be the collection of all finite complex measure sequences $\{\omega_m : m = 0, \pm 1, \pm 2, \ldots\}$ which satisfy the following condition:

$$\sum_{m=-\infty}^{\infty} |m| \int (1 + |s|)(1 + |t|) d|\omega_m(t, s)| < \infty. \tag{2.10}$$

For a sequence $\{\omega_m : m = 0, \pm 1, \pm 2, \ldots\}$ in $\mathcal{M}(R \times S^1 \times R)$, the characteristic function of $\{\omega_m\}$ is defined by

$$f(x,\tau,\lambda)=\sum_{m=-\infty}^{\infty}e^{im\theta}\int e^{itx+is\lambda}d\omega_m(t,s). \tag{2.11}$$

Denote all the characteristic functions by $\hat{\mathcal{M}}(R\times S^1\times R)$. Then for any $f(x,\tau,\lambda)\in\hat{\mathcal{M}}(R\times S^1\times R)$, we follow [8] and define the functional calculus by the following:

$$f(P,W,H_2)=\sum_{m=-\infty}^{\infty}W^m\int e^{itP+isH_2}d\omega_m(t,s). \tag{2.12}$$

This functional calculus is well defined modulo the trace class ideal when all the commutators of $\{P,W,H_2\}$ are in trace class. That is, the map $f\to f(P,W,H_2)$ is a *-homomorphism from $\hat{\mathcal{M}}(R\times S^1\times R)$ to the algebra of all bounded operators modulo the trace class ideal. It follows from the homomorphism property that the commutator $[f(P,W,H_2),h(P,W,H_2)]$ is in trace class for two arbitrary elements f,h in $\hat{\mathcal{M}}(R\times S^1\times R)$.

There is a similar smooth functional calculus for a pair of contraction and self-adjoint operators, which we will need in this work. Let $H_{2,a}$ be the restriction of H_2 to its absolutely continuous subspace. One notes that $[T,T^*]$ and $[T,H_2]$ are in trace class by Lemma 2.1 below. Let $\hat{\mathcal{M}}_T(S^1\times S^1\times R)$ be the family of functions of the form:

$$f(e^{i\theta},e^{i\phi},x)=\sum_{m,n=0}^{\infty}e^{-i(m\theta+n\phi)}\int e^{ixt}d\omega_{m,n}(t),$$

where $\{\omega_{m,n}(t):m,n=0,1,2,\ldots\}$ are complex measures satisfying

$$\sum_{m,n=0}^{\infty}(m+n)||T^{*m}T^n||\int(1+|t|)d|\omega_{m,n}|(t)<\infty.$$

We define

$$f(T^*, T, H_{2,a}) = \sum_{m,n}^{\infty} T^{*m}T^n \int e^{itH_{2,a}} d\omega_{m,n}(t). \tag{2.13}$$

This is also a *-homomorphism from $\hat{\mathcal{M}}(S^1 \times S^1 \times R)$ to the algebra of bounded operators modulo the trace ideal class ideal.

2.4 The structure of W

Let

$$E(l, z) = \exp \frac{1}{2\pi i} \int\int g(x, \lambda) \frac{dx}{x - l} \frac{d\lambda}{\lambda - z}. \tag{2.14}$$

Then there is a pair of possibly unbounded self-adjoint operators $\{H_1, H_2\}$ on a Hilbert space $\mathcal{H}$ having $g(x, \lambda)$ as its principal function, See [26], such that

$$E(l, z) = 1 + \frac{1}{\pi i} k(H_1 - l)^{-1}(H_2 - z)^{-1}k^*, \tag{2.15}$$

where k^* is a map of the complex numbers into a one dimensional subspace of the Hilbert space $\mathcal{H}$ and k is the adjoint of k^* which maps $\mathcal{H}$ back into the complex numbers.

Furthermore, the resolvent of $\{H_1, H_2\}$ has one dimensional commutator, that is

$$(H_1 - l)^{-1}(H_2 - z)^{-1} - (H_2 - z)^{-1}(H_1 - l)^{-1}$$
$$= \frac{1}{\pi i}(H_2 - z)^{-1}(H_1 - l)^{-1}k^*k(H_1 - l)^{-1}(H_2 - z)^{-1}. \tag{2.16}$$

and the pair is determined up to unitary equivalence by $g(x, \lambda)$, the principal function of $\{H_1, H_2\}$. Let $\mathcal{H}_a$ be the absolutely subspace of H_2. Then it is known (see [24]) that $\mathcal{H}_a$ is the smallest invariant subspace of H_2 which contains the range of k^*. And there is a unitary map of $\mathcal{H}_a$ to $L_2(E)$ which takes $H_2|_{\mathcal{H}_a}$ to M, the multiplication operator by the position function on $L_2(E)$. That unitary map carries k^* into the multiplication operator by $k(\lambda)$, and carries k to the integral operator $\int \cdot k(\lambda) d\lambda$. In this representation, $k(\lambda)$ is a basis for the range of the commutator $[H_1, H_2]$. This result is from [26].

Now form the Cayley transform $W_l = (H_1 - l)(H_1 - l)^{-1}$, and put $T_l = PW_lP$. By the results of [24] we know that W_l is the minimal unitary dilation of T_l. We note that $W_i = W$ and $T_i = T$.

According to [22], the minimal unitary dilation W_l of T_i has the following matrix representation:

$$W_l = \begin{pmatrix} \dots & 0 & I & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & D_{T_l} & -T_l^* & 0 & \dots \\ \dots & 0 & 0 & \underline{T_l} & D_{T_l^*} & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & I & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix} \tag{2.17}$$

on

$$\mathcal{H} = \dots \oplus \mathcal{D}_{T_l}^{(-2)} \oplus \mathcal{D}_{T_l}^{(-1)} \oplus \underline{\mathcal{H}_a} \oplus \mathcal{D}_{T_l^*}^{(1)} \oplus \mathcal{D}_{T_l^*}^{(2)} \oplus \mathcal{D}_{T_l^*}^{(3)} \oplus \dots,$$

where $\mathcal{D}_{T_l}^{(-n)}$ and $\mathcal{D}_{T_l^*}^{(n)}$ are respectively the ranges of $\mathrm{D}_{T_l} = (I - T_l^* T_l)^{\frac{1}{2}}$ and $\mathrm{D}_{T_l^*} = (I - T_l T_l^*)^{\frac{1}{2}}$, and $\mathcal{H}_a$ is the absolutely continuous subspace of H_2. The underlined element or operator indicates the center element or operator.

Lemma 2.1 *$[P, W]$ is of finite rank, and therefore is in trace class .*

Proof: It follows immediately from the matrix representation above that $[P, W]$ is of finite rank because $\mathcal{D}_{T_l}$ and $\mathcal{D}_{T_l^*}$ have finite dimensional range. See [24].

We will let $Q(l)$ denote the smallest invariant subspace of H_2 which contains the vector $(H_1 - \bar{l})^{-1}k^*$, where k^* corresponds to the function $\bar{k}(\lambda)$ introduced earlier as the representative of $\bar{k}$ in the spectral representation space of the absolutely continuous subspace of H_2. It is known from [24] that $Q(-i)$ contains the the absolutely continuous subspace $\mathcal{H}_a$ of H_2. Then we have

Lemma 2.2

$$Q(l) = \mathcal{D}_{T_l}^{(-1)} \oplus \mathcal{H}_a \quad \textit{for} \quad \Im l < 0$$

and

$$Q(l) = \mathcal{H}_a \oplus \mathcal{D}_{T_l^*}^{(1)} \quad \textit{for} \quad \Im l > 0.$$

Proof: We will give the proof for $l = -i$.

First we prove that $\mathcal{D}_T^{(-1)} \oplus \mathcal{H}_a \subset Q(-i)$. It is clear from the matrix representation (1.17) of W_l that

$$W_l(\mathcal{H}_a \oplus \mathcal{D}_{T_l^*}^{(1)}) = \mathcal{D}_{T_l}^{(-1)} \oplus \mathcal{H}_a \qquad \text{and} \qquad W_l\mathcal{D}_{T_l^*}^{(1)} \subset \mathcal{H}_a.$$

Thus it is enough to prove $W_l\mathcal{H}_a \subset Q(-i)$ since $\mathcal{H}_a \subset Q(-i)$. Recall from [24] that $\{H_2^n k^*, n = 0, 1, \ldots\}$ is dense in $\mathcal{H}_a$. Thus we need only prove that $(H_1 - i)^{-1}H_2^n k^* \in Q(-i)$, because $WH_2^n k^* = (H_1 + i)(H_1 - i)^{-1}H_2^n k^* = H_2^n k^* + 2i(H_1 - i)^{-1}H_2^n k^*$, and $H_2^n k^* \in L_2(E) = \mathcal{H}_a \subset Q(-i)$. See [24].

To prove that $(H_1 - i)^{-1}H_2^n k^* \in Q(i)$ we will use induction on n. This is trivial for $n = 0$. For $n \geq 1$ assume that $(H_1 - i)^{-1}H_2^{n-1}k^* \in Q(-i)$.

Now

$$(H_1 - i)^{-1}H_2^n - H_2^n(H_1 - i)^{-1}$$
$$= (H_1 - i)^{-1}(H_2^n(H_1 - i) - (H_1 - i)H_2^n)(H_1 - i)^{-1}$$
$$= (H_1 - i)^{-1}(H_2^n H_1 - H_1 H_2^n)(H_1 - i)^{-1}$$
$$= (H_1 - i)^{-1}(H_2^{n-1}[H_2, H_1] + H_2^{n-2}[H_2, H_1]H_2 + \cdots$$
$$+[H_2, H_1]H_2^{n-1})(H_1 - i)^{-1}.$$

Thus, by the commutator relation (2.16), we have

$(H_1 - i)^{-1}H_2^n k^* = H_2^n(H_1 - i)^{-1}k^* + \frac{1}{\pi i}(H_1 - i)^{-1}H_2^{n-1}k^*k(H_1 - i)^{-1}k^* + \frac{1}{\pi i}(H_1 - i)^{-1}H_2^{n-2}k^*kH_2(H_1 - i)^{-1}k^* + \cdots + \frac{1}{\pi i}(H_1 - i)^{-1}H_2k^*kH_2^{n-2}(H_1 - i)^{-1}k^* + \frac{1}{\pi i}(H_1 - i)^{-1}k^*kH_2^{n-1}k^*.$

In the above expression, $(H_1 - i)^{-1}k^* \in Q(-i)$, and $Q(-i)$ is invariant under H_2. Thus, the first term above, $H_2^n(H_1-i)^{-1}k^* \in Q(-i)$. The remaining terms are just constants times the factor $(H_1-i)^{-1}H_2^j k^*$ for $j < n$. By induction $(H_1 - i)^{-1}H_2^j k^* \in Q(-i)$. Therefore $(H_1 - i)^{-1}H_2^n k^* \in Q(-i)$.

The other inclusion $Q(-i) \subset \mathcal{D}_T^{(-1)} \oplus \mathcal{H}_a$ is proved using the same identity, which gives an expression for $H_2^n(H_1 - i)^{-1}k^*$ in "lower" order terms. Also note that $\{H_2^n(H_1 - i)^{-1}k^*\}$ generates $Q(-i)$. This concludes the proof.

Recall that it was shown in [24] that it is possible to choose an orthogonal basis $\{e_1^{(-1)}, \ldots, e_m^{(-1)}\}$ for $\mathcal{D}_T^{(-1)}$ such that $H_2 e_j^{(-1)} = \lambda_j^- e_j^{(-1)}$. We now fix $\lambda_0 \in \{\lambda_j^-\}$ and define $e_j^{(-n)} = W^n e_j^{(-1)}$ for $n > 0$. Then we have

Lemma 2.3 *If $x \in \mathcal{H}$ is an eigenvector of H_2 with eigenvalue λ_0. Then Wx is also an eigenvector of H_2 having eigenvalue λ_0. In particular $H_2 e_j^{(-n)} = \lambda_j^- e_j^{(-n)}$, for $n > 0$.*

Proof: To prove this lemma, we need another representation for W and H_2 given in [24].

Let $\mathcal{H}_L = \mathcal{H}_s((H_2+D_{-i})|_{Q(-i)})$ and $\mathcal{H}_R = \mathcal{H}_s(H_2|_{Q(-i)})$, where $\mathcal{H}_s$ denotes the spectrally singular subspace of the indicated operator; and $D_{-i} = d_{-i} \otimes d_{-i}$ with $d_{-i} = \frac{1}{\pi}(H_1 - i)^{-1}k^*$.

Then let

$$\hat{\mathcal{H}} = \cdots \oplus \mathcal{H}_L \oplus \mathcal{H}_L \oplus \underline{Q(-i)} \oplus \mathcal{H}_R \oplus \mathcal{H}_R \oplus \cdots. \tag{2.18}$$

For any $\hat{x} \in \hat{\mathcal{H}}$ of the form $\hat{x} = (\ldots, x_{-2}, x_{-1}, \underline{x_0}, x_1, \ldots)$, where P_R is the projection of $Q(-i)$ to $\mathcal{H}_R$, let

$$\hat{W}\hat{x} = (\ldots, x_{-2}, x_{-1}, \underline{x_1 + W_0 x_0}, P_R x_0, x_1 \ldots), \tag{2.19}$$

where $W_0 = W_- f$, $W_- = \lim_{t\to\infty} e^{it(H_2+D_{-i})}e^{-itH_2}P$ is the wave operator of the perturbation problem $H_2 \to H_2 + D_{-i}$, and $f(\lambda) = S_-(H_2, W)(\lambda)$ is the symbol of W, which is a unimodular function. Also define $\hat{H}_2$ by setting

$$\hat{H}_2\hat{x} = (\ldots, (H_2 + D_{-i})x_{-2}, (H_2 + D_{-i})x_{-1}, \underline{H_2 x_0}, H_2 x_1, H_2 x_2, \ldots).$$

Then it follows, again from the basic fact that the principal function is a complete unitary invariant, that the operator pair $\{\hat{W}, \hat{H}_2\}$ is unitarily equivalent to the pair $\{W, H_2\}$, see [1] and the references given there. This means that there is a unitary operator $U : \hat{\mathcal{H}} \to \mathcal{H}$ such that $U\hat{W}U^* = W$ and $U\hat{H}_2U^* = H_2$. But the lemma just above shows that $UQ(-i) = \mathcal{D}_T^{(-1)} \oplus \mathcal{H}_a$.

Now suppose that $\hat{x} \in \hat{\mathcal{H}}$ and that $H_2\hat{x} = \lambda_0\hat{x}$. We may take $\hat{x}$ of the form $\hat{x} = (\ldots, 0, 0, \underline{x_0}, x_1, \ldots)$. Thus $H_2 x_j = \lambda_0 x_j$ for $j = 0, 1, 2, \ldots$.

But $\hat{W}\hat{x} = (\ldots, 0, \underline{W_0 x_0}, P_R x_0, x_1, \ldots) = (\ldots, 0, \underline{W_0 x_0}, x_0, x_1, \ldots)$ while $\hat{H}_2 \hat{W}\hat{x} = (\ldots, 0, \underline{H_2 W_0 x_0}, H_2 x, H_2 x_1, \ldots) = (\ldots, 0, \underline{H_2 W_0 x_0}, \lambda_0 x, \lambda_0 x_1, \ldots)$.

Set $y_0 = U x_0$ and note that $\hat{W} = U^* W U$. Thus $U^* W U(\ldots, 0, \underline{x_0}, \ldots) = U^* W y_0$. But $y_0 \in \mathcal{D}_T^{(-1)} \oplus \mathcal{H}_a$. And because λ_0 is an eigenvalue of $\hat{H}_2$, y_0 is an eigenvector of H_2 on $\mathcal{H}$. So $y_0 \in \mathcal{D}_T^{(-1)}$. By the matrix representation (1.10), $W y_0 \in \mathcal{D}_T^{(-2)}$. This implies that $(W y_0, y_{(-1)}) = 0$, for any $y_{(-1)} \in \mathcal{D}_T^{(-1)}$ and $(W y_0, y) = 0$ for any $y \in \mathcal{H}_a$. Thus $(U \hat{W} U^* y_0, y_{(-1)}) = 0$ and $(\hat{W} x_0, U^* y) = 0$. Therefore $(\hat{W} x_0, U^*(y_{(-1)} + y)) = (\hat{W} x_0, U^* y_{(-1)}) + (\hat{W} x_0, U^* y) = 0$. Since $y_{(-1)}$ and y are arbitrary we will have $(\hat{W} x_0, x) = 0$, for any $x \in Q(-i)$.

But $\hat{W} x_0 = (, \ldots, 0, \underline{W_0 x_0}, x_0, \ldots)$, and for any $x \in Q(-i)$, we have $(W_0 x_0, x) = (\hat{W} x_0, x) = 0$. Thus $W_0 x_0 = 0$. And finally we have

$$\hat{H}_2 \hat{W}\hat{x} = \lambda_0(\ldots, 0, \underline{0}, x_0, x_1, \ldots) = \lambda_0(\ldots, 0, \underline{W_0 x_0}, x_0, x_1, \ldots) = \lambda_0 \hat{W}\hat{x}.$$

Chapter 3

The Principal Current for $\{P, W, H_2\}$

In this chapter, we first compute the principal function $G_P(x,\tau)$ for the self-adjoint and unitary operator pair $\{P, W\}$, and then use the principal functions $G_P(x,\tau)$ and $G(\tau,\lambda)$ to explicitly construct the principal current for the operator triple $\{P, W, H_2\}$. Then, we derive a normal operator perturbation problem and compute the phase shift for this perturbation problem as well as a displacement formula similar to (2.5) in the introduction.

3.1 The principal function of $\{P, W\}$

We still fix λ_0 to be one of $\{\lambda_j^-\}$ as in section 2.4. Let P_0 be the projection to the eigenspace of H_2 with eigenvalue λ_0. By Lemma 2.3, P_0 has the following matrix representation

$$P_0 = \ldots \oplus \tilde{P}_0 \oplus \tilde{P}_0 \oplus \underline{0} \oplus 0 \oplus 0 \ldots \tag{3.1}$$

where

$$\tilde{P}_0 = \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}_{m\times m} . \tag{3.2}$$

From this we can see directly by matrix computation that

$$P_0W - WP_0 = \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \tilde{P}_0 D_T & -\tilde{P}_0 T^* & 0 & \dots \\ \dots & 0 & 0 & \underline{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \ddots \end{pmatrix} . \tag{3.3}$$

Because $\tilde{P}_0$ is a one dimensional projection, $P_0W - WP_0$ is of finite rank. Therefore there is a principal function $G_0(x,\tau)$ associated to the self-adjoint and unitary operator pair $\{P_0, W\}$ which is supported on the cylinder $R \times S^1$. We wish to calculate this principal function.

Let us therefore examine the polar symbols of P_0 relative to W

$$S_{\pm}(W, P_0) = s - \lim_{n \to \pm\infty} W^n P_0 W^{-n} P_a(W). \qquad (3.4)$$

where $P_a(W)$ is the projection to the absolutely continuous subspace of W. Let Ω_0 be the one dimensional subspace of $\mathcal{H}$ generated by $e_0^{(-1)}$. We may then form $M(\Omega_0) = \bigoplus_{n=-\infty}^{\infty} W^n \Omega_0$.

Note that Ω_0 is a wandering subspace of W since $\Omega_0 \subset \mathcal{D}_T^{(-1)}$ which is a wandering subspace of W. See [22]. Now we will prove the validity of the following observation.

Lemma 3.1 *$S_-(W, P_0) = P_{\Omega_0}$, where P_{Ω_0} is the projection to $M(\Omega_0)$; and $S_+(W, P_0) = 0$.*

Proof: Because W is absolutely continuous (see [28] and [24]), we have $S_+(W, P_0) = s - \lim_{n \to \infty} W^n P_0 W^{-n}$. By direct matrix calculation we see that

$$W P_0 W^{-1} == \ldots \tilde{P}_0 \oplus \tilde{P}_0 \oplus 0 \oplus \underline{0} \oplus 0 \ldots.$$

And

$$W^n P_0 W^{-n} = \ldots \tilde{P}_0 \oplus \tilde{P}_0 \oplus \overbrace{0 \oplus 0 \ldots \oplus 0}^{n \text{ zeros}} \oplus \underline{0} \oplus 0 \ldots.$$

Thus it is clear that $S_+(W, P_0) = 0$, since the limit we are taking is in the strong operator topology and $\tilde{P}_0$ is a finite dimensional projection.

On the other hand

$$W^{-n}P_0W^n = \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \tilde{P}_0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \tilde{P}_0 & \underline{*} & * & * & \dots \\ \dots & 0 & 0 & * & * & * & \dots \\ \dots & 0 & 0 & * & * & * & \ddots \end{pmatrix}. \tag{3.5}$$

It follows immediately that $S_-(W,P_0)|_{\Omega_0} = \tilde{P}_0 = P_{\Omega_0}|_{\Omega_0}$.

But both $S_-(W,P_0)$ and P_{Ω_0} commute with W. Thus $S_-(W,P_0)|_{M(\Omega_0)} = P_{\Omega_0}|_{M(\Omega_0)} = P_{\Omega_0}$.

Now take $x \in \mathcal{H}, x \perp M(\Omega_0)$. That is, $(x, W^m e_0^{(-1)}) = 0$ for any integer m. Fix $n \geq 0$ and for any $y \in \mathcal{H}$, consider the inner product $(W^{-n}P_0W^n x, y)$.

$$(W^{-n}P_0W^n x, y) = (x, W^{-n}P_0W^n y) = \lim_{k\to\infty}\sum_{j=1}^{k}(x, W^{-n}\tilde{P}_0^{(-j)}y),$$

where $\tilde{P}_0^{(-j)} = \ldots \oplus \overset{(-j\ location)}{\tilde{P}_0} \oplus 0 \ldots \oplus \underline{0} \oplus 0 \ldots.$

But $W^{-n}\tilde{P}_0^{(-j)} = W^{-n}W^j\tilde{P}_0^{(-1)}$, and $\tilde{P}_0^{(-1)}W^n y \in \Omega_0$, for any n. Thus $(x, W^{-n}\tilde{P}_0^{(-j)}W^n y) = (x, W^{-n+j}\tilde{P}_0^{(-1)}W^n y) = 0$, because we have assumed that $(x, W^i e_0^{(-1)}) = 0$. and the range of the projection $\tilde{P}_0^{(-1)}$ is $e_0^{(-1)}$. Thus we have shown that $\lim_{n\to\infty} W^{-n}P_0W^n|_{M(\Omega_0)^\perp} = 0$, and thus we have proved that $S_-(W,P_0) = P_{\Omega_0}$.

Theorem 3.2 *The principal function, $G_0(x,\tau)$, of the pair $\{P_0, W\}$ is* *the negative of the characteristic function of the set* $D = \{(x,\tau) : 0 \leq x \leq 1\}$ *on* $R \times S^1$.

Proof: We first note that W restricted to $M(\Omega_0)$ is a bilateral shift of multiplicity one. Now note that in the direct integral diagonalizing space for W, $S_\pm(W,P_0) = \int_{S^1} \oplus S_\pm(W,P_0)(\tau)d\tau$. By Lemma 3.1, $S_-(W,P_0)(\tau) = 0$, and $S_+(W,P_0)(\tau)$ is a one dimensional projection.

Consider the perturbation : $S_+(W,P_0)(\tau) \longrightarrow S_-(W,P_0)(\tau)$ and denote the corresponding phase shift by $\delta_\tau(x)$, in the fiber space of the direct integral Hilbert space for W. Thus we have

$$\det(S_-(W,P_0)(\tau) - z)(S_+(W,P_0)(\tau) - z)^{-1} = \exp \int_0^1 \frac{\delta_\tau(x)}{x - z} dx. \qquad (3.6)$$

Through this perturbation, the dimension of the eigenspace increases by one at the eigenvalue 0, while it decreases by one at the eigenvalue 1, cf. [15], The known relationship between the phase shift for symbol perturbation problems and principal functions then gives us that $G_0(x,\tau) = -\delta_\tau(x) = -1$ for $0 \leq x \leq 1$.

We note that here we use the ordering (x,τ) as providing the orientation for the pair $\{P_0, W\}$ and this is the opposite from the orientation which is used in [5] for the self-adjoint unitary operator pair $\{W, P_0\}$.

Remark It is clear that if we replace λ_0 by one of the $\{\lambda_r^+\}$ we will get a principal function for the new pair $\{P_0, W\}$ which takes the value one for $0 \leq x \leq 1$.

Let $P_r^+, (P_j^-)$ be the projections to the eigenvector spaces of H_2 with eigenvalues $\lambda_r^+, (\lambda_j^-)$. Then $P = I - \sum_r P_r^+ - \sum_j P_j^-$, and $S_\pm(W,P) = I - \sum_r S_\pm(W,P_r^+) - \sum_j S_\pm(W,P_j^-)$.

Theorem 3.3 *The principal function $G_P(x,\tau)$ for the pair $\{P,W\}$ is $-(m-n)$ times the characteristic function of the set $\{(x,\tau): 0 \le x \le 1\}$ on $R \times S^1$.*

Proof: If we consider the fiber perturbation problem $S_+(W,P)(\tau) \longrightarrow S_-(W,P)(\tau)$, we see immediately that the principal function for the pair $\{P,W\}$ is the negative of the algebraic sums of the principal function for the pairs $\{P_r^+, W\}$ and $\{P_j^-, W\}$. That is, $G_P(x,\tau) = -(m-n)$ for $0 \le x \le 1$.

3.2 Trace Identities

In this section, we prove certain trace identities which we will need in order to prove the main theorem of this chapter.

Suppose $f_1(\tau), h_1(\tau)$ and $f_2(\lambda), h_2(\lambda)$ are smooth functions on the unit circle and real line respectively. The functional calculus $f_1(W), h_1(W), f_2(H_2)$ and $h_2(H_2)$ are defined in the sense of section 2.2. In the following calculations of this and next section, we will use f_i, h_i to denote both the scalar valued functions and the corresponding operators under the functional calculus. The context will make the meaning clear.

Lemma 3.5

(i) $\operatorname{tr}[Pf_1f_2, Ph_1h_2] = \operatorname{tr}[Pf_1f_2, h_1h_2]$

(ii) $\operatorname{tr}[Pf_1f_2, h_1h_2] = \operatorname{tr}[Pf_1, h_1h_2]f_2 + \operatorname{tr}[f_2, h_1h_2]Pf_1$

(iii) $\operatorname{tr}[Pf_1, h_1h_2]f_2 = \operatorname{tr}[Pf_1, h_1]h_2f_2 + \operatorname{tr}[Pf_1, h_2]f_2h_1$

(iv) $\operatorname{tr}[f_1f_2, h_1h_2] = \operatorname{tr}[f_1, h_2]f_2h_1 + \operatorname{tr}[f_2, h_1]h_2f_1$

Proof:

(i) $\text{tr}\,[Pf_1f_2, Ph_1h_2] - \text{tr}[Pf_1f_2, h_1h_2]$

$= \text{tr}[Pf_1f_2Ph_1h_2 - Ph_1h_2Pf_1f_2 - Pf_1f_2h_1h_2 + h_1h_2Pf_1f_2]$

$= \text{tr}(Pf_1f_2(P-I)h_1h_2) + \text{tr}((I-P)h_1h_2Pf_1f_2)$

$+\text{tr}((P-I)h_1h_2Pf_1f_2) + \text{tr}((I-P)h_1h_2Pf_1f_2)$

$= 0.$

(ii) $\text{tr}[Pf_1f_2, h_1h_2] - \text{tr}[Pf_1, h_1h_2]f_2 - \text{tr}[f_2, h_1h_2]Pf_1$

$= \text{tr}(Pf_1f_2h_1h_2 - h_1h_2Pf_1f_2 - Pf_1h_1h_2f_2$

$+h_1h_2Pf_1f_2 - f_2h_1h_2Pf_1 + h_1h_2f_2Pf_1)$

$= \text{tr}Pf_1(f_2h_1h_2 - h_1h_2f_2) + \text{tr}(h_1h_2f_2 - f_2h_1h_2)Pf_1$

$= 0.$

(iii) $\text{tr}[Pf_1, h_1h_2]f_2 - \text{tr}[Pf_1, h_1]h_2f_2 - \text{tr}[Pf_1, h_2]f_2h_1$

$= \text{tr}(Pf_1h_1h_2f_2 - h_1h_2Pf_1f_2 - Pf_1h_1h_2f_2$

$+h_1Pf_1h_2f_2 - Pf_1h_2f_2h_1 + h_2Pf_1f_2h_1)$

$= \text{tr}h_1(Pf_1h_2f_2 - h_2Pf_1f_2) + \text{tr}(h_2Pf_1f_2 - Pf_1h_2f_2)h_1$

$= 0$

In the last step, we used the fact that $[f_1, h_1] = 0$ and $[f_2, h_2] = 0$.

(iv) $\text{tr}[f_1f_2, h_1h_2] - \text{tr}[f_1, h_2]f_2h_1 - \text{tr}[f_2, h_1]h_2f_1$

$= \text{tr}[f_1f_2, h_1h_2] - \text{tr}[f_1, h_2]f_2h_1 - \text{tr}h_2f_1[f_2, h_1]h_2f_1$

$= \text{tr}(f_1f_2h_1h_2 - h_1h_2f_1f_2 - f_1h_2f_2h_1 + h_2f_1h_1f_2)$

$= \text{tr}f_1(f_2h_1h_2 - h_2f_2h_1) + \text{tr}(h_2f_1h_1 - h_1h_2f_1)f_2$

$= \text{tr}(f_2h_1h_2 - h_2f_2h_1)f_1 + \text{tr}f_2(h_2f_1h_1 - h_1h_2f_1)$

$= 0.$

We will need to express the above traces in another form which involves $I - P$.

Lemma 3.6

(i) $\text{tr}[Pf_1, h_1]h_2f_2 = \text{tr}[Pf_1, h_1](I-P)h_2f_2 +$

$$\text{tr}[Ph_1, f_1](I-P)f_2h_2 + \text{tr}(I-P)f_1P[h_2f_2, h_1]$$

(ii) $\text{tr}[h_2f_2, h_1](I-P)f_1P = \text{tr}[f_2, h_1](I-P)f_1Ph_2$

$$+\text{tr}[h_2, h_1]f_2(I-P)f_1P$$

(iii) $\text{tr}[Pf_1, h_2]f_2h_1 + \text{tr}(I-P)f_1P[h_2f_2, h_1] + \text{tr}[f_2, h_1h_2]Pf_1 - \text{tr}[f_1f_2, h_1h_2]$

$$= \text{tr}f_2(I-P)[h_2, f_1](I-P)h_1 - \text{tr}[f_2, h_1](I-P)f_1(I-P)h_2.$$

Proof:

(i) $\text{tr}[Pf_1, h_1]h_2f_2$

$= \text{tr}[Pf_1, h_1](I-P)h_2f_2 + \text{tr}[Pf_1, h_1]Ph_2f_2$

But, for the second term in the above expression, we have

$\text{tr}[Pf_1, h_1]Ph_2f_2 = \text{tr}P(f_1h_1 - h_1Pf_1)f_2h_2$

$= \text{tr}Ph_1(I-P)f_1f_2h_2$

$= \text{tr}(I-P)f_1Ph_1f_2h_2 + \text{tr}(I-P)f_1P[h_2f_2, h_1]$

$= -\text{tr}[Ph_1, f_1](I-P)f_2h_2 + \text{tr}(I-P)f_1P[h_2f_2, h_1].$

(ii) $\text{tr}[h_2f_2, h_1](I-P)f_1P - \text{tr}[f_2, h_1](I-P)f_1Ph_2 - \text{tr}[h_2, h_1]f_2(I-P)f_1P$

$= \text{tr}[h_2f_2, h_1](I-P)f_1P - \text{tr}h_2[f_2, h_1](I-P)f_1P - \text{tr}[h_2, h_1]f_2(I-P)f_1P$

$= 0.$

(iii) Let $\mathcal{S}$ denote the left hand of (iii). By using (iv) of lemma 3.5 and (ii) above, and noting that P commutes with h_2, we will have

$$\mathcal{S} = \text{tr}[f_1, h_2]f_2h_1P + \text{tr}[f_2, h_1](I-P)f_1Ph_2 + \text{tr}[h_2, h_1]f_2(I-P)f_1P$$
$$+\text{tr}[f_2, h_1]h_2Pf_1 - \text{tr}[f_1, h_2]f_2h_1 - \text{tr}[f_2, h_1]h_2f_1$$
$$= -\text{tr}[f_1, h_2]f_2h_1(I-P) - \text{tr}[f_2, h_1]h_2(I-P)f_1$$
$$+\text{tr}[f_2, h_1](I-P)f_1Ph_2 + \text{tr}[h_2, h_1]f_2(I-P)f_1P.$$

Now we write

$$\mathrm{tr}[f_2,h_1]h_2(I-P)f_1 = \mathrm{tr}[f_2,h_1](I-P)f_1f_2 + \mathrm{tr}[f_2,h_1](I-P)[h_2,f_1].$$

Then

$$\begin{aligned}
\mathcal{S} &= -\mathrm{tr}[f_1,h_2]f_2h_1(I-P) - \mathrm{tr}[f_2,h_1](I-P)f_1(I-P)f_2 \\
&\qquad -\mathrm{tr}[f_2,h_1](I-P)[h_2,f_1] + \mathrm{tr}[h_2,h_1]f_2(I-P)f_1P \\
&= -\mathrm{tr}[f_1,h_2](I-P)f_2h_1(I-P) - \mathrm{tr}[f_2,h_1](I-P)[h_2,f_1] \\
&\qquad -\mathrm{tr}[f_2,h_1](I-P)f_1(I-P)f_2 + \mathrm{tr}[h_2,h_1]f_2(I-P)f_1P \\
&= -\mathrm{tr}[f_2,h_1](I-P)f_1(I-P)h_2 - \mathrm{tr}[f_2,h_1](I-P)f_1(I-P)f_2 \\
&\qquad +\mathrm{tr}[h_2,h_1]f_2(I-P)f_1P \\
&= -\mathrm{tr}[f_2,h_1](I-P)f_1(I-P)h_2 + \mathrm{tr}(I-P)(h_2f_1h_1 - f_1h_2h_1 \\
&\qquad +f_1Ph_2h_1 - f_1Ph_1h_2)f_2 \\
&= -\mathrm{tr}[f_2,h_1](I-P)f_1(I-P)h_2 + \mathrm{tr}f_2(I-P)(h_2f_1Ph_1 + h_2f_1(I-P)h_1 \\
&\qquad -f_1(I-P)h_2h_1 - f_1Ph_1h_2) \\
&= -\mathrm{tr}[f_2,h_1](I-P)f_1(I-P)h_2 + \mathrm{tr}f_2(I-P)h_2f_1Ph_1 \\
&\qquad -\mathrm{tr}f_2(I-P)f_1Ph_1h_2 + \mathrm{tr}f_2(I-P)(h_2f_1 - f_1h_2)(I-P)h_1 \\
&= \mathrm{tr}f_2(I-P)[h_2,f_1](I-P)h_1 - \mathrm{tr}[f_2,h_1](I-P)f_1(I-P)h_2.
\end{aligned}$$

Let $\lambda_j = \lambda_j^-$ for $j = 1,2.\dots,m$, and let $\lambda_j = \lambda_{j-m}^+$ for $j = m+1, m+2,\dots,m+n$. Similarly, set $P_j = P_j^-$ for $j \le m$ and $P_j = P_{j-m}^+$ for $j = m+1, m+2,\dots,m+n$. Then $I-P = \sum_{j=1}^{m+n} P_j$.

Lemma 3.7

$$\mathrm{tr}[h_2,f_1](I-P)h_1f_2(I-P) - \mathrm{tr}[f_2,h_1](I-P)f_1h_2(I-P)$$

$$= \sum_{j=1}^{m+n} h_2(\lambda_j)f_2(\lambda_j)\mathrm{tr}(f_1P_jh_1P - [P_j,h_1]f_1 - f_1Ph_1P_j).$$

Proof:

$$\mathrm{tr}([h_2,f_1](I-P)h_1f_2(I-P)) - \mathrm{tr}([f_2,h_1](I-P)f_1h_2(I-P))$$

$$= \mathrm{tr}\big(\textstyle\sum_{i,j}(h_2f_1 - f_1h_2)P_ih_1f_2P_j - \sum_{i,j}(f_2h_1 - h_1f_2)P_if_1P_jh_2\big)$$

$$= \textstyle\sum_{i,j} \mathrm{tr}((h_2(\lambda_j)f_1P_i - P_jf_1P_ih_2(\lambda_i))h_1f_2(\lambda_j)P_j$$

$$- \textstyle\sum_{i,j} \mathrm{tr}(P_j(f_2(\lambda_j)h_1P_i - P_jh_1f_2(\lambda_i))P_if_1P_jh_2(\lambda_j))$$

$$= \mathrm{tr}\textstyle\sum_{i\neq j}(h_2f_2)(\lambda_j)f_1P_ih_1P_j - \mathrm{tr}\sum_{i\neq j} h_2(\lambda_i)f_2(\lambda_j)P_jf_1P_ih_1$$

$$-\mathrm{tr}\textstyle\sum_{i\neq j}(h_2f_2)(\lambda_j)f_1P_ih_1P_j - \sum_{i\neq j} \mathrm{tr}(h_2f_2)(\lambda_j)P_ih_1P_jf_1$$

$$= \textstyle\sum_j(h_2f_2)(\lambda_j)\mathrm{tr}f_1(I - P - P_j)h_1P_j - \sum_j(h_2f_2)(\lambda_j)\mathrm{tr}f_1P_jh_1(I - P - P_j)$$

$$= \textstyle\sum_j(h_2f_2)(\lambda_j)\mathrm{tr}(f_1P_jh_1P - f_1Ph_1P_j - [P_j, h_1]f_1).$$

Using Lemmas 3.5, 3.6 and 3.7, we can now prove another trace identity.

Lemma 3.8

$$\mathrm{tr}[Pf_1f_2, h_1h_2] = \mathrm{tr}[f_1f_2, h_1h_2] - \sum_j (h_2f_2)(\lambda_j)\mathrm{tr}[P_jf_1, h_1].$$

Proof:

$$\mathrm{tr}[Pf_1f_2, h_1h_2]$$

$$= \mathrm{tr}[Pf_1, h_1](I - P)f_2h_2 - \mathrm{tr}[Ph_1, f_1](I - P)f_2h_2 + \mathrm{tr}[f_1f_2, h_1h_2]$$

$$+\mathrm{tr}[h_2, f_1](I - P)h_1f_2(I - P) - \mathrm{tr}[f_2, h_1](I - P)f_1(I - P)f_2$$

$$= \mathrm{tr}[f_1f_2, h_1h_2] + \textstyle\sum_j(h_2f_2)(\lambda_j)[Pf_1, h_1]P_j - \sum_j(h_2f_2)(\lambda_j)[Ph_1, f_1]P_j$$

$$+ \textstyle\sum_j(f_2h_2)(\lambda_j)(f_1P_jh_1P - f_1Ph_1P_j - [P_jf_1, h_1])$$

$$= \mathrm{tr}[f_1f_2, h_2h_1] - \textstyle\sum_j(h_2f_2)(\lambda_j)\mathrm{tr}[P_j, h_1]f_1$$

$$= \mathrm{tr}[f_1f_2, h_1h_2] - \textstyle\sum_j(h_2f_2)(\lambda_j)\mathrm{tr}[P_jf_1, h_1].$$

We prove another lemma to complete this section.

Lemma 3.9

$$\mathrm{tr}[P_jf_1, h_1] = -\mathrm{trace}\,[f_1, P_jh_1].$$

Proof: We note that P_j is one of the P_r^+, P_j^-. By Theorem 3.2 and the remark after the theorem, the principal function of the pair $\{P_j, W\}$ is either

-1 or $+1$ on the cylinder $[0,1]\times S^1$. Without loss of generality, consider the second case. Note that we are integrating on the unit circle S^1. We have

$$\mathrm{tr}[P_j f_1, h_1] = \int_{S^1}\int_0^1 f_1(\tau)h_1'(\tau)dx\,d\tau = \int_{S^1} f_1(\tau)h_1'(\tau)d\tau$$

$$= f_1(\tau)h_1(\tau)\Big|_{S^1} - \int_{S^1} f_1'(\tau)h_1(\tau)d\tau$$

$$= -\int_{S^1}\int_0^1 f_1'(\tau)h_1(\tau)dx\,d\tau = -\mathrm{tr}[f_1, P_j h_1].$$

3.3 Proof of Theorem 1.1

We are now able to begin the proof of Theorem 1.1. Suppose that $F(x,\tau,\lambda)$, $H(x,\tau,\lambda)$ are smooth functions in $\hat{\mathcal{M}}(R\times S^1\times R)$. We start To evaluate $\mathrm{tr}[F(x,\tau,\lambda), H(x,\tau,\lambda)]$. It is enough to take special functions of the form $F(x,\tau,\lambda) = f_0(x)f_1(\tau)f_2(\lambda)$ and $H(x,\tau,\lambda) = h_0(x)h_1(\tau)h_2(\lambda)$ since these functions are dense in the algebra $\hat{\mathcal{M}}(R\times S^1\times R)$. Further, for the same reason we can assume h_0, f_0 are polynomials. Then it is clear that $f_0(P) = c_1 + c_2P$, $h_0(P) = d_1 + d_2P$, where c_1, c_2, d_1 and d_2 are constants. Therefore we only need to examine

$$\mathcal{I} = \mathrm{tr}[(c_1+c_2P)f_1(W)f_2(H_2), (d_1+d_2P)h_1(W)h_2(H_2)].$$

Now

$$\mathcal{I} = \mathrm{tr}[(c_1+c_2P)f_1f_2, (d_1+d_2P)h_1h_2]$$

$$= c_1d_1\mathrm{tr}[f_1f_2, g_1g_2] + c_1d_2\mathrm{tr}[f_1f_2, Pg_1g_2]$$

$$+c_2d_1\mathrm{tr}[Pf_1f_2, g_1g_2] + c_2d_2\mathrm{tr}[Pf_1f_2, Pg_1g_2]$$

$$= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.$$

$\mathcal{I}_1$ can be computed immediately from the principal function of the pair $\{W, H_2\}$.

$\mathcal{I}_3$ and $\mathcal{I}_4$ are equal according to Lemma 3.5.

Lemma 3.5 and 3.8 imply that

$\mathcal{I}_2 = c_1 d_2 \mathrm{tr}[f_1 f_2, P h_1 h_2] = -c_1 d_2 \mathrm{tr}[P h_1 h_2, f_1 f_2]$

$= -c_1 d_2 \mathrm{tr}[h_1 h_2, f_1 f_2] - c_1 d_2 \sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j h_1, f_1]$

$= c_1 d_2 \mathrm{tr}[f_1 f_2, h_1 h_2] + c_1 d_2 \sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1].$

Thus

$$
\begin{aligned}
\mathcal{I} &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 \\
&= c_1 d_1 \mathrm{tr}[f_1 f_2, h_1 h_2] + c_1 d_2 \mathrm{tr}[f_1 f_2, h_1 h_2] + c_1 d_2 \textstyle\sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1] \\
&\qquad + c_2 d_1 \mathrm{tr}[f_1 f_2, h_1 h_2] + c_2 d_1 \textstyle\sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1] \\
&\qquad\qquad + c_2 d_2 \mathrm{tr}[f_1 f_2, h_1 h_2] + c_2 d_2 \textstyle\sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1] \\
&= (c_1 d_2 + c_1 d_2 + c_2 d_1 + c_2 d_2) \mathrm{tr}[f_1 f_2, h_1 h_2] \\
&\qquad + (c_1 d_2 + c_2 d_1 + c_2 d_2)) \textstyle\sum_j (f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1] \\
&= f_0(1) h_0(1) \mathrm{tr}[f_1 f_2, h_1 h_2] \\
&\qquad + \textstyle\sum_j (f_0(1) h_0(1) - f_0(0) h_0(0))(f_2 h_2)(\lambda_j) \mathrm{tr}[P_j f_1, h_1] \\
&= \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2\pi i} f_0(1) h_0(1) \int\int \frac{\partial(f_1 f_2, h_1 h_2)}{\partial(\tau, \lambda)} G(\tau, \lambda) d\tau d\lambda \\
&= \frac{1}{2\pi i} \int\int \frac{\partial(F, H)}{\partial(\tau, \lambda)} (1, \tau, \lambda) G(\tau, \lambda) d\tau d\lambda.
\end{aligned}
$$

Let η_1 be the orientation (the unit normal) vector of the support of $G(\tau, \lambda)$ on the cylinder $S^1 \times R$ embedded in $R \times S^1 \times R$ as the surface $\{1\} \times S^1 \times R$.

Then

$$\mathcal{J}_1 = \frac{1}{2\pi}\int\int < dF \wedge dH, \eta_1 > G(\tau,\lambda) d\tau d\lambda. \tag{3.7}$$

Now we take another normalized orienting vector η_2 on the surface $\{\lambda_j\} \times S^1 \times [0,1]$ in $R \times S^1 \times R$ such that η_2 points - in the negative λ-axis direction when $\lambda_j = \lambda_\tau^+$, and η_2 points to the positive λ-axis direction when $\lambda_j = \lambda_j^-$. Then if $G_j(x,\tau)$ denotes the principal function for the pair $\{P_j, W\}$ we will have

$$2\pi i \mathcal{J}_2 = \sum_j (f_2h_2)(\lambda_j)((f_0h_0)(1) - (f_0h_0)(0)) \int f_1(\tau)h_1'(\tau)\, G_j(x,\tau)\, dx\, d\tau$$

$$= \sum_j (f_2h_2)(\lambda_j) \int_0^1 (f_0'(x)h_0(x) + f_0(x)h_0'(x))\, dx \cdot \int f_1(\tau)h_1'(\tau)(\int G_j(x,\tau)dx)\, d\tau.$$

Note that

$$\int_0^1 G_j(x,\tau)\, dx = \begin{cases} -1 & \text{for} \quad j \le m \\ 1 & \text{for} \quad j > m. \end{cases}$$

Thus

$$2\pi i \mathcal{J}_2 = \sum_j \int\int \Big(f_0'(x)f_1(\tau)f_2(\lambda_j)h_0(x)h_1'(\tau)h_2(\lambda_j)$$

$$- f_0(x)f_1'(\tau)f_2(\lambda_j)h_0'(x)h_1(\tau)h_2(\lambda_j)\Big) \cdot G_j(x,\tau)\, dx d\tau$$

$$= \sum_j \int\int \frac{\partial(F,H)}{\partial(x,\tau)}(x,\tau,\lambda_j)G_j(x,\tau)\, dx\, d\tau.$$

If we let $G(x,\tau,\lambda) = G_j(x,\tau)$ for $\lambda_j = \lambda_j^+$, while $G(x,\tau,\lambda) = -G_j(x,\tau)$ for $\lambda_j = \lambda_j^-$, then

$$\mathcal{J}_2 = \frac{1}{2\pi}\int\int_\sigma (dF \wedge dH, \eta_2)\, G(x,\tau,\lambda)\, d\mathcal{H}^2. \tag{3.8}$$

trace $[f(T,T^*,H_{2,a}),h(T,T^*,H_{2,a})]$

$$= \frac{1}{2\pi i}\int\int G_T(e^{i\theta},x)\frac{\partial(f(e^{i\theta},e^{-i\theta},x),h(e^{i\theta},e^{-i\theta},x))}{\partial(\theta,x)}dxd\theta, \qquad (3.10)$$

where $f,h\in\mathcal{M}_T(S^1\times S^1\times R)$*, and* $G_T(\tau,\lambda)\equiv G(\tau,\lambda)+\{\#\ \ of\ \ \lambda_j^-:\lambda_j^-<\lambda\}-\{\#\ of\ \ \lambda_r^+:\lambda_r^+<\lambda\}$.

Proof: We first look at $\mathrm{tr}[F(T,H_{2,a}),H(T,H_{2,a})]$. As before it will be enough to prove the theorem in the case where $F(\tau,\lambda)=f_1(\tau)f_2(\lambda), H(\tau,\lambda)=h_1(\tau)h_2(\lambda)$ and f_i, h_i are real analytic in τ.

In this case, because we know from [24] that W is the minimal unitary dilation of T, we have

$$F(T,H_{2,a})=Pf_1(W)Pf_2(H_{2,a}) \text{ and } H(T,H_{2,a})=Ph_1(W)Ph_2(H_{2,a}).$$

Thus by noting that $[P,W]$ is in trace class

$$\begin{aligned}&\mathrm{tr}[F(T,H_{2,a}),H(T,H_{2,a})]\\&=\mathrm{tr}[Pf_1(W)Pf_2(H_{2,a}),Ph_1(W)Ph_2(H_{2,a})]\\&=tr[Pf_1(W)f_2(H_{2,a}),Ph_1(W)h_2(H_{2,a})].\end{aligned}$$

For simplicity, and without real loss of generality, we can now examine just the case where T is a (1,1) contraction, and $\lambda^-<\lambda^+$.

In this case, Theorem 1.1 asserts that

$$\begin{aligned}&2\pi i\,\mathrm{tr}\,[F(T,H_{2,a}),H(T,H_{2,a})]\\&=\int\int\frac{\partial(F,H)}{\partial(\tau,\lambda)}G(\tau,\lambda)\,d\tau d\lambda+f_2(\lambda^+)h_2(\lambda^+)\int f_1(\tau)h_1'(\tau)\,d\tau\\&\qquad -f_2(\lambda^-)h_2(\lambda^-)\int f_1(\tau)h_1'(\tau)\,d\tau.\end{aligned}$$

On the other hand

$$\int\int_{\lambda^-}^{\lambda^+}f_1'(\tau)f_2(\lambda)h_1(\tau)h_2'(\tau)d\tau d\lambda$$

$$= \int_{\lambda^-}^{\lambda^+} f_2(\lambda)h_2'(\lambda)d\lambda \int f_1'(\tau)h_1(\tau)d\tau$$

$$= \left(f_2(\lambda^+)h_2(\lambda^+) - f_2(\lambda^-)h_2(\lambda^-) - \int_{\lambda^-}^{\lambda^+} f_2'(\lambda)h_2(\lambda)d\lambda\right) \cdot \int f_1'(\tau)h_1(\tau)d\tau.$$

After a τ integration by parts, taking the periodicity into account, this becomes

$$((f_2h_2)(\lambda^+) - (f_2h_2)(\lambda^-)) \int f_1(\tau)h_1'(\tau)d\tau + \int\int f_1(\tau)f_2'(\lambda)h_1'(\tau)h_2(\lambda)d\tau d\lambda.$$

Thus

$$((f_2h_2)(\lambda^+) - (f_2h_2)(\lambda^-)) \int f_1(\tau)h_1'(\tau)d\tau$$

$$= \int\int \frac{\partial((f_1f_2)(\tau,\lambda),(h_1h_2)(\tau,\lambda))}{\partial(\tau,\lambda)} \tilde{G}(\tau,\lambda)d\tau d\lambda,$$

where $\tilde{G}(\tau,\lambda) = 1 \quad \text{for} \quad \lambda^- < \lambda < \lambda^+$.

Setting $G_T(\tau,\lambda) = \tilde{G}(\tau,\lambda) + G(\tau,\lambda)$, we see that $G_T(\tau,\lambda)$ is the principal function defined in the introduction and we will obtain

$$\operatorname{tr}[F(T,H_{2,a}),H(T,H_{2,a})] = \frac{1}{2\pi i}\int\int \frac{\partial(F,H)}{\partial(\tau,\lambda)}\tilde{G}(\tau,\lambda)d\tau d\lambda$$

$$+\frac{1}{2\pi i}\int\int \frac{\partial(F,H)}{\partial(\tau,\lambda)}G(\tau,\lambda)d\tau d\lambda$$

$$= \frac{1}{2\pi i}\int\int \frac{\partial(F,H)}{\partial(\tau,\lambda)}G_T(\tau,\lambda)d\tau d\lambda.$$

To complete the proof of Theorem 3.10, it is only necessary to note that $f(T^*,T,H_{2,a}) = F(T,H_{2,a})$ for some function F in two variables because $T = PWP$, $T^* = PW^*P$, $[P,W]$ is in trace class, and the functional calculus is only defined modulo the trace class. Thus, for $f = f(e^{i\theta},e^{i\phi},\lambda)$ and $h = h(e^{i\theta},e^{i\phi},\lambda)$ in $\hat{\mathcal{M}}_T(S^1\times S^1\times R)$, we let $F(\tau,\lambda) = f(e^{i\theta},e^{-i\theta},\lambda)$, and $H(\tau,\lambda) = h(e^{i\theta},e^{-i\theta},\lambda)$; for $\tau = e^{i\theta}$. Since $I - T^*T$ and $I - TT^*$ are in trace class, we have that both $f(T,T^*,H_{2,a}) - F(T,H_{2,a})$ and $h(T,T^*,H_{2,a}) - H(T,H_{2,a})$ are in trace class.

Thus

$$\begin{aligned}&\mathrm{tr}[f(T,T^*,H_{2,a}),h(T,T^*,H_{2,a})]\\&= \mathrm{tr}[F(T,H_{2,a}),H(T,H_{2,a})]\\&= \frac{1}{2\pi i}\int\int \frac{\partial(F,H)}{\partial(\tau,\lambda)}G_T(\tau,\lambda)d\tau d\lambda.\end{aligned}$$

But an elementary direct calculation shows that

$$\frac{\partial(F,H)}{\partial(\tau,\lambda)}d\tau d\lambda = \frac{\partial(f,h)}{\partial(\theta,\lambda)}d\theta d\lambda. \tag{3.11}$$

We have therefore established the asserted equality.

We will call $G(x,\tau,\lambda)$ the full principal function for the operator triplet $\{P,W,H_2\}$. In an obvious sense, usual for the principal current theory, the "projection" of this function (we mean the principal function associated to the projected principal current here and in the following sentences) to the (τ,λ) cylinder will give the principal function of the original pair $\{W,H_2\}$ while the "projection" to the (x,τ) also gives a principal function $G_P(x,\tau)$. The "projection" of the full principal function to the (x,λ) plane is almost everywhere equal to zero. This reflects the fact that the operators P and H_2 commute.

In this connection we remark that there is a normal perturbation problem in the (x,λ) plane that deserves some discussion. Let $\mathcal{C}$ be the curve in the (x,λ) plane which consists of the points $(1,\lambda)$ for $\lambda\in\sigma(H_2)$, together with the segments connecting $(0,\lambda_j)$ to $(1,\lambda_j)$; where λ_j runs through $\{\lambda_r^+\}$ and $\{\lambda_j^-\}$, that is the projection of the full current to the (x,λ) plane.

Define $\delta(x,\lambda)$ as follows:

$$\delta(x,\lambda_j) = \frac{1}{2\pi}\int_0^{2\pi} G(x,\tau,\lambda_j)\frac{d\tau}{\tau} \quad \& \quad \delta(1,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} G(1,\tau,\lambda)\frac{d\tau}{\tau}. \quad (3.12)$$

Consider the normal perturbation problem $P+iH_2 \to W(P+iH_2)W^*$. Alternatively consider the trace class perturbation problem for pair of operators: $(P,H_2) \to (WPW^*, WH_2W^*)$.

Theorem 3.11

$$\operatorname{tr}(f(P,H_2) - f(WPW^*, WH_2W^*)) = \int_{\mathcal{C}} < df, \sigma > \delta(x,\lambda) d\mathcal{H}^1,$$

where $d\mathcal{H}^1$ is Hausdorff one measure, and σ is the tangent vector to $\mathcal{C}$ chosen in a way compatible with the orientation vector in Theorem 1.1

Proof:

$$\operatorname{tr}(f(P,H_2) - f(WPW^*, WH_2W^*)) = \operatorname{tr}[f(P,H_2)W^*, W]$$

$$= \frac{1}{2\pi i}\int\int f_\lambda(1,\lambda)G(1,\tau,\lambda)\frac{1}{\tau}d\lambda d\tau$$

$$+\frac{1}{2\pi i}\sum_{r=1}^{n}\int\int f_x(x,\lambda_r^+)G(x,\tau,\lambda_r^+)\frac{d\tau}{\tau}dx$$

$$-\frac{1}{2\pi i}\sum_{j=1}^{m}\int\int f_x(x,\lambda_j^-)G(x,\tau,\lambda_j^-)\frac{d\tau}{\tau}dx$$

$$= \int_{\mathcal{C}} < df, \sigma > \delta(x,\lambda) d\mathcal{H}^1.$$

Chapter 4

The Characteristic Operator function and its Determinant

In this chapter, we compute the characteristic operator function of T defined by (1.6). If T has equal deficiency indices, we express the determinant of the characteristic function in terms of the metric intersection geometry of the principal current. Then we conclude that the class of contraction operators we study are quasi-similar but not similar to unitary operators. We note that in a paper [37] with Pincus, a wider class of contraction operators is studied which includes operators similar to unitary operators.

4.1 The characteristic operator function

We require some further results from [24]. Recall that in chapter 1, we introduced the operator H_1. Now let $d_l = (\frac{|l-\bar{l}|}{2})^{\frac{1}{2}}(H_1 - \bar{l})^{-1}k^*$ for $\Im l > 0$. Then define the rank one operator $D(l) = \frac{1}{\pi} d_l \otimes d_l$, and consider the perturbation problem

$$H_2 \longrightarrow H_2 + D(l). \tag{4.1}$$

Let $h_l(t)$ be the corresponding phase shift function. Then there will be unique positive measures $d\mu_l^{\pm}$ such that

$$\exp \int \frac{h_l(t)}{t-z}\,dt = 1 + \frac{1}{\pi}\int \frac{d\mu_l^+(t)}{t-z} \tag{4.2}$$

and

$$\exp - \int \frac{h_l(t)}{t-z}\,dt = 1 - \frac{1}{\pi}\int \frac{d\mu_l^-(t)}{t-z}. \tag{4.3}$$

The measures μ^+, μ^- are the spectral measures of H_2 and $H_2 + D(l)$ respectively. In [24], the phase shift function $h_l(\lambda)$ is calculated in terms of the symbols $A(\lambda) \pm |k(\lambda)|^2$ for $l = \zeta + i\eta$:

$$h_l(\lambda) = \frac{1}{\pi}\tan^{-1}(\frac{A(\lambda) + |k(\lambda)|^2 - \zeta}{\eta}) - \frac{1}{\pi}\tan^{-1}(\frac{A(\lambda) - |k(\lambda)|^2 - \zeta}{\eta}). \tag{4.4}$$

Note that $\mu_i^{\pm} = \mu^{\pm}$ in the notation of chapter 1. It is part of lemma 2 in [1] that the singular parts of $\mu_l^{\pm}$ all consist of atoms at $\{\lambda_r^+\}$ and $\{\lambda_j^-\}$. Furthermore, in [24], basis vectors $\{\omega_{l,r}^+\}$ and $\{\omega_{l,j}^-\}$ for $\mathcal{D}_{T_l^*}$ and for $\mathcal{D}_{T_l}$, were constructed from the principal function, so that

$$(\alpha) \qquad T\omega_{l,j}^- = \frac{1}{\pi}\sum_{r=1}^{n} \frac{\mu_l^+(\lambda_r^+)}{\lambda_r^+ - \lambda_j^-}\omega_{l,r}^+ \qquad j = 1, \ldots, m.$$

$$(\beta) \qquad T^*\omega_{l,r}^+ = \frac{1}{\pi}\sum_{j=1}^{m} \frac{\mu_l^-(\lambda_j^-)}{\lambda_r^+ - \lambda_j^-}\omega_{l,j}^- \qquad r = 1, \ldots, n.$$

We are thus able to state

Theorem 4.1 *For* $|\mu| < 1$, $\det \Theta^*(\mu)\Theta(\mu) = \det \left(\Theta_{s,t}(\mu)\right)_{s,t=1}^{m}$, *where*

$$\Theta_{s,t}(\mu) = \frac{1}{\pi^2}\sum_{r=1}^{n} \frac{\mu_l^+(\lambda_r^+)\cdot\mu_l^-(\lambda_t^-)}{(\lambda_r^+ - \lambda_s^-)(\lambda_r^+ - \lambda_t^-)}$$

and $l = i\dfrac{1+\bar{\mu}}{1-\bar{\mu}}$.

Proof: For any $\mu \in \mathcal{D}$, the unit disc, let $T(\mu) = (T-\mu)(1-\bar{\mu}T)^{-1}$. We know, see [22], that there exist isometries $Z(\mu) : \mathcal{D}_{T(\mu)} \longrightarrow \mathcal{D}_T$, and $Z_*(\mu) : \mathcal{D}_{T^*(\mu)} \longrightarrow \mathcal{D}_{T^*}$ such that

$$Z_*(\mu)\Theta_{T(\mu)}(a)Z^{-1}(\mu) = \Theta_T(\tilde{a}). \tag{4.5}$$

for $\tilde{a} = \dfrac{a+\mu}{1+\bar{\mu}a}$.

But $T(\mu) = (T-\mu)(1-\bar{\mu}T)^{-1} = P(W-\mu)(1-\bar{\mu}W)^{-1}P$, since W is the minimal unitary dilation of T. Thus

$$T(\mu) = \frac{1-\mu}{1-\bar{\mu}}T_l \quad \text{for} \quad l = i\frac{1+\bar{\mu}}{1-\bar{\mu}}. \tag{4.6}$$

Therefore by (4.5) and (4.6), we obtain

$\Theta^*(\mu)\Theta(\mu) = Z(\mu)\Theta^*_{T(\mu)}(0)Z_*(\mu)^*Z_*(\mu)\Theta_{T(\mu)}(0)Z^*(\mu)$

$= Z(\mu)\Theta^*_{T(\mu)}(0)\Theta_{T(\mu)}(0)Z(\mu)^* = Z(\mu)T^*(\mu)T(\mu)Z^*(\mu)$.

Thus we have proved

$$\det\ \Theta^*(\mu)\Theta(\mu) = \det\ T_l^*T_l.$$

But (α) and (β) above together gives us

$$T_l^*T_l\,\omega_{l,s}^- = \frac{1}{\pi^2}\sum_{j=1}^{n}\sum_{r=1}^{m} \frac{\mu_l^+(\lambda_r^+)\cdot\mu_l^-(\lambda_j^-)}{(\lambda_r^+ - \lambda_s^-)(\lambda_r^+ - \lambda_j^-)}\omega_{l,j}^-.$$

This completes the proof

For the remainder of this section we will take the case where T has equal deficiency indices i.e. $m = n$.

Theorem 4.2 *If T has equal deficiency indices (m, m), then we have, for $l = i\dfrac{1+\bar{\mu}}{1-\bar{\mu}}$,*

$$\det\, \Theta^*(\mu)\Theta(\mu) = \frac{1}{\pi^{2m}} \prod_{r,j=1}^{m} \left(\mu_l^+(\lambda_r^+)\mu_l^-(\lambda_j^-)\right) \prod_{\substack{s\neq r\\ t\neq j}} \frac{(\lambda_r^+ - \lambda_s^+)^2(\lambda_j^- - \lambda_t^-)^2}{(\lambda_r^+ - \lambda_j^-)^2}.$$

Proof: We first decompose the determinant into the product of two simpler determinants.

Let

$$A_m(\mu_1, \ldots, \mu_m) = \begin{pmatrix} \frac{\mu_1}{\lambda_1^+ - \lambda_1^-} & \frac{\mu_1}{\lambda_1^+ - \lambda_2^-} & \cdots & \frac{\mu_1}{\lambda_1^+ - \lambda_m^-} \\ \frac{\mu_2}{\lambda_2^+ - \lambda_1^-} & \frac{\mu_2}{\lambda_2^+ - \lambda_2^-} & \cdots & \frac{\mu_2}{\lambda_2^+ - \lambda_m^-} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\mu_m}{\lambda_m^+ - \lambda_1^-} & \frac{\mu_m}{\lambda_m^+ - \lambda_2^-} & \cdots & \frac{\mu_m}{\lambda_m^+ - \lambda_m^-} \end{pmatrix}. \tag{4.7}$$

Then it is clear that

$$\det\, \Theta^*(\mu)\Theta(\mu) = \frac{1}{\pi^{2m}} \det\, A_m^*(\mu_1^-, \ldots, \mu_m^-)A_m(\mu_1^+, \ldots, \mu_m^+). \tag{4.8}$$

Let $B_m = A_m(1, 1, \ldots, 1)$. This is a nice enough classical determinant, the so called double alternate of Sylvester. The determinant B_m is evaluated, for example, in [20]. We can also easily verify, by induction on m say, that

$$\det\, B_m = \prod_{r,j=1}^{m} \frac{\prod_{s\neq r}(\lambda_r^+ - \lambda_s^+)\prod_{t\neq j}(\lambda_t^- - \lambda_j^-)}{\lambda_r^+ - \lambda_j^-}. \tag{4.9}$$

Since

$$\det\, A_m(\mu_1^{\pm},\ldots,\mu_m^{\pm}) = (\prod_{j=1}^{m} \mu_j^{\pm}) \det\, B_m,$$

we are done.

4.2 The proof of Theorem 1.2

We give the proof of Theorem 1.2 in this section. Through out this section, we always assume that T has equal deficiency indices, that is, $m = n$. We start with a lemma.

Lemma 4.3 *Let $l = \zeta + i\eta$. then*

$$\mu_l^+(\lambda_r^+) = \pi \lim_{y\to 0} y \exp \int h_l(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda$$

and

$$\mu_l^-(\lambda_j^-) = \pi \lim_{y\to 0} y \exp - \int h_l(\lambda) \frac{\lambda - \lambda_j^-}{(\lambda - \lambda_j^-)^2 + y^2}\, d\lambda.$$

Proof: Consider the first equation. We let

$$\phi_l(z) = 1 + \frac{1}{\pi} \int \frac{d\mu_l(t)}{t - z} = \exp \int h_l(t) \frac{dt}{t - z}. \qquad (4.10)$$

By the Stieltjes inversion formula.

$\frac{1}{\pi}\mu_l^+(\lambda_r^+) = \lim_{y\to 0}(-iy)\phi_l(z)$

$$= -i \lim_{y\to 0} y \exp \Big(\int h_l(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda + i \int h_l(\lambda) \frac{y}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda\Big).$$

But it was proved in [24] that $h_l(\cdot)$ has left Lebesgue value 0 and right Lebesgue value 1 at λ_r^+. Thus by the Fatou theorem for Poissson integrals we have

$$i \int h_l(\lambda) \frac{y}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda \longrightarrow \frac{\pi}{2} i.$$

Hence

$$\frac{1}{\pi}\mu_l(\lambda_r^+) = \lim_{y \to 0} \exp \int h_l(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda.$$

The second equation can be proven in the same way.

Lemma 4.4 *For $l = \zeta + i\eta$ we have*

$$\lim_{\eta \to 0} \lim_{y \to 0} \int_{-\infty}^{\infty} \tan^{-1} \frac{2\eta |k(\lambda)|^2}{\eta^2 + (A(\lambda) - \zeta)^2 - |k(\lambda)|^4} \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda = 0$$

for almost all ζ. A corresponding result holds for λ_j^-.

Proof: For the proof we first note that for any $\epsilon < 0$, there is an open set Ω such that $K_\zeta \subset \Omega$ and $m(K_\zeta) < \epsilon$ for almost all ζ when $K_\zeta = \{\lambda : A(\lambda) = \zeta \text{ or } A(\lambda) - \zeta = \pm |k(\lambda)|^2\}$.

In fact, let $F_\delta = \{\zeta : m(K_\zeta) \geq \delta\}$. If $A(\cdot)$ and $k(\cdot)$ are continuous, then K_ζ is closed and $K_\zeta \neq K'_\zeta$ if $\zeta \neq \zeta'$. Thus $m(F_\delta) = 0$. On the other hand $F_\delta \subset F'_\delta$ if $\delta < \delta'$. Therefore $m(F_\delta) = \lim_{\delta \to 0} m(F_\delta) = 0$. Thus for almost all ζ, $m(K_\zeta) = 0$.

If $A(\cdot), k(\cdot)$ are not continuous, then by Lusin's theorem they can be approximated uniformly by continuous functions outside a set of arbitrarily small measure.

Thus, without loss of generality, we can assume that $\zeta = 0$ and $K_0 \subset \Omega$ and $m(\Omega) < \epsilon$ for arbitrary small ϵ.

Furthermore we can assume that $\lambda_r^+ \notin \Omega$ by adjusting Ω a little.

Now we let

$$\tilde{h}_\eta(\lambda) = \begin{cases} \tan^{-1} \dfrac{2\eta|k(\lambda)|^2}{\eta^2 + A(\lambda)^2 - |k(\lambda)|^4} & \text{for} \quad \lambda \in \sigma(H_2) \\ 0 & \text{for} \quad \lambda \in \sigma(H_2), \end{cases}$$

then

$$\left| \int_\Omega \tilde{h}_\eta(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2} \, d\lambda \right| \leq \int_\Omega d\lambda < Cm(\Omega) < \epsilon. \tag{4.11}$$

where C is a constant. We will let C denote different constants throughout the following.

Now we need only show that

$$\left| \lim_{y \to 0} \int_{R/\Omega} \tilde{h}_\eta(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2} \; d\lambda \; \right| < \epsilon, \tag{4.12}$$

when η is small enough.

We know from lemma 2 of [24] that $|k(\lambda)|^2$ has left Lebesgue value 0 at λ_r^+. As $\lambda \to \lambda_r^+ - 0$, we can estimate $\tan^{-1}(x)$ by x and write

$$|\tilde{h}_\eta(\lambda)| \leq \eta |k(\lambda)^2|. \tag{4.13}$$

Also for a range of $N > 0$, by the eigenvalue criterion (2.6), we have

$$\int_{\lambda_r^+ - N}^{\lambda_r^+} \frac{|k(\lambda)|^2}{\lambda - \lambda_r^+} \, d\lambda < \infty \tag{4.14}$$

and

$$\left| \int_{\lambda +}^{\lambda_r^+ + N} \frac{1}{|k(\lambda)|^2} \frac{1}{\lambda - \lambda_r^+} \, d\lambda \right| < \infty. \tag{4.15}$$

Therefore there exists a $\delta_1 > 0$ such that

$$\left| \int_{\lambda_r^+ - \delta_1}^{\lambda_r^+} \tilde{h}_\eta(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2} \, d\lambda \right| < \epsilon. \tag{4.16}$$

We also know from [24] that $A(\lambda) \pm |k(\lambda)|^2$ has right Lebesgue value $\pm\infty$ as $\lambda \to \lambda_r^+ + 0$. Thus

$$|\tilde{h}_\eta(\lambda)| < C\frac{\eta|k(\lambda)|^2}{|k(\lambda)|^4} \leq \frac{C}{|k(\lambda)|^2} \tag{4.17}$$

in some interval to the right of λ_r^+. Now (4.15) and (4.17) imply that there is a $\delta_2 > 0$ such that

$$\left| \int_{\lambda_r^+}^{\lambda_r^+ + \delta_2} \tilde{h}_\eta(\lambda) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2} \, d\lambda \right| < \epsilon. \tag{4.18}$$

Finally for $\lambda \in \mathcal{V} = R \backslash (\Omega \cup (\lambda_r^+ - \delta_1, \lambda_r^+ + \delta_2))$, we have $|A(\lambda)^2 - |k(\lambda)|^4 > C$ and

$$|\tilde{h}_\eta(\lambda)| \leq \frac{2\eta|k(\lambda)|^2}{|\eta^2 + A(\lambda)^2 - |k(\lambda)|^4|} \leq C\eta|k(\lambda)|^2. \tag{4.19}$$

Thus as $\eta \to 0$

$$\left| \int_{\mathcal{V}} \frac{\tilde{h}_\eta(\lambda)}{\lambda - \lambda_r^+} \, d\lambda \right| < c\eta \int_{\mathcal{V}} \frac{|k(\lambda)|^2}{|\lambda - \lambda_r^+|^2} \, d\lambda < c\eta < \epsilon. \tag{4.20}$$

The proof is completed by combining (4.16), (4.18), and (4.20).

Now we come to the proof of Theorem 1.2.

Define $\delta(l) = \{\lambda : |h_l(\lambda)| \geq \frac{\pi}{2}\}$, for $l = \zeta + i\eta$, $\eta > 0$.

By (4.4) and the difference of two angles formula for the tangent function, we get the following

$$h_l(\lambda) = \begin{cases} \tilde{h}_l(\lambda) & \text{for} \quad \lambda \notin \delta(l) \\ \\ \pi + \tilde{h}_l(\lambda) & \text{for} \quad \lambda \in \delta(l). \end{cases}$$

But when $\eta > 0$, we have

$$\tan^{-1}\frac{A(\lambda)+|k(\lambda)|^2-\zeta}{\eta} \geq \tan^{-1}\frac{A(\lambda)-|k(\lambda)|^2-\zeta}{\eta}.$$

Therefore $\delta(l) = \{\lambda : h_l(\lambda) \geq \frac{\pi}{2}\}$.

Thus if we let

$$\delta(\zeta) = \{A(\lambda)+|k(\lambda)|^2-\zeta \geq 0\} \cup \{\lambda : A(\lambda)-|k(\lambda)|^2-\zeta \leq 0\}. \qquad (4.21)$$

Then as $\eta \to 0$ we have

$$\delta(l) \to \delta(\zeta).$$

It is clear that $\delta(\zeta)$ is the intersection of the line $y = \zeta$ with the essential support of the principal function $g(\zeta, \lambda)$ for the pair $\{H_1, H_2\}$.

But the transformation law of principal functions gives us $G(\frac{\zeta+i}{\zeta-i}, \lambda) = g(\zeta, \lambda)$. So we see that $\delta(\zeta)$ is the λ-support, $\mathcal{R}_\tau$, of $G(\tau, \lambda)$ at height $\tau = \frac{\zeta+i}{\zeta-i}$.

If we use the notations of the introduction, we can write $\delta(\zeta) = \cup_{j=1}^{m(\tau)}(\lambda_j^+(\tau), \lambda_j^-(\tau))$. But because $\delta(l) \to \delta(\zeta)$ as $l \to \zeta$, we can assume, for l close enough to ζ that

$$\delta(l) = \cup_{j=1}^{m}(a_j(l), \lambda_j^-) \cup (\cup_{j=1}^{m}(\lambda_j^+, b_j(l)) \cup (_{j=m+1}^{m(\tau)-m}(a_j(l), b_j(l))$$

where $a_j(l) \to a_j(\zeta) \equiv a_j(\tau)$ and $b_j(l) \to b_j(\zeta) \equiv b_j(\tau)$.

Denote $\tilde{\delta}(\zeta) = \cup_{j=1}^{m(\tau)-m}(a_j(\tau), b_j(\tau))$. Then by Lemmas 4.3 and 4.4 above we have

$$\lim_{y\to 0}\mu_l(\lambda_\tau^+) = \lim_{\eta\to 0}\lim_{y\to 0}\exp\frac{1}{\pi}\int h_l(\lambda)\frac{\lambda-\lambda_\tau^+}{(\lambda-\lambda_\tau^+)^2+y^2}\,d\lambda$$

$$= \lim_{\eta\to 0}\lim_{y\to 0} y\exp\int_{\delta(l)}\frac{\lambda-\lambda_\tau^+}{(\lambda-\lambda_\tau^+)^2+y^2}\,d\lambda.$$

Let us look at the integral in the exponential:

$$\int_{\delta(l)} \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda$$

$$= \Big(\sum_{j=1}^{m} \int_{a_j(l)}^{\lambda_j^-} + \sum_{j=1}^{m} \int_{\lambda_j^+}^{b_j(l)} + \int_{\text{other}}\Big) \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda$$

$$= \frac{1}{2}\Big(\sum_{j=1}^{m} \log \frac{(\lambda_j^- - \lambda_r^+)^2 + y^2}{(a_j(l) - \lambda_r^+)^2 + y^2)} + \sum_{j=1}^{m} \log \frac{(b_j(l) - \lambda_r^+)^2 + y^2}{(\lambda_j^+ - \lambda_r^+)^2 + y^2}\Big)$$

$$+ \int_{\text{other}} \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda$$

$$= -\log|y| + \log \prod_{j=1}^{m} ((\lambda_j^- - \lambda_r^+)^2)^{\frac{1}{2}} - \log \prod_{j \neq r} ((\lambda_j^+ - \lambda_r^+)^2 + y^2)^{\frac{1}{2}}$$

$$+ \sum_{j=1}^{m} \int_{a_j(l)}^{b_j(l)} \frac{(\lambda - \lambda_r^+)}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda + \int_{\text{other}} \frac{(\lambda - \lambda_r^+)}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda$$

$$= \log\Big(\frac{1}{|y|} \frac{\prod_{j=1}^{m} ((\lambda_j^- - \lambda_r^+)^2 + y^2)^{\frac{1}{2}}}{\prod_{j \neq r} ((\lambda_j^+ - \lambda_r^+)^2 + y^2)^{\frac{1}{2}}}\Big) + \int_{\tilde{\delta}(l)} \frac{\lambda - \lambda_r^+}{(\lambda - \lambda_r^+)^2 + y^2}\, d\lambda\Big).$$

Thus

$$\frac{1}{\pi} \lim_{\eta \to 0} \mu_l(\lambda_r^+) = \frac{\prod_{j=1}^{m} |\lambda_j^- - \lambda_r^+|}{\prod_{j \neq r} |\lambda_j^+ - \lambda_r^+|} \exp \int_{\tilde{\delta}(l)} \frac{d\lambda}{\lambda - \lambda_r^+} \tag{4.22}$$

Or

$$\frac{1}{\pi} \lim_{\eta \to 0} \mu_{\zeta + i\eta}(\lambda_r^+) = \frac{\prod_{j=1}^{m} |\lambda_j^- - \lambda_r^+|}{\prod_{j \neq r} |\lambda_j^+ - \lambda_r^+|} \cdot \prod_{k=1}^{m(\tau)} \prod_{r=1}^{m} \frac{|b_k(\tau_\zeta) - \lambda_r^+|}{|a_k(\tau_\zeta) - \lambda_r^+|}, \tag{4.23}$$

where $\tau_\zeta = \dfrac{\zeta + i}{\zeta - i}$.

Thus we have proved (1.7) in Theorem 1.2. By Theorem 4.2, $\det T^*T = \det \Theta^*(0)\Theta(0) \neq 0$. So a result of Sz. Nagy and C. Foias implies that T is invertible. Now because T has finite deficiency indices, T is a weak contraction and $\sigma(T) \subset S^1$. On the other hand, T is a finite perturbation of an isometry with finite deficiency indices. Therefore the essential spectrum of T is the full circle and $\sigma(T) \subset \sigma_e(T) = S^1$. Thus $\sigma(T) = S^1$.

This completes the proof of Theorem 1.2.

At the end of this section, we remark that the finiteness assumption for the multiplicity of W is not essential. It is made only for the convenience of expressing the Riemann-Hilbert barrier (1.7) in terms of the quotient of certain distances explicitly. Indeed, if we form the following Riemann-Hilbert barrier in general:

$$S(\tau) = \exp\Big(\sum_{r=1}^{n}\sum_{k=1}^{\infty}\int_{F_\tau^k}\frac{d\lambda}{\lambda-\lambda_r^+}\Big)\cdot\exp\Big(\sum_{j=1}^{m}\sum_{k=1}^{\infty}\int_{F_\tau^k}\frac{d\lambda}{\lambda_j^+-\lambda}\Big), \qquad (4.24)$$

where F_τ^k are defined in the introduction and the sum over k is a finite sum because we only have a finite number eigenvalues $\{\lambda_j^\pm\}$. Without essential change for the proof given above, one can easily see that Theorem 1.2 is still true for new barrier (4.24).

4.3 Some Consequences

In this last section of this chapter we derive a few consequences from the determinant expression of T in Theorem 1.2.

Theorem 4.5 *Suppose that the deficiency indices of T are $(m, m), m < \infty$. Then $\Theta_T(e^{it})$ is an outer function.*

Proof: By a well known theorem in [22], it suffices to show, since $\Theta_T(e^{it})$ is invertible, that $\det\Theta_T(e^{it})$ is outer. This is an almost immediate consequence of the preceding analysis. It is well known [14] that an analytic function on the unit disc, $F(\mu)$, is outer if and only if

$$\ln|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\ln|F(e^{i\theta})|\,d\theta. \qquad (4.25)$$

Thus, setting $F(\mu) = \Theta_T(\mu)$, for $|\mu| < 1$ and by Theorem 1.2, we have

$$|F(\mu)|^2 = C \prod_{r,j=1}^{m} \mu_l^+(\lambda_r^+)\mu_l^-(\lambda_j^-),$$

where C is constant in μ, and $l = i(1+\bar{\mu})(1-\bar{\mu})^{-1}$

From this, we see that $F(\mu)$ has no zeros in the unit disc. Therefore it will suffice for us to demonstrate simply that

$$\mu_i^+(\lambda_r^+) = \frac{1}{2\pi}\int_0^{2\pi} \mu_\zeta^+(\lambda_r^+)\,d\theta \tag{4.26}$$

and

$$\mu_{-i}^+(\lambda_j^-) = \frac{1}{2\pi}\int_0^{2\pi} \mu_\zeta^-(\lambda_j^-)\,d\theta. \tag{4.27}$$

where $\zeta = i(1+e^{-i\theta})(1-e^{-i\theta})^{-1}$, and $\mu_\zeta^\pm(\lambda_i^\pm) = \lim_{\eta\to 0}\mu_l^\pm(\lambda_i^\pm)$ for $l = \zeta + i\eta, \eta > 0$. The proof of the first of these assertions follows at once from the observation that

$$\frac{1}{\pi}\mu_l^+(\lambda_r^+) = \lim_{y\to 0}(-iy)\phi_l(z) \tag{4.28}$$

with $z = \lambda_r^+ + iy$.

But by (2.15)

$$\phi_l(z) = 1 + \frac{1}{\pi}\int \frac{d\mu_l(z)}{t-z} = \exp\int h_l(t)\frac{dt}{t-z} = E(l,z)\bar{E}(l,\bar{z}),$$

where G(ν,μ) is the characteristic function of the set

$$\{(\nu,\mu) : A(\mu) - \nu - |k(\mu)|^2 < \nu < A(\mu) - \nu + |k(\mu)|^2\}.$$

Clearly $\phi_l(z)$ is harmonic in l and an easy estimate shows that the limit above is uniform on any compact l - set in the upper half plane. Thus $\mu_l(\lambda_r^+)$ is harmonic.

Theorem 4.6 *T is a weak contraction and is quasi-similar to a unitary operator, $\sigma(T) = S^1$; and T is not similar to a unitary operator.*

Proof: By the previous theorems $\Theta(\mu)$ is invertible for all $|\mu| < 1$. Thus $\sigma(T) \subset S^1$. Indeed by [22] T is quasi-similar to the residual part of its minimal unitary dilation. But by the theorem above and lemma 2 in [24] we know that $\det \Theta^*(\mu)\Theta(\mu) \longrightarrow 0$ as $\mu \longrightarrow 1$. Thus by the well known criterion of Sz-Nagy and C. Foias, T is not similar to a unitary operator.

Finally we have

Theorem 4.7 *T is completely non unitary if and only if $m = m(\tau)$.*

Proof: The proof is quite simple. The spectral multiplicity, see [22], of the minimal dilation W of a completely non-unitary T is related to the deficiency index m by $m(\tau) = m + \text{rank } F(\tau)$, where $F^2(\tau) = I - \Theta^*(\tau)\Theta(\tau)$. But Theorem 1.2 and its proof show that $F(\tau)$ must have full rank, m, for almost all τ. Thus when T is completely non-unitary we have $m(\tau) = m + m$. Of course if there were a unitary part to T this would produce additional spectral multiplicity in W so the converse also follows.

Chapter 5

Unimodular Singular Integral Operators

The contraction T has the the singular integral representation (1.8). It has unimodular symbols and is the product of the compression of the wave operator and a unitary operator. In this chapter we study more general singular integral operators (1.9) with unimodular symbols.

5.1 unitary singular integral operators

Let $\alpha(\lambda), \beta(\lambda)$ be square integrable functions. Consider the following formal singular integral operator in $L_2(E)$:

$$Uf(\lambda) = f(\lambda) + \frac{1}{\pi}\bar{\alpha}(\lambda)\int \beta(t)f(t)\frac{dt}{t-(\lambda - i0)}, \tag{5.1}$$

where

$$\int g(t)\frac{dt}{t-(\lambda - i0)} = \lim_{\epsilon \to 0}\int g(t)\frac{dt}{t-(\lambda - i\epsilon)}.$$

In general a domain can be found so that U is densely defined. But in certain cases, the closure may be bounded and have $L_2(E)$ as its domain. In this section, we determine when the closure is defined everywhere and is a unitary operator. In the following, when we say U is unitary we always mean the closure of U is unitary.

Using the Plemelj formula, U can be equivalently written as:

$$Uf(\lambda) = (1 - i\bar{\alpha}(\lambda)\beta(\lambda))f(\lambda) + \frac{1}{\pi}\bar{\alpha}(\lambda)\int \beta(t)f(t)\frac{dt}{t-\lambda}. \tag{5.2}$$

Let $C^2 = ||\alpha|| \cdot ||\beta||^{-1}$, and $\alpha'(\lambda) = C^{-1}\alpha(\lambda), \beta'(\lambda) = C\beta(\lambda)$. Then we rewrite (5.1) as:

$$Uf(\lambda) = f(\lambda) + \frac{1}{\pi}\bar{\alpha}'(\lambda)\int \beta'(t)f(t)\frac{dt}{t-(\lambda - i0)}. \tag{5.3}$$

where $\alpha'(\lambda), \beta'(\lambda)$ have equal L_2−norm. Therefore, throughout this chapter we assume that $||\alpha|| = ||\beta||$ in (5.1).

Theorem 5.1 *If U is a unitary operator, then one of the following is true:*

(i) $|1 - 2i\bar{\alpha}\beta| = 1$ *and*

$$(1 + \frac{1}{\pi}\int \frac{|\alpha(\lambda)|^2}{t-z}\,dt)\,(1 - \frac{1}{\pi}\int \frac{|\beta(\lambda)|^2}{t-z}\,dt) = 1. \tag{5.4}$$

(ii) $|1 - 2i\bar{\alpha}\beta| = 1$ *and*

$$(1 - \frac{1}{\pi}\int \frac{|\alpha(\lambda)|^2}{t-z}\,dt)\,(1 + \frac{1}{\pi}\int \frac{|\beta(\lambda)|^2}{t-z}\,dt) = 1. \tag{5.5}$$

Proof: Let M be the multiplication operator by the position function on $L_2(E)$, i.e. $Mf(\lambda) = \lambda f(\lambda)$. Then M is a simple self-adjoint operator.

Suppose now that U is a unitary operator. Then

$$[M, U] = UM - MU = \frac{1}{\pi}(\cdot, \bar{\beta})\bar{\alpha} = \frac{1}{\pi}\bar{\alpha} \otimes \bar{\beta},$$

that is

$$UM = (M - \frac{1}{\pi}(\cdot, U\bar{\beta})\bar{\alpha})U. \tag{5.6}$$

Let $D = -\frac{1}{\pi}(\cdot, U\bar{\beta})\bar{\alpha}$. Then $D = UMU^* - M$ is a self-adjoint operator. But $D^* = -\frac{1}{\pi}(\cdot, \bar{\alpha})U\bar{\beta}$. Thus $U\bar{\beta} = C\bar{\alpha}$ for some real number C. But the fact that U is unitary and $||\alpha|| = ||\beta||$ imply that $||U\bar{\beta}|| = ||\bar{\beta}|| = ||\alpha||$. Therefore we conclude that $C = \pm 1$.

We first assume that $C = -1$. In this case $D = \frac{1}{\pi}(\cdot, \bar{\alpha})\bar{\alpha}$ is a positive rank one operator. For the triplet $\{U, M, D\}$, by a result of Pincus, cf. [5], there is a principal function $G(\zeta, \lambda)$ for the triplet, which is a completly unitary invariant. Furthermore, since $D \geq 0$, for fixed λ, $G(\zeta, \lambda)$ is the characteristic function of the positive arc extended from $S_+(M, U)(\lambda)$ to $S_-(M, U)(\lambda)$.

But it is clear that:

$$S_+(M, U)(\lambda) = 1 - 2i\bar{\alpha}\beta \quad \text{and} \quad S_-(M, U)(\lambda) = 1. \tag{5.7}$$

Let

$$\delta(\lambda) = \frac{1}{2\pi i}\int G(\zeta, \lambda)\frac{d\zeta}{\zeta} \tag{5.8}$$

be the average of $G(\zeta, \lambda)$. Then $\delta(\lambda)$ is the phase shift function of the perturbation problem

$$M \to M + \frac{1}{\pi}\bar{\alpha} \otimes \bar{\alpha}. \tag{5.9}$$

By (5.8), we have $S_+(M, U)(\lambda) = e^{-2\pi i\delta(\lambda)}$.

Forming the perturbation determinant, we get

$$\det(M + D - z)(M - z)^{-1} = 1 + \frac{1}{\pi}\int \frac{d(E_\lambda\bar{\alpha}, \bar{\alpha})}{\lambda - z} = \exp\int \frac{\delta(\lambda)}{\lambda - z}\, d\lambda.$$

where E_λ is the spectral resolution of M.

Since M is simple absolutely continuous, $d(E_\lambda \bar{\alpha},\, \bar{\alpha}) = |\alpha(\lambda)|^2 d\lambda$. So we conclude

$$1 + \frac{1}{\pi}\int \frac{|\alpha(\lambda)|^2}{\lambda - z} d\lambda = \exp \int \frac{\delta(\lambda)}{\lambda - z}\, d\lambda. \tag{5.10}$$

On the other hand, we can rewrite the commutator relation (5.6) as

$$MU^* = U^*M + \frac{1}{\pi}(\cdot, \bar{\alpha})U^*\bar{\alpha} = (M - \frac{1}{\pi}(\cdot, \bar{\beta})\bar{\beta})U^*. \tag{5.11}$$

Now consider the perturbation problem

$$M \to M - \frac{1}{\pi}\bar{\beta} \otimes \bar{\beta}.$$

By a similar argument to the above, we get

$$1 - \frac{1}{\pi}\int \frac{d(E_\lambda \bar{\beta},\, \bar{\beta})}{\lambda - z} = \exp \int \frac{\tilde{\delta}(\lambda)}{\lambda - z}\, d\lambda. \tag{5.12}$$

where $\tilde{\delta}(\lambda) = \frac{1}{2\pi i}\int \tilde{G}(\zeta, \lambda)\frac{d\zeta}{\zeta}$ and $\tilde{G}(\zeta, \lambda)$ is the principal function associated the triplet $\{U^*, M, -\frac{1}{\pi}\bar{\beta} \otimes \bar{\beta}\}$. By the transformation property of the principal function, we see that $\tilde{G}(\zeta, \lambda) = G(\bar{\zeta}, \lambda)$. Thus $\tilde{\delta}(\lambda) = -\delta(\lambda)$ and

$$1 - \frac{1}{\pi}\int |\beta(\lambda)|^2 \frac{d\lambda}{\lambda - z} = \exp - \int \frac{\delta(\lambda)}{\lambda - z}\, d\lambda. \tag{5.13}$$

Combining (5.10) and (5.11), we get (i).

In the case $C = 1$, a similar argument applies and we conclude that (ii) is satisfied.

In the proof of the above Theorem, we see that if U is unitary , then

$$U\bar{\beta} = \bar{\alpha} \quad \text{or} \quad U\bar{\beta} = -\bar{\alpha}.$$

Actually this characterizes the unitary property of U.

Theorem 5.2 *U is unitary if and only if one of the following are true:*

(i) $U\bar{\beta} = -\bar{\alpha}$ *and* $U^*\bar{\alpha} = -\bar{\beta}$.

(ii) $U\bar{\beta} = \bar{\alpha}$ *and* $U^*\bar{\alpha} = \bar{\beta}$.

Proof: The necessarity is proved in Theorem 5.1. We only need to prove the sufficiency here.

Suppose (i) is true, then $U^*U\bar{\beta} = \bar{\beta}$. We will prove the following by induction on n:

$$U^*UM^n\bar{\beta} = M^n\bar{\beta} \qquad n \geq 0 \tag{5.14}$$

(5.14) is true for $n = 0$ by the assumption. Now assume it is true for n. Then by the commutator relation (5.6), (5.11), we have

$$U^*UM^{n+1}\bar{\beta} = U^*(MU - \tfrac{1}{\pi}(\cdot,\bar{\beta})\bar{\alpha})M^n\bar{\beta} = U^*MUM^n\bar{\beta} + \tfrac{1}{\pi}(\cdot,\bar{\beta})M^n\bar{\beta}$$
$$= (MU^* + \tfrac{1}{\pi}(\cdot,\bar{\alpha})\bar{\beta})UM^n\bar{\beta} + \tfrac{1}{\pi}(M^n\bar{\beta},\bar{\beta})\bar{\beta} = M^{n+1}\bar{\beta}.$$

But $M^{n+1}\bar{\beta}$ are dense in $L_2(E)$, so the closure of U is a unitary operator.

Now we consider the converse of Theorem 5.1. Let's define the conjugate operator U' of U as following:

$$U'f(\lambda) = 1 - \frac{1}{\pi}\alpha(\lambda)\int \bar{\beta}(t)f(t)\frac{dt}{t-(\lambda - i0)}. \tag{5.15}$$

Theorem 5.3 *If the coefficients of U satisfy the following;*

(i) $|1 - 2i\bar{\alpha}(\lambda)\beta(\lambda)| = 1$,

(ii) $(1 + \frac{1}{\pi}\int |\alpha(\lambda)|^2 \frac{d\lambda}{\lambda - z})(1 - \frac{1}{\pi}\int |\beta(\lambda)|^2\frac{d\lambda}{\lambda - z}) = 1.$

Then U or U′ is unitary.

Proof: Let $\delta(\lambda)$ be the unique real function $0 \leq \delta(\lambda) \leq 1$, such that

$$1 + \frac{1}{\pi}\int |\alpha(\lambda)|^2\frac{d\lambda}{\lambda - z} = \exp\int\frac{\delta(\lambda)}{\lambda - z}. \tag{5.16}$$

Then by (ii)

$$1-\frac{1}{\pi}\int|\beta(\lambda)|^2\frac{d\lambda}{\lambda-z}=\exp-\int\frac{\delta(\lambda)}{\lambda-z}. \tag{5.17}$$

Suppose M is still the position operator on $L_2(E)$. Consider the one dimensional perturbation problem $M\to\tilde{M}=M+\frac{1}{\pi}\bar{\alpha}\otimes\bar{\alpha}$. By (5.16) and (5.17), $\delta(\lambda)$ is the phase shift of the perturbation problem and $-\delta(\lambda)$ is the the phase shift of the inverse perturbation problem. And $|\beta(\lambda)|^2d\lambda=d(\tilde{E}_\lambda\beta,\beta)$ where $\tilde{E}_\lambda$ is the spectral resolution of $\tilde{M}$. Thus $\tilde{M}$ is absolutely continuous. Therefore the wave operators $W_\pm$ defined by

$$W_\pm=\lim_{t\to\pm\infty}e^{it\tilde{M}}e^{-itM}P_a(M)$$

are unitary operators. Moreover W_- has the following singular integral representation (see [5])

$$W_-f(\lambda)=f(\lambda)+\frac{1}{\pi}\bar{\alpha}\int\frac{\alpha(t)}{\det(t+i0)}\frac{f(t)dt}{t-(\lambda-i0)}. \tag{5.18}$$

where $\det(t+i0)=\lim_{\epsilon\to 0}\det(\tilde{M}-(t+i\epsilon))(M-(t-i\epsilon))^{-1}$.

Let $\tilde{\beta}(\lambda)=\alpha(\lambda)(\det(\lambda+i0))^{-1}$ and apply Theorem 5.1 to W_-, we have

$$1-\frac{1}{\pi}\int|\tilde{\beta}(\lambda)|^2\frac{d\lambda}{\lambda-z}=\exp-\int\frac{\delta(\lambda)}{\lambda-z}. \tag{5.19}$$

Thus by assumption (ii), we have

$$1-\frac{1}{\pi}\int|\tilde{\beta}(\lambda)|^2\frac{d\lambda}{\lambda-z}=1-\frac{1}{\pi}\int|\beta(\lambda)|^2\frac{d\lambda}{\lambda-z}.$$

Taking the residues at infinity, we obtain $|\beta(\lambda)|=|\tilde{\beta}(\lambda)|$.

Let

$$\beta(\lambda)=\tilde{\beta}(\lambda)e^{i\theta(\lambda)}. \tag{5.20}$$

Apply Theorem 5.1 to W_- again, we have

$$1 = |1 - 2i\bar{\alpha}(\lambda)\tilde{\beta}(\lambda)| = |1 - 2i\bar{\alpha}(\lambda)\beta(\lambda)e^{-i\theta(\lambda)}|. \tag{5.21}$$

But the assumption (i) says

$$|1 - 2i\bar{\alpha}(\lambda)\beta(\lambda)| = 1. \tag{5.22}$$

By simple elementary geometry, (5.21) and (5.22) are true if and only if

(a) $\theta(\lambda) = 0$

or

(b) $\overline{2i\bar{\alpha}(\lambda)\beta(\lambda)} = 2i\bar{\alpha}(\lambda)\beta(\lambda)e^{-i\theta(\lambda)}$.

If (a) is true, $U = W_-$ is unitary.

If (b) is true, we have

$$-\alpha(\lambda)\bar{\beta}(\lambda) = \bar{\alpha}(\lambda)\beta(\lambda)e^{-i\theta(\lambda)}.$$

So we let $\phi(\lambda) = \bar{\beta}(\lambda)\beta^{-1}(\lambda)e^{i(\pi+\theta(\lambda))} = \bar{\alpha}(\lambda)\alpha^{-1}(\lambda)$ and M_ϕ be the multiplication operator by $\phi(\lambda)$ on $L_2(E)$. Then M_ϕ is unitary since $\phi(\lambda)$ is a unimodular function. Therefore

$$M_\phi U' M_\phi^* = 1 - \frac{1}{\pi}\bar{\alpha}(\lambda)\int \beta(\lambda)e^{-i(\pi+\theta)}\frac{dt}{t-\lambda}$$

$$= 1 + \frac{1}{\pi}\bar{\alpha}(\lambda)\int \beta(\lambda)e^{-i(\pi+\theta)}\frac{dt}{t-\lambda} = W_-.$$

Thus U' is unitary and the proof is complete.

In the proof above, we see that U is unitary if $U = W_-$. In this case, $\beta(\lambda) = \tilde{\beta}(\lambda)$. Then $\alpha(\lambda)\beta^{-1}(\lambda) = \det(\lambda - i0)$. Because $\det(z)$ has positive imaginary part, we conclude that $\alpha(\lambda)\beta^{-1}(\lambda)$ has positive imaginary part.

On the other hand, if U is a unitary operator, then $U = W_-$ since the pairs $\{U, M\}$ and $\{W_-, M\}$ have the same principal function and U, W_- have the same coefficient $\alpha(\lambda)$. So $\alpha(\lambda)\beta^{-1}(\lambda) = \det(\lambda - i0)$ has positive imaginary part.

Similarly, one can see that U' is unitary if and only if $\alpha(\lambda)\beta^{-1}(\lambda)$ has negative imaginary part.

Therefore we conclude that

Theorem 5.4 *Let U be the singular integral operator in (5.1) with coefficients $\alpha(\lambda), \beta(\lambda)$ satisfying the conditions of Theorem 5.3, Then U is unitary if and only if $\alpha(\lambda)\beta^{-1}(\lambda)$ has positive imaginary part. And U' is unitary if and only if $\alpha(\lambda)\beta^{-1}(\lambda)$ has negative imaginary part. Furthermore, if U satisfies the conditions of Theorem 5.3 and the imaginary part of $\alpha(\lambda)\beta^{-1}(\lambda)$ is positive, then we have the following inversion formula:*

$$U^{-1}f(\lambda) = f(\lambda) + \frac{1}{\pi}\bar{\beta}(\lambda)\int \alpha(\lambda)f(t)\frac{dt}{t-(\lambda - i0)}. \tag{5.23}$$

Now, we construct a unitary operator of the form (5.1) with a given $\alpha(\lambda) \in L_2(E)$. By a unitary transformation–the multiplication of the argument of $\alpha(\lambda) - U$ in (5.1) is unitarily equivalent to a singular integral operator of the same form with coefficients $|\alpha(\lambda)|$. Therefore for the rest of this section we assume that the $\alpha(\lambda)$ in (5.1) is always non-negative.

Suppose $\alpha(\lambda)$ is a L_2-integral function. Let $\delta(\lambda)$ be the unique function satisfying:

$$1 + \frac{1}{\pi}\int |\alpha(\lambda)|^2 \frac{d\lambda}{\lambda - z} = \exp\int \frac{\delta(\lambda)}{\lambda - z}d\lambda. \tag{5.24}$$

Let

$$\Phi(\lambda) = 1 + \frac{1}{\pi}\int |\alpha(\lambda)|^2 \frac{d\lambda}{\lambda - z}. \tag{5.25}$$

Solve for $|\alpha(\lambda)|^2$ by using the Plemelj formula to get

$$|\alpha(\lambda)|^2 = \frac{1}{2i}(\Phi(\lambda + i0) - \Phi(\lambda - i0)). \tag{5.26}$$

On the other hand, $\Phi(z) = e^{2\pi i \Psi(z)}$, for $\Psi(z) = \frac{1}{2\pi i}\int \frac{\delta(\lambda)}{\lambda - z} d\lambda$.

Let

$$Hf(\lambda) = \frac{1}{\pi i}\int \frac{f(t)}{t - \lambda} dt \tag{5.27}$$

be the usual Hilbert transform. Then by apply the Plemelj formula to $\Psi(z)$. We have

$$\Phi(\lambda + i0) = e^{1\pi i \Psi(\lambda + i0)} = e^{\pi i H\delta(\lambda) + \pi i \delta(\lambda)}.$$

Similarly

$$\Phi(\lambda - i0) = e^{\pi i H\delta(\lambda) - \pi i \delta(\lambda)}.$$

Thus

$$|\alpha(\lambda)|^2 = \tfrac{1}{2i} e^{\pi i H\delta(\lambda)}(e^{\pi i \delta(\lambda)} - e^{-\pi i \delta(\lambda)}) = e^{\pi i H\delta(\lambda)} \sin \pi\delta(\lambda).$$

Therefore

$$\alpha(\lambda) = (e^{\pi i H\delta(\lambda)} \sin \pi\delta(\lambda))^{\frac{1}{2}}. \tag{5.28}$$

Now we solve for $\beta(\lambda)$ in $1 - 2i\alpha(\lambda)\beta(\lambda) = e^{-2\pi i\delta(\lambda)}$, to get

$$\beta(\lambda) = (e^{-\pi i H\delta(\lambda)} \sin \pi\delta(\lambda))^{\frac{1}{2}} e^{-\pi i \delta(\lambda)}. \tag{5.29}$$

It is clear that $\alpha(\lambda), \beta(\lambda)$ satisfy the assumption of Theorem 5.3 and $\alpha(\lambda)\beta^{-1}(\lambda) = e^{2\pi i \delta(\lambda)}$ has positive imaginary part. By Theorem 5.4, the singular integral operator U in 5.1 with coefficients $\alpha(\lambda), \beta(\lambda)$ constructed above is unitary. In fact, it is of the following form:

$$Uf(\lambda) = 1 + \frac{1}{\pi}(e^{\pi i H\delta(\lambda)} \sin \pi\delta(\lambda))^{\frac{1}{2}} \int (e^{-\pi i H\delta(t)} \sin \pi\delta(t))^{\frac{1}{2}} e^{-\pi i \delta(t)} \frac{f(t)dt}{t-(\lambda - i0)}. \tag{5.30}$$

In particular, if $\delta(\lambda) = c < 1$ and the measurable set $E = (a, b)$, then (5.30) gives a unitary operator since there is an $\alpha(\lambda) \in L_2(E)$ satisfying (5.28). In the case $c = \frac{1}{2}$, (5.30) becomes

$$Uf(\lambda) = f(\lambda) + \frac{i}{\pi}|\frac{\lambda - b}{\lambda - a}|^{\frac{1}{4}} \int |\frac{t-a}{t-b}|^{\frac{1}{4}} f(t) \frac{dt}{t-(\lambda - i0)}. \tag{5.31}$$

If we rewrite the singular integral above in terms of the principal value by the Plemelj formula, we will get

$$Uf(\lambda) = \frac{1}{\pi i}|\frac{\lambda - b}{\lambda - a}|^{\frac{1}{4}} \int |\frac{t-a}{t-b}|^{\frac{1}{4}} f(t) \frac{dt}{t-\lambda}. \tag{5.32}$$

Let $\psi(\lambda) = |\frac{\lambda-b}{\lambda-a}|^{\frac{1}{4}}$ and M_ψ be the multiplication operator by $\psi(\lambda)$, that is $M_\psi f(\lambda) = \psi(\lambda)f(\lambda)$. Then $U = M_\psi H M_\psi^{-1}$. This says, by an "unbounded similarity" transformation, the unitary operator U in (5.32) is transformed into the Hilbert transform operator on the interval (a, b). One can also easily see that in this case $U' = U$ and this is the only case that both U and U' are unitary operators.

5.2 Contractive Singular Integral Operators

In the last section, the unitary operator U is completely determined by a scalar function $0 \leq \delta(\lambda) \leq 1$. But in order for U to be unitary, $\delta(\lambda)$ has to satisfy (2.6) in chapter 2.

In general, for an arbitrary L_1-function $\delta(\lambda)$ on E, we will have positive measures $\mu^{\pm}$ such that

$$1 + \frac{1}{\pi}\int \frac{d\mu^{\pm}(\lambda)}{\lambda - z} = \exp \pm \int \frac{\delta(\lambda)}{\lambda - z} d\lambda. \tag{5.33}$$

Then by the canonical model construction ([4], [5]), there is a rank one self-adjoint operator perturbation problem $H \to H + \frac{1}{\pi} \otimes d$ with the given $\delta(\lambda)$ as its phase shift and $d\mu^{+}(\lambda) = d(E_\lambda d, d)$. The absolutely continuous part of H is represented as the multiplication by the position function on $L_2(E)$. Let us call the Radon-Nikodym derivative of μ^{+} with respect to Lebesgue measure by $|\alpha(\lambda)|^2$ and let μ_s be the singular part of μ. the wave operator W_- of the perturbation problem $H \to H + \frac{1}{\pi} d \otimes d$ with the given $\delta(\lambda)$ then has the following singular integral representation on $L_2(E)$

$$W_- f(\lambda) = f(\lambda) + \frac{1}{\pi}\bar{\alpha}(\lambda) \int \frac{\alpha(\lambda)}{\det(t + i0)} \frac{f(t)dt}{\lambda - (t - i0)}. \tag{5.34}$$

where $\det(t - i0) = \lim_{\epsilon \to 0} \det(H + \frac{1}{\pi} d \otimes d - (t + i\epsilon))(H - (t - i\epsilon))^{-1}$. Furthermore, the wave operator W_- has the dimensions of $L_2(\mu^-), L_2(\mu^+)$ as its defficiency indices, and $1 - 2i\bar{\alpha}(\lambda)\beta(\lambda) = e^{-2\pi i \delta(\lambda)}$ since the symbols of W_- are 1 and $e^{-2\pi i \delta(\lambda)}$.

Now we rewrite (5.33)

$$1 + \frac{1}{\pi}\int |\alpha(\lambda)|^2 \frac{d\lambda}{\lambda - z} + \frac{1}{\pi}\int \frac{d\mu_s(\lambda)}{\lambda - z} = \exp \int \frac{\delta(\lambda)}{\lambda - z} d\lambda. \tag{5.35}$$

As in the last section, we solve $|\alpha(\lambda)|^2$ from (5.35) by Plemelj formula to get

$$|\alpha(\lambda)|^2 = (e^{\pi i H \delta(\lambda)} \sin \pi \delta(\lambda))^{\frac{1}{2}}.$$

and solve for $\beta(\lambda)$ in $1 - 2i\bar{\alpha}(\lambda)\beta(\lambda) = e^{2\pi i\delta(\lambda)}$ to get:

$$\beta(\lambda) = (e^{-\pi i H\delta(\lambda)} \sin \pi\delta(\lambda))^{\frac{1}{2}} e^{-\pi i\delta(\lambda)}.$$

Therefore by a unitary transform—the multiplication of the argument of $\alpha(\lambda)$—we see that W_- is unitarily equivalent to the following singular integral contraction operator

$$Uf(\lambda) = 1 + \frac{1}{\pi}(e^{\pi i H\delta(\lambda)} \sin \pi\delta(\lambda))^{\frac{1}{2}} \int (e^{-\pi i H\delta(t)} \sin \pi\delta(t))^{\frac{1}{2}} e^{-\pi i\delta(t)} \frac{f(t)dt}{t - (\lambda - i0)}. \tag{5.36}$$

If we assume that the singular parts $\mu^{\pm}$ only consist of finite number of masses, then U has finite indices and by Theorem 1.1, $\sigma_e(U) = S^1$, the unit circle. The index of U equals $\dim(L_2(d\mu^-)) - \dim(L_2(d\mu^+))$. By the geometric characterization of the eigenvalue criterion (lemma 2 in [24]), This number is certain weighting winding number of the images of the symbols $S_{\pm}(H, W_-)$ around the origin on the cylinder $S^1 \times R$ if the symbols are continuous. Of course, here the function $\delta(\lambda)$ is an arbitrary L_1-function and may not be smooth. For example, if $e^{-2\pi i\delta(\lambda)}$ has a jump at some point λ_0, then we fill in the jump by an arc on the cylinder.

The classical singular integral operator theory ([19], [21]) studies singular integral operators of the form:

$$Lf(\lambda) = a(\lambda) + b(\lambda) \int f(t) \frac{dt}{\lambda - (t - i0)}. \tag{5.37}$$

The essential spectrum of such an operator is shown to be the image of the symbols and the index is then the winding number of the images of the quotient of symbols $a(\lambda) \pm b(\lambda)$ around the origin if the symbols are continuous.

If the symbols $a(\lambda) \pm b(\lambda)$ are discontinuous, but only have finite number of jumps, then the index of L is computed by a modified symbols, which is the original image together with segments connecting these jumps.

The class of singular integral operators in (5.1) we study is a different class. Unlike the operator L in (5.37), the symbols in the singular integral operator (5.1) does not determine the operator completely. The unitary property of the operator U gives a symmetric decomposition of the symbols.

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