Toeplitz and Hankel Operators on the Bergman Spaces of Bounded Symmetric Domains and the Bargman-Fock-Segal Spaces, and Some Disk Algebras

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Abstract of dissertation

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In this dissertation we study Toeplitz and Hankel operators on the Bergman space of bounded symmetric domains. We employ the Jordan triple product associated to those domains to obtain a precise formula for the Bergman metric near the boundary of such domains, and we are able to characterize the Hankel operators H_f and $H_{\overline{f}}$ with integrable symbol f which are in p-Schatten class $2 \leq p < \infty$.

On the unit ball of n-dimensional complex space we study properties of semi-commutators of Toeplitz operators and also commuting Toeplitz operators with bounded pluriharmonic symbols on the Bergman space.

On n-dimensional complex space, we use the Berezin transform to define the mean oscillation of square integrable functions. We discuss certain spaces BMO_∞ and VMO_∞. Using BMO_∞, VMO_∞ and an associated mean oscillation, we characterize those Hankel operators H_f and $H_{\bar{f}}$ on the Bargman-Fock-Segal space which are either bounded or compact or belong in to the p-Schatten class for $2 \leq p < \infty$. A conjecture of Berger-Coburn is established.

On the unit disk, the theory of Toeplitz operators and Hankel operators has a deep relationship with function algebras. We study the Bourgain algebras of some subalgebras on the unit disk to shed light on the theory of these algebras on the disk.

GUIHUA and XUAN

Contents

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Introduction

In this chapter we will give some background information and list the major results of this thesis. The thesis consists of four parts.

On Bounded Symmetric Domains

Let $\mathcal{D}$ be a domain in a finite dimensional complex vector space C^n, for dA(w) the usual Euclidean volume measure on $\mathrm{C}^n=\mathrm{R}^{2n}$, normalized so that $\mathrm{A}(\mathcal{D})=1$, we consider the Hilbert space of square-integrable complex-valued functions $L^2 = L^2(\mathcal{D},\mathrm{dA})$ and the Bergman space $L_a^2 = L_a^2(\mathcal{D},\mathrm{dA})$ of holomorphic functions in L^2. Since the evaluation at any fixed point of $\mathcal{D}$ is a bounded functional on L_a^2, there is a function K(z,w), the so-called Bergman reproducing kernel, in L_a^2 such that

$$f(z) = < f, K(z,) > \tag{0.1}$$

for all f in L_a^2.

As is well known, for any orthogonal basis $\{e_n(w)\}$ of the Bergman space, K(z,w) can be represented as

$$K(z,w) = \sum_1^\infty \overline{e_n(z)} e_n(w) \tag{0.2}$$

where the sum converges pointwise to K(z,w). The Bergman kernel K(z,w) is holomorphic on $\mathcal{D} \times \overline{\mathcal{D}}$ and clearly satisfies $\overline{K(z,w)} = K(w,z)$, $K(z,z) \geq 0$. A fundamental property of the Bergman kernel is its transformation rule under the action of automorphisms; namely,

$$J\phi(z)K(\phi(z),\phi(w))\overline{J\phi(w))} = K(z,w) \tag{0.3}$$

for all $\phi \in Aut(\mathcal{D})$ and z, w in $\mathcal{D}$. Here $J\phi(z) = det(\phi'(z))$ is the complex Jacobian of ϕ at z.

Suppose $\mathcal{D}$ is bounded. Then K(z,z)> 0, and the formula

$$h_z(u,v) = \partial_u \partial_{\bar{v}} log K(z,z) \tag{0.4}$$

defines a Kaehler metric on $\mathcal{D}$, the Bergman metric ([Hel]). From (0.3) it follows that h is invariant under the group $Aut(\mathcal{D})$ of biholomorphic automorphisms of $\mathcal{D}$.

A bounded domain $\mathcal{D}$ in C^n is symmetric if every point z in $\mathcal{D}$ is an isolated fixed point of a biholomorphic automorphism ϕ_z of $\mathcal{D}$ of period two (i.e. $\phi_z \circ \phi_z$ =I). Since ϕ_z leaves the Bergman metric invariant, it follows easily that it is the geodesic symmetry around z, and thus $\mathcal{D}$ is a hermitian symmetric space in the sense of E. Cartan. The Bergman metric is then complete since any geodesic may be extended indefinitely by repeated geodesic symmetries. Moreover, $\mathcal{D}$ is homogeneous (Aut($\mathcal{D}$) acts transitively on) as one sees by joining two points by a geodesic and reflecting in the midpoint. Let G be the connected component of the identity in $Aut(\mathcal{D})$. Both $Aut(\mathcal{D})$ and G are semi-simple real lie groups. Let $K = G \bigcap GL(C^n)$ be the subgroup of linear automorphisms in G. By Cartan's linearity theorem, $K = \{\phi \in G; \phi(0) = 0\}$; K is known to be a maximal compact subgroup of G. All other maximal compact subgroups of G are conjugate to K; thus they are the isotropy subgroups of points in $\mathcal{D}$. The evaluation map $G \to \mathcal{D}$ by $\to \phi(0)$ realizes $\mathcal{D}$ as the quotient G/K.

A domain in C^n is called circled (with respect to 0) if $0 \in \mathcal{D}$, and $ze^{it} \in \mathcal{D}$ for all $z \in \mathcal{D}$, $t \in R$. E. Cartan first proved that every bounded symmetric domain in C^n is isomorphic to a bounded symmetric and circled domain which is unique up to a linear isomorphism of C^n. $\mathcal{D}$ is reducible if it is biholomorphically isomorphic to a product of two nontrivial domains. Otherwise $\mathcal{D}$ is irreducible. The irreducible bounded symmetric domains were completely classified up to a biholomorphic isomorphism by E. Cartan.

The following is a list of Cartan domains:

- Type $I_{n,m}(n \leq m) = \{Z \in M_{n\times m}; Z^*Z < I_m\}$;
- Type II_n $(5 \leq n) = \{Z \in M_{n\times n}; Z^* = -Z,\ Z^*Z < I_n\}$;
- Type III_n $(2 \leq n)$ $= \{Z \in M_{n\times n}; Z^* = Z,\ Z^*Z < I_n\}$;
- Type IV_n $(5 \leq n)$ $(the\ Lie\ ball)$ $= \{z \in C^n; ((\sum_{j=1}^n |z_j|^2)^2 - |(\sum_{j=1}^n z_j^2)^2|^2)^{1/2} < 1 - \sum_{j=1}^n |z_j|^2\}$;
- Type V= {1 × 2 matrices z over the 8-dimensional Cayley algebra, with $Z^*Z < I_2\}$;
- Type VI= {3 × 3 Hermitian matrices z with entries in the 8-dimensional Cayley algebra, with $Z^*Z < I_3\}$.

The Cartan domains of type I-IV are called classical, while the domains of types V-VI (of dimension 16 and 27 respectively) are exceptional. An important property of the Cartan domains is that they are convex. Therefore a

Cartan domain is the open unit ball of a certain complex Banach space. So it carries a unique triple product $C^n \times C^n \times C^n \to C^n$ which is induced by the Bergman kernel as follows.

Let $E_1, \cdots, E_n$ be the standard basis of C^n. Define the structure constants $C_{i,j,k,l}$ by

$$C_{i,j,k,l} = \frac{\partial^4 log K(z,z)}{\partial z_i \partial \overline{z}_j \partial z_k \partial \overline{z}_l}|_{z=0} \quad (0.5)$$

and for u,v,w in $V = C^n$ define the triple product $\{u\overline{v}w\}$ by

$$\{u\overline{v}w\} = \sum_{i,j,k,l} C_{i,j,k,l} u_i \overline{v}_j w_k E_l \quad (0.6)$$

Clearly the triple product $\{u\overline{v}w\}$ is C-linear and symmetric in u and w and C-antilinear in v. It turns out that it satisfies the Jordan triple identity

$$\{x\overline{y}\{u\overline{v}w\}\} - \{u\overline{v}\{x\overline{y}w\}\} = \{\{x\overline{y}u\}\overline{v}w\} - \{u\{\overline{y\overline{x}v}\}w\} \quad (0.7)$$

and the positivity condition

$$\{u\overline{u}u\} = \lambda u (\lambda \in C) \Rightarrow \lambda > 0 \quad (0.8)$$

for all 0≠u inV. So (V,{.,.}) is a positive hermitian Jordan triple system.

The automorphism groups of the classical Cartan domains admit realizations as classical groups of matrices, see [Hu], [Kn], and [Py]. In a type $I_{n,m}$, $G = SU(n,m)$ and $K = S(U(n) \times U(m))$. The action of G on $\mathcal{D}$ is by Potapov-Mobius transformations: If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in $SU(n,m)$ (where $a \in M_{n,n}(C)$, $b \in M_{n,m}(C)$, $c \in M_{m,n}(C)$, and $d \in M_{m,m}(C)$) then for z in $\mathcal{D}$

$$Az = (az+b)(cz+d)^{-1}. \tag{0.9}$$

For a type II_n domain, $G = SO(2n)$, and for type III_n, $G = Sp(n,R)$. In both cases $K = \{ \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} ; a \in U(n)\}$ and the action of G is via (0.9).

The Bergman kernel is closely related to the determinant function as the following examples show:

- Type $I_{n,m}$: $K(z,w) = det(I_n I_n - zw^*)^{-(n-m)}$
- Type II_n: $K(z,w) = det(I_n - zw^*)^{-(n-1)}$
- Type III_n: $K(z,w) = det(I_n - zw^*)^{-(n+1)}$
- Type IV_n: $K(z,w) = (1 + \sum_{j=1}^n z_j^2 \sum_{j=1}^n \overline{w}_j^2 - 2\sum_{j=1}^n z_j \overline{w}_j)^{-n}$.

In the matrix factors of type I-III the triple product is given by

$$\{abc\} = (ab^*c + cb^*a)/2. \tag{0.10}$$

In the Cartan factor of type IV, (the spin factor), the triple product is given by

$$\{abc\} = (a,b)c + (c,b)a - (a,\overline{c})\overline{b}. \tag{0.11}$$

Let us turn to operator theory on the Bergman space of bounded symmetric domains.

Let P be the self-adjoint projection from L^2 onto L^2_a. For f and g in L^2, we consider the multiplication operator M_f on L^2 given by

$$M_f g = fg \tag{0.12}$$

and the Hankel operator H_f on L^2_a given by

$$H_f = (I - P)M_f P \tag{0.13}$$

and the Toeplitz operator T_f on L^2_a given by

$$T_f = PM_f P. \tag{0.14}$$

The commutator $[M_f, P] = M_f P - PM_f$ is densely defined on L^2 and it is easy to check that

$$[M_f, P] = H_f \bigoplus(-H^*_{\overline{f}}). \tag{0.15}$$

So studying the properties of $[M_f, P]$ is equivalent to studying the properties of both H_f and $H_{\overline{f}}$

An operator T on Hilbert space H is said to be in the Schatten p-class if T^*T is compact

$$\sum_{i=1}^{\infty} s_i^p < \infty \tag{0.16}$$

where $(T^*T)^{1/2} = \sum_i^\infty s_i e_i \otimes e_i$ if $\{e_i\}$ are an orthogonal basis of H. We use S_p to denote the set of all operators in Schatten p-class for $p > 0$.

Toeplitz operators on the unit disk have long been studied because of their importance in the theory of integral equations (Wiener-Hopf equations) and algebraic topology (index theory [D]). More recently, Toeplitz operators were also considered over bounded symmetric domains ([U][BBCZ]). The structure of Toeplitz operators and Toeplitz C*-algebras over symmetric domains is closely related to the Jordan algebraic structure underlying these domains ([U]).

In the first chapter we are going to study the behavior of the Bergman metric near the boundary of $\mathcal{D}$ and to get a precise formula for $h_{tz}(z,z)$ in terms of tripotent frame as follows.

Theorem 1.4. Suppose that $\mathcal{D}$ is a bounded symmetric domain with rank r in C^n. If we associate to $\mathcal{D}$ a Jordan pair as above. For z in $\mathcal{D} \subset C^n$, and t in [0,1] we have

$$h_{tz}(z,z) = g/2 \sum_{i=1}^{r} \frac{|\lambda_i|^2}{(1-t^2|\lambda_i|^2)^2}$$

if z $=\sum_{i=1}^{r} \lambda_i e_i$ and e_i is primitive. So

$$h_{tz}(z,z) \le gr/2 \frac{\|z\|^2}{(1-t^2\|z\|^2)^2}.$$

Using the formula we characterize those functions f such that both H_f and $H_{\overline{f}}$ are in S_p for $2 \le p < +\infty$. We obtain in this way a proof of a conjecture of K. Zhu ([Zh1]). In order to state our result more precisely we define the Berezin transform of f in L^2 by

$$\tilde{f}(z) = < fk_z, k_z > \tag{0.17}$$

where k_z is the normalized Bergman kernel $K(z,z)^{-1}K(z,)$. It follows from known properties of k_z that $\tilde{f}$ is defined and smooth (C^∞) everywhere on $\mathcal{D}$. Using the boundedness of the k_z, the Berezin transform extends to all f in L^1 by the formula

$$\tilde{f}(z) = \int_{\mathcal{D}} f(w)|k_z(w)|^2 dA(w). \tag{0.18}$$

For f in L^2, we define

$$MO(f)(z) = [\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2]^{1/2}. \tag{0.19}$$

Roughly speaking, MO(f)(z) is the mean oscillation under the Bergman metric on $\mathcal{D}$. It is easy to check that the Berezin transform and MO(f) commutes with the G-action L^1 in the sense that

$$\widetilde{f \circ g}(z) = \tilde{f}(gz). \tag{0.20}$$

Our result is stated precisely as the following theorem.

Theorem 1.17. Suppose $2 \leq p < +\infty$. For f in L^2, then both H_f and $H_{\bar{f}}$ are in S_p if and only if

$$\int_{\mathcal{D}} MO(f)^p(z) d\mu(z) < +\infty$$

where $d\mu(z)$ is the volume in the Bergman metric, $K(z,z)dA(z)$.

On the Unit Ball B_n

Let us focus on bounded symmetric domains of rank 1. It is well-known that these are isomorphic to the unit ball B_n of $(C^n, \| \ \|_2)$. So we consider operator theory and function theory on the unit ball. The boundary of the unit ball B_n is the unit sphere S_n.

Let $H^\infty(S_n)$ denote the subalgebra of $L^\infty(S_n)$ which contains the holomorphic functions in B_n. For $p \geq 1$, $H^p(S_n)$ is the Banach space of holomorphic functions in B_n with norm defined by

$$\|f\|_p = Sup\{|\int_{S_n} |f(rz)|^p d\sigma(z)|^{1/p}; 0 < r < 1\}. \tag{0.21}$$

In fact $H^2(S_n)$ is a subspace of the Hilbert space $L^2(S_n, d\sigma)$ which is called the Hardy space.

Let $\mathcal{M}$ be the maximal ideal space of $H^\infty(S_n)$. This is defined to be the set of multiplicative linear maps from $H^\infty(S_n)$ onto the field of complex numbers. Each multiplicative linear functional $\phi \in \mathcal{M}$ has norm 1 (as an element of the dual of $H^\infty(S_n)$). If we think of $\mathcal{M}$ as a subset of the dual space $H^\infty(S_n)$ with weak-star topology then $\mathcal{M}$ becomes a compact Hausdoff space. For $z \in B_n$ the evaluation functional

$$f \rightarrow f(z) \tag{0.22}$$

is a multiplicative functional. So we can think of B_n as a subset of $\mathcal{M}$.

For m, $\tau \in \mathcal{M}$ the pseudohyperbolic distance between m and τ, denoted by $\rho(m, \tau)$, is defined by

$$\rho(m, \tau) = Sup\{|m(f)| : f \in H^\infty(B_n), \|f\| < 1, \tau(f) = 0\}. \tag{0.23}$$

For m in $\mathcal{M}$, the Gleason part P(m) of m is defined by

$$P(m) = \{\tau \in \mathcal{M} | \rho(m, \tau) < 1\} \tag{0.24}$$

For z in B_n, we can think of the Mobius transformation ϕ_z as a map from B_n to $\mathcal{M}$ since B_n is a subset of $\mathcal{M}$. Because $\mathcal{M}$ is compact, Tychonoff 's theorem in topology tells us that for any net of maps $\{\phi_{z_\alpha}\}$ there is a subnet $\{\phi_{z_\beta}\}$ such that ϕ_{z_β} converges to some ϕ which is a map from B_n to $\mathcal{M}$. Let Φ denote the set of limits of $\{\phi_z\}_{z \in B_n}$ except $\{\phi_z\}_{z \in B_n}$. We define the ϕ-part by

$$G(\phi) = \phi(B_n) \tag{0.25}$$

for $\phi \in \Phi$.

On the unit disk Hoffman has shown that ϕ has many remarkable properties. For example, every Gleason part is a ϕ-part, and ϕ is either constant or injective. In the latter case G(ϕ) is called an analytic disk. However on the unit ball for $n > 1$, it is not known what Gleason parts look like since the Corona problem is unsolved.

A C^2- function f: $B_n \to C$ is said to be pluriharmonic if for every complex line $l = \{a + bz\}$ the function $z \to f(a + bz)$ is harmonic on the set $(B_n)_l = \{z \in C : a + bz \in B_n\}$; f is said to $\mathcal{M}$-harmonic if $\tilde{\Delta} f = 0$; f is said to be harmonic if $\Delta f = 0$ where Δ is the usual Laplacian operator

$$\Delta = 4 \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z_i}} \tag{0.26}$$

and $\tilde{\Delta}$ is the Laplace-Beltrami operator for the Bergman metric of B_n, which is defined by

$$\tilde{\Delta} = \sum g^{ij}(z) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \tag{0.27}$$

in terms of the metric tensor on B_n. Explicitly,

$$\tilde{\Delta} = c_n(1 - \|z\|^2)[\sum_k \frac{\partial^2}{\partial z_k \partial \overline{z}_k} - \sum_{ij} z_i \overline{z}_j \frac{\partial^2}{\partial z_i \partial \overline{z}_j}] \tag{0.28}$$

for some constant c_n which depends on n.

For any f in $C^2(B_n)$, the gradient of f is given by

$$\nabla_z(f)(w) = (\frac{\partial f}{\partial z_1}(w), \cdots, \frac{\partial f}{\partial z_n}(w)). \tag{0.29}$$

Let f: $B_n \to C$ be holomorphic function on B_n. As in [T1], for z in B_n, $Q_f(z)$ is defined by

$$Q_f(z) = Sup\{|(\nabla_z f, x)| / H_z(x,x)^{1/2} \mid 0 \neq x \in C^n\}. \tag{0.30}$$

In the second chapter of this thesis we take up the subject of semicommutators of Toeplitz operators and commuting Toeplitz operators with pluriharmonic symbols over the unit ball B_n.

As is well known for f and g in $L^\infty(S_1)$, Axler, Chang and Sarason [ACS] and Volberg [V] have shown that $T_{\bar{f}}T_g - T_{\bar{f}g}$ on the Hardy space $H^2(S_1)$ is compact if and only if either $f|_S \in H^\infty(S_1)|_S$ or $g|_S \in H^\infty(S_1)|_S$ for each support set S. In the case every function in $L^\infty(S_1)$ extends as a bounded harmonic function on the unit disk B_1 via the Possion integral formula. Axler, Gorkin [AG1] and the author [Z1] have shown that for the bounded harmonic functions f and g , $T_{\bar{f}}T_g - T_{\bar{f}g}$ on the Bergman space $L^2_a(B_1)$ is compact if and only if $f|_{G(m)} \in H^\infty(S_1)|_{G(m)}$ or $g|_{G(m)} \in H^\infty(S_1)|_{G(m)}$ for each Gleason part G(m).

On the Hardy space of the unit circle A. Brown and P. Halmos ([BH]) characterize commuting Toeplitz operators with symbols in $L^\infty(S_1)$ by examining the matrix products of Toeplitz operators on the Hardy space. On the Bergman space $L^2_a(B_1)$ of the unit disk, Toeplitz operators do not have nice matrices. However S. Axler and P. Gorkin [AG1] used the theory of function algebra to get some partial results on commuting Toeplitz operators on the Bergman space of the unit disk. The author [Z1] also got some partial results on the problem by means of function theory on the unit disk. Recently, S. Axler and Z. Cuckovic [AG1] completely characterized commuting Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk to make use of the mean value property of harmonic functions.

We will show that for the bounded pluriharmonic functions f and g, $T_{\bar{f}}T_g -$

$T_{\overline{f}g}$ on the Bergman space $L_a^2(B_n)$ if and only if $f|_{G(\phi)} \in H^\infty(S_n)|_{G(\phi)}$ or $g|_{G(\phi)} \in H^\infty(S_n)|_{G(\phi)}$ for each ϕ-part. In the case of the unit disk the Gleason part and the ϕ-part are the same. Also by means of the characterization of $\mathcal{M}$-harmonic function $\overline{f_1}g_1$ for holomorphic functions f_1 and g_1 we prove that for the bounded pluriharmonic functions f and g, $T_{\overline{f}}T_g = T_{\overline{f}g}$ on the Bergman space $L_a^2(B_n)$ or the Hardy space $H^2(S_n)$ if and only if either f or g is holomorphic in B_n.

Our main results in the second chapter are stated as follows.

Theorem 2.20. Suppose that ϕ and ψ are bounded pluriharmonic functions on the unit ball. Then

$$T_\phi T_\psi = T_\psi T_\phi$$

on the Bergman space $L_a^2(B_n)$ of the unit ball if and only if ϕ and ψ satisfy one of the following conditions:

(1) Both ϕ and ψ are holomorphic on the unit ball B_n;

(2) Both $\overline{\phi}$ and $\overline{\psi}$ are holomorphic on the unit ball B_n;

(3) Either ϕ or ψ is constant;

(4) There is a nonzero constant b such that ϕ -bψ is constant.

It is important to note that Theorem 2.20 does not hold if "pluriharmonic" is replaced by " measurable " or even "continuous". For example, it is easy to extend P. Bourdan's example in [AG1] on the unit ball of C^n. i.e. If ϕ and ψ are bounded measurable on the unit ball B_n and invariant under the action of unitary matrices, then $T_\phi T_\psi = T_\psi T_\phi$.

Theorem 2.22. Let f and g be two bounded pluriharmonic functions on

B_n. Then $T_{\overline{f}}T_g = T_{\overline{f}g}$ on the Bergman space if and only if f or g is holomorphic on B_n.

Theorem 2.24. Let f and g be two bounded pluriharmonic functions on B_n. Then the following are equivalent:

(1) $T_{\overline{f}}T_g - T_{\overline{f}g}$ is compact;

(2) $H_{\overline{f}}^{*}H_g$ is compact;

(3)

$$\lim_{\|z\|\to 1} min\{max_{E(z,r)}(1-\|s\|^2)\|\nabla_z\overline{f}(s)\|, max_{E(z,r)}(1-\|t\|^2)\|\nabla_z\overline{g}(t)\|\} = 0$$

for all fixed r, $0 < r < 1$;

(4)

$$\lim_{\|z\|\to 1} min\{\|f\circ\phi_z - P(f\circ\phi_z)\|_2, \|g\circ\phi_z - P(g\circ\phi_z)\|_2\} = 0;$$

(5)

$$\lim_{\|z\|\to 1} min\{Q_{P\overline{f}}(z), Q_{P\overline{g}}(z)\} = 0;$$

(6) Either foϕ or goϕ is holomorphic for every $\phi \in \Phi$ where Φ is the set of ϕ-parts.

Since the Hardy space $H^2(S_n)$ is a subspace of $L^2(S_n)$, there is an orthogonal projection P from $H^2(S_n)$ onto $L^2(S_n)$, the so-called Szego projection. As on the Bergman space we can define Toeplitz operators and Hankel operators on the Hardy spaces.

Theorem 2.25. Let f and g be two bounded pluriharmonic functions on B_n. Then $T_{\overline{f}}T_g = T_{\overline{f}g}$ on the Hardy space if and only if f or g is holomorphic on B_n.

Theorem 2.26. Suppose that ϕ and ψ are bounded pluriharmonic functions on the unit ball. Then

$$T_\phi T_\psi = T_\psi T_\phi$$

on the Hardy space of the unit sphere if and only if ϕ and ψ satisfy one of the following conditions:

(1) Both ϕ and ψ are holomorphic on the unit ball B_n;

(2) Both $\overline{\phi}$ and $\overline{\psi}$ are holomorphic on the unit ball B_n;

(3) Either ϕ or ψ is constant;

(4) There is a nonzero constant b such that ϕ -bψ is constant.

On the Bargman-Fock-Segal Space

Let C^n be the n-dimensional complex space, $d\mu(z)=e^{-\|z\|^2/2}dV(z)/(2\pi)^n$, the Gaussian measure where dV(z) is the Lebesgue measure on C^n. The Bargman-Fock-Segal space $H^2(C^n, d\mu)$ is the space of Gaussian square-integrable entire functions on C^n. Clearly $H^2(C^n, d\mu)$ is a closed subspace of $L^2(C^n, d\mu)$ with Bergman reproducing kernel functions

$$K(z, w) = e^{<z,w>/2} \tag{0.31}$$

and orthogonal projection from $L^2(C^n, d\mu)$ onto $H^2(C^n, d\mu)$ is given by

$$Pf(z) = \int_{C^n} f(w)K(z, w)d\mu(w). \tag{0.32}$$

For g so that gK(.,a) is in $L^2(C^n, d\mu)$ for all a in C^n we can consider the Toeplitz operator T_g with symbol g on $H^2(C^n, d\mu)$

$$T_g f(z) = \int_{\mathcal{D}} g(w)K(z, w)f(w)d\mu(u), \tag{0.33}$$

and the Hankel operator H_g with symbol g is defined by

$$H_g f = \int_{\mathcal{D}} (g(z) - g(w)) K(z,w) f(w) d\mu(u). \qquad (0.34)$$

Generally speaking, both Toeplitz operator and Hankel operator may be unbounded. Our starting point is the observation that there is a natural isometry B from $L^2(R^n, dV)$ onto $H^2(C^n, d\mu)$, called the Bargman transform under which

$$B[M_{x_j} - i\frac{\partial}{\partial x_j}]B^{-1} = T_{z_j} \qquad (0.35)$$

$$B[M_{x_j} + i\frac{\partial}{\partial x_j}]B^{-1} = T_{\overline{z}_j} \qquad (0.36)$$

The Bargman transform is represented as

$$Bf(z) = \int f(x) B(z,x) dx \qquad (0.37)$$

where B(z, x) is the Bargman kernel. It is easy to check that

$$T_{\overline{z}_j} = 2\frac{\partial}{\partial z_j}. \qquad (0.38)$$

The complex representation of the Hamiltonian of the harmonic oscillator on $L^2(R^n, dV)$ is

$$B\frac{1}{2}\{-\Delta + M_{|x|^2} - nI\}B^{-1} = 1/2 \sum_{j=1}^{n} T_{z_j} T_{\overline{z}_j}. \qquad (0.39)$$

The commutator of the unbounded operator T_z

$$[T_{z_j}, T_{\overline{z}_j}] = H^*_{\overline{z}_j} H_{\overline{z}_j}. \qquad (0.40)$$

can be extended a bounded operator even though T_{z_j} is not bounded since it is easy to check that $H_{\overline{z}_j}$ are bounded operator with unbounded symbol $\overline{z}_j$ on C^n.

The rigorous development of the representation of the Heisenberg group on $H^2(C^n, d\mu)$ and the intertwining operator B is due to Bargman [Bar]; the same ideas also appear in work of Segal [Se], independently at about the same time.

The map $g \to T_g$ is a natural "quantization" and has been studied by Berezin, Berger-Coburn, Guilleman, Howe, Shubin and Folland [Be1], [BC1,2], [Gu], [How], [Sh], and [F].

On the other hand, as observed by Berezin and Guilleman [Ber2], [Gu], there is a natural identification

$$BW_{f_{\frac{1}{4}}}B^{-1} = T_f \tag{0.41}$$

between the Berezin quantization T_f and the Weyl (pseudo-differential) quantization $W_{f_{\frac{1}{4}}}$ where

$$f_t(a) = (4\pi t)^{-n} \int_{C^n} f(z) e^{-\frac{|z-a|^2}{4t}} dV(z) \tag{0.42}$$

is the heat semigroup and W_F is a pseudo-differential operator with symbol $F(z) = F(x + i\xi)$ given by

$$W_F h(x) = \int\int F(i\xi + \frac{1}{2}(x+y)) e^{2\pi i(x-y)\xi} h(y) dy d\xi, \tag{0.43}$$

for h in $L^2(R^n)$.

The problem of deciding if $W_{FG} - W_F W_G$ (F, G smooth) is compact was studied by Hormander [Hor]. But the Weyl and Toeplitz symbol calculus problems are not simply equivalent because $(fg)_t \neq f_t g_t$. Berger and Coburn studied the compactness problem of both $T_{|f|^2} - T_f T_{\overline{f}}$ and $T_{|f|^2} - T_{\overline{f}} T_f$ for f in $L^\infty(C^n, dV)$.

We will study when $T_{|f|^2} - T_f T_{\overline{f}}$ and $T_{|f|^2} - T_{\overline{f}} T_f$ are bounded or compact or in p-Schatten class for p≥ 1 in the third chapter.

Recall that

$$k_a(z) = e^{\overline{a}z/2 - |a|^2/4} \tag{0.44}$$

is the normalized reproducing kernel for the functional of "evaluation at a" on $H^2(C^n, d\mu)$. For f in $L^1(C^n, d\mu)$ such that $f|k_a|^2$ is in $L^1(C^n, d\mu)$ for all a in C^n, we define the Berezin transform

$$\tilde{f}(a) = \int_{C^n} f(z)|k_a(z)|^2 d\mu(z) = f_{1/2}(a). \tag{0.45}$$

The Berezin transform yields a measure of mean oscillation,

$$MO(f)(a) = \{\widetilde{|f|^2}(a) - |\tilde{f}(a)|^2\}^{1/2}. \tag{0.46}$$

We say that a function f is in BMO_∞ if

$$\sup_z MO(f)(a) < \infty \tag{0.47}$$

and f is in VMO_∞ if f is in BMO_∞ and

$$\lim_{a\to\infty} MO(f)(a) = 0. \tag{0.48}$$

We are able to prove the following:

Theorem 3.1. Suppose that f(z) and f(z+w) are in $L^2(C^n, d\mu)$ for every w in C^n. Then both H_f and $H_{\overline{f}}$ are bounded if and only if

$$Sup_{z\in C^n} MO(f)(z) < +\infty,$$

namely , f is in BMO_∞

Theorem 3.14. Suppose that f is in BMO_∞. Both $T_{|f|^2} - T_f T_{\overline{f}}$ and $T_{|f|^2} - T_{\overline{f}} T_f$ are compact if and only if $\lim_{z\to\infty} MO(f)(z) = 0$, i.e. f is in VMO_∞.

Now applying Theorems 3.1 and 3.14 to Hankel operators with symbols in the conjugate of entire functions we have

Theorem 3.15. Suppose that for any fixed w in C^n , f(z) and f(z+w) are in $H^2(C^n, d\mu)$. Then

1). $H_{\overline{f}}$ is bounded if only if f is an affine function, i.e. there are a constant vector A and a constant B such that f(z)=(z,A)+B.

2). $H_{\overline{f}}$ is compact if and only if f is constant.

Theorem 3.21. Suppose that f is in BMO_∞. For $p \geq 1$, both $T_{|f|^2} - T_f T_{\overline{f}}$ and $T_{|f|^2} - T_{\overline{f}} T_f$ are in p-Schatten class if and only if

$$\int_{C^n} MO(f)^{2p}(z) dV(z) < \infty.$$

Some Function Algebras on the Unit Disk

On the unit disk the theory of Toeplitz operators and Hankel operators has a deep relation with function algebras. In the fourth chapter we study some function algebras on the unit disk.

Let A be a Banach algebra and B be a linear subspace of A. Recall that A has the Dunford-Pettis property if whenever $f_n \to 0$ weakly in A and $x_n \to 0$ weakly in A^* then $x_n(f_n) \to 0$. Bourgain [Bou] showed that H^∞ has the Dunford-Pettis property using the theory of ultraproducts. The Dunford-Pettis Property is related to the notion of Bourgain algebra. In [CT], Cima and Timoney introduced the concept of Bourgain algebra B_b of a linear space

B of a Banach algebra A, which is the set of f in A such that if $f_n \to 0$ weakly in B, then $dist_A(ff_n, B) \to 0$, and they showed that if B is an algebra then $B \subset B_b$. In [CJY], Cima, Janson and Yale described the Bourgain algebra of $H^\infty(\partial D)$ in $L^\infty(\partial D)$ using Fefferman's duality theorem. Gorkin, Izuchi and Morntini [GIM] studied the Bourgain algebras of Douglas algebras in $L^\infty(\partial D)$. There has been further study of Bourgain algebras on the bi-torus and the polydisk [Y].

Let $H^\infty(D)$ be the algebra of bounded analytic functions on the unit disk D. Let $\mathcal{M}$ denote the maximal ideal space of $H^\infty(D)$. Then $H^\infty(D)$ can be thought of as a subalgebra of the algebra C($\mathcal{M}$) of continuous functions on $\mathcal{M}$. Sometimes we use H^∞ to denote $H^\infty(D)$ for simplicity. In [GSZ] we described the Bourgain algebras of $H^\infty(D)$ and $H^\infty(D) + UC(D)$ in C($\mathcal{M}$). In the fourth chapter we study the Bourgain algebras on the disk in order to shed light on the [illegible] algebras on the disk. So far on the unit disk there is no any analogy to Chang-Marshall's Theorem on the circle [Ch], [Mar] , it is more difficult to study the algebras on the disk than to study the Douglas algebras on the circle. However there are some interesting results on some algebras on the disk [AG1,2]. In fact there are very important relationships between some algebras on the disk and operator theory on the Bergman space [AG1,Z1, Z2].

As defined in [AG1], AOP (which stands for "analytic on parts") is the closed algebra defined by

$$AOP = \{u \in C(\mathcal{M}) : u \circ L_\varphi \in H^\infty,\ for\ every\ \varphi \in \mathcal{M} - D\}. \qquad (0.49)$$

COP (which stands for "constant on parts") is the algebra defined by

$$COP = \{u \in C(\mathcal{M}) : u \textit{ is constant on } P(\varphi) \textit{ for } \varphi \in \mathcal{M} - D\}. \qquad (0.50)$$

Now we define another algebra HCOP which is given by

$$HCOP = \{u \in C(\mathcal{M}) : u \circ L_\varphi \in H^\infty + UC(D)\} \qquad (0.51)$$

for all thin parts m in $\mathcal{M}$.

We [Z2] proved that for f in C($\mathcal{M}$), H_f is compact iff f is in AOP, and both H_f and $H_{\overline{f}}$ are compact iff f is in COP.

Theorem 4.5. Suppose that f is in H^∞. If its complex conjugate $\overline{f}$ is in either $(AOP)_b$ or $(H^\infty + COP)_b$, then

$$\lim_{z \to \partial D}(1 - |z|^2)f'(z) = 0.$$

Theorem 4.9. $(H^\infty + COP)_b$ and $(AOP)_b$ are proper subset of HCOP.

Theorem 4.10. $(HCOP)_b = HCOP.$

Theorem 4.15. Suppose that A is a subalgebra of C(M) and contains H^∞. If its Bourgain algebra A_b is C(M), then $A|_X$ must be $L^\infty(\partial D)$.

Chapter 1

Operators on Bounded Symmetric Domains

Let $\mathcal{D}$ be a bounded symmetric (Cartan) domain of rank r with its standard (Harish-Chandra) realization in C^n ([Hel],[Hua],[Sa]). We may assume that $\mathcal{D}$ is circled, irreducible and contains the origin $0 \in C^n$. Let G denote the connected component of the biholomorphic automorphism group of $\mathcal{D}$ and K its isotropic group at 0 in G. Then $\mathcal{D}$ = G/K. For a suitable subspace C^r of C^n, as in [Kob] $\mathcal{D} \bigcap C^r$ is a polydisk D^r and $\mathcal{D}$ can be written as a union of polydisks KD^r, where K is considered as a subgroup of U(n) of C^n. So we have

$$D^r \hookrightarrow \mathcal{D} \hookrightarrow D^n \tag{1.1}$$

where i:$D^r \hookrightarrow \mathcal{D}$ is a holomorphic embedding.

We recall that the Bergman metric $h_z(u,v)$ for z in $\mathcal{D}$ and u, v in C^n is defined by

$$h_z(u,v) = \sum_{i,j} \frac{\partial}{\partial z_i}\frac{\partial}{\partial \overline{z_j}} log K(z,z) u_i \overline{v}_j \tag{1.2}$$

Then $\mathcal{D}$ is a complete Hermitian symmetric space of noncompact type with the Bergman metric which gives the usual topology on $\mathcal{D}$. By definition, the

Bergman distance β(z,w) is given by

$$\beta(z,w) = inf_\gamma \int_0^1 \sqrt{h_{\gamma(t)}(\gamma'(t),\gamma'(t))}dt \tag{1.3}$$

where the inf is taken over all geodesics in $\mathcal{D}$ which connect z and w.

For a bounded symmetric domain, for fixed w in $\mathcal{D}$, as z goes to the topological boundary,

$$K(z,z) \to +\infty \tag{1.4}$$

and

$$\beta(z,w) \to +\infty. \tag{1.5}$$

Moreover K(z,w) and β(z,w) have the following invariance properties

$$K(ka,kb) = K(a,b) \tag{1.6}$$

for all k in K and

$$\beta(ga,gb) = \beta(a,b) \tag{1.7}$$

for all g in G. For each a in $\mathcal{D}$, there is a biholomorphic automorphism ϕ_a of $\mathcal{D}$ (ϕ_a in G') with the properties

$$(1)\ \phi_a(a) = 0 \tag{1.8}$$

$$(2)\ \phi_a \circ \phi_a = Id.$$

ϕ_a is determined uniquely up to composition with an element of K.

In this chapter we study the Bergman metric and characterize those functions f such that both H_f and $H_{\overline{f}}$ are in S_p for $2 \leq p < +\infty$. We prove a conjecture of K.Zhu, established by him for the special case of unit ball in [Zh1]. It was shown in [AFP] that for holomorphic function f on the unit disk

D, $H_{\overline{f}}$ is in S_p for $1 < p < \infty$ if and only if f is in the Besov space B_p. We will present two proofs. One proof uses the Jordan theoretic characterization of bounded symmetric domains. i.e., every bounded symmetric domain $\mathcal{D}$ can be realized as the open unit ball of a uniquely determined Jordan triple system $V \approx C^n$ for the so called spectral norm ([L1],[Sa]) . The second proof will reduce the problem to the polydisk case using the Schwarz Lemma. In this way, inspired by Zhu's method in [Zh1], we can completely prove his conjecture on bounded symmetric domains.

1.1 Jordan triple systems and bounded symmetric domains

In order to get a formula for the Bergman metric on bounded symmetric domain we need some more Jordan theoretic tools. The material presented here is studied in detail in [L1,2] and [U].

As defined in Chapter 0 we can associated a Jordan triple system with a bounded symmetric domain on $V = C^n$. Then the Jordan triple system satisfies the following properties.

- $\{uvw\}$ is bilinear and symmetric in u, w and conjugate linear in v.
- for all x,y, u,v,w$\in V$ the Jordan triple identity holds:

$$\{\{uvw\}yx\} + \{\{uvx\}yw\} - \{uv\{wyx\}\} = \{w\{vuy\}x\}. \qquad (1.9)$$

We define operators $D(x,\overline{y})$ and Q(x,z) by

$$D(x,\overline{y})z = Q(x,z)y = \{x\overline{y}z\}. \qquad (1.10)$$

It follows from (1) that Q(x,z) is bilinear and symmetric in x and z, and D(x,$\overline{y}$) is the endomorphism z $\rightarrow$ $\{x\overline{y}z\}$ of V. Define

$$(z) = \frac{1}{2}Q(z,z), \tag{1.11}$$

operator $D(v)$ by

$$D(v) = D(v,v), \tag{1.12}$$

and the Bergman operator or the operator triple norm of V

$$B(x,y) = Id - D(x,\overline{y}) + Q(x)\overline{Q(y)} \tag{1.13}$$

for x,y, z in V.

Then it is obvious that D(x,$\overline{y}$),Q(x,z) and the Jordan triple system completely determine each other. The following theorem gives the relations between the Bergman metric, Bergman kernel and Jordan triple system, this is very useful in computing the Bergman metric.

Theorem 1.1. (a) The Bergman kernel function of $\mathcal{D}$ is

$$K(x,y) = detB(x,y)^{-1}; \tag{1.14}$$

(b) The Bergman metric at 0 is

$$h_0(u,v) = traceD(u,\overline{v}) \tag{1.15}$$

and at an arbitrary point z $\in \mathcal{D}$

$$h_z(u,v) = h_0(B(z,z)^{-1}u,v); \tag{1.16}$$

(c) The curvature tensor of the Bergman metric at 0 is

$$R_0(u,v)w = -\{u\overline{v}w\} + \{v\overline{u}w\}; \tag{1.17}$$

(d) K acts by automorphisms of the Jordan structure. Namely, for x,y $\in \mathcal{D}$ and k $\in \mathcal{D}$

$$k(Q(x)\overline{y}) = Q(kx)\overline{ky} \tag{1.18}$$

(e) For x,y $\in \mathcal{D}$, g $\in$ G, we have

$$B(gx, gy) = dg(x)B(x,y)dg(y)^* \tag{1.19}$$

An element v in V is tripotent if $\{vvv\} = v$. In the Cartan factors of types I-III the tripotents are simply the partial isometries. Two tripotents v, u are orthogonal if $D(v,v) = 0$. It follows easily from the Jordan triple identity that orthogonality is a symmetric relation. If v is a tripotent then by the Jordan triple identity, the operator $D(v)$ satisfies

$$D(v)(2D(v) - I)(D(v) - I) = 0. \tag{1.20}$$

It follows that the spectrum of D(v) is contained in $\{0, 1/2, 1\}$ and V admits a dirct sum decomposition

$$V = V_2(v) \oplus V_1(v) \oplus V_0(v) \tag{1.21}$$

called the Peirce decomposition associated with v, where $V_j(v)$ is the eigen space of D(v) corresponding to the eigenvalue j/2, $j = 0, 1, 2$. So every x in V has a "spectral decomposition"

$$x = \lambda_1 e_1 + \cdots + \lambda_m e_m \tag{1.22}$$

where the e_i are orthogonal tripotents and the eigenvalues λ_i are positive. Moreover, the spectral norm $\|x\| = max|\lambda_i|$ is a norm on V.

Theorem 1.2 (Peirce decomposition). Let (V,$\overline{V}$) be a Jordan pair with hermitain involution τ, and let e be a tripotent of V.

(a) V decomposes

$$V = V_2 \bigoplus V_1 \bigoplus V_0$$

where V_α is the α-eigenspace of $D(e, \overline{e})$. In fact the V_α are orthogonal with respect to any associative scalar product and satisfy the multiplication rules

$$\{V_\alpha, V_\beta, V_\gamma\} \subset V_{\alpha-\beta+\gamma} \tag{1.23}$$

$$\{V_2, V_0, V\} = \{V_0, V_2, V\} = 0. \tag{1.24}$$

(b) let $e_1, \cdots, e_n$ be orthogonal tripotents of V. Then

$$V = \sum_{0 \leq i \leq j \leq n} V_{ij} \tag{1.25}$$

(the direct sum of subspaces) where

$$V_{ii} = V_2(e_i)$$

$$V_{ij} = V_{ji} = V_1(e_i) \bigcap V_1(e_j)$$

$$V_{i0} = V_{0i} = V_1(e_i) \bigcap (\textstyle\bigcap_{j \neq i} V_0(e_j))$$

$$V_{00} = V_0(e_1) \bigcap \cdots \bigcap V_0(e_n).$$

(c) Let x=$\lambda_1 e_1 + \cdots + \lambda_n e_n$ where $\lambda_i \in C$, set $\lambda_0 = 0$. Let $y_{ij} \in V_{ij}$. Then

$$D(x, \overline{x}) y_{ij} = (|\lambda_i^2| + |\lambda_j^2|) y_{ij} \tag{1.26}$$

$$B(x, x) y_{ij} = (1 - |\lambda_i^2|)(1 - |\lambda_j^2|) y_{ij}. \tag{1.27}$$

A tripotent is called primitive if it cannot be written as a sum of orthogonal tripotents in a non-trivial way. A tripotent v in V is minimal if $V_2(v) = Cv$. Since V is finite dimensional it contains minimal tripotents. A maximal family of orthogonal minimal tripotents is called a frame. It is known that if $\{e_j\}_{j=1}^r$ and $\{v_j\}_{j=1}^s$ are two frames then $r = s$ and there is k in K so that $ke_j = v_j$ for all $1 \leq j \leq r$. In fact the rank of the bounded symmetric domain is r. The following theorem tells us that a tripotent is primitive iff a tripotent v is minimal.

Theorem 1.3. (a) A tripotent e is primitive if and only if $V_2(e) = Re \oplus Re$.

(b) If $r(\mathcal{D}) = r$, there is an orthogonal system of tripotents $\{e_1, \cdots, e_r\}$ such that e_i are primitive and $Re_1 + \cdots + Re_r$ is a maximal flat subspace of V. Moreover every z in $\mathcal{D}$ can be represented as

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r$$

where $\{e_1, \cdots, e_r\}$ is a orthogonal system of tripotents and λ_i are real numbers.

Fix a frame $\{e_j\}_{j=1}^r$ and let $e = \sum_{j=1}^r e_j$. Clearly, e is a maximal tripotent. Let $V = \sum_{0 \leq i \leq j \leq r} \oplus V_{i,j}$ be the joint Pierce decomposition associated with $\{e_j\}_{j=1}^r$. We define a parameter $a = a(D)$ by $a = 0$ if $r = 1$ and $a = dim_C(V_{i,j})$ for some $1 \leq i < j \leq r$ if $r > 1$. The parameter a is well defined and independent of (i, j) and of the frame $\{e_j\}_{j=1}^r$ because of transitivity of the action of K on the frames.

We define a parameter $b = b(D)$ by $b = dim(V_{0,j})$. Again, b is well defined invariant of D. The genus of D is $g = g(D) = (r-1)a + b + 2$.

For every x in V, let $\|x\|$ denote the largest eigenvalue of x. Then $\|.\|$ is an Aut(V)-invariant norm on V, called the spectral norm. So a circled bounded symmetric domain is the open unit ball of the associated Jordan pair with involution. Thus it is convex.

Now we are going to prove the following theorem which gives a precise formula for the Bergman metric on a bounded symmetric domain.

Theorem 1.4. Suppose that $\mathcal{D}$ is a bounded symmetric domain with rank r in V=C^n. We associate to $\mathcal{D}$ a Jordan pair as above. For z in $\mathcal{D} \subset V$, and t in [0,1] we have

$$h_{tz}(z,z) = g/2 \sum_{i=1}^{r} \frac{|\lambda_i|^2}{(1-t^2|\lambda_i|^2)^2} \tag{1.28}$$

if z =$\sum_{i=1}^{r} \lambda_i e_i$ and e_i is primitive. So

$$h_{tz}(z,z) \le gr/2 \frac{\|z\|^2}{(1-t^2\|z\|^2)^2}. \tag{1.29}$$

Proof. It follows from (b) of Theorem 1.1 that for any z in $\mathcal{D}$

$$h_{tz}(z,z) = h_0(B(tz,tz)^{-1}z,z).$$

Since for any z in $\mathcal{D}$, there is an orthogonal system $\{e_1, \cdots, e_r\}$ of primitive tripotents such that

$$z = \sum_{i=1}^{r} \lambda_i e_i$$

where $\lambda \in$[0,1].

Using (c) of Theorem 1.2 we get

$$B(tz,tz)e_i = (1-t^2|\lambda_i|^2)^2 e_i$$

since e_i is in $V_2(e_i)$. So

$$B(tz,tz)^{-1}e_i = (1-t^2|\lambda_i|^2)^{-2}e_i.$$

Thus

$$B(tz,tz)^{-1}z = \sum \lambda_i(1-t^2|\lambda_i|^2)^{-2}e_i.$$

Therefore

$$\begin{aligned} h_{tz}(z,z) &= \textstyle\sum_{i,j} \lambda_i\overline{\lambda_j}(1-t^2|\lambda_i|^2)^{-2}h_0(e_i,e_j) \\ &= \textstyle\sum_{i,j} \lambda_i\overline{\lambda_j}(1-t^2|\lambda_i|^2)^{-2}traceD(e_i,\overline{e_j}) \\ &= \textstyle\sum_i |\lambda_i|^2(1-t^2|\lambda_i|^2)^{-2}traceD(e_i,\overline{e_i}) \end{aligned}$$

since e_i is orthogonal to e_j if $i \neq j$. Now we turn to computing $trD(e_i,\overline{e}_i)$. From Theorems 1.2 and 1.3 we can decompose V as

$$V = \sum_{0\leq i\leq j\leq r} V_{ij}$$

and $V_{00}=0$. For $x_{kl} \in V_{kl}$

$$D(e_i,\overline{e}_i)x_{kl} = \begin{cases} x_{kl} & k\neq i, l=i \quad or k=i, l\neq i \\ \frac{1}{2}x_{kl} & k=l=i \\ 0 & otherwise \end{cases}$$

Moreover Theorem 1.3 (a) tells us that $\dim_C V_2(e_i) = 1$. Thus

$$trD(e_i,\overline{e_i}) = 1 + \frac{1}{2}dim_C V_1(e_i)$$

Since

$$V_1(e_i) = \oplus_{0<j\neq i} V_{ij} \oplus V_{0i}$$

then

$$trD(e_i, \overline{e_i}) = 1 + \frac{1}{2}((r-1)a + b) = \frac{g}{2}$$

Thus

$$h_{tz}(z, z) = \frac{g}{2} \sum \frac{|\lambda_i|^2}{(1-t^2|\lambda_i|^2)^2}$$

$$\leq \frac{g}{2} \frac{\|z\|^2}{(1-t^2\|z\|^2)^2}$$

since $\lambda_i \leq \|z\|$.

1.2 Schatten p-class Hankel operators

First we collect some known results.

Lemma 1.5. Suppose A is a positive or trace class operator on L_a^2, then

$$tr(A) = \int_{\mathcal{D}} < Ak_z, k_z > K(z, z) dA(z) \tag{1.30}$$

The above result was proved in [AFP] for the unit disk. But it is easy to extend the proof to any Bergman spaces.

Proposition 1.6. Suppose A is a positive operator on L_a^2 and f is a unit vector in L_a^2, then

$$< Af, f >^p \leq < A^p f, f > \tag{1.31}$$

for all $1 \leq p$.

This result was proven in [AFP]. It follows directly from the spectral decomposition of the positive operator A.

Proposition 1.7. If f is in L_a^2 and z $\in \mathcal{D}$, then

$$1/2(\|H_f k_z\| + \|H_{\bar{f}} k_z\|) \leq MO(f)(z) \leq \|H_f k_z\| + \|H_{\bar{f}} k_z\|. \tag{1.32}$$

It follows directly from the proof of Theorem F in [BBCZ] that

Proposition 1.8. Suppose $\gamma(t)$ (t $\in$[0,1]) is a smooth curve in $\mathcal{D}$, then

$$|d\tilde{f}(\gamma(t))| \leq 2^{3/2} MO(f)(\gamma(t)) \sqrt{h_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt \tag{1.33}$$

where $h_z(u, v)$ is the Bergman metric defined in Section 1.

Proposition 1.9. Suppose $f \geq 0$ and $1 \leq p < +\infty$, then $\tilde{f}(z) \in L^2(\mathcal{D}, d\mu)$ if and only if T_f is in S_p.

Proposition 1.9 was proved in [Zh2].

Proposition 1.10. Suppose $2 \leq p < +\infty$ and A_G is the integral operator on L^2 defined by

$$A_G f(z) = \int_{\mathcal{D}} G(z, w) K(z, w) f(w) dA(w)$$

If

$$\int_{\mathcal{D}} \int_{\mathcal{D}} |G(z, w)|^p |K(z, w)|^2 dA(z) dA(w) \leq +\infty \tag{1.34}$$

then A_G is in S_p .

Proof. The case p=2 is well known. If G(z,w) is bounded on $\mathcal{D} \times \mathcal{D}$, it follows from the proof of Theorem 21 of [BBCZ] that A_G is a bounded linear operator on L^2. In order to make this clear, we give the details of the proof that A_G is bounded. First we show that

$$Af(z) = \int_{\mathcal{D}} |K(z, w)| |f(w)| dA(w)$$

is bounded on L^2. In fact from the proof of Theorem 21 of [BBCZ] we can see that for small $\varepsilon > 0$, the integral operator with kernel

$$K(\varphi_z(w), \varphi_z(w))^\varepsilon |K(z,w)|$$

is bounded on L^2.

Since

$$|K(z,w)| \leq K(z,z)^{1/2} K(w,w)^{1/2},$$

we have

$$K(\varphi_z(w), \varphi_z(w))^\varepsilon = (\frac{K(z,z)K(w,w)}{|K(z,w)|^2})^\varepsilon \geq 1.$$

Then

$$Af(z) \leq \int K(\varphi_z(w), \varphi_z(w))^\varepsilon |K(z,w)||f(w)|dA(w).$$

Thus A is bounded on L^2.

Now we consider the linear mapping

$$F : L^2(\mathcal{D} \times \mathcal{D}, d\eta) + L^\infty(\mathcal{D} \times \mathcal{D}, d\eta) \to \mathcal{D}B(L^2)$$

given by

$$F(G) = A_G$$

where $d\eta = |K(z,w)|^2 dA(z)dA(w)$. Then

$$F : L^2(\mathcal{D} \times \mathcal{D}, d\eta) \to S_2$$

$$F : L^\infty(\mathcal{D} \times \mathcal{D}, d\eta) \to S_\infty$$

are both bounded. By interpolation [BL], we have

$$F : L^p(\mathcal{D} \times \mathcal{D}, d\eta) \to S_p$$

is also bounded for all $2 \leq p < \infty$. In particular if

$$\int_{\mathcal{D}} \int_{\mathcal{D}} |G(z,w)|^p |K(z,w)|^2 dA(z) dA(w) < \infty$$

then A_G is in S_p

Theorem 1.11 will be used in the proof of Theorem 1.12.

Theorem 1.11. Let $Tf(z) = \tilde{f}(z)$ for f in L^2. Then T can be extended to be a bounded linear operator from L^p to L^p for $2 \leq p \leq \infty$.

Proof. For $f \in L^\infty(\mathcal{D}, dA)$, we have

$$|Tf(z)| \leq |\int f|k_z|^2 dA| \leq \|f\|_\infty \tag{1.35}$$

and moreover for f $\in L^2$

$$|Tf(z)| \leq \int |f(w)||k_z(w)|^2 dA(w)$$

$$= \int |f(w)| \frac{|K(z,w)|^2}{K(z,z)} dA(w).$$

Since $K(z,w) = \frac{K(z,z)}{K(z,\varphi_z(w))}$ and $K(z,w)^{-1}$ is continuous on $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, there is a constant $C > 0$ such that

$$|\frac{K(z,w)}{K(z,z)}| \leq C.$$

Therefore

$$|Tf(z)| \leq C \int |f(w)||K(z,w)| dA(w).$$

According to the proof of Proposition 1.10, T is a bounded operator on L^2.

Now we define a linear mapping

$$F : L^2(\mathcal{D}, dA) + L^\infty(\mathcal{D}, dA) \to L^2(\mathcal{D}, dA) + L^\infty(\mathcal{D}, dA)$$

we just show that

$$F : L^2(\mathcal{D}, dA) \to L^2(\mathcal{D}, dA)$$

and

$$F : L^\infty(\mathcal{D}, dA) \to L^\infty(\mathcal{D}, dA)$$

are bounded. By interpolation [BL]

$$F : L^p(\mathcal{D}, dA) \to L^p(\mathcal{D}, dA)$$

is bounded for $2 \leq p \leq \infty$.

Theorem 1.12. If $2 \leq p \leq \infty$ and

$$\|f \circ \varphi_z - P(f \circ \varphi_z)\|_2$$

is in $L^p(\mathcal{D}, d\mu)$, then $H_{f-\tilde{f}}$ is in S_p.

Proof. It is easy to check that

$$H_g^* H_g = T_{|g|^2} - T_{\bar{g}} T_g \leq T_{|g|^2}$$

for all g in L^2. Let $g = f - \tilde{f}$, then

$$\widetilde{|g|^2}(z) = \int_{\mathcal{D}} |f(w) - \tilde{f}(w)|^2 |k_z(w)|^2 dA$$

$$= \int_{\mathcal{D}} |f \circ \varphi_z(w) - \tilde{f} \circ \varphi_z(w)|^2 dA$$

$$\leq \|f \circ \varphi_z - P(f \circ \varphi_z)\|^2 + \int_{\mathcal{D}} |(\widetilde{P(f \circ \varphi_z)} - f \circ \varphi_z)(w)|^2 dA.$$

It follows from Theorem 1.11 that there is a positive constant C such that

$$\widetilde{|g|^2}(z) \leq (1 + C)\|f \circ \varphi_z - P(f \circ \varphi_z)\|^2. \tag{1.36}$$

Then

$$\widetilde{|g|^2}(z)^{1/2} \leq C\|f \circ \varphi_z - P(f \circ \varphi_z)\|.$$

So if $\|f \circ \varphi_z - P(f \circ \varphi_z)\|$ is in $L^p(\mathcal{D}, d\mu)$, then $\widetilde{|g|^2}(z)^{1/2}$ is in $L^p(\mathcal{D}, d\mu)$. It follows from Proposition 3.6 that $T_{|g|^2}$ is in $S_{p/2}$. Therefore $H_{f-\tilde{f}}$ is in S_p.

Combining the above theorem with Proposition 3.3 we have the following corollary

Corollary. For $2 \leq p < \infty$, if $MO(f)(z)$ is in $L^p(\mathcal{D}, d\mu)$, then $H_{f-\tilde{f}}$ and $H_{\overline{f}-\overline{\tilde{f}}}$ are both in S_p.

Theorem 1.13. For any $1 \leq p < \infty$ there exists a positive constant C_p (independent of f) such that

$$\int_{\mathcal{D}} |\tilde{f}(0) - \tilde{f}(w)|^p dA \leq C_p \int_{\mathcal{D}} \int_{\mathcal{D}} MO(f)(w)^p \|w\|^{-(2n-1)} dA(w) \qquad (1.37)$$

where $\|.\|$ denotes the Hilbert norm of C^n.

This theorem is the key for us to prove Theorem 1.17. Here we give two proofs. One is based on the estimate of the Bergman metric through an analyse of the Jordan structure on bounded symmetric domains. Another depends on an integration formula in polar coordinates and the Schwarz Lemma on bounded symmetric domains.

Proof 1. First we use Theorem 1.6 to prove the theorem. Suppose $\mathcal{D}$ is a bounded symmetric circled domain. We associate to $\mathcal{D}$ a Jordan pair as in Section 1. So for every point z in $\mathcal{D}$ we can write z

$$z = \lambda_1 e_1 + \cdots + \lambda_r e_r$$

where $\{e_1, \cdots, e_r\}$ is a maximal orthogonal system of tripotents of V=$(C^n, \{,.\})$.

Then Theorem 1.4 says that

$$h_{tz}(z,z) \leq (n+2-r)\frac{\|z\|^2}{(1-t^2\|z\|^2)^2}.$$

Since $\mathcal{D}$ is convex, we define a curve in $\mathcal{D}$, which connects 0 and z, given by

$$\gamma(t) : [0,1] \to \mathcal{D}$$

and $\gamma(t) = tz$.

It follows from Proposition 1.8 that

$$|d\tilde{f}(\gamma(t))| \leq 2^{3/2} MO(f)(\gamma(t))\sqrt{h_{\gamma(t)}(\gamma'(t),\gamma'(t))}dt$$

$$= 2^{3/2} MO(f)(tz)\sqrt{h_{tz}(z,z)}dt$$

$$\leq \sqrt{2gr} MO(f)(tz)\tfrac{\|z\|}{1-t^2\|z\|^2}.$$

Therefore

$$|\tilde{f}(z) - \tilde{f}(0)| \leq \textstyle\int_0^1 |d\tilde{f}(\gamma(t))|$$

$$\leq \sqrt{2gr} \textstyle\int_0^1 MO(f)(tz)\tfrac{\|z\|}{1-t^2\|z\|^2}dt$$

For the case $1 < p < \infty$, we write

$$1 - t\|z\| = (1-t\|z\|)^{1/2p}(1-t\|z\|)^{1/2q+1/2}$$

where 1/q + 1/q =1, and apply the Holder inequality to get

$$|\tilde{f}(z) - \tilde{f}(0)| \leq C[\int_0^1 \frac{\|z\|dt}{(1-t\|z\|)^{1/2+q/2}}]^{1/q}[\int_0^1 \frac{MO(f)(tz)^p dt}{\sqrt{1-t\|z\|}}]^{1/p}.$$

Simplifying the right hand side of the above inequality we get

$$|\tilde{f}(z)-\tilde{f}(0)|^p \leq \frac{C}{\sqrt{1-\|z\|}}\int_0^1 \frac{MO(f)(tz)^p dt}{\sqrt{(1-t\|z\|)}}.$$

So

$$\int_{\mathcal{D}} |\tilde{f}(z)-\tilde{f}(0)|^p dA(z)$$

$$\leq C\int_{\mathcal{D}}\int_0^1 \frac{MO(f)(tz)^p}{\sqrt{(1-\|z\|)(1-t\|z\|)}}dA(z)dt$$

$$\leq C\int_{t\mathcal{D}}\int_0^1 \frac{MO(f)(z)^p}{\sqrt{(1-\|z\|/t)}\sqrt{(1-t\|z\|)}}dA(z)\frac{dt}{t^{2n}}$$

$$\leq C\int_{\mathcal{D}}\int_{\|z\|}^1 \frac{MO(f)(z)^p}{\sqrt{(1-\|z\|/t)}\sqrt{(1-t\|z\|)}}dA(z)\frac{dt}{t^{2n}}$$

$$= C\int_{\mathcal{D}} \frac{MO(f)(z)^p dA(z)}{\sqrt{(1-\|z\|)}}\int_{\|z\|}^1 \frac{dt}{t^{2n}\sqrt{(1-t\|z\|)}}.$$

A change of variable shows that

$$\int_{\|z\|}^1 \frac{dt}{t^{2n}\sqrt{(1-t\|z\|)}} \leq \frac{2\sqrt{1-\|z\|}}{\|z\|^{(2n-1)}}.$$

Therefore

$$\int_{\mathcal{D}} |\tilde{f}(z)-\tilde{f}(0)|^p dA(z)$$

$$\leq C\int_{\mathcal{D}} \frac{MO(f)(z)^p dA(z)}{\sqrt{(1-\|z\|)}}\frac{2\sqrt{1-\|z\|}}{\|z\|^{(2n-1)}}$$

$$\leq C\int_{\mathcal{D}} MO(f)(z)^p \frac{dA(z)}{\|z\|^{2n-1}}.$$

For the case p=1, we consider

$$\int_{\mathcal{D}} |\tilde{f}(z) - \tilde{f}(0)| dA(z)$$

$$\leq \int_{\mathcal{D}} \int_0^1 MO(f)(tz) \frac{\|z\|}{1-t^2\|z\|^2} dA(z)dt$$

$$= \int_{t\mathcal{D}} \int_0^1 MO(f)(z) \frac{\|z\|}{1-\|z\|^2} dA(z)dt/t^{2n+1}$$

$$= \int_{\mathcal{D}} \int_{\|z\|}^1 MO(f)(z) \frac{\|z\|}{1-\|z\|^2} dA(z)dt/t^{2n+1}$$

$$= \int_{\mathcal{D}} \frac{MO(f)(z)\|z\|}{1-\|z\|^2} \frac{1-\|z\|^{2n}}{\|z\|^{2n}} dA(z)$$

$$\leq C \int_{\mathcal{D}} MO(f)(z) dA(z)/\|z\|^{2n-1}.$$

Since any two norms on C^n are equivalent we have finished the proof.

Before going to the second proof we need an integration formula and the Schwarz Lemma on bounded symmetric domains. As in ([FK], [BBCZ]), let $\mathcal{D} = G/K$ be a symmetric bounded domain of rank r in C^n (in its natural realization). We may assume that $\mathcal{D} \bigcap C^r$ is a polydisk D^r and $\mathcal{D}$ can be written as a union of polydisks KD^r, where K is considered as a subgroup of U(n) of C^n.

There is a formula for integration in polar coordinates, a technique developed in [Hua] for the classical Cartan domains and in [Hel] in the case of bounded symmetric domains and homogeneous cones. We state the formula as follows.

Proposition 1.14. If $\mathcal{D}$ is a bounded symmetric domain with rank r in

C^n, then there is a function G(z) on $\mathcal{D}$ such that

$$\int_{\mathcal{D}} f(w)dA(w) = \int_{D^r}\int_K f(kz)G(z)dkdA(z) \tag{1.38}$$

where

$$G(z) = \prod_{j=1}^{r} |z_j|^{2b+1} \prod_{1\leq j<l\leq r} ||z_i|^2 - |z_l|^2|^a \tag{1.39}$$

and dA(w) is the Euclidean volume element of C^n, and dk the normalized Haar measure of K.

Proposition 1.15. Let D^r be a polydisk of dimension r with Bergman metric $ds^2_{D^r}$ and let M an n-dimensional Hermitian manifold whose holomorphic sectional curvature is bounded above by a negative constant -B . Then every holomorphic mapping j:$D^r \to M$ satisfies

$$j^*(ds^2_M) \leq A/Bds^2_{D^r} \tag{1.40}$$

where -A is the curvature of ds^2_D.

Proposition 1.15 is just Theorem 3.1 in [Kob]. If $\mathcal{D}$ is a bounded symmetric domain with Bergman metric, then $\mathcal{D}$ is a Hermitian manifold. Also

$$i : D^r \hookrightarrow \mathcal{D}$$

is holomorphic. With respect to a canonical Hermitian metric, the holomorphic sectional curvature of $\mathcal{D}$ lies between -A and -A/r for a suitable positive constant A. So $\mathcal{D}$ satisfies the condition of Proposition 1.15. Therefore

$$i^* ds^2_{\mathcal{D}} \leq A/Bds^2_{D^r}$$

Proof 2. Suppose that $\mathcal{D}$ can be written as a union of polydisks KD^r. Let

$$D^r \hookrightarrow \mathcal{D} \hookrightarrow D^n$$

and

$$(D^r, 0) = \mathcal{D} \bigcap C^r \times 0$$

be the holomorphic embedding and every element k in K is unitary with respect to the norm $\|.\|_2$ of C^n.

Fix k in K and z in D^r, let

$$\gamma(t) = tz.$$

Proposition 1.8 implies

$$|d\widetilde{f \circ k}(\gamma(t))| \leq 2^{3/2} MO(f \circ k)(\gamma(t)) \sqrt{i^* ds^2_{\mathcal{D}}}.$$

Since the Berezin tra[illegible] commutes with the K-action, we have

$$|d\tilde{f}(k\gamma(t))| \leq 2^{3/2} MO(f)(k\gamma(t)) \sqrt{i^* ds^2_{\mathcal{D}}}.$$

Since $i : D^r \hookrightarrow \mathcal{D}$ is holomorphic, Proposition 1.15 implies

$$|d\tilde{f}(k\gamma(t))| \leq 2^{3/2} C MO(f)(k\gamma(t)) \sqrt{ds^2_{D^r}}.$$

It is easy to check that

$$ds^2_{D^r} = \sum_{i=1}^{r} \frac{dz_i \overline{dz_i}}{(1 - |z_i|^2)^2}. \tag{1.41}$$

Then

$$\sqrt{ds^2_{D^r}(\gamma(t))} \leq r^{1/2} \frac{\|z\|_\infty}{1 - t^2 \|z\|^2_\infty}.$$

Therefore

$$|\tilde{f}(kz) - \tilde{f}(0)| \leq C \int_0^1 MO(f)(tkz) \frac{\|z\|_\infty}{1 - t^2 \|z\|^2_\infty} dt.$$

As in Proof 1 , in the case $1 < p < \infty$, we have

$$|\tilde{f}(kz) - \tilde{f}(0)|^p \leq \frac{C}{\sqrt{1 - \|z\|_\infty}} \int_0^1 \frac{MO(f)(tkz)^p}{\sqrt{1 - t\|z\|_\infty}} dt.$$

So

$$\int_{D^r} |\tilde{f}(kz) - \tilde{f}(0)|^p G(z) dA(z)$$

$$\leq C \int_{D^r} \int_0^1 \frac{MO(f)(tkz)^p G(z)}{\sqrt{1-t\|z\|_\infty}\sqrt{1-\|z\|_\infty}} dA(z) dt$$

$$\leq C \int_{tD^r} \int_0^1 \frac{MO(f)(kz)^p G(z/t)}{\sqrt{1-\|z\|_\infty}\sqrt{1-\|z\|_\infty/t}} dA(z) dt / t^{2r}.$$

Since $G(z/t) = t^{-(2n-2r)} G(z)$, then we have

$$\int_{D^r} |\tilde{f}(kz) - \tilde{f}(0)|^p G(z) dA(z)$$

$$\leq C \int_{tD^r} \int_0^1 \frac{MO(f)(kz)^p G(z)}{\sqrt{1-\|z\|_\infty}\sqrt{1-\|z\|_\infty/t}} dA(z) dt / t^{2n}$$

$$\leq C \int_{D^r} \int_{\|z\|_\infty^1} \frac{MO(f)(kz)^p G(z)}{\sqrt{1-\|z\|_\infty}\sqrt{1-\|z\|_\infty/t}} dA(z) dt / t^{2n}$$

$$\leq C \int_{D^r} \frac{MO(f)(kz)^p G(z) \sqrt{1-\|z\|_\infty}}{\sqrt{1-\|z\|_\infty} \|z\|_\infty^{2n-1}} dA(z)$$

$$\leq C \int_{D^r} MO(f)(kz)^p \frac{dA(z)}{\|z\|_\infty^{2n-1}}.$$

In addition, in the case p=1, we have

$$\int_{D^r} |\tilde{f}(kz) - \tilde{f}(0)|G(z)dA(z)$$

$$C \int_{D^r} \int_0^1 MO(f)(tkz)G(z)\frac{\|z\|_\infty}{1-t^2\|z\|_\infty^2} dA(z)dt$$

$$C \int_{tD^r} \int_0^1 MO(f)(kz)G(z)\frac{\|z\|_\infty}{1-\|z\|_\infty^2} dA(z)dt/t^{2n+1}$$

$$= C \int_{D^r} \int_{\|z\|_\infty}^1 MO(f)(kz)G(z)\frac{\|z\|_\infty}{1-\|z\|_\infty^2} dA(z)dt/t^{2n+1}$$

$$\leq C \int_{D^r} \frac{MO(f)(kz)G(z)}{\|z\|_\infty^{2n-1}} dA(z).$$

Combining the above two estimates with Proposition 2.1 implies that

$$\int_{\mathcal{D}} |\tilde{f}(w) - \tilde{f}(0)|^p dA(w) \leq C \int_K dk \int_{D^r} \frac{MO(f)^p(kz)G(z)}{\|z\|_\infty^{2n-1}} dA(z).$$

Since any two norms of C^n are equivalent and $\|z\|_2$ is the K-invariant norm, then

$$\int_{\mathcal{D}} |\tilde{f}(w) - \tilde{f}(0)|^p dA(w) \leq C \int_{\mathcal{D}} \frac{MO(f)^p(w)}{\|w\|^{2n-1}} dA(w).$$

So we have completed Proof 2.

Proposition 1.16. There is a positive constant M such that

$$\int_{\mathcal{D}} |k_z(w)|^2 \frac{dA(w)}{\|\varphi_w(z)\|_2^{2n-1}} \leq M \qquad (1.42)$$

for all z in $\mathcal{D}$.

Proof. Since the Bergman metric induces the usual topology on $\mathcal{D}$ for any $\varepsilon > 0$, there are positive constants δ_1 and δ_2 such that

$$\{w : \|w\|_2 < \delta_1\} \subset \{w : \beta(o, w) < \varepsilon\} \subset \{w : \|w\| < \delta_2\}$$

Moreover since the Bergman metric is complete on $\mathcal{D}$, on any compact set K of $\mathcal{D}$, the Bergman distance is equivalent to Euclidean distance, and $\{w, \beta(0,w) \leq \varepsilon\}$ is compact. Thus

$$\int_{\beta(z,w)<\varepsilon} |k_z(w)^2| \frac{dA(w)}{\|\varphi_w(z)\|^{2n-1}}$$

$$\leq C \int_{\beta(z,w)<\varepsilon} |k_z(w)^2| \frac{dA(w)}{\beta(0,\varphi_w(z))^{2n-1}}$$

$$\leq C \int_{\beta(z,w)<\varepsilon} |k_z(w)^2| \frac{dA(w)}{\beta(w,z)^{2n-1}}$$

$$\leq C \int_{\beta(z,w)<\varepsilon} |k_z(w)^2| \frac{dA(w)}{\beta(\varphi_z(w),0)^{2n-1}}$$

$$\int_{\beta(z,w)<\varepsilon} |k_z(w)^2| \frac{dA(w)}{\|\varphi_z(w)\|^{2n-1}}$$

$$= \int_{\beta(0,w)<\epsilon} \frac{dA(w)}{\|w\|_2^{2n-1}} \leq C_1.$$

Also

$$\int_{\beta(z,w)>\varepsilon} |k_z(w)^2| \frac{dA(w)}{\|\varphi_w(z)\|^{2n-1}}$$

$$= \int_{\beta(\varphi_w(z),0)>\varepsilon} |k_z(w)^2| \frac{dA(w)}{\|\varphi_w(z)\|^{2n-1}}$$

$$\leq \int_{\|\varphi_w(z)\|_2>\delta_1} |k_z(w)^2| \frac{dA(w)}{\|\varphi_w(z)\|^{2n-1}}$$

$$\leq \delta_1^{-2n+1} \int_{\mathcal{D}} |k_z(w)|^2 dA(w) = \delta_1^{-2n+1}.$$

So we choose $M \geq \delta_1^{-2n+1} + C_1$ to complete the proof.

Now we are ready to prove our main result.

Theorem 1.17. Suppose $2 \leq p < +\infty$. For f in L^2, both H_f and $H_{\bar{f}}$ are in S_p if and only if

$$\int_{\mathcal{D}} MO(f)^p(z)d\mu(z) < +\infty \tag{1.43}$$

where $d\mu(z)$ is the volume in the Bergman metric, which is $K(z,z)dA(z)$.

Proof. The only if part is easy using Propositions 1.5 and 1.6. For completeness we give a proof here. Since $H_f \in S_p$, Proposition 1.5 says

$$\int_{\mathcal{D}} < (H_f^* H_f)^{p/2} k_z, k_z > d\mu = tr((H_f^* H_f)^{p/2}) < \infty$$

Since $p/2 \geq 1$, and each k_z is a unit vector, Proposition 1.16 implies

$$\int_{\mathcal{D}} < H_f^* H_f k_z, k_z >^{p/2} d\mu < \infty$$

i.e.

$$\int_{\mathcal{D}} \|H_f k_z\|^{p/2} d\mu < \infty.$$

Similarly we can get

$$\int_{\mathcal{D}} \|H_{\bar{f}} k_z\|^{p/2} d\mu < \infty.$$

Combining Proposition 1.7 with above two estimates implies that $MO(f)(z)$ is in $L^p(\mathcal{D}, d\mu)$.

Now we turn to the proof of sufficiency. By means of $H_f = H_{f-\tilde{f}} + H_{\tilde{f}}$ and Theorem 1.12, it suffices to prove that $H_{\tilde{f}}$ and $H_{\bar{\tilde{f}}}$ are in S_p if MO(f) is in $L^p(\mathcal{D}, d\mu)$. In fact $H_{\tilde{f}}$ is an integral operator represented as

$$H_{\tilde{f}} g(z) = \int_{\mathcal{D}} (\tilde{f}(w) - \tilde{f}(z)) K(z,w) dA(w)$$

By Proposition 1.10 $H_{\tilde{f}}$ will be in S_p if we can show that

$$M = \int_{\mathcal{D}} \int_{\mathcal{D}} |\tilde{f}(w) - \tilde{f}(z)|^p |K(z,w)|^2 dA(w) dA(z) < \infty$$

Using Fubini's theorem and making a change of variable in the inner integral, we get

$$M = \int_{\mathcal{D}} d\mu(z) \int_{\mathcal{D}} |\widetilde{f \circ \varphi_z}(0) - \widetilde{f \circ \varphi_z}(w)|^p dA(w).$$

By means of Theorem 1.13, we have

$$\begin{aligned} M &\leq C_p \int_{\mathcal{D}} d\mu(z) \int_{\mathcal{D}} MO(f)(\varphi_z(w))^p \frac{dA(w)}{\|w\|^{2n-1}} \\ &\leq C_p \int_{\mathcal{D}} d\mu(z) \int_{\mathcal{D}} MO(f)(w)^p |k_z(w)|^2 \frac{dA(w)}{\|\varphi_z(w)\|^{2n-1}} \\ &= C_p \int_{\mathcal{D}} dA(w) \int_{\mathcal{D}} MO(f)(w)^p |K(w,z)|^2 \frac{dA(z)}{\|\varphi_z(w)\|^{2n-1}} \\ &\leq C_p \int_{\mathcal{D}} d\mu(w) \int_{\mathcal{D}} MO(f)(w)^p |k_w(z)|^2 \frac{dA(z)}{\|\varphi_z(w)\|^{2n-1}}. \end{aligned}$$

By Proposition 1.16 we have

$$M \leq C \int_{\mathcal{D}} MO(f)(w)^p d\mu(w) < \infty.$$

Thus $H_{\tilde{f}}$ is in S_p. Similarly we can prove that $H_{\overline{\tilde{f}}}$ is in S_p. So the proof is completed.

Chapter 2

Operator theory and function theory on the unit ball

In this chapter we consider the question of when the semi-commutator $T_{\overline{f}}T_g - T_{\overline{f}g}$ of the Toeplitz operators $T_{\overline{f}}$ and T_g on the Bergman space $L_a^2(B_n, dA)$ is compact for bounded pluriharmonic functions f and g. First we want to mention that our problem can be reformulated as a problem about Hankel operators for which the product $H_f^* H_g$ of two Hankel operators on the Bergman space $L_a^2(B_n, dA)$ is compact since it is easy to check that

$$T_{\overline{f}}T_g - T_{\overline{f}g} = H_f^* H_g. \tag{2.1}$$

We will also study commuting Toeplitz operators on the Bergman space of the unit ball or on the Hardy space of the unit sphere in higher dimensional complex space.

Although there are some results on commuting Toeplitz operators and compact semicommutators of Toeplitz operators on the unit disk, translating those results from the unit disk to the unit ball is more involved than merely saying " Now let n> 1". The transition from statements and their proofs on the Hardy space on the unit circle and the Bergman space on the unit disk to the unit ball involves many remarkable properties of the harmonic functions such as maximal property and the mean value property , and the theory of function algebras. On the unit ball there are three different concepts harmonicity, $\mathcal{M}$-harmonicity and pluriharmonicty. On the disk those concepts coincide and harmonic functions have a very special relationship with holomorphic functions. On the unit ball, the pluriharmonic functions enjoy an elevated status, but the $\mathcal{M}$-harmonic functions do not have any useful special relationships with holomorphic functions, neither do the harmonic functions. In addition the theory of function algebra on the sphere S_n is very complicated and even it is not known whether the Corona theorem is true for $H^\infty(S_n)$ and what the Gleason parts of $H^\infty(S_n)$ look like, for $n > 1$. So it seems that it is very hard to use , "soft analysis", the theory of function algebras to work on the Toeplitz operators and the Hankel operators in several dimensions complex space. The theory of several complex variables plays a very important role in this chapter. Motivated by the results on the unit disk we just consider the Toeplitz operators and Hankel operators with pluriharmonic symbols. However the invariant mean value property does not completely characterize the pluriharmonic functions. But it completely characterizes $\mathcal{M}$-harmonic functions. So we have to study the $\mathcal{M}$-harmonic functions.

2.1 Function theory on the unit ball

This section contains the materials on the function theory of the unit ball , which will be needed later on. Good references for this are Rudin's book [R] and Krantz's book [Kra]

For z in B_n there is a Moebius transformation ϕ_z: $B_n \to B_n$ given by

$$\phi_z(w) = \frac{z - P_z w - (1 - \|z\|^2)^{1/2} Q_z w}{1 - (w, z)} \tag{2.2}$$

where P_z is the orthogonal projection from C^n onto the subspace of C^n spanned by z, and $Q_z = I - P_z$.

Since each $\phi \in Aut(B_n)$ acts on B_n as an isometry in the Bergman metric, we have

$$Q_{f o \phi}(z_0) = Q_f(\phi(z_0)) \tag{2.3}$$

for any $\phi \in Aut(B_n)$ and z_0 in B_n. A holomorphic function f is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = Sup_z Q_f(z) < \infty. \tag{2.4}$$

Timoney ([T1] and [T2]) proved that on the unit ball $\|f\|_{\mathcal{B}}$ is equivalent to

$$Sup_z(1 - \|z\|^2)\|\nabla_z f\|. \tag{2.5}$$

Although the definition of $\tilde{\Delta}$ in Chapter 0 is slightly different from the so-called invariant Laplacian operator in [R], the concept of $\mathcal{M}$-harmonic functions here is the same as in [R].

The Moebius transformation has the following properties:

Proposition 2.1. For every a $\in B_n, \phi_a$ has the following properties

(1.1) $\phi_a(0) = a$ and $\phi_a(a) = 0$;

(1.2) $\phi_a'(0) = -s^2P_a - sQ_a$ and $\phi_a'(a) = -s^{-2}P_a - s^{-1}Q_a$ where $s = (1-|a|^2)^{1/2}$;

(1.3) ϕ_a is an involution; $\phi_a(\phi_a(z)) = z$;

(1.4) ϕ_a is a homeomorphism of B_n onto B_n, and $\phi_a \in Aut(B_n)$. Moreover $Aut(B_n) = \{\phi_z \circ U : z \in B_n, U \in \mathcal{U}(n)\}$

For a $\in B_n$, fixed r, $0 < r < 1$, and define

$$E(a,r) = \phi_a(rB_n).$$

Since ϕ_a is an involution, z in E(a,r) if and only if $|\phi_a(z)| < r$. It is easy to check that

$$E(a,r) = \{z \in B_n : \frac{|P_az - c|^2}{r\rho^2} + \frac{|Q_az|^2}{r^2\rho} < 1\} \tag{2.6}$$

where

$$c = \frac{(1-r^2)a}{1-r^2|a|^2}, \rho = \frac{1-|a|^2}{1-r^2|a|^2}.$$

Thus E(a,r) is an ellipsoid with center at c, this is close to a when r is small and the volume of E(a,r) equals to $r^{2n}\rho^{n+1}$.

There are other several characterizations of pluriharmonic functions, which give a relation between pluriharmonic functions and holomorphic functions.

Proposition 2.2. The function u is pluriharmonic if and only if u satisfies one of the following conditions:

(a) There are two holomorphic functions f and g on B_n such that

$$u = f + \overline{g}; \tag{2.7}$$

(b) u satisfies the n^2-differential equations

$$\frac{\partial^2}{\partial z_i \partial \overline{z}_j} u = 0. \tag{2.8}$$

$\mathcal{M}$-harmonic functions have many useful properties, we state some of them.

Proposition 2.3. (1) **Maximum Principle.** Suppose G is an open subset of B_n and u $\in C(\overline{G}), \tilde{\Delta}u = 0$ in G and $u \leq 0$ on ∂ G. Then $u \leq 0$ in G.

(2) **The invariant mean value property.** Suppose f is in $C^2(B_n)$. Then f is $\mathcal{M}$-harmonic if and only if f has the invariant mean value property. i.e. for every $\psi \in Aut(B_n)$ and $0 < r < 1$

$$f(\psi(0)) = \int_{S_n} f(\psi(r\varsigma))d\sigma(\varsigma). \tag{2.9}$$

(3) **The volume version of invariant mean value property.** If f is in $C(\overline{B_n})$ and has the mean-valued property

$$f(\psi(0)) = \int_B f \circ \psi dV \tag{2.10}$$

for every $\psi \in Aut(B_n)$, then f is $\mathcal{M}$-harmonic on B_n.

Let $\mathcal{U} = \mathcal{U}$ (n) be the group of all unitary operators on the Hilbert space C^n. Clearly, $\mathcal{U}$ is a compact subgroup of O(2n). It is well-known that there is a Haar measure $d\mathcal{U}$ on $\mathcal{U}$. Since $\mathcal{U}$(n) acts transitively on the unit ball or sphere we can represent an integration of a function over the unit sphere as an integration over the compact group $\mathcal{U}$(n).

Proposition 2.4. If f is measurable function on B_n, the identity

$$\int_S f d\sigma = \int_{\mathcal{U}} f(\mathcal{U}\eta)d\mathcal{U} \tag{2.11}$$

holds for any $\eta \in S_n$.

We will need the following proposition to identify a holomorphic function in the Hardy space $H^2(S_n)$.

Proposition 2.5. For every multi-index α

$$\int_{S_n} |(\eta)^\alpha|^2 d\sigma(\eta) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}. \tag{2.12}$$

Proposition 2.6. If f is a bounded pluriharmonic function on B_n, then there are functions f_1 and f_2 in both the Bloch space and $H^p(S_n)$ for all $p > 1$ such that

$$f = f_1 + \overline{f}_2.$$

Proof. Without loss of generality we consider just the case that f is real-valued. It follows from Proposition 2.2 that there is a holomorphic function f_1 on B_n such that

$$f = f_1 + \overline{f}_1.$$

Because f is in $L^\infty(B_n)$ then f_1 is in $L^2_a(B_n)$. In addition it is known that the Bergman projection P is a map from $L^\infty(B_n)$ to the Bloch space. Thus $f_1 + \overline{f}_1(0) = P(f)$ is a Bloch function. This implies that f_1 is a Bloch function. Now we consider functions $f_r(\xi) = f(r\xi)$ on S for $0 < r < 1$. Then

$$\|f\|_\infty \geq \int_S \|f_r(\xi)\|^2 d\sigma(\xi).$$

Thus f_1 is in $H^2(S)$. Since

$$f = f_1 + \overline{f}_1$$

$f_1(z) + \overline{f_1(0)} = S(f)(z)$. It is well-known that the Hardy projection is bounded on $L^p(S_n)$ for any $p > 1$. Because f is bounded, f_1 is in $H^p(S_n)$ for any $p > 1$.

2.2 Maximal ideal space of $H^\infty(S_n)$ and ϕ-parts

The structure of the maximal ideal space of $H^\infty(B_n)$ is very complicated The Corona problem is the question: Is the unit ball dense in the maximal ideal space $\mathcal{M}$? This is still an open problem for $n > 1$. In the case of the unit disk, Carleson solved the problem [Car1].

By using the Gelfand transform, we can think of $H^\infty(S_n)$ as a subset of C($\mathcal{M}$), the continuous, complex-valued functions on the maximal ideal space of $H^\infty(S_n)$. Explicitly, for f$\in H^\infty(S_n)$, we extend f from B_n to $\mathcal{M}$ by defining

$$f(\tau) = \tau(f) \tag{2.13}$$

for every $\tau \in \mathcal{M}$. Note that this definition is consistent with our earlier identification of B_n with a subset of $\mathcal{M}$. Now we will prove that each bounded complex-valued pluriharmonic function on B_n can be extended to a continuous complex-valued functions on the maximal ideal space $\mathcal{M}$. The following proposition extends Lemma 4.4 in [H].

Proposition 2.7. The following algebras of complex-valued functions on B_n are identical.

(1) The algebra of (bounded continuous) functions on B_n which admit continuous extensions to $\mathcal{M}$;

(2) The complex algebra generated by $H^\infty(B_n)$ and the conjugate of $H^\infty(B_n)$ in $C(\mathcal{M})$;

(3) The complex algebra generated by bounded pluriharmonic functions on B_n.

Proof. Since $\mathcal{M}$ is maximal ideal space of $H^\infty(B_n)$, then $H^\infty(B_n)$ separates points of $\mathcal{M}$. Thus the Stone-Weierstrass theorem guarantees that (1) and (2) describe the same algebra. Obviously (3) describes a larger algebra than does (2). Therefore our task is merely to show that each bounded pluriharmonic function extends continuously to $\mathcal{M}$. Because f is pluriharmonic if and only if both real and image parts of f are bounded and pluriharmonic. Now we can assume that f is real-valued. Proposition 2.2 implies that there is a holomorphic function g on B_n such that f= $\Re g$. Let

$$h = e^g.$$

Then h is in $H^\infty(B_n)$ and $|h|$ is bounded away from 0. The continuous extension of f to $\mathcal{M}$ is

$$\widehat{f} = log|\widehat{h}|.$$

The following proposition tells us that $\phi-$maps are analytic in the sense that $f \circ \phi(z)$ is bounded and holomorphic if f is bounded and holomorphic on B_n.

Proposition 2.8. If a net $\{\phi_{z_\alpha}\} \subset B_n^{B_n} \subset \mathcal{M}^{B_n}$ converges to some $\phi \in \Phi$ then for any bounded pluriharmonic function f on B_n, $f \circ \phi_{z_\alpha}(z)$ converges uniformly to fo$\phi(z)$ on every compact subset of B_n. Consequently fo$\phi(z)$ is pluriharmonic in B_n.

Proof. Proposition 2.7 tells us that every bounded pluriharmonic function can be approximated in the norm of $L^\infty(B_n)$ by the sum of products of bounded holomorphic functions and the conjugate of bounded functions. Then it suffices to prove that for a holomorphic function g , goϕ_{z_α} normally converges to goϕ. It is well-known that for any bounded holomorphic function g, there is a constant $C_g > 0$ depending only on g such that for any z and w in B_n

$$|g(z) - g(z)| < C_g \beta(z, w).$$

Because the Bergman metric $\beta(z, w)$ is invariant under the action of the automorphism group $\mathrm{Aut}(B_n)$, we have for any $\psi \in \mathrm{Aut}(B_n)$,

$$|g \circ \psi(z) - g \circ \psi(w)| < C_g \beta(z, w).$$

Hence $\{g \circ \phi_{z_\alpha}\}$ is equicontinuous and uniformly bounded. So $g\circ\phi_{z_\alpha}$ normally converges to $g\circ\phi$.

2.3 Hankel operators as integral operators

In this section we study Hankel operators as integral operators to find a sufficient condition for the compactness of $H_{\overline{f}}^* H_{\overline{g}}$ for the Bloch functions f and g on B_n. It is easy to check that $H_{\overline{f}}^* H_{\overline{g}}$ is an integral operator

$$Th(z) = \int (\overline{g(z)} - \overline{g(w)})(f(z) - f(w))\overline{K}(z.w)h(w)dA(w) \tag{2.14}$$

on the Bergman space $L_a^2(B_n)$. Now we apply the Schur test [HS] as stated in the following proposition to estimate the distance between T and the set of compact operators.

Schur test. Let (X, $d\mu$) be a measure space and L(x,y) be a measurable function on X×X. Suppose there is a positive measurable function u on X and positive numbers α and β such that

$$\int_X |L(x, y)| u(y) d\mu(y) \leq \alpha u(x) \tag{2.15}$$

and

$$\int_X |L(x,y)|u(x)d\mu(x) \leq \beta u(y). \tag{2.16}$$

Then

$$Af(x) = \int_X L(x,y)f(y)d\mu(y) \tag{2.17}$$

for f $\in L^2(X,\mu)$ and $x \in$X, defines a bounded linear operator from $L^2(X,\mu)$ into itself. Moreover, $\|A\|^2 \leq \alpha\beta$.

Theorem 2.9. Suppose that f and g are Bloch functions on B_n. The distance d(T, $\mathcal{K}$) between T and the set of compact operators is bounded by

$$d(T,\mathcal{K}) \leq C \limsup_{\|z\|\to 1} \|(\overline{g}\circ\phi_z - \overline{g}(z))(f\circ\phi_z - f(z))\|_p^{1/2} \tag{2.18}$$

for some $p > 2$ and the positive constant C which depend on f and g

Proof. For $0 < r < 1$, it is easy to show that the following integral operator T_r defined by

$$T_r h(z) = \int_{B_n} (\overline{g}(z) - \overline{g}(w))(f(z) - f(w))\overline{K(z,w)}\chi_{rB_n}(z)h(w)dA(w)$$

is a Hilbert-Schmidt operator because the kernel of T_r is in $L^2(B_n \times B_n)$ where $\chi_C(z)$ denotes the characteristic function of the set C. Then for any h in $L_a^2(B_n)$ and z in B_n,

$$(T - T_r)h(z) = \int_{B_n} (\overline{g}(z) - \overline{g}(w))(f(z) - f(w))\overline{K(z,w)}\chi_{(1-r)B_n}(z)h(w)dA(w)$$

Now we are going to estimate the norm of T-T_r by using the Schur test. Let L(z,w) be the kernel of T-T_r

$$(\overline{g}(z) - \overline{g}(w))(f(z) - f(w))\overline{K(z,w)}\chi_{(1-r)B_n}(z).$$

The Forelli-Rudin's inequality [R] tells us that there are positive constants ε, $1 < p < \infty$ and C which depend on f and g such that $u(z) = K(z,z)^{\varepsilon}$ satisfies the conditions of the Schur test and

$$\alpha \leq C' sup_{\|z\| \geq r} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_p,$$

and β is bounded. The Schur test gives

$$\|T - T_r\|^2 \leq C sup_{\|z\| \geq r} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_p.$$

So

$$d(T, \mathcal{K}) \leq \limsup_{\|z\| \to 1} \|T - T_r\| \leq$$

$$C \limsup_{\|z\| \to 1} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_p^{1/2}$$

which completes the proof.

Immidiately we have following corollary which gives a sufficient condition for the compactness of $H_{\overline{f}}^* H_{\overline{g}}$.

Theorem 2.10. Suppose that f and g are Bloch functions on B_n. If

$$\lim_{\|z\| \to 1} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_p = 0 \tag{2.19}$$

for some $p \geq 1$ then $H_{\overline{f}}^* H_{\overline{g}}$ is compact.

Proof. Since any Bloch functions are in L^p, for $p \geq 1$ and space of the Bloch functions is invariant under the action of $\text{Aut}(B_n)$, then the family of functions

$$\{(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\}$$

are uniformly bounded by some constant C_p in L^p.

On the other hand for $1 \leq p < p_1 < \infty$, and any function h$\in \bigcap L^p$, using Holder inequality we have

$$\|h\|_p \leq \|h\|_{p_1} \leq \|h\|_p^{1/p_1} \|h\|_{q(p_1-1)}^{(p_1-1)/p_1}$$

where $1/p + 1/q = 1$.

So

$$\lim_{\|z\|\to 1} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_p = 0$$

is equivalent to

$$\lim_{\|z\|\to 1} \|(\overline{g} \circ \phi_z - \overline{g}(z))(f \circ \phi_z - f(z))\|_{p_1} = 0.$$

The result follows from Theorem 2.8.

Remark. In [I] Izuchi gave a sufficient condition for the compactness of $H_{\overline{f}}^* H_{\overline{g}}$ for bounded holomorphic functions f and g.

We will use those results above to show several equivalent conditions about f and g such that $T_{\overline{f}} T_g - T_{\overline{f}g}$ is compact in Section 2.6.

2.4 $\mathcal{M}$-harmonic functions

In this section we will characterize functions f, g, h and l in $H^2(S_n)$ such that $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic . From now on in this we assume that f, g h and l are holomorphic on the unit ball B_n.

A polynomial P in C^n is said to be homogeneous of degree s if P(tz)=t^sP(z). If f(z) is holomorphic in a neighborhood of the origin in C^n, then the power series of f can be written in the grouped form

$$f(z) = \sum_{s=0}^{\infty} f_s(z)$$

where $f_s(z)$ is homogeneous polynomial of degree s. This is the homogeneous expansion of f.

For our convenience we define a differential operator by

$$\mathcal{L}(f,g,h,l.z.w) = (\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i})}$$
$$-(\sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial l}{\partial w_i})} - (\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}\overline{\frac{\partial g}{\partial w_i}} - \sum_{i=1}^{n} \frac{\partial h}{\partial z_i}\overline{\frac{\partial l}{\partial w_i}}) \tag{2.20}$$

for z and w in B_n.

Lemma 2.11. $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic if and only if

$$\mathcal{L}(f,g,h,l.z.w) = 0 \tag{2.21}$$

Proof. First we claim that $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic if and only if

$$\mathcal{L}(f,g,h,l.z.z) = 0 \tag{2.22}$$

It follows from the definition of $\tilde{\Delta}$ that

$$\tilde{\Delta}(f\overline{g} - h\overline{l}) = c_n(1-\|z\|^2)\mathcal{L}(f,g,h,l.z.z)$$

since f , g , h and l are holomorphic on B_n. As the factor $(1-\|z\|^2)$ is non-zero on B_n, $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic if and only if $\mathcal{L}(f,g,h,l.z.z) = 0$.

Now write f, g, h and l as the homogeneous expansions

$$f(z) = \sum_{s=1}^{\infty} f_s(z), \quad g(z) = \sum_{s=1}^{\infty} g_s(z)$$

and

$$h(z) = \sum_{s=1}^{\infty} h_s(z), \quad l(z) = \sum_{s=1}^{\infty} l_s(z).$$

Euler's theorem implies that

$$\sum_{i=1}^{n} z_i \frac{\partial f_s}{\partial z_i} = s f_s(z).$$

So the equation $\mathcal{L}(f,g,h,l.z.z) = 0$ becomes

$$\sum_{s,t=1} st(f_s(z)\overline{g_t(z)} - h_s(z)\overline{l_t(z)}) = \sum_{s,t=0}^{\infty} \sum_{i=1}^{n} (\frac{\partial f_s}{\partial z_i}\overline{\frac{\partial g_t}{\partial z_i}} - \frac{\partial h_s}{\partial z_i}\overline{\frac{\partial l_t}{\partial z_i}})$$

Let $z = \xi w$ for ξ in the unit disk and $w \in B_n$. We have

$$\sum_{s,t=1}^{\infty} st\xi^s\overline{\xi^t}(f_s(w)\overline{g_t(w)} - h_s(w)\overline{l_t(w)}) =$$

$$\sum_{s,t=0}^{\infty} \sum_{i=1}^{n} \xi^{s-1}\overline{\xi^{t-1}}(\frac{\partial f_s}{\partial w_i}\overline{\frac{\partial g_t}{\partial w_i}} - \frac{\partial h_s}{\partial w_i}\overline{\frac{\partial l_t}{\partial w_i}}).$$

Comparing the coefficients of $\xi^s\overline{\xi^t}$ implies

$$st(f_s(w)\overline{g_t(w)} - h_s(w)\overline{l_t(w)}) = \sum_{i=1}^{n}(\frac{\partial f_{s+1}}{\partial w_i}\overline{\frac{\partial g_{t+1}}{\partial w_i}} - \frac{\partial h_{s+1}}{\partial w_i}\overline{\frac{\partial l_{t+1}}{\partial w_i}}).$$

for any s and t. To take derivatives with respect to $\overline{w^\alpha}$ with $|\alpha| = t$, we see

$$st(f_s(w)\overline{D^\alpha g_t(w)} - h_s(w)\overline{D^\alpha l_t(w)}) = \sum_{i=1}^{n} \frac{\partial f_{s+1}}{\partial w_i}\overline{D^\alpha \frac{\partial g_{t+1}}{\partial w_i}} - \frac{\partial h_{s+1}}{\partial w_i}\overline{D^\alpha \frac{\partial l_{t+1}}{\partial w_i}}).$$

Since for any homogeneous polynomial P of degree t

$$P(w) = \sum_{|\alpha|=t} \frac{1}{\alpha!} D^\alpha P w^\alpha,$$

above equation implies

$$st(f_s(w)\overline{g_t(z)} - h_s(w)\overline{l_t(z)}) = \sum_{i=1}^{n} \frac{\partial f_{s+1}}{\partial w_i}\overline{\frac{\partial g_{t+1}}{\partial z_i}} - \frac{\partial h_{s+1}}{\partial w_i}\overline{\frac{\partial l_{t+1}}{\partial z_i}}.$$

To take the sum s and t we have the equation $\mathcal{L}(f,g,h,l.z.w) = 0$ for any z and w in B_n.

Lemma 2.12. For any unitary matrix U in $\mathcal{U}$(n) and f holomorphic in B_n, the gradient of foU is expressed as

$$\nabla(f \circ U)(z) = U^T(\nabla f) \circ U(z) \tag{2.23}$$

Proof. For U in $\mathcal{U}$(n), U induces a linear map from B_n onto itself. Let w=Uz. The chain rule gives

$$\frac{\partial(f \circ U)}{\partial z_i} = \sum_j^n \frac{\partial f}{\partial w_j}\frac{\partial w_j}{\partial z_i}.$$

Hence

$$\nabla(f \circ U)(z) = (\frac{\partial w_i}{\partial z_j})(\nabla f)(w)$$

$$= U^T(\nabla f)(w) = U^T(\nabla f) \circ U(z).$$

Lemma 2.13. Let H=$\{a \in C^n : a^*\nabla f(w) = 0 \ for \ w \in C^n\}$. Then H is a subspace of C^n and there is an orthonormal base $\{e_{k+1}, \cdots, e_n\}$ of H which extends as an orthonormal base $\{e_1, \cdots, e_n\}$ of C^n. Let $U = \overline{(e_1, \cdots, e_n)}$ which is a unitary matrix. Then $\frac{\partial(f \circ U)}{\partial z_i} = 0$ for $i > k$ and $\frac{\partial(f \circ U)}{\partial z_1}, \cdots \frac{\partial(f \circ U)}{\partial z_k}$ are linearly independent over C^n

Proof. From the definition of H we see that H is a subspace of C^n. So there is an orthonormal base $\{e_{k+1}, \cdots, e_n\}$ of H which extends as an orthonormal base $\{e_1, \cdots, e_n\}$ of C^n. Let $U = \overline{(e_1, \cdots, e_n)}$ which is a unitary matrix. Lemma 2.12 tells us that $\frac{\partial(f \circ U)}{\partial z_i} = e_i^*(\nabla f) \circ (Uz)$ for any i . Hence $\frac{\partial(f \circ U)}{\partial z_i} = 0$ for $i > k$. The rest is to show that $\frac{\partial(f \circ U)}{\partial z_1}, \cdots \frac{\partial(f \circ U)}{\partial z_k}$ are linearly independent over C^n. Suppose that there are constants c_i such that

$$\sum_{i=1}^k c_i \frac{\partial(f \circ U)}{\partial z_i} = 0.$$

Then

$$\sum_{i=1}^{k} c_i e_i^*(\nabla f)(Uz) = 0.$$

Therefore $\sum_{i=1}^{k} \overline{c_i} e_i \in$ H, but e_i is orthogonal to H for $i = 1, \cdots, k$. So $c_i = 0$ for $i = 1, \cdots, k$, which completes the proof.

Lemma 2.14. If $c = (c_1, \cdots, c_k) \in C^k$ and

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = \sum_{i=1}^{k} c_i \frac{\partial f}{\partial z_i}, \tag{2.24}$$

then there is a unitary matrix

$$U = U_k \bigoplus I_{n-k}$$

such that

$$(Uz)^T \nabla (f \circ U^*)(Uz) = \|c\| \frac{\partial (f \circ U^*)}{\partial z_1}. \tag{2.25}$$

Proof. It follows from linear algebra that there is a $k \times k$ unitary matrix U_k such that

$$C^T (U_k^*)^T = (\|c\|, 0, \cdots, 0).$$

Set

$$U = U_k \bigoplus I_{n-k}.$$

Lemma 2.13 implies

$$(Uz)^T \nabla (f \circ U^*)(Uz) = z^T \nabla f(z)$$

$$= (c^T, 0) \nabla f(z) = (c^T, 0)(U^*)^T \nabla (f \circ U^*)(Uz)$$

$$= (\|c\|, 0) \nabla (f \circ U^*)(Uz) = \|c\| \frac{\partial (f \circ U^*)}{\partial z_1}.$$

Lemma 2.15. The equation

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = \sum_{i=1}^{\infty} c_i \frac{\partial f}{\partial z_i} \tag{2.26}$$

does not have nontrivial solutions in $H^{2n}(B_n)$ if $c = (c_1, \cdots, c_n)$ is in the closure of the unit ball B_n.

Proof. Case 1. If $c = (c_1, \cdots, c_n)$ is in B_n, suppose that the equation has a holomorphic solution f on B_n. Making the change of variables $w = z - c$ we have

$$\sum_{i=1}^{n} w_i \frac{\partial f}{\partial w_i}(w + c) = 0.$$

Set $g(w) = f(w + c)$, then the above equation becomes

$$\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(w) = 0.$$

Because f is holomorphic on B_n and c is in B_n, there is a neighborhood N of 0 in B_n such that g is holomorphic on N and satisfies

$$\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}(w) = 0$$

in N. If we write g as a homogeneous expansion, Euler's theorem implies

$$\sum_{s=0}^{\infty} s g_s(w) = 0$$

for w in N. Hence $g_s(w) = 0$ for $s > 0$. This means that g is constant on N. Since f is holomorphic on B_n and N is an open subset of B_n, f is constant on B_n.

Case 2. If c is in S_n, by Lemma 2.14 we consider the following equation

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial z_1}. \tag{2.27}$$

f has a homogeneous expansion

$$f(z) = \sum_{t=0}^{\infty}\sum_{s=0}^{\infty} f_{t,s}(z_2, \cdots, z_n) z_1^t$$

where $f_{t,s}(z_2, \cdots, z_n)$ is homogeneous with degree s in $z_2, \cdots, z_n$. The Euler's theorem and Equation (2.27) imply that

$$\sum_{t=0}^{\infty}\sum_{s=0}^{\infty}(s+t) f_{t,s}(z_2, \cdots, z_n) z_1^t = \sum_{t=0}^{\infty}\sum_{s=0}^{\infty} t f_{t,s}(z_2, \cdots, z_n) z_1^{t-1}.$$

Comparing the coefficients of powers of z_1 in the above equation gives

$$(s+t) f_{t,s}(z_2, \cdots, z_n) = (t+1) f_{t+1,s}(z_2, \cdots, z_n).$$

So we have the following reduction formula for t

$$f_{t+1,s}(z_2, \cdots, z_n) = \frac{s+t}{t+1} f_{t,s}(z_2, \cdots, z_n).$$

Iterating the formula we get

$$f_{t,s}(z_2, \cdots, z_n) = \frac{(s+t-1)!}{t!s!} f_{0,s}(z_2, \cdots, z_n).$$

Now f can be expressed as

$$\begin{aligned} f(z) &= \sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \frac{(s+t-1)!}{t!s!} f_{0,s}(z_2, \cdots, z_n) z_1^t \\ &= \sum_{s=0}^{\infty}\left(\sum_{t=0}^{\infty} \frac{(s+t-1)!}{t!s!} z_1^t\right) f_{0,s}(z_2, \cdots, z_n) \\ &= \sum_{s=0}^{\infty} \frac{f_{0,s}(z_2, \cdots, z_n)}{(1-z_1)^s}. \end{aligned}$$

Fo the sake of simplicity, we let f_s be $f_{0,s}$.

Since $f_s(z_2, \cdots, z_n)$ is a homogeneous polynomial, we can write it as

$$f_s(z_2, \cdots, z_n) = \sum_{|\alpha|=s} a_{\alpha s} z^{(0,\alpha)}.$$

So

$$\|f\|_2^2 = \sum_{s=0}^{\infty} \sum_{|\alpha|=s} \sum_{t=0}^{\infty} (\frac{(s+t-1)!}{t!s!})^2 |a_{\alpha s}|^2 \|z^{(t,\alpha)}\|_2^2.$$

It follows from Proposition 2.5 that

$$\|f\|_2^2 = \sum_{s=0}^{\infty} \sum_{|\alpha|=s} \sum_{t=0}^{\infty} (\frac{(s+t-1)!}{t!s!})^2 \frac{(n-1)!\alpha!t!}{(n-1+s+t)!} |a_{\alpha s}|^2. \tag{2.28}$$

As t is very large,

$$(\frac{(s+t-1)!}{t!})^2 \frac{\alpha!t!}{(n-1+s+t)!}$$

is asymptotically equivalent to

$$\frac{1}{t^{n-s+1}}.$$

In order that the series (2.28) converges , the p-series theorem implies $n-s+1>1$. Therefore

$$f(z) = \sum_{s \leq s_0 < n} \frac{f_s(z_2, \cdots, z_n)}{(1-z_1)^s}.$$

On the other hand f^n is in $H^2(S_n)$. However the term with the highest degree of $\frac{1}{(1-z_1)}$ in f^n is

$$\frac{f^n_{s_0}(z_2, \cdots, z_k)}{(1-z_1)^{s_0(n)}}.$$

So $s_0 n < n$, thus $s_0 = 0$. This means that f is constant.

Lemma 2.16. The following equation

$$\frac{\partial}{\partial z_j} (\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}) \overline{(\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i})} = \frac{\partial}{\partial z_j} \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \overline{\frac{\partial g}{\partial w_i}} \tag{2.29}$$

does not have solutions in $H^{2n}(B_n)$ unless either $\frac{\partial f}{\partial z_j} = 0$ is zero or g is constant for any fixed j.

Proof. If g is not constant, then there is a vector a$\in C^n$, which is orthogonal to A=$\{c \in C^n : c^*\nabla\frac{\partial f}{\partial z_j} = 0\}$, such that

$$\frac{\partial}{\partial z_j}(\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}) = a^*\nabla\frac{\partial f}{\partial z_j}. \tag{2.30}$$

On the other hand, from Equation (2.29) we see that

$$a^*\nabla\frac{\partial f}{\partial z_j}\overline{(\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i})} = \nabla g(w)^*\nabla\frac{\partial f}{\partial z_j}.$$

So

$$a\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i} - \nabla g(w)$$

is in A for any w in the unit ball B_n. Since a is orthogonal to A, we have

$$\|a\|^2\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i} - a^*\nabla g(w) = 0.$$

Lemma 2.15 implies that $\frac{a}{\|a\|^2}$ is not in B_n unless g is constant. If g is not constant, then $\frac{a}{\|a\|^2}$ is not in the closure of the unit ball B_n. This implies a is in B_n. However Equation (9) implies

$$\frac{\partial}{\partial z_j}\sum_{i=1}^{n}(z_i - \overline{a_i})\frac{\partial f}{\partial z_i} = 0.$$

Making change of variables by $w = z - \overline{a}$ we have

$$\frac{\partial}{\partial w_j}\sum_{i=1}^{n} w_i \frac{\partial f}{\partial w_i}(w + \overline{a}) = 0.$$

Set $F(w) = f(w + \overline{a})$, then the above equation becomes

$$\frac{\partial}{\partial w_j}\sum_{i=1}^{n} w_i \frac{\partial F}{\partial w_i}(w) = 0.$$

Since f is holomorphic on the unit ball B_n and a is in B_n, there is a neighborhood N of 0 in B_n such that F is holomorphic on N and satisfies

$$\frac{\partial}{\partial w_j}\sum_{i=1}^{n} w_i \frac{\partial F}{\partial w_i}(w) = 0$$

on N. If we write F as a homogeneous expansion, Euler's theorem implies

$$\frac{\partial}{\partial w_j}\sum_{s=0}^{\infty} sF_s(w) = 0.$$

Thus $\frac{\partial F_s}{\partial w_j}(w) = 0$ on N. Since f is holomorphic on the unit ball and N is an open subset of the unit ball, then $\frac{\partial f}{\partial z_j}(z) = 0$ on the unit ball. So this completes the proof.

Lemma 2.17. Suppose that neither g nor l is constant and $\mathcal{L}(f, g, h, l, z.w) = 0$. For any vector a in C^n, $a^*\nabla f = 0$ implies $a^*\nabla h = 0$.

Proof. Lemma 2.13 implies that it suffices to show that $\frac{\partial f}{\partial z_1} = 0$ implies $\frac{\partial h}{\partial z_1} = 0$ since the space of $\mathcal{M}$-harmonic functions is invariant under the action of the unitary matrices. To take derivative with respect to z_1 in the equation $\mathcal{L}(f, g, h, l, z.w) = 0$ we have

$$\frac{\partial}{\partial z_1}(\sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial l}{\partial w_i})} = \frac{\partial}{\partial z_1}\sum_{i=1}^{n}\frac{\partial h}{\partial z_i}\overline{\frac{\partial l}{\partial w_i}}$$

since $\frac{\partial f}{\partial z_1} = 0$. It follows from Lemma 2.16 that $\frac{\partial h}{\partial z_1} = 0$ because l is not constant.

Lemma 2.18. If $\mathcal{L}(f, g, h, l, z, w) = 0$, then there is a unitary matrix U such that

$$(\sum_{i=1}^{n} z_i \frac{\partial (f \circ U)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial (g \circ U)}{\partial w_i})} - (\sum_{i=1}^{n} z_i \frac{\partial (h \circ U)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial (l \circ U)}{\partial w_i})} \quad (2.31)$$
$$= \sum_{i=1}^{r}(\frac{\partial (f \circ U)}{\partial z_i}\overline{\frac{\partial (g \circ U)}{\partial w_i}} - \frac{\partial (h \circ U)}{\partial z_i}\overline{\frac{\partial (l \circ U)}{\partial w_i}})$$

and $\{\frac{\partial(f\circ U)}{\partial z_1}, \cdots, \frac{\partial(f\circ U)}{\partial z_r}\}$ are linearly independent and $\{\frac{\partial(g\circ U)}{\partial w_1}, \cdots, \frac{\partial(g\circ U)}{\partial w_r}\}$ linearly independent where $0 \leq r \leq n$.

Proof. Lemma 2.13 says that there is a unitary matrix V such that $\{\frac{\partial(f\circ V)}{\partial z_1}, \cdots, \frac{\partial(f\circ V)}{\partial z_{r_1}}\}$ are linearly independent and $\frac{\partial(f\circ V)}{\partial z_i} = 0$ for $i > r_1$. So Lemma 2.17 implies

$$(\sum_{i=1}^{n} z_i \frac{\partial(f\circ V)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial(g\circ V)}{\partial w_i})} - (\sum_{i=1}^{n} z_i \frac{\partial(h\circ V)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial(l\circ V)}{\partial w_i})}$$

$$= \sum_{i=1}^{r_1}(\frac{\partial(f\circ V)}{\partial z_i}\overline{\frac{\partial(g\circ V)}{\partial w_i}} - \frac{\partial(h\circ V)}{\partial z_i}\overline{\frac{\partial(l\circ V)}{\partial w_i}}).$$

If $\{\frac{\partial(g\circ V)}{\partial w_1}, \cdots, \frac{\partial(g\circ V)}{\partial w_{r_1}}\}$ are linearly independent, then we are done. Otherwise let $A = \{a \in C^{r_1} : \sum_{i=1}^{r_1} a_i \frac{\partial(g\circ V)}{\partial w_i} = 0\}$. Repeating the argument in the proof of Lemma 2.13 implies that there is a r_1 by r_1 unitary matrix U_2 such that $\{\frac{\partial(g\circ V)\circ U_1}{\partial w_1}, \cdots, \frac{\partial(g\circ V)\circ U_1}{\partial w_r}\}$ are linearly independent and $\frac{\partial(g\circ V)\circ U_1}{\partial w_i} = 0$ for $i = r + 1 tor_1$ where $U_1 = U_2 \bigoplus I$. It is also easy to check that $\{\frac{\partial(f\circ V)\circ U_1}{\partial z_1}, \cdots, \frac{\partial(f\circ V)\circ U_1}{\partial z_r}\}$ are linearly independent. Then L[illegible]

$$(\sum_{i=1}^{n} z_i \frac{\partial(f\circ U)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial(g\circ U)}{\partial w_i})} - (\sum_{i=1}^{n} z_i \frac{\partial(h\circ U)}{\partial z_i})\overline{(\sum_{i=1}^{n} w_i \frac{\partial(l\circ U)}{\partial w_i})}$$

$$= \sum_{i=1}^{r}(\frac{\partial(f\circ U)}{\partial z_i}\overline{\frac{\partial(g\circ U)}{\partial w_i}} - \frac{\partial(h\circ U)}{\partial z_i}\overline{\frac{\partial(l\circ U)}{\partial w_i}})$$

where $U = V \circ U_1$.

Theorem 2.19. Suppose that f, g, h, and l are in $H^{2n}(B_n)$. Then $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic on the unit ball if and only if f, g, l, and l satisfy one of the following conditions

(1) Both f and h are constants;

(2) Both f and l are constants;

(3) Both g and l are constants;

(4) Both g and h are constants;

(5) There is a nonzero constant b such that bg-l and $f - \bar{b}h$ are constant.

Proof. It follows from Lemmas 1 and 2 that $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic if and only if

$$\mathcal{L}(f, g, h, l, z, w) = 0.$$

It is easy to see that if these four functions satisfy one of the conditions in the theorem , they are solutions of $\mathcal{L}(f, g, h, l, z, w) = 0$. So we are going to show that any solutions in $H^{2n}(B_n)$ of $\mathcal{L}(f, g, h, l, z, w) = 0$ must satisfy one of the conditions in the theorem.

By Lemma 2.18 we may assume that $\{\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_r}\}$ are linearly independent and $\{\frac{\partial g}{\partial w_1}, \cdots, \frac{\partial g}{\partial w_r}\}$ are linearly independent and

$$\mathcal{L}(f, g, h, l, z, w) = 0$$

is equivalent to

$$(\sum_{i=1}^{n} \frac{\partial f}{\partial z_i})\overline{(\sum_{i=1}^{n} \frac{\partial g}{\partial w_i})} - (\sum_{i=1}^{n} \frac{\partial h}{\partial z_i})\overline{(\sum_{i=1}^{n} \frac{\partial l}{\partial w_i})}$$

$$= \sum_{i=1}^{r}(\frac{\partial f}{\partial z_i}\overline{\frac{\partial g}{\partial w_i}} - \frac{\partial h}{\partial z_i}\overline{\frac{\partial l}{\partial w_i}}).$$

Let $H = \{c \in C^r : c^*\nabla_r f \in span\{\frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_r}\}\}$ where $\nabla_r f = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_r})$.

Now we consider three cases.

Case 1. If $dimH_1 = r$, there is a matrix A such that

$$\nabla_r f = A\nabla_r h.$$

The Jordan reduction to form implies that there is a unitary matrix U such that

$$U^T A (U^T)^* = J_1 \bigoplus \cdots \bigoplus J_k$$

where J_i is a s_i by s_i matrix

$$\begin{pmatrix} c_i & 0 & 0 & \cdots & 0 & 0 \\ 1 & c_i & 0 & \cdots & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & c_i \end{pmatrix}$$

and $\sum_{i=1}^{k} s_i = n$. For the sake of simplicity we still use f, g, h, and l to denote respectively $f \circ U, g \circ U, h \circ U$ and $l \circ U$. So Combining Lemma 2.12 with above equations gives

$$\frac{\partial f}{\partial z_1} = c_1 \frac{\partial h}{\partial z_1}$$

$$\frac{\partial f}{\partial z_2} = c_1 \frac{\partial h}{\partial z_2} + \frac{\partial h}{\partial z_1}$$

$$\vdots$$

To take derivative with respect to z_1 we have the following equation

$$\left(\frac{\partial}{\partial z_1} \sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i}\right)\overline{\left(\sum_{i=1}^{n} w_i \frac{\partial (c_1 g - l)}{\partial w_i}\right)} = \frac{\partial}{\partial z_1} \sum_{i=1}^{n} \frac{\partial h}{\partial z_i} \overline{\frac{\partial (c_1 g - l)}{\partial w_i}}.$$

It follows from Lemma 2.16 that either $c_1 g - l$ is constant or $\frac{\partial h}{\partial z_1} = 0$. Thus if

$c_1 g - l$ is not constant then $\frac{\partial h}{\partial z_1} = 0$. So

$$\frac{\partial f}{\partial z_2} = c_1 \frac{\partial h}{\partial z_2}.$$

Using induction we can show that both f and h are constants if $c_i g - l$ are not constants for all i. So f,g,h,and l satisfy (1)

In the case $c_j g - l$ is constant for some j and g is not constant, then $\mathcal{L}(f, g, h, l, z, w) = 0$ implies

$$(\sum_{i=1}^{n} z_i \frac{\partial f - \overline{c_j} h}{\partial z_i})(\overline{\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}}) = \frac{\partial}{\partial z_1} \sum_{i=1}^{n} \frac{\partial f - \overline{c_j} h}{\partial z_i} \overline{\frac{\partial g}{\partial w_i}}.$$

It follows again from Lemma 2.16 that $f - \overline{c_j} h$ is constant.

If the $c_j \neq 0$ then f, g, h, and l satisfy (4).

If the $c_j = 0$ then l is constant and either g or f is constant. i.e. f, g, h, and l satisfy either (2) or (3).

(b) If r is zero then we have the following equation

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} \overline{\sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i}} - \sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i} \overline{\sum_{i=1}^{n} w_i \frac{\partial l}{\partial w_i}} = 0.$$

So if neither g nor l is constant and neither f nor g is constant, there is a nonzero constant b such that

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i} = \overline{b} \sum_{i=1}^{n} \frac{\partial h}{\partial z_i}$$

and

$$b \sum_{i=1}^{n} \frac{\partial g}{\partial w_i} = \sum_{i=1}^{n} \frac{\partial l}{\partial w_i}$$

Therefore Euler's theorem implies that both bg-l and $f - \overline{b} h$ are constant.

Therefore we consider the situation where these four functions satisfy one of the conditions in the theorem.

Case 2. $dimH < r-1$. In the case we may assume that $\{\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_k}\}$ are linearly independent of $\{\frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_r}\}$ and $\{\frac{\partial f}{\partial z_{k+1}}, \cdots, \frac{\partial f}{\partial z_r}\}$ are in the space spaned by $\{\frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_r}\}$ for some $k > 1$.

So there are vectors a , a′, c , c′ and a matrix B such that

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = a^* \nabla_k f + c^* \nabla_r h$$

$$\sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i} = (a')^* \nabla_k f + (c')^* \nabla_r h$$

and

$$(\frac{\partial f}{\partial z_{k+1}}, \cdots, \frac{\partial f}{\partial z_n})^T = (\nabla_r h)^T B^*.$$

Since $\mathcal{L}(f,g,h,l,z,w) = 0$, then

$$a \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i} - a' \sum_{i=1}^{n} w_i \frac{\partial l}{\partial w_i} = \nabla_k g$$

and

$$c \sum_{i=1}^{n} w_i \frac{\partial g}{\partial w_i} - c' \sum_{i=1}^{n} w_i \frac{\partial l}{\partial w_i} = B(\frac{\partial g}{\partial w_{k+1}}, \cdots, \frac{\partial w}{\partial w_n})^T - \nabla_r l.$$

From the above equations we see that $\{\frac{\partial l}{\partial w_1}, \cdots, \frac{\partial l}{\partial w_r}\}$ are in the span$\{\frac{\partial g}{\partial w_1}, \cdots, \frac{\partial g}{\partial w_r}\}$. So this case is reduced to case 1 if we replace f and h respectively by g and l.

Case 3. $dimH = r-1$.

In the case by Le[illegible] may assume that $\frac{\partial f}{\partial z_1}$ is independent of $\{\frac{\partial h}{\partial z_1}, \cdots, \frac{\partial h}{\partial z_r}\}$ which are linearly independent, and there is a (r-1)×r matrix A such that

$$(\frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_r})^T = A \nabla_r h.$$

Since $\mathcal{L}(f,g,h,l,z,w) = 0$, if g and l are not affinely dependent, then there are constants c_1, c_2 and vectors a and a′ such that

$$\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = c_1 \frac{\partial f}{\partial z_1} + a^* \nabla_r h$$

and

$$\sum_{i=1}^{n} z_i \frac{\partial h}{\partial z_i} = c_2 \frac{\partial f}{\partial z_1} + (a')^* \nabla_r h.$$

We will reduce this case to Case 1. Before doing this we extend a holomorphic function f on B_n to a holomorphic function $\tilde{f}$ on B_{n+1} by

$$\tilde{f}(z_1, \cdots, z_n, z_{n+1}) = f(z_1, \cdots, z_n).$$

The following claim easily follows from Lemma 2.11.

Claim. If $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic, then $\tilde{f}\overline{(\tilde{g})} - \tilde{h}\overline{(\tilde{l})}$ is $\mathcal{M}$-harmonic.

For c in the unit disk we define

$$\phi_c(z_1, \cdots, z_n, z_{n+1}) = (\frac{z_1 s}{1 - \overline{c}z_{n+1}}, \cdots, \frac{z_n s}{1 - \overline{c}z_{n+1}}, \frac{c - z_{n+1}}{1 - \overline{c}z_{n+1}})$$

where $s = (1 - |c|^2)^{1/2}$, thus $\phi_c \in Aut(B_{n+1})$. Since the space of $\mathcal{M}$-harmonic functions is invariant under the action of $Aut(B_{n+1})$, then $\tilde{f} \circ \phi_c \overline{\tilde{g} \circ \phi_c} - \tilde{h} \circ \phi_c \overline{\tilde{l} \circ \phi_c}$ is $\mathcal{M}$-harmonic.

On the other hand it is easy to check that

$$\frac{\partial(\tilde{f} \circ \phi_c)}{\partial z_{n+1}} = \sum_{i=1}^{n} \frac{\overline{c} s z_i}{(1 - \overline{c}z_{n+1})^2} \frac{\partial \tilde{f}}{\partial z_i} \circ \phi_c,$$

then

$$\frac{\partial(\tilde{f} \circ \phi_c)}{\partial z_{n+1}} = \frac{c}{(1 - \overline{c}z_{n+1})} \sum_{i=1}^{n} (\phi_c)_i \frac{\partial f}{\partial z_i}((\phi_c(z))_i, \cdots, (\phi_c(z))_n).$$

Because $\sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i} = c_1 \frac{\partial f}{\partial z_1} + a^* \nabla_r h$, we have

$$\sum_{i=1}^{n} (\phi_c)_i \frac{\partial f}{\partial z_i}((\phi_c(z))_i, \cdots, (\phi_c(z))_n) = c_1 \frac{\partial \tilde{f}}{\partial z_1} \circ \phi_c + a^* \nabla_r h((\phi_c(z))_i, \cdots, (\phi_c(z))_n).$$

So

$$\frac{\partial(\tilde{f} \circ \phi_c)}{\partial z_{n+1}} = \frac{\overline{c}}{s} c_1 \frac{\partial(\tilde{f} \circ \phi_c)}{\partial z_1} + \frac{\overline{c}}{s} a^* \nabla_r (\tilde{h} \circ \phi_c).$$

Similarly we can show

$$\frac{\partial(\tilde{h}\circ\phi_c)}{\partial z_{n+1}} = \frac{\overline{c}}{s}c_2\frac{\partial(\tilde{f}\circ\phi_c)}{\partial z_1} + \frac{\overline{c}}{s}(a')^*\nabla_r(\tilde{h}\circ\phi_c).$$

(a) If $c_2 \neq 0$ set $F = \tilde{f}\circ\phi_c$, $G = \tilde{g}\circ\phi_c$, and $H = \tilde{h}\circ\phi_c$, $L = \tilde{l}\circ\phi_c$, then the above two equations tell us that $\frac{\partial F}{\partial z_i}$ is in the span$\{\frac{\partial H}{\partial z_1}, \cdots, \frac{\partial H}{\partial z_r}, \frac{\partial H}{\partial z_{n+1}}\}$ for $i = 1, \cdots, r$, and $n+1$, and

$$\mathcal{L}(F, G, H, L, \tilde{z}, \tilde{w}) = \sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial z_i}\overline{\sum_{i=1}^{n+1} w_i \frac{\partial G}{\partial w_i}} - \sum_{i=1}^{n+1} z_i \frac{\partial H}{\partial z_i}\overline{\sum_{i=1}^{n+1} w_i \frac{\partial L}{\partial w_i}}$$

$$-\sum_{i=1}^{r}\left(\frac{\partial F}{\partial z_i}\overline{\frac{\partial G}{\partial w_i}} - \frac{\partial H}{\partial z_i}\overline{\frac{\partial L}{\partial w_i}}\right) - \left(\frac{\partial F}{\partial z_{n+1}}\overline{\frac{\partial G}{\partial w_{n+1}}} - \frac{\partial H}{\partial z_{n+1}}\overline{\frac{\partial L}{\partial w_{n+1}}}\right).$$

where $\tilde{z} = (z, z_{n+1})$. So repeating the argument in Case 1 we see that F,G, H, and G must satisfy one of the conditions of Theorem 2.19. Since f, g, h, and l are holomorphic on the unit ball then they must satisfy one of the conditions of the theorem.

(b) If $c_2 = 0$, then

$$\frac{\partial\tilde{h}\circ\phi_c}{\partial z_{n+1}} = \frac{c}{s}(a')^*\nabla_r(\tilde{h}\circ\phi_c).$$

So we have

$$\overline{c_1}\sum_{i=1}^{n} w_i\frac{\partial g}{\partial w_i} = \frac{\partial g}{\partial w_1}$$

and

$$a\sum_{i=1}^{n} w_i\frac{\partial g}{\partial w_i} - a'\sum_{i=1}^{n} w_i\frac{\partial l}{\partial w_i} =$$

$$A^*\left(\frac{\partial g}{\partial w_2}, \cdots, \frac{\partial g}{\partial w_r}\right)^T - \nabla_r l.$$

Repeating the argument in (a) we have

$$\overline{c_1}\frac{\partial(\tilde{g}\circ\phi_c)}{\partial w_{n+1}} = \frac{\overline{c}}{s}\frac{\partial(\tilde{g}\circ\phi_c)}{\partial w_1}.$$

If neither f nor h is constant it follows from Lemma 2.17 that

$$\overline{c_1}\frac{\partial(\tilde{l}\circ\phi_c)}{\partial w_{n+1}} = \frac{\overline{c}}{s}\frac{\partial(\tilde{l}\circ\phi_c)}{\partial w_1}.$$

So

$$\overline{c_1}\sum_{i=1}^{n} w_i\frac{\partial l}{\partial w_i} = \frac{\partial l}{\partial w_1}.$$

Therefore

$$(\frac{a}{\overline{c_1}}, -A^*)\nabla_r g = \begin{pmatrix} \frac{a_1'}{\overline{c_1}} - 1 & 0 & 0 & \cdots & 0 \\ \frac{a_2'}{\overline{c_1}} & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_r'}{\overline{c_1}} & 0 & 0 & \cdots & -1 \end{pmatrix}\nabla_r l$$

If $\frac{a_1'}{\overline{c_1}} - 1 \neq 0$, from the above equations we see that $\frac{\partial l}{\partial w_i}$ is in the span$\{\frac{\partial g}{\partial w_1}, \cdots, \frac{\partial g}{\partial w_1}\}$, so this is reduced to Case 1 for g and l.

If $\frac{a_1'}{\overline{c_1}} - 1 = 0$, it follows from above equations that

$$\frac{a_1}{\overline{c_1}}\frac{\partial g}{\partial w_1} - \overline{a_{21}}\frac{\partial g}{\partial w_2} - \cdots - \overline{a_{r1}}\frac{\partial g}{\partial w_r} = 0.$$

Since $\{\frac{\partial g}{\partial w_1}, \cdots, \frac{\partial g}{\partial w_r}\}$ are linearly independent then $a_{i1} = 0$ for $i = 2, \cdots, r$. Thus

$$(\frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_r})^T = (0, A_{(r-1)\times(r-1)})\nabla_r h.$$

So

$$(\frac{\partial f}{\partial z_2}, \cdots, \frac{\partial f}{\partial z_r})^T = A_{(r-1)\times(r-1)}(\frac{\partial h}{\partial z_2}, \cdots, \frac{\partial h}{\partial z_r})^T.$$

Repeating the argument in Case 1, we see that either both g and l are constants, or there is a nonzero constant b such that $\overline{b}g - l$ is constant, or $\frac{\partial f}{\partial z_2} = \cdots = \frac{\partial f}{\partial z_r}$. By our hypothesis that $\{\frac{\partial f}{\partial z_1} \cdots \frac{\partial f}{\partial z_r}\}$ are linearly independent we have that both g and l are constants.

2.5 Commuting Toeplitz operators on the Bergman space

Theorem 2.20. Suppose that f , g, h and l are in $H^2(S)$ with the property

$$\int_{B_n} (f\overline{g} - h\overline{l}) \circ \psi(w)|k_z(w)|^2 dA(w) = (f\overline{g} - h\overline{l}) \circ \psi(z) \tag{2.32}$$

for all $\psi \in Aut(B_n)$. Then $f\overline{g} - h\overline{l}$ is $\mathcal{M}$- harmonic on the unit ball B_n.

Proof: Let d$\mathcal{U}$ be Haar measure on the unitary group $\mathcal{U}$. Set

$$G(z) = \int_{\mathcal{U}} (f\overline{g} - h\overline{l})(Uz) d\mathcal{U}$$

for z in B_n. For any f in $H^2(S)$, f can be expressed in power series

$$f(z) = \sum_{\alpha} a_\alpha(f) z^\alpha$$

with

$$\sum_{\alpha} |a_\alpha(f)|^2 I(\alpha) < +\infty$$

where $I(\alpha) = \|z^\alpha\|^2_{H^2(S)}$.

By the hypothesis we have

$$\int_B G(w)|k_z(w)|^2 dA(w) = G(z). \tag{2.33}$$

It is easy to check that

$$G(z) = \int_S (f\overline{g} - h\overline{l})(\|z\|\xi)d\sigma(\xi)$$

$$= \sum_\alpha (a_\alpha(f)\overline{a(g)_\alpha} - a_\alpha(h)\overline{a(l)_\alpha})I(\alpha)\|z^\alpha\|^{2|\alpha|}$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$.

Since f , g, h and l are in $H^2(S)$ then G(z) is continuous on $\overline{B}_n$ and $Aut(B_n) = \{\phi : B_n \to B_n | \phi(z) = U\phi_a(z)\}$ where U is a n by n unitary matrix and ϕ_a is a mobius transformation , which depends on ϕ. The mean value theorem tells us that $\tilde{\Delta}G = 0$. In addition G(z) is constant on S. It follows from Maximal Principle that G(z) is constant on B_n. Thus

$$\sum_{|\alpha|=1} a_\alpha(f)\overline{a(g)}_\alpha - a_\alpha(h)\overline{a(l)}_\alpha = 0. \tag{2.34}$$

For convenience we use $a_i(f)$ to denote $a_{(0,\cdots,0,1,0,\cdots,0)}(f)$. Then (2.34) becomes

$$\sum_i a_i(f)\overline{a(g)}_i - a_i(h)\overline{a(l)}_i = 0.$$

It is to easy check that $a_i(f) = \frac{\partial f}{\partial z_i}(0)$. Set $a_i(f)(w) = \frac{\partial f \circ \phi_w}{\partial z_i}(0)$. Since $Aut(B_n)$ is a group, similarly we can prove that

$$\sum_i a_i(f)(z)\overline{a(g)}_i(z) - a_i(h)(z)\overline{a(l)}_i(z) = 0. \tag{2.35}$$

Computing $a_i(f)(w)$ implies that

$$a(f)(z) = s(1-s)P_{\overline{z}}\nabla_z f - s\nabla_z f \tag{2.36}$$

where

$$a(f)(z) = \begin{pmatrix} a_1(f)(z) \\ \vdots \\ a_n(f)(z) \end{pmatrix}.$$

Replacing $a_i(f)(z)$ in (2.35) by the right hand side of (2.36) gives

$$0 = [s(1-s)P_{\overline{z}}\nabla_z g - \nabla_z g]^*[s(1-s)P_{\overline{z}}\nabla_z f$$

$$-s\nabla_z f] - [s(1-s)P_{\overline{z}}\nabla_z l - s\nabla_z l]^*[s(1-s)P_{\overline{z}}\nabla_z h - s\nabla_z h]$$

$$= s^2(1-s)(-1-s)[\nabla g]^* P_{\overline{z}}\nabla f + s^2[\nabla g]^*\nabla f$$

$$-s^2(1-s)(-1-s)[\nabla l]^* P_{\overline{z}}\nabla h + s^2[\nabla l]^*\nabla h$$

Thus

$$-s^2(1-s^2) < \overline{z}, \nabla g >< \nabla f, \overline{z} > +s^2(1-s^2) < \nabla f, \nabla g >=$$

$$-s^2(1-s^2) < \overline{z}, \nabla l >< \nabla h, \overline{z} > +s^2(1-s^2) < \nabla h, \nabla l >$$

Multiplying by $\frac{1}{s^2(1-s^2)}$ we get

$$< \overline{z}, \nabla g >< \nabla f, \overline{z} > - < \overline{z}, \nabla l >< \nabla h, \overline{z} >=$$

$$< \nabla f, \nabla g > - < \nabla h, \nabla l >$$

which is

$$(\sum_j z_j \frac{\partial f}{\partial z_j})(\sum_i \overline{z_i \frac{\partial g}{\partial z_i}}) - (\sum_j z_j \frac{\partial h}{\partial z_j})(\sum_i \overline{z_i \frac{\partial l}{\partial z_i}}) =$$

$$\sum_j \frac{\partial f}{\partial z_j} \overline{\frac{\partial g}{\partial z_j}} - \sum_j \frac{\partial h}{\partial z_j} \overline{\frac{\partial l}{\partial z_j}}$$

It follows from Lemma 2.11 that $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic.

Theorem 2.21. Suppose that ϕ and ψ are bounded pluriharmonic functions on the unit ball. Then

$$T_\phi T_\psi = T_\psi T_\phi \tag{2.37}$$

on the Bergman space $L_a^2(B_n)$ of the unit ball if and only if ϕ and ψ satisfy one of the following conditions:

(1) Both ϕ and ψ are holomorphic on the unit ball B_n;

(2) Both $\overline{\phi}$ and $\overline{\psi}$ are holomorphic on the unit ball B_n;

(3) Either ϕ or ψ is constant;

(4) There is a nonzero constant b such that ϕ -bψ is constant.

Proof. Suppose that ϕ and ψ are bounded and pluriharmonic on the unit ball. It follows from Proposition 2.6 that there are four functions f, g, h, and l in $H^{2n}(S_n)$ such that

$$\phi = f + \overline{l}$$

and

$$\psi = h + \overline{g}.$$

First we prove that if ϕ and ψ satisfy one of the conditions in the theorem , then T_ϕ commutes with T_ψ.

If ϕ is holomorphic on B_n, then T_ϕ is the operator on the Bergman space $L_a^2(B_n)$ of multiplication by ϕ. So if ϕ and ψ satisfy (1), then T_ϕ commutes with T_ψ. Since the adjoint of T_ϕ on the Bergman space is $T_{\overline{\phi}}$, then T_ϕ commutes with T_ψ if ϕ and ψ satisfy (2).

If ϕ and ψ satisfy (3), either T_ϕ or T_ψ is a scalar operator, then T_ϕ commutes with T_ψ.

If ϕ and ψ satisfy (4), let $c = b\psi - \phi$ be a constant. Thus $T_\phi = bT_\psi + c$. So T_ϕ commutes with T_ψ.

Now we are going to prove "only if" part. First we prove that f, g, h, and l satisfy the hypothesis of Theorem 2.20. For every $\tau \in Aut(B_n)$, we define a unitary operators on the Bergman space $L_a^2(B_n)$ by

$$U_\tau u = u \circ \tau \mathcal{J}(\tau) \tag{2.38}$$

Immediately we see that

$$U_\tau^* T_f U_\tau = T_{f \circ \tau}. \tag{2.39}$$

For z in B_n, it is easy to check that

$$(U_\tau^*(T_\phi T_\psi - T_\psi T_\phi)U_\tau k_z, k_z) = ((T_{\phi\circ\tau} T_{\psi\circ\tau} - T_{\psi\circ\tau} T_{\phi\circ\tau})k_z, k_z)$$

$$= \int_{B_n} (f \circ \tau(w)\overline{g \circ \tau(w)} - h \circ \tau(w)\overline{l \circ \tau(w)})|k_z(w)|^2 dA(w)$$

$$-(f \circ \tau(z)\overline{g \circ \tau(z)} - h \circ \tau(z)\overline{l \circ \tau(z)}).$$

So if T_ϕ commutes with T_ψ, then

$$\int_{B_n}(f\circ\tau(w)\overline{g\circ\tau(w)} - h\circ\tau(w)\overline{l\circ\tau(w)})|k_z(w)|^2 dA(w)$$

$$= f\circ\tau(z)\overline{g\circ\tau(z)} - h\circ\tau(z)\overline{l\circ\tau(z)}.$$

Therefore Theorem 2.20 implies that $f\overline{g} - h\overline{l}$ is $\mathcal{M}$- harmonic. On the other hand f, g, h, and l are in $H^{2n}(S_n)$. So Theorem 2.19 tells us that f, g, h, and l must satisfy one of the conditions in Theorem 2.19. It is easy to see that this is equivalent to the statement that ϕ and ψ satisfy one of the conditions (1) to (4). So the proof of Theorem 2.21 is finished.

2.6 Semi-commutators on the Bergman space

In this section we will prove Theorems 2.22 and 2.23.

Theorem 2.22. Let f and g be two bounded pluriharmonic functions on B_n. Then $T_{\overline{f}}T_g = T_{\overline{f}g}$ on the Bergman space if and only if f or g is holomorphic on B_n.

Proof. Since f and g are bounded pluriharmonic on B_n there are f_1, f_2, g_1 and g_2 in $H^p(S)$ for $p > 1$ such that

$$f = f_1 + \overline{f}_2,\ g = g_1 + \overline{g}_2.$$

First we prove that f_2 and g_2 satisfy the hypothesis of Theorem 2.20. For every $\psi \in Aut(B_n)$ we define a unitary operator on $L^2(B_n)$ by

$$U_\psi h = h\circ\psi J(\psi).$$

Immediately we see that

$$U_{\psi}^{*}T_{f}U_{\psi} = T_{f\circ\psi}.$$

Thus

$$T_{f\circ\psi}^{*}T_{g\circ\psi} = T_{\overline{f\circ\psi}g\circ\psi}.$$

For z in B_n it is easy to check

$$(T_{\overline{f}\circ\psi}T_{g\circ\psi} - T_{\overline{f}\circ\psi g\circ\psi}k_z, k_z) = (H_{f\circ\psi}^{*}H_{g\circ\psi}k_z, k_z)$$

$$= ((\overline{g}_2 \circ \psi - \overline{g}_2 \circ \psi)(z)k_z, (\overline{f}_2 \circ \psi - \overline{f}_2 \circ \psi)(z)k_z)$$

$$= \int \overline{g}_2 \circ \psi f_2 \circ \psi |k_z|^2 dA - \overline{g}_2 \circ \psi(z) f_2 \circ \psi(z).$$

Then

$$\int \overline{g}_2 \circ \psi f_2 \circ \psi |k_z|^2 dA = \overline{g}_2 \circ \psi(z) f_2 \circ \psi(z).$$

So it follows from Theorem 2.20 that $f_2\overline{g_2}$ is $\mathcal{M}$-harmonic .

If in addition, f_2 and g_2 are in $H^{2n}(S_n)$, Theorem 2.19 implies that either f_2 or g_2 is constant. This means that either f or g is holomorphic on B_n

Now using remarkable properties of the ϕ-map we will reduce the compactness of $T_{\overline{f}}T_g - T_{\overline{f}g}$ to a statement that $T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f\circ\phi}g\circ\phi} = 0$ for every ϕ-map since the Bergman kernel k_z weakly converges to zero as z goes to the boundary S_n of B_n.

Theorem 2.23. If f and g are bounded pluriharmonic on B_n and $T_{\overline{f}}T_g - T_{\overline{f}g}$ is compact then

$$T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f}\circ\phi g\circ\phi} = 0 \tag{2.40}$$

for every $\phi \in \Phi$.

Proof Since $\phi \in \Phi$ there is a net $\{w_\alpha\} \subset B_n$ such that $\{\phi_{w_\alpha}\}$ converges to ϕ as $\|w_\alpha\| \to 1$.

For fixed z in B_n and h in $H^\infty(B_n)$ we have

$$((T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f}\circ\phi g\circ\phi})k_z, h) =$$

$$(H^*_{\overline{f}\circ\phi}H_{g\circ\phi}k_z, h)$$

$$= \lim_{w_\alpha}(H^*_{\overline{f}\circ\phi_{w_\alpha}}H_{g\circ\phi_{w_\alpha}}k_z, h)$$

$$= \lim_{w_\alpha}(H^*_{\overline{f}}H_g(k_z \circ \phi_{w_\alpha}k_{w_\alpha}), h \circ \phi_{w_\alpha}k_{w_\alpha}).$$

Since $k_z \circ \phi_{w_\alpha}k_{w_\alpha}$ weakly converges to zero, the compactness of $H^*_{\overline{f}}H_g$ implies

$$((T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f}\circ\phi g\circ\phi})k_z, h) = 0$$

Because $H^\infty(B_n)$ is dense in $L^2_a(B_n)$ we have

$$(T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f}\circ\phi g\circ\phi})k_z = 0$$

Since $\{k_z\}$ span $L^2_a(B_n)$ this means that

$$T_{\overline{f\circ\phi}}T_{g\circ\phi} - T_{\overline{f}\circ\phi g\circ\phi} = 0$$

for every $\phi \in \Phi$. The proof is completed.

Theorem 2.24. Let f and g be two bounded pluriharmonic functions on B_n. Then the following are equivalent:

(1) $T_{\overline{f}}T_g - T_{\overline{f}g}$ is compact;

(2) $H^*_{\overline{f}}H_g$ is compact;

(3)

$$\lim_{\|z\|\to 1} min\{max_{E(z,r)}(1-\|s\|^2)\|\nabla_z$$
$$\overline{f}(s)\|, max_{E(z,r)}(1-\|t\|^2)\|\nabla_z\overline{g}(t)\|\} = 0 \tag{2.41}$$

for all fixed r, $0 < r < 1$;

(4)

$$\lim_{\|z\|\to 1} min\{\|f\circ\phi_z - P(f\circ\phi_z)\|_2, \|g\circ\phi_z - P(g\circ\phi_z)\|_2\} = 0; \tag{2.42}$$

(5)

$$\lim_{\|z\|\to 1} min\{Q_{P\overline{f}}(z), Q_{P\overline{g}}(z)\} = 0; \tag{2.43}$$

(6) Either foϕ or goϕ is holomorphic for every $\phi \in \Phi$ where Φ is the set of ϕ-parts.

Proof. It is easy to see that (1) is equivalent to (2).

Since f and g are bounded and pluriharmonic, it follows from Proposition 2.6 that there are functions f_1, f_2, g_1 and g_2 in both the Bloch space and the Hardy space $H^2(S_n)$ such that

$$f = f_1 + \overline{f}_2$$

and

$$g = g_1 + \overline{g}_2.$$

First we prove that (2) $\Rightarrow$ (6).

If the semi-commutator $T_{\overline{f}}T_g - T_{\overline{f}g}$ is compact, it follows from Theorems 2.22 and 2.23 that either $f\circ\phi$ or $g\circ\phi$ is constant for every $\phi \in \Phi$.

Second we prove that (3) $\Rightarrow$ (4) .

For w $\in$E(z,r), we have

$$f_2(w) - f_2(z) = \int_0^1 \nabla_z f_2(tw + (1-t)z)(w-z)dt.$$

Thus

$$|f_2(w) - f_2(z)| \leq C max_{\overline{E(z,r)}}(1 - \|s\|^2)\|\nabla_z f_2(s)\|.$$

Therefore

$$\|f \circ \phi_z - P(f \circ \phi_z)\|_2 = \|f_2 \circ \phi_z - f_2(z)\|_2$$

$$\leq C\|f_2\|_{\mathcal{B}}(1 - r^{2n})^{1/2} + max_{\overline{E(z,r)}}(1 - \|s\|^2)\|\nabla_z f_2(s)\|.$$

Similarly we can also get

$$\|g \circ \phi_z - P(g \circ \phi_z)\|_2 \leq$$

$$C\|g_2\|_{\mathcal{B}}(1 - r^{2n})^{1/2} + max_{\overline{E(z,r)}}(1 - \|s\|^2)\|\nabla_z g_2(s)\|.$$

So (3) implies

$$\lim_{\|z\| \to 1} min\{\|f \circ \phi_z - P(f \circ \phi_z)\|_2, \|g \circ \phi_z - P(g \circ \phi_z)\|_2\}$$

$$\leq C(\|f_2\|_{\mathcal{B}} + \|g_2\|_{\mathcal{B}})(1 - r^{2n})^{1/2}$$

for any $0 < r < 1$. So (4) is true.

Third we want to show that (4) $\Rightarrow$ (5)

Since the Bergman kernel $K(z, w) = (1 - (w, z))^{-n-1}$ is the reproducing kernel of the Bergman space, for every function h$\in L_a^2(B_n)$ and z$\in B_n$

$$h(z) = \int h(w)K(w, z)dA(w).$$

Then

$$\frac{\partial h}{\partial z_i}(0) = -(n+1)\int h(w)w_i dA(w).$$

Therefore

$$|\frac{\partial h}{\partial z_i}(0)| \leq (n+1)\|h\|_2.$$

So we have

$$|\nabla h(0)| \leq n(n+1)\|h\|_2.$$

It follows from the definition of $Q_h(z)$ that there is a positive constant C such that

$$Q_h(0) \leq C\|h\|_2$$

Replacing h by $h \circ \phi_z - h(z)$ gives

$$Q_{h\circ\phi_z}(0) \leq C\|h \circ \phi_z - h(z)\|_2.$$

Since

$$Q_{h\circ\phi_z}(0) = Q_h(\phi_z(0)) = Q_h(z)$$

then

$$Q_h(z) \leq C\|h \circ \phi_z - h(z)\|_2.$$

So we have

$$min\{Q_{P\overline{f}}(z), Q_{P\overline{g}}(z)\} \leq$$

$$min\{Q_{f_2}(z), Q_{g_2}(z)\} \leq$$

$$Cmin\{\|f_2 \circ \phi_z - f_2(z)\|_2, \|g_2 \circ \phi_z - g_2(z)\|_2\}$$

Therefore (4) implies (5).

We are going to prove that (5) $\Rightarrow$(6)

It is easy to check that

$$Q_{P\overline{f}}(\phi_z(\lambda)) = Q_{P(\overline{f})\circ\phi_z}(\lambda) = \mu(\lambda)\|\nabla(P(\overline{f})\circ\phi_z)(\lambda)\|_2$$

$$= \mu(\lambda)\|\nabla(\overline{f}\circ\phi_z)(\lambda)\|_2.$$

for z and $\lambda \in B_n$. So (5) implies (6).

Then we show that (6) $\Rightarrow$ (3)

Suppose (3) is false. Then there is a sequence $\{z_k\} \subset B_n$ such that for some $\varepsilon > 0$

$$\liminf_{\|z_k\|\to 1} min\{max_{E(z,r)}(1-\|s\|^2)\|\nabla_z\overline{f}(s)\|, \\ max_{E(z,r)}(1-\|t\|^2)\|\nabla_z\overline{g}(t)\|\} = \varepsilon \qquad (2.44)$$

Let ϕ be in the closure of $\{\phi_{z_k}\}$ which a subnet $\{\phi_{z_{k_\alpha}}\}$ of $\{\phi_{z_k}\}$ converges to ϕ. Without loss generality we assume that $f\circ\phi$ is holomorphic. Let w_{k_α} be in $\overline{E}(z_\alpha, r)$ such that

$$(1-\|w_{k_\alpha}\|^2)\|\nabla_z f_2(w_{k_\alpha})\| = max_{s\in\overline{E}(z_{k_\alpha},r)}(1-\|s\|^2)\|\nabla_z f_2(s)\|.$$

There are λ_{k_α} in $\overline{rB_n}$, such that

$$w_{k_\alpha} = \phi_{z_{k_\alpha}}(\lambda_{k_\alpha}).$$

Since there is an $\mu > 0$ such that

$$H_\lambda(u,u) \geq \mu^2$$

for λ in $\overline{rB_n}$ and any unit vector u in C^n then

$$Q_{f_2\circ\phi_{z_{k_\alpha}}}(\lambda_{k_\alpha}) \leq \frac{1}{\mu}\|\nabla(f_2\circ\phi_{z_{k_\alpha}})(\lambda_{k_\alpha})\|.$$

In addition, because the Bergman metric is invariant under the action of $\mathrm{Aut}(B_n)$ we have

$$Q_{f_2 \circ \phi_{z_{k_\alpha}}}(\lambda_{k_\alpha}) = Q_{f_2}(\phi_{z_{k_\alpha}}(\lambda_{k_\alpha})) = Q_{f_2}(w_{k_\alpha})$$

$$\geq (\frac{2}{n+1})^{1/2} \|(\nabla_z f_2)(w_{k_\alpha})\|(1 - \|w_{k_\alpha}\|^2).$$

Therefore

$$\|\nabla_{\overline{z}}(f \circ \phi_{z_{k_\alpha}})(\lambda_{k_\alpha})\| \geq \mu(\frac{2}{n+1})^{1/2} \|(\nabla_z f_2)(w_{k_\alpha})\|(1 - \|w_{k_\alpha}\|^2)$$

$$= \mu(\frac{2}{n+1})^{1/2} \|(\nabla_{\overline{z}} f)(w_{k_\alpha})\|(1 - \|w_{k_\alpha}\|^2).$$

Since $\overline{rB_n}$ is compact there is a subnet $\{\lambda_{k_{\alpha_\beta}}\}$ of $\{\lambda_{k_\alpha}\}$ converging to some λ in $\overline{rB_n}$ and the subnet $\{\phi_{\lambda_{k_\alpha}}\}$ converges ϕ. It follows from Proposition 2.8 that

$$\lim_{z_{\alpha_\beta}} \|\nabla_{\overline{z}}(f \circ \phi_{z_{k_{\alpha_\beta}}})(\lambda_{z_{k_{\alpha_\beta}}})\| = \|\nabla_{\overline{z}}(f \circ \phi)(\lambda)\|.$$

Since foϕ is holomorphic we have

$$\lim_{z_{\alpha_\beta}} \|\nabla_{\overline{z}}(f \circ \phi_{z_{k_{\alpha_\beta}}})(\lambda_{z_{k_{\alpha_\beta}}})\| = 0.$$

So $\lim_{w_{\alpha_\beta}} \|\nabla_{\overline{z}} f(w_{z_{k_{\alpha_\beta}}})\|(1 - \|w_{\alpha_\beta}\|^2) = 0$. This contradicts (2.44).

Using the Cauchy-Schwarz inequality we have

$$\int_{B_n} |f_2(w) - f_2(z)||g_2(w) - g_2(z)||k_z(w)|^2 dA(w)$$

$$\leq \|f_2 \circ \phi_z - f_2(z)\|_2 \|g_2 \circ \phi_z - g_2(z)\|_2$$

$$\leq (\|f_2\|_{\mathcal{B}} + \|g_2\|_{\mathcal{B}})C min\{\|f_2 \circ \phi_z - f_2(z)\|_2, \|g_2 \circ \phi_z - g_2(z)\|_2\}.$$

It follows from Theorem that (4) implies (2).

Combining Theorems 2.23 and 2.10 we have that (1) implies (6).

2.7 Toeplitz operators on the Hardy space

In this section we still use T_f to denote Toeplitz operator on the Hardy space. The main task is to prove Theorems 2.25 and 2.26. Since every function in $H^2(S_n)$ extends holomorphicly on B_n and the evaluation at z is a bounded linear functional on $H^2(S_n)$, there is a function S_z in the Hardy space $H^2(S_n)$ such that

$$f(z) = < f, S_z >_{H^2(S_n)} \tag{2.45}$$

S_z is called the Szego-Hardy kernel and is , in fact,

$$S_z(w) = (1 - (w, z))^{-n} \tag{2.46}$$

for z in B_n and w in S_n.

Theorem 2.25. Let f and g be two bounded pluriharmonic functions on B_n. Then $T_{\overline{f}}T_g - T_{\overline{f}g} = 0$ on the Hardy space if and only if f or g is holomorphic on B_n.

Proof . Since f and g are bounded pluriharmonic on B_n there are f_1, f_2, g_1 and g_2 in $H^p(S)$ for $p > 1$ such that

$$f = f_1 + \overline{f}_2, \ g = g_1 + \overline{g}_2.$$

If the semi-commutator $T_{\overline{f}}T_g - T_{\overline{f}g} = 0$, then

$$< (T_{\overline{f}}T_g - T_{\overline{f}g})S_z, S_z >= 0$$

for all z in B_n. It is easy to verify that the right hand side of the above equation gives

$$\int_{S_n} f_2\overline{g_2}\frac{|S_z|^2}{\|S_z\|^2}d\sigma = f_2(z)\overline{g_2(z)}. \tag{2.47}$$

On the other hand, it follows from [R] that the above equation defines a $\mathcal{M}$-harmonic on B_n. So $f_2(z)\overline{g_2(z)}$ is $\mathcal{M}$-harmonic function on B_n. Since f_2 and g_2 are in $H^p(S_n)$ for any $p > 1$, Theorem 2.20 tells us that either f_2 or g_2 is constant.

This means that either f or g is holomorphic.

Theorem 2.26. Suppose that ϕ and ψ are bounded pluriharmonic functions on the unit ball. Then

$$T_\phi T_\psi = T_\psi T_\phi \tag{2.48}$$

on the Hardy space of the unit sphere if and only if ϕ and ψ satisfy one of the following conditions:

(1) Both ϕ and ψ are holomorphic on the unit ball B_n;

(2) Both $\overline{\phi}$ and $\overline{\psi}$ are holomorphic on the unit ball B_n;

(3) Either ϕ or ψ is constant;

(4) There is a nonzero constant b such that ϕ -bψ is constant.

Proof. The proof of the "if" part is similar to that of Theorem 2.25, so it is omitted. It remains to prove the "only if" part.

Since ϕ and ψ are bounded and pluriharmonic on the unit ball, it follows from Proposition 2.6 that there are four functions f, g, h, and l in $H^{2n}(S_n)$ such that

$$\phi = f + \overline{l}$$

and

$$\psi = h + \overline{g}.$$

Every functions in $L^p(S_n)$, via the Possion integral , extend $\mathcal{M}$-harmonic functions on the unit ball. So for the fixed z in the unit ball and u in $H^2(S_n)$, we have

$$< (T_\phi T_\psi - T_\psi T_\phi)\frac{S_z}{\|S_z\|}, \frac{S_z}{\|S_z\|} >=$$

$$\int_{S_n} (f(w)\overline{g(w)} - h(w)\overline{l(w)})\frac{|S_z(w)|^2}{\|S_z\|^2}d\sigma(w) - (f(z)\overline{g(z)} - h(z)\overline{l(z)}).$$

So if T_ϕ commutes with T_ψ, then

$$\int_{S_n} (f(w)\overline{g(w)} - h(w)\overline{l(w)})\frac{|S_z(w)|^2}{\|S_z\|^2}d\sigma(w) = f(z)\overline{g(z)} - h(z)\overline{l(z)}.$$

We know that if the right hand side of the above equation defines an $\mathcal{M}$-harmonic function on the unit ball, then $f\overline{g} - h\overline{l}$ is $\mathcal{M}$-harmonic. In addition, f, g, h, and l are in $H^{2n}(S_n)$, so Theorem 2.19 implies that f, g, h, and l satisfy one of the conditions in Theorem 2.19. This implies that ϕ and ψ satisfy one of the conditions of the theorem.

Chapter 3

Operators on the Bargman-Fock-Segal space

In this chapter we characterize those functions f, for which f and $f \circ \tau_z$ are in $L^2(C^n, d\mu)$, such that both H_f and $H_{\overline{f}}$ are bounded or compact or in p-Schatten class for p$\geq$ 2. In [BC1,BC2] Berger and Coburn considered Hankel operators on the Bargman-Segal space using the Berezin transform and then characterized bounded functions f such that both H_f and $H_{\overline{f}}$ are compact. However on the Bargman-Segal space it is more natural and important to consider Hankel operators and Toeplitz operators with unbounded symbols. The commutator of the unbounded operator T_z

$$[T_{z_j}, T_{\overline{z}_j}] = H^*_{\overline{z}_j} H_{\overline{z}_j}. \tag{3.1}$$

can be extended a bounded operator even though T_{z_j} is not bounded since it is easy to check that $H_{\overline{z}_j}$ are bounded operator with unbounded symbol $\overline{z}_j$ on C^n.

3.1 Boundedness

Using the Berezin transform we define a measure of mean oscillation

$$MO(f)(z) = \{\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2\}^{1/2} \tag{3.2}$$

Our main result of the section is the following theorem

Theorem 3.1. Suppose that f and f$\circ\tau_z$ are in $L^2(C^n, d\mu)$ for every z in C^n. Then both H_f and $H_{\overline{f}}$ are bounded if and only if

$$Sup_{z\in C^n} MO(f)(z) < +\infty. \tag{3.3}$$

This result was conjectured by L.A. Coburn in [C]. In order to prove the theorem we need the following lemmas.

Lemma 3.2. Suppose that f and f$\circ\tau_z$ are in $L^2(C^n, d\mu)$. Then

$$\|H_f k_z\| = \|f \circ \tau_z - P(f \circ \tau_z)\|_2. \tag{3.4}$$

This lemma was proved in [Str].

Lemma 3.3. Suppose that f and f$\circ\tau_z$ are in $L^2(C^n, d\mu)$. Then

$$MO(f)(z) \leq \|H_f k_z\| + \|H_{\overline{f}} k_z\|. \tag{3.5}$$

Proof. It follows from Lemma 3.2 that

$$\|H_f k_z\| = \|f \circ \tau_z - P(f \circ \tau_z)\|_2$$

$$\|H_{\overline{f}} k_z\| = \|\overline{f} \circ \tau_z - P(\overline{f} \circ \tau_z)\|_2.$$

Also it is easy to check that

$$MO(f)(z) = \|f \circ \tau_z - \tilde{f}(z)\|_2.$$

Combining the above equations we have

$$MO(f)(z) \leq \|f \circ \tau_z - P(f \circ \tau_z)\|_2 + \|P(f \circ \tau_z) - \tilde{f}(z)\|_2$$

$$\leq \|H_f k_z\| + \|P(f \circ \tau_z) - \tilde{f}(z)\|_2.$$

On the other hand, a change of variables gives

$$\tilde{f}(z) = P(\overline{P(\overline{f} \circ \tau_z)}).$$

Thus

$$\|P(f \circ \tau_z) - \tilde{f}(z)\|_2 = \|P(f \circ \tau_z - \overline{P(\overline{f} \circ \tau_z)})\|_2$$

$$\leq \|f \circ \tau_z - \overline{P(\overline{f} \circ \tau_z)}\| = \|H_{\overline{f}} k_z\|.$$

Hence

$$MO(f)(z) \leq \|H_f k_z\| + \|H_{\overline{f}} k_z\|,$$

we complete the proof.

In order to prove our main results we have to vie estimates of $\tilde{f}$(z) from works on the bounded symmetric domain [BBCZ], [Z3].

Theorem 3.4. For fixed z in C^n, we have

$$|\frac{d}{dt}\tilde{f}(tz)| \leq MO(f)(tz)\|z\|(\|z\| + C) \tag{3.6}$$

for some positive constant C which does not depend on f or z.

Proof. Since the reproducing kernel of the Bargman-Segal space can be written as $e^{\frac{\overline{z}w}{2}}$ we can directly compute $\frac{d}{dt}\tilde{f}(tz)$. For t in [0,1], z in C^n,

$$\frac{d}{dt}\tilde{f}(tz) = \int_{C^n} f(w)Re(\frac{dk_{tz}}{dt}(w)\overline{k}_{tz}(w))d\mu(w).$$

Since

$$\int_{C^n} k_{tz}(w)\overline{k}_{tz}(w)d\mu(w) = 1$$

then

$$\int_{C^n} Re(\frac{dk_{tz}}{dt}(w)\overline{k}_{tz}(w))d\mu(w) = 0.$$

So

$$\frac{d}{dt}\tilde{f}(tz) = \int_{C^n}(f(w) - \tilde{f}(tz))Re(\frac{dk_{tz}}{dt}(w)\overline{k}_{tz}(w))d\mu(w).$$

Using Holder inequality we get

$$|\frac{d}{dt}\tilde{f}(tz)| = MO(f)(tz)[\int_{C^n} |\frac{dk_{tz}}{dt}(w)|^2 d\mu(w)]^{1/2}.$$

Since

$$k_{tz}(w) = e^{\frac{t\overline{z}w}{2} - \frac{t^2\|z\|^2}{4}},$$

then

$$\frac{dk_{tz}}{dt}(w) = k_{tz}(w)(-\frac{\overline{z}w + t\|z\|^2}{2}).$$

Hence

$$|\frac{dk_{tz}}{dt}(w)| = |k_{tz}(w)|\|z\|(\frac{\|w\| + \|z\|}{2}).$$

So

$$|\tfrac{d}{dt}\tilde{f}(tz)| \leq MO(f)(tz)\|z\|[\textstyle\int_{C^n} |k_{tz}(w)|^2(\frac{\|w\|+\|z\|}{2})^2 d\mu]^{1/2}$$

$$\leq MO(f)(tz)\|z\| \textstyle\int_{C^n}(\frac{\|tz+w\|+\|z\|}{2})^2 d\mu]^{1/2}$$

$$\leq MO(f)(tz)\|z\|[\textstyle\int_{C^n}(\|z\| + \|w\|/2)^2 d\mu]^{1/2} \leq MO(f)(tz)\|z\|(\|z\| + C)$$

for some positive constant C which does not depend on f or z.

Consequently we have the following corollary

Corollary. For z and w in C^n,

$$|\tilde{f}(z) - \tilde{f}(w)| \leq C \int_0^1 MO(f \circ \tau_z)(t(w-z))dt\|z-w\|(\|z-w\|+C)$$

for some positive constant C.

Proof. It follows from Theorem 3.4 that

$$|\tilde{f}(w-z) - \tilde{f}(0)| \leq C \int_0^1 MO(f)(t(w-z))dt\|z-w\|(\|z-w\|+C).$$

Replacing f by f$\circ\tau_z$ implies

$$|\tilde{f}(z) - \tilde{f}(w)| \leq C \int_0^1 MO(f \circ \tau_z)(t(w-z))dt\|z-w\|(\|z-w\|+C).$$

Theorem 3.5. If $Sup_{z \in C^n} MO(f)(z)$ is finite then both $H_{\tilde{f}}$ and $H_{\overline{\tilde{f}}}$ are bounded.

Proof. For b in the domain of $H_{\tilde{f}}$, we have

$$H_{\tilde{f}} b(z) = \int_{C^n} (\tilde{f}(z) - \tilde{f}(w)\overline{K}_z(w))b(w)d\mu(w)$$

which is an integral operator with kernel

$$\mathcal{K} = (\tilde{f}(z) - \tilde{f}(w))\overline{K}_z(w)$$

By the corollary of Theorem 3.4, it suffices to check that the integral operator T with kernel

$$\mathcal{M}(z,w) = \|z-w\|(\|z-w\| + C\|)\overline{K}_z(w)$$

is bounded. It is easy to verify

$$\int_{C^n} \|z-w\|(\|z-w\|+C)|K_z|K_w(w)^{1/2}d\mu(w)$$

$$= K_z(z)^{1/2}\int_{C^n} \|z-w\|(\|z-w\|+C)e^{-\frac{\|z-w\|}{4}}dV(w)$$

$$= K_z(z)^{1/2}\int_{C^n} \|w\|(\|w\|+C)e^{-\frac{\|w\|}{4}}dV(w)$$

$$= C_1K_z(z)^{1/2}$$

where C_1 is a positive constant which does not depend on z. By the Schur lemma, T is bounded. Thus $H_{\tilde{f}}$ is bounded. Similarly we can prove that $H_{\overline{\tilde{f}}}$ is bounded.

As in [BC2], we define a unitary operator from $L^2(C^n, d\mu)$ to $L^2(C^n, \frac{dV}{(2\pi)^{2n}})$ given by

$$Ug(z) = e^{-\frac{\|z\|^2}{4}}g(z).3.7 \tag{3.7}$$

The following lemma was proved in [BC2]. We will use it in the next sections.

Lemma 3.6. For b in $L^2(C^n, \frac{dV}{(2\pi)^{2n}})$,

$$UPM_{|g|^2}PU^*b(z) = (2\pi)^{-2n}\int_{C^n} h(|g|^2, z, w)b(w)dV(w)$$

where

$$h(|g|^2, z, w) = (2\pi)^{-n}e^{-\frac{\|z-w\|^2}{8}}\int_{C^n} |g(w)|^2 exp\{-\frac{\|u-\frac{z+w}{2}\|^2}{2} + iIm\frac{z-w}{2}\overline{u}\}dV(u).$$

From the lemma we see that

$$|UPM_{|g|^2}PU^*b(z)| \le (2\pi)^{-n}\int_{C^n} e^{-\frac{\|z-w\|^2}{8}}\widetilde{|g|^2}(\frac{z+w}{2})b(w)dV(w).$$

Theorem 3.7. For f in $L^2(C^n, d\mu)$, if $\widetilde{|f|^2}$ is bounded, then M_fP is bounded. Moreover

$$\|M_fP\| \leq C\|\widetilde{|f|^2}\|_\infty^{1/2} \tag{3.8}$$

for some constant C which does not depend on f.

Proof. Let B_N be the ball with center 0 and radius N. For any g in $L^2(C^n, d\mu)$, we are going to estimate the following integral

$$\int_{B_N} |f|^2 |\int g(w) e^{z\overline{w}/2} e^{-\|w\|^2/2} dV(w)|^2 e^{-\|z\|^2/2} dV(z)$$

$$= \int_{B_N} \int \int |g(u)||g(w)||f(z)|^2 \times$$

$$exp\{-\|z-(u+w)/2\|^2/2 - \|u-w\|^2/8 - \|u\|^2/4 - \|w\|^2/4\} dV(u)dV(w)dV(z)$$

$$= \int \int |g(u)||g(w)| \int_{B_N} |f(z)|^2 exp\{-\|z-(u+w)/2\|^2/2\} dV(z) \times$$

$$exp\{-\|u-w\|^2/8 - \|u\|^2/4 - \|w\|^2/4\} dV(u)dV(w)$$

$$\leq \|\widetilde{|f|^2}\|_\infty \int \int |g(u)||g(w)| \times$$

$$exp\{-\|u-w\|^2/8 - \|u\|^2/4 - \|w\|^2/4\} dV(u)dV(w)$$

$$\leq \|\widetilde{|f|^2}\|_\infty [\int \int |g(u)|^2 exp\{-\|u-w\|^2/8 - \|u\|^2/2\} dV(u)]^{1/2} \times$$

$$\int \int |g(w)|^2 exp\{-\|u-w\|^2/8 - \|w\|^2/2\} dV(w)]^{1/2}$$

$$\leq \|\widetilde{|f|^2}\|_\infty \|e^{-\|.\|^2/8}\|_2^2 \|g\|^2.$$

Let N go to $+\infty$, then

$$\|M_f P g\|^2 \leq \|\widetilde{|f|^2}\|_\infty \|e^{-\|\cdot\|^2/8}\|_2^2 \|g\|^2.$$

So $M_f P$ is bounded and

$$\|M_f P\| \leq \|\widetilde{|f|^2}\|_\infty^{1/2} \|e^{-\|\cdot\|^2/8}\|_2.$$

Theorem 3.8. If $\mathrm{Sup}_{z\in C^n} MO(f)(z)$ is finite, then both $\mathrm{H}_{f-\tilde{f}}$ and $\mathrm{H}_{\overline{f-\tilde{f}}}$ are bounded.

Proof. Let g=f -$\tilde{f}$, then

$$\widetilde{|g|^2}(z) = \int_{C^n} |f(w) - \tilde{f}(w)|^2 e^{-\frac{\|z-w\|^2}{2}} dV(w)$$

$$\leq \int_{C^n} |f(w) - \tilde{f}(z)|^2 e^{-\frac{\|z-w\|^2}{2}} dV(w) + \int_{C^n} |\tilde{f}(z) - \tilde{f}(w)|^2 e^{-\frac{\|z-w\|^2}{2}} dV(w)$$

$$\leq MO(f)(z)^2 + \int_{C^n} (Sup_{u\in C^n} MO(f)(u))^2$$

$$\|z-w\|^2 (\|z-w\| + C)^2 e^{-\frac{\|z-w\|^2}{2}} dV(w)$$

$$\leq C_1 Sup_{z\in C^n} MO(f)(z)^2.$$

Hence $\widetilde{|g|^2}$ is bounded. It follows from Theorem 3.7 that $M_g P$ is bounded. So is H_g since $H_g = (I-P)M_f P$. Thus H_g is bounded. Using the same method we can prove that $H_{\overline{g}}$ is bounded.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. If both H_f and $\mathrm{H}_{\overline{f}}$ are bounded, it follows from Lemma 3.3 that

$$MO(f)(z) \leq \|H_f k_z\| + \|H_{\overline{f}} k_z\| \leq \|H_f\| + \|H_{\overline{f}}\|$$

since k_z is normalized.

On the other hand if $Sup_{z\in C^n}$ MO(f)(z) is finite, Theorem 3.5 says that both $H_{\bar{f}}$ and $H_{\overline{\bar{f}}}$ are bounded. In addition, Theorem 3.7 implies that both $H_{f-\tilde{f}}$ and $H_{\overline{f-\tilde{f}}}$ are bounded. So H_f and $H_{\overline{f}}$ are bounded.

For our convenience in the next sections we define $BMO_\infty(C^n)$ by

$$BMO_\infty(C^n) = \{f \in L^2(C^n, d\mu) | Sup_{z\in C^n} MO(f)(z) < \infty\}.$$

The seminorm for f in $BMO_\infty(C^n)$ is defined by

$$\|f\|_{BMO} = Sup_{z\in C^n} MO(f)(z). \tag{3.9}$$

3.2 Compactness

First we need the following well-known lemma.

Lemma 3.9. For some measure $d\nu$ on C^n, if the integral operator on $L^2(C^n, d\nu)$ with kernel $\mathcal{K}$(z,w) is in $L^2(C^n \times C^n, d\nu \times d\nu)$, then the integral operator is a Hilbert-Schmidt operator.

Consequently we immediately see that the following integral operator

$$Th(z) = \int_{C^n} \chi_R(z)(\tilde{f}(z) - \tilde{f}(w))e^{\frac{\overline{w}z}{2}} h(w) d\mu(w) \tag{3.10}$$

is a Hilbert-Schimdt operator on $L^2(C^n, d\mu)$ where χ_R(z) denotes the characteristic function of the ball in C^n with center 0 and radius R.

Lemma 3.10. A convolution operator T on $L^2(R^n)$ is given by

$$Th(z) = \int K(z-w)h(w)dA(w) = K * h(z).$$

If K is in $L^1(R^{2n})$, then

$$\|T\| \leq \|K\|_1.$$

Lemma 3.11. Suppose that f is in $BMO_\infty(C^n)$. If $\lim_{z\to\infty} MO(f)(z) = 0$, then

$$\lim_{z\to\infty} \widetilde{|f-\tilde{f}|^2}(z) = 0. \tag{3.11}$$

Proof. For $z \in C^n$, we have

$$\widetilde{|f-\tilde{f}|^2}(z) \leq MO(f)^2(z) + \int |\tilde{f}(z) - \tilde{f}(w)|^2 e^{\frac{-\|z-w\|}{2}} dA(w).$$

Since

$$\int_{C^n} \|w\|^2(\|w\| + C)^2 e^{\frac{-\|w\|^2}{2}} dA(w) < \infty$$

then for any $\varepsilon > 0$, there exists R> 0 such that

$$\int_{C^n/B(0,R)} \|w\|^2(\|w\| + C)^2 e^{\frac{-\|w\|^2}{2}} dA(w) < \varepsilon.$$

Using Theorem 3.4 we have

$$\int_{C^n} |\tilde{f}(z) - \tilde{f}(w)|^2 e^{\frac{-\|z-w\|^2}{2}} dA(w)$$

$$\leq \int_{C^n} \|z-w\|^2(\|z-w\| + C)^2 sup_{u\in[z,w]} MO^2(f)(u) e^{\frac{-\|z-w\|^2}{2}} dA(w)$$

$$\leq \int_{B(z,R)} \|z-w\|^2(\|z-w\|+C)^2 \times$$

$$sup_{u\in[z,w]} MO^2(f)(u)e^{\frac{-\|z-w\|^2}{2}} dA(w)+$$

$$\int_{C^n/B(z,R)} \|z-w\|^2(\|z-w\|+C)^2 \times$$

$$sup_{u\in[z,w]} MO^2(f)(u)e^{\frac{-\|z-w\|^2}{2}} dA(w)$$

$$\leq \int_{B(0,R)} \|w\|^2(\|w\|+C)^2 \times$$

$$sup_{\|z-u\|\leq\|w\|} MO^2(f)(u)e^{\frac{-\|w\|^2}{2}} dA(w)+$$

$$\int_{C^n/B(0,R)} \|w\|^2(\|w\|+C)^2 sup_{u\in C^n} MO^2(f)(u)e^{\frac{-\|w\|^2}{2}} dA(w)$$

$$\leq sup_{\|u-z\|\leq R} MO(f)^2(u)C_1 + \varepsilon\|f\|^2_{BMO}$$

where $C_1 = \|\|w\|(\|w\|+C))\|_2$. Thus

$$\overline{\lim}_{z\to\infty} |\widetilde{f-\tilde{f}}|^2(z)$$

$$\leq \lim_{z\to\infty} MO(f)^2(z) + (\lim_{z\to\infty} sup_{\|u-z\|\leq r} MO(f)^2(u))C_1 + \varepsilon\|f\|_{BMO}$$

$$\leq \varepsilon\|f\|^2_{BMO}.$$

Since ε is arbitrary, then

$$\lim_{z\to\infty} |\widetilde{f-\tilde{f}}|^2(z) = 0.$$

Theorem 3.12. Suppose that f is in $\mathrm{BMO}_\infty(C^n)$. If

$$\lim_{z\to\infty} MO(f)(z) = 0, \tag{3.12}$$

then both $H_{f-\tilde{f}}$ and $H_{\overline{f-\tilde{f}}}$ are compact.

Proof. First we prove that $T_{|g|^2}$ is compact provided that $\tilde{|g|^2}$ is bounded and $\lim_{z\to\infty} \widetilde{|g|^2}(z) = 0$. It follows from Lemma 3.6 that for $b \in L^2(C^n, dV)$,

$$UPM_{|g|^2}PU^*b(z) = \int_{C^n} h(|g|^2, z, w) b(w) dv(w).$$

Let T_R be the following operator

$$T_R b(z) = \int_{C^n} \chi_R(z) h(|g|^2, z, w) b(w) dv(w).$$

It is easy to check that the integral operator has the kernel in $L^2(C^n, dv)$. Then Lemma 3.9 implies that T_R is compact for any R> 0. Now we estimate

$$|(UPM_{|g|^2}PU^* - T_R)b(z)| \le (1-\chi_R)(z) \textstyle\int_{C^n} e^{-\frac{\|z-w\|^2}{8}} \widetilde{|g|^2}(\frac{z+w}{2})|b(w)|dA(w)$$

$$\le (1-\chi_R)(z) \textstyle\int_{B(z,r)} e^{-\frac{\|z-w\|^2}{8}} \widetilde{|g|^2}(\frac{z+w}{2})|b(w)|dA(w) +$$

$$(1-\chi_R)(z) \textstyle\int_{C^n/B(z,r)} e^{-\frac{\|z-w\|^2}{2}} \widetilde{|g|^2}(\frac{z+w}{2})|b(w)|dA(w)$$

$$\le (1-\chi_R)(z) sup_{\|u-z\|\le\frac{r}{2}} \widetilde{|g|^2}(u) \textstyle\int_{B(z,r)} e^{-\frac{\|z-w\|^2}{8}} |b(w)|dA(w) +$$

$$(1-\chi_R)(z) \|\widetilde{|g|^2}\|_\infty \textstyle\int_{C^n/B(z,r)} e^{-\frac{\|z-w\|^2}{8}} |b(w)|dA(w).$$

We know that the norm of the convolution operator

$$Ab(z) = \int e^{-\frac{\|z-w\|}{8}} b(w) dA(w)$$

is $\|e^{-\frac{\|z\|}{8}}\|_2$. We choose r so large that

$$(\int_{C^n/B(0,r)} e^{-\frac{\|w\|^2}{4}} dA(w))^{1/2} < \varepsilon.$$

Then

$$\|(UPM_{|g|^2}PU^* - T_R)\| \leq \varepsilon \|\widetilde{|g|^2}\|_\infty + \|A\| sup_{\|z-u\| \leq r, \|z\| \geq R} \widetilde{|g|^2}(u).$$

Hence

$$\overline{\lim}_{R\to\infty} \|(UPM_{|g|^2}PU^* - T_R)\| \leq \varepsilon \|\widetilde{|g|^2}\|_\infty + \lim_{z\to\infty} C_1 \widetilde{|g|^2}(z) \|A\|$$

$$\leq \varepsilon \|\widetilde{|g|^2}\|_\infty.$$

Since ε is arbitrary, then

$$\lim_{z\to\infty} \|(UPM_{|g|^2}PU^* - T_R)\| = 0.$$

In addition, T_R is compact, thus $UPM_{|g|^2}PU^*$ is compact. This implies that $T_{|g|^2}$ is compact.

Let g=f-$\tilde{f}$. Then Lemma 3.11 implies that

$$\lim_{z\to\infty} \widetilde{|g|^2}(z) = 0.$$

We have just shown that $T_{|g|^2}$ is compact. On the other hand

$$H_g^* H_g = T_{|g|^2} - T_g T_{\overline{g}} \leq T_{|g|^2}.$$

Then H_g is compact. So $H_{f-\tilde{f}}$ is compact. Using the same method we can prove that $H_{\overline{f-\tilde{f}}}$ is compact

Theorem 3.13. Suppose that f is in BMO$_\infty$. If $\lim_{z\to\infty} MO(f)(z) = 0$, then both H$_{\tilde f}$ and H$_{\bar{\tilde f}}$ are compact.

Proof. For b$\in H^2(C^n, d\mu)$, H$_{\tilde f}$ can be written as

$$H_{\tilde f}b(z) = \int_{C^n} (\tilde f(z) - \tilde f(w))e^{\frac{\overline{w}z}{2}} b(w)d\mu(w)$$

Since for any ε, we can find R> 0 such that

$$\int_{C^n/B(0,R)} \|w\|^2(\|w\| + C)^2 d\mu(w) < \varepsilon$$

using corollary of Theorem 3.4, we get

$$\chi_{C^n/B(0,R)}(z)|H_{\tilde f}b(z)| \leq$$

$$\int_{C^n/B(z,r)} \|z-w\|^2(\|z-w\| + C)^2 Sup_{u\in[z,w]} MO(f)(u)|b(w)||e^{\frac{z\overline{w}}{2}}|d\mu(w) +$$

$$\chi_{C^n/B(0,R)}(z) \int_{B(z,r)} \|z-w\|^2(\|z-w\| + C)^2 Sup_{u\in[z,w]} MO(f)(u)|b(w)||e^{\frac{z\overline{w}}{2}}|d\mu(w)$$

$$\leq \|f\|_{BMO} e^{\frac{\|z\|}{4}} \int_{C^n}(1 - \chi_r(z-w))\|z-w\|^2(\|z-w\| + C)^2|b(w)|$$

$$e^{-\frac{\|w\|^2}{4}} e^{-\frac{\|z-w\|^2}{4}} dV(w) +$$

$$Sup_{\|z-u\|<r, \|z\|\geq R} \int_{B(z,r)} e^{\frac{\|z\|}{4}} \|z-w\|^2(\|z-w\| + C)^2|b(w)|e^{-\frac{\|w\|^2}{4}} e^{-\frac{\|z-w\|^2}{4}} dV(w).$$

Now we consider the following operators

$$T_1b(z) = e^{\frac{\|z\|^2}{4}} \int_{C^n}(1-\chi_r(z-w))\|z-w\|^2(\|z-w\|+C)^2|b(w)|e^{-\frac{\|w\|^2}{4}} e^{-\frac{\|z-w\|^2}{4}} dV(w)$$

and

$$T_2b(z) = e^{\frac{\|z\|}{4}} \int_{C^n} \|z-w\|^2(\|z-w\| + C)^2|b(w)|e^{-\frac{\|w\|^2}{4}} e^{-\frac{\|z-w\|^2}{4}} dV(w)$$

It is to check that these two operators T_1 and T_2 are equivalent to some convolution operators on $L^2(C^n, dV)$ with the norms satisfying

$$\|T_1\| \leq \|(1-\chi_r)\|w\|(\|w\|+C)\|_{L^2(C^n, e^{-\frac{\|z\|^2}{4}}dV)}$$

and

$$\|T\|_2 \leq \|(\|w\|(\|w\|+C)\|_{L^2(C^n, e^{-\frac{\|z\|^2}{4}}dV)}.$$

Since for any ε, we can find r> 0 such that

$$\int_{C^n/B(0,r)} \|w\|^2(\|w\|+C)^2 d\mu(w) < \varepsilon$$

we get $\|T_1\| \leq \varepsilon$. Let

$$T_R b(z) = \chi_R(z) H_{\tilde{f}} b(z)$$

So T_R is a Hilbert-Schimdt integral operator. Thus

$$(H_{\tilde{f}} - T_R)b(z) = (1-\chi_R(z)) \int_{C^n} (\tilde{f}(z) - \tilde{f}(w)) e^{\frac{\overline{w}z}{2}} b(w) d\mu(w)$$

We have shown above that

$$\|H_{\tilde{f}} - T_R\| \leq C Sup_{\|z-u\|<r, \|z\|\geq R} MO(f)(u) + \varepsilon\|f\|_{BMO}$$

Then

$$\overline{\lim}_{R\to\infty} \|H_{\tilde{f}} - T_R\|$$

$$\leq \lim_{u\to\infty} MO(f)(u) + \varepsilon\|f\|_{BMO}$$

$$\leq \varepsilon\|f\|_{BMO}$$

Since ε is arbitrary we get $\lim_{R\to\infty} \|H_{\tilde{f}} - T_R\| = 0$. So $H_{\tilde{f}}$ is compact. Similarly we can prove that $H_{\overline{\tilde{f}}}$ is compact.

Combining Lemma 3.3 with Theorem 3.12 and Theorem 3.13 we get the main result in this section

Theorem 3.14. Suppose that f is in BMO_∞. Both H_f and $H_{\overline{f}}$ are compact if and only if $\lim_{z\to\infty} MO(f)(z) = 0$.

Now we define $\mathrm{VMO}_\infty(C^n) = \{f \in BMO_\infty | \lim_{z\to\infty} MO(f)(z) = 0\}$. In [BC2] it was proved that for $f \in L^\infty(C^n, d\mu)$, f is in VMO_∞ if and only if H_f is compact. However this is not true for Hankel operators with unbounded symbols, for example, $\mathrm{MO}(\overline{z})(z) = 2^{1/2}$, so $H_{\overline{z}}$ is bounded but not compact and H_z is zero. This example shows that the Hankel operators with unbounded symbols are quite different from the Hankel operators with bounded symbols.

To end the section we deal with the Hankel operators with symbols co-analytic on C^n.

Theorem 3.15. Suppose that f and $f \circ \tau_z$ are in $H^2(C^n, d\mu)$. Then

1). $H_{\overline{f}}$ is bounded if only if f is an affine function, i.e. there are constant vector A and constant B such that f(z)=(z,A)+B.

2). $H_{\overline{f}}$ is compact if and only if f is constant.

Proof. By Theorems 1.1 and 2.6 we are going to compute MO(f)(z) if f is analytic on C^n.

Since f is analytic on C^n, for any fixed z in C^n, we can write f(z+w) as a power series of w

$$f(z+w) = \sum_\alpha f^\alpha(z) w^\alpha$$

where α is multiple index and $f^\alpha(z)$ is the α derivative of f. Then

$$MO(f)(z)^2 = \sum_\alpha |f^\alpha(z)|^2 \|w^\alpha\|_2^2 - |f(z)|$$

So if $Sup_{z\in C^n} MO(f)(z) < \infty$ then

$$Sup_{z\in C^n} \sum_{\alpha\neq 0} |f^\alpha(z)|^2 \|w^\alpha\|_2^2 < \infty.$$

Therefore the norm of the gradient of f is bounded on C^n. So the gradient is constant since there is not any bounded nonconstant analytic function on C^n. Thus there are constant vector A and constant B such that

$$f(z) = (z, A) + B,$$

so we prove 1) since if f is an affine function , MO(f)(z) is bounded.

If $\lim_{z\to\infty} MO(f)(z) = 0$, then the gradient of f goes to 0 as z goes to ∞. Thus f is constant. It is easy to see that $H_{\bar{f}}$ is zero if f is constant.

3.3 p-Schatten Class

Lemma 3.16. Suppose $2\leq p < +\infty$ and A_G is the integral operator on $L^2(C^n, d\mu)$ defined by

$$A_G f(z) = \int_{C^n} G(z,w)K(z,w)f(w)d\mu(w)$$

If

$$\int_{C^n}\int_{C^n} |G(z,w)|^p |K(z,w)|^2 d\mu(z)d\mu(w) < +\infty \tag{3.13}$$

then A_G is in S_p where K(z,w) is the reproducing kernel.

Proof. The case p=2 is well-known. If G(z,w) is bounded on $C^n \times C^n$, we are going to show that A_G is bounded. On $H^2(C^n, d\mu)$, we know that the reproducing kernel

$$K(z,w) = e^{\frac{\bar{z}w}{2}}.$$

So

$$|A_G f(z)| \leq \|G\|_\infty \int_{C^n} |f(w)| exp\{\frac{\|z\|^2}{4} - \frac{\|w\|^2}{4} - \frac{\|z-w\|^2}{4}\} dV(w)$$

As in Section 2, we have proved that

$$\|A_G\| \leq \|G\|_\infty \int_{C^n} e^{-\frac{\|z\|^2}{4}} dV(w)$$

Now we consider the linear mapping

$$F : L^2(C^n \times C^n, d\eta) + L^\infty(C^n \times C^n, d\eta) \to \mathcal{B}(L^2(C^n, d\mu))$$

given by

$$F(G) = A_G$$

where $d\eta = |K(z,w)|^2 d\mu(z) d\mu(w)$. Then

$$F : L^2(\mathcal{D} \times \mathcal{D}, d\eta) \to S_2$$

$$F : L^\infty(\mathcal{D} \times \mathcal{D}, d\eta) \to S_\infty$$

are both bounded. By interpolation [BL], we have

$$F : L^p(C^n \times C^n, d\eta) \to S_p$$

is also bounded for all $2 \leq p < \infty$. In particular if

$$\int_{C^n} \int_{C^n} |G(z,w)|^p |K(z,w)|^2 d\mu(z) d\mu(w) < \infty$$

then A_G is in S_p.

Similarly we can prove the following lemma

Lemma 3.17. Suppose $2\leq p < +\infty$ and A_G is the integral operator on $L^2(C^n, dV)$, defined by

$$A_G f(z) = \int_{C^n} e^{-\frac{\|z-w\|^2}{8}} G(z,w) f(w) d(w)$$

If

$$\int_{C^n}\int_{C^n} |G(z,w)|^p e^{-\frac{\|z-w\|^2}{4}} dV(z) dV(w) < \infty \tag{3.14}$$

then A_G is in S_p.

Theorem 3.18. Suppose that f is in $BMO_\infty(C^n)$ and $p\geq 2$. If $\int_{C^n} MO(f)^p(z) < \infty$, then both $H_{f-\tilde{f}}$ and $H_{\overline{f-\tilde{f}}}$ are in S_p-class.

Proof It follows from Lemma 3.6 that for b in $L^2(C^n, dV)$ and g in $L^2(C^n, d\mu)$

$$|UPM_{|g|^2}PU^*b(z)| \leq \int_{C^n} e^{-\frac{\|z-w\|^2}{8}} \widetilde{|g|^2}(\frac{z+w}{2})|b(w)|dV(w).$$

In order to to show that $T_{|g|^2}$ is in $S_{p/2}$, we prove the integral operator

$$Tb(z) = \int_{C^n} e^{-\frac{\|z-w\|^2}{8}} \widetilde{|g|^2}(\frac{z+w}{2}) b(w) dV(w)$$

to be in $S_{p/2}$. From Lemma 3.17 it suffices to show that

$$\int_{C^n}\int_{C^n} \widetilde{|g|^2}^{p/2}(\frac{z+w}{2}) e^{-\frac{\|z-w\|^2}{4}} dV(z) dV(w) < \infty.$$

Let g = f - $\tilde{f}$, then Theorem 3.4 implies

$$\widetilde{|g|^2}(z) \leq MO^2(f)(z) + \textstyle\int_{C^n} |\tilde{f}(z) - \tilde{f}(w)|^2 e^{-\frac{\|z-w\|^2}{2}} dV(w)$$

$$\leq MO(f)^2(z) + \textstyle\int_{C^n}\int_0^1 MO(f)^2(z + t(w-z))\|z-w\|^2 \times$$

$$(\|z-w\| + C)^2 e^{-\frac{\|z-w\|^2}{2}} dV(w) dt.$$

So

$\int_{C^n}\int_{C^n}\widetilde{|g|^2}^{p/2}(\frac{z+w}{2})e^{-\frac{\|z-w\|^2}{4}}dV(z)dV(w)$

$\leq \int_{C^n}\int_{C^n}\widetilde{|g|^2}^{p/2}(w)e^{-\|z-w\|^2}dV(w)dV(z)$

$\leq \int_{C^n}\int_{C^n} MO(f)^p(w)e^{-\|z-w\|^2}dV(w)dV(z)+$

$C_1\int_{C^n}\int_{C^n}\int_{C^n}\int_0^1 MO(f)^p(z+tw)\|w\|^p(\|w\|+C)^p\times$

$exp\{-\frac{p\|w\|^2}{4}-\|z-u\|^2\}dV(z)dV(w)dV(u)dt \leq C_1\int_{C^n} MO(f)^p(w)dV(w)+$

$C_2\int_{C^n}\int_{C^n}\int_{C^n}\int_0^1 MO(f)^p(z)\|w\|^p(\|w\|+C)^p\times$

$exp\{-\frac{p\|w\|}{4}-\|z-tw-u\|^2\}\times dV(z)dV(w)dV(u)dt$

$\leq C_1\int_{C^n} MO(f)^p(w)dV(w)+C_2\int_{C^n} MO(f)^p(z)dV(z)$

$\leq C_3\int_{C^n} MO(f)^p(w)dV(w)<\infty$

where C_1, C_2, and C_3 are some positive constants.

On the other hand it is well-known that

$$H_g^*H_g = T_{|g|^2} - T_gT_{\overline{g}} \leq T_{|g|^2}. \tag{3.15}$$

Thus H_g is in S_p. So $\mathrm{H}_{f-\tilde{f}}$ is in S_p. Similarly we can prove that $\mathrm{H}_{\overline{f-\tilde{f}}}$ is in S_p.

Theorem 3.19. Suppose that f is in $\mathrm{BMO}_\infty(C^n)$ and p$\geq$ 2. If

$$\int_{C^n} MO(f)^p(z) < \infty, \tag{3.16}$$

then $H_{\tilde{f}}$ and $H_{\overline{\tilde{f}}}$ are in S_p.

Proof. Since for b in $H^2(C^n, d\mu)$, the Hankel operator can be written as the following integral operator

$$H_{\tilde{f}}b(z) = \int_{C^n} (\tilde{f}(z) - \tilde{f}(w))e^{\frac{z\overline{w}}{2}} d\mu(w),$$

Lemma 3.16 tells us that to prove the theorem it is sufficient to show that

$$\int_{C^n}\int_{C^n} |\tilde{f}(z) - \tilde{f}(w)|^p |e^{\frac{z\overline{w}}{2}}|^2 d\mu(w) d\mu(z) < \infty.$$

It follows from Theorem 3.4 that

$$\int_{C^n}\int_{C^n} |\tilde{f}(z) - \tilde{f}(w)|^p |e^{\frac{z\overline{w}}{2}}|^2 d\mu(w) d\mu(z)$$

$$\leq \int_{C^n}\int_{C^n}\int_0^1 MO(f)^p(z+t(w-z))\|z-w\|^p(\|z-w\|+C)^p e^{-\frac{\|z-w\|^2}{2}} dV(z)dV(w)dt$$

$$\leq \int_{C^n}\int_{C^n}\int_0^1 MO(f)^p(z+tw)\|w\|^p(\|w\|+C)^p e^{-\frac{\|w\|^2}{2}} dV(z)dV(w)dt$$

$$\leq \left(\int_{C^n} MO(f)^p(z)dV(z)\right)\left(\int_{C^n} \|w\|^p(\|w\|+C)^p e^{-\frac{\|w\|^2}{2}} dV(w)\right) < \infty.$$

In the same way we can prove that $H_{\overline{\tilde{f}}}$ is in S_p.

Lemma 3.20. Suppose A is a positive operator on $H^2(C^n, d\mu)$, then for $p \geq 1$

$$tr(A^p) = \int_{C^n} < A^p k_z, k_z > dV(z)$$
$$\geq \int_{C^n} < A k_z, k_z >^p dV(z).$$

The proof of the lemma is easy so we omit it. Summarizing all above results we state the following theorem.

Theorem 3.21. Suppose that f is in $BMO_\infty(C^n)$ and $p \geq 2$. Both H_f and $H_{\overline{f}}$ are in p-Schatten class if and only if

$$\int_{C^n} MO(f)^p(z) < \infty. \tag{3.17}$$

Combining Lemma 3.3 and Lemma 3.20 gives the proof of the only if part of the theorem.

Chapter 4

Some function algebras on the unit disk

In this chapter we will study the Bourgain algebras of AOP, $H^\infty + COP$, HCOP and the countably generated algebra $H^\infty[\mathcal{A}]$ where A is a subset of the complex conjugate of H^∞. Some properties of subalgebra A of C($\mathcal{M}$) are given if $A_b = C(\mathcal{M})$. Thin Blaschke products play a very important role since they have good properties on the Gleason parts.

4.1 The maximal ideal space of $H^\infty(D)$ and Blaschke products

The maximal ideal space of $H^\infty(D)$ is defined to be of multiplicative linear maps from $H^\infty(D)$ onto the field of complex numbers. If we think of $\mathcal{M}$ as a subset of the dual of $H^\infty(D)$ with the weak-star topology, then $\mathcal{M}$ becomes a compact Hausdorff space but is not a metric space.

If w is a point in the unit disk D, then the point evaluation at w is a multiplicative linear functional on H^∞, and so we can think of w as an element of $\mathcal{M}$. Thus we will freely think of the unit disk D as a subset of the maximal ideal space $\mathcal{M}$. Carleson's Corona Theorem [Car2] states that the disk D is dense in $\mathcal{M}$.

By using the Gelfand transform, we can think of H^∞ as a subset of C($\mathcal{M}$), the continuous, complex-valued functions on the maximal ideal space of H^∞. In fact Hoffman [Ho1] showed that C($\mathcal{M}$) is the algebra generated by the bounded harmonic functions on the unit disk D. Since Jones [J] proved that interpolating Blaschke products separate the maximal ideal space $\mathcal{M}$ of H^∞, the Stone-Weierstrass Theorem tells us that C($\mathcal{M}$) is a C*-algebra generated by interpolating Blaschke products although it is not known whether H^∞ is generated by interpolating Blaschke products as a uniform algebra.

For $\varphi, \tau \in \mathcal{M}$, the pseudohyperbolic distance between φ and τ, denoted by $\rho(\varphi, \tau)$, is defined by

$$\rho(\varphi, \tau) = sup\{|\varphi(f)| : f \in H^\infty, \|f\| < 1, and\ \tau(f) = 0\} \tag{4.1}$$

The Gleason part of φ is denoted by P(φ), and is defined by

$$P(\varphi) = \{\tau \in \mathcal{M} : \rho(\varphi, \tau) < 1\}. \tag{4.2}$$

For each $\varphi \in \mathcal{M}$, Hoffman [Ho2] constructed a fundamental canonical map L_φ of the unit disk D onto the part $P(\varphi)$. This map is defined by taking a net $\{w_\alpha\}$ in D such that $w_\alpha \to \varphi$ and defining

$$f \circ L_\varphi(z) = \lim_\alpha f(\frac{w_\alpha + z}{1 + \overline{w_\alpha} z}) \tag{4.3}$$

for z$\in D$ and f $\in H^\infty$, the above limit exists and is independent of the net $\{w_\alpha\}$ provided that $w_\alpha \to \varphi$. Budde [Bu] extended the map L_φ from the maximal ideal space $\mathcal{M}$ onto the closure of the part $P(\varphi)$ in $\mathcal{M}$.

For each bounded harmonic function f, the composition $f \circ L_\varphi$ is bounded and harmonic. Moreover

$$(D_{\bar{z}}^n D_z^m f \circ L_{w_\alpha})(z) \to (D_{\bar{z}}^n D_z^m f \circ L_\varphi)(z) \tag{4.4}$$

uniformly on any compact subset of D provided that $w_\alpha \to \varphi$.

Viewing H^∞ functions as continuous functions over the Shilov boundary

$$X = \mathcal{M}(L^\infty(\partial D)) \tag{4.5}$$

of H^∞, we can represent an element φ of $\mathcal{M}$ as integration over X against a positive measure $d\mu_\varphi$:

$$f(\varphi) = \int f d\mu_\varphi. \tag{4.6}$$

This representation allows us to extend φ to $L^\infty(\partial D)$ in such a manner that the Gelfand transforms of functions are also continuous on $\mathcal{M}$. The measure $d\mu_\varphi$ is called a representing measure, and its support is known as a support set, denoted by Suppμ_φ. The support set Suppμ_φ is a weak peak set for H^∞ ([Hol]).

An interpolating sequence is a sequence $\{w_n\}$ in D such that for every bounded sequence $\{c_n\}$ of complex numbers there is a function f$\in H^\infty$ such that $f(w_n) = c_n$ for every positive integer n. An interpolating Blaschke product b is a function on D of the form

$$b(z) = \prod_{n=1}^{\infty} \frac{|w_n|}{w_n} \frac{w_n - z}{1 - \overline{w_n} z} \tag{4.7}$$

which is associated with an interpolating sequence $\{w_n\}$. Carleson [Carl] proved that a sequence $\{w_n\} \subset D$ is interpolating if and only if

$$\inf_n \prod_{m=1, m \neq n}^{\infty} \left|\frac{w_m - w_n}{1 - w_m \overline{w_n}}\right| > 0. \tag{4.8}$$

An infinite sequence $\{w_n\}$ in D is called thin if

$$\prod_{m=1,m\neq n}^{\infty} |\frac{w_m - w_n}{1 - w_m \overline{w_n}}| \to 1 \tag{4.9}$$

as n$\to \infty$. The Blaschke product associated with a thin sequence is called a thin Blaschke product. $\varphi \in \mathcal{M} - D$ is called thin if φ is a cluster point of some thin sequence. For an interpolating Blaschke product b associated with $\{w_n\}$, we use Z(b) to denote the closure of $\{w_n\}$ in $\mathcal{M} - D$. Hoffman proved that for every thin point φ in $\mathcal{M} - D$, its support set is maximal, i.e. it is not properly contained in any other support set . So for any two distinct points x, y in Z(b), where b is thin, Supp$\mu_x \bigcap$Supp$\mu_y = \phi$.

4.2 AOP, $H^\infty + COP$, and HCOP

First we need the following proposition due to Izuchi.

Proposition 4.1. For any infinte sequence $\{x_n\} \subset$Z(b) where b is a thin Blaschke product, there is sequence $\{f_n\} \subset H^\infty$ such that

1.$|f_n(y_n)| > 1 - \varepsilon_n$ for some subsequence $\{y_n\} \subset \{x_n\}$ and $\varepsilon_n \to 0$;

2.$f_n \to 0$ weakly in H^∞;

3.$\|f\| \leq 1$.

The proof of Proposition 4.1 involves the fact that the support set of a thin point is maximal and any support sets are weak peak sets for H^∞. In fact using the idea in [AG2] Gorkin and Izuchi have a stronger result as follows: (For completeness we include a brief proof.)

Proposition 4.2. Let $\{\varphi_n\}$ be a sequence in $\mathcal{M}$ such that for distinct m and n, φ_m and φ_n are contained in distinct Gleason parts. Then there are a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ and a sequence $\{f_n\}$ in H^∞ such that

$$|f_k(\varphi_{n_k})| > \frac{3}{4}$$

and

$$\sum_{k=1}^{\infty} |f_k(z)| < 2$$

for all z in D.

Proof. By the nested intersection property, there exists φ in $\mathcal{M}$ such that

$$\varphi \in \bigcap_{n=1}^{\infty} \overline{\{\varphi_n, \varphi_{n+1}, \cdots\}}.$$

Our hypothesis implies that $\varphi_n \in P(\varphi)$ for at most one n. Thus from the proof of Theorem 3 in [AG2] we see that there are functions F_n and G_n such that for some subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$,

$$\sum_{n=1}^{\infty} |F_n(z) \prod_{j=1}^{n-1} G_j(z)| < 2$$

and

$$F_k(\varphi_{n_k}) = 1$$

and

$$|1 - (\prod_{j=1}^{k-1} G_j)(\varphi_{n_k})| < \frac{1}{4}.$$

Let $f_k = F_k(z) \prod_{j=1}^{k-1} G_j(z)$. Then

$$\sum_{k=1}^{\infty} |f_k(z)| < 2$$

and

$$|1 - f_k(\varphi_{n_k})| < \frac{1}{4}.$$

So $|f_k(\varphi_{n_k})| > \frac{3}{4}$.

We often use the following proposition in [5, 16] which describes the properties of thin Blaschke products on parts.

Proposition 4.3. Suppose that b is a thin Blaschke product. For any Gleason part P(m), either b is constant with modulus 1 on P(m) or there is a real number θ such that

$$b \circ L_m(z) = e^{i\theta} z$$

for z∈D.

Axler and Gorkin [AG1] proved the following proposition which is one of the interesting results on algebras on the unit disk.

Proposition 4.4. Let f be in H^∞. Let $\{w_n\}$ be a thin sequence in D and let b the associated Blaschke product. If

$$\lim_{n\to\infty} inf(1-|w_n|^2)|f'(w_n)| > 0 \tag{4.10}$$

then the complex conjugate of b is contained in $H^\infty[\overline{f}]$.

It is easy to see that neither AOP nor H^∞+COP contains any complex conjugates of interpolating Blaschke products. We first show the following theorem which implies that neither $(AOP)_b$ nor $(H^\infty + COP)_b$ contains any complex conjugates of interpolating Blaschke products.

Theorem 4.5. Suppose that f is in H^∞. If its complex conjugate $\overline{f}$ is in either $(AOP)_b$ or $(H^\infty + COP)_b$, then

$$\lim_{z\to\partial D}(1-|z|^2)f'(z) = 0. \tag{4.11}$$

Proof. Here we deal with just $(AOP)_b$ since from the following proof we will see that it is easy to get results on $(H^\infty + COP)_b$.

Step 1. First we show that $(AOP)_b$ does not contain any complex conjugates of thin Blaschke products.

Let b be a thin Blaschke product. Suppose that the complex conjugate $\bar{b}$ is in $(AOP)_b$. For any sequence $\{f_n\}$ which weakly converges to 0 in AOP, there is a sequence $\{g_n\} \subset AOP$ such that

$$\|\bar{b} f_n - g_n\| < \varepsilon_n \to 0.$$

Then for any Gleason part P(m),

$$\|\bar{b} \circ L_m f_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n.$$

Since $X = \mathcal{M}(L^\infty(\partial D))$ is contained in $\mathcal{M}$, then

$$\|\bar{b} \circ L_m f_n \circ L_m - g_n \circ L_m\|_X \leq \varepsilon_n.$$

It follows from Proposition 4.2 that $b \circ L_m$ has modulus 1 on X. So

$$\|f_n \circ L_m - b \circ L_m g_n \circ L_m\|_X \leq \varepsilon_n.$$

Thus on $\mathcal{M}$-D

$$\|f_n - b g_n\|_{\mathcal{M}-D} \leq \varepsilon_n.$$

Of course on Z(b) we have

$$\|f_n\|_{Z(b)} \leq \varepsilon_n. \tag{4.12}$$

On the other hand it follows from Proposition 4.1 that we can find a sequence $\{f_n\} \subset H^\infty$ such that

1. $|f_n(y_n)| > 1 - \varepsilon_n$ for some $y_n \in$Z(b) ;

2. $f_n \to 0$ weakly in H^∞.

For the sequence we have

$$\|f_n\|_{Z(b)} \geq 1 - \varepsilon_n$$

This contradicts (4.12). So far we have proved that $(AOP)_b$ does not contain the complex conjugates of any thin Blaschke products.

Step 2. Suppose that $(AOP)_b$ contains the complex conjugate $\overline{f}$ of f for f in H^∞ such that

$$\lim_{z \to \partial D} sup(1 - |z|^2)|f'(z)| > 0.$$

Without loss of generality we may choose a thin sequence $\{w_n\}$ in D such that

$$\lim_n (1 - |w_n|^2)|f'($$ [illegible]

Let b be the associated Blaschke product. It follows from Proposition 4 that $H^\infty[\overline{f}]$ contains the complex conjugate of b. So $(AOP)_b$ contains the conjugate of b since $(AOP)_b$ contains $H^\infty[\overline{f}]$. But this contradicts Step 1.

Let B_0 denote the little Bloch space. Immediately we have the following corollaries.

Corollary 4.6. $(H^\infty + COP)_b \bigcap \overline{H^\infty} = (AOP)_b \bigcap \overline{H^\infty} = \overline{B_0} \bigcap \overline{H^\infty}$.

Corollary 4.7. Neither $(AOP)_b$ nor $(H^\infty + COP)_b$ contains the conjugates of any interpolating Blaschke products.

The following lemma appeared in a proof in [CJY]. For completeness we state it again here.

Lemma 4.8. Suppose that $\{f_n\}$ is a sequence of H^∞ functions such that $\sum_{n=1}^{\infty} |f_n(z)| < M$ for all z in D. Then $f_n \to 0$ weakly in H^∞.

Proof. As in [CJY], if $\varphi \in (H^\infty)^*$ and $a_n = \overline{sgn < \varphi, f_n >}$ then for any positive integer N we have

$$\sum_{n=1}^{N} | < \varphi, f_n > | = \sum_{n=1}^{N} a_n < \varphi, f_n >$$

$$= < \varphi, \sum_{n=1}^{N} a_n f_n > \leq \|\varphi\| \| \sum_{n=1}^{N} a_n f_n \| \leq \|\varphi\| M$$

So $< \varphi, f_n > \to 0$ since the series $\sum_{n=1}^{\infty} | < \varphi, f_n > |$ converges.

Theorem 4.9. $(H^\infty + COP)_b$ and $(AOP)_b$ are proper subset of HCOP.

Proof. We still deal with only the case that $(AOP)_b$ is a proper subset of HCOP.

Suppose that there is a function f$\in (AO$ [illegible] HCOP. Then there is a thin part m such that $f \circ L_m$ is [illegible] $g = f \circ L_m|_X$. If g is not in $H^\infty + C(\partial D)$, Chang-Marshall's Theorem ([8, 12, 19]) tells us that $H^\infty[g]$ contains the complex conjugate of an interpolating Blaschke product ϕ. Since f is in $(AOP)_b$ and the Hoffman map L_m is bijective, for any sequence $\{f_n\} \subset AOP$ weakly converging to zero, there is a sequence $\{g_n\} \subset AOP$ such that

$$\|\overline{\phi} f_n \circ L_m - g_n \circ L_m\|_X \leq \varepsilon_n \to 0$$

as n goes to ∞. So

$$\|f_n \circ L_m - \phi g_n \circ L_m\|_X \leq \varepsilon_n.$$

Since $f_n \circ L_m$ and $g_n \circ L_m$ are in H^∞ and X is the Shilov boundary of H^∞, then

$$\|f_n \circ L_m - \phi g_n \circ L_m\| \leq \varepsilon_n.$$

Let $\{z_n\}$ be the zeros of ϕ in D. Beurling's Interpolation Theorem ([Ga]) implies that there is a sequence $\{F_n\}$ of H^∞ functions such that

1. $F_n(z_k) = \delta_{nk}$

2. $\sum |F_n(z)| < M$ *for all z in D*

for some constant M. Since m is thin, there is a thin Blaschke product b such that

$$b \circ L_m(z) = z.$$

Let $f_n = F_n \circ b$. Then

$$\sum |f_n(z)| = \sum |F_n(b(z))| < M$$

for all z in D. Lemma 4.8 tells us that $f_n \to 0$ weakly [illegible] here is a sequence $\{g_n\}$ in AOP such that

$$\|f_n \circ L_m - \phi g_n \circ L_m\| \leq \varepsilon_n.$$

On the zero set $Z(\phi) \bigcup \{z_n\}$ we have

$$\|f\|_{Z(\phi) \bigcup \{z_n\}} \leq \varepsilon_n \tag{4.13}$$

But $f_n \circ L_m(z_k) = F_n(z_k) = \delta_{nk}$, so $\|f\|_{Z(\phi) \bigcup \{z_n\}} = 1$. this contradicts (4.13). So $g \in H^\infty + C(\partial D)$.

We can naturally e[illegible] to D, denoted by G, which is harmonic in D. Let $F = f \circ L_m(z) - G(z)$. It is easy to see that F vanishes on X. Now we are going to prove that F is in $C_0(\overline{D})$. Otherwise, suppose that there is a sequence $\{z_n\} \subset D$ such that

$$1. z_n \to \partial D$$

$$2.|F(z_n)| > \delta$$

for some positive constant δ.

First we have to prove that for any sequence $\{f_n\}$ weakly converging to 0 in AOP there is a sequence $\{g_n\} \subset AOP$ such that

$$\|Gf_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

Since g is in $H^\infty + C(\partial D)$, then G is in $H^\infty + UC$. In addition, polynomials of z and the conjugate of z are dense in UC so it is sufficient to show that for any sequence $\{f_n\}$ weakly converging to 0 in AOP and any sequence $\{\varepsilon_n\}$ of positive numbers converging to 0 there is a sequence $\{g_n\} \subset AOP$ such that

$$\|\overline{z^l} f_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n$$

for all positive integers l. For simplicity we simply estalish the case for which $l = 1$.

Let $G_n = \frac{f_n \circ L_m - f_n(m)}{z}$. Then G_n is in H^∞ and G_n converges to 0 pointwise on D since $f \circ L_m$ converges to 0 pointwise on D. So both $f \circ L_m$ and G_n converge to 0 uniformly on any compact subset of D. On ∂D we have $\overline{z} f_n \circ L_m - G_n = e^{-i\theta} f_n(m)$. Hence

$$\|\overline{z} f_n \circ L_m - G_n\|_{\partial D} \to 0.$$

So

$$\|f_n \circ L_m - zG_n\| \to 0.$$

Clearly

$$\|\overline{z} f_n \circ L_m - \overline{z} z G_n\| \to 0.$$

It follows easily that

$$\|\overline{z}f_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n$$

if let $g_n = G_n \circ b$.

So for any sequence $\{f_n\}$ weakly converging to zero in AOP there is a sequence $\{g_n\} \subset AOP$ such that

$$\|Ff_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

On X , F vanishes. So $\|g_n \circ L_m\|_X \leq \varepsilon_n$. Since $g_n \circ L_m \in H^\infty$, then $\|g_n \circ L_m\|_X \leq \varepsilon_n$. Thus

$$\|Ff_n \circ L_m\| \leq \varepsilon_n.$$

On the other hand we can find a sequence $\{f_n\}$ weakly converging to 0 in H^∞ such that

$$f_n \circ L_m(z_k) = \delta_{nk}$$

Then $|F(z_n)f_n \circ L_m(z_n)| \geq \delta$ for all n. So $\delta \leq 2\varepsilon_n$. This is impossible since ε_n goes to 0. Hence $f \circ L_m - G = F \in C_0(D)$. So $f \circ L_m \in H^\infty + UC$ for every thin part m. So far we have proved that $(AOP)_b \subset HCOP$.

On the other hand it follows from Proposition 3 that HCOP contains the complex conjugates of every thin Blaschke product. However Theorem 4.5 tells us that $(AOP)_b$ does not contains the complex conjugates of any thin Blaschke product. So $(AOP)_b$ is a proper subset of HCOP.

In [GSZ] we showed that $(H^\infty + UC)_b = H^\infty + UC$. Now we can prove the following theorem

Theorem 4.10. $(HCOP)_b = HCOP$.

Proof. It is sufficient to show that $(HCOP)_b \subset HCOP$. Let f be in $(HCOP)_b$ and m be a thin point. Write $g = f \circ L_m|_X$.

Now we are going to show that g is in $H^\infty + C(\partial D)$. If g is not in $H^\infty + C(\partial D)$, Chang-Marshall's theorem implies that the Douglas algebra $H^\infty[g]$ contains the complex conjugate of some thin Blaschke product b. Since Hoffman's map is bijective, for any sequence $\{f_n\}$ weakly converging to zero in HCOP there is a sequence $\{g_n\}$ in HCOP such that

$$\|\overline{b}f_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n \to 0$$

then

$$\|f_n \circ L_m - bg_n \circ L_m\|_X \leq \varepsilon_n \to 0.$$

Because $f_n \circ L_m$ and $g_n \circ L_m$ are in $H^\infty + UC$, on the maximal ideal space $\mathcal{M} - D$ of $H^\infty + C(\partial D)$ we have

$$\|f_n \circ L_m - bg_n \circ L_m\|_{\mathcal{M}-D} \leq \varepsilon_n$$

So on Z(b)

$$\|f_n \circ L_m\|_{Z(b)} \leq \varepsilon_n \tag{4.14}$$

However since Proposition 4.3 implies that any two points in Z(b) lie in different Gleason parts and

$$\rho(L_m(\tau), L_m(\varphi)) = \rho(\tau, \varphi) = 1$$

for τ and φ in Z(b) it follows from Proposition 4.2 that we can find a sequence $\{f_n\}$ in H^∞ such that

$$\|f_n \circ L_m\|_{Z(b)} > \frac{3}{4} \tag{4.15}$$

and

$$\sum_{n=1}^{\infty} |f_n(z)| < 2.$$

By Lemma 4.8, the sequence $\{f_n\}$ weakly converges to 0 in H^∞. Thus (4.14) contradicts (4.15). So g is in $H^\infty + C(\partial D)$. We extend g to be harmonic on D, denoted by G. Let $F = f \circ L_m - G$. We claim that F is in $C_0(\overline{D})$. If not we may choose a thin sequence $\{z_n\}$ in D satisfying $|F(z_n)| \geq \delta$ for some positive constant δ. Let ϕ be the Blaschke product associated with zeros $\{z_n\}$.

First we note that in the proof of Theorem 4.9 we have proved that for any sequence $\{f_n\}$ weakly converging to 0 in H^∞ there is a sequence $\{g_n\}$ in H^∞ such that

$$\|Gf_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

So for the sequence $\{f_n\}$ there is a sequence $\{G_n\}$ in HCOP such that

$$\|Ff_n \circ L_m - G_n \circ L_m\| \leq \varepsilon_n$$

since f is in $(HCOP)_b$. As F vanishes on X, we have

$$\|G_n \circ L_m\|_X \leq \varepsilon_n.$$

If we extend $G_n \circ L_m|_X$ harmonic on D, denoted by H_n, then

$$\|H_n\| \leq \varepsilon_n.$$

Since $G_n \circ L_m$ is in $H^\infty + UC$, then $G_n \circ L_m = H_n$ on $\mathcal{M} - D$. So we have

$$\|Ff_n \circ L_m\|_{\mathcal{M}-D} \leq 2\varepsilon_n.$$

Hence

$$\|f_n \circ L_m\|_{Z(\phi)} \leq \frac{2\varepsilon_n}{\delta}.$$

But, as in the above argument, we can find a sequence $\{f_n\}$ such that f_n weakly converges to 0 in H^∞ and

$$\|f_n \circ L_m\|_{Z(\phi)} \geq \frac{3}{4}.$$

So $f \circ L_m - G$ is in $C_0(\overline{D})$. Hence $f \circ L_m$ is in $H^\infty + UC$ for every thin point m. Therefore f is in HCOP.

4.3 The Bourgain algebras of $H^\infty[\mathcal{A}]$

Let A be a subset of the complex conjugate of H^∞. We consider the algebra $H^\infty[\mathcal{A}]$ which is generated by A over H^∞. So we can think of $H^\infty[\mathcal{A}]$ as a closed subalgebra of the algebra $C(\mathcal{M})$ of continuous functions on $\mathcal{M}$. Since A is a subset of the complex conjugate of H^∞ it is easy to see that the maximal ideal space of $H^\infty[\mathcal{A}]$ is $\mathcal{M}$. This implies that the algebras on the disk cannot be determined by their maximal ideal space although the Douglas algebras are determined by their maximal ideal space by Chang-Marshall's theorem. We say $H^\infty[\mathcal{A}]$ to be countable generated if A is a countable subset of the complex conjugate of H^∞. In the section we show a countably generated algebra $H^\infty[\mathcal{A}]$ contains the conjugates of every thin Blaschke products in its Bourgain algebra . To do this we use Bishop's antisymmetric decomposition theorem ([Ga]) as in [AG1].

A subset S of maximal ideal space of a uniform algebra A is called antisymmetric for A if every function g in A which is real-valued on S must be constant on S.

Proposition 4.11. (Bishop Antisymmetric Decomposition) Let A be a uniform algebra on U. Let $\{E_\alpha\}$ be the family of maximal sets of antisymmetric for A. The E_α are closed disjoint subsets of U whose union is U. Each E_α is a p-set. If f$\in C(U)$, and $f|_{E_\alpha} \in A|_{E_\alpha}$ for all E_α, then $f \in A$.

Let A be a subset of the complex conjugate of H^∞. We define

$$E(\mathcal{A}) = \{m \in \mathcal{M} : f \circ L_m \textit{ is not constant for some } f \in \mathcal{A}\}. \tag{4.16}$$

First we are going to stre n a result in the proof of Theorem 1 in [AG1].

Lemma 4.12. For naximal antisymmetric set S of $H^\infty[\mathcal{A}]$, if the intersection $S \bigcap E(\mathcal{A})$ of d $E(\mathcal{A})$ is not empty, then S contains only one point.

Proof. Suppose t $E(\mathcal{A})$ is not empty. Then there is a function $f \in \mathcal{A}$ such that $S \bigcap E($ iot empty. From the proof of Theorem 4.5 [AG1] it follows that there is a rpolating Blaschke product b with zeros $\{z_n\}$ in D such that

$$S \subset Z(b).$$

Let $\varphi \in S$. Now we p at S is in the closure of the Gleason part $P(\varphi)$. If this is not true there oint $\tau \in S$ but τ is not in in the closure of $P(\varphi)$. Then there is an oper borhood V of τ such that $\overline{V} \bigcap \overline{P(\varphi)} = \phi$. Let $\{z_{n_k}\} = \{z_n\} \bigcap V$ and ie Blaschke product associated with $\{z_{n_k}\}$. Then b can be factored as t duct $b_1 b_2$ of two interpolating Blaschke products. So

$$= S \bigcap Z(b_1) \bigcup S \bigcap Z(b_2).$$

Since b_1 and b_2 are f of the inerpolating Blaschke product b, we can use the fact that Hoffman ([Ho2]) showed that $Z(b_1)$ and $Z(b_2)$ are disjoint. Thus either $S \subset Z(b_1)$ or $S \subset Z(b_2)$ since S is connected. This contradicts $\varphi \in S \bigcap Z(b_1)$ and $\tau \in S \bigcap Z(b_2)$. So $S \subset \overline{P(\varphi)}$.

As b is an interpolating Blaschke product, there is a nonvanishing function F and a Blaschke product ϕ with zeros $\{z_n\}$ in H^∞ on D such that

$$b \circ L_\varphi(z) = \phi(z)F(z).$$

In order to show that $S = \{\varphi\}$ it is sufficient to show that F does not vanish on the maximal ideal space $\mathcal{M}$ and $\{z_n\} = \{0\}$.

Since F is nonvanishing on D, if F vanishes at some point m$\in \mathcal{M} - D$, then F vanishes on the part P(m). Then $b \circ L_\varphi(m) = 0$. Thus $L_\varphi(m) \in Z(b)$. So $P(L_\varphi(m))$ is nontrivial. P. Budde ([Bu]) showed that $L_\varphi|_{P(m)}$ is a bijection from the part P(m) onto the part $P(L_\varphi(m))$. So b vanishes on $P(L_\varphi(m))$. This is impossible since b is an interpolating Blaschke product.

Because $P(\varphi)$ is a nontrivial part, Hoffman ([Ho2]) proved that there are an interpolating Blaschke product ψ and constants $0 < \delta < 1$ and $0 < \eta < 1$ such that

$$P(\varphi) \bigcap \{m : |\psi(m)| < \delta\} \subset \{L_\varphi(z) : |z| < \eta\}.$$

Since b is an interpolating Blaschke product, if $\{z_n\}$ is infinite, then $|z_n| \to 1$ as n goes to ∞. So if $|z_n| \geq \eta$, then

$$|\psi(L_\varphi(z_n))| \geq \delta.$$

Thus for τ in Z(ϕ) we have

$$|\psi(L_\varphi(\tau))| \geq \delta.$$

So

$$S \bigcap \{m : |\psi(m)| < \delta\} \subset (L_\varphi(Z(B) \bigcup \{z_n\}) \bigcap \{m : |\psi(m)| < \delta\}$$

$$= \{L_\varphi(z_n) : |z_n| < \eta\}.$$

Hence $S \bigcap \{m : |\psi(m)| < \delta\}$ is finite. This set is closed and open in S. Since S is connected, thus $S = \{\varphi\}$.

Theorem 4.13. Let A be a countably generated subalgebra of $C(\mathcal{M})$ and b be a thin Blaschke product. If $\bar{b}$ is in the Bourgain algebra A_b, then $\bar{b}$ is in A.

Proof. Let A be a countable subset $\{\overline{f_n}\}$ of the complex conjugate of H^∞. Let $A = H^\infty(\mathcal{A})$. Define

$$E(\mathcal{A}) = \{m \in \ldots f \circ L_m \textit{ is not constant for some } f \in \mathcal{A}\}.$$

Let b be the thin Blaschke product associated with $\{z_n\}$ in D such that its complex conjugate is in the Bourgain algebra A_b.

Step 1 We are trying to show that $Z(b) \subset E(\mathcal{A})$. If this is not true, there is a point m in Z(b) but m is not in $E(\mathcal{A})$. Then for all n , $f_n \circ L_m$ is constant on D.

For z in D we define

$$g_{n,l}(z) = (1 - |z|^2)^l f_n^{(l)}(z). \tag{4.17}$$

Hoffman showed that $g_{n,l}$ extends continuously to the maximal ideal space $\mathcal{M}$. Define

$$G(z) = \sum_{n,l=0}^{\infty} \frac{|g_{n,l}(z)|}{2^{n+l} \|g_{n,l}\|}. \tag{4.18}$$

Since for $\alpha \in D$ and f in H^∞ there are constants c_k such that

$$(f \circ L_\alpha)^{(l)}(0) = \sum_{k=1}^{l} c_k \overline{\alpha^{l-k}} (1 - |z|^2)^\alpha f^{(k)}(\alpha) \tag{4.19}$$

then every functions in A are constant on the part P(m) if and only if $g_{n,l}(m) = 0$ for all n, l if and only if $G(m) = 0$.

As m is in Z(b), there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{n_k} G(z_{n_k}) = 0$. Let b_1 be the thin Blaschke product associated with $\{z_{n_k}\}$. Then for any m in $Z(b_1)$, we have $G(m) = 0$, so every function in A is constant on the part P(m) for m in $Z(b_1)$.

On the other hand since $H^\infty[\overline{b}] \subset A_b$, it follows from Proposition 4.4 that the complex conjugate $\overline{b_1} \in A_b$. So for any sequence $\{F_n\}$ weakly converging to 0 in A, there is a sequence $\{g_n\}$ in A such that

$$\|\overline{b_1} F_n - g_n\| \leq \varepsilon_n \to 0.$$

Hence for m in $Z(b_1)$,

$$\|\overline{b_1 \circ L_m} F_n \circ L_m - g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

Thus

$$\|\overline{b_1 \circ L_m} F_n \circ L_m - g_n \circ L_m\|_X \leq \varepsilon_n \to 0.$$

By Proposition 4.3, $b_1 \circ L_m$ has modulus 1 on X. So

$$\|F_n \circ L_m - b_1 \circ L_m g_n \circ L_m\|_X \leq \varepsilon_n \to 0.$$

Since every function in A is constant on the part P(m), we have $F_n \circ L_m$ and $g_n \circ L_m$ are in H^∞. So

$$\|F_n \circ L_m - b_1 \circ L_m g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

If m is not in $Z(b_1)$, then $b_1 \circ L_m$ is constant with modulus 1. Thus we still have that

$$\|F_n \circ L_m - b_1 \circ L_m g_n \circ L_m\| \leq \varepsilon_n \to 0.$$

So

$$\|F_n - b_1 g_n\|_{\mathcal{M}-D} \leq \varepsilon_n \tag{4.20}$$

On the other hand, it follows from Proposition 4.1 that there is a sequence $\{F_n\}$ such that F_n weakly goes to 0 in H^∞ and

$$\|F_n\|_{Z(b_1)} \geq 1 - \varepsilon_n.$$

But (4.20) implies that

$$\|F_n\|_{Z(b_1)} \leq \varepsilon_n.$$

So this is not possible. Thus $Z(b) \subset E(\mathcal{A})$.

Step 2. Using Bishop's antisymmetric decomposition theorem we will show that $\overline{b} \in A$.

If $S \bigcap E(\mathcal{A})$ is not empty, Lemma 4.12 implies that S contains just one point. Clearly $\overline{b}_S \in A|_S$.

If $S \bigcap E(\mathcal{A})$ is empty, in Step 1 we have proved that $Z(b) \subset E(\mathcal{A})$. Then b cannot vanish on S. Because S is a weak-peak set for the algebra A, the maximal ideal space of the Banach algebra $A|_S$ equals S. Thus $b|_S$ does not vanish on the maximal ideal space of $A|_S$, and so $\frac{1}{b}|_S \in A|_S$. Since S does not contain any zeros of b, Proposition 4.3 gives us that b must have modulus 1 on S. Thus

$$\overline{b}|_S = \frac{1}{b}|_S \in A|_S.$$

This completes the proof.

In [GIM] Gorkin, Izuchi and Mortini showed that for any Douglas algebra if its Bourgain algebra B_b is $L^\infty(\partial D)$, then it must be $L^\infty(\partial D)$. Here we will give a different proof which leads us to study subalgebras A of $C(\mathcal{M})$ such that its Bourgain algebra $A_b = C(\mathcal{M})$.

Theorem 4.14. Let B be a Douglas algebra. If its Bourgain algebra in $L^\infty(\partial D)$ is $L^\infty(\partial D)$, then B equals $L^\infty(\partial D)$.

Proof. Suppose that there is a Douglas algebra B such that $B_b = L^\infty(\partial D)$ but B does not equal $L^\infty(\partial D)$. Chang-Marshall's Theorem tells us that there exists an inner function b such that its complex conjugate $\overline{b}$ is not in B. Let

$$K = \{x \in \mathcal{M}(B) : |b(x)| < 1\} \tag{4.21}$$

then $b(K) = D$. Otherwise there is a c in D such that c is not in b(K). Define

$$I = \frac{b - c}{1 - \overline{c}b}. \tag{4.22}$$

Then I is an inner function and invertible on $\mathcal{M}(B)$, Chang-Marshall's Theorem implies that $\overline{I} \in B$. So $\overline{b} \in B$. This contradicts that $\overline{b}$ is not in B.

Let $b(m_n) = z_n$ for $m_n \in K$ such that $\{z_n\}$ is an interpolating sequence in D, Beurling's Theorem tells us that there is a sequence $\{f_n\}$ in H^∞ such that

$$f_n(z_k) = \delta_{nk}$$

and

$$\sum_{n-1}^{\infty} |f_n(z)| < M \ for\ all\ z \in D$$

for some positive constant M. Let $F_n = f_n \circ b$. Then

$$\sum_{n=1}^{\infty} |F_n(z)| = \sum_{n-1}^{\infty} |f_n(b(z))| < M$$

It follows from Lemma 4.8 that F_n weakly converges to 0 in H^∞. Let ϕ be the Blaschke product associated with the interpolating sequence $\{z_n\}$. Now we consider

$$dist_{\partial D}(\overline{\phi \circ b}F_n, B) = dist_{\partial D}(F_n, (\phi \circ b)B)$$

$$\geq Sup_k|F_n \circ b(m_k)| = Sup_k|f_n(z_k)| = 1.$$

This means that $\overline{\phi \circ b}$ is not in B_b. So B_b does not equal $L^\infty(\partial D)$. Hence B must be $L^\infty(\partial D)$ if its Bourgain algebra B_b equals $L^\infty(\partial D)$.

Since X is a subset of the maximal ideal space $\mathcal{M}$ of H^∞, for every subalgebra A of $C(\mathcal{M})$, $A|_X$ can be viewed as a subalgebra of $L^\infty(\partial D)$

Theorem 4.15. Suppose that A is a subalgebra of $C(\mathcal{M})$ and contains H^∞. If its Bourgain algebra A_b is $C(\mathcal{M})$, then $A|_X$ must be $L^\infty(\partial D)$.

Proof Since A contains H^∞, then $B = A|_X$ is a Douglas algebra. By Chang-Marshall's Theorem we just show that complex conjugates of interpolating Blaschke products are in B. Otherwise let b be an interpolating Blaschke product such that its complex conjugate $\overline{b}$ is not in B

Let

$$K = \{x \in \mathcal{M}(B) : |b(x)| < 1\}$$

then $b(K) = D$. Otherwise there is a c in D such that c is not in b(K). Define

$$I = \frac{b - c}{1 - \overline{c}b}.$$

Then I is an inner function and invertible on $\mathcal{M}(B)$, Chang-Marshall's Theorem implies that $\overline{I} \in B$. So $\overline{b} \in B$. This contradicts that $\overline{b}$ is not in B.

Let $b(m_n) = z_n$ for $m_n \in K$ such that $\{z_n\}$ is an interpolating sequence in D, Beurling's Theorem tells us that there is a sequence $\{f_n\}$ in H^∞ such that

$$f_n(z_k) = \delta_{nk}$$

and

$$\sum_{n-1}^{\infty} |f_n(z)| < M \ for \ all \ z \in D$$

for some positive constant M. Let $F_n = f_n \circ b$. Then

$$\sum_{n=1}^{\infty} |F_n(z)| = \sum_{n-1}^{\infty} |f_n(b(z))| < M$$

It follows from Lemma 4.8 that F_n weakly converges to 0 in H^∞. Let ϕ be the Blaschke product associated with the interpolating sequence $\{z_n\}$. Since A_b equals $C(\mathcal{M})$, then $\overline{\phi \circ b}$ is in A_b. So

$$dist_{\mathcal{M}}(\overline{\phi \circ b} F_n, A) \to 0.$$

On the other hand we have that

$$dist_{\mathcal{M}}(\overline{\phi \circ b} F_n, A)$$

$$\geq dist_X(\overline{\phi \circ b} F_n, B)$$

$$= dist_X(F_n, (\phi \circ b)B)$$

$$\geq dist_K(f_n \circ b, (\phi \circ b)B)$$

$$\geq Sup_k |f_n(b(m_k))| = 1.$$

This contradicts to that $\overline{\phi \circ b}$ is in A_b.

Since $L^\infty(\partial D)$ is not a countably generated Douglas algebra then from Theorem 4.15 we get the following corollary

Corollary. There is no a countably generated subalgebra such that its Bourgain algebra is $C(\mathcal{M})$.

Since Jones proved that interpolating Blaschke products separate the maximal ideal space $\mathcal{M}$ of H^∞, we see that the algebra $C(\mathcal{M})$ of the continuous functions on $\mathcal{M}$ is generated by the complex conjugates of interpolating Blaschke products over H^∞. So we are going to study the subalgebra $H^\infty[\mathcal{A}]$ such that its Bourgain algebra is $C(\mathcal{M})$ for some subset A of the complex conjugate of H^∞.

Theorem 4.16. Let A be a subset of the complex conjugate of H^∞ and $A = H^\infty[\mathcal{A}]$. If the Bourgain algebra A_b is $C(\mathcal{M})$, then A contains the complex conjugates of every thin Blaschke products

Proof. We use Bishop's antisymmetric decomposition theorem as in the proof of Theorem 4.13 to prove Theorem 4.16. Let b be a thin Blaschke product. First we are going to prove that $Z(b) \subset E(\mathcal{A})$ where

$$E(\mathcal{A}) = \{m : f \circ L_m \; is \; not \; constant \; for \; some \; f \; in \mathcal{A}\}.$$

If this is not true there is a point m in Z(b) such that every function in A is constant on the Gleason part P(m). It follows from Proposition 4.3 that $b \circ L_m(z) = e^{i\theta} z$ for z in D . Let $\{z_n\}$ be an interpolating sequence in D. Define $w_n = e^{-i\theta} z_n$. Then $b \circ L_m(w_n) = z_n$. By the Beurling interpolation theorem there is a sequence $\{f_n\}$ in H^∞ such that $f_n(z_k) = \delta_{n,k}$ and $\sum_{n=1}^{\infty} |f_n(z)| < M$ for some positive constant M. Define $F_n = f_n \circ b$. Lemma 4.8 implies that F_n weakly converges to zero in H^∞. Let ϕ be the Blaschke product associated with $\{z_n\}$. Then $dist_{\mathcal{M}}(\overline{\phi \circ b} F_n, A)$ goes to 0 as n goes to ∞.

But

$$dist_{\mathcal{M}}(\overline{\phi \circ b} F_n, A)$$

$$\geq dist_{P(m)}(\overline{\phi \circ b} F_n, A)$$

$$\geq dist_D(\overline{\phi \circ b \circ L_m} f_n \circ b \circ L_m, A \circ L_m)$$

$$\geq dist_X(\overline{\phi \circ b \circ L_m} f_n \circ b \circ L_m, H^\infty)$$

$$= dist_X(f_n \circ b \circ L_m, (\phi \circ b \circ L_m) H^\infty)$$

$$= dist_D(f_n \circ b \circ L_m, (\phi \circ b \circ L_m) H^\infty)$$

$$\geq Sup_k |f_n \circ b \circ L_m(w_k)| = 1.$$

This contradicts that $dist_{\mathcal{M}}(\overline{\phi \circ b} F_n, A)$ goes to 0 as n goes to ∞. So $Z(b) \subset E(\mathcal{M})$.

Let S be any maximal antisymmetric set of A. We are going to show that $\overline{b}|_S$ is in $A|_S$.

If $S \bigcap E(\mathcal{A})$ is not empty, it follows from Lemma 4.12 that S contains only one point. Clearly $\overline{b}|_S$ is in $A|_S$.

If $S \bigcap E(\mathcal{A})$ is empty, so is $S \bigcap Z(b)$ since we have proved above that $Z(b) \subset E(\mathcal{A})$. As in the proof of Theorem 4.13 we see that $\overline{b}|_S = \frac{1}{b}|_S \in A|_S$. By Bishop's antisymmetric decomposition theorem we have that $\overline{b} \in A$.

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