Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms

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Abstract of dissertation

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We are concerned with the stability of flows of an ideal fluid in Lagrangian coordinates. We find conditions for linear and nonlinear stability of flows in the Lagrangian sense and construct classes of stable and unstable flows. Our method involves the study of the geometry of $\mathcal{D}_\mu(M)$-the group of volume preserving diffeomorphisms of a smooth, compact Riemannian manifold M, which is the region filled with fluid. Related to stability is the existence of conjugate points on $\mathcal{D}_\mu(M)$. Using Jacobi fields we construct such points for spheres S^2, S^3 and ball B^3.

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Chapter 1

Introduction

We are concerned with the stability of flows of an ideal fluid in Lagrangian coordinates. We find conditions for linear and nonlinear stability of flows in the Lagrangian sense and construct classes of stable and unstable flows. Our method involves the study of the geometry of $\mathcal{D}_\mu(M)$ - the group of volume preserving diffeomorphisms of a smooth, compact Riemannian manifold M, which is the region filled with fluid. Related to stability is the existence of conjugate points on $\mathcal{D}_\mu(M)$. Using Jacobi fields we construct such points for spheres S^2, S^3 and the ball B^3.

In the Lagrangian formulation of hydrodynamics of ideal fluids, as developed by V. Arnold [A], D. Ebin and J. Marsden [EM], $\mathcal{D}_\mu(M)$ can be considered an infinite dimensional manifold equipped with a weak Riemannian structure. It is well known that fluid flows in M (with or without boundary) correspond to geodesics in $\mathcal{D}_\mu(M)$. Using the weak Riemannian structure we introduce the notion of Lagrangian stability of fluid flows in M saying that a flow $\eta(t)$ is stable if all geodesics in $\mathcal{D}_\mu(M)$ with sufficiently close initial conditions at t=0 remain close for all t$\geq$0.

It must be emphasized that stability in Lagrangian coordinates is not the same as stability in Eulerian coordinates. Roughly speaking in the Lagrangian case one is concerned with positions of the fluid particles, whereas in the Eulerian case with their velocities considered as functions of their position in space. Thus a velocity field $u(t)$ of the fluid on M is stable in the Eulerian sense if small changes in initial conditions $u(0)$ result in small changes in $u(t)$ for all later times. The classical result of Rayleigh (cf [Li]) gives conditions for (linear) Eulerian stability of stationary plane parallel flows. We construct examples of flows which are stable in the Eulerian but unstable in the Lagrangian sense and also flows which are unstable in both senses.

In section 2 we describe the functional analytic setting following [EM]. In section 3 we show that the (weak) curvature operator of $\mathcal{D}(M)$ and the (weak) second fundamental form of $\mathcal{D}_\mu(M)$ are bounded in the (strong) H^s topology, $s > \frac{n}{2} + 1$. Next in section 4 we prove the existence and uniqueness of Jacobi fields on $\mathcal{D}_\mu(M)$. In section 5 we use the Gauss' equation to compute the (weak) curvature of $\mathcal{D}_\mu(M)$ and obtain results on linear stability of fluid flows while in section 6 we treat the nonlinear case. Finally in section 7 we give examples of conjugate points on $\mathcal{D}_\mu(M)$.

Chapter 2

A weak Riemannian structure for $\mathcal{D}^s$ and $\mathcal{D}^s_\mu$

The proofs of all the main results in this section may be found in either [EM] or [E1].

We begin with $\mathcal{D}(M)$ - the group of all diffeomorphisms of a compact Riemannian manifold M. $\mathcal{D}(M)$ can be considered a smooth, infinite dimensional manifold modelled locally on a Frechet space $C^\infty(TM)$. To avoid Frechet spaces one can enlarge $\mathcal{D}$ to include all bijective maps $\eta : M \to M$ such that η and η^{-1} are of Sobolev class H^s. If $s > \frac{n}{2}+1$ this enlarged set $\mathcal{D}^s(M)$ becomes a smooth manifold which now locally, around each of its points η, looks like a Hilbert space $H^s_\eta(TM) = \{V : M \to TM : V \in H^s, \pi \circ V = \eta\}$, where $\pi : TM \to M$. A chart at η, $\Phi : H^s_\eta(TM) \to \mathcal{D}^s$ is defined by $\Phi(X) = \exp \circ X$ where exp is the exponential map of M. That Φ is a local homeomorphism and the overlap maps are smooth follows from the usual properties of exp. Furthermore, $\mathcal{D}^s$ can be given a group structure with multiplication being the composition of two H^s diffeomorphisms, and then be continuously embedded in the group of C^1 diffeomorphisms by the Sobolev lemma. Right multiplication in $\mathcal{D}^s$ is smooth but the left multiplication is only continuous in the H^s topology.

If M has a nonempty boundary ∂M, we embedd it in its double $\hat{M}$. We choose a metric on $\hat{M}$ for which ∂M is totally geodesic and let $\mathcal{D}^s(M)$ consist of all H^s bijections mapping M to M with H^s inverses. The construction of charts is now analogous to the one described previously except this time the exponential map comes from the metric on $\hat{M}$. Consult [EM] for additional details.

Using the L^2 inner product we can equip $\mathcal{D}^s(M)$ with a weak Riemannian metric given by

$$(V,W)_\eta = \int_M \langle V(x), W(x)\rangle_{\eta(x)}\mu(x) \tag{2.1}$$

where $\eta \in \mathcal{D}^s$, $V, W \in T_\eta\mathcal{D}^s(M) = H^s_\eta(TM)$ and $\langle\cdot,\cdot\rangle$ and μ are the Riemannian metric and the volume element of M.

The weak Riemannian connection $\bar{\nabla}$ associated with $(\cdot,\cdot)$ on $\mathcal{D}^s$ can be obtained as follows. Let $K : T^2M \to TM$ be the connector induced by ∇, the Riemannian connection of M. Then if X, Y are smooth vector fields on $\mathcal{D}^s$ define

$$\bar{\nabla}_X Y = K \circ (TY(X))$$

This connection is preserved under right multiplication by $\mathcal{D}^s$. The geodesics of $\bar{\nabla}$ are all those curves $\eta(t)$ in $\mathcal{D}^s$ which for each $x \in M$ are geodesics $t \to \eta(t)(x)$ in M.

Similarly we define $\mathcal{D}^s_\mu(M)$ to be the completion of $\mathcal{D}_\mu(M)$ - the group of all volume preseving diffeomorphisms of M - in the H^s topology. From the implicit function theorem it follows that $\mathcal{D}^s_\mu$ is a submanifold of $\mathcal{D}^s$. It is also a subgroup. For each $\eta \in \mathcal{D}^s_\mu$ we have a smooth map given by

$$P_\eta : T_\eta \mathcal{D}^s \to T_\eta \mathcal{D}^s_\mu$$

$$P_\eta(X) = (P_e X \circ \eta^{-1}) \circ \eta$$

where P_e is the orthogonal projection onto the divergence free part in the Weyl decomposition:

$$H^s(TM) = \mathrm{div}^{-1}(0) \oplus_{L^2} \mathrm{grad} H^{s+1}(M)$$

where $\mathrm{div}^{-1}(0) = \{u \in H^s(TM) : \mathrm{div}\, u = 0, u \text{ tangent to } \partial M\}$.

In order to provide a formula for P_e we must further decompose $H^s(TM)$. Given $u \in H^s(TM)$ let p be the solution of the Dirichlet problem

$$\begin{aligned} \Delta p &= \mathrm{div}\, u\,, \\ \mathrm{supp}\, p &\subset M\,, \end{aligned} \tag{2.2}$$

and let $\mathcal{HE}(u)$ be the solution of the Neumann problem

$$\begin{aligned} \Delta \mathcal{HE}(u) &= 0\,, \\ \langle \mathrm{grad}\, \mathcal{HE}(u), \nu \rangle &= \langle u - \mathrm{grad}\, p, \nu \rangle\,, \end{aligned} \tag{2.3}$$

where ν is the outer unit normal field on ∂M.

It is easily seen that $\mathrm{grad}\, p$ and $\mathrm{grad}\, \mathcal{HE}(u)$ are L^2 orthogonal and that the projection onto $\mathrm{div}^{-1}(0)$ summand is now given by

$$P_e(u) = u - \text{grad}\Delta^{-1}\text{div}\, u - \text{grad}\mathcal{HE}(u)$$

where $\Delta^{-1}\text{div}\, u$ denotes the solution of the Dirichlet problem above.

We shall also denote the orthogonal projection onto $\text{grad}H^{s+1}(M)$ by

$$Q_e(u) = \text{grad}\Delta^{-1}\text{div}\, u + \text{grad}\mathcal{HE}(u) \tag{2.4}$$

$\mathcal{D}^s_\mu(M)$ becomes now a weak Riemannian submanifold of $\mathcal{D}^s$ with the metric (2.1) and the Riemannian connection $\tilde{\nabla} = P \circ \bar{\nabla}$ inherited from $\mathcal{D}^s$, where P is the projection defined above. The metric on $\mathcal{D}^s_\mu$ as well as its connection $\tilde{\nabla}$ are right invariant and the geodesics in $\mathcal{D}^s_\mu(M)$ correspond to fluid flows in the following sense. If $\eta(t)$ is a geodesic then $u(t) = \dot{\eta}(t) \circ \eta^{-1}(t)$ is a vector field on M which satisfies the Euler equations of an ideal fluid

$$\partial_t u(t) + \nabla_{u(t)} u(t) = \text{grad}\, p(t)$$

$$\text{div}\, u(t) = 0, \; u(0) = u_0$$

where $u(t)$ is tangent to ∂M and $p(t)$ is the pressure function which can be determined from $u(t)$.

Chapter 3

Curvature and second fundamental form

In this section we use the weak Riemannian structures of $\mathcal{D}^s_\mu(M)$ and $\mathcal{D}^s(M)$ to introduce the weak Riemannian analogues of finite dimensional geometrical invariants.

But first we shall obtain a more convienient, for our purposes, formula for $\bar{\nabla}$. For $\eta \in \mathcal{D}^s$, $X_\eta \in T_\eta\mathcal{D}^s$ and $Y \in C^\infty(T\mathcal{D}^s)$ let $t \to \varphi(t)$ be an integral curve of X_η in $\mathcal{D}^s$ passing through η i.e. $\varphi(0) = \eta$, $\frac{d\varphi}{dt}(0) = X_\eta$. Then since $Y_{\varphi(t)} \circ \varphi^{-1}(t)$ is a curve in T_xM for each $x \in M$, we have from the definition of a connector (cf for example [E1])

$$\begin{aligned} (\bar{\nabla}_X Y)_\eta &= K \circ (\frac{d}{dt}(Y_{\varphi(t)}))|_{t=0} = K \circ \frac{d}{dt}(Y_{\varphi(t)} \circ \varphi(t)^{-1} \circ \varphi(t))|_{t=0} = \\ &= \frac{d}{dt}(Y_{\varphi(t)} \circ \varphi^{-1}(t))|_{t=0} \circ \eta + (\nabla_{X_\eta \circ \eta^{-1}} Y_\eta \circ \eta^{-1}) \circ \eta \end{aligned} \tag{3.1}$$

Alternatively we can write

$$(\bar{\nabla}_X Y)_\eta = \frac{d}{dt}(Y_{\varphi(t)})|_{t=0} + \Gamma_\eta(X_\eta, Y_\eta) \tag{3.2}$$

where $\Gamma_\eta : T_\eta\mathcal{D}^s \times T_\eta\mathcal{D}^s \to T_\eta\mathcal{D}^s$ is the map which in a local chart (U, x^i) on

M^n is defined by

$$\Gamma_\eta(X_\eta, Y_\eta) = (\sum_{i,j,k} \Gamma^i_{jk} X^j \circ \eta^{-1} Y^k \circ \eta^{-1} \frac{\partial}{\partial x^i}) \circ \eta$$

Using the connections $\bar{\nabla}$ and $\tilde{\nabla}$ we first introduce the second fundamental form S of $\mathcal{D}^s_\mu(M)$, by

$$S_\eta : T_\eta \mathcal{D}^s_\mu \times T_\eta \mathcal{D}^s_\mu \rightarrow \nu_\eta \mathcal{D}^s_\mu$$

$$S_\eta(X_\eta, Y_\eta) = Q_\eta(\bar{\nabla}_X Y) \tag{3.3}$$

where $\eta \in \mathcal{D}^s_\mu, X, Y$ are extensions of X_η, Y_η to C^∞ vector fields on $\mathcal{D}^s_\mu(M)$, $\nu\mathcal{D}^s_\mu$ is the normal bundle of $\mathcal{D}^s_\mu(M)$ with respect to the weak Riemannian metric (2.1) and

$$Q_\eta(X_\eta) = (Q_e(X \circ \eta^{-1})) \circ \eta$$

can be computed from (2.4).

Thus S_η is a bilinear vector valued map which being in fact the difference of two Riemannian connections $\bar{\nabla}_X Y - \tilde{\nabla}_X Y$ is also symmetric.

We next define the (weak) Riemannian curvature tensor $\bar{R}$ by

$$\bar{R}_\eta : T_\eta \mathcal{D}^s \times T_\eta \mathcal{D}^s \times T_\eta \mathcal{D}^s \rightarrow T_\eta \mathcal{D}^s$$

$$\bar{R}_\eta(X_\eta, Y_\eta) Z_\eta = (\bar{\nabla}_X \bar{\nabla}_Y Z)_\eta - (\bar{\nabla}_Y \bar{\nabla}_X Z)_\eta - (\bar{\nabla}_{[X,Y]} Z)_\eta$$

where $\eta \in \mathcal{D}^s$ and X, Y, Z are smooth extensions of vectors X_η, Y_η, Z_η to a neighbourhood of η.

Proposition 3.4 *The curvature $\bar{R}$ of $\mathcal{D}^s(M)$ is completely determined by the curvature of M. Furthermore S_η and $\bar{R}_\eta$ are invariant, with respect to the right multiplication by the elements of $\mathcal{D}^s_\mu$, multilinear maps which are bounded in the H^s topology if $s > \frac{n}{2} + 1$.*

Proof. Let $X_\eta, Y_\eta, Z_\eta \in T_\eta\mathcal{D}^s$, $\eta \in \mathcal{D}^s$ and let X^R, Y^R, Z^R be smooth right invariant vector fields such, that $X^R(\eta) = X_\eta, Y^R(\eta) = Y_\eta, Z^R(\eta) = Z_\eta$. Then, using the right invariance of $\bar{\nabla}$, $\bar{R}$ can be computed as follows.

$$\bar{R}(X_\eta, Y_\eta)Z_\eta =$$

$$= (\bar{\nabla}_{X^R}\bar{\nabla}_{Y^R}Z^R)(\eta) - (\bar{\nabla}_{Y^R}\bar{\nabla}_{X^R}Z^R)(\eta) - (\bar{\nabla}_{[X^R,Y^R]}Z^R)(\eta) =$$

$$= (\nabla_{X_e^R}\nabla_{Y_e^R}Z_e^R)\circ\eta - (\nabla_{Y_e^R}\nabla_{X_e^R}Z_e^R)\circ\eta - (\nabla_{[X_e^R,Y_e^R]}Z_e^R)\circ\eta =$$

$$= (R(X_e^R, Y_e^R)Z_e^R)\circ\eta = (\bar{R}_e(X_\eta\circ\eta^{-1}, Y_\eta\circ\eta^{-1})Z_\eta\circ\eta^{-1})\circ\eta \tag{3.5}$$

where ∇, R are the Riemannian connection and the curvature of M. Right invariance of $\bar{R}$ thus follows from the right invariance of $\bar{\nabla}$. Similarly for S_η. Now choosing a chart (U, x^i) on M and denoting by the same letter the representative of $\eta \in \mathcal{D}^s(M)$ in U we have by the composition lemma for H^s maps ([E1], chapter 4) and the fact that H^s functions form a Schauder ring if $s > \frac{n}{2}$ ([E1], chapter 2)

$$\|\bar{R}_\eta(X_\eta, Y_\eta)Z_\eta\|_{H^s} = \|(R(X_\eta\circ\eta^{-1}, Y_\eta\circ\eta^{-1})Z_\eta\circ\eta^{-1})\circ\eta\|_{H^s} \le$$

$$\le \sum_{ijk} C\|R^i_{jkl}X^j_\eta\circ\eta^{-1}Y^k_\eta\circ\eta^{-1}Z^l_\eta\circ\eta^{-1}\frac{\partial}{\partial x^i}\|_{H^s}|max J(\eta)|^{\frac{1}{2}}(\|\eta\|^s_{H^s}+1) \le$$

$$\le C\|X_\eta\|_{H^s}\|Y_\eta\|_{H^s}\|Z_\eta\|_{H^s}$$

where C denotes any constant which may depend on s, η and the derivatives of the metric g_{ij} of M. Let now $\eta \in \mathcal{D}^s_\mu$ and let $X_\eta, Y_\eta \in T_\eta\mathcal{D}^s_\mu$. From (2.2), (2.3) and [ADN] (section 14), we obtain

$$\|S_\eta(X_\eta, Y_\eta)\|_{H^s} = \|(Q_e(\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}))\circ\eta\|_{H^s} \le$$

$$\leq C\|\mathrm{grad}\Delta^{-1}\mathrm{div}\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}\|_{H^s}+C\|\mathrm{grad}\mathcal{HE}(\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1})\|_{H^s}\leq$$

$$\leq C\|\mathrm{div}\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}\|_{H^{s-1}(M)}+C\|\langle\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}-\mathrm{grad}p,\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}$$

where $p=\Delta^{-1}\mathrm{div}\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}$ and C is as above.

For the boundary term we observe that ∂M is a smooth $n-1$ dimensional Riemannian submanifold of M, whose second fundamental form can be written as

$$\langle\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1},\nu\rangle=X_\eta\circ\eta^{-1}\cdot\langle Y_\eta\circ\eta^{-1},\nu\rangle-\langle Y_\eta\circ\eta^{-1},\nabla_{X_\eta\circ\eta^{-1}}\nu\rangle=$$

$$=-\langle Y_\eta\circ\eta^{-1},\nabla_{X_\eta\circ\eta^{-1}}\nu\rangle$$

Next we note that $s>\frac{n-1}{2}+\frac{1}{2}=\frac{n}{2}$ and thus $H^{s-\frac{1}{2}}(\partial M)$ also forms a Schauder ring. Now using the trace theorem (cf [P], chapter 10)

$$\|\langle Y_\eta\circ\eta^{-1},\nabla_{X_\eta\circ\eta^{-1}}\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}\leq C\|Y_\eta\|_{H^s}\|X_\eta\|_{H^s}$$

$$\|\langle\mathrm{grad}p,\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}\leq C\|\mathrm{grad}p\|_{H^s}$$

where C depends on ν.

Since $X_\eta\circ\eta^{-1}$ and $Y_\eta\circ\eta^{-1}$ are divergence free direct calculation shows that

$$\mathrm{div}\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}=\mathrm{Ric}(X_\eta\circ\eta^{-1},Y_\eta\circ\eta^{-1})+\mathrm{tr}(\nabla(X_\eta\circ\eta^{-1})\cdot\nabla(Y_\eta\circ\eta^{-1}))$$

From the assumption $s>\frac{n}{2}+1$, thus $H^{s-1}(M)$ is also a Schauder ring and so

$$\|\mathrm{div}\nabla_{X_\eta\circ\eta^{-1}}Y_\eta\circ\eta^{-1}\|_{H^{s-1}(M)}\leq$$

$$\leq\|\mathrm{Ric}_{ij}X^i_\eta\circ\eta^{-1}Y^j_\eta\circ\eta^{-1}\|_{H^{s-1}}+\|\nabla_i(X^j_\eta\circ\eta^{-1})\nabla_j(Y^i_\eta\circ\eta^{-1})\|_{H^{s-1}}\leq$$

$$\leq C\|X_\eta\|_{H^s}\|Y_\eta\|_{H^s}$$

where C depends on s, η and the derivatives of g_{ij}. The Proposition follows.

Finally let us turn to the (weak) curvature $\tilde{R}$ of $\mathcal{D}^s_\mu(M)$.

$$\tilde{R}_\eta : T_\eta\mathcal{D}^s_\mu \times T_\eta\mathcal{D}^s_\mu \times T_\eta\mathcal{D}^s_\mu \to T_\eta\mathcal{D}^s_\mu$$

$$\tilde{R}_\eta(X_\eta, Y_\eta)Z_\eta = (\tilde{\nabla}_X\tilde{\nabla}_Y Z)_\eta - (\tilde{\nabla}_Y\tilde{\nabla}_X Z)_\eta - (\tilde{\nabla}_{[X,Y]}Z)_\eta$$

where $\eta \in \mathcal{D}^s_\mu$ and X, Y, Z are as before.

The right invariance of $\tilde{R}$ follows again from the the right invariance of the connection $\tilde{\nabla}$.

From the formulas for the second fundamental form of $\mathcal{D}^s_\mu$ and the curvature of $\mathcal{D}^s$ we are now able to compute $\tilde{R}$ using, as in the finite dimensional case, the Gauss' equation.

Let X, Y be two smooth vector fields on $\mathcal{D}^s_\mu(M)$, then for any $\eta \in \mathcal{D}^s_\mu$ we have

$$\begin{aligned}(\tilde{R}_\eta(X,Y)Y,X)_\eta &= (\bar{R}_\eta(X,Y)Y,X)_\eta + (S_\eta(X,X), S_\eta(Y,Y))_\eta - \\ &- (S_\eta(X,Y), S_\eta(Y,X))_\eta \qquad (3.6)\end{aligned}$$

We shall use this formula in the stability computations in section 5.

Chapter 4

Jacobi equation

In the next section we shall consider the linear theory of stability according to which stability is governed by the linearized geodesic equation - the Jacobi equation. Here we shall derive this equation and show the existence and uniqueness of Jacobi fields.

Let M^n be a compact Riemannian manifold with boundary. The geodesic equation on $\mathcal{D}^s_\mu(M)$ can be written as

$$\bar{\nabla}_{\dot{\xi}}\dot{\xi} = S_\xi(\dot{\xi},\dot{\xi}) \tag{4.1}$$

Equivalently, reducing to the first order system

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} \dot{\xi} \\ \\ Z(\xi,\dot{\xi}) \end{pmatrix} \tag{4.2}$$

where $Z(\xi,\dot{\xi}) = -\Gamma_\xi(\dot{\xi},\dot{\xi}) + S_\xi(\dot{\xi},\dot{\xi})$ is smooth in ξ and bilinear in $\dot{\xi}$. Let η be a geodesic in $\mathcal{D}^s_\mu(M)$. We shall linearize (4.1) in the neighbourhood of η and refer to the resulting equation as the Jacobi equation.

Theorem 4.3 *Let η be a geodesic in $\mathcal{D}^s_\mu$. The linearization of* (4.1) *about* $(\eta,\dot\eta) \in \mathcal{D}^s_\mu(M) \times T_\eta\mathcal{D}^s_\mu \subset H^s(M,M) \times H^s(M,R^n)$ *in the direction* $(Y,\dot Y) \in H^s(M,R^n) \times H^s(M,R^n)$ *is given by*

$$\frac{d}{dt}\begin{pmatrix} Y \\ \dot Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A_1(\eta,\dot\eta) & A_2(\eta,\dot\eta) \end{pmatrix}\begin{pmatrix} Y \\ \dot Y \end{pmatrix} \tag{4.4}$$

where

$$A_1(\eta,\dot\eta)(Y) = (D\Gamma)_\eta(\dot\eta,Y;\dot\eta) + \Gamma_\eta(Y,\Gamma_\eta(\dot\eta,\dot\eta)) - \Gamma_\eta(\dot\eta,\Gamma_\eta(\dot\eta,Y)) + \bar R_\eta(\dot\eta,Y)\dot\eta +$$

$$+[\langle Y\circ\eta^{-1},\mathrm{grad}\rangle,\mathrm{grad}\Delta^{-1}]_\eta(\mathrm{div}\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta +$$

$$+2\mathrm{grad}_\eta\Delta_\eta^{-1}\theta_\eta(\dot\eta,Y) + \langle Y\circ\eta^{-1},\mathrm{grad}\rangle_\eta\mathrm{grad}_\eta\mathcal{HE}_\eta(\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta -$$

$$-\mathrm{grad}_\eta\Delta_\eta^{-1}\langle Y\circ\eta^{-1},\mathrm{grad}\rangle_\eta\Delta_\eta\mathcal{HE}_\eta(\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta +$$

$$+\mathrm{grad}_\eta\mathcal{HE}_\eta(\Theta_\eta(\dot\eta,Y))$$

$$A_2(\eta,\dot\eta)(\dot Y) = 2\mathrm{grad}_\eta\Delta_\eta^{-1}(\mathrm{div}\nabla_{\dot Y\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta - 2\Gamma_\eta(\dot Y,\dot\eta) +$$

$$+\mathrm{grad}_\eta\mathcal{HE}_\eta((\nabla_{\dot Y\circ\eta^{-1}}\dot\eta\circ\eta^{-1} + \nabla_{\dot\eta\circ\eta^{-1}}\dot Y\circ\eta^{-1})\circ\eta)$$

and where θ_η and Θ_η are defined in (4.7) *and* (4.6) *below and $\mathcal{HE}$ was defined in* (2.3).

If $s > \frac{n}{2}+1$, then given $Y_e,\dot Y_e \in T_e\mathcal{D}^s$, there exists a unique H^s vector field $Y(t)$ along η which is a solution to (4.4) *with $Y(0) = Y_e, \dot Y(0) = \dot Y_e$*

Proof. Let $\xi(t,s) : [0,t_0] \times (-\epsilon,\varepsilon) \to \mathcal{D}^s_\mu$ be a smooth family of curves such that each $\xi(t,s)$, for fixed s, is a geodesic starting at e and $\xi(t,0) = \eta(t)$.

Let $\dot{\xi}(t,s) = \xi_*\partial_t$ and $Y(t,s) = \xi_*\partial_s$. Since $[\dot{\xi}, Y] = 0$ ($\bar{\nabla}$ is symmetric) and $\bar{\nabla}_{\dot{\xi}}\dot{\xi} = S_\xi(\dot{\xi},\dot{\xi})$ along $\xi(t,s)$, we have: $\bar{\nabla}_{\dot{\xi}}\bar{\nabla}_{\dot{\xi}}Y = \bar{\nabla}_{\dot{\xi}}\bar{\nabla}_Y\dot{\xi} = \bar{R}_\xi(\dot{\xi},Y)\dot{\xi} + \bar{\nabla}_Y S_\xi(\dot{\xi},\dot{\xi})$. Using the formula (3.2) for $\bar{\nabla}$ we obtain:

$$\ddot{Y} = -\frac{d}{dt}(\Gamma_\xi(Y,\dot{\xi})) - \Gamma_\xi(\dot{Y},\dot{\xi}) - \Gamma_\xi(\dot{\xi},\Gamma_\xi(\dot{\xi},Y)) + \bar{R}_\xi(\dot{\xi},Y)\dot{\xi} +$$

$$\frac{d}{ds}(S_\xi(\dot{\xi},\dot{\xi})) + \Gamma_\xi(Y, S_\xi(\dot{\xi},\dot{\xi}))$$

To proceed with the proof we need the following

Lemma 4.5 $\frac{d}{dt}(\Gamma_\xi(Y,\dot{\xi})) = \Gamma_\xi(\ddot{\xi},Y) + \Gamma_\xi(\dot{\xi},\dot{Y}) + (D\Gamma)_\xi(\dot{\xi},Y;\dot{\xi})$

$$\frac{d}{ds}(S_\xi(\dot{\xi},\dot{\xi})) = [\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\mathrm{grad}\Delta^{-1}]_\xi(\mathrm{div}\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi +$$
$$+2\mathrm{grad}_\xi\Delta_\xi^{-1}((\mathrm{div}\nabla_{\dot{Y}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi + \theta_\xi(\dot{\xi},Y)) +$$
$$+[\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\mathrm{grad}]_\xi\mathcal{HE}_\xi(\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi -$$
$$-\mathrm{grad}_\xi\Delta_\xi^{-1}[\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\Delta]_\xi\mathcal{HE}_\xi(\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi +$$
$$+\mathrm{grad}_\xi\mathcal{H}_\xi\langle Y\circ\xi^{-1},\mathrm{grad}\rangle_\xi\mathcal{HE}_\xi(\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi +$$
$$+\mathrm{grad}_\xi\mathcal{HE}_\xi((\nabla_{\dot{Y}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1} + \nabla_{\dot{\xi}\circ\xi^{-1}}\dot{Y}\circ\xi^{-1})\circ\xi + \Theta_\xi(\dot{\xi},Y))$$

Proof. We use the summation convention here. The proof follows [E2]. Using a local chart (U,x^i) on M^n and the definition of Γ_ξ

$$\frac{d}{dt}(\Gamma_\xi(Y,\dot{\xi})) = \frac{d}{dt}((\Gamma^i_{jk}\dot{\xi}^j\circ\xi^{-1}\dot{Y}^k\circ\xi^{-1}\frac{\partial}{\partial x^i})\circ\xi) =$$

$$= \frac{d}{dt}(\Gamma^i_{jk}\dot{\xi}^j\circ\xi^{-1}\dot{Y}^k\circ\xi^{-1}\frac{\partial}{\partial x^i})\circ\xi + (\dot{\xi}^l\circ\xi^{-1}\partial_l(\Gamma^i_{jk}\dot{\xi}^j\circ\xi^{-1}\dot{Y}^k\circ\xi^{-1})\frac{\partial}{\partial x^i})\circ\xi$$

From $\xi\circ\xi^{-1} = e$ we find that

$$\frac{d}{dt}(\xi^{-1m}) = -\partial_k\xi^{-1m}\dot{\xi}^k\circ\xi^{-1}$$

and hence

$$\frac{d}{dt}(\dot{\xi}^j \circ \xi^{-1}) = \ddot{\xi}^j \circ \xi^{-1} - \partial_k(\dot{\xi}^j \circ \xi^{-1})\dot{\xi}^k \circ \xi^{-1}$$

$$\frac{d}{dt}(Y^j \circ \xi^{-1}) = \dot{Y}^j \circ \xi^{-1} - \partial_k(Y^j \circ \xi^{-1})\dot{\xi}^k \circ \xi^{-1}$$

Thus we get

$$\frac{d}{dt}(\Gamma_\xi(Y,\dot{\xi})) = (\Gamma^i_{jk}\ddot{\xi}^j \circ \xi^{-1} Y^k \circ \xi^{-1}\frac{\partial}{\partial x^i} + \Gamma^i_{jk}\dot{\xi}^j \circ \xi^{-1}\dot{Y}^k \circ \xi^{-1}\frac{\partial}{\partial x^i} +$$

$$+\dot{\xi}^l \circ \xi^{-1}\partial_l \Gamma^i_{jk}\dot{\xi}^j \circ \xi^{-1} Y^k \circ \xi^{-1}\frac{\partial}{\partial x^i})$$

the desired result for Γ.

From (3.3) and (2.4) we have

$$S_\xi(\dot{\xi},\dot{\xi}) = \mathrm{grad}_\xi \Delta_\xi^{-1}(\mathrm{div}\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi + \mathrm{grad}_\xi \mathcal{H}\mathcal{E}_\xi(\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi$$

where for an arbitrary operator L and $\xi \in \mathcal{D}^s_\mu$, we denote $L_\xi = R_\xi \circ L \circ R_{\xi^{-1}}$, R_ξ being the appropriate right translation. We first find that

$$\frac{d}{ds}(\xi^{-1m}) = -\partial_k \xi^{-1m} Y^k \circ \xi^{-1}$$

$$\frac{d}{ds}(\dot{\xi}^j \circ \xi^{-1}) = \dot{Y}^j \circ \xi^{-1} - \partial_k(\dot{\xi}^j \circ \xi^{-1})Y^k \circ \xi^{-1}$$

Using the above formulas and computing as before we have

$$\frac{d}{ds}((\nabla_{\dot{\xi}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1})\circ\xi) = (\nabla_{\dot{Y}\circ\xi^{-1}}\dot{\xi}\circ\xi^{-1} + \nabla_{\dot{\xi}\circ\xi^{-1}}\dot{Y}\circ\xi^{-1})\circ\xi + \Theta_\xi(\dot{\xi},Y) \quad (4.6)$$

where Θ_ξ is defined by

$$\Theta_\xi(\dot{\xi},Y) = (-\dot{\xi}^j \circ \xi^{-1}\partial_m(\dot{\xi}^i \circ \xi^{-1})\partial_j(Y^m \circ \xi^{-1})\frac{\partial}{\partial x^i} +$$

$$+Y^l \circ \xi^{-1}\dot{\xi}^j \circ \xi^{-1}\dot{\xi}^k \circ \xi^{-1}\partial_l \Gamma^i_{jk}\frac{\partial}{\partial x^i})\circ\xi$$

Next we find that

$$\frac{d}{ds}((\mathrm{div}\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi) = \frac{d}{ds}((\nabla_i(\dot\xi^j\circ\xi^{-1})\nabla_j(\dot\xi^i\circ\xi^{-1})+Ric_{ij}\dot\xi^i\circ\xi^{-1}\dot\xi^j\circ\xi^{-1})\circ\xi)$$

since $\dot\xi\circ\xi^{-1}$ is divergence free. Similar computations as above together with the formulas for $\frac{d}{ds}(\xi^{-1m})$ and $\frac{d}{ds}(\dot\xi^j\circ\xi^{-1})$ give

$$\frac{d}{ds}((\mathrm{div}\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi) = 2(\mathrm{div}\nabla_{\dot Y\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi + 2\theta_\xi(\dot\xi, Y) \tag{4.7}$$

where θ_ξ is

$$\theta_\xi(\dot\xi,Y) = (-\partial_k(\dot\xi^j\circ\xi^{-1})\partial_i(Y^k\circ\xi^{-1})\partial_j(\dot\xi^i\circ\xi^{-1})-$$

$$-\partial_m(\dot\xi^j\circ\xi^{-1})\partial_i(Y^m\circ\xi^{-1})\dot\xi^k\circ\xi^{-1}\Gamma^i_{jk} + \partial_i(\dot\xi^j\circ\xi^{-1})\dot\xi^k\circ\xi^{-1}Y^m\circ\xi^{-1}\partial_m\Gamma^i_{jk}+$$

$$+\dot\xi^k\circ\xi^{-1}\dot\xi^l\circ\xi^{-1}Y^m\circ\xi^{-1}\partial_m\Gamma^i_{jl} + \frac{1}{2}\partial_k\mathrm{Ric}_{ij}\dot\xi^i\circ\xi^{-1}\dot\xi^j\circ\xi^{-1}Y^k\circ\xi^{-1})\circ\xi$$

Let now $f\in C^\infty(M)$, then we get

$$\frac{d}{ds}(\mathrm{grad}_\xi f) = \frac{d}{ds}(\mathrm{grad}(f\circ\xi^{-1}))\circ\xi + (Y^k\circ\xi^{-1}\partial_k(\mathrm{grad}(f\circ\xi^{-1}))^j\frac{\partial}{\partial x^i})\circ\xi =$$

$$= (-\partial_i(\partial_k(f\circ\xi^{-1})Y^k\circ\xi^{-1})g^{ij}\frac{\partial}{\partial x^j} + Y^k\circ\xi^{-1}\partial_k(\partial_i(f\circ\xi^{-1}g^{ij})\frac{\partial}{\partial x^j})\circ\xi =$$

$$= [\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\mathrm{grad}]_\xi f \tag{4.8}$$

where $\langle Y\circ\xi^{-1},\mathrm{grad}\rangle$ is the differential operator $Y^k\circ\xi^{-1}\partial_k$.

We shall now compute $\frac{d}{ds}\Delta_\xi$ and therefore must first obtain $\frac{d}{ds}\mathrm{div}_\xi$. Let $W\in C^\infty(TM)$ then from the formula for $\frac{d}{ds}(\xi^{-lm})$ we have

$$\frac{d}{ds}(\mathrm{div}_\xi W) = \frac{d}{ds}(\partial_k W^i\partial_i(\xi^{-1k})\circ\xi + W^j\Gamma^i_{ij}\circ\xi) = (-\partial_l(W^i\circ\xi^{-1})\partial_i(Y^l\circ\xi^{-1})+$$

$$+W^i\circ\xi^{-1}Y^l\circ\xi^{-1}\partial_l\Gamma^i_{ij})\circ\xi = [\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\mathrm{div}]_\xi W$$

Therefore

$$\frac{d}{ds}(\Delta_\xi) = \frac{d}{ds}(\mathrm{div}_\xi)\mathrm{grad}_\xi + \mathrm{div}_\xi \frac{d}{ds}(\mathrm{grad}_\xi) =$$

$$= [\langle Y \circ \xi^{-1}, \mathrm{grad}\rangle, \mathrm{div}]_\xi \mathrm{grad}_\xi + \mathrm{div}_\xi [\langle Y \circ \xi^{-1}, \mathrm{grad}\rangle, \mathrm{grad}]_\xi =$$

$$= [\langle Y \circ \xi^{-1}, \mathrm{grad}\rangle, \Delta]_\xi \tag{4.9}$$

Because of the boundary condition the calculation of $\frac{d}{ds}\Delta_\xi^{-1}$ requires more consideration. Let $f \in C^\infty(M)$, then

$$\Delta_\xi \Delta_\xi^{-1} f = f \tag{4.10}$$

$$\Delta_\xi^{-1} \Delta_\xi f = f - \mathcal{H}_\xi f \tag{4.11}$$

where $\mathcal{H}_\xi f = g$ is defined by

$$\Delta(g \circ \xi^{-1}) = 0$$

$$\langle \mathrm{grad}(g \circ \xi^{-1}), \nu\rangle = \langle \mathrm{grad}(f \circ \xi^{-1}), \nu\rangle$$

and ν is the outer unit normal field on ∂M.

Using (4.10), (4.11) and the fact that $\mathcal{H}_\xi \Delta_\xi^{-1} = 0$ we obtain

$$\frac{d}{ds}(\Delta_\xi^{-1}) = -\Delta_\xi^{-1}\frac{d}{ds}(\Delta_\xi)\Delta_\xi^{-1} - \frac{d}{ds}(\mathcal{H}_\xi)\Delta_\xi^{-1}$$

To compute $\frac{d}{ds}(\mathcal{H}_\xi)$ we use $\Delta_\xi \mathcal{H}_\xi = 0$ and (4.11) again. If $g \in C^\infty(M)$, then

$$\frac{d}{ds}(\mathcal{H}_\xi)g = -\Delta_\xi^{-1}\frac{d}{ds}(\Delta_\xi)\mathcal{H}_\xi g + \mathcal{H}_\xi \frac{d}{ds}(\mathcal{H}_\xi g) \tag{4.12}$$

$\mathcal{H}_\xi \frac{d}{ds}(\mathcal{H}_\xi g)$ is harmonic with the same boundary values as $\frac{d}{ds}(\mathcal{H}_\xi g)$. Consider $\mathcal{H}_\xi g = h_s$. It is harmonic and

$$\langle \mathrm{grad}(h_s \circ \xi^{-1}), \nu\rangle = \langle \mathrm{grad}(g \circ \xi^{-1}), \nu\rangle$$

Applying $\frac{d}{ds}$ to both sides we see that the boundary values of $\frac{d}{ds}(\mathcal{H}_\xi g)$ satisfy

$$\langle \mathrm{grad}(\frac{d}{ds}(h_s) \circ \xi^{-1}), \nu \rangle =$$

$$= \langle \mathrm{grad}((\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi h_s) \circ \xi^{-1}), \nu \rangle - \langle \mathrm{grad}((\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi g) \circ \xi^{-1}), \nu \rangle$$

Therefore $\mathcal{H}_\xi \frac{d}{ds}(\mathcal{H}_\xi g) = -\mathcal{H}_\xi(\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi \Delta_\xi^{-1} \Delta_\xi g)$ and letting now $g = \Delta_\xi^{-1} f$ we obtain from (4.12) and (4.10)

$$\frac{d}{ds}(\mathcal{H}_\xi)\Delta_\xi^{-1} f = -\mathcal{H}_\xi(\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi \Delta_\xi^{-1} f)$$

and thus

$$\frac{d}{ds}(\Delta_\xi^{-1}) = -\Delta_\xi^{-1} \frac{d}{ds}(\Delta_\xi)\Delta_\xi^{-1} + \mathcal{H}_\xi \langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi \Delta_\xi^{-1} =$$

$$-\Delta_\xi^{-1}[\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle, \Delta]_\xi \Delta_\xi^{-1} + \mathcal{H}_\xi \langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi \Delta_\xi^{-1} =$$

$$= [\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle, \Delta^{-1}]_\xi \tag{4.13}$$

Finally if $W \in C^\infty(TM)$ and $\mathcal{HE}_\xi(W) = h_s$, then from (2.3)

$$\Delta(h_s \circ \xi^{-1}) = 0$$

$$\langle \mathrm{grad}(h_s \circ \xi^{-1}), \nu \rangle = \langle W - \mathrm{grad}(p_W), \nu \rangle$$

where p_W is determined from (2.2). As before we compute

$$\frac{d}{ds}\mathcal{HE}_\xi(W) = -\Delta_\xi^{-1} \frac{d}{ds}(\Delta_\xi)\mathcal{HE}_\xi(W) + \mathcal{H}_\xi \frac{d}{ds}(\mathcal{HE}_\xi W)$$

Differentiating the boundary condition gives

$$\langle \mathrm{grad}(\frac{d}{ds}(h_s) \circ \xi^{-1}), \nu \rangle - \langle \mathrm{grad}((\langle Y \circ \xi^{-1}, \mathrm{grad} \rangle_\xi h_s) \circ \xi^{-1}), \nu \rangle = 0$$

Therefore

$$\begin{aligned}\frac{d}{ds}\mathcal{H}\mathcal{E}_\xi(W) &= -\Delta_\xi^{-1}[\langle Y\circ\xi^{-1},\mathrm{grad}\rangle,\Delta]_\xi\mathcal{H}\mathcal{E}_\xi(W)+\\ &+ \mathcal{H}_\xi\langle Y\circ\xi^{-1},\mathrm{grad}\rangle_\xi\mathcal{H}\mathcal{E}_\xi(W) \qquad (4.14)\end{aligned}$$

Since

$$\frac{d}{ds}(S_\xi(\dot\xi,\dot\xi)) = \frac{d}{ds}(\mathrm{grad}_\xi)\Delta_\xi^{-1}(\mathrm{div}\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi+$$

$$+\mathrm{grad}_\xi\frac{d}{ds}(\Delta_\xi^{-1})(\mathrm{div}\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi+\mathrm{grad}_\xi\Delta_\xi^{-1}\frac{d}{ds}((\mathrm{div}\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi)+$$

$$+\frac{d}{ds}(\mathrm{grad}_\xi)\mathcal{H}\mathcal{E}_\xi(\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi+\mathrm{grad}_\xi\frac{d}{ds}(\mathcal{H}\mathcal{E}_\xi)(\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi+$$

$$+\mathrm{grad}_\xi\mathcal{H}\mathcal{E}_\xi\frac{d}{ds}((\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi)$$

the lemma follows from (4.6), (4.7), (4.8), (4.13), (4.14) and the above.

From Lemma 4.5, evaluating at $s=0$, using the fact that η is a geodesic in $\mathcal{D}^s_\mu$ and rewriting some of the commutators we get

$$\ddot Y = -(D\Gamma)_\eta(\dot\eta,W;\dot\eta)-2\Gamma_\eta(\dot Y,\dot\eta)+\Gamma_\eta(Y,\Gamma_\eta(\dot\eta,\dot\eta))-\Gamma_\eta(\dot\eta,\Gamma_\eta(\dot\eta,Y))+$$

$$+\tilde R_\eta(\dot\eta,Y)\dot\eta+([\langle Y\circ\eta^{-1},\mathrm{grad}\rangle,\mathrm{grad}\Delta^{-1}]_\eta(\mathrm{div}\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta+$$

$$+2\mathrm{grad}_\eta\Delta_\eta^{-1}((\mathrm{div}\nabla_{\dot Y\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta+\theta_\eta(Y,\dot\eta))+$$

$$+\langle Y\circ\eta^{-1},\mathrm{grad}\rangle_\eta\mathrm{grad}_\eta\mathcal{H}\mathcal{E}_\xi(\nabla_{\dot\xi\circ\xi^{-1}}\dot\xi\circ\xi^{-1})\circ\xi+$$

$$+\mathrm{grad}_\eta\mathcal{H}\mathcal{E}_\xi((\nabla_{\dot Y\circ\xi^{-1}}\dot\xi\circ\xi^{-1}+\nabla_{\dot\xi\circ\xi^{-1}}\dot Y\circ\xi^{-1})\circ\eta+\Theta_\eta(\dot\eta,Y))$$

To show that the right hand side of this expression defines a bounded operator in $H^s\oplus H^s$ we need another

Lemma 4.15 *Let η be a geodesic in $\mathcal{D}^s_\mu(M^n), s > \frac{n}{2}+1$, $Y, \dot{Y} \in T_\eta\mathcal{D}^s_\mu$. Then*

$$\left\| \begin{pmatrix} 0 & 1 \\ A_1(\eta,\dot\eta) & A_2(\eta,\dot\eta) \end{pmatrix} \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix} \right\|^2_{\eta,H^s} \leq C(\eta, g_{ij}, s) \left\| \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix} \right\|^2_{\eta,H^s} \tag{4.16}$$

Proof. From their expressions in local coordinates and Proposition 3.4 we see that the first five terms can be estimated using the composition lemma and the Schauder ring property by $C(\eta)\|Y\circ\eta^{-1}\|_{e,H^s}$. For the remaining terms we use again the elliptic estimates.

Let $v = \nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1}$, then

$$\|([\langle Y\circ\eta^{-1}, \mathrm{grad}\rangle, \mathrm{grad}\Delta^{-1}]\mathrm{div}\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta\|_{H^s} \leq$$

$$\leq C_\eta\|Y\circ\eta^{-1}\|_{H^s}\|\mathrm{grad}\Delta^{-1}\mathrm{div} v\|_{H^{s+1}} + C_\eta\|\mathrm{grad}\Delta^{-1}\langle Y\circ\eta^{-1}, \mathrm{grad}\rangle \mathrm{div} v\|_{H^s}$$

$$\leq C_\eta\|Y\circ\eta^{-1}\|_{e,H^s} + C_\eta\|\mathrm{div} v\|_{H^s}\|Y\circ\eta^{-1}\|_{H^{s-1}}$$

Similarly

$$\|(\langle Y\circ\eta^{-1}, \mathrm{grad}\rangle \mathrm{grad}\mathcal{HE}(v))\circ\eta\|_{H^s} \leq C(\eta)\|Y\circ\eta^{-1}\|_{H^s}$$

Using the formula for θ in (4.7) and $\mathrm{div}(\dot\eta\circ\eta^{-1}) = 0$

$$\|\mathrm{grad}\Delta^{-1}\mathrm{div}_{\dot{Y}\circ\eta^{-1}}\dot\eta\circ\eta^{-1} + \theta(\dot\eta\circ\eta^{-1}, Y\circ\eta^{-1})\|_{H^s} \leq$$

$$\leq C_\eta\|\dot{Y}\circ\eta^{-1}\|_{H^s}\|\dot\eta\circ\eta^{-1}\|_{H^s} + C_\eta\|\dot\eta\circ\eta^{-1}\|^2_{H^s}\|Y\circ\eta^{-1}\|_{H^s}$$

where the constant C_η depends again on η, g_{ij} and its derivatives.

From (2.3) and [ADN] and proceeding as in the proof of Proposition 3.4

$$\|(\mathrm{grad}\mathcal{HE}(\nabla_{\dot{Y}\circ\eta^{-1}}\dot{\eta}\circ\eta^{-1}+\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{Y}\circ\eta^{-1}+\Theta(\dot{\eta}\circ\eta^{-1},Y\circ\eta^{-1}))\circ\eta\|_{H^s}\leq$$

$$\leq C_\eta\|\langle\nabla_{\dot{Y}\circ\eta^{-1}}\dot{\eta}\circ\eta^{-1}+\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{Y}\circ\eta^{-1},\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}+$$

$$+C_\eta\|\langle\Theta(\dot{\eta}\circ\eta^{-1},Y\circ\eta^{-1}),\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}+C_\eta\|\langle\mathrm{grad}p,\nu\rangle\|_{H^{s-\frac{1}{2}}(\partial M)}\leq$$

where $p=\Delta^{-1}\mathrm{div}(\nabla_{\dot{Y}\circ\eta^{-1}}\dot{\eta}\circ\eta^{-1}+\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{Y}\circ\eta^{-1}+\Theta(\dot{\eta}\circ\eta^{-1},Y\circ\eta^{-1}))$. Here p is H^{s+1} if $\dot{Y}$ is H^s by the divergence free condition, also

$$\langle\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{Y}\circ\eta^{-1},\nu\rangle=-\langle\dot{Y}\circ\eta^{-1},\nabla_{\dot{\eta}\circ\eta^{-1}}\nu\rangle$$

and since $H^{s-\frac{1}{2}}(\partial M)$ is a Schauder ring ($s-\frac{1}{2}>\frac{n}{2}$), using the trace theorem [P], we obtain

$$C(\eta)\|Y\circ\eta^{-1}\|_{H^s(M)}+C(\eta)\|\dot{Y}\circ\eta^{-1}\|_{H^s(M)}$$

where we used the linear dependence of Θ on $Y\circ\eta^{-1}$ and the previous estimates to estimate the p term. Put together the above estimates prove the lemma.

The last assertion of Theorem 4.3 follows now from the existence and uniqueness theorem for ordinary differential equations on a Hilbert manifold.

Remark Using the connection $\tilde{\nabla}$ and the curvature $\tilde{R}$ of $\mathcal{D}^s_\mu$, equation (4.4) is

$$\tilde{\nabla}_{\dot{\eta}}\tilde{\nabla}_{\dot{\eta}}Y+\tilde{R}_\eta(Y,\dot{\eta})\dot{\eta}=0 \tag{4.17}$$

Chapter 5

Stability and curvature

In this section we introduce the notion of linear stability and construct examples of stable and unstable fluid flows. Our main tool will be the Gauss' equation relating the curvatures $\tilde{R}$ and $\bar{R}$ with the second fundamental form S. Roughly speaking curvature of $\mathcal{D}^s_\mu$ controls, as in finite dimensions, the behaviour of Jacobi fields, which are solutions of the linearized geodesic equation and thus control the behaviour of geodesics.

Following [A] we say that a fluid motion $\eta(t)$ is (Lagrangian) stable if the curvatures in the directions of all two dimensional planes containing $\dot{\eta}(t)$ are positive along $\eta(t)$.

We first examine flows whose pressure function $p(t)$ remains constant throughout. Such flows correspond to those geodesics in $\mathcal{D}^s_\mu(M)$ which are also geodesics in $\mathcal{D}^s(M)$. This is evident in the proof of the following

Theorem 5.1 *Let M^n be a compact Riemannian manifold of nonpositive sectional curvature. If η is a pressure constant flow in M^n, then η is not stable*

Proof. Since η is a geodesic in $\mathcal{D}^s_\mu(M)$, from the Euler equations, the formulas for the second fundamental form S and the connection $\bar{\nabla}$

$$S_\eta(\dot\eta,\dot\eta) = Q_\eta(\bar\nabla_{\dot\eta}\dot\eta) = Q_\eta(\frac{d}{dt}(\dot\eta\circ\eta^{-1})\circ\eta + (\nabla_{\dot\eta\circ\eta^{-1}}\dot\eta\circ\eta^{-1})\circ\eta) =$$

$$= (\operatorname{grad} p)\circ\eta = 0$$

Then for any vector field $X(t)$ along η the Gauss' equation (3.6) gives

$$(\tilde{R}_\eta(X,\dot\eta)\dot\eta, X)_\eta = (\bar{R}_\eta(X,\dot\eta)\dot\eta, X)_\eta - \|S_\eta(\dot\eta, X)\|^2_{L^2} \leq 0$$

The last inequality follows from the right invariance of $\bar{R}_\eta$, (3.5) and the assumption about the curvature of M.

In fact it also follows from this Proposition that any geodesic with initial conditions sufficiently close to those of a pressure constant η will diverge from η at least as fast as in the flat case.

In [A] Arnold used Lie group methods to compute the curvature of $\mathcal{D}^s_\mu(T^2)$ at the identity, where T^2 is the flat two torus. He found, that in few directions the sectional curvature was positive. Later Lukatski [Lu] used the same approach to study the field $v = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$ on the two shere S^2 and proved that sectional curvatures at the identity containing v were non-negative.

Below we show that any infinitesimal isometry of M generates a fluid flow η such that the curvatures along η are non-negative (we shall call such η weakly stable). We have

Theorem 5.2 *Let v be a Killing vector field on a compact Riemannian manifold M^n. Then v generates a stationary weakly stable fluid flow such that*

$$(\tilde{R}_e(v,w)w,v)_e = \|P_e(\nabla_w v)\|^2_{L^2}$$

for any $w \in T_e\mathcal{D}_\mu(M^n)$.

Proof For v a Killing field $\operatorname{div} v = 0$ and for any $w \in C^\infty(TM)$

$$\langle \nabla_v v + \frac{1}{2}\mathrm{grad}|v|^2, w\rangle = \langle \nabla_v v, w\rangle + \langle v, \nabla_w v\rangle = (L_v g)(v,w) = 0$$

where $g = \langle\cdot,\cdot\rangle$ is the metric on M.

Therefore

$$Q_e(\nabla_v v) = -\frac{1}{2}\mathrm{grad}|v|^2 = \nabla_v v$$

and since $\partial_t v = 0$, v solves the Euler equations on M.

Let $\eta(t)$ be the corresponding geodesic in $\mathcal{D}^s_\mu(M)$. Since $\eta(t)$ is stationary it suffices to do the computations at the identity.

Now $\dot\eta(t)\circ\eta^{-1}(t) = v \in T_e\mathcal{D}_\mu(M)$ being Killing also implies

$$\langle R(v,w)w, v\rangle + \langle \nabla_v v, \nabla_w w\rangle = -w\cdot\langle \nabla_w v, v\rangle + \langle \nabla_w v, \nabla_w v\rangle$$

for every smooth vector field w on M.

By stationarity $Q_e(\nabla_v v) = \nabla_v v$, so that from Gauss' equation (3.6) we get

$$(\tilde{R}_e(v,w)w,v)_e = (\bar{R}_e(v,w)w,v)_e + (Q_e(\nabla_v v), Q_e(\nabla_w w))_e -$$

$$-(Q_e(\nabla_w v), Q_e(\nabla_w v))_e =$$

$$= \int_M \{\langle R(v,w)w,v\rangle + \langle \nabla_v v, \nabla_w w\rangle\} - (Q_e(\nabla_w v), Q_e(\nabla_w v))_e =$$

$$= -\int_M w\cdot\langle \nabla_w v, v\rangle + \int_M \langle \nabla_w v, \nabla_w v\rangle - (Q_e(\nabla_w v), Q_e(\nabla_w v))_e =$$

$$= (\nabla_w v, \nabla_w v)_e - (Q_e(\nabla_w v), Q_e(\nabla_w v))_e =$$

$$= (P_e(\nabla_w v), P_e(\nabla_w v))_e \geq 0$$

The last two equalities follow from $(Q_e(u), P_e(u))_e = 0$ for any vector field u on M, and from the fact for any $u \in T_e\mathcal{D}_\mu(M)$ and $f \in C^\infty(M)$

$$\int_M u \cdot f = \int_M \langle u, \operatorname{grad} f\rangle = \int_{\partial M} f\langle u, \nu\rangle - \int_M f \cdot \operatorname{div} u = 0.$$

Before considering examples we make few remarks.

Remark It is well known that if M is compact without boundary and with strictly negative Ricci curvature, then there are no Killing fields on M. They can be easily found however on other manifolds, for example rotation fields $x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}$, $1 \leq i, j \leq n$ on S^n of any dimension n are Killing.

Remark Perhaps the simplest examples of pressure constant geodesics are provided by flows generated by parallel vector fields on M. If M is compact without boundary a vector field is parallel if and only if it is harmonic and Killing. Again there are no harmonic fields on such M if the Ricci curvature is strictly positive.

From Theorem 5.1 pressure constant flows on nonpositively curved manifolds are necessarily unstable. Not so if the curvature is positive.

Example 5.3 Let M be a compact Lie group with a bi-invariant metric. Then any left invariant vector field generates a weakly stable, stationary, pressure constant fluid flow on M. This is because here left translations are isometries and

$$Q_e\nabla_v v = \frac{1}{2}Q_e[v, v] = 0$$

Furthermore, if w is another left invariant vector field on M such that $[w, v] \neq$

0, then also $Q_e \nabla_w v = 0$ so by Theorem 5.2

$$(\tilde{R}_e(v,w)w,v)_e = \|P_e(\nabla_w v)\|_{L^2}^2 = \frac{1}{4}\int_M |[w,v]|^2 > 0$$

and thus the nearby geodesics $\xi(s,t) = \tilde{\exp}t(v+sw)$ will pull toward η at least initially ($\tilde{\exp}$ is the exponential map of the weak Riemannian metric on $\mathcal{D}_\mu^s$).

Example 5.4 A simple nonstationary example of a pressure constant flow in $\mathcal{D}_\mu(T^2)$, T^2 the flat two torus, is given by

$$\eta(t)(x_1,x_2) = (x_1 + th(x_2), x_2 + ct)$$

where c is a constant and h is a smooth, periodic function. In this case the geodesic equation (4.1) simplifies to

$$\ddot{\xi} \circ \xi^{-1} = \mathrm{grad}\Delta^{-1}\mathrm{tr}(D\dot{\xi} \circ \xi^{-1})^2$$

Clearly $\dot{\eta}(t)(x_1,x_2) = (h(x_2),c)$ and $\ddot{\eta}(t) = (0,0)$ and since a straightforward computation shows that $\mathrm{tr}(\mathrm{D}\dot{\eta} \circ \eta^{-1})^2 = 0$, $\eta(t)$ is a geodesic in $\mathcal{D}_\mu(T^2)$. Furthermore using (3.2) we get

$$S_\eta(\dot{\eta},\dot{\eta}) = Q_\eta(\bar{\nabla}_{\dot{\eta}}\dot{\eta}) = Q_\eta(\ddot{\eta}) = (0,0)$$

On the other hand

$$P_e(\nabla_{\dot{\eta}\circ\eta^{-1}}\dot{\eta} \circ \eta^{-1}) = \nabla_{\dot{\eta}\circ\eta^{-1}}\dot{\eta} \circ \eta^{-1} = (ch'(x_2 - ct), 0)$$

thus $\eta(t)$ is pressure constant but not stationary.

Our next example points out to the difference between the Eulerian and the Lagrangian notions of stability of fluid flows. As mentioned in the Introduction, in the Lagrangian case one is concerned with positions of the fluid particles, whereas in the Eulerian case with their velocities considered as functions of their position in space. Thus a time dependent vector field $u(t)$ on M satisfying Euler equations is stable in the Eulerian sense if any small change in the initial conditions $u(0)$ results in small changes in $u(t)$ for all later times. The example below shows that it is possible to have a flow stable in Eulerian but not in Lagrangian sense, as well as a flow unstable in both senses.

Example 5.5 Let us consider a curve in $\mathcal{D}_\mu(S^1 \times [0,\pi])$

$$\eta(t) = (x_1 + t\sin x_2, x_2)$$

i.e. a periodic plane parallel flow in a strip. The cylinder being locally flat, the geodesic equation in $\mathcal{D}_\mu$ simplifies as in Example 5.4 except here to compute Δ^{-1} we must solve a boundary value problem. As before we find that $\dot{\eta} \circ \eta^{-1}(x_1, x_2) = (\sin x_2, 0)$ and show that η is a stationary geodesic. The classical stability criterion of Rayleigh (cf [Li]) states that a necessary condition for (Eulerian) instability of a plane parallel flow is that its velocity profile have an inflection point. For symmetric velocity profiles this condition is also sufficient.

$\sin x_2$ has no points of inflection on $[0,\pi]$, so η is stable in the Eulerian sense. Similar computations to those in Example 5.4 show however that $S_\eta(\dot{\eta}, \dot{\eta}) = 0$ thus it is not stable in the Lagrangian sense. In fact if we take $X_e \in T_e\mathcal{D}_\mu$, $X_e = (\cos x_1 \sin 2x_2, \frac{1}{2}\sin x_1(1-\cos 2x_2))$ then from (3.6) we obtain

$$(\tilde{R}_e(X_e, \dot{\eta}(0))\dot{\eta}(0), X_e)_e =$$

$$= \int_0^\pi \int_0^{2\pi} |\mathrm{grad}\Delta^{-1}\frac{1}{4}\cos x_1(\cos x_2 - \cos 3x_2)|^2 dx_1 dx_2 =$$

$$= -\frac{3\pi^2}{160}$$

Thus at least initially in some directions the Lagrangian instability is even exponential.

Consider another curve in $\mathcal{D}_\mu(S^1 \times [0,\pi])$

$$\gamma(t)(x_1, x_2) = (x_1 + t\cos 2x_2, x_2)$$

As above we can show that γ is a stationary, pressure constant geodesic. This flow has a symmetrical velocity profile with two inflection points, hence the classical criterion implies Eulerian instability. To show γ's exponential (Lagrangian) instability we may take for example

$$X_e = (2\cos x_1 \cos 2x_2, \sin x_1 \sin 2x_2)$$

to get $(\tilde{R}_e(X_e, \dot{\eta}(0))\dot{\eta}(0), X_e)_e < 0$ as above.

Finally we shall show that there are manifolds which do not admit pressure constant flows

Theorem 5.6 *Let M^2 be a compact Riemannian manifold without boundary and with nowhere vanishing Gaussian curvature $K \neq 0$. Then there are no pressure constant flows on M^2*

Proof. If η is a pressure constant flow, then $v = \dot{\eta} \circ \eta^{-1}$ satisfies

$$\partial_t v + \nabla_v v = 0$$

$$\operatorname{div} v = 0$$

Applying div to both sides of the equation above and using the divergence condition gives

$$\operatorname{div}\nabla_v v = \operatorname{Ric}(v,v) + \operatorname{tr}(\nabla v \cdot \nabla v) = 0$$

Let $p \in M$ be the point where $|v|^2$ achieves its supremum, then $d|v|^2(p) = 0$. Choosing normal coordinates at p we obtain

$$\mathrm{div}\nabla_v v(p) = K(p)|v|^2(p) - 2\mathrm{det}Dv(p) = 0$$

$$d|v|^2(p) = (v \cdot Dv)(p) = 0$$

But if $v(p) \neq 0$ the second of the above eqations implies that $Dv(p)$ is singular, hence $\mathrm{det}Dv(p) = 0$, a contradiction with the first equation and the assumption on the curvature.

Remark Example 5.3 above shows that pressure constant flows can be found on manifolds with nonvanishing sectional curvature if the dimension n is greater than 2, for example if $M^n = S^3$. We shall return to this in section 7.

Chapter 6

Nonlinear stability

Our method here is based on the linearization method. We will use a different notion of (uniform) stability, which is more suitable in the nonlinear case.

Consider a differential equation in a Banach space B with norm $|\cdot|$

$$\frac{dx}{dt} = f(t, x)$$

where $f(t, 0) = 0, t \geq 0$. A solution $x(t)$ is uniformly stable on $[0, \infty)$, if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such, that for any $t_1 \geq 0$ and any other solution $\bar{x}(t)$, $|x(t_1) - \bar{x}(t_1)| < \delta$ implies that $|x(t) - \bar{x}(t)| < \epsilon$ for all $t \geq t_1$.

We shall also make use of the following concept (cf [DK]). Let

$$\frac{dx}{dt} = A(t)x$$

be a linear, nonstationary differential equation in B. The greatest lower bound of all numbers ρ for which there exist numbers N_ρ such, that

$$|x(t)| \leq N_\rho e^{\rho(t-\tau)} |x(\tau)|$$

for any $t \geq \tau \geq 0$, is called the (upper) Bohl exponent of the equation.

Consider the geodesic equation (4.2) in $\mathcal{D}^s_\mu$. Let η be a solution of this equation and let

$$V(t) = \xi(t) - \eta(t)$$

$$\xi_s(t) = \eta(t) + sV(t),\ 0 \leq s \leq 1$$

Then from Theorem 4.3 we have

$$\frac{d}{dt}\begin{pmatrix} V \\ \dot{V} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix} - \frac{d}{dt}\begin{pmatrix} \eta \\ \dot{\eta} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ A_1(\eta,\dot{\eta}) & A_2(\eta,\dot{\eta}) \end{pmatrix}\begin{pmatrix} V \\ \dot{V} \end{pmatrix} + \begin{pmatrix} 0 \\ \int_0^1 \int_0^s D^2 Z(\xi_r(t), \dot{\xi}_r(t))(V,\dot{V})^2 \end{pmatrix} \tag{6.1}$$

where Z, A_1 and A_2 were defined in (4.2), (4.4).

It may be expected that the stability behaviour of $(V, \dot{V})$ approximates that of $(Y, \dot{Y})$, the solution of the Jacobi equation, provided a suitable estimate can be found for the perturbation term on the right. In fact we shall prove the following

Proposition 6.2 *Let M^n be a compact Riemannian manifold and let $s > \frac{n}{2}+1$. Suppose that $\eta(t)$ is a geodesic which stays in a compact set in $\mathcal{D}^s_\mu(M^n)$. Then if there is a Jacobi field $Y(t)$ along $\eta(t)$ such, that $Y(0) = 0, \dot{Y}(0) = \dot{Y}_e$ and the two dimensional curvature of the plane spanned by $Y(t), \dot{\eta}(t)$ is nonpositive for all t, then $\eta(t)$ is not uniformly stable.*

Proof We need three lemmas.

Lemma 6.3 *Let* η *be a geodesic which stays in a compact set in* $\mathcal{D}^s_\mu(M^n)$, $s > \frac{n}{2} + 1$. *Then there exists* $C(V, \dot{V})$ *with* $\lim_{(V,\dot{V})\to 0} C(V, \dot{V}) = 0$ *and such, that*

$$\|D^2 Z(\xi(r), \dot{\xi}(r))(V, \dot{V})^2\|_s \leq C(V, \dot{V}) \|(V, \dot{V})\|_s$$

uniformly for all $(\xi, \dot{\xi})$ *in some neighbourhood of* $(\eta, \dot{\eta})$.

Proof The Lemma follows immediately from the fact that $Z(\xi, \dot{\xi})$ is C^2 and bilinear in $\dot{\xi}$.

Remark In this case the perturbation term in (6.1) involving $Z(\xi, \dot{\xi})$ is said to satisfy the quasilinearity condition [DK].

Next we have

Lemma 6.4 *Let* η *be a geodesic in* $\mathcal{D}^s_\mu$ *and let* $Y(t)$ *be a nonzero Jacobi field as in the* Proposition *above. Then the Bohl exponent of the Jacobi equation is non-negative.*

Proof. Since by taking inner product with $Y(t)$ we may rewrite the Jacobi equation as

$$\frac{1}{2}\frac{d^2}{dt^2}\|Y\|^2_{L^2} = \|\tilde{\nabla}_{\dot{\eta}} Y\|^2_{L^2} - (\tilde{R}_\eta(Y, \dot{\eta})\dot{\eta}, Y)_\eta$$

the nonpositive curvature assumption implies that $\|Y\|^2_{L^2}(t)$ is convex. It follows that since $Y(0) = 0$, there exists $t_1 > 0$ such, that $\frac{d}{dt}\|Y\|^2_{L^2}(t_1) = c > 0$ for some c. Therefore

$$\|Y\|^2_{L^2}(t) \geq c(t - t_1)\|\dot{Y}_e\|^{-2}_{H^s}\|\dot{Y}_e\|^2_{H^s}$$

whenever $t \geq t_1$. For each $m > 0$ let $t - t_1 = mc^{-1}\|\dot{Y}_e\|^2_{H^s}$, then

$$\left\| \begin{pmatrix} Y \\ \\ \dot{Y} \end{pmatrix} \right\|^2_{H^s} (t) \geq \left\| \begin{pmatrix} Y \\ \\ \dot{Y} \end{pmatrix} \right\|^2_{H^s} (0) \cdot m \tag{6.5}$$

which implies the non-negativeness of the Bohl exponent.

Finally we use the following

Lemma 6.6 ([DK]) *In order for the zero solution of equation* (5.1) *to be uniformly stable for any perturbation satisfying the quasilinearity condition it is necessary and sufficient that the corresponding unperturbed equation have a negative Bohl exponent.*

Proposition 6.2 follows.

As an immediate corollary we have

Theorem 6.7 *If η is a periodic, pressure constant flow on a manifold M^n with nonpositive sectional curvature, then η is not uniformly stable.*

Proof. This is obvious since by Theorem 5.1 the curvatures along any such $\eta(t)$ are nonpositive for all t.

Chapter 7

Examples of conjugate points in $\mathcal{D}_\mu$

The question of whether there exist conjugate points on $\mathcal{D}_\mu(M)$ is of particular interest in that it provides some information about the stability behaviour of geodesics.

Let η be a geodesic in $\mathcal{D}_\mu(M)$. We say that two points $\eta(t_1)$ and $\eta(t_2)$ are conjugate with respect to η if there exists a nonzero Jacobi field $Y(t)$ along η such, that $Y(t_1) = Y(t_2) = 0$.

We first note the following consequence of the Gauss' equation.

Corollary 7.1 *Let η be a pressure constant geodesic in $\mathcal{D}_\mu(M)$. If M has nonpositive sectional curvature, then there are no conjugate points along η.*

Next we construct three examples of stationary flows in M, which have conjugate points. We begin with the boundary case.

Example 7.1 Let $M = B^3$, the three dimensional ball in R^3. Consider a one parameter family of rotations

$$\gamma(s,t)(x) =$$

$$(x_1 \cos t - (x_2 \cos s + x_3 \sin s) \sin t,$$

$$(x_1 \sin t + (x_2 \cos s + x_3 \sin s) \cos t) \cos s - (-x_2 \sin s + x_3 \cos s) \sin s,$$

$$(x_1 \sin t + (x_2 \cos s + x_3 \sin s) \cos t) \sin s + (-x_2 \sin s + x_3 \cos s) \cos s)$$

Then $\gamma(0,t)(x) = (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3)$ is a rotation about the x_3-axis and $\gamma(s,0)(x) = (x_1, x_2, x_3)$. We shall show that for each s $\gamma(s,t)$ is a family of geodesics in $\mathcal{D}_\mu(B^3)$ and compute its variation field along $\gamma(0,t)$.

We first find that

$$v(s,t)(x) = \dot{\gamma}(s,t) \circ \gamma^{-1}(s,t)(x) = (-x_2 \cos s - x_3 \sin s, x_1 \cos s, x_1 \sin s)$$

which implies that $\mathrm{tr}(Dv(x))^2 = -2$. From the Euler equation the pressure function $p(t)$ must satisfy the Neumann boundary condition, thus

$$\Delta_\nu^{-1}(-2) = -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \cos^2 s - x_2 x_3 \sin s \cos s - \frac{1}{2}x_3^2 \sin^2 s$$

and we have that

$$\ddot{\gamma}(s,t) \circ \gamma^{-1}(s,t)(x) =$$

$$= (-x_1, -x_2 \cos^2 s - x_3 \sin s \cos s, -x_2 \sin s \cos s - x_3 \sin^2 s) =$$

$$= \mathrm{grad}\Delta_\nu^{-1}\mathrm{tr}(Dv(s)(x))^2$$

which means that $\gamma(s,t)$ satisfies the geodesic equation in $\mathcal{D}_\mu(B^3)$. The variation field of this family is

$$Y(t)(x) = \frac{d}{ds}(\gamma(s,t)(x))|_{s=0} = (-x_3 \sin t, x_3 \cos t - x_3, x_1 \sin t + x_2 \cos t - x_2)$$

and clearly vanishes for $t = 0$ and $t = 2\pi$.

Basically the same idea works in the next example.

Example 7.2 Let now $\xi(s,t)$ be a family of rotations of a unit sphere in R^3. Then $\xi(s,t)$ is again given by the same formula as γ in Example 7.1 above, this time with $|x| = 1$. We see that $v = \dot{\xi} \circ \xi^{-1}$ is a time-independent Killing vector field on S^2 and therefore by Theorem 5.2 satisfies the Euler equations.

Thus $\xi(s,t)$ is a family of geodesics in $\mathcal{D}_\mu(S^2)$. The Jacobi field along $\xi(0,t)$ can be now computed as in the previous example and shown to be zero at $t = 0$ and $t = 2\pi$.

Remark Theorem 5.2 shows that the above example generalizes to the case of a general flow by isometries on a compact manifold of positive curvature.

Example 7.3 Let $M = S^3$. It will be convenient to think of S^3 as the unit quaternions. Let

$$\zeta(s,t)(x) = (\cos t + i \sin t \cos s + j \sin t \sin s) \cdot x$$

where $x = x_1 + ix_2 + jx_3 + kx_4$, $|x|^2 = x \cdot \bar{x} = 1$.

We have $Jac\zeta(s,t) = 1$ for any s and t. Also, the inverse is

$$\zeta^{-1}(s,t)(x) = (\cos t - i \sin t \cos s - j \sin t \sin s) \cdot x$$

and we find that

$$v(s)(x) = \dot{\zeta}(s,t) \circ \zeta^{-1}(s,t)(x) = (i \cos s + j \sin s) \cdot x$$

is a right invariant vector field on S^3. The metric on S^3 is bi-invariant and thus $\nabla_{v(s)(x)}v(s)(x) = 0$, therefore $\zeta(s,t)$ are stationary geodesics in $\mathcal{D}_\mu(S^3)$ parametrized by s. The variation field of this family along $\zeta(0,t)$ is

$$Y(t)(x) = j \cdot x \sin t$$

and $Y(0) = Y(\pi) = 0$, thus $\zeta(0,0)$ and $\zeta(0,\pi)$ are conjugate to each other in $\mathcal{D}_\mu(S^3)$.

Remark The Jacobi fields computed in the above examples can be interpreted as stable perturbations of the initial flow. It is interesting to note that the first two examples involve flows of variable pressure, while the flow on S^3 is pressure constant (compare Example 5.3).

Bibliography

[ADN] S.Agmon, A.Douglis and L.Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions 1*, Comm. Pure Appl. Math. 12 (1959).

[A] V.Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits*, Ann. Inst. Grenoble 16 (1966).

[DK] Yu.L.Daleckii and M.G.Krein, *Stability of solutions of differential equations in Banach space* Amer. Math. Soc., Providence R.I. 1974.

[E1] D.G.Ebin, *Espace des Metriques Riemanniennes et Mouvement des Fluides via les Varietes d'Applications*, Ecole Polytechnique, Paris 1972.

[E2] D.G.Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed*, Comm. Partial Diff. Eq. 12 (1987).

[EM] D.G.Ebin and J.Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. 92 (1970).

[KN] S.Kobayashi and K.Nomizu, *Foundations of Differential Geometry*, Interscience, New York 1963, 1969.

[Li] C.C.Lin, *The Theory of Hydrodynamic Stability*, Cambridge 1955.

[Lu] A.M.Lukatskii, *Curvature of groups of diffeomorphisms preserving the measure of the 2-sphere*, Funkt. Analiz i Ego Prilozh. 13 (1979).

[P] R.S.Palais, *Seminar on the Atiyah - Singer Index Theorem*, Ann. Math. Studies 57, Princeton 1965.