The Euler-Chow series for Toric Varieties

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York
at
Stony Brook

August 1992

State University of New York
at Stony Brook

The Graduate School

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Abstract of the Dissertation
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Let X be a smooth n-dimensional projective variety. Consider the space $\mathcal{C}_\lambda$ of all p-dimensional effective cycles on X with a given homology class λ. We define the Euler-Chow series E of X as the element $\sum_{\lambda \in H_{2p}(X)} \chi(\mathcal{C}_\lambda)\, \lambda \ \in \mathbb{Z}^H$. If we choose a basis $\mathcal{A}$ for $H_{2p}(X, \mathbb{Z})$, E can be written as $E = \sum_{\lambda \in \mathbb{Z}^m} \chi(\mathcal{C}_\lambda)\, t^\lambda$ where m is the rank of $H_{2p}(X)$. We introduce a concept of rationality in $\mathbb{Z}^H$ and prove that this does not depend of the basis $\mathcal{A}$. It is shown that if X is a smooth projective toric variety the Euler-Chow series is rational and an explicit computation is given in terms of an arbitrary basis $\mathcal{A}$. In particular, we compute the euler characteristic

of $\mathcal{C}_\lambda$, for all λ in $H_{2p}(X, \mathbb{Z})$. The idea behind the proof is to define the Equivariant Euler-Chow series E_T for X, using equivariant cohomology instead of singular cohomology. Making use of properties of toric varieties, it is not complicate to compute explicity E_T. Then, there is a map I from $\mathbb{Z}^H$ to $\mathbb{Z}^{H_T}$, induced by the projection map π from $H_T^{2p}(X, \mathbb{Z})$ to $H^{2p}(X, \mathbb{Z})$ which sends E_T to E. The computation for E follows since E_T and I are known.

Para Alicia, mi inseparable media naranja.

Para mi madre Arcelia.

A la memoria de Guillermo Torres.

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Acknowledgements

I would like to express my gratitude to Blaine Lawson and Emili Bifet. I have truely benefited from Blaine's unique understanding of mathematics. He introduced me to a wonderful problem. It was my first serious encounter with the beautiful field of Algebraic Geometry. Emili showed me many of its jewels, among them, Toric Varieties and Equivariant Cohomology.

I would not have been able to finish without the support and love of "mi indispensable media naranja," Alicia.

Life without friends is not life. I was really lucky for having many of them, special mention to Paulo, Eleonora, Dany, Munir, Nita, Angelica, Leticia, Emili, Albert, Margarita, Pablo, Roberto, Pilar, Pacelli, Alberto, Daniel, Maggie, Luis, and Pawel.

I like also to thank all the secretaries in the mathematics department, they were great with me: Amy, Lucille, Joan, Gerry, Barbara, and Grace.

Finally, I thank to the "Instituto de Matemáticas " at the National University of Mexico for the support given to me along all these years.

Chapter 1

Introduction

Algebraic geometers have long been interested in the space of all effective algebraic cycles on a given projective algebraic variety. It is well known that each connected component of this space is itself a projective algebraic set which is not in general irreducible. Many years have been devoted to understanding the structure of this space. However, there are still many important and fundamental questions which have not been answered. For instance, we do not know the Betti numbers and in general the Euler characteristic of these algebraic sets. In order to state our results we need to give a more precise meaning to "the space of all effective algebraic cycles".

Let X be a fixed n-dimensional variety contained in $\mathbb{P}^N$. For each pair of integers p and d, with $d \geq 1$ and $0 \leq p < n$, let $\mathcal{C}_{p,d}(X)$ be the set of all finite formal sums

$$c = \sum n_\alpha V_\alpha$$

where n_α is a positive integer, V_α is an irreducible algebraic p-dimensional variety of X and

$$\deg(c) \stackrel{def}{=} \sum n_\alpha \deg(V_\alpha) = d$$

where $\deg(V_\alpha)$ is the homological degree of V_α as a subvariety of $\mathbb{P}^N$. Each space $\mathcal{C}_{p,d}(X)$ has the structure of a projective algebraic set and is called a **Chow variety.** The inclusion map $i : X \hookrightarrow \mathbb{P}^N$ induces a map $i_* : H_{2p}(X,\mathbb{Z}) \to H_{2p}(\mathbb{P}^N,\mathbb{Z})$. For each homology class λ in $H_{2p}(X,\mathbb{Z})$ with $i_*\lambda = d[\mathbb{P}^p]$, the space $\mathcal{C}_\lambda$ of all elements of $\mathcal{C}_{p,d}(X)$ with homology class λ, is a subvariety of $\mathcal{C}_{p,d}(X)$. We are interested in computing the Euler characteristic of $\mathcal{C}_\lambda$, for all λ, and in looking for relations among them. The case of $X = \mathbb{P}^N$ was worked out in a paper by Blaine Lawson and Stephen S. Yau. They prove the following equality

$$\sum_{d=0}^{\infty} \chi\left(\mathcal{C}_{p,d}(X)\right) t^d = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}$$

We would like to state the problem for an arbitrary algebraic projective variety, and solve it for the case of a smooth projective toric variety. The series that we obtain is not in general a formal power series. We have to clarify, and give to it, the meaning of being rational. Homology and cohomology are considered without torsion.

Definition 1.0.1 . *For each nonegative integer p, the* **Euler-Chow series** *E^p of X is the element*

$$E^p = \sum_\lambda \chi(\mathcal{C}_\lambda)\,\lambda \quad \in \quad \mathbb{Z}^{H_{2p}(X,\mathbb{Z})}.$$

where $\mathbb{Z}^{H_{2p}(X,\mathbb{Z})}$ is the set of all functions from $H_{2p}(X,\mathbb{Z})$ to $\mathbb{Z}$.

If we fix a basis $\{x_1, \ldots x_m\}$ for $H_{2p}(X, \mathbb{Z})$ and if λ is written in terms of the basis as $\lambda = \sum_{i=1}^{m} \alpha_i x_i$, then the Euler-Chow series takes the form

$$E^p = \sum_{\lambda} \chi(\mathcal{C}_\lambda)\, t^{\alpha(\lambda)} \quad \in \mathbb{Z}^{\mathbb{Z}^m}.$$

where $t^{\alpha(\lambda)} = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$. Since we can have negative powers, E^p is not necessary a formal power series, so it does not make sense in general to ask if E^p is rational. In the case of $\mathbb{P}^N$, the series runs over positive numbers, so it is itself a formal power series, and it is rational. However, we will be able to define a similar concept for E^p. In order to do this, we observe that there is a map p, from a subgroup of $\mathbb{Z}[[t_1, \ldots, t_m, \overline{t}_1, \ldots, \overline{t}_m]]$ to $\mathbb{Z}^{\mathbb{Z}^m}$, which sends the series $S_\alpha = \sum_{\alpha \in \mathbb{N}^{2m}} n_\alpha t_1^{\alpha_1} \cdots t_m^{\alpha_m} (\overline{t}_1)^{\overline{\alpha}_1} \cdots \overline{t}_m^{\overline{\alpha}_m}$ to $S_\beta = \sum_{\beta \in \mathbb{Z}^m} n_\beta t_1^{\beta_1} \cdots t_m^{\beta_m}$ where $\beta_i = \alpha_i - \overline{\alpha}_i$, and the sum $n_\beta = \sum_\alpha n_\alpha$ is taken over the α's with the property $\beta_i = \alpha_i - \overline{\alpha}_i$. We have that S_α is in the domain of p if, for all i, $\beta_i = \alpha_i - \overline{\alpha}_i$ with $n_\beta \neq 0$, has a finite number of solutions on β. We arrive to the following definition.

Definition 1.0.2 . *An element E of $\mathbb{Z}^{H_{2p}(X,\mathbb{Z})}$ is called* **rational** *if there is a rational function S_α in the domain of p such that $p(S_\alpha) = E$.*

Here, we are just associating to the Euler-Chow series a rational function in the variables $\{t_1, \ldots, t_m\}$. We show that this definition does not depend on the basis, in other words, rational is an intrinsic concept.

Theorem 1.0.1 . *Let $\mathcal{A}$ and $\mathcal{B}$ be two integral basis of $H_{2p}(X, \mathbb{Z})$. An element E of $\mathbb{Z}^{H_{2p}(X,\mathbb{Z})}$ is rational with respect to the basis $\mathcal{A}$ if and only if it is with respect to the basis $\mathcal{B}$.*

We recall that a **Toric Variety** X, is a normal variety that contains a torus T as an open dense set, and the action of the torus on itself can be extended to the whole variety. In order to study the Euler-Chow series for a smooth projective toric variety we need to define the equivariant Euler-Chow series. From now on, toric variety means smooth toric variety.

Definition 1.0.3 . *Let X be a toric variety, and let $H_T^{2p}(X,\mathbb{Z})$ be the equivariant cohomology of X. The* **equivariant Euler-Chow series** *of X is the element*

$$E_T^p = \sum_{\xi} \chi((\mathcal{C}^T)_\xi)\,\xi \quad \in \mathbb{Z}^{H_T^{2p}(X,\mathbb{Z})}$$

where $\xi \in H_T^{2p}(X,\mathbb{Z})$ and $(\mathcal{C}^T)_\xi$ is the space of all invariant effective cyles on X, with equivariant cohomology class ξ.

Consider a basis $\mathcal{A}$ for $H_T^{2p}(X,\mathbb{Z})$ which contains as elements, the cohomology class of the closure of the orbits under the action of the torus T on X. The equivariant Euler-Chow series is a rational function, and its expression with respect to the basis $\mathcal{A}$ is

$$E_T^p = \prod_{i=1}^{N} \left(\frac{1}{1-s_i}\right)$$

where N is the number of orbits of dimension p. The map π, from the equivariant cohomology to the singular cohomology, induces a map I from a subset $\mathcal{D}$, containing E_T^p, of $\mathbb{Z}^{H_T^{2p}(X,\mathbb{Z})}$ to $\mathbb{Z}^{H^{2p}(X,\mathbb{Z})}$. From the definition of I, it easy to see that this map sends the equivariant Euler-Chow series E_T^p to the Euler-Chow series E^p. All these results, together with the important fact that $\mathcal{C}_\lambda$ is compact, allow us to prove the following theorem.

Theorem 1.0.2 . *For any nonsingular projective toric variety X over $\mathbb{C}$, the Euler-Chow series is rational. In fact, if $(\mathcal{O}_1, \ldots, \mathcal{O}_N)$ denotes all the orbits of codimension 2p, and if $\mathcal{B} = \{x_1, \ldots, x_m\}$ is a basis for $H^{2p}(X, \mathbb{Z})$. Then*

$$E^p = \prod_{i=1}^{N} \left(\frac{1}{1 - \prod_{j=1}^{m} t_j^{a_{ji}}} \right)$$

where $[\overline{\mathcal{O}}_i] = \sum_{j=1}^{m} a_{ji} x_j$ is the expression, in terms of the basis $\mathcal{B}$, of the cohomology class of the closure $\overline{\mathcal{O}}_i$ of the orbit $\mathcal{O}_i$.

An important remark that should be made here is that the matrix $A = (a_{ji})$ is directly computable in terms of the fan of X. This allows us to compute easily calculate many examples. We recover the results of B. Lawson and S.S. Yau. Chapter 2 contains the definition of the Euler-Chow series, the definition of rationality for an element in $\mathbb{Z}^{\mathbb{Z}^m}$, and the proof that this definition does not depends on the basis.

Chapter 3 contains a review of Toric varieties. The most important result that will be used later is the last theorem of the chapter. It is characterized by the cohomology ring of a smooth projective toric variety.

Chapter 4 contains the main resultsof this work we find an explicit expression for the equivariant Euler-Chow which we use to compute the Euler-Chow for a smooth projective toric variety. The appendix contains a letter from Emili Bifet. He suggests how to reformulate, in an intrinsic way, the results contained in this thesis. I consider the approach suggested for him very elegant. Unfortunaly, this letter was written one week before the defense, and it was very hard to rewrite everything in the terms suggested by him.

Chapter 2

The Euler-Chow Series and the Space of Cycles

2.1 Definitions and Properties

In this chapter we define the Euler-Chow series and state the problem in which we are interested. At the end we discuss a few examples. All the definitions and results are restricted to the complex numbers, unless otherwise stated.

Throughout this chapter all homology and cohomology are considered modulo torsion.

A subset X of $\mathbf{P}^N$ is an **algebraic set** if there exists a finite number of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ such that X is the set of common zeros of these polynomials. X is always provided with an embedding $j : X \to \mathbf{P}^N$.

X is called **irreducible** if it is not a non-trivial finite union of algebraic sets, no one of of which contained in another. We define the **Zariski topology** on

$\mathbb{P}^N$ by taking the open sets to be the complements of algebraic sets.

A **projective algebraic variety** is an irreducible algebraic set of $\mathbb{P}^N$ where its dimension is defined as that of a topological space. A **subvariety** Y of X is a subset of X which is itself a projective algebraic variety. Let (X,j) be an algebraic subset. The **degree**, deg(V), of an irreducible subvariety V $\subseteq \mathbb{P}^n$ of dimension p is the number of points in the intesection of V with a generic (N-p)-dimensional linear subspace of $\mathbb{P}^N$, and it is the same as the degree of itsfundamental class in $H_{2p}(\mathbb{P}^N, \mathbb{Z}) = \mathbb{Z}$ (see [GH78])

Definition 2.1.1 *An* **effective p-cycle** *c on X is a finite (formal) sum* $c = \sum n_s V_s$ *where each* n_s *is a nonegative integer and each* V_s *is an irreducible p-dimensional subvariety of X. We define the degree of any effective cycle* $c = \sum n_s V_s$ *as* $\deg(c) = \sum n_s \deg(V_s)$. *From now on, we shall use the term* **cycle** *for* **effective cycle**.

For any algebraic set $j : X \hookrightarrow \mathbb{P}^N$ and integers $p \geq 0, d \geq 0$, we denote by $\mathcal{C}_{p,d}(X)$ the space of all cycles of dimension p and degree d on X. We set $\mathcal{C}_{p,0}(X) = \{\emptyset\}$ by convention. It is well known that $\mathcal{C}_{p,d}(X)$ is a projective algebraic set of $\mathbb{P}^{N(p,d)}$ and that $\mathcal{C}_{p,d}(X)$ is an algebraic subset of $\mathcal{C}_{p,d}(\mathbb{P}^N)$ (see [Sam55] ,[Sha74]). Let λ be an element in $H_{2p}(X, \mathbb{Z})$, and denote by $\mathcal{C}_\lambda(X)$ the set of all cycles on X whose homology class is λ. Note that $\mathcal{C}_\lambda(X)$ is contained in $\mathcal{C}_{p,d}(X)$ by the inclusion $j : X \hookrightarrow \mathbb{P}^N$ where $j_* \lambda = d[\mathbb{P}^p]$. Since any algebraic set is the finite union of irreducible components (see [Har77]), we have $\mathcal{C}_{p,d}(X) = \cup_{i=1}^{M} \mathcal{C}_{p,d}^i(X)$, where $\mathcal{C}_{p,d}^i(X)$ are its irreducible components. Suppose $\mathcal{C}_\lambda(X) \cap \mathcal{C}_{p,d}^{i_o}(X) \neq \emptyset$. Since any two cycles in $\mathcal{C}_{p,d}^{i_o}$ are algebraically equivalent, (

see [Ful84]) they represent the same element of homology, so $\mathcal{C}^{i_o}_{p,d}(X) \subset \mathcal{C}_\lambda(X)$ for some λ. Therefore, $\mathcal{C}_\lambda(X) = \cup_{j=1}^{l} \mathcal{C}^{i_j}_{p,d}(X)$ where $\{i_1, \ldots, i_l\}$ is a subset of $\{1,\ldots,m\}$ with $\mathcal{C}_\lambda(X) \cap \mathcal{C}^{i_o}_{p,d}(X) \neq \emptyset$. We have proved the following lemma:

Lemma 2.1.1 *Let λ be an element of $H_{2p}(X, \mathbb{Z})$, then $\mathcal{C}_\lambda(X)$ is a projective algebraic set.*

As a consequence of this lemma we have that $\mathcal{C}_\lambda(X)$ has a finite triangulation (see for example [Hir75]).

2.2 The Euler-Chow Series

In this section we define the Euler-Chow series of X in dimension p, and discuss some of its properties.

Let H be a free abelian group. We denote by $\mathbb{Z}^H$ the set of all functions from H to $\mathbb{Z}$. We shall write the elements of $\mathbb{Z}^H$ as $\sum_{h \in \mathrm{H}} m(h)\, h$ where $m(h) \in \mathbb{Z}$ for each h in H. We are ready for the following definition.

Definition 2.2.1 *Let p be a fixed nonnegative integer. The* **Euler-Chow series** *of X in dimension p is the element*

$$E^p = \sum_\lambda \chi\left(\mathcal{C}_\lambda\right) \lambda \quad \in \quad \mathbb{Z}^{H_{2p}(X,\mathbb{Z})} \ ,$$

where $\mathcal{C}_\lambda(X)$ is the space of all effective cycles on X with homology class λ and where $\chi\left(\mathcal{C}_\lambda(X)\right)$ is the Euler characteristic of $\mathcal{C}_\lambda$. If $\mathcal{C}_\lambda$ is the empty set, by convention its Euler characteristic is zero.

Let $P = \mathbb{Z}[t_1, \ldots, t_r]$ be the ring of polynomials with coefficients in $\mathbb{Z}$. Let S be the multiplicative set of all polynomials of P whose constant term is ± 1. We denote by P_S the localisation of P with respect S. There is an injective morphism

$$\mathbb{Z}[t_1, \ldots, t_r]_S \overset{\phi}{\hookrightarrow} \mathbb{Z}[[t_1, \ldots, t_r]]$$

from the localisation P_S to the ring of formal power series in the same variables, given by the universal property of P_S. We call an element of $\mathbb{Z}[[t_1, \ldots, t_r]]$ a **rational function** if it is in the image of the morphism ϕ.

Let H be a free abelian group and m its rank. Let $\mathcal{A}$ be an integral basis of H. This gives us an isomorphism

$$F : \mathbb{Z}^m \longrightarrow H$$

and an isomorphism of groups

$$\Psi_{\mathcal{A}} : \mathbb{Z}^H \longrightarrow \mathbb{Z}^{\mathbb{Z}^m} \quad .$$

We shall express any element of $a \in \mathbb{Z}^{\mathbb{Z}^m}$ in exponential notation as

$$a = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha$$

where $a_\alpha = a(\alpha)$ and $t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m)$. Observe that the elements of $\mathbb{Z}^{\mathbb{Z}^m}$ are not in general formal power series, therefore we can not say, or ask, if they are rational functions. However, we will give a definition of rational, and will prove it is an intrinsic concept.

Let $t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m$ be variables and denote by $\mathcal{R}$ the ring of formal power series $\mathbb{Z}[[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]]$ in these variables. We write each element

σ of $\mathcal{R}$ as

$$\sigma = \sum_{(\beta,\overline{\beta})\in \mathbb{N}^{2m}} n_{\beta\overline{\beta}}\, t^{\beta}\,\overline{t}^{\overline{\beta}} \ ,$$

where

$$(\beta, \overline{\beta}) = (\beta_1, \ldots, \beta_m, \overline{\beta}_1, \ldots, \overline{\beta}_m) \ \text{and} \ \ t^{\beta} = t_1^{\beta_1} \cdots t_m^{\beta_m}, \ \ \overline{t}^{\overline{\beta}} = \overline{t}_1^{\,\overline{\beta}_1} \cdots \overline{t}_m^{\,\overline{\beta}_m} .$$

Let $\sigma = \sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\, t^{\beta}\ \overline{t}^{\overline{\beta}}$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ be elements in $\mathcal{R}$ and $\mathbb{N}^m$, respectively. We write

$$N_{\sigma,\alpha} = \text{card}\ \{(\beta, \overline{\beta}) \in \mathbb{N}^{2m} \,|\, n_{\beta\overline{\beta}} \neq 0 \text{ and } \ \alpha = \beta - \overline{\beta}\}.$$

Let $\mathcal{D}$ be the set of all elements $\sigma \in \mathcal{R}$ with the property that the number $N_{\sigma,\alpha}$ is finite for all α in $\mathbb{Z}^m$. We have a map

$$p : \mathcal{D} \subset \mathcal{R} \longrightarrow \mathbb{Z}^{\mathbb{Z}^m}$$

defined by

$$\sum_{\beta\overline{\beta}\in \mathbb{N}^{2m}} n_{\beta\overline{\beta}}\, t^{\beta}\,\overline{t}^{\overline{\beta}} \ \stackrel{p}{\longmapsto} \ \sum_{\alpha} v_{\alpha}\, t^{\alpha} \tag{2.1}$$

where

$$v_{\alpha} = \sum_{\beta-\overline{\beta}=\alpha} n_{\beta\overline{\beta}}.$$

We arrive at the following definition.

Definition 2.2.2 *Let $\mathcal{A}$ be a basis for H. We say that E in $\mathbb{Z}^H$ is $\mathcal{A}$*-rational *if there is a rational function $\sigma \in \mathcal{D}$, such that $p(\sigma) = \Psi_{\mathcal{A}}(E)$.*

Observe that in this definition $\Psi_{\mathcal{A}} : \mathbb{Z}^H \longrightarrow \mathbb{Z}^{\mathbb{Z}^m}$ is the isomorphism defined above in terms of the basis $\mathcal{A}$. The next theorem shows that this definition is intrinsic, in other words, it does not depend on the choice of basis.

Theorem 2.2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two integral bases for H. An element E of $\mathbb{Z}^H$ is $\mathcal{A}$-rational if and only if it is $\mathcal{B}$-rational.*

Before we go on with the proof of the theorem, we give the following definition as a consequence of the theorem.

Definition 2.2.3 *An element E of $\mathbb{Z}^H$ is* **rational** *if it is $\mathcal{A}$-rational for some basis $\mathcal{A}$ of H.*

Proof of the theorem

The heart of the argument is to show that the following diagram commutes

$$\begin{array}{ccc} \mathcal{R} \supset \mathcal{D} & \xrightarrow{p} & \mathbb{Z}^{\mathbb{Z}^m} \\ \downarrow \overline{\Phi} & & \downarrow \Phi \\ \mathcal{R} \supset \mathcal{D} & \xrightarrow{p} & \mathbb{Z}^{\mathbb{Z}^m} \end{array},$$

where $\overline{\Phi}$ and Φ are maps induced by the change of bases. We start with a description of these two maps. For any $a \in \mathbb{Z}$, let

$$a^+ = \max\{a, 0\} \quad \text{and} \quad a^- = \max\{-a, 0\} \quad \text{and} \quad a = a^+ - a^-.$$

Let $A = (a_{ij})$ be the matrix representing the change of bases. Then $A = A^+ - A^-$ where $A^+ = (a_{ij}^+)$ and $A^- = (a_{ij}^-)$. The isomorphism A induces two maps. The first map is the isomorphism of groups

$$\Phi : \mathbb{Z}^{\mathbb{Z}^m} \longrightarrow \mathbb{Z}^{\mathbb{Z}^m}$$

given by

$$\Phi\left(\sum_\alpha n_\alpha\, t^\alpha\right) = \sum_\alpha n_\alpha\, t^{A(\alpha)} = \sum_\gamma n_{A^{-1}(\gamma)}\, t^\gamma \ .$$

The second one is the isomorphism

$$\overline{\Phi} : \mathcal{R} \longrightarrow \mathcal{R}$$

defined at each generator t_i and $\bar{t}_i$ by

$$\overline{\Phi}\,(t_i) = \prod_{j=1}^m t_j^{a_{ji}^+}\, \bar{t}_j^{a_{ji}^-} \ \text{ and } \ \overline{\Phi}\,(\bar{t}_i) = \prod_{j=1}^m t_j^{a_{ji}^-}\, \bar{t}_j^{a_{ji}^+} \ .$$

To see that $\overline{\Phi}$ is an isomorphism one checks that for any two basis $\mathcal{A}$ and $\mathcal{B}$, we have $\overline{\Phi_{\mathcal{A}} \circ \Phi_{\mathcal{B}}} = \overline{\Phi_{\mathcal{A}}} \circ \overline{\Phi_{\mathcal{B}}}$. Observe that By the universal property of localisation, the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S & \overset{\phi}{\hookrightarrow} & \mathcal{R} = \mathbb{Z}[[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]] \\
\downarrow \overline{\Phi} & & \downarrow \overline{\Phi} \\
\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S & \overset{\phi}{\hookrightarrow} & \mathcal{R} = \mathbb{Z}[[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]] \ ,
\end{array}$$

where $\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]_S$ is the localisation of the ring of polynomials $\mathbb{Z}[t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_m]$ with respect to the multiplicative set S which consists of all polynomials with constant term 1 or -1. Consequently $\overline{\Phi}$ sends rational functions into rational functions.

Next, we would like to prove that $\overline{\Phi}(\mathcal{D}) \subset \mathcal{D}$. Let $\sigma \in \mathcal{D}$ be an element in the domain of $\overline{\Phi}$. We write σ as $\sigma = \sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\, t^\beta\, \bar{t}^{\overline{\beta}}$. By the definition of $\mathcal{D}$, for each α the number of solutions $N_{\sigma,\alpha}$ of

$$\alpha = \beta - \overline{\beta} \quad \text{with} \quad n_{\beta\overline{\beta}} \neq 0 \tag{2.2}$$

is finite. From the definition of $\overline{\Phi}$ we obtain

$$\overline{\Phi}\,(\sigma) \;=\; \overline{\Phi}\left(\sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\; t^{\beta}\; \overline{t}^{\overline{\beta}}\right) = \sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\; t^{(A^{+}\beta+A^{-}\overline{\beta})}\; \overline{t}^{(A^{-}\beta+A^{+}\overline{\beta})} \quad .$$

Observe that the system

$$\delta \;=\; A^{+}\beta + A^{-}\overline{\beta} \quad \text{and} \quad \overline{\delta} \;=\; A^{-}\beta + A^{+}\overline{\beta} \;\text{ with }\; (\beta,\overline{\beta}) \in \mathbb{N}^{2m} \tag{2.3}$$

has a unique solution, since $\overline{\Phi}$ is an isomorphism. We write

$$\begin{aligned}\overline{\Phi}(\sigma) \;&=\; \overline{\Phi}\left(\sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\; t^{\beta}\; \overline{t}^{\overline{\beta}}\right) = \\ &=\; \sum_{\beta\overline{\beta}} n_{\beta\overline{\beta}}\; t^{(A^{+}\beta+A^{-}\overline{\beta})}\; t^{(A^{-}\beta+A^{+}\overline{\beta})} \;=\; \sum_{\delta\overline{\delta}} u_{\delta\overline{\delta}}\; t^{\delta}\overline{t}^{\overline{\delta}} \;=\; \overline{\sigma}\,.\end{aligned}$$

Where

$$u_{\delta\overline{\delta}} \;=\; n_{\beta\overline{\beta}} \quad \text{with} \quad (\delta,\overline{\delta}) \;=\; (A^{+}\beta + A^{-}\overline{\beta},\; A^{-}\beta + A^{+}\overline{\beta})$$

We now prove that $\overline{\sigma} \in \mathcal{D}$. In order to do that, we have to show that for any $\gamma \in \mathbb{Z}^{m}$, there is a finite number of solutions on $(\delta,\overline{\delta})$ to the system

$$\gamma \;=\; \delta - \overline{\delta} \quad \text{with} \quad u_{\delta,\overline{\delta}} \neq 0 \;\; . \tag{2.4}$$

However, any solution $(\delta,\overline{\delta})$ to this sytem, we have

$$\gamma \;=\; \delta - \overline{\delta} \;=\; A^{+}\beta + A^{-}\overline{\beta} - A^{-}\beta - A^{+}\overline{\beta} \;=\; A\,(\beta - \overline{\beta}) \quad \text{with } n_{\beta\overline{\beta}} \neq 0\,.$$

This is equivalent to the system

$$A^{-1}(\gamma) \;=\; A^{-1}A(\beta - \overline{\beta}) \;=\; \beta - \overline{\beta} \quad \text{with} \quad n_{\beta\overline{\beta}} \neq 0. \tag{2.5}$$

However, this is the system 2.2, which has a finite number $N_{A^{-1}(\sigma),\sigma}$ of solutions. Then 2.4 has a finite number of solutions. Therefore, $\overline{\Phi}(\sigma) = \overline{\sigma} \in \mathcal{D}$ and

$$p\left(\overline{\Phi}(\sigma)\right) = p\left(\sum_{\delta\overline{\delta}} u_{\delta\overline{\delta}}\, t^{\delta}\overline{t}^{\overline{\delta}}\right) = \sum_{\gamma\in\mathbb{Z}^m} w_\gamma t^\gamma \tag{2.6}$$

where

$$w_\gamma = \sum_{\beta-\overline{\beta}=A^{-1}(\gamma)} n_{\beta\overline{\beta}}\,.$$

Next, from the definition of p (see 2.1), we have

$$p(\sigma) = \sum_{\alpha\in\mathbb{Z}^m} v_\alpha\, t^\alpha\,, \quad \text{where } v_\alpha = \sum_{\alpha=\beta-\overline{\beta}} n_{\beta\overline{\beta}}\,.$$

Then,

$$\Phi\left(p(\sigma)\right) = \Phi\left(\sum_{\alpha\in\mathbb{Z}^m} v_\alpha\, t^\gamma\right) = \sum_{\gamma\in\mathbb{Z}^m} N_{A^{-1}(\gamma)}\, t^\alpha \tag{2.7}$$

with

$$v_{A^{-1}(\gamma)} = \sum_{A^{-1}(\gamma)=\beta-\overline{\beta}} n_{\beta\overline{\beta}}\,.$$

Finally, comparing 2.6 and 2.7 we obtain

$$p\left(\overline{\Phi}(\sigma)\right) = \Phi\left(p(\sigma)\right).$$

∎

2.3 Examples

I) The projective space $P^n(\mathbb{C})$.

Let $[x_0 : \ldots : x_{n+1}]$ be homogenous coordinates for $P^n(\mathbb{C})$.

For each (p+1)-tuple of integers $\alpha = (\alpha_0, \ldots, \alpha_p)$ with $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_p \leq n$, consider the coordinate (p+1)-plane

$$\mathbb{C}_\alpha^{p+1} = \{z \in \mathbb{C}^{n+1} \mid z_i = 0 \text{ if } i \neq \alpha_i \text{ for some } j\} \subset \mathbb{C}^{n+1}$$

and denote by

$$\mathbb{P}_\alpha^p \subset \mathbb{P}^n$$

the corresponding projective p-plane. It is not hard to see that any p-dimensional invariant cycle on $\mathbb{P}^n$ has the form

$$V = \sum \eta_\alpha \mathbb{P}_\alpha^p \quad \text{for nonegative integers } \eta_\alpha.$$

In [LY87] it is proved that

$$\chi(\mathcal{C}_{p,d}) = \text{card}\,\{(m_1, \ldots, m_\nu) \in \mathbb{Z}^\nu \,:\, m_j \geq 0 \text{ and } \sum m_j = d\} = \binom{\nu+d-1}{d},$$

where $\nu = \binom{n+1}{p+1}$. Therefore,

$$E^p = \sum_{d=0}^{\infty} \chi(\mathcal{C}_{p,d})\, t^d = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}.$$

II) $\mathbb{P}^n \times \mathbb{P}^m$.

Using the same method as in the last example, and the canonical decomposition

$$H_{2p}(X, \mathbb{Z}) = \bigoplus_{k+l=p} H_{2k}(\mathbb{P}^n) \otimes H_{2l}(\mathbb{P}^m)$$

we obtain, for any integer p, and $0 \leq p \leq n+m$

$$E^p = \prod_{k+l} (1 - t_{k,l})^{-\binom{n+1}{k+1}\binom{m+1}{l+1}}.$$

III)Zero cycles.

Let X be a projective algebraic variety. We denote by $SP^d(X)$ the d-fold symmetric product on X. Let b_0, b_1, ... be the Betti numbers of X. MacDonald proves (see [Mac62]) that the k^{th} Betti number of $SP^d(X)$ is the coefficient of $x^k\,t^d$ in the power series expansion of

$$\frac{(1+xt)^{b_1}\,(1+x^3t)^{b_3}\,\cdots}{(1-t)^{b_0}\,(1-x^2t)^{b_2}\,(1-x^4t)^{b_4}\,\cdots}\ ,$$

and therefore the Euler-Poincaré characteristic of $SP^d(X)$ is the coefficient of t^n. Finally we have

$$E^p = \sum_{d=0}^{\infty} \chi\left(SP^d(X)\right)\,t^d = \frac{1}{(1-t)^{\chi(X)}}\ .$$

Chapter 3

Toric Varieties

3.1 Strongly Convex Rational Polyhedral Cones

In this section we state some definitions and theorems without proof about convex polyhedral cones, we refer the reader to ([Ful89], [Oda88], [Oda91]). We start with the following definition,

Definition 3.1.1 *A subset $\sigma \subset V$ of the vector space V is callled a* **convex polyhedral cone** *if*

$$\sigma = \mathbb{R}_{\geq 0}V_1 + \ldots + \mathbb{R}_{\geq 0}V_s \ = \ \{r_1V_1 + \cdots + r_sV_s | \, r_i \in \mathbb{R}, r_i \geq 0, \ V_i \in V\}$$

$\{V_i\}$ (sometimes $\mathbb{R}_{\geq 0}V_i$) are called **generators** *of σ.*

The **dimension** of σ is the dimension of the subspace $\mathbb{R}\sigma = \sigma + (-\sigma)$ of V. The first Lemma that we will need later is due to Carathéodory:

Lemma 3.1.1 *(Carathéodory)*
Let σ be a convex Polyhedral cone with generators $V_1, \ldots, V_s$ the any nonzero

vector V in σ can be written as $V = \sum_{i=1}^{l} r_i V_{j_i}$ with $l \leq dim\ \sigma$ and $\{V_{j_i}\}$ is a subset of linearly independent vectors of $\{V_1, \ldots, V_s\}$.

Let V^* be the dual space with dual pairing denoted by

$$<,>: V^* \times V \longrightarrow \mathbb{R}$$

The dual cone σ^* of σ is the subset of V^* defined by

$$\sigma^* = \{u \in V^* | < u, v > \geq 0 \ \ \forall\, v \in \sigma\}$$

and it is easy to see that σ^* is also a convex polyhedral cone.

A cone σ is called **Strictly Convex** if it contains no vector subspace but $\{0\}$, in other words, if $\sigma + \{-\sigma\} = \{0\}$. In fact, σ is strictly convex if and only if σ^* spans V^*. For any cone σ, the cone

$$\sigma^{\perp} = \{u \in V^* | < u, v >= 0 \ \ \forall\, v \in \sigma\} = \sigma^* \cap (-\sigma)^*$$

is a vector space.

Definition 3.1.2 *A subset τ of σ is called a* **face** *if there exists $u_o \in V^*$ such that*

$$\tau = \sigma \cap u_o^{\perp}.$$

τ itself is a convex polyhedral cone.

Definition 3.1.3 *σ is* **simplicial** *if it is spanned by linearly independent generators.*

Let N be a lattice, there is r such that N $\cong \mathbb{Z}^r$. Let $M = Hom(N, \mathbb{Z})$ be the dual lattice with pairing

$$<,>: M \times N \longrightarrow \mathbb{Z}$$

Let $N_{\mathbb{R}}=N \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $M_{\mathbb{R}}=M \otimes_{\mathbb{Z}} \mathbb{R}$) be the vector space over $\mathbb{R}$ generated by N (resp. M).

Definition 3.1.4 *A strictly convex polyhedral cone is* **rational** *if it has generators $V_1, \ldots, V_s$ with each V_i in N*

Everything said up to this point preserves rationality. From now on, a cone means a strictly convex rational polyhedral cone. The following proposition is one of the main keys to construct toric varieties.

Proposition 3.1.1 *Let σ be a cone in $N_{\mathbb{R}}$. Then*

(1) $S_\sigma := M \cap \sigma^* = \{m \in M \mid < m, v > \geq 0 \; \forall \, v \in \sigma\}$ is an additive sub semigroup, containing 0, of M.

(2)S_σ is finitely generated as an additive semigroup, in othe words, there exist $m_1, \ldots, m_p \in S_\sigma$ such that $S_\sigma = \mathbb{Z}_{\geq 0} m_1 + \ldots + \mathbb{Z}_{\geq 0} m_p$.

(3)S_σ generates M as a group, i.e. $S_\sigma + (-S_\sigma) = M$.

(4)S_σ is saturated, i.e. $cm \in S_\sigma$; $m \in M$, $c \in \mathbb{Z}_{\geq 0} \Rightarrow m \in S_\sigma$.

Conversely, for any additive semigroup S of M satisfying the properties (1) through (4), there exists a unique strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ such that $S = S_\sigma$.

We are going to prove only (2), which is known as Gordan's lemma.

Proof of (2): Because of Carathéodory's lemma, we can assume the cone σ^*

is generated by linearly independent vectors, $u_i \in \sigma^* \cap$M. Now

$$K = \{\sum r_i u_i \mid 0 \leq r_i \leq 1\} \cap M$$

is compact and discrete, so finite, and since any $u \in \sigma^* \cap$M has the form $\sum(a_i + r_i)\, u_i$ for $a_i \in \mathbb{Z}$ and $r_i \in$ [0,1], then K generates $\sigma^* \cap$M as semigroup.

The next lemma is the last one in this introductory section.

Lemma 3.1.2 *If τ is a face of σ and if $S_\sigma = \sigma^* \cap M$, the there is a $u \in \sigma^* \cap M$ with $\tau = \sigma \cap \tau^\perp$ and*

$$\tau^* \cap M = S_\sigma + \mathbb{Z}_{\geq 0}(-u).$$

For a proof, see ([Oda88], chap1.).

3.2 Affine Toric Varieties

Let $\mathbb{C}[S]$ be the commutative $\mathbb{C}$-algebra determined by a semigroup S. As a vector space, $\mathbb{C}[S]$ has a basis $\{\chi^s | s \in S\}$ and multiplication given by $\chi^s \chi^{s'} = \chi^{s+s'}$. If $\{s_i\}$ is a set of generators for S, then $\{\chi^{s_i}\}$ is a set of generators for $\mathbb{C}[S]$. Now, consider Spec ($\mathbb{C}[S]$), here, when we speak of a point p of Spec ($\mathbb{C}[S]$) we understand that p is a closed point. We recall that any homomorphism A $\rightarrow$ B of $\mathbb{C}$-algebras determines a morphism Spec (B) $\rightarrow$ Spec (A) of varieties, in particular we have that closed points correspond to $\mathbb{C}$-algebra homomorphisms from A to $\mathbb{C}$. And if X=Spec (A), for each element x$\in$ Spec (A), x$\neq$0, the principal open subset X_g=Spec (A_g) $\subset$ X=Spec (A) correspond to the localisation homomorphism A$\rightarrow A_g$.

An important example is the torus:

$S = M$ is a lattice. Choosing an isomorphism $M \cong \mathbb{Z}^n$ determines 2n semigroup generators

$$(\pm 1, 0, \ldots, 0), (0, \pm 1, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1).$$

Letting $X_1 = \chi^{(1,0,\ldots,0)}$, so $X_1^{-1} = \chi^{(-1,0,\ldots,0)}$. and similarly for other basis elements, we have

$$\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$$

$$\mathbb{C}[X_1, \ldots, X_n]_{X_1 \cdot \ldots \cdot X_m}$$

the ring of laurant polynomials in n variables. So

$$\text{Spec }(\mathbb{C}[M]) = \mathbb{C}^* \times \cdots \times \mathbb{C}^* = (\mathbb{C})^n = T$$

is an affine algebraic torus. Any point of T can be identify with a homomorphism from M to $\mathbb{C}^*$ as a group, i.e. $T^n \cong \text{Hom}_{gr}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. we denote this group by T_N. Furthemore, for S a semigroup, the points of Spec $(\mathbb{C}[S])$ correspond to semigroup homomorphisms from S to $\mathbb{C}$:

$$\text{Spec }(\mathbb{C}[S]) = \text{Hom}_{sg}(S, \mathbb{C}).$$

Definition 3.2.1 *Let σ be a rational strictly convex polyhedral cone in a lattice N. Let S_σ be the semigroup $\sigma^* \cap M$ and let $A_\sigma := \mathbb{C}[S_\sigma]$. Then*

$$U_\sigma = Spec(A_\sigma) = Hom_{sg}(S_\sigma, \mathbb{C})$$

is called the **Affine Toric Variety** *determined by σ*

Example

$N = \mathbb{Z}^n$, $\{e_1, \dots, e_n\}$ a basis for N.

$$\sigma = \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_k, \quad k \leq n$$

and

$$\sigma^* \cap M = \mathbb{Z}_{\geq 0}e_1^* + \cdots + \mathbb{Z}_{\geq 0}e_k^* + \mathbb{Z}_{\not\geq 0}e_{k+1}^* + \cdots + \mathbb{Z}_{\not\geq 0}e_n^*.$$

$A_\sigma = \mathbb{C}[x_1, \dots, x_k, x_{k+1}, x_{k+1}^{-1}, \dots, x_n, x_n^{-1}]$ so

$$U_\sigma = \mathbb{C} \times \cdots \times \mathbb{C}^* \cdots \times \mathbb{C}^* = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$$

and $U_\sigma$is nonsingular. Let τ be a face of σ, then from the previous section we have that there exists $u \in \sigma^* \cap$ M such that $\tau = \sigma \cap u^\perp$ and $\tau^* \cap M = \sigma^* \cap$ $u + \mathbb{Z}_{\geq 0}(-u)$. Then each element of $\mathbb{C}[\tau^* \cap \mathrm{M}]$ has the form $\chi^{w-pu} = \chi^w/(\chi^u)^p$ for $w \in \sigma^* \cap$ M. Therefore $A_\tau = (A_\sigma)_{\chi^u}$ and we have that $U_\tau = \text{Spec }(A_\tau) \subset$ Spec $(A_\sigma) = U_\sigma$ is a rational open subset.

The process of associating the toric variety U_σ with the pair (N,σ) is a contravariant functor since, if $\varphi : N' \to N$ is a homomorphism of lattices that maps $\sigma' \subset N'$ to $\sigma \subset N$ then dual $\varphi^* : M \to M'$ maps σ^* to σ'^*, determining a homomorphism $A_\sigma \to A_{\sigma'}$ and hence $U_{\sigma'} \to U_\sigma$.

Finally, it is easy to see that we have an action of T_N over U_σ which extends the action of T_N on itself, given by multiplication. More precisely, a point $t \in T_N$ is represented by a map $M \to \mathbb{C}^*$ of groups, and $\chi \in U_\sigma$ by a map $S_\sigma \to \mathbb{C}$ of semigroups. The product $t \cdot \chi$ is the map of semigroups $S_\sigma \to \mathbb{C}$ given by $s \mapsto t(s)\chi(s)$ and the dual map on algebras $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$ is given by $\chi^s \mapsto \chi^s \otimes \chi^s$ for $s \in S_\sigma$.

Now let us go to see some basic properties of affine varieties.

Proposition 3.2.1 *An affine toric variety U_σ is nonsingular if and only if σ is generated by part of a basis for the lattice N, in which case*

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, \;\; k = dim(\sigma).$$

Proof

$\Leftarrow$) See example above.

$\Rightarrow$) We can assume k=n. Let $\mathcal{M}$ be the maximal ideal in A_σ generated by all χ^u for $u \in S_\sigma - \{0\}$. Then $\mathcal{M}/\mathcal{M}^2$ is generated by all χ^u such that u in $S_\sigma - \{0\}$ and u is not the sum of two elements in $S_\sigma - \{0\}$. Let $P_\mathcal{M}$ be the point in U_σ that correspond to $\mathcal{M}$, then if U_σ is not singular at $P_\mathcal{M}$, the cotangent space $\mathcal{M}/\mathcal{M}^2$ is n-dimensional, then σ^* have n edges and the minimal generators along these generate S_σ, but S_σ generates M as a group so the dual σ is generated by a part of a basis for N, therefore $U_\sigma \cong \mathbb{C}^n$ ∎

Proposition 3.2.2 *U_σ is normal.*

Proof

Let $A_\sigma = \mathbb{C}[\sigma^* \cap \; M]$, we have to show that A_σ isd integrally closed. Let $\tau_1, \ldots, \tau_n$ be the faces of codimension 1 of σ. Then we have that $\Sigma^* = \cap_{i=1}^n \tau_i^*$. Therefore $A_\sigma = \cap A_{\tau_i}$ is isomorphic to $\mathbb{C}[x_1, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ which is integrally closed. So is A_σ.

∎

3.3 Toric Varieties and their Orbits

Definition 3.3.1 *A* **Fan** *Δ is a set of cones σ in N such that*

1) if τ is a face of $\sigma \in \Delta$ then $\tau \in \Delta$,

2) if $\sigma, \sigma' \in \Delta$ then $\sigma \cap \sigma'$ is a face of both, σ and σ'.

We are going to consider only finite fans. For each $\sigma \in \Delta$ consider its respective U_σ. We can glue all the U_σ in the following way: take σ and τ in Δ, then $\sigma \cap \tau$ is a face of both σ and τ. Then $U_{\sigma\cap\tau}$ is identified as an open subvariety of U_σ and of U_τ. We glue U_σ and U_τ by identifying this common subvariety $U_{\sigma\cap\tau}$. Furthermore, it is proven easily that the action of T_N is compatible.

Now, suppose $\varphi : N' \to N$ is a homomorphism of lattices and that $\Delta \subset N$, $\Delta' \subset N'$ are two fans in N and N' respectively. Assume that for each cone σ' in Δ' there exists a cone σ in Δ such that $\varphi(\sigma') \subset \sigma$, then the dual

$$\mathbb{Z} - \text{homomorphism } \varphi^* : M' \to M$$

induces a semigroup homomorphism

$$\varphi^* : M' \cap (\sigma')^* \to M \cap \sigma^*$$

which induces an equivariant morphism $\varphi_* : U_\sigma \to U'_{\sigma'}$. We can glue equivariant morphisms of this form for toric varieties and obtain a general equivariant morphism. In other words:

Theorem 3.3.1 *Via a covariant functor*

$$(N, \Delta) \mapsto X(\Delta) and\, \varphi \mapsto \varphi_*,$$

the category of fans together with maps of fans is equivalent to the category of toric varieties over $\mathbb{C}$ together with equivariant morphisms.

The main reason for introducing fans is that the algebro-geometric properties of toric varieties can be interpreted (in a combinatorial way) using the fan associated to them. The next two proposition are examples of this fact.

Proposition 3.3.1 *The toric variety $X(\Delta)$ is nonsingular iff each $\sigma \in \Delta$ is generated by elements of a basis of N.*

We have already given a proof of this result for the affine case.

Proposition 3.3.2 *$X(\Delta)$ is compact iff Δ is finite and complete, where complete means that $N_{\mathcal{R}} = \cup_{\sigma \in \Delta} \sigma$.*

For a complete proof of these two facts see [Oda88], [Ful89].

ORBITS:

We would like to describe the orbits of X(Δ) under the action of T_N. For each $\tau \in \Delta$, let N_τ be the lattice generated by $\tau \cap N$.

Consider $N(\tau) := N/N_\tau$ and its dual $M(\tau) = \tau^\perp \cap M$. Define

$$O_\tau := T_{N_\tau} = \text{Spec } (\mathbb{C}[M(\tau)]) = \text{Hom } (M(\tau), \mathbb{C}^*) = N(\tau) \otimes_Z \mathbb{C}^*.$$

This is the torus associated to the lattice N_τ. Its dimension is n-k, where n=dim Δ and k=dim τ. Observe that T_N acts transitively via the projection $T_N \to T_{N_\tau}$, i.e. O_τ is an orbit.

Let us describe the closure $V(\tau)$ of O_τ. Consider the following set $A := \{\sigma \in \Delta | \tau < \sigma\}$. For any $\sigma \in A$ we denote the image of σ in N_τ by $\overline{\tau}$. The set $\{\overline{\tau} | \tau < \sigma\}$ forms a fan st(τ) in N_τ. Set $V(\tau) = X(st(\tau))$; here O_τ corresponds to the cone $\{0\}=\overline{\tau}$ in N_τ. This toric variety $V(\tau)$ is covered by $\{U_\sigma(\tau)\}_{\sigma\in\Delta,\tau<\sigma}$ and

$$U_\sigma(\tau) = Spec(\mathbb{C}[\overline{\tau}^* \cap M(\tau)]) = Spec(\mathbb{C}[\sigma^* \cap \tau^\perp \cap M]).$$

We have the projection map from $\sigma^* \cap M$ to $\mathbb{C}[\sigma^* \cap \tau^\perp \cap M]$, both considered as semigroups, the latter with the multiplication operation, this map sends $\sigma^* \cap \tau^\perp \cap M$ to the identity and the complement goes to zero. This induces a surjective map

$$\mathbb{C}[\sigma^* \cap M] \to \mathbb{C}[\sigma^* \cap \tau^\perp \cap M]$$

and this gives us a map

$$U_\sigma(\tau) = \text{Spec }(\mathbb{C}[\sigma^* \cap \tau^\perp \cap M]) \hookrightarrow \text{Spec }(\mathbb{C}[\sigma^* \cap M]) = U_\sigma.$$

In other words, we have a closed embedding from $U_\sigma(\tau)$ to U_σ, for each $\sigma \succ \tau$. It is easy to see that these maps are compatible on $\{U_\sigma(\tau)\}_{\sigma\in\Delta,\tau\in\sigma}$ then we have a closed emmbedding $V(\tau) \hookrightarrow X(\Delta)$.
Finally, we have an ordering reversing correspondende from cones $\{\tau\}$ in Δ to closure of orbits $V(\tau)$ in Δ.

To finish this section, we state the relations among orbits O_τ, orbit closures $V(\tau)$, and the affine open sets U_σ.

Proposition 3.3.3 *Let U_σ $V(\tau)$ and O_τ as before, then*

I) $U_\sigma = \coprod_{\tau\prec\sigma} U_\tau$

II) $V(\tau) = \coprod_{\gamma\succ\tau} V_\gamma$

III) $O_\tau = V(\tau) - \cup_{\gamma\succ\tau} V(\gamma)$

In particular, it says that $X(\Delta)$ is a disjoint union of the O_τ, which are the orbits of T_n and $O_\sigma \subset \overline{O} = V(\sigma)$ if an only if σ is a face of τ.

sketch of the proof

Take $\varphi \in U_\sigma$, in other words, $\varphi \in Hom_{sg}(s_\sigma, \mathbb{C})$. Using lemma 3.1.2, and the fact that two elements of σ^* can not be in $\varphi^{-1}(\mathbb{C}^*)$ unless both are in $\varphi^{-1}(\mathbb{C}^*)$ we can prove that the set $\{m \in s_\sigma | \varphi^{-1}(m) \in \varphi^{-1}(\mathbb{C}^*)$ is of the form $M \cap \sigma^* \cap \tau^\perp$, for a unique face τ of σ. In other words, for each $\varphi \in U_\sigma$, $\varphi^{-1}(\mathbb{C}^*) = M \cap \sigma^* \cap \tau^\perp$ and this means that φ corresponds to a point of O_τ.This prove 1. For 3, if we go to $N(\tau)$, i.e. in $V(\tau)$ we may assume $\tau = 0$, we have $T_n = X(\Delta) - \cup_{\sigma \neq \{0\}} V(\sigma)$, now, intersecting with $U(\sigma)$ three follows from one, and two follows from 3 using induction on dimension.

3.4 Cohomology

Let $A^p(X(\Delta))$ be the group of the formal finite $\mathbb{Z}$-linear combinations of closed irreducible subvarieties of $X(\Delta)$ of codimension p, modulo rational equivalence. The intersection of cycles induces a product

$$A^p(X(\Delta)) \times A^q(X(\Delta)) \longrightarrow A^{p+q}(X(\Delta))$$

so

$$A^*(X(\Delta)) = \oplus_{i=0}^{\infty} A^i(X(\Delta))$$

is a graded ring of $X(\Delta)$.

For each closed irreducible subvariety we can associate to it its fundamental cohomology class and this map induces an additive homomorphism

$$A^p(X(\Delta)) \longrightarrow H^{2p}(X(\Delta), \mathbb{Z})$$

which doubles the degrees.

Let $\Delta(J)$ be the set of J-dimensional cones in Δ. For each $\sigma \in \Delta(j)$ let us denote by $v(\sigma) \in A^j(X)$ the rational equivalence class of the irreducible cycle $V(\sigma)$ on X of codimension j. For distinct $D_1, \ldots, D_s$ in $\Delta(1)$ we have

$$v(D_1)\cdots v(D_s) = \begin{cases} D_1 + \cdots + D_s & \text{if} \quad D_1 + \cdots + D_s \in \Delta \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, each $m \in M$ determines a divisor that corresponds to the rational function χ_m which is rationally equivalent to zero

$$\sum_{i=1}^{r} < m, m(D_i) > v(D_i) = 0 \qquad \text{for all} \quad m \in M$$

Here, r is the number of invariant divisors in $X(\Delta)$ or equivalently, the cardinallity of $\Delta(1)$. For each $D_i \in \Delta(1)$, we introduce the variable t_i and consider the polynomial ring $S = \mathbb{Z}[t_1, \ldots, t_r]$ which is the symmetric algebra associated to the $\mathbb{Z}$-module $T_N Div(X(\Delta))$ of T_N-invariant divisors on $X((\Delta))$.

Let J be the ideal in S generated by

$$\{t_{i_1} \cdots t_{i_s} \,|\, t_{i_l} \neq t_{i_m} \text{ if } l \neq m \text{ and } D_1 + \cdots + D_s \notin \Delta\}$$

and let I be the eideal in S generated by

$$\{\sum_{j=1}^{r} < m, m(D_J) > v(D_i) \,|\, m \in M\}$$

we have the following homomorphisms of graded rings

$$S/(I+J) \longrightarrow A^*(X(\Delta)) \longrightarrow H^*(X(\Delta), \mathbb{Z})$$

The main theorem of this section is due to Jurkiewicz and Danilov:

Theorem 3.4.1 *Jurkiewicz-Danilov's theorem.*

Let $X(\Delta)$ be an n-dimensional compact toric variety then,

I) If $X(\Delta)$ is nonsingular, the homomorphisms of graded rings described above are in fact Isomorphisms

$$S/(I+J) \xrightarrow{\sim} A^*(X(\Delta)) \xrightarrow{\sim} H^*(X(\Delta), \mathbb{Z}).$$

II) If Δ is simplicial, we can define the chow ring with coefficients in $\mathbb{Q}$ as, $A^*(X(\Delta)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and again we get isomorphisms of $\mathbb{Q}$-algebras

$$S/(I+J) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A^*(X(\Delta)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} H^*(X(\Delta), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and again we refer to the reader to [Dan78] for the proof.

Chapter 4

The Euler-Chow Series for Toric Varieties

4.1 General properties

Through this chapter **toric variety** means smooth projective toric variety, unless otherwise stated. Poincaré duality allow us to express the Euler-Chow series in terms of the cohomology groups of the variety, i.e., we can write

$$E = \sum_{\lambda} \chi(\mathcal{C}_\lambda)\,\lambda \qquad \lambda \in H^{2p}(X, \mathbb{Z})$$

where $\mathcal{C}_\lambda$ is the space of all effective cycles whose homology class is the Poincaré dual of λ.

Let $X(\Delta)$ be the toric variety associated to the fan Δ. Let m be the rank of $H^{2p}(X, \mathbb{Z})$. We know that

$$m = \sum_{i=1}^{n} (-1)^{i-p} \tbinom{p}{i}\, d_{n-i}$$

where d_k is the number of cones of dimension k in Δ and n is the complex dimension of $X(\Delta)$. The following lemma describes the invariant effective cycles of X.

Lemma 4.1.1 *Let $X(\Delta)$ be a nonsingular projective toric variety. Then any irreducible subvariety V of $X(\Delta)$ which is invariant under the torus action is the closure of an orbit. Therefore, any invariant cycle has the form $c = \sum n_i \overline{\mathcal{O}}_i$ where each n_i is a nonnegative integer and each $\overline{\mathcal{O}}_i$ is the closure of the orbit $\mathcal{O}_i$.*

Proof Since the variety is compact, the fan Δ is finite. Hence there are a finite number of cones and, therefore a finite number of orbits.
Let V be an invariant irreducible subvariety of $X(\Delta)$, then V is the closure of the union of orbits. Since there are a finite number of them we must have that

$$V = \overline{\mathcal{O}}_1 \cup \overline{\mathcal{O}}_2 \cup \cdots \cup \overline{\mathcal{O}}_N$$

where $\overline{\mathcal{O}}_i$ is the closure of an orbit. Finally since V is irreducible, there must be i_0 such that $V = \overline{\mathcal{O}}_{i_0}$. ∎

Let $\mathcal{C}_\lambda^T$ and $\mathcal{C}_\lambda$ be the spaces of all effective invariant cycles and effective cycles respectively, with homology class λ. It is proved in [LY87] that

$$\chi\left(\mathcal{C}_\lambda^T\right) = \chi\left(\mathcal{C}_\lambda\right). \tag{4.1}$$

We end this section with the following important lemma,

Lemma 4.1.2 *Let λ be an element in $H^{2p}(X, \mathbb{Z})$. Then $\mathcal{C}_\lambda$ is a finite set.*

Proof. Since any invariant effective cycle c in $\mathcal{C}_\lambda^T$ has the form $c = \sum_{i=1}^N \beta_i \overline{\mathcal{O}}_i$ with $\beta_i \in \mathbb{N}$, we obtain that $\mathcal{C}_\lambda^T$ has a countable number of elements. By Lemma 2.1.1, $\mathcal{C}_\lambda$ is a projective algebraic variety, and since $\mathcal{C}_\lambda^T$ is Zariski closed (see [Hum85], pag.59), we have that $\mathcal{C}_\lambda^T$ is finite. ∎

4.2 The equivariant Euler-Chow series

In this section we introduce the equivariant cohomology and the equivariant Euler-Chow series. We show it is rational, and find a rational function associated to it. We should start with the definition of equivariant cohomology. We refer the reader to ([AB84], [Bif92]). For any topological group G the classifying space BG is the base of a certain principal G-bundle $EG \to BG$ whose total space is contractible. If G acts on a space X, we define $X_G = EG \times_G X$ as the associated bundle over BG with fiber X. The group G acts on EG from the right, on X from the left, and $X_G = EG \times_G X$ is constructed by identifying (eg, x) with (e, gx), for all $g \in G$. The equivariant cohomology of X is defined as

$$H_G^*(X) = H^*(X_G).$$

Observe that the inclusion $X \hookrightarrow X_G$ gives an homomorphism

$$\pi : H_G^*(X, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z}) \tag{4.2}$$

Let $\overline{\mathcal{O}}$ be an irreducible invariant cycle in a smooth toric variety. Since $\overline{\mathcal{O}} \subset X$ is smooth, we have an equivariant Thom-Gysin sequence

$$H_T^{i-2cod\,\overline{\mathcal{O}}}(\overline{\mathcal{O}}) \longrightarrow H_T^i(X) \longrightarrow H_T^i(X - \overline{\mathcal{O}}) \longrightarrow$$

and we define $[\overline{\mathcal{O}}]_T$ as the image of 1 under

$$H_T^0(\overline{\mathcal{O}}) \longrightarrow H_T^{2cod\,\overline{\mathcal{O}}}(X).$$

Let $\{D_1, \ldots, D_K\}$ be the set of T-invariant divisors on X. To each D_i we associate the variable t_i in the polynomial ring $\mathbb{Z}[t_1, \ldots, t_K]$. Let $\mathcal{I}$ be the ideal generated by the (square free) monomials $\{t_{i_1}, \ldots, t_{i_l} | i_m \neq i_n, m \neq n \text{ and } D_{i_1} + \cdots + D_{i_l} \notin \Delta\}$.

It is proved in [BDP90] that

$$\mathbb{Z}[t_1, \ldots, t_K]/\mathcal{I} \cong H_T(X, \mathbb{Z}) \tag{4.3}$$

The arguments given there also prove the following.

Proposition 4.2.1 *For any T-orbit $\mathcal{O}$ in X, one has*

$$[\overline{\mathcal{O}}]_T = \prod_{\overline{\mathcal{O}} \subset D_i} [D_i]_T.$$

Furthermore if $\mathcal{O}$ and $\mathcal{O}'$ are distinct orbits, then

$$[\overline{\mathcal{O}}]_T \neq [\overline{\mathcal{O}'}]_T.$$

It is natural to define the cohomology class for any effective invariant cycle $V = \sum m_i \overline{\mathcal{O}}_i$ as $[V]_T = \sum_i m_i [\overline{\mathcal{O}}_i]_T$. Where $\mathcal{O}_i \neq \mathcal{O}_j$ if $i \neq j$. We are ready for the following definition

Definition 4.2.1 *Let X be a smooth projective toric variety, and let $H_T^{2p}(X, \mathbb{Z})$ be the equivariant cohomology of X. Let us denote by $\mathcal{C}_\xi^T$ the space of all invariant effective cycles on X whose equivariant cohomology class is ξ. The* **equivariant Euler-Chow series** *of X is the element*

$$E_T = \sum_\xi \chi\left(\mathcal{C}_\xi^T\right) \cdot \xi \quad \in \quad \mathbb{Z}^{H_T^{2p}(X, \mathbb{Z})}$$

where the sum is over $\xi \in H_T^{2p}(X, \mathbb{Z})$.

Next, we compute E_T for a certain basis.

Theorem 4.2.1 *Let X be a smooth projective toric variety of dimension n. Let $\{\mathcal{O}_1, \ldots, \mathcal{O}_N\}$ be the set of all distinct orbits on X of codimension p. Let $\mathcal{B} = \{y_1, \ldots, y_r\}$ be an integral basis for $H_T^{2p}(X, \mathbb{Z})$ such that $y_i = [\overline{\mathcal{O}}_i]_T$ for $i = 1, 2, \ldots, N$. Then with respect to this basis*

$$E_T = \prod_{i=1}^{N} \frac{1}{(1 - s_i)}.$$

Proof

Two invariant cycles $V_1 = \sum m_i \overline{\mathcal{O}}_i$ and $V_2 = \sum n_i \overline{\mathcal{O}}_i$ represent the same equivariant cohomology class if and only if m_i is equal to n_i. In other words

$$\chi(\mathcal{C}_\xi^T) = \begin{cases} 1 & \text{if } \xi = \sum m_i [\overline{\mathcal{O}}_i]_T \quad \text{where} \quad m_i \geq 0 \text{ for all } \; i \\ \\ 0 & \text{otherwise.} \end{cases}$$

and so

$$E_T = \sum_{m \in \mathbb{N}^N} s_1^{m_1} \cdots s_N^{m_N} = \prod_{i=1}^{N} \frac{1}{(1 - s_i)}.$$

∎

4.3 The Euler-Chow series

In this section we prove that the Euler-Chow series is rational. In order to do that, we need to define a map I from $\mathcal{D}_0 \subset \mathbb{Z}^{H_T^{2p}(X,\mathbb{Z})}$ to $\mathbb{Z}^{H^{2p}(X,\mathbb{Z})}$. Let us begin by giving a description of the domain of I. Let $\mathcal{D}_0$ be the subgroup of $\mathbb{Z}^{H_T^{2p}(X,\mathbb{Z})}$ whose elements $\sum_\xi n_\xi \cdot \xi$ have the property that for any $\lambda \in H^{2p}(X, \mathbb{Z})$, there are only a finite number of ξ's with $n_\xi \neq 0$, and $\pi(\xi) = \lambda$, where π is the homomorphism defined in equation 4.2. Now, we can define the map I as follows.

Definition 4.3.1 *Let*

$$I : \mathcal{D}_0 \subset \mathbb{Z}^{H_T^{2p}(X,\mathbb{Z})} \longrightarrow \mathbb{Z}^{H^{2p}(X,\mathbb{Z})}$$

be the map defined by

$$I\left(\sum_{\xi} n_\xi\, \xi\right) = \sum_{\lambda}\left(\sum_{\pi(\xi)=\lambda} n_\xi\right)\lambda$$

The first interesting property of I is given in the following proposition.

Proposition 4.3.1 *Let E_T and E be the Euler-Chow series for the equivariant cohomology and for the ordinary cohomology respectively. Then*

$$E_T \in \mathcal{D}_0 \quad \textit{and} \quad I(E_T) = E.$$

Proof. We start proving that $E_T \in \mathcal{D}_0$. In the proof of 4.2.1, we found that $\chi(\mathcal{C}_\xi^T) \neq 0$ if $\mathcal{C}_\xi^T$ is the set which contains the unique element $\xi = \sum_{i=1}^N n_i\,[\overline{\mathcal{O}}_i]_T$. Since $\pi([\overline{\mathcal{O}}_i]_T) = [\overline{\mathcal{O}}_i]$ we obtain that

$$\pi(\xi) = \sum_{i=1}^{N} n_i\,[\overline{\mathcal{O}}_i].$$

By Lemma 4.1.2 we know that $\mathcal{C}_\lambda^T$ is finite. Therefore, there is a finite number of ξ such that $\pi(\xi) = \lambda$. The second part of the theorem is proved by the following equalities,

$$I\left(\sum_{\xi}\chi(\mathcal{C}_\xi^T)\,\xi\right) = \sum_{\lambda}\left[\sum_{\pi(\xi)=\lambda}\chi(\mathcal{C}_\xi^T)\right]\lambda = \sum_{\lambda}\chi(\mathcal{C}_\lambda^T)\,\lambda = \sum_{\lambda}\chi(\mathcal{C}_\lambda)\,\lambda$$

∎

We are ready for the main theorem of this chapter.

Theorem 4.3.1 *For any nonsingular projective toric variety X over $\mathbb{C}$, the Euler-Chow series is rational. In fact, if $(\mathcal{O}_1, \dots, \mathcal{O}_N)$ denote all the orbits of codimension p, and if $\mathcal{A} = \{x_1, \dots, x_m\}$ is a basis for $H^{2p}(X, \mathbb{Z})$. Then*

$$E^p = \prod_{i=1}^{N} \left(\frac{1}{1 - \prod_{j=1}^{m} t_j^{a_{ji}}} \right)$$

where $[\overline{\mathcal{O}}_i] = \sum_{j=1}^{m} a_{ji} x_j$ is the expression, in terms of the basis $\mathcal{A}$, of the cohomology class of the closure $\overline{\mathcal{O}}_i$ of the orbit $\mathcal{O}_i$.

Proof.

We start by choosing a basis $\mathcal{B}$ for $H_T^{2p}(X)$ and by expressing the map I in terms of the bases $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{B} = \{y_1, \dots, y_r\}$ be an integral basis for $H_T^{2p}(X, \mathbb{Z})$, such that $y_i = [\overline{\mathcal{O}}_i]_T$, for $i = 1, \dots, N$. It was proved in theorem 4.2.1 that

$$E_T = \prod_{i=1}^{N} \frac{1}{(1 - s_i)} = \sum_{i=1}^{N} s_1^{\beta_1} \cdots s_N^{\beta_N}.$$

Let

$$I_{\mathcal{AB}} : \mathbb{Z}^{\mathbb{Z}^r} \longrightarrow \mathbb{Z}^{\mathbb{Z}^m}$$

be the map I with respect to the bases $\mathcal{A}$ and $\mathcal{B}$. Consider the following diagram

$$\begin{tikzcd}
\mathbb{Z}[[s_1, \dots, s_N]] \arrow[r, hook, "p"] \arrow[d, "\Phi"] & \mathbb{Z}^{\mathbb{Z}^r} \arrow[d, "I_{\mathcal{AB}}"] \\
\mathbb{Z}[[t_1, \dots, t_m, \bar{t}_1, \dots, \bar{t}_m]] \supset \mathcal{D} \arrow[r, "p"] & \mathbb{Z}^{\mathbb{Z}^m}
\end{tikzcd}$$

where Φ is defined on each generator s_i as

$$\Phi(s_i) = \prod_{j=1}^{m} t_j^{a_{ji}^+} \bar{t}_j^{a_{ji}^-}.$$

We recall that for any integer number a we defined $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. Before we prove that the diagram commutes, we take a closer look at the number of effective invariant cycles in a given cohomology class. Let $\lambda \in H^{2p}(X)$, and denote by $\gamma = (\gamma_1, \ldots, \gamma_m)$ the coordinates of λ in terms of the basis $\mathcal{A}$. Let $c = \sum_{i=1}^{N} \beta_i \overline{\mathcal{O}}_i$ be an invariant effective cycle with cohomology class $[c] = \lambda$. Then,

$$\begin{aligned}
[c] &= \sum_{i}^{N} \beta_i [\overline{\mathcal{O}}_i] = \sum_{i}^{N} \beta_i \left(\sum_{j=1}^{m} a_{ji}\, x_j \right) \\
&= \sum_{j=1}^{m} \left(\sum_{i}^{N} a_{ji}\, \beta_i \right) x_j = \sum_{j=1}^{m} \gamma_j\, x_j \\
&= \lambda.
\end{aligned}$$

Therefore, the number $\chi(\mathcal{C}_\lambda)$ of invariant cycles with cohomology class λ is equal to the number N_λ of solutions to the system

$$A\,\beta = \gamma \quad \text{for} \quad \beta \in \mathbb{N}^N. \tag{4.4}$$

But it was proved in Lemma 4.1.2 that this number is finite. Therefore, the equality $I(E_T) = E$ proved in the last proposition becomes

$$I_{\mathcal{AB}}(E_T) = I_{\mathcal{AB}}\Big(\sum_{\beta \in \mathbb{N}^N} s^\beta \Big) = \sum_{\gamma \in \mathbb{Z}^m} N_\gamma t^\gamma = E.$$

Next, we show that $\Phi(\prod_{i=1}^{N} \frac{1}{1-s_i}) \in \mathcal{D}$. We have

$$\begin{aligned}
\Phi\left(\prod_{i=1}^{N}\frac{1}{1-s_i}\right) &= \prod_{i=1}^{N}\left(\frac{1}{1-\prod_{j=1}^{m} t_j^{a_{ji}^{+}}\bar{t}_j^{a_{ji}^{-}}}\right)\\
&= \sum_{\beta\in\mathbf{N}^N}\left(\prod_{j=1}^{m} t_j^{a_{j1}^{+}}\bar{t}_j^{a_{j1}^{-}}\right)^{\beta_1}\cdots\left(\prod_{j=1}^{m} t_j^{a_{jN}^{+}}\bar{t}_j^{a_{jN}^{-}}\right)^{\beta_N}\\
&= \sum_{\beta\in\mathbf{N}^N} t_1^{\sum a_{1i}^{+}\beta_i}\cdots t_m^{\sum a_{mi}^{+}\beta_i}\,\bar{t}_m^{\sum a_{1i}^{-}\beta_i}\cdots\bar{t}_m^{\sum a_{mi}^{-}\beta_i}\\
&= \sum_{\beta\in\mathbf{N}^N} t^{A^+\beta}\,\bar{t}^{A^-\beta},
\end{aligned}$$

and we know that for each $\gamma \in \mathbb{Z}^m$ the number of solutions to

$$\gamma = A^+\beta - A^-\beta = A\beta \quad \text{for } \beta \in \mathbf{N}^N$$

is finite, and equal to N_γ. Therefore, $\Phi(\prod_{i=1}^{N}\frac{1}{1-s_i}) = \sum_\beta t^{A^+\beta}\,\bar{t}^{A^-\beta} \in \mathcal{D}$. Furthermore, we have that

$$\begin{aligned}
p\left(\Phi\left(\sum_{\beta\in\mathbf{N}^N} s^\beta\right)\right) &= p\left(\Phi\left(\prod_{i=1}^{N}\frac{1}{1-s_i}\right)\right)\\
&= p\left(\sum_\beta t^{A^+\beta}\,\bar{t}^{A^-\beta}\right)\\
&= \sum_\beta t^{A\beta}\\
&= \sum_\gamma N_\gamma t^\gamma = E.
\end{aligned}$$

Finally,

$$E = \sum_\gamma N_\gamma t^\gamma = \sum_\gamma N_\gamma t^{\gamma_1}\cdots t^{\gamma_m}$$

$$= \sum_{\beta \in \mathbf{N}^N} t_1^{\sum a_{1i}\beta_i} \cdots t_m^{\sum a_{mi}\beta_i}$$

$$= \sum_{\beta \in \mathbf{N}^N} \left(\prod_{j=1}^{m} t_j^{a_{j1}} \right)^{\beta_1} \cdots \left(\prod_{j=1}^{m} t_j^{a_{jN}} \right)^{\beta_N}$$

$$= \prod_{i=1}^{N} \left(\frac{1}{1 - \prod_{j=1}^{m} t_j^{a_{ji}}} \right).$$

∎

4.4 Examples

I) The projective space $\mathbf{P}^n$

Let $X = \mathbf{P}^n$ be the complex projective space of dimension n. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbf{R}^n$. Consider $A = \{e_1, \ldots, e_{n+1}\}$ a set of generators of the fan Δ where $e_{n+1} = -\sum_{i=1}^{n} e_i$. We have the following equality

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_{n+1}]/I$$

where I is the ideal generated by

$$i) \quad t_1 \cdots t_{n+1}$$

and

$$ii) \quad \sum_{j=1}^{n+1} e_i^*(e_j)\, t_j \quad \text{for } i+1, \ldots, n+1\,,$$

where $e_i^* \in (\mathbf{R})^*$ is the element dual to e_i.

However $ii)$ says that $t_i \sim t_j$ for all $i, j = 1, \ldots, n+1$. Therefore

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[t]/t^{n+1}.$$

Consequently any two cones of dimension p represent the same element in cohomology, and Theorem 4.3.1 implies, that

$$\prod_{i=1}^{\binom{n+1}{p}} \left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p}} = E^p.$$

where, as always, $\alpha \in H^{2p}(X, \mathbb{Z})$. Note that the power $\binom{n+1}{p}$ is different from the power $\binom{n+1}{p+1}$ obtained in Example I, chapter II. This is because we work with cohomology while in chapter II we worked with homology. If we would consider cones of dimension $n-p$ instead of cones of dimension p, then we would get

$$\left(\frac{1}{1-t}\right)^{\binom{n+1}{n-p}} = E^p$$

since $\binom{n+1}{n-p} = \binom{n+1}{p+1}$, we obtain the same answer as in Chapter II.

II) $P^n \times P^m$

Recall that $X(\Delta') \cong X(\Delta) \times X(\Delta')$. Using the same notation as that in I) we have that a set of generators of $\Delta \times \Delta'$ is given by

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}, e_{n+m+1}, e_{n+m+2}\}$$

with $e_{n+m+1} = -\sum_{i=1}^{n} e_i$ and $e_{n+m+2} = -\sum_{i=n+1}^{n+m} e_i$. and $\{e_1, \ldots, e_{n+m}\}$ is a basis for $P^n \times P^m$. Then

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_{n+m+2}] / I$$

where I is the ideal generated by

$$i) \ \ \{t_1 \cdots t_n t_{n+m+1}, \ \ t_{n+m+1} \cdots t_{n+m+2}, \ \ \prod_{i=1}^{n+m+2} t_i\}$$

and

$$ii) \quad \sum_{j=1}^{n+m+2} e_i^*(e_j)\, t_j \qquad i = 1, \ldots, n+m\,.$$

From $ii)$ we obtain,

$$t_i \sim t_{n+m+1} \qquad \text{if} \quad 1 \leq i \leq n$$

$$t_j \sim t_{n+m+2} \qquad \text{if} \quad n+1 \leq j \leq n+m$$

The number of cones of dimension p is equal to $\sum_{k+l=p} \binom{n+1}{k}\binom{m+1}{l}$.

Denote by $t_{IJ} = t_{i_1} \cdots t_{i_k} t_{j_1} \cdots t_{j_l}$. Then

$$\prod_{k+l=p} \left(\frac{1}{1 - t_{k,l}}\right)^{\binom{n+1}{k}\binom{m+1}{l}} = E^p$$

where $t_{k,l} = t_{n+m+1}^k\, t_{n+m+2}^l$.

III) Blow up of $\mathbb{P}^n$

Let Δ be the fan associated to P^n. We know by example 1 that Δ is the generated by $\{e_1, \ldots, e_{n+1}\}$ where $e_{n+1} = -\sum_{i=1}^n e_i$ and $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. The fan $\tilde{\Delta}$ associated to the blow up $\tilde{\mathbb{P}^n}$ of the projective space at the fixed point given by the cone $\mathbb{R}^+ e_2 + \cdots + \mathbb{R}^+ e_{n+1}$ is generated by $\{e_1, \ldots, e_{n+1}, e_{n+2}\}$ where $e_{n+2} = -e_1$. Denote by D_i the 1-dimensional cone $\mathbb{R}^+ e_i$ and by s_i its class in cohomology where

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[s_1, \ldots, s_{n+2}] / I$$

and I is the ideal generated by

$$i) \quad \{s_{i_1} \cdots s_{i_k} \,:\, D_{i_1} \cdots D_{i_k} \text{ is not in } \tilde{\Delta}\}$$

and

$$ii) \quad \sum_{j=1}^{n+2} e_i^*(e_j)\, s_j \qquad i = 1, \ldots, n\,.$$

However $ii)$ is equivalent to

$$ii) \;\; s_2 \sim \cdots s_{n+1} \quad \text{and} \quad s_1 \sim s_{n+1} + s_{n+2}$$

Note that a p-dimensional cone can not contain D_{n+2} and D_1. The reason is that D_{n+2} is generated by $-e_1$ and D_1 by e_1, but by definition, a cone does not contain a subspace of dimension greater than 0. We would like to find a basis for $H^*(\tilde{\mathbb{P}^n})$ and write any monomial of degree p in terms of it. Consider the monomial $s_{i_1} \cdots s_{i_p}$. There are three possible situations:

1) s_{i_j} is different from both s_{n+2} and s_{n+1}. In this situation we have from $ii)$ that $s_{i_1} \cdots s_{i_p} = s_{n+1}^p$.

2) s_{n+2} is equal to s_{i_j} for some, $j = 1, \ldots, p$. Then from $ii)$ we obtain that $s_{i_1} \cdots s_{i_p} = s_{n+1}^{p-1} s_{n+2}$.

3) s_1 is equal to s_{i_j} for some $j = 1, \ldots, p$. Then from $ii)$ we obtain $s_{i_1} \cdots s_{i_p} = (s_{n+1} + s_{n+2})\, s_{n+1}^{p+1} = s_{n+1}^p + s_{n+2}\, s_{n+1}^{p-1}$ which is the sum of 1) and 2).

We conclude that s_{n+1}^p and $s_{n+2} s_{n+1}^{p-1}$ form a basis for H^{2p} if $p \leq n$. If $p = n$ then $s_{n+1}^p = 0$ and the only generator is $s_{n+2} s_{n+1}^{p-1}$. Let us call s_{n+1} by t_1 and $s_{n+2} s_{n+1}^{p-1}$ by t_2. The Euler-Chow series for $\tilde{\mathbb{P}^n}$ is:

$$E^p = \left(\frac{1}{1-t_1}\right)^{\binom{n}{p}} \left(\frac{1}{1-t_1 t_2}\right)^{\binom{n}{p-1}} \left(\frac{1}{1-t_2}\right)^{\binom{n}{p-1}} \quad if \;\; p \leq n\,.$$

$$E^p = \left(\frac{1}{1-t_2}\right)^{\binom{n+2}{p}} \quad if \;\; p = n\,.$$

IV) Hirzebruch surfaces

A set of generators for the fan Δ, in $\mathbb{R}^2$, that represents the Hirzebruch surface $X(\Delta)$ is given by, $\{e_1, \ldots, e_4\}$ with $\{e_1, e_2\}$ the standard basis for $\mathbb{R}^2$, and $e_3 = -e_1 + ae_2, \;\; a > 1$ and $e_4 = -e_2$.

With the same notation as in last examples, we have,

$$H^*(x(\Delta)) = \mathbf{Z}[t_1, \ldots, t_4] \,/\, I$$

where I is generateded by

$$i) \quad \{t_1t_3,\, t_2t_4,\, t_{i_1}t_{i_2}t_{i_3},\, t_1t_2t_3t_4 \,:\, i_j \neq i_{j'} \text{for } j \neq j'\}$$

and

$$ii) \quad \{t_1 - t_3,\, t_2 + at_3 - t_4\}$$

from $ii)$ we have the following conditions for the t_i's in $H^*(X)$

$$(*) \quad t_1 \sim t_3 \text{ and } t_2 \sim t_4 - at_3.$$

A basis for $H^*(x(\Delta))$ is given by $\{\{0\}, t_3, t_4, t_4t_1\}$ (see [Dan78]). The Euler-Chow series for each dimension is:

1) Codimension 2: There are four orbits (four cones of dimension 2), and all of them are equivalent in homology. From Theorem 4.3.1 we obtain

$$E^2 = \left(\frac{1}{1-t}\right)^4$$

2) Codimension 1: Again, there are four orbits (four cones of dimension 1), and the relation among them, in homology is given by $(*)$. From Theorem 4.3.1 we obtain

$$E^1 = \left(\frac{1}{1-t_3}\right)^2 \left(\frac{1}{1-t_4}\right) \left(\frac{1}{1-t_3^{-a}t_4}\right).$$

3) Codimension 0: The only orbit is the torus itself, from Theorem 4.3.1 we have

$$E^0 = \left(\frac{1}{1-t}\right).$$

Appendix A

Letter from Emili Bifet

As I mentioned before, Emili's letter gives a sketch of how to reformulate, in intrinsic form, the work done in this thesis. I received it one week before the defense, so it was hard to rewrite everything in the terms he had suggested. Since Emili's approach is very elegant, I decided it would be worth to add the letter as an appendix.

August 1, 1992

Dear Blaine and Javier,

Here follows what I hope is an intrinsic reformulation of Javier's results (I have not checked everything in full detail, but it seems to work.)

1. H (resp. H_T) denotes the $2p$ cohomology (resp. equivariant cohomology) group.

2. $C \subset H$ (resp. $C_T \subset H_T$) denotes the sub-monoid of cohomology (resp. equivariant cohomology) classes of effective (resp. invariant effective) p-cycles.

3. $\pi : H_T \to H$ denotes the standard surjection.

4. $\pi : C_T \to C$ is surjective with finite fibres (the Chow variety is projective and there are countably many fixed points.)

5. $+ : C_T \times C_T \to C_T$ has finite fibres ($C_T \simeq \mathbf{N}^N$, where N is the number of p-orbits.)

6. The following diagram commutes:

$$\begin{tikzcd} C_T \times C_T \arrow[r, "+"] \arrow[d, "\pi \times \pi"'] & C_T \arrow[d, "\pi"] \\ C \times C \arrow[r, "+"'] & C \end{tikzcd}$$

(π is a homomorphism.)

7. $+ : C \times C \to C$ has finite fibres (follows from 4, 5 and 6.)

8. $\mathcal{A} \subset \mathbf{Z}^H$ (resp. $\mathcal{A}_T \subset \mathbf{Z}^{H_T}$) denotes the functions with support in C (resp. C_T.)

9. $\mathcal{A}$ (resp. $\mathcal{A}_T$) is a commutative ring (from 7 and 5.)

10. $I : \mathcal{A}_T \to \mathcal{A}$ (defined by $I(u)(\lambda) = \sum_{\xi \in \pi^{-1}(\lambda)} u(\xi)$) is a ring homomorphism.

11. E (resp. E_T) denotes the Euler-Chow (resp. equivariant Euler-Chow) invariant.

12. $E_T \in \mathcal{A}_T$ and $I(E_T) = E$.

13. Define for each p-orbit $\mathcal{O}_i$ an element $f_i \in \mathcal{A}$ (resp. $f_i^T \in \mathcal{A}_T$) by:

$$f_i(\lambda) = \begin{cases} 1 & \text{if } \lambda = n \cdot [\overline{\mathcal{O}}_i],\ n \geq 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(resp.

$$f_i^T(\xi) = \begin{cases} 1 & \text{if } \xi = n \cdot [\overline{\mathcal{O}}_i]_T,\ n \geq 0 \\ \\ 0 & \text{otherwise.)} \end{cases}$$

We also denote e_λ (resp. e_ξ) the characteristic function of $\{\lambda\}$ (resp. $\{\xi\}$.) Note in particular that $1 = e_0$.

14. $I(f_i^T) = f_i$ and $I(e_\xi) = e_{\pi(\xi)}$.

15. $\pi[\overline{\mathcal{O}}_i]_T = [\overline{\mathcal{O}}_i]$.

16. $E_T = \prod_{1 \leq i \leq N} f_i^T$ and $(1 - e_{[\overline{\mathcal{O}}_i]_T}) \cdot f_i^T = 1$.

17. $E = \prod_{1 \leq i \leq N} f_i$ and $(1 - e_{[\overline{\mathcal{O}}_i]}) \cdot f_i = 1$ (apply I to 16.)

18. Finally:

$$E = \frac{1}{\prod_{1 \le i \le N} (1 - e_{[\overline{\mathcal{O}}_i]})} .$$

19. The ring $\mathcal{A}$ contains the monoid-ring $\mathbf{Z}[C]$ (= functions with finite support.) It is perhaps natural to say that an element of $\mathcal{A}$ is "rational" provided it is the quotient of two elements of $\mathbf{Z}[C]$. If one wants to consider bases, one will perhaps have to use Gordan's lemma and ...

20. It is tempting to especulate about the general case (with H being the homology group and C the sub-monoid of homology classes of effective cycles.) When is $E \in \mathcal{A}$ rational in the above sense? Well ...

Sincerely,

Emili

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