L^2-Analytic Torsions, Equivariant Cyclic Cohomology and the Novikov Conjecture

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Donggeng Gong

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Joel Pincus
Professor of Mathematics
Dissertation Director

Blaine Lawson, Jr.
Professor of Mathematics
Chairman of Defense

Chih-Han Sah
Professor of Mathematics

Martin Roček
Professor of Physics
SUNY at Stony Brook
Outside member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of dissertation

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In this dissertation we study L^2-analytic torsions, equivariant cyclic cohomology and the Novikov conjecture from a functional analysis point of view, closely related to geometry and topology. We discuss the L^2-analytic torsion functions for homogeneous spaces of Lie groups. The properties of this torsion function are similar to that of the Ray-Singer analytic torsion. We introduce both L^2-analytic and K-theory torsions for (semifinite) von Neumann algebras, which share most properties of geometric ones. An equivariant bivariant cyclic theory is also developed here. We prove two of the remarkable properties

of this bivariant cyclic theory, namely the excision and Chern characters for equivariant p-quasi-homomorphisms. We construct the Chern characters for both odd and even p-summable Fredholm modules and for even θ-summable Fredholm modules. We obtain the pairing of equivariant cyclic cohomology with equivariant K-theory. We also find an index theorem for equivariant θ-summable Fredholm modules and a higher index theorem for homogeneous spaces. A higher equivariant index map is defined, which is considered as a possible tool to solve the pairing version of the equivariant Novikov conjecture proposed in this dissertation. Finally, we give a survey of the current status of the Novikov conjecture and prove an equivariant Connes-Gromov-Moscovici theorem and equivariant Novikov conjecture for groups acting on euclidean buildings. This dissertation lays a foundation for new research areas which we state in the section on Open Problems.

To my parents and Mei

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Introduction

The starting point of this dissertation is Cyclic Cohomology. Motivated by the work of Carey-Pincus [CaP 1], Douglas-Voiculescu [DoV] and Helton-Howe [HeH], Connes [Con 1-3] developed Cyclic Cohomology as a powerful vehicle to drive index theory beyond the scope of the Atiyah-Singer index theorems. Meanwhile, Loday-Quillen [LoQ] and Tsygan [Tsy] (see also [FeT]) developed cyclic homology from a Lie algebraic homology and additive K-theory point of view. The important properties of cyclic (co-)homology are Wodzicki's excision theorem [Wod 1-2] (see also [Gong 2] for a six term exact sequence for periodic cyclic cohomology), Chern characters and the pairings with K-theory. Because of the analogy between cyclic (co-)homology and K-theory, most results about cyclic (co-)homology are obtained in parallel to those of K-theory. In particular, cyclic (co-)homology was extended by Jones-Kassel [JoK] to the bivariant case. Our motivation in the study of cyclic (co-)homology here is its application to equivariant index theory. Keeping this in mind , we develop in Part II an equivariant version of (bivariant) cyclic (co-)homology, which is motivated by our work [Gong 1-2]. We obtain the Chern characters and the pairings with equivariant K-theory which are used to get the equivariant index theorem. Our further motivation for developing the equivariant cyclic(co-)homology is the hope that we may obtain a higher equivariant index theorem and its application to the equivariant Novikov conjecture. This point needs to be fully investigated elsewhere.

The important application of cyclic (co-)homology to the Novikov conjecture ([CoM 1], [CGM 1]) also stimulated our study on this well-known conjecture. This is the main topic of Part III. To see the link of the Novikov conjecture with topology, geometry and functional analysis, we use one chapter to survey the current status of the conjecture. In particular, this conjecture is related to the Thurston geometralization conjecture. We employ equivariant Hilsum-Skandalis technique to obtain equivariant Connes-Gromov-Moscovici theorem on the homotopy invariance of (higher) signature with coefficients in (almost) flat C^*-algebra bundles. This result is a crucial tool for proving the equivariant Novikov conjecture for groups acting on euclidean buildings and enables us to improve the Rosenberg-Weinberger theorem about the equivariant Novikov conjecture for groups acting on manifolds of nonpositive curvature. Our method is quite different from the usual one which relys on the Miscenko symmetric signature invariant.

Because the discussion of the Novikov conjecture involves the signature operators on the universal covering spaces, the recent work of Carey-Mathai [CaM], Lott [Lott 3] and Lück-Rothenberg [LüR] motivates our study of L^2-analytic torsions in Part I. We use a semifinite trace on type II_∞ von Neumann algebras to consider the L^2-analytic torsions for homogeneous spaces. By borrowing this geometric idea, we introduce L^2-analytic torsion for n-tuples of commuting elements in semifinite von Neumann algebras. The case of finite von Neumann algebras is treated in [GoP 1], where we find also a new homotopy invariant for compact Riemannian manifolds by using this new analytic torsion and the result in Part III of this dissertation. We also consider the K-theory torsion for n-tuples of commuting elements in finite von Neumann algebras.

The dissertation is organized in the reversed order of the above discussion as follows. We begin with the L^2-analytic torsions in Chapter 1 where a torsion function on homogeneous spaces is introduced to avoid the positivity condition on the Novikov-Shubin invariants. The main properties of the torsion function are stated in Theorem 1.1 and Propositions 1.2 and 1.4. A formula for the torsion function is given in terms of group structure and harmonic analysis. We also compute this torsion function for hyperbolic spaces. In Chapter 2 we use the geometric idea of Chapter 1 and follow [GoP 1] to discuss the determinant and L^2-analytic functions of n-tuples of commuting elements in semifinite von Neumann algebras. The main results are Theorems $2.1-2.2$ and Propositions $2.6-2.7$. We compute the torsion function for compact Riemann surfaces. The general case of Riemann surfaces of finite volume with cusp singularity will be treated in another paper. We consider in Chapter 3 a K-theory torsion for n-tuples of commuting elements in finite von Neumann algebras. The advantage of this K-theory torsion over the analytic one is that there are no restrictions in the definition. The main properties of this torsion are given in Theorems $3.1-3.4$. In single operator case, the torsion and operator are determined with each other up to certain factors and weak isomorphisms.

The second part of the dissertation is about equivariant cyclic (co-)homology and index theory. We follow [Gong 1-2] to define equivariant bivariant cyclic theory in Chapter 4. Our goal is to construct bivariant Chern character for equivariant p-quasi-homomorphisms without using the excision property of bivariant cyclic theory, inspired by [Nis]. In Chapter 5 we carry out the construction of the pairings of equivariant (entire) cyclic cohomology with equivariant K-theory and the Chern characters in equivariant (entire)cyclic (co-)homology. These pairings and corresponding Chern characters are used, motivated by ([Con 3], [JLO], [GeS]), to get an index theorem for even equivariant θ-summable Fredholm modules. Chapter 6 is devoted to the higher equivariant analytic index map for Riemannian manifolds. An equivariant Alexander-Spanier cohomology is developed here. The main results of this chapter are Theorem 6.1 and Proposition 6.3. We should point out that this chapter is motivated by the desire to establish the higher equivariant index theorem and to use it to verify the pairing version of the equivariant Novikov conjecture for certain groups, which we will investigate elsewhere (the nice book of Berline-Getzler-Vergne [BGV] might be helpful for this problem). But for homogeneous spaces we are able to prove the higher equivariant index theorem in Chapter 7, since this is a special case of the general higher equivariant index problem and we can transfer the problem to the one studied by Connes-Moscovici [CoM 2].

The third part of the dissertation is about the Novikov conjecture. Beginning with a survey of this conjecture in Chapter 8, we present two pictures of the conjecture: group picture-which tells us which groups satisfy the Novikov conjecture; and manifold picture- which gives us the manifolds whose fundamental groups satisfy the conjecture. We will see that many conjectures, which theory in Chapter 4. Our goal is to construct bivariant Chern character for equivariant p-quasi-homomorphisms without using the excision property of bivariant cyclic theory, inspired by [Nis]. In Chapter 5 we carry out the construction of the pairings of equivariant (entire) cyclic cohomology with equivariant K-theory and the Chern characters in equivariant (entire)cyclic (co-)homology. These pairings and corresponding Chern characters are used, motivated by ([Con 3], [JLO], [GeS]), to get an index theorem for even equivariant θ-summable Fredholm modules. Chapter 6 is devoted to the higher equivariant analytic index map for Riemannian manifolds. An equivariant Alexander-Spanier cohomology is developed here. The main results of this chapter are Theorem 6.1 and Proposition 6.3. We should point out that this chapter is motivated by the desire to establish the higher equivariant index theorem and to use it to verify the pairing version of the equivariant Novikov conjecture for certain groups, which we will investigate elsewhere (the nice book of Berline-Getzler-Vergne [BGV] might be helpful for this problem). But for homogeneous spaces we are able to prove the higher equivariant index theorem in Chapter 7, since this is a special case of the general higher equivariant index problem and we can transfer the problem to the one studied by Connes-Moscovici [CoM 2].

The third part of the dissertation is about the Novikov conjecture. Beginning with a survey of this conjecture in Chapter 8, we present two pictures of the conjecture: group picture-which tells us which groups satisfy the Novikov conjecture; and manifold picture- which gives us the manifolds whose fundamental groups satisfy the conjecture. We will see that many conjectures, which are the ones of current most active research areas, are proposed in the same fashion as the Novikov conjecture. In particular, for discrete groups realized as fundamental groups of 3-dimensional compact manifolds the Novikov conjecture is related to the Thurston geometralization conjecture. Chapter 9 is the most technical part of the dissertation where we carry out the equivariant Hilsum-Skandalis technique. The main results of this chapter are Theorems 9.1 – 9.2. These results together with Kasparov-Skandalis theorem [KaS] enable us to prove in Chapter 10 the equivariant Novikov conjecture for groups acting on euclidean buildings. We also remove an important condition in the Rosenberg-Weinberger theorem [RoW 2]. Finally, we end the dissertation with a list of open problems which form a natural continuation of our work.

Chapter 1

L^2-Analytic Torsion Functions For Homogeneous Spaces

The Ray-Singer L^2-analytic torsion [RaS 1] on a compact Riemannian manifold M has been extended to the covering space $\tilde{M}$ of M. One uses the trace on finite von Neumann algebra generated by the fundamental group $\pi_1(M)$ to define the L^2-analytic torsion on $\tilde{M}$. From the functional analysis point of view, this situation involves only the finite von Neumann algebras. There are other geometric cases which are related to type II_∞-von Neumann algebras. In this chapter we study such a special case, namely homogeneous spaces of Lie groups, where semifinite traces on type II_∞-von Neumann algebra play a central role. In section 1.1 we define the ζ-(regularized) determinant function for equivariant elliptic differential operators on a homogeneous space G/H with G a unimodular semisimple Lie group and H a compact subgroup of G. Some elementary properties of this determinant function are discussed. In section 1.2 we use the ζ-determinant function to define an L^2-analytic torsion function $T(G/H, \lambda)$ on G/H. The Ray-Singer torsion is a special value of this torsion function at $\lambda = 0$. $T(G/H, \lambda)$ has the main properties of the Ray-Singer torsion. The main point in introducing a parameter λ in $T(G/H, \lambda)$ is to get rid of the decay condition on the trace of the heat kernel of the Laplacian. Such a condition is necessary if one wants to define the L^2-analytic torsion on non compact manifolds [Lott 3], though there is a conjecture that this decay condition is always satisfied for the covering spaces of compact Riemannian manifolds (cf. Open Problems). But this parameter in some cases makes the computation of L^2-analytic torsion more complicated. In section 1.3 we use harmonic analysis to give a formula for computing the L^2-analytic torsion function when G/H is the universal covering of compact symmetric spaces of negative curvature. This formula can be easily obtained from [MoS]. We then calculate in particular the L^2-analytic torsion functions for hyperbolic spaces. The computation of the torsion function for G/H being the universal covering space of locally symmetric spaces with finite volume remains to be done.

1.1 ζ-Determinant Functions

let G be a unimodular Lie group with countably many connected components and H a compact subgroup of G. Denote by $M = G/H$ the homogeneous space of left cosets $gH, g \in G$. Suppose that α is a finite dimensional unitary representation of H on a complex vector space E. Let $\mathcal{E}$ be the associated homogeneous vector bundle over M, $\mathcal{E} = G \times_\alpha E$, and $C^\infty(M.\mathcal{E})$ (resp. $C_c^\infty(M, \mathcal{E})$) the space of all C^∞-sections (resp. with compact support) of $\mathcal{E}$. Using the right (resp. left) regular representation R (resp. L) of G on $C^\infty(G)$

defined by $(R(h)f)(g) = f(gh)$ (resp. $(L(h)f)(g) = f(h^{-1}g)$), $g, h \in G$, and the diagonal action of H on $C^\infty(G) \otimes E$, one has $C^\infty(M, \mathcal{E}) \simeq (C^\infty(G) \otimes E)^H$ and $C_c^\infty(M, \mathcal{E}) = (C_c^\infty(G) \times E)^H$. Choosing a Haar measure dg on G and the normalized Haar measure dh on H, we can endow M with a G-invariant measure dg/dh and E with an H-invariant inner product, and then define a global inner product on $C_c^\infty(M, \mathcal{E})$. Denote by $L^2(M, \mathcal{E})$ the completion of $C_c^\infty(M, \mathcal{E})$ with respect to this inner product. We have $L^2(M, \mathcal{E}) = (L^2(G) \times E)^H$.

We recall below some notations. Let $\Psi^n(M; \mathcal{E})$ be the space of all differential operators P of order n in $L^2(M, \mathcal{E})$ with distributional kernels K_P in $(C^{-\infty}(G \times G) \otimes Hom(E, E))^{H \times H}$ consisting of all elements $K \in C^{-\infty}(G \times G) \otimes Hom(E, E)$ such that $\alpha(h)K(xh, yg)\alpha(g)^{-1} = K(x, y)$, $x, y \in G$, and $g, h \in H$. Let $\Psi_{cc}^n(M; \mathcal{E})$ (resp. $\Psi_c^n(M; \mathcal{E})$) be the subspace of $\Psi^n(M; \mathcal{E})$ consisting of all $P \in \Psi^n(M; \mathcal{E})$ with (resp. G-) compactly supported kernels K_P, i.e., $K_P(x, y) \neq 0$ only for (x, y) (resp. $x^{-1}y$) in a compact subset of $G \times G$ (resp. G). A differential operator $P \in \Psi^n(M; \mathcal{E})$ is G-invariant if $L(g)PL(g)^{-1} = P, \forall g \in G$. The kernel of a G-invariant operator P is determined by an element $k_P \in ((C^{-\infty}(G) \otimes Hom(E, E))^{H \times H}$. This means that k_P satisfies $k_P(x) = \alpha(h)K(h^{-1}xg)\ \alpha(g)^{-1}, \forall x \in G, g, h \in H$, and $K_P(x, y) = k_P(x^{-1}y)$. Let $\Psi^n(M; \mathcal{E})^G$ be the space of all G-invariant operators in $\Psi^n(M; \mathcal{E})$. Thus an operator $P \in \Psi_c^n(M; \mathcal{E})^G$ can be written as

$$Pu = Av(\tilde{P})u \stackrel{\text{def}}{=} \int_G L(g)\tilde{P}L(g)^{-1}u dg,\ u \in dom(P),$$

for $\tilde{P} = fP \in \Psi_{cc}^n(M; \mathcal{E})$ with a cut-off function f. Note that $f \in C_c^\infty(G)$ is called a cut-off function if (1) $f \geq 0$; (2) $f(gh) = f(g), g \in G, h \in H$; (3) $\int_G f(g)dg = 1$.

One of the basic notations in this chapter is the trace Tr_G in type II_∞-von Neumann algebra $R_G(\mathcal{E})$ which is defined below. Let $\mathcal{B}(L^2(G))$ be the space of all bounded operators on $L^2(G)$ and let R_G be the commutant in $\mathcal{B}(L^2(G))$ of the left regular representation L. There is a faithful semifinite normal trace Tr_G on R_G determined uniquely by

$$Tr_G(R(\varphi)^*R(\varphi)) = \int_G |\varphi(g)|^2 dg$$

for $\varphi \in L^2(G)$ and the right regular representation R with $R(\varphi) \in \mathcal{B}(L^2(G))$. Let $R_G(\mathcal{E})$ be the commutant of the restriction to $L^2(M, \mathcal{E})$ of the representation $L \otimes I$ of G on $L^2(G) \otimes E$. There is therefore a natural trace on $R_G(\mathcal{E})$, denoted also by Tr_G, which is obtained from the trace Tr_G on R_G and the ordinary trace on $End(E)$.

It is an important fact about the trace Tr_G that it is linked in a simple way with the ordinary trace Tr on $\mathcal{B}(L^2(M, \mathcal{E}))$. There is faithful semifinite normal operator valued weight Av_G from the positive part $\mathcal{B}^+(L^2(M, \mathcal{E}))$ to the extended positive part $\tilde{R}_G^+(\mathcal{E})$ where Tr_G can be extended,

$$Av_G(T) = \int_G L(g)TL(g)^{-1}dg, \;\; T \in \mathcal{B}^+(L^2(M, \mathcal{E})),$$

such that $Tr_G(Av_G(T)) = Tr(T)$ for $T \in \mathcal{B}^+(L^2(M, \mathcal{E}))$. Here the domain of $Av_G(T)$ is

$$dom(Av_G) = \{\sum_{i=1}^n \lambda_i T_i : T_i \in \mathcal{B}^+(L^2(M, \mathcal{E})), \|Av_G(T_i)\| < \infty, \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$$

and $Av_G(PTQ) = PAv_G(T)Q$ for $P, Q \in R_G(\mathcal{E})$ and $T \in dom(Av_G)$. In particular, if $P \in \Psi_c^{-\infty}(M; \mathcal{E})^G$, then

$$Tr_G(P) = Tr(fP) = \int_G f(x)Tr(K_P(x,x))dx = Tr(k_P(e)),$$

for any cut-off function $f \in C_c^\infty(G)$.

With these preliminaries we now prove a basic lemma.

Lemma 1.1 *Let $P \in \Psi^\infty(M;\mathcal{E})^G$ be a selfadjoint elliptic differential operator of order n with positive definite principal symbol $\sigma_0(P)$ on $T^*M \setminus M$.*
(1) The spectrum $Sp(P)$ of P is contained in $[-a, \infty)$ for some real number a,
(2) $Tr_G(e^{-tP}) \sim \sum_{j=0}^\infty t^{\frac{j-m}{n}} \alpha_j, \ \ t \to 0,$
where $m = dim M$.

Proof. (1) let f be a cut-off function such that

$$P = Av(\tilde{P}) = \int_G L(g) f P L(g)^{-1} dg.$$

Let $G_0 = Supp(f), Mes(G_0) < \infty$. If for some real number a_0 we have $(fPu, u) \geq -a_0(u,u), u \in dom(fP)$, then

$$\begin{aligned} (Pu, u) &= \int_G (fPL(g)^{-1}u, L(g)^{-1}u) dg \\ &\geq -a_0 \int_{G_0} (L(g)^{-1}u, L(g)^{-1}u) dg = -a_0 mes(G_0)\|u\|^2. \end{aligned}$$

Thus it suffices to prove that $\tilde{P} = fP$ is bounded below. Since the principal symbol $\sigma_0(fP) = f\sigma_0(P) \geq 0$, we can construct $Q_0 \in \Psi_{cc}^n(M;\mathcal{E})$ such that $\sigma_0(Q_0) = \sqrt{f\sigma_0(P)}$. Let $\tilde{Q} = Q_0^* Q_0$. Then $\sigma_0(fP - \tilde{Q}) = 0$ and $fP - \tilde{Q} \in S^{n-1}(M_0;\mathcal{E})$ for some compact $M_0 \subset M$. Here $S^k(M_0, \mathcal{E})$ is the usual symbol space. The rest of the proof is to estimate $fP - \tilde{Q}$. This estimate follows from ([Gil], P. 45).

(2) Note that part (2) of the lemma means that $|Tr_G(e^{-tP}) - \sum_{j=0}^{m_1} t^{\frac{j-m}{n}} \alpha_j| = O(t^{\frac{m_1-m}{n}})$ as $t \to 0$. This assertion is proved for the Laplace operator on M

in [CoM 1]. We follow ([CoM 1], [Gil]) to include a proof here for the general case.

Let $U \subset M$ be a relative compact open set which serves also as a coordinate chart of M. Let O be the region in $\mathbb{C}$ bounded by two infinite rays $\nu_\pm$ beginning at $-a-1 : \nu_\pm = \{z = x+iy \in \mathbb{C} : x+(a+1)(t-1), y = \pm ta_1, t \geq 0\}$ for some $a_1 > 0$, such that $\mathbb{R}^+ \not\subset O$. Then $(P - \lambda I)^{-1}$ is an analytic function in U. But $(P-\lambda I)^{-1}$ is not an elliptic operator. Yet given a cut-off function f with support in U mod H we can choose as in ([Gil] P.52) an analytic family of pseudo-differential operators R_λ on U whose kernel is supported in $U \times U$ such that $fR_\lambda = R_\lambda$ and $\sigma_0(fR_\lambda(P-\lambda I)-f) \sim 0$, and in terms of the norms $\|..\|_{k,U}$ on Sobolev spaces $H_k(U, \mathcal{E})$,

$$\|fR_\lambda - f(P-\lambda I)^{-1}\|_{k,U} \leq c_k(1+|\lambda|)^{-k-1}, \lambda \in U, \tag{1.1}$$

where c_k is the constant independent of $\lambda \in U$. Using the nuclearity of the inclusion operator I_k from $L^2(U, \mathcal{E})$ to $H_k(U, \mathcal{E})$ for $k > m = dim M$, we get that $Q_\lambda = fR_\lambda(P-\lambda I) - f$ is a trace class operator on $L^2(U, \mathcal{E})$ and $\|Q_\lambda\| \leq \|I_k\|_{Tr}\|Q_\lambda\|_{k,U}$. Since R_λ and Q_λ are compactly supported, we can extend them trivially to M, denoted also by R_λ and Q_λ. Now let $E_t = \frac{1}{2\pi i}\int_{\nu_- \cup \nu_+} e^{-t\lambda} R_\lambda d\lambda$. By the Cauchy theorem and (1.1), we have

$$\begin{aligned}
\|E_t - fe^{-tP}\|_{Tr} &= \|\int_{\nu_\pm} \frac{1}{2\pi i} e^{-t\lambda}(R_\lambda - f(P-\lambda I)^{-1})d\lambda\|_{Tr} \\
&\leq \frac{1}{2\pi}\int_{\nu_\pm} |e^{-\lambda}| c_k (1 + \frac{|\lambda|}{t})^{-k-1}\frac{d|\lambda|}{t} \\
&\leq \frac{1}{2\pi}\int_{\nu_\pm} |e^{-\lambda}| c_k |\lambda|^{-k-1} t^k |d\lambda|, \; as \; t \to 0.
\end{aligned} \tag{1.2}$$

Furthermore, we know that R_λ is a smooth operator ([Gil], P.53). Q_λ is thus in $dom(Av_G) \cap dom(Tr)$ and (1.2) shows that $fe^{-tP} \in dom(Tr)$. Clearly,

$fe^{-tP} \in dom(Av_G)$. Therefore,

$$\begin{aligned} |Tr_G e^{-tP} - Tr E_t| &\leq \|Av_G(fe^{-tP} - E_t)\|_{Tr_G} \leq 2\|fe^{-tP} - E_t\|_{Tr} \\ &\leq c_0(t^k), \quad t \to 0, k > 0. \end{aligned} \tag{1.3}$$

On the other hand, we have the asymptotic expansion [Gil]

$$Tr E_t \sim \sum_{j > -m} t^{j/n} \alpha_j, t \to 0.$$

This together with (1.3) proves part (2) of the lemma. Q.E.D.

Let us remark that by (1.1) and the proof of (1.2),

$$\|E_t - fe^{-tP}\|_{Tr} \leq e^{t(a+\varepsilon)}, t \to 0,$$

for arbitrary $\varepsilon > 0$ and constant $c(\varepsilon) > 0$, provided that two rays $\nu_\pm$ are equal to $\nu = \{\lambda \in \mathbb{C} : dis(\lambda, [-a, \infty)) = \varepsilon\}$. We see therefore that $|e^{-\lambda t} Tr_G(e^{-tP})| \leq c(\lambda) e^{-(\Re\lambda - a)t}$ for $\Re\lambda > a$. It is thus appropriate to define ζ-determinant function of P.

Definition 1.1 Let P be a G-invariant elliptic differential operator on M with positive definite principal symbol $\sigma_0(P)$ on $T^*M \setminus M$. Suppose that the spectrum $Sp(P)$ of P is contained in $[-a, \infty)$. Then the ζ-determinant function $D_\zeta(P, \lambda)$ of P is defined by

$$\log D_\zeta(P, \lambda) = -\frac{d}{ds}(\zeta_P(s, \lambda))_{s=0},$$

where $\zeta_P(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr_G(e^{-tP}) e^{-t\lambda} dt$, $\Re\lambda > a$, with $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

Remark 1.1 *The integral defining $\zeta_P(s, \lambda)$ is convergent for $\Re\lambda > a$ as we remarked before. By Lemma 1.1, $\zeta_P(s, \lambda)$ has a meromorphic extension to $\mathbb{C}$ and the extended function is analytic near $s = 0$. $\frac{d}{ds}(\zeta_P(s, \lambda))_{s=0}$ is well defined.*

We state the basic properties of the ζ-determinant function in the following proposition.

Proposition 1.1 *Let P be a G-invariant elliptic differential operator on M with positive definite principal symbol $\sigma_0(P)$ on $T^*M \setminus M$. Suppose that the spectrum $Sp(P)$ of P is contained in $[-a, \infty)$ for some $a \geq 0$.*
(1) $\log D_\zeta(cP, \lambda) = \log D_\zeta(P, \frac{\lambda}{c}) + \zeta_P(0, \frac{\lambda}{c}) \log(c), \quad c > 0, \quad \Re\lambda > ac,$
(2) $\log D_\zeta(U^*PU, \lambda) = \log D_\zeta(P, \lambda)$ *for G-invariant unitary operator U on $L^2(M, \mathcal{E})$ and $\Re\lambda > a$.*

Proof. (1) We have

$$\begin{aligned} \zeta_{cP}(s, \lambda) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr_G(e^{-tP}) e^{-t\lambda/c} c^{-s} dt \\ &= c^{-s} \zeta_P(s, \frac{\lambda}{c}). \end{aligned}$$

Hence the result follows.

(2) follows immediately from $Tr_G(e^{-tU^*PU}) = Tr_G(e^{-tP})$. Q.E.D.

1.2 L^2-Analytic Torsion Functions

We begin with the Ray-Singer analytic torsion for a closed oriented Riemannian manifold M of dimension m. Let $\tilde{M}$ be the universal covering space of M. Denote by $\wedge^j(\tilde{M})$ and $\wedge^j(M)$ the spaces of $L^2 - j$ forms on $\tilde{M}$ and M, resp.. Let Γ be the fundamental group $\pi_1(M)$ of M and $l^2(\Gamma)$ the Hilbert space of square integrable functions on Γ. Γ acts on $l^2(\Gamma)$ via the left (resp. right) regular representation $L(g)$ (resp. $R(g)$). The finite von Neumann algebra R_Γ generated by $R(g), g \in \Gamma$ in $\mathcal{B}(l^2(\Gamma))$ will play an important role via its finite trace $Tr_\Gamma(R(g)) = \delta_{g,e}$. In fact, let $\mathcal{F}$ be a fundamental domain for the action of Γ on $\tilde{M}$, we have $\wedge^j(\tilde{M}) \simeq \wedge^j(\mathcal{F}) \otimes l^2(\Gamma)$, and the space of Γ-invariant bounded operators on $\wedge^j(\tilde{M})$ can be identified with $R_\Gamma \otimes \mathcal{B}(\wedge^j(\mathcal{F}))$. Thus the trace Tr_Γ on R_Γ together with the usual trace on $\mathcal{B}(\wedge^j(\mathcal{F}))$ produces a trace (denoted also by Tr_Γ) on $R_\Gamma \otimes \mathcal{B}(\wedge^j(\mathcal{F}))$. Let $\tilde{\Delta}_j = \tilde{d}_j^* \tilde{d}_j + \tilde{d}_{j-1}\tilde{d}_{j-1}^*$ be the Laplace operator on $\tilde{M}, j = 0, 1, \ldots, m$, with domain equal to the Sobolev space of all j-forms on $\tilde{M}$. Here $\tilde{d}$ is the exterior differential on $\tilde{M}$. $\tilde{\Delta}_j$ is Γ-invariant and selfadjoint and for $t > 0$ $e^{-t\tilde{\Delta}_j}$ is a bounded Γ-operator on $\wedge^j(\tilde{M})$ and has a Schwartz kernel on $\tilde{M} \times \tilde{M}$. Furthermore, $Tr_\Gamma(e^{-t\tilde{\Delta}_j}) < \infty$.

Let $\Delta_j = d_j^* d_j + d_{j-1} d_{j-1}^*$ be the Laplace operator on M, and $P_j : \wedge^j(M) \to Ker\Delta_j$ (resp. $\tilde{P}_j : \wedge^j(\tilde{M}) \to Ker(\tilde{\Delta}_j)$) be the orthogonal projection. Denote by $\Delta_j' = \Delta_j|_{(Ker\Delta_j)^\perp}$ (resp. $\tilde{\Delta}_j' = \tilde{\Delta}_j|_{(Ker\tilde{\Delta}_j)^\perp}$). By the Seeley expansion theorem for heat kernels, we get that the zeta function $\zeta_j = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-t\Delta_j'}) dt$, defined for $\Re s - 1 > m/2$, extends to an analytic function near $s = 0$.

Definition 1.2 The Ray-Singer analytic torsion $T(M)$ of M is defined by

$$\log T(M) = \frac{1}{2} \sum_{j=0}^{m} (-1)^j j \zeta_j'(0).$$

$T(M)$ depends in general on the Riemannian metric on M. But we can define the torsion $T'(M)$ of M as

$$\log T'(M) = \sum_{j=0}^{m} (-1)^j (\frac{1}{2} j \zeta_j'(0) + \log |A^j(\eta^j)/(h^j)|),$$

where η^j and h^j are two orthonormal bases for $\mathcal{H}^j(M) = Ker\Delta_j$ and for $H^j(M) = Kerd_j/imd_{j-1}$, $A^j : \mathcal{H}^j(M) \to H^j(M)$ is the de Rham map given by $A^j(\alpha) = \int_\nu \alpha$ for $\alpha \in Ker\Delta_j$ and j-chain ν in M, $A^j(\eta^j)$ is a basis of $H^j(M)$

and $|A^j(\eta^j)/(h^j)|$ stands for the determinant of the change between the two bases. Then $T'(M)$ is independent of the metric on M [RaS 2].

To recall the definition of L^2-analytic torsion on $\tilde{M}$ [Lott 3], we need the Novikov- Shubin invariants given by

$$\alpha_j = \sup\{\beta_j \geq 0 : Tr_\Gamma(e^{-t\tilde{\Delta}'_j}) \sim O(t^{-\beta_j/2}) \ as \ \ t \to \infty\}, j = 0, 1, \ldots, m.$$

They are homotopy invariants [GrS].

Definition 1.3 The L^2-analytic torsion $T(\tilde{M})$ of $\tilde{M}$ is defined by

$$\log T(\tilde{M}) = \frac{1}{2}\sum_{j=0}^{m}(-1)^j j(\zeta'_{j,1}(0) + \int_1^\infty t^{-1}Tr_\Gamma(e^{-t\tilde{\Delta}'_j})dt),$$

Provided the Novikov-Shubin invariants are positive, where

$$\zeta_{j,1} = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}Tr_\Gamma(e^{-t\tilde{\Delta}'_j})dt, \quad \Re s > \frac{m}{2} + 1,$$

also extends to an analytic function near $s = 0$

The condition that $\alpha_j > 0$ is imposed to guarantee the convergence of the integral in $\log T'(\tilde{M})$. This condition is satisfied for hyperbolic manifolds and for M with abelian fundamental groups [Lott 3]. W. Lück has conjectured this is always true, i.e., $\alpha_j > 0, j = 0, 1, \ldots, m$.

We now come to the L^2-analytic torsion function for the homogeneous space $M = G/H$. Let $\alpha : H \to Aut(E)$ be a unitary representation of H on finite dimensional vector space E. The induced representation $Ind_H^G\alpha$ of G on $L^2(M, \mathcal{E})$ is given by

$$\begin{aligned} ((Ind_H^G\alpha)(g)(\varphi))(x) &\stackrel{\text{def}}{=} R_{H,\alpha}(g)(\varphi)(x) = \varphi(gx), \\ \varphi \in L^2(M, \mathcal{E}) &= \{\varphi \in L^2(G, E) : \varphi(gh) = \alpha(h)\varphi(g)\}, \end{aligned}$$

where $\mathcal{E} = G \times_\alpha E$ is the homogeneous vector bundle associated with α. Let $\mathcal{P}$ be the tangent space to M at eH. Then $(L^2(M,\mathcal{E}) \otimes_{\mathbb{R}} \wedge^j \mathcal{P})^H$ can be identified with $\mathcal{E}$-valued $L^2 - j$ forms. Define the differential operators $d(\alpha)$ and $d^*(\alpha)$ by

$$\begin{aligned} d(\alpha) &= \sum_j R_{H,\alpha}(x_j) \otimes \varepsilon(x_j), \\ d^*(\alpha) &= \sum_j R_{H,\alpha}(x_j) \otimes i(x_j), \end{aligned}$$

where $\{x_j\}$ is an orthonormal basis of $\mathcal{P}$ and ε (resp. i) is the exterior (resp. interior) multiplication of $\wedge^* \mathcal{P}$. Then we define the Laplace operator $\Delta_j(\alpha) = d_j^*(\alpha) d_j(\alpha) + d_{j-1}(\alpha) d_{j-1}^*(\alpha), j = 0, 1, \ldots, m$. Let $\Delta_j'(\alpha) = \Delta_j(\alpha)|_{(Ker \Delta_j(\alpha))^\perp}$. By Lemma 1.1, we can define the L^2-analytic torsion function on M as follows.

Definition 1.4 The L^2-analytic torsion function $T(M,\lambda)$ of M is defined by

$$\log T(M,\lambda) = \frac{1}{2} \sum_{j=0}^{m} (-1)^{j+1} j \log D_\zeta(\Delta_j'(\alpha), \lambda), \Re\lambda > 0,$$

where

$$\log D_\zeta(\Delta_j'(\alpha), \lambda) = -\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr_G(e^{-t\Delta_j'(\alpha)}) e^{-t\lambda} dt \right)_{s=0}. \tag{1.4}$$

Remark 1.2 (1) *Unlike the L^2-analytic torsion on covering spaces, where a positive decay rate of $Tr_G(e^{-t\Delta_j'})$ is required, we use the parameter λ to control the integral in* (1.4).

(2) *When $M = G/H$ is the universal covering space of a compact symmetric space, the value of $T(M,\lambda)$ at $\lambda = 0$ differs from the L^2-analytic torsion [Lott 3] by a volume constant.*

The L^2-analytic torsion function $T(M,\lambda)$ has similar properties to the Ray-Singer torsion. We first prove the vanishing result of $T(M,\lambda)$ for an even dimensional homogeneous space $M = G/H$.

Proposition 1.2 *If the $dimM = m$ is even, then $T(M,\lambda) = 0$ for all $\Re\lambda > 0$.*

Proof. The proof is a routine computation. Let $\wedge^j(M,\mathcal{E})$ be the space of $\mathcal{E}$-valued $L^2 - j$ forms on M. The Hodge theorem which is valid for M shows

$$\wedge^j(M,\mathcal{E}) = \overline{d\wedge^{j-1}} \oplus \overline{d^*\wedge^{j+1}} \oplus \mathcal{H}^j(M,\mathcal{E}),$$

where $Ker\Delta_j(\alpha) = \mathcal{H}^j(M,\mathcal{E})$. We have $(Ker\Delta_j(\alpha))^\perp = \overline{d\wedge^{j-1}} \oplus \overline{d^*\wedge^{j+1}}$ and $\Delta_j'(\alpha) = d_j^*(\alpha)d_j(\alpha)|_{\overline{d^*\wedge^{j+1}}} + d_{j-1}(\alpha)d_{j-1}^*(\alpha)|_{\overline{d\wedge^{j-1}}}$. This implies

$$Tr_G(e^{-t\Delta_j'(\alpha)}) = Tr_G(e^{-td_j^*(\alpha)d_j(\alpha)}|_{\overline{d^*\wedge^{j+1}}}) + Tr_G(e^{-td_{j-1}(\alpha)d_{j-1}^*(\alpha)}|_{\overline{d\wedge^{j-1}}}).$$

We omit α from the notations and get

$$\begin{aligned}
Tr_G(e^{-td_{j-1}d_{j-1}^*}) &= \int_t^\infty Tr_G(d_{j-1}e^{-td_{j-1}^*d_{j-1}}d_{j-1}^*)dt \\
&= \int_t^\infty Tr_G(d_{j-1}^*d_{j-1}e^{-td_{j-1}^*d_{j-1}})dt \\
&= -\int_t^\infty \frac{d}{dt}Tr_G(e^{-td_{j-1}^*d_{j-1}})dt = Tr_G(e^{-td_{j-1}^*d_{j-1}}), \quad t > 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{j=0}^m (-1)^j j Tr_G(e^{-t\Delta_j'(\alpha)}) &= \sum_{j=0}^m (-1)^j j(Tr_G(e^{-td_{j-1}d_{j-1}^*}) + Tr_G(e^{-td_j^*d_j})) \\
&= \sum_{j=0}^m (-1)^j j(Tr_G(e^{-td_{j-1}^*d_{j-1}}) + Tr_G(e^{-td_j^*d_j})) \\
&= \sum_{j=0}^{m-1} (-1)^{j+1} Tr_G(e^{-td_j^*d_j}) \\
&= \sum_{j=1}^m (-1)^j Tr_G(e^{-td_{j-1}d_{j-1}^*}). \qquad (1.5)
\end{aligned}$$

On the other hand, since $dimM$ is even, Hodge duality implies $*(d_{j-1}d_{j-1}^*) = d_{m-j}^*d_{m-j}*$, where $*$ is the Hodge operator, an isometry. We obtain

$$\sum_{j=1}^m (-1)^j Tr_G(e^{-td_{j-1}d_{j-1}^*}) = \sum_{j=1}^m (-1)^j Tr_G(e^{-td_{m-j}^*d_{m-j}}) = \sum_{j=0}^{m-1} (-1)^j Tr_G(e^{-td_j^*d_j}).$$

This together with (1.5) shows that $\sum_{j=0}^{m}(-1)^j jTr_G(e^{-t\Delta'_j(\alpha)}) = 0$. Hence, $\log T(M,\lambda) = 0$. Q.E.D.

In order to consider the dependence of $T(M,\lambda)$ on the metric in M, let us first prove the following.

Proposition 1.3 *Let $g(u) = g_0u+g_1(1-u)$ be a family of G-invariant metrics on M. Suppose that $\Delta_j(u)$ are the Laplace operators on M associated with $g(u)$. Then*

$$\frac{d}{du}(\sum_{j=0}^{m}(-1)^j jTr_G(e^{-t\Delta_j(\alpha)}))_{u=u_0} = -t\sum_{j=0}^{m}(-1)^j jTr_G(\frac{d}{du}(\Delta_j(u))_{u=u_0}e^{-t\Delta_j(u_0)}).$$

Proof. By the Duhamel principal, we have

$$\begin{aligned}
& Tr_G(\sum_{j=0}^{m}(-1)^j j(e^{-(t-\varepsilon)\Delta_j(u)}e^{-\varepsilon\Delta_j(u_0)} - e^{-\varepsilon\Delta_j(u)}e^{-(t-\varepsilon)\Delta_j(u_0)})) \\
= \; & Tr_G(-\int_{\varepsilon}^{t-\varepsilon}\frac{d}{ds}(\sum_{j=0}^{m}(-1)^j j(e^{-(t-s)\Delta_j(u)}e^{-s\Delta_j(u_0)})ds) \\
= \; & Tr_G(-\int_{\varepsilon}^{t-\varepsilon}\sum_{j=0}^{m}(-1)^j j(\Delta_j(u)e^{-(t-s)\Delta_j(u)}e^{-s\Delta_j(u_0)} - e^{-(t-s)\Delta_j(u)}e^{-s\Delta_j(u_0)}\Delta_j(u_0))ds) \\
= \; & -\int_{\varepsilon}^{t-\varepsilon}\sum_{j=0}^{m}(-1)^j jTr_G(\Delta_j(u)-\Delta_j(u_0))e^{-(t-s)\Delta_j(u)}e^{-s\Delta_j(u_0)})ds. \qquad (1.6)
\end{aligned}$$

By taking the derivatives with respect to u on both sides of (1.6) and then letting $u = u_0$ and $\varepsilon \to 0$, we obtain that the right hand side of (1.6) is

$$\begin{aligned}
& \lim_{\varepsilon\to 0}(-\int_{\varepsilon}^{t-\varepsilon}\sum_{j=0}^{m}(-1)^j jTr_G(\frac{d}{du}(\Delta_j(u))e^{-(t-s)\Delta_j(u)}e^{-s\Delta_j(u_0)})_{u=u_0}ds) \\
= \; & -\lim_{\varepsilon\to}(t-2\varepsilon)\sum_{j=0}^{m}(-1)^j jTr_G(\frac{d}{du}(\Delta_j(u))_{u=u_0}e^{-t\Delta_j(u_0)}) \\
= \; & -t\sum_{j=0}^{m}(-1)^j jTr_G(\frac{d}{du}(\Delta_j(u))_{u=u_0}e^{-t\Delta_j(u_0)}).
\end{aligned}$$

On the other hand, the left hand side of (1.6) is

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{d}{du}\Big(\sum_{j=0}^{m}(-1)^j j Tr_G(e^{-(t-\varepsilon)\Delta_j(u)}e^{-\varepsilon\Delta_j(u_0)} - e^{-\varepsilon\Delta_j(u)}e^{-(t-\varepsilon)\Delta_j(u_0)})\Big)_{u=u_0} \\
= \;& \lim_{\varepsilon\to 0}\sum_{j=0}^{m}(-1)^j j\frac{d}{du}Tr_G(e^{-(t-\varepsilon)\Delta_j(u)}e^{-\varepsilon\Delta_j(u)})_{u=u_0} \\
= \;& \sum_{j=0}^{m}(-1)^j j\frac{d}{du}Tr_G(e^{-t\Delta_j(u)})_{u=u_0}.
\end{aligned}$$

Here we used the fact that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{j=0}^{m}(-1)^j j Tr_G\Big(\frac{d}{du}(e^{-\varepsilon\Delta_j(u)})_{u=u_0}e^{-(t-\varepsilon)\Delta_j(u_0)}\Big) \\
= \;& \lim_{\varepsilon\to 0}(-\varepsilon)\sum_{j=0}^{m}(-1)^j j Tr_G\Big(\frac{d}{du}(\Delta_j(u))_{u=u_0}e^{-t\Delta_j(u_0)}\Big) = 0.
\end{aligned}$$

Combining these two identities, we complete the proof. Q.E.D.

Theorem 1.1 *With the notations in Proposition 1.3, then for $dimM = m$ odd,*

$$\begin{aligned}
\frac{d}{du}\log T_u(M,\lambda) \;=\; & \frac{1}{2(\lambda+1)}\sum_{j=0}^{m}(-1)^{j+1}Tr_G(\nu(u)P_j(u)) \\
- \;& \frac{\lambda}{2}\Big(\int_0^\infty e^{-\lambda t}\sum_{j=0}^{m}(-1)^j Tr_G(e^{-t(\Delta_j(u)+P_j(u))}\nu(u))t^s dt\Big)_{s=0},
\end{aligned}$$

where $\nu(u) = (\frac{d}{du}(*_u))*_u^{-1}$ and $P_j(u)$ is the orthogonal projection of $\wedge^j(M,\mathcal{E})$ onto $Ker\Delta_j(u)$.

Proof. Since the Hodge operator $*_u$ associated with the metrics $g(u)$ satisfies $*_u^2 = \pm 1$, $\frac{d}{du}(*_u)*_u + *_u\frac{d}{du}(*_u) = 0$. Then

$$\begin{aligned}
\frac{d}{du}(\Delta_j(u)) \;=\; & \pm\frac{d}{du}(*_u d(u) *_u d(u)) \pm \frac{d}{du}(d(u) *_u d(u) *_u) \\
= \;& \pm\Big(\frac{d}{du}(*_u) *_u^{-1} *_u d(u) *_u d(u) + *_u d(u) *_u *_u^{-1}\frac{d}{du}(*_u)d(u)\Big) \\
\pm \;& \Big(d(u)\frac{d}{du}(*_u) *_u^{-1} *_u d(u) *_u + d(u) *_u d(u) *_u *_u^{-1}\frac{d}{du}(*_u)\Big) \\
= \;& \nu(u)d^*(u)d(u) - d^*(u)\nu(u)d(u) + d(u)\nu(u)d^*(u) - d(u)d^*(u)\nu(u).
\end{aligned}$$

Here we omit the subindex for $d(u)$. Clearly,

$$\begin{aligned}
Tr_G(\nu(u)d^*(u)d(u)e^{-t\Delta_j(u)}) &= Tr_G(d^*(u)d(u)e^{-t\Delta_j(u)}\nu(u)),\\
Tr_G(-d^*(u)\nu(u)d(u)e^{-t\Delta_j(u)}) &= -Tr_G(d(u)e^{-t\Delta_j(u)}d^*(u)\nu(u)),\\
Tr_G(d(u)\nu(u)d^*(u)e^{-t\Delta_j(u)}) &= Tr_G(d^*(u)e^{-t\Delta_j(u)}d(u)\nu(u)),
\end{aligned}$$

and

$$Tr_G(-d(u)d^*(u)\nu(u)e^{-t\Delta_j(u)}) = -Tr_G(e^{-t\Delta_j(u)}d(u)d^*(u)\nu(u)).$$

Using the identities $d_j(u)\Delta_j(u) = \Delta_{j+1}(u)d_j(u)$ and $d^*_{j-1}(u)\Delta_j(u) = \Delta_{j-1}(u)d^*_{j-1}(u)$, we get

$$\begin{aligned}
Tr_G(\frac{d}{du}(\Delta_j(u))e^{-t\Delta_j(u)}) = {}& Tr_G(e^{-t\Delta_j(u)}d^*(u)d(u)\nu(u)) - Tr_G(e^{-t\Delta_{j+1}(u)}d(u)d^*(u)\nu(u))\\
&+Tr_G(e^{-t\Delta_{j-1}(u)}d^*(u)d(u)\nu(u)) - Tr_G(e^{-t\Delta_j(u)}d(u)d^*(u)\nu(u)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{m}(-1)^j j Tr_G(\frac{d}{du}(\Delta_j(u))e^{-t\Delta_j(u)})\\
= {}& \sum_{j=0}^{m}(-1)^{j+1}(Tr_G(e^{-t\Delta_j(u)}d_j^*(u)d_j(u)\nu(u)) + Tr_G(e^{-t\Delta_j(u)}d_{j-1}(u)d^*_{j-1}(u)\nu(u)))\\
= {}& \sum_{j=0}^{m}(-1)^{j+1}Tr_G(e^{-t\Delta_j(u)}\Delta_j(u)\nu(u)) = \frac{d}{dt}(\sum_{j=0}^{m}(-1)^j Tr_G(e^{-t\Delta_j(u)}\nu(u))).
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&\frac{d}{du}(\log T_u(M,\lambda))\\
= {}& \frac{1}{2}\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\frac{d}{du}(\sum_{j=0}^{m}(-1)^j j Tr_G(e^{-t\Delta_j(u)} - P_j(u)))dt)_{s=0}\\
= {}& \frac{1}{2}\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\sum_{j=0}^{m}(-1)^j j(-t)Tr_G(\frac{d}{du}(\Delta_j(u))e^{-t\Delta_j(u)})dt)_{s=0}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{d}{ds}\Big(\frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-\lambda t}\frac{d}{dt}\Big(\sum_{j=0}^m(-1)^j Tr_G(e^{-t\Delta_j(u)}\nu(u))\Big)dt\Big)_{s=0}\\
&= \frac{1}{2}\frac{d}{ds}\Big(\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\sum_{j=0}^m(-1)^j Tr_G((e^{-t(\Delta_j(u)+P_j(u))}-e^{-t}P_j(u))\nu(u))dt\\
&- \frac{\lambda}{\Gamma(s)}\int_0^\infty t^s e^{-\lambda t}\sum_{j=0}^m(-1)^j Tr_G((e^{-t(\Delta_j(u)+P_j(u))}-e^{-t}P_j(u))\nu(u))dt\Big)_{s=0}\\
&= \frac{1}{2}\frac{d}{ds}\Big(\frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(\lambda+1)t}dt\sum_{j=0}^m(-1)^j Tr_G(-P_j(u)\nu(u))\\
&- \frac{\lambda}{\Gamma(s)}\int_0^\infty t^s e^{-(\lambda+1)t}dt\sum_{j=0}^m(-1)^j Tr_G(-P_j(u)\nu(u))\\
&- \frac{\lambda}{\Gamma(s)}\int_0^\infty t^s e^{-\lambda t}\sum_{j=0}^m(-1)^j Tr_G(e^{-t(\Delta_j(u)+P_j(u))}\nu(u))dt\Big)_{s=0}\\
&= \frac{1}{2}\Big(1-\frac{\lambda}{\lambda+1}\Big)\sum_{j=0}^m(-1)^j Tr_G(-P_j(u)\nu(u))\\
&- \frac{\lambda}{2}\Big(\int_0^\infty t^s e^{-\lambda t}\sum_{j=0}^m(-1)^j Tr_G(e^{-t(\Delta_j(u)+P_j(u))}\nu(u))dt\Big)_{s=0}.
\end{aligned}
$$

Here we have used the fact that the zeta function $\zeta_{j,u}(s,\lambda)$ of $\Delta_j(u)$ can be analytically extended to a neighborhood of $s=0$ which follows from Lemma 1.1. In fact, we have the following asymptotic expansion for m odd,

$$Tr_G(e^{-t\Delta_j(u)}) \sim \sum_{j=0}^\infty t^{j-\frac{m}{2}}\alpha_j, \quad t\to 0.$$

Q.E.D.

In the above proof we also used the fact that $Tr_G(P_j(u))$ is independent of the metric on M, since $Ker\Delta_j(u)\simeq\mathcal{H}^j(M,\mathcal{E})$ is independent of the metric. Note that when $\lambda=0$ the formula in Theorem 1.1 reduces to that of the L^2-analytic torsion [RaS 1].

We now consider the product formula for the L^2-analytic torsion function. let G_1 and G_2 be two unimodular Lie groups with countably many connected components, H_1 and H_2 two compact subgroups of G_1 and G_2, resp.. Suppose α_1 and α_2 are two unitary representations of H_1 and H_2 on finite dimensional vector spaces E_1 and E_2, resp.. Then $\alpha = \alpha_1 \times \alpha_2$ is a unitary representation of $H = H_1 \times H_2$ on $E = E_1 \times E_2$, and $\mathcal{E} = G \times_\alpha E = (G_1 \times_{\alpha_1} E_1) \times (G_2 \times_{\alpha_2} E_2) = \mathcal{E}_1 \times \mathcal{E}_2$, where $G = G_1 \times G_2$ and $\mathcal{E}_i = G_i \times_{\alpha_i} E_i$. We can thus consider the L^2-analytic torsion functions $T(G/H, \lambda)$ and $T(G_i/H_i, \lambda)$. These torsion functions are related by the following formula.

Proposition 1.4 *Let $\chi(G_i/H_i)$ be the Euler characteristic number of G_i/H_i. Then*

$$\log T(G/H, \lambda) \;=\; \chi(G_1/H_1) \log T(G_2/H_2, \lambda) + \chi(G_2/H_2) \log T(G_1/H_1, \lambda).$$

Proof. We have $M = G/H = M_1 \times M_2$, where $M_i = G_i/H_i$. Then $\oplus_{k+l=j} \wedge^k (M_1) \otimes \wedge^l (M_2)$ is dense in $\wedge^j(M)$. The Laplace operators are related by $\Delta(M) = \Delta(M_1) \otimes I + I \otimes \Delta(M_2)$.

$$\begin{aligned}
\sum_{j=0}^{m} (-1)^j j Tr_G(e^{-t\Delta'_j(M)}) &= \sum_{j=0}^{m} \sum_{k+l=j} (-1)^{k+l}(k+l)(Tr_{G_1}(e^{-t\Delta'_k(M_1)}) Tr_{G_2}(e^{-t\Delta'_l(M_2)}) \\
&+ Tr_{G_1}(e^{-t\Delta_k(M_1)}|_{Ker\Delta_k(M_1)}) Tr_{G_2}(e^{-t\Delta'_l(M_2)}) \\
&+ Tr_{G_1}(e^{-t\Delta'_k(M_1)}) Tr_{G_2}(e^{-t\Delta_l(M_2)}|_{Ker\Delta_l(M_2)})) \\
&= \chi(M_1) \sum_{l=0}^{m_2} (-1)^l l Tr_{G_2}(e^{-t\Delta'_l(M_2)}) \\
&+ \chi(M_2) \sum_{k=0}^{m_1} (-1)^k k Tr_{G_1}(e^{-t\Delta'_k(M_1)}),
\end{aligned}$$

since $\chi(M_i) = \sum_{j=0}^{m_i} (-1)^j j Tr_{G_i}(P_j(M_i))$ and $\sum_{j=0}^{m_i} (-1)^j Tr_{G_i}(e^{-t\Delta'_j(M_i)}) = 0$. The latter follows from the step preceding (1.5). See also [GoP 1]. The rest of proof is clear. Q.E.D.

We state below a result about how the L^2-analytic torsion function varies with respect to the induced representation. Thus let H_1 and H_2 be two compact subgroups of G. Suppose that H_1 is a normal subgroup of H_2 and has finite index in H_2. Then for a finite dimensional unitary representation $\alpha_1 : H_1 \to Aut(E_1)$ we can associate an induced unitary representation $Ind_{H_1}^{H_2}\alpha_1 : H_2 \to Aut(E_2)$ by

$$Ind_{H_1}^{H_2}\alpha_1(h_2)f(h) = f(hh_2),\ h, h_2 \in H_2,$$

where

$$E_2 = \{\varphi \in L^2(H_2, E_1), \varphi(h_1 h) = \alpha(h_1)\varphi(h),\ h_1 \in H_1, h \in H_2\}.$$

Let $T(G/H_i, \lambda)$ be the L^2-analytic torsion functions on G/H_i corresponding to the representations α_1 and $Ind_{H_1}^{H_2}\alpha_1$.

Proposition 1.5 $\log T(G/H_1, \lambda) = \log T(G/H_2, \lambda)$.

Proof. See [RaS 1] for the proof. Q.E.D.

We close this section by the following results.

Proposition 1.6 $\log T(M, \lambda)$ *is an analytic function of* λ *for* $\Re\lambda > 0$.

Proof. By Lemma 1.1, we see that the first integral in the right side of the equality

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}Tr_G(e^{-t\Delta'_j})dt = \frac{1}{\Gamma(s)}(\int_0^\delta + \int_\delta^\infty)t^{s-1}e^{-\lambda t}Tr_G(e^{-t\Delta'_j})dt$$

has the extension to a neighborhood of $s = 0$ which is analytic for both s and λ. Obviously, the second integral in the right side of the above equality can be extended to an analytic function for both s near zero and λ with $\Re\lambda > 0$. $\log T(M,\lambda)$ is thus an analytic function of λ for $\Re\lambda > 0$. Furthermore, for m odd,

$$\begin{aligned}\frac{d}{d\lambda}\log T(M,\lambda) &= \frac{1}{2}\frac{d}{ds}\Big(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}(-t)\sum_{j=0}^{m}(-1)^j j Tr_G(e^{-t\Delta_j'})\Big)_{s=0} \\ &= -\frac{1}{2}\frac{d}{ds}\Big(\frac{s}{\Gamma(s+1)}\sum_{j=0}^{m}(-1)^j j\int_0^\infty t^s e^{-\lambda t}Tr_G(e^{-t\Delta_j'})dt\Big)_{s=0} \\ &= -\frac{1}{2}\sum_{j=0}^{m}(-1)^j j\Big(\int_0^\infty t^s e^{-\lambda t}Tr_G(e^{-t\Delta_j'})dt\Big)_{s=0}. \qquad Q.E.D.\end{aligned}$$

The following proposition shows that the L^2-analytic torsion function defines an additive map on the representation group $R(H)$ of H.

Proposition 1.7 *Let α_1 and α_2 be two representations of H on finite dimensional vector spaces E_1 and E_2, resp.. Let $T_{\alpha_i}(M,\lambda)$ be the L^2-analytic torsion functions corresponding to $\alpha_i, i = 0,1,2, \alpha_0 = \alpha_1 \oplus \alpha_2$.*

(1) $\log T_{\alpha_i}(M,\lambda) = \log T_{\alpha_1}(M,\lambda) + \log T_{\alpha_2}(M,\lambda)$,

(2) *If α_1 and α_2 are unitarily equivalent, then* $\log T_{\alpha_1}(M,\lambda) = \log T_{\alpha_2}(M,\lambda)$.

Proof. (1) We have $d(\alpha_0) = d(\alpha_1) \oplus d(\alpha_2)$ and $\Delta(\alpha_0) = \Delta(\alpha_1) \oplus \Delta(\alpha_2)$. Assertion (1) follows easily from the definition of the L^2-analytic torsion function.

(2) Since α_1 is unitarily equivalent to α_2, the induced representations $Ind_H^G\alpha_1$ and $Ind_H^G\alpha_2$ are also unitarily equivalent. In fact, if $\varphi : G \to E_1$ is such that $\varphi(gh) = \alpha_1(h)\varphi(g), h \in H, g \in G$, then for $\alpha_2 = U\alpha_1 U^*$, $U : E_1 \to E_2$ unitary, $(U\varphi)(gh) = U\alpha_1(h)\varphi(g) = \alpha_2(h)(U\varphi)(g)$. Hence, $d(\alpha_2)$ and $d(\alpha_1)$ (resp. $\Delta(\alpha_1)$ and $\Delta(\alpha_2)$) are unitarily equivalent. By Proposition 1.1, We get the assertion. Q.E.D.

As a consequence, $\log T_\alpha(M,\lambda) : R(H) \to \mathcal{F}(\Re\lambda > 0)$ is an additive map, where $\mathcal{F}(\Re\lambda > 0)$ denotes the set of all analytic functions on $\Re\lambda > 0$.

1.3 Computations

To calculate effectly the L^2-analytic torsion function $T(M,\lambda)$, let us note that the trace of the heat kernel $Tr_G(e^{-t\Delta'_j})(x,x)$ is independent of $x \in G$ and for any cut-off function f,

$$\begin{aligned} Tr_G(e^{-t\Delta'_j}) &= \int_G f(x) Tr(e^{-t\Delta'_j})(x,x)dx \\ &= \int_G f(x)(Tr(e^{-t\Delta_j})(x,x) - Tr(P_j)(x,x))dx \\ &= Tr(e^{-t\Delta_j})(e,e) - Tr(P_j)(e,e) \stackrel{\text{def}}{=} Tr(e^{-t\Delta_j})(e) - Tr(P_j)(e), \end{aligned}$$

where Tr is the trace on $End(E)$ of the finite dimensional space E. We get

$$\log T(M,\lambda) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\sum_{j=0}^{m}(-1)^j j Tr_G(e^{-t\Delta_j} - P_j)dt\right)_{s=0}$$

$$= \frac{1}{2}\frac{d}{ds}\left(\sum_{j=0}^{m}(-1)^j j\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}Tr(e^{-t\Delta'_j})(e)dt - \lambda^{-s}Tr(P_j)(e)\right)\right)_{s=0}. \quad (1.7)$$

From now on we asume that G is a connected semisimple Lie group and H is a maximal compact subgroup of G. Let $\mathcal{G}$ and $\mathcal{H}$ be the Lie algebras of G and H, resp.. We can write $\mathcal{G}$ as $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$ with respect to the Cartan form. $\mathcal{P}$ can be identified with the tangent space of G/H at eH. Choose a maximal abelian subalgebra $\mathcal{A} \subset \mathcal{P}$. Let $\mathcal{Q} = \mathcal{M}_q \oplus \mathcal{A}_q \oplus \mathcal{N}_q$ be a standard cuspidal parabolic subalgebra of $\mathcal{G}$ with $\mathcal{A}_q \subset \mathcal{A}$. Suppose Q is the normalizer of $\mathcal{Q}$ in G. Q can be written as a Langlands decomposition $Q = M_Q A_Q N_Q$. Recall that the representation $\pi_{\zeta,\nu}$ of Q associated with an irreducible unitary representation (ζ, W_ζ) of M_Q and a quasi-character e^ν of A_Q is defined by $\pi_{\zeta,\nu} = Ind_Q^G \zeta \otimes e^\nu \otimes I$ acting on

$$H_{\zeta,\nu} = \{\varphi : G \to W_\zeta : \varphi(gman) = e^{-(\nu+\rho_Q)\log a}\zeta(m)^{-1}\varphi(g)\}$$

with the norm $\int_H |\varphi(h)|^2_{W_\zeta} dh$. Let $M_Q^+ = M_Q^0 C$ with the identity component M_Q^0 of M_Q and $C = KerAd|_{M_Q}$. We also use $\mathcal{E}_2(M_Q)$ to denote the set of equivalent classes of discrete representations of M_Q^0. With theses notations we now state

Proposition 1.8 (1) *If G has no simple factor locally isomorphic to $SO_e(p,q)$, pq odd, or $SL(3,\mathbb{R})$, then $\log T(M,\lambda) = 0, \Re\lambda > 0$.*

(2) *If G is not contained in* (1), *then*

$$\begin{aligned}\log T(M,\lambda) &= \frac{1}{2}\frac{d}{ds}\Big(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\sum_{j=0}^m(-1)^j jTr(e^{-t\Delta_j})(e) \\ &- \sum_{j=0}^m(-1)^j jTr(P_j)(e)\lambda^{-s}\Big)_{s=0},\end{aligned}$$

with

$$\begin{aligned}&\sum_{j=0}^m(-1)^j jTr(e^{-t\Delta_j})(e) = dimE\cdot\sum_{l=0}^{2m}(-1)^l\int_{-\infty}^\infty e^{-t(\nu^2+(m-l)^2)}\,. \\ &\cdot\Big(\sum_{\xi\in\mathcal{E}_2(M_Q^0)} dim(W_\zeta\otimes(\Lambda^{odd}\mathcal{P}_m - \Lambda^{ev}\mathcal{P}_m)\otimes\Lambda^l\mathcal{N})^{H\cap M_Q^0}\Big)d\mu(\xi,\nu),\end{aligned}\quad (1.8)$$

where $\mathcal{P}_m = \mathcal{P}\cap\mathcal{M}$ is a component of $\mathcal{P}$ in the Iwasawa decomposition, and $d\mu(\xi,\nu)$ is the Plancherel measure

$$\mu(\xi,\nu) = c\prod_{\alpha\in\Delta_M^+}(\Lambda_\xi,\alpha)\prod_{\nu\in\Delta_{\mathbb{C}}^+} < (\Lambda_\xi,\nu),\nu > .$$

Proof. The results follow from the proofs of Proposition 2.9 and corollary 2.2 in [MoS]. (We use only $\tilde{K}_t(e)$ in the notation of this reference.) Q.E.D.

We thus see the computation of the L^2-analytic torsion function on G/H is much easier than that of the L^2 -analytic torsion on $\Gamma \setminus G/H$ for some cocompact torsion free discrete group Γ of G, since the heat kernel $K_t(x,y)$ on $\Gamma \setminus G/H$ involves the action of Γ, namely, $K_t(x,y) = \tilde{K}_t(e) + \sum_{\nu\in\Gamma} \tilde{K}_t(y^{-1}\nu x)$.

We now calculate the L^2-analytic torsion function for the hyperbolic space $M = G/H$ with $G = SO(m,1)$ and $H = S(O(m) \times O(1))$. In this case, $P_j = 0$ [Don]. We can further simplify (1.7) by the step preceding (1.5),

$$\log T(M,\lambda) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\sum_{j=0}^{m}(-1)^{j+1} j Tr(e^{-td_j^* d_j})(e)dt\right)_{s=0}.$$

To compute $Tr(e^{-td_j^*d_j})(e)$, we use the Plancherel formula [Wil] for $f_j(x) = Tr(e^{-td_j^*d_j})(x)$,

$$f_j(e) = \sum_{\omega\in\hat{G}_d} d(\omega)\Theta_\omega(f_j) + \sum_{\sigma\in\hat{M}_1}\int_{-\infty}^{\infty}\Theta_{\pi_{\sigma,x\lambda}}(f_j)\mu_\sigma(x\lambda)dx, \tag{1.9}$$

where $M_1 = SO(m-1)$. Since m is odd, the discrete series representations $\omega \in \hat{G}_d$ do not contribute to the heat kernel. The first term in (1.9) dropped out. Note also that $\overline{imd_j^*}$ corresponds to the standard representation σ_i of M_1 on $\wedge^j(\mathbb{R})$ [Mill]. Thus only σ_i contributes to the second term in (1.9). We have ([Fre], [MoS]),

$$\Theta_{\pi_{\sigma,x\lambda}}(f_j) = e^{-t(x^2+(\frac{m-1}{2}-j)^2)}.$$

Hence,

$$f_j(e) = \int_{-\infty}^{\infty} e^{-t(x^2+(\frac{m-1}{2}-j)^2)}\mu_{\sigma_j}(x\lambda)dx.$$

Using the explicit formula for the Plancherel measure $\nu_\sigma(x\lambda)$ in [Mia], we get

$$f_j(e) = \frac{(4\pi)^{-\frac{m}{2}}}{\Gamma(\frac{m}{2})}\binom{m-1}{j}\int_{-\infty}^{\infty} e^{-t(x^2+(\frac{m-1}{2}-j)^2)}\left(\prod_{l=0}^{(m-1)/2}(x^2+l^2)\right)\left(x^2+\left(\frac{m-1}{2}-j\right)^2\right)^{-1}dx \tag{1.10}$$

See also [Lott 3] for this formula. Therefore we obtain

Proposition 1.9 *Let $G = SO(m,1)$ and $H = S(O(m) \times O(1))$ with m odd. Then*

$$\log T(M,\lambda) = \frac{1}{2}\frac{d}{ds}\Big(\sum_{j=0}^{m-1}(-1)^{j+1}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}f_j(e)dt\Big)_{s=0},$$

where $f_j(e)$ is given by (1.10).

Let us calculate some examples.

Example 1.1 *Let $m = 3$ in Proposition* 1.9, $f_0(e) = \frac{e^{-t}}{(4\pi t)^{3/2}}, f_1(e) = \frac{2(1+2t)}{(4\pi t)^{3/2}}$, $f_2(e) = f_0(e)$.

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty \sum_{j=0}^{2}(-1)^j f_j(e)t^{s-1}e^{-t\lambda}dt &= \frac{1}{\Gamma(s)}\frac{2}{(4\pi)^{3/2}}((1+\lambda)^{-(s-3/2)}\Gamma(s-\frac{3}{2}) \\ &- (\lambda^{-(s-\frac{3}{2})}\Gamma(s-\frac{3}{2}) + 2\lambda^{-(s-\frac{1}{2})}\Gamma(s-\frac{1}{2}))).\end{aligned}$$

Hence,

$$\begin{aligned}\log T(M,\lambda) &= -(4\pi)^{-3/2}\frac{d}{ds}\Big(\frac{1}{\Gamma(s)}((1+\lambda)^{-(s-3/2)}\Gamma(s-\frac{3}{2}) \\ &- (\lambda^{-(s-\frac{3}{2})}\Gamma(s-\frac{3}{2}) + 2\lambda^{-(s-\frac{1}{2})}\Gamma(s-\frac{1}{2})))\Big)_{s=0} \\ &= -(4\pi)^{-3/2}((1+\lambda)^{3/2}\Gamma(-3/2) - \lambda^{3/2}\Gamma(-3/2) - 2\lambda^{\frac{1}{2}}\Gamma(-\frac{1}{2})) \\ &= \frac{1}{2\pi}(\frac{1}{3}\lambda^{3/2} - \lambda^{\frac{1}{2}} - \frac{1}{3}(1+\lambda)^{\frac{3}{2}}),\end{aligned}$$

where we used the formula $\Gamma(\frac{1}{2}-n) = \frac{(-2)^n\sqrt{\pi}}{(2n-1)!!}, n = 0, 1, \ldots$. *In particular,* $\log T(M,0) = \frac{-1}{6\pi}$.

Example 1.2 *For $m = 5$ in Proposition* 1.9, *we have*

$$f_j(e) = \frac{1}{(4\pi)^{5/2}}\frac{1}{\Gamma(\frac{5}{2})}\binom{4}{j}\int_{-\infty}^{\infty} e^{-t(x^2+(2-j)^2)}\frac{x^2(x^2+1)(x^2+2)}{(x^2+(2-j)^2)}dx.$$

$$f_0(e) = \frac{1}{(4\pi)^{5/2}}\frac{1}{\Gamma(\frac{5}{2})}\binom{4}{0}\int_{-\infty}^{\infty} e^{-t(x^2+4)}x^2(x^2+1)dx$$

$$= \frac{1}{(4\pi)^{5/2}}\frac{2e^{-4t}}{3t^{3/2}}(1+\frac{3}{2t}),$$

where we used $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$ *and* $\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$.

$$f_1(e) = \frac{1}{(4\pi)^{5/2}}\frac{1}{\Gamma(\frac{5}{2})}\binom{4}{1}\int_{-\infty}^{\infty} e^{-t(x^2+1)}x^2(x^2+2)dx$$

$$= \frac{1}{(4\pi)^{5/2}}\frac{8e^{-t}}{3t^{3/2}}(2+\frac{3}{2t}).$$

$$f_2(e) = \frac{1}{(4\pi)^{5/2}}\frac{1}{\Gamma(\frac{5}{2})}\binom{4}{2}\int_{-\infty}^{\infty} e^{-t(x^2+(2-j)^2)}(x^2+2)(x^2+1)dx$$

$$= \frac{4}{(4\pi)^{5/2}\sqrt{t}}(\frac{3}{2t^2}+\frac{3}{t}+4).$$

Since $f_4(e) = f_0(e)$ *and* $f_3(e) = f_1(e)$, *we obtain*

$$\begin{aligned}
\log T(M,\lambda) &= \frac{1}{2}\frac{d}{ds}(\sum_{j=0}^{4}(-1)^{j+1}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}f_j(e)dt)_{s=0} \\
&= \frac{1}{2}\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}(-2\frac{1}{(4\pi)^{5/2}}\frac{2e^{-4t}}{3t^{3/2}}(1+\frac{3}{2t}) \\
&+ 2\frac{1}{(4\pi)^{5/2}}\frac{8e^{-t}}{3t^{3/2}}(2+\frac{3}{2t}) - \frac{4}{(4\pi)^{5/2}\sqrt{t}}(\frac{3}{2t^2}+\frac{3}{t}+4))dt)_{s=0} \\
&= \frac{1}{96\pi^2}(-\frac{8}{3}(\lambda+4)^{3/2}+\frac{8}{5}(\lambda+4)^{5/2}+\frac{64}{3}(\lambda+1)^{3/2} \\
&- \frac{32}{5}(\lambda+1)^{5/2}+\frac{24}{5}\lambda^{5/2}-24\lambda^{3/2}+48\lambda^{1/2}),
\end{aligned}$$

where we used $\Gamma(-\frac{3}{2}) = \frac{4\sqrt{\pi}}{3}, \Gamma(-\frac{5}{2}) = -\frac{8\sqrt{\pi}}{15}$ *and* $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. *In particular,* $\log T(M,0) = \frac{7}{15\pi^2}$.

Chapter 2

L^2-Analytic Torsion Functions for Semifinite von Neumann Algebras

In [GoP 1] we studied an L^2-analytic torsion for n-tuples of commuting elements in finite von Neumann algebras which is motivated by the work on the L^2-analytic torsion for covering spaces ([CaM], [Lott 3], [LüR]). We now extend the L^2-analytic torsion to semifinite von Neumann algebras. A natural geometric example of using semifinite von Neumann algebras is the homogeneous spaces discussed in Chapter 1. As we know, the Fuglede-Kadison determinant is fundamental for the discussion of the L^2-analytic torsion in finite von Neumann algebras. For the semifinite von Neumann algebras there is a Fredholm type determinant [Brow] which is not good enough for our purpose. We thus introduce a ζ-(regularized) determinant in semifinite von Neumann algebras either by means of s-numbers or by using the usual ζ-regularization furnished by the Mellin transformation of the trace of restricted heat kernels. In section 2.1 we collect some basic properties of this ζ determinant (function). We examine the relation of the ζ-determinant function with the Fredholm type determinant, which is used to derive a mapping property of the L^2-analytic torsion function in section 2.3. In section 2.2 we utilize the ζ-determinant to study the L^2-analytic torsion for n-tuples of commuting elements in semifinite von Neumann algebras. This analytic torsion shares most properties of the torsion in geometry. In particular, we obtain a similarity formula which compares the torsions of two similar n-tuples. There are some conditions initially needed for the definition of the ζ-determinant and torsion which we will remove in section 2.3 by introducing a parameter in the L^2-analytic torsion, as we did in the previous chapter. But this parameter causes some difficulty in the computation of the L^2-analytic torsion as we see in section 2.4 where we use the Selberg trace formula and Zeta function to calculate the ζ-determinant and torsion functions of the Laplace-like operators on compact Riemann surfaces of constant negative curvature. The L^2-analytic torsion function for general Riemann surfaces will be discussed elsewhere.

2.1 ζ-Determinants

Let $\mathcal{A}$ be a semifinite von Neumann algebra acting on a Hilbert space H. This means that there is a faithful semifinite normal trace τ on $\mathcal{A}$. Recall that the s-numbers $S_T(t)$ of $T \in \mathcal{A}$ are defined in [Fac 2] as

$$S_T(t) = \inf\{s \geq 0 : \tau(1 - e_s) \leq t\}$$

for the spectral family $\{e_\lambda\}$ of $|T| = \int \lambda de_\lambda$, where $|T| = (T^*T)^{\frac{1}{2}} \in \mathcal{A}$. As the type I factor case, we are going to use the s-numbers to define the ζ-determinant. Thus for $T \in \mathcal{A}$ let P_T be the orthogonal projection of H onto the kernel $Ker(T)$ of T. Denote $T' = (I - P_T)T(I - P_T) \in \mathcal{A}_T = (I - P_T)\mathcal{A}(I - P_T)$, where $\mathcal{A}_T$ is also a semifinite von Neumann algebra with trace $\tau_T(x) = \tau((I - P_T)x)$ ([Dis], P. 114). Without confusion we do not distinguish τ_T and τ, $\mathcal{A}$ and $\mathcal{A}_T$. Let $L^p(\mathcal{A})$ $(1 \leq p < \infty)$ be the set of all $T \in \mathcal{A}$ such that $\int_0^\infty |S_{T'}(t)|^p dt < \infty$. We use $L_\theta(\mathcal{A})$ to denote the set of all injective selfadjoint elements $T \in \mathcal{A}$ such that $e^{-tT^{-1}} \in \mathcal{A}$ and $\tau(e^{-tT^{-1}}) < \infty$, $\forall t > 0$. Clearly, if $T \in L^p(\mathcal{A})$, then $T' \in L^p(\mathcal{A})$, since $T' \preceq T$ and then $S_{T'}(t) \leq S_T(t)$ by Lemma 4 [BrK]. We define the zeta function $\zeta_{T'}(s)$ for $T \in L^p(\mathcal{A})$ as

$$\zeta_{T'}(s) = \int_0^\infty |S_{T'}(t)|^s dt, \ \ \Re s \geq p.$$

Since $S_{T'}(t) \leq \|T'\|$ and $S_{T'}(t) \to 0$ as $t \to \infty$,

$$|\zeta_{T'}(s)| \leq \int_{S_{T'}(t) \geq 1} |S_{T'}(t)|^{\Re s} dt + \int_{S_{T'}(t) < 1} |S_{T'}(t)|^p dt, \ \ \Re s > p.$$

We see that the integral defining $\zeta_{T'}(s)$ is absolutely convergent and the convergence is uniform for any bounded strip in $\Re s \geq p$. Hence $\zeta_{T'}(s)$ is analytic for $\Re s > p$.

We now want to extend $\zeta_{T'}(s)$ to an analytic function near $s = 0$. Such extendibility of $\zeta_{T'}(s)$ is satisfied by the resolvents of elliptic pseudo-differential operators on compact Riemannian manifolds. To formulate a general sufficient condition regarding this, we first prove the following lemma.

Lemma 2.1 *Let $T \in \mathcal{A}$ be such that $|T'| \in L^p(\mathcal{A})$.*
(1) $|T'| \in L_\theta(\mathcal{A})$ *and*

$$\zeta_{T'}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \tau(e^{-t|T'|^{-1}}) dt, \ \Re s \geq p.$$

(2) *If* $\tau(e^{-|T'|^{-1}}) \sim \sum_{j\geq 0}\alpha_{l_j}t^{l_j}$ *with* $l_j \to \infty$ *as* $j \to \infty$, *namely, for* $\varepsilon > 0$ *there is* $K > 0$ *such that* $|\tau(e^{-t|T'|^{-1}}) - \sum_{j=0}^{K}\alpha_{l_j}t^{l_j}| < O(t^{\varepsilon})$ *as* $t \to 0$, *then* $\zeta^1_{T'}(s) = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\tau(e^{-t|T'|^{-1}})dt$ *can be extended to an analytic function near* $s = 0$.

Proof. (1) Using the spectral representation of $|T'|$, $|T'| = \int \lambda de_\lambda$, we have

$$\begin{aligned}
\tau(|T'|^s) &= \int \lambda^s d\tau(e_\lambda) = \frac{1}{\Gamma(s)}\int(\int_0^\infty \lambda^s u^{s-1}e^{-u}du)d\tau(e_\lambda) \\
&= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}(\int e^{-t\lambda^{-1}}d\tau(e_\lambda)) \\
&= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\tau(e^{-t|T'|^{-1}})dt. \qquad (2.1)
\end{aligned}$$

Since $|T'| \in L^p(\mathcal{A})$, we get $\tau(|T'|^s) < \infty$ for $\Re s \geq p$. In fact, for s real number, $s \geq p$, Propositions 1.6 and 1.11 [Fac 2] show that

$$\tau(|T'|^s) = \int_0^\infty (S_{T'}(t))^s dt < \infty.$$

By the analyticity of $\tau(|T'|^s)$ and $\zeta_{T'}(s)$ for $\Re s > p$, we obtain that $\tau(|T'|^s) = \zeta_{T'}(s)$ for all s with $\Re s \geq p$. It follows from (2.1) that $|T'| \in L_\theta(\mathcal{A})$ and (1) follows.

(2) By part (1),

$$\begin{aligned}
\zeta^1_{T'}(s) &= \frac{1}{\Gamma(s)}(\int_0^\delta + \int_\delta^1)t^{s-1}\tau(e^{-|T'|^{-1}})dt \\
&= \frac{1}{\Gamma(s)}(\int_0^\delta \sum_{j=0}^{K}\alpha_{l_j}t^{l_j+s-1}dt + \int_0^\delta c(t)t^{s-1}dt + \int_\delta^1 t^{s-1}\tau(e^{-t|T'|^{-1}})dt) \\
&= \frac{1}{\Gamma(s)}\sum_{j=0}^{K}\alpha_{l_j}\frac{\delta^{s+l_j}}{l_j+s} + \frac{1}{\Gamma(s)}(\int_0^\delta c(t)t^{s-1}dt + \int_\delta^1 t^{s-1}\tau(e^{-t|T'|^{-1}})dt),
\end{aligned}$$

where $|c(t)| < O(t^\varepsilon)$ for $\varepsilon > 0$. Thus the last two integrals define a function which can be extended to an analytic function near $s = 0$. Clearly, the first term has an analytic extension to a neighborhood of $s=0$. Hence $\zeta^1_{T'}(s)$ can be extended to an analytic function near $s=0$. Q.E.D.

We say $\zeta_{T'}(s)$ is extendible if it can be analytically extended to a neighborhood of $s=0$.

Definition 2.1 (1) Let $T \in \mathcal{A}$ be such that $|T'| \in L^p(\mathcal{A}), 1 \leq p < \infty$. If $\zeta_{T'}(s)$ is extendible, then the ζ-determinant $\Delta_\zeta(T)$ of T is defined as

$$\log \Delta_\zeta(T) = \frac{d}{ds}(\zeta_{T'}(s))_{s=0}.$$

(2) For $T \in \mathcal{A}$ with $|T'| \in L_\theta(\mathcal{A})$, if $\zeta^\infty_{T'}(0) = \int_1^\infty t^{-1}\tau(e^{-t|T'|^{-1}})dt < \infty$ and $\zeta^1_{T'}(s)$ is extendible, then the ζ-determinant $D_\zeta(T)$ of T is defined as

$$\log D_\zeta(T) = \frac{d}{ds}(\zeta^1_{T'}(s))_{s=0} + \int_1^\infty t^{-1}\tau(e^{-t|T'|^{-1}})dt.$$

Let $dom(\Delta_\zeta) = \{T \in \mathcal{A} : |T'| \in L^p(\mathcal{A}) \;\; for \;\; some \;\; 1 \leq p < \infty, \zeta_{T'}(s) \;\; is \;\; extendible\}$ and $dom(D_\zeta) = \{T \in \mathcal{A} : |T'| \in L_\theta(\mathcal{A}), \zeta^\infty_{T'}(0) < \infty \;\; and \;\; \zeta^1_{T'}(s) \;\; is \;\; extendible\}$.

Remark 2.1 (1) *Let $T \in dom(\Delta_\zeta)$ and $\zeta^\infty_{T'}(0) < \infty$. Then $T \in dom(D_\zeta)$ and $\log D_\zeta(T) = \log \Delta_\zeta(T)$. In fact, $\zeta^\infty_{T'}(0) = \frac{d}{ds}(\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\tau(e^{-t|T'|^{-1}})dt)_{s=0}$.*

(2) The determinant $D_\zeta(T)$ is in particular useful when $\tau(e^{-t|T'|^{-1}})$ has a slow decay as $t \to \infty$, since $\zeta_{T'}(s)$ may not be well defined for $\Re s \gg 0$.

(3) If $\mathcal{A} = \mathcal{L}(H)$, the type I_∞-algebra of bounded linear operators on H, the determinant $\Delta_\zeta(T)$ is discussed in [KKW]. See also [Sim]. In this case, if $|T'| \in L^p(\mathcal{A})$ for $1 \leq p < \infty$, then $|T'| \in L_\theta(\mathcal{A})$ and $\zeta^\infty_{T'}(0) < \infty$.

(4) The ζ-determinants $\Delta_\zeta(T)$ and $D_\zeta(T)$ are different from the Fredholm type determinant $\Delta(T)$, which is defined as follows. Let $T \in I + L^1(\mathcal{A})$, i.e.,

$T = I + T_1$ for some $T_1 \in L^1(\mathcal{A})$. Then for $|T| = \int \lambda e_\lambda(|T|)$,

$$\log \Delta(T) = \int \ln \lambda d\tau(e_\lambda(|T|)).$$

$\Delta(T)$ has many nice properties [Brow]. We list only the mapping formula, namely for $T \in L^p(\mathcal{A}), 0 \leq p < \infty$,

$$\log \Delta(f(T)) = \int \ln |f(\lambda)| d\mu_\lambda(T)$$

with f holomorphic near the spectrum $Sp(T)$ of T and $f(0) - 1$ vanishing to order at leat k ($k \geq p$), where $\mu_\lambda(T)$ is the measure associated with T [Brow]. $\mu_\lambda(T)$ is defined by $d\mu_\lambda(T) = d\mu_0(\frac{1}{\lambda})$ and μ_0 is the Riesz measure of harmonic function $u(z) = \log \Delta(g_k(zT))$ near $z = 0$ and $g_k(z) = (1 - z)\exp(z + \ldots + \frac{1}{k-1} z^{k-1}), k \geq p$. We thank Joel Pincus for telling us this measure.

The relation between $\Delta_\zeta(T)$ and $\Delta(T)$ is given by the following.

Proposition 2.1 *Let $T_t : [0,1] \to dom(\Delta_\zeta)$ be a differentiable family such that T_t is injective and $|T_t|^{-1}\frac{d}{dt}(|T_t|) \in L^1(\mathcal{A})$. Then*
(1) $\frac{d}{dt} \log \Delta_\zeta(T_t) = -\tau(|T_t|^{-1}\frac{d}{dt}(|T_t|))$.
(2) $\frac{\Delta_\zeta(T_t)}{\Delta_\zeta(T_0)} = -\Delta(T_0^{-1}T_t)$, provided $\log(|T_0|^{-1}|T_t|) \in L^1(\mathcal{A})$.

Proof. Assume that the spectrum $Sp(|T_t|)$ of $|T_t|$ is contained in $[-\delta, a]$ and Ω is the closed curver $\{z \in \mathbb{C} : dis(z, [-\delta, a]) = \varepsilon\}$ with $\varepsilon > 0$. Then

$$|T_t|^s = -\frac{1}{2\pi i} \int_\Omega \lambda^s (|T_t| - \lambda)^{-1} d\lambda.$$

We obtain by integration by parts

$$\frac{d}{dt} \log \Delta_\zeta(T_t) = \frac{d}{dt}\frac{d}{ds}(\tau(|T_t|^s))_{s=0}$$

$$\begin{aligned}
&= \frac{d}{ds}(\tau(\frac{d}{dt}(|T_t|)\frac{1}{2\pi i}\int_\Omega \lambda^s(|T_t|-\lambda)^{-2}d\lambda))_{s=0} \\
&= \frac{d}{ds}(\tau(\frac{d}{dt}(|T_t|)\frac{1}{2\pi i}\int_\Omega s\lambda^{s-1}(|T_t|-\lambda)^{-1}d\lambda))_{s=0} \\
&= -\tau(|T_t|^{-1}\frac{d}{dt}(|T_t|)).
\end{aligned}$$

(2) Since $|T_t|^{-1}\frac{d}{dt}(|T_t|) \in L^1(\mathcal{A})$, Lemma 1.1 [Brow] implies

$$\frac{d}{dt}\log\Delta(|T_0|^{-1}|T_t|) = \frac{d}{dt}\tau(\log(|T_0|^{-1}|T_t|)) = \tau(|T_t|^{-1}\frac{d}{dt}(|T_t|)).$$

Using part (1), we get

$$\frac{d}{dt}\log\Delta_\zeta(T_t) = -\frac{d}{dt}\log\Delta(|T_0'|^{-1}|T_t'|).$$

This yields

$$\log\frac{\Delta_\zeta(T_t)}{\Delta_\zeta(T_0)} = -\log\frac{\Delta(|T_0'|^{-1}|T_t'|)}{\Delta(I)}.$$

Q.E.D.

We now collect some properties of the ζ-determinants $\Delta_\zeta(T)$ and $D_\zeta(T)$ which are similar to those in the case of $\mathcal{A} = \mathcal{L}(H)$. See ([For], [KKW], [Sim], [Vor]). The following is a perturbation property of the ζ-determinants.

Lemma 2.2 *Let $T \in dom(D_\zeta)$ be injective, $T \geq 0$. Let $T_0 \in \mathcal{A}$ be nonnegative such that $T+T_0$ is injective and nonnegative. Suppose that $\zeta^\infty_{T+T_0}(0) < \infty$ and $T^{-1}T_0T^{-1}(I+T_0T^{-1})^{-1}T^\alpha \in L^1(\mathcal{A})$ for some $0<\alpha<1$. Then*
(1) $T+T_0 \in dom(D_\zeta)$,
(2) $\log D_\zeta(T+T_0) = \log\Delta(I+T_1) + \log D_\zeta(T)$, Provided $T_0 = T_1T$ for some $T_1 \in L^1(\mathcal{A})$.

Proof. (1) By the Duhamel principal,

$$e^{-t(T+T_0)^{-1}} - e^{-tT^{-1}} = -\int_0^t e^{-s(T+T_0)^{-1}}((T+T_0)^{-1}-T^{-1})e^{-(t-s)T^{-1}}ds$$

$$= -\int_0^t e^{-(t-u)(T+T_0)^{-1}}((T+T_0)^{-1} - T^{-1})T^{\alpha}T^{-\alpha}e^{-uT^{-1}}du.$$

This implies for $t > 0$

$$\begin{aligned}\|e^{-t(T+T_0)^{-1}} - e^{-tT^{-1}}\| &\leq \int_0^t \|e^{-(t-u)(T+T_0)^{-1}}\| \|((T+T_0)^{-1} - T^{-1})T^{\alpha}\|_1 \\ &\cdot \|T^{-\alpha}e^{-uT^{-1}}\|du \leq c_1 \int_0^t u^{-\alpha}\|((T+T_0)^{-1} - T^{-1})T^{\alpha}\|_1 du \\ &= \frac{c_1}{1-\alpha}t^{1-\alpha}\|T^{-1}T_0T^{-1}(I+T_0T^{-1})^{-1}T^{\alpha}\|_1 < \infty, \quad (2.2)\end{aligned}$$

where c_1 is a constant. Hence $(T+T_0) \in L_\theta(\mathcal{A})$ and $\zeta^1_{T+T_0}(s)$ is extendible.

(2) By (2.2) we have

$$\log D_\zeta(T+T_0) - \log D_\zeta(T) = \int_0^\infty t^{-1}\tau(e^{-t(T+T_0)^{-1}} - e^{-tT^{-1}})dt.$$

It thus suffices to check that the last integral is $\log \Delta(I + T_1)$. In fact, let

$$\varphi(s) = \int_0^\infty t^{-1}\tau(e^{-t(T+sT_0)^{-1}} - e^{-tT^{-1}})dt, \; s > 0.$$

Then

$$\begin{aligned}\frac{d}{ds}\varphi(s) &= \int_0^\infty \tau((T+sT_0)^{-1}T_0(T+sT_0)^{-1}e^{-t(T+sT_0)^{-1}})dt \\ &= \tau((T+sT_0)^{-1}T_0) = \tau((I+sT_1)^{-1}T_1) \\ &= \frac{d}{ds}\log\Delta(I+sT_1), \quad s > 0.\end{aligned}$$

It follows that $\varphi(1) - \varphi(0) = \log\Delta(I + T_1) - 1$. But $\varphi(0) = 0$. We get the result. Q.E.D.

The basic properties of the ζ- determinants are stated in the following proposition.

Proposition 2.2 (1) *If $T \in dom(\Delta_\zeta)$ is injective and normal, then*

$$\log \Delta_\zeta(T^m) = m \log \Delta_\zeta(T), m \in \mathbb{N}.$$

(2) *If* $T \in dom(\Delta_\zeta)$, *then* $\log \Delta_\zeta(\lambda T) = (\ln|\lambda|)\zeta_{|T'|}(0) + \log \Delta_\zeta(T), \lambda \in \mathbb{C}, \lambda \neq 0$. *Similarly for* $T \in dom(D_\zeta), \log D_\zeta(\lambda T) = \zeta^1_{|T'|}(0)\ln|\lambda| + \log D_\zeta(T)$.

(3) $\log \Delta_\zeta(UTU^*) = \log \Delta_\zeta(T)$ *for* $T \in dom(\Delta_\zeta)$ *and* $U \in \mathcal{A}$ *unitary operator. This is true also for* $T \in dom(D_\zeta)$.

(4) *If* $T \in dom(\Delta_\zeta)$, *then*

$$\log \Delta_\zeta(T^*) = \log \Delta_\zeta(T) = \frac{1}{2}\log \Delta_\zeta(T^*T) = \frac{1}{2}\log \Delta_\zeta(|T|^2) = \frac{1}{2}\log \Delta_\zeta(|T'|^2).$$

(5) $\log D_\zeta(T_1 \oplus T_2) = \log D_\zeta(T_1) + \log D_\zeta(T_2)$ *for* $T_i \in dom(D_\zeta)$.

Proof. (1) Note that $|T'| = |T|$ and $|T^m| = |T|^m$ since T is injective and normal. Using the relation $S_{(T')^m}(t) = (S_{T'}(t))^m$, we have $\zeta_{(T')^m}(s) = \zeta_{T'}(ms)$. This proves part (1).

(2) We have $S_{(\lambda T)'}(t) = |\lambda| S_{T'}(t)$. This implies $\zeta_{(\lambda T)'}(s) = |\lambda|^s \zeta_{T'}(s)$, which proves the first part of (2). For $T \in dom(D_\zeta)$ we get

$$\begin{aligned}
\zeta^1_{(\lambda T)'}(s) &= \frac{1}{\Gamma(s)}|\lambda|^s \int_0^{|\lambda|} u^{s-1}\tau(e^{-u|T'|^{-1}})du, \\
\zeta^\infty_{(\lambda T)'}(0) &= \int_{|\lambda|}^\infty u^{-1}\tau(e^{-u|T'|^{-1}})du.
\end{aligned}$$

Hence,

$$\begin{aligned}
\log D_\zeta(\lambda T) &= (\ln|\lambda|)(\zeta^1_{T'}(s) - \frac{1}{\Gamma(s)}\int_{|\lambda|}^1 u^{s-1}\tau(e^{-u|T'|^{-1}})du)_{s=0} \\
&+ \frac{d}{ds}(\zeta^1_{T'}(s) - \frac{1}{\Gamma(s)}\int_{|\lambda|}^1 u^{s-1}\tau(e^{-u|T'|^{-1}})du)_{s=0} \\
&+ \int_{|\lambda|}^\infty u^{-1}\tau(e^{-u|T'|^{-1}})du = \zeta^1_{T'}(0)\ln|\lambda| + \log D_\zeta(T).
\end{aligned}$$

(3) This part is clear since $S_{UT'U^*}(t) = S_{T'}(t)$ and $\tau(e^{-t|UT'U^*|^{-1}}) = \tau(e^{-t|T'|^{-1}})$.

(4) Since $I - P_T$ is the support of T, we get by Propositions 1.5 and 1.6 [Fac 2] $S_{(T^*)'}(t) = S_{T^*}(t) = S_T(t) = S_{T'}(t)$, It follows that $\zeta_{T'}(t) = \zeta_{(T^*)'}(t)$. Hence $\log \Delta_\zeta(T^*) = \log \Delta_\zeta(T)$. Also by Proposition 1.6 [Fac 2],

$$S_{(T^*T)'}(t) = S_{T^*T}(t) = S_{|T|^2}(t) = (S_{|T|}(t))^2 = (S_T(t))^2 = (S_{T'}(t))^2.$$

Thus, $\zeta_{(T^*T)'}(s) = \zeta_{T'}(2s) = \zeta_{|T|}(2s)$. The result then follows.

(5) This part is trivial since $\tau(e^{-t|(T_1 \oplus T_2)'|^{-1}}) = \tau(e^{-t|T_1'|^{-1}}) + \tau(e^{-t|T_2'|^{-1}})$.

Q.E.D.

Observe that the conditions in part (1) can be dropped out by the proof of part (4).

We now consider the ζ-determinant function to avoid the condition $\zeta_{T'}^{\infty}(0) < \infty$ in the definition of $D_\zeta(T)$. Suppose that $T \in \mathcal{A}$ satisfies

$$\begin{aligned} |T'| &\in L_\theta(\mathcal{A}), && (2.3)\\ \tau(e^{-t|T'|^{-1}}) &\sim \sum_{j=0}^{\infty} \alpha_{l_j} t^{l_j} \textit{ as } t \to 0 \textit{ and } l_j \nearrow \infty \textit{ as } j \to \infty, && (2.4)\\ \tau(e^{-t|T'|^{-1}}) &\sim t^{-\alpha} e^{ta} \textit{ as } t \to \infty \textit{ for some } a \geq 0 \textit{ and } \alpha \in \mathbb{R}. && (2.5) \end{aligned}$$

Conditions (2.3) and (2.4) are satisfied for the resolvents of elliptic differential operators on (the universal coverings of) compact Riemannian manifolds and homogeneous spaces of Lie groups. Condition (2.5) is much weaker than $\zeta_{T'}^{\infty}(0) < \infty$. Define

$$\zeta_{T'}(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} \tau(e^{-t|T'|^{-1}}) dt, \ \ \Re s > -l_0, \ \Re \lambda > a.$$

By (2.4), we obtain for $\delta, \varepsilon > 0$ and $r(t)$ with $r(t) \sim O(t^\varepsilon)$ as $t \to 0$,

$$\zeta_{T'}(s, \lambda) \ = \ \frac{1}{\Gamma(s)} (\sum_{j=0}^{K} \alpha_{l_j} \int_0^\delta t^{l_j + s - 1} e^{-\lambda t} dt + \int_0^\delta t^{s-1} r(t) e^{-\lambda t} dt$$

$$
\begin{aligned}
&+ \int_\delta^\infty t^{s-1}e^{-\lambda t}\tau(e^{-t|T'|^{-1}})dt) \\
&= \sum_{j=0}^{K}\alpha_{l_j}\frac{\Gamma(l_j+s)}{\Gamma(s)}\lambda^{-(s+l_j)} - \sum_{j=0}^{K}\alpha_{l_j}\frac{1}{\Gamma(s)}\int_\delta^\infty t^{l_j+s-1}e^{-\lambda t}dt \\
&+ \frac{1}{\Gamma(s)}(\int_0^\delta t^{s-1}r(t)e^{-\lambda t}dt + \int_\delta^\infty t^{s-1}e^{-\lambda t}\tau(e^{-t|T'|^{-1}})dt). \quad (2.6)
\end{aligned}
$$

Hence, $\zeta_{T'}(s)$ is extendible.

Definition 2.2 Assume that $T \in \mathcal{A}$ satisfies (2.3) – (2.5). The ζ-determinant function $D_\zeta(T, \lambda)$ of T is defined as

$$
\log D_\zeta(T,\lambda) = \frac{d}{ds}(\zeta_{T'}(s,\lambda))_{s=0}, \;\; \Re\lambda > a.
$$

Remark 2.2 (1) *$D_\zeta(T,0) = D_\zeta(T)$ for $T \in dom(D_\zeta)$.*

(2) *$\frac{d}{d\lambda}\log D_\zeta(T,\lambda) = -\zeta_{T'}(1,\lambda)$, $\Re\lambda > a$. In fact, for $\Re\lambda > a$,*

$$
\begin{aligned}
\frac{d}{d\lambda}\log D_\zeta(T,\lambda) &= \frac{d}{ds}(-\frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-\lambda t}\tau(e^{-t|T'|^{-1}})dt)_{s=0} \\
&= \frac{d}{ds}(\frac{-s}{\Gamma(s+1)}\int_0^\infty t^s e^{-\lambda t}\tau(e^{-t|T'|^{-1}})dt)_{s=0} = -\zeta_{T'}(1,\lambda).
\end{aligned}
$$

Here the analyticity of $\log D_\zeta(T,\lambda)$ as a function of λ is guaranteed by (2.5).

Proposition 2.3 *Let $T \in \mathcal{A}$ satisfies (2.3) – (2.5). Then $D_\zeta(T,\lambda)$ has the following expansion as $\lambda \to \infty$,*

$$
\begin{aligned}
\log D_\zeta(T,\lambda) \;\sim\; & \sum_{l_j \neq 0,-1,-2\ldots}(\alpha_{l_j}(\Gamma(l_j)\lambda^{-l_j} - \int_\delta^\infty t^{l_j-1}e^{-\lambda t}dt) \\
& - \sum_{j=0}^{[-l_0]}\alpha_{-j}((\log\lambda - \sum_{i=1}^{j}\frac{1}{i})\frac{(-\lambda)^j}{j!} - \int_\delta^\infty t^{j-1}e^{-\lambda t}dt),
\end{aligned}
$$

where $\alpha_{-j} = \alpha_{l_j}$ for $j = l_j$, and 0 otherwise.

Proof. By (2.6),

$$\begin{aligned}\zeta_{T'}(s,\lambda) &= \sum_{j=0}^{\infty} \alpha_{l_j} (\frac{\Gamma(l_j+s)}{\Gamma(s)} \lambda^{-(s+l_j)} - \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} t^{l_j+s-1} e^{-\lambda t} dt) \\ &+ \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} t^{s-1} e^{-\lambda t} \tau(e^{-t|T'|^{-1}}) dt, \ \ \Re\lambda > a.\end{aligned}$$

For l_j negative we use the identity $\frac{\Gamma(l_j+s)}{\Gamma(s)} = ((s-l_j)\dots(s-1))^{-1}$ and $Res(\Gamma(s),-j) = \frac{(-1)^j}{j!}$. Then

$$\begin{aligned}\frac{d}{ds}(\zeta_{T'}(s,\lambda))_{s=0} &= \sum_{l_j \neq 0,-1,-2\dots} (\alpha_{l_j}(\Gamma(l_j)\lambda^{-l_j} - \int_{\delta}^{\infty} t^{l_j-1} e^{-\lambda t} dt) \\ &+ \sum_{j=0,-1,\dots} \alpha_{l_j}((-\ln\lambda)\lambda^{-l_j} \frac{(-1)^{l_j}}{(l_j)!} + \lambda^{-l_j} \sum_{i=1}^{l_j} \frac{(-1)^{l_j}}{(l_j)! i} \\ &- \int_{\delta}^{\infty} t^{l_j-1} e^{-\lambda t} dt) + \int_{\delta}^{\infty} t^{-1} e^{-\lambda t} \tau(e^{-t|T'|^{-1}}) dt.\end{aligned}$$

Here we used the computation

$$\frac{d}{ds}(\frac{\Gamma(s-j)}{\Gamma(s)})_{s=0} = \sum_{i=1}^{j} \frac{(-1)^j}{j! i}.$$

The result then follows easily. Q.E.D.

We now consider the θ-determinant of T. Assume that $T \in \mathcal{A}$ satisfies $(2.3)-(2.5)$, and $|T'| \in L^p(\mathcal{A})$. Define the θ-determinant $D_\theta(I+\lambda T)$ of $I+\lambda T$ as

$$\log D_\theta(I+\lambda T) = \int_0^\lambda FP(\zeta_{T'}(1,u))du,$$

subject to $\log D_\theta(I+\lambda T)|_{\lambda=0} = 0$. Here the finite part $FP(f(x))$ of a meromorphic function $f(x)$ is defined by

$$FP(f(x)) = \begin{cases} f(x), & x \ is \ not \ \ a \ pole, \\ \lim_{\varepsilon\to 0}(f(x+\varepsilon) - \frac{Res_{u=x} f(u)}{\varepsilon}), & x \ \ is \ \ a \ \ pole.\end{cases}$$

Proposition 2.4 *Assume that* $T \in \mathcal{A}$ *satisfies* (2.3) – (2.5) *and* $|T'| \in L^p(\mathcal{A})$, $1 \le p < \infty$. *Then*

(1) $\log D_\theta(I + \lambda T) = -\log D_\zeta(T, \lambda) + \log D_\zeta(T) + c\sum_{j=0}^{[-l_0]} \alpha_{-j}\frac{(-\lambda)^j}{j!} - c\alpha_{-0}$ *with the Euler constant* c.

(2) *for* $p = 1$,

$$\frac{d}{d\lambda}\log D_\theta(I + \lambda T) = \frac{d}{d\lambda}\log \Delta(I + \lambda|T'|).$$

Proof. Using the identity $\frac{d}{d\lambda}(\Gamma(s)\zeta_{T'}(s,\lambda)) = -\Gamma(s+1)\zeta_{T'}(s+1,\lambda)$, we have

$$\begin{aligned}\log D_\theta(I + \lambda T) &= \int_0^\lambda FP(-\frac{d}{du}(\Gamma(s)\zeta_{T'}(s,u))_{s=0})du \\ &= FP(\Gamma(s)\zeta_{T'}(s,0))_{s=0} - FP(\Gamma(s)\zeta_{T'}(s,\lambda))_{s=0}.\end{aligned}$$

As shown in [Vor],

$$\begin{aligned}FP(\Gamma(s)\zeta_{T'}(s,\lambda))_{s=0} &= \zeta'_{T'}(0,\lambda) - c\zeta_{T'}(0,\lambda) \\ &= \log D_\zeta(T,\lambda) - c\sum_{j=0}^{[-l_0]} \alpha_{-j}\frac{(-1)^j}{j!}\end{aligned}$$

Combining these tow identities together, we get part (1).

(2) Clearly, $\Gamma(s)\zeta_{T'}(s,\lambda)$ is analytic near $s = 1$.

$$\begin{aligned}\frac{d}{d\lambda}\log D_\theta(I + \lambda T) &= FP(\zeta_{T'}(1,\lambda)) = \int_0^\infty e^{-\lambda t}\tau(e^{-t|T'|^{-1}})dt \\ &= \int(\lambda + u)^{-1}d\tau(e_u(|T'|^{-1})) = \tau((\lambda + |T'|^{-1})^{-1}) \\ &= \frac{d}{d\lambda}\tau(\log(I + \lambda|T'|)).\end{aligned}$$

Part (2) then follows. Q.E.D.

One refers to ([MoS], [Vos]) for the similar formulas of Propositions 2.3 and 2.4. Part (1) of Proposition 2.4 provides a link between the ζ-determinant function and the Fredholm type determinant which will be used in section 2.3.

2.2 L^2-Analytic Torsions

We now apply the ζ-determinants to the n-tuple $T$of commuting elements in $\mathcal{A}$. To guarantee the finiteness of $\log D_\zeta$ for the Laplace operators associated with the n-tuple T, we introduce spectral invariants. We then focus on the L^2-analytic torsion of commuting n-tuples.

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting elements in $\mathcal{A}$. This means that $T_i T_j = T_j T_i$ and $T_i \in \mathcal{A}$, $i = 1, \ldots, n$. Denote by $\sigma = (\sigma_1, \ldots, \sigma_n)$ the n-indeterminants. Let $\wedge^j[\sigma]$ be the exterior space generated by $\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_j}, 1 \leq i_1 < \ldots < i_j \leq n$. Let $C_j(T) = H \otimes \wedge^j[\sigma]$, $j = 0, \ldots, n$. We can associate a Koszul complex $\{C_*(T), d_*(T)\}$ with T by defining

$$d_j : C_j(T) \to C_{j+1}(T),\ d_j(T) = \sum_{j=1}^{n} T_j S_j,$$

where $S_j : \wedge^j[\sigma] \to \wedge^{j+1}[\sigma]$ is given by $S_j(\xi) = \sigma_j \wedge \xi,\ \xi \in \wedge^j[\sigma], j = 1, \ldots, n$. Since T is a commuting tuple and $S_i S_j + S_j S_i = 0$, $d_{j+1}(T) d_j(T) = 0$. Note that the adjoint S_j^* of S_j is given by

$$S_j^*(\xi_1 + \sigma_j \wedge \xi_2) = \xi_2,\ \sigma_j \wedge \xi_l \neq 0,\quad l = 1, 2$$

and $S_i S_j^* + S_j^* S_i = 1$ for $i = j$, and 0 for $i \neq j$. We define the Laplace operators $\Delta_j(T)$ associated with T as

$$\Delta_j(T) = d_j^*(T) d_j(T) + d_{j-1}(T) d_{j-1}^*(T),\quad j = 0, \ldots, n,$$

with convention $d_{-1}(T) = 0 = d_n(T)$. Since $\Delta_j(T)$ can be expressed as a matrix of elements in $\mathcal{A}$, we see $\Delta_j(T) \in \mathcal{A} \otimes M_{m_j} \stackrel{\text{def}}{=} \mathcal{A}_{m_j}$, where M_{m_j} is the algebra of all $m_j \times m_j$ matrices over $\mathbb{C}$ and m_j is the dimension of

$\wedge^j[\sigma]$. There is a faithful normal semifinite trace $\tau \otimes Tr$ on $\mathcal{A}_{m_j}$, denoted also by τ without confusion. We will thus not distinguish $\mathcal{A}$ and $\mathcal{A}_{m_j}$. Let $\Delta_j'(T) = (I-P_j)\Delta_j(T)(I-P_j)$ for the projection P_j of $C_j(T)$ onto $Ker\Delta_j(T)$. $\Delta_j'(T)$ is injective. $(\Delta_j'(T))^{-1}$ may be unbounded. But it affiliates with $\mathcal{A}$, i.e.,$S(\Delta_j'(T))^{-1} \subset (\Delta_j'(T))^{-1}S$ for $S \in \mathcal{A}'$, the commutant of $\mathcal{A}$. Clearly, the spectrum $Sp((\Delta_j'(T))^{-1})$of $(\Delta_j'(T))^{-1}$ is contained in the interval $[\|\Delta_j'(T)\|^{-1}, \infty)$ for $\Delta_j'(T) \neq 0$. Let

$$(\Delta_j'(T))^{-1} = \int_{\alpha_1}^{\alpha_2} \lambda de_\lambda(T)$$

be the spectral representation of $(\Delta_j'(T))^{-1}$, where $\alpha_1 = \|\Delta_j'(T)\|^{-1}$ and $\alpha_2 = \infty$ if $0 \in Sp(\Delta_j'(T))$, $\alpha_2 < \infty$ if $0 \notin Sp(\Delta_j'(T))$. We have $e_\lambda(T) \in \mathcal{A}$. Then for $\alpha_2 < \infty$ and $\tau(I) = \infty$,

$$\tau(e^{-t(\Delta_j'(T))^{-1}}) = \int_{\alpha_1}^{\alpha_2} e^{-t\lambda} d\tau(e_\lambda(T)) \geq e^{-t\alpha_2} \int_{\alpha_1}^{\alpha_2} d\tau(e_\lambda(T)) = \infty.$$

Thus throughout this section we assume $\alpha_2 = \infty$ for $\tau(I) = \infty$. Let

$$\begin{aligned}
\alpha_{j,\infty}(T) &= \sup\{\beta \geq 0 : \tau(e^{-t(\Delta_j'(T))^{-1}}) \sim O(t^{-\beta/2}),\ t \to \infty\}, \\
\Theta_{j,0}(T) &= \sup\{|\beta| : \tau(e^{-t(\Delta_j'(T))^{-1}}) \sim O(t^{\beta/2}),\ t \to 0\}, \\
\Theta_{j,\infty}(T) &= \sup\{\beta \geq 0 : \tau(e_t(T)) \sim O(t^{\beta/2}),\ t \to \infty\}.
\end{aligned}$$

By the Karamata Tauberian theorem [Shu], we have $\Theta_{j,0}(T) = \Theta_{j,\infty}(T)$ provided $\Theta_{j,0}(T) < \infty$. Assume $\Theta_{j,\infty}(T) < \infty$,

$$\begin{aligned}
\tau(e^{-t(\Delta_j'(T))^{-1}}) &= \int_{\alpha_1}^{\infty} e^{-t\lambda} d\tau(e_\lambda(T)) \leq -e^{-t\alpha_1}\tau(e_{\alpha_1}(T)) + t\int_{\alpha_1}^{K} e^{-t\lambda}\tau(e_\lambda(T))d\lambda \\
&+ t^{-\beta/2}\int_{tK}^{\infty} e^{-u} a_1 u^{\beta/2} du \leq a_2 t^{-\beta/2},\ t > 0
\end{aligned}$$

for some constants a_i. This implies $\Theta_{j,0}(T) \leq \beta/2$ and $\alpha_{j,\infty}(T) \geq \beta/2$. Hence $\Theta_{j,0}(T) = \Theta_{j,\infty}(T)$.

We point out the convention that $\alpha_{j,\infty}(T) = 0$ and $\Theta_{j,0}(T) = \infty$ if there are no β satisfying the condition in the definition. $\alpha_{j,\infty}(T)$ are the invariants introduced by Novikov and Shubin [NoS] for the Laplace operators on smooth manifolds. They are homotopy invariants. We will be concerned with $\Theta_{j,\infty}(T)$ for the n-tuple T in contrast to the Laplace operators on manifolds for which $\Theta_{j,\infty}(T)$ does not contain much information.

We call $\{\Theta_{j,\infty}(T)\}_{j=0}^{n}$ the spectral invariants of T. The important fact about the spectral invariants is that if $0 < \Theta_{j,\infty}(T) < \infty$, then $\zeta^{\infty}_{\Delta'_j(T)}(0) < \infty$, and $\Delta'_j(T) \in dom(D_\zeta), j = 0, 1, \ldots, n$. The following is a basic property of the spectral invariants.

Proposition 2.5 *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting elements in $\mathcal{A}$ such that $\Delta'_j(T) \in L_\theta(\mathcal{A}), j = 0, 1, \ldots$.*

(1) $\Theta_{j,0}(zT) = \Theta_{j,0}(T), \quad z \in \mathbb{C}, z \neq 0.$

(2) $\Theta_{n-j,0}(T^*) = \Theta_{j,0}(T)$.

(3) $\{\Theta_{j,0}(T)\}$ *are unitary invariants, i.e.,* $\Theta_{j,0}(U^*TU) = \Theta_{j,0}(T)$ *for unitary element* $U \in \mathcal{A}$.

Proof. (1) We have $\Delta_j(zT) = |z|^2\Delta_j(T)$ and $\Delta'_j(zT) = |z|^2\Delta'_j(T)$. This implies $\tau(e^{-t(\Delta'_j(zT))^{-1}}) = \tau(e^{-|z|^{-2}t(\Delta'_j(T))^{-1}})$. Part (1) is clear.

(2) Let $\sharp : C_*(T) \to C_*(T)$ be the unitary map defined by

$$\sharp(\xi) = i^{n(n-1)/2} \sum_{j_1 < \ldots < j_p} S^*_{j_1} \ldots S^*_{j_p} S_1 \ldots S_n x_{j_1 \ldots j_p} \in C_{n-p}(T),$$

for $\xi = \sum_{j_1 < \ldots < j_p} S_{j_1} \ldots S_{j_p} x_{j_1 \ldots j_p} \in C_p(T)$ [Vas]. We get $\Delta_j(T) = \sharp^*\Delta_{n-j}(T^*)\sharp$ and $\Delta'_j(T) = \sharp^*\Delta'_{n-j}(T^*)\sharp$. It follows that $\tau(e^{-t(\Delta'_j(T))^{-1}}) = \tau(e^{-t(\sharp^*\Delta'_{n-j}(T)\sharp)^{-1}}) =$

$\tau(e^{-t(\Delta'_{n-j}(T^*))^{-1}})$. This proves part (3). Part (4) also follows from this argument. Q.E.D.

It would be interesting to investigate further these spectral invariants under similarity.

We now use the ζ-determinant to study the L^2-analytic torsion of commuting n-tuples. Let $D(\mathcal{A})$ be the set of all n-tuples of commuting elements in $\mathcal{A}$ such that $\Delta'_j(T) \in dom(D_\zeta)$. As we remarked before, $\Delta'_j(T) \in dom(D_\zeta)$ provided $0 < \Theta_{j,0}(T) < \infty$.

Definition 2.3 Let $T \in D(\mathcal{A})$. The L^2-analytic torsion $\tau_{\mathcal{A}}(T)$ of T is defined as

$$\log \tau_{\mathcal{A}}(T) = \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j \log D_\zeta(\Delta'_j(T)).$$

It is easy to see by Proposition 2.2 that $\tau_{\mathcal{A}}(T)$ is a unitary invariant, namely, $\tau_{\mathcal{A}}(U^*TU) = \tau_{\mathcal{A}}(T)$ for $U \in \mathcal{A}$ unitary.

Proposition 2.6 *Let* $T \in D(\mathcal{A})$.

(1) $\log \tau_{\mathcal{A}}(T) = \sum_{j=0}^{n}(-1)^{j+1} j \zeta^1_{\Delta'_j(T)}(0) \ln|z| + \log \tau_{\mathcal{A}}(T), z \in \mathbb{C}, z \neq 0$.

(2) $\log \tau_{\mathcal{A}}(T) = (-1)^{n+1} \log \tau_{\mathcal{A}}(T^*)$.

(3) *If* $T^{(1)}$ *and* $T^{(2)}$ *are in* $D(\mathcal{A})$, *then* $\log \tau_{\mathcal{A}}(T^{(1)} \oplus T^{(2)}) = \log \tau_{\mathcal{A}}(T^{(1)}) + \log \tau_{\mathcal{A}}(T^{(2)})$.

Proof. (1) follows from Proposition 2.2.

(2) Since $\Delta'_j(T) = \sharp^* \Delta'_{n-j}(T^*) \sharp$, $\tau(e^{-t(\Delta'_j(T))^{-1}}) = \tau(e^{-t(\Delta'_{n-j}(T^*))^{-1}})$. We have $T^* \in D(\mathcal{A})$. Furthermore,

$$\log \tau_{\mathcal{A}}(T) = \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j \log D_\zeta(\Delta'_{n-j}(T^*))$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+2+n}(j\log D_\zeta(\Delta_j'(T^*)) - n\log D_\zeta(\Delta_j'(T^*))) \\
&= (-1)^{n+1}\log\tau_{\mathcal{A}}(T^*),
\end{aligned}
$$

since $\sum_{j=0}^{n}(-1)^j \log D_\zeta(\Delta_j'(T^*)) = 0$ [GoP 1].

(3) We have $\Delta_j'(T^{(1)}\oplus T^{(2)}) = \Delta_j'(T^{(1)})\oplus\Delta_j'(T^{(2)})$. Part (3) follows from Proposition 2.2. Q.E.D.

The following is the similarity property of the L^2-analytic torsion.

Theorem 2.1 *Let $T^{(k)} \in D(\mathcal{A})$ be similar, $k = 1, 2$, i.e., there is an invertible $U \in \mathcal{A}$ such that $T^{(2)} = UT^{(1)}U^{-1}$. Suppose that the L^2 -cohomology of the Koszul complex $\{C_*(T^{(1)}), d_*(T^{(1)})\}$ is zero. If $0 \le \Theta_{j,0}(T^{(2)}) \le \infty$, then*

$$
\log\tau_{\mathcal{A}}(T^{(2)}) = \log\tau_{\mathcal{A}}(T^{(1)}) + \Big(\sum_{j=0}^{n}\frac{(-1)^{j+1}}{2\Gamma(s)}\int_0^1 dt t^{s-1}\int_0^1 du\,\tau(\nu_u e^{-t\Delta_j(T_u^{(2)})^{-1}})\Big)_{s=0} \tag{2.7}
$$

where $\nu_u = UU^* - I$ for $u = 0$, $I - (U^*)^{-1}U^{-1}$ for $u = 1$, and $\frac{1}{u}(UU^* + \frac{1-u}{u})^{-1}(UU^* - I)$ for $0 < u < 1$. $T_u^{(2)} = T^{(2)}$ on Hilbert space H with inner product $<\cdot,\cdot>_u = u<\cdot,\cdot> + (1-u)<U^{-1}\cdot, U^{-1}\cdot>$.

Proof. We will prove (2.7) by endowing H with a new equivalent inner product $<\cdot,\cdot>_u$ which does not effect the structure of $\mathcal{A}$.

First of all, with respect to the new equivalent inner product $<\cdot,\cdot>_n = <U^{-1}\cdot, U^{-1}\cdot>$ on H, U is a unitary operator. Thus $T^{(1)}$ on H is unitarily equivalent to $\tilde{T}^{(2)} = UT^{(1)}U^{-1}$ on H with $<\cdot,\cdot>_n$. This implies that $\Delta_j'(T^{(1)})$ and $\Delta_j'(\tilde{T}^{(2)})$ are unitarily equivalent. Hence by Proposition 2.2, $\log\tau_{\mathcal{A}}(T^{(1)}) = \log\tau_{\mathcal{A}}(\tilde{T}^{(2)})$. It is therefore enough to prove (2.7) for $T^{(2)}$ and $\tilde{T}^{(2)}$, namely we need to find the relation between two L^2-analytic torsions of one n-tuple with respect to two equivalent inner products on H.

Let $i_u : (H, < \cdot, \cdot >) \to (H, < \cdot, \cdot >_u)$ be the identity map. We have $(T_u^{(2)})^* = (i_u^*)^{-1}(T^{(2)})^* i_u^*$ and $\Delta_j(T_u^{(2)}) = (i_u^*)^{-1} d_j^*(T^{(2)}) i_u^* d_j(T^{(2)}) + d_{j-1}(T^{(2)})(i_u^*)^{-1}$ $d_{j-1}^*(T^{(2)}) i_u^*$. Using the notation $\nu_u = (i_u^*)^{-1}\frac{d}{du}(i_u^*)$, we get $\frac{d}{du}((i_u^*)^{-1}) = -\nu_u (i_u^*)^{-1}$ and

$$\begin{aligned}
\frac{d}{du}(\Delta_j(T_u^{(2)})) &= -\nu_u d_j^*(T_u^{(2)}) d_j(T_u^{(2)}) + d_j^*(T_u^{(2)}) \nu_u d_j(T_u^{(2)}) \\
&- d_{j-1}(T_u^{(2)}) \nu_u d_{j-1}^*(T_u^{(2)}) + d_{j-1}(T_u^{(2)}) d_{j-1}^*(T_u^{(2)}) \nu_u .
\end{aligned}$$

Note that $d_j(T_u^{(2)})\Delta_j(T_u^{(2)}) = \Delta_{j+1}(T_u^{(2)}) d_j(T_u^{(2)})$ and $d_{j-1}^*(T_u^{(2)})\Delta_j(T_u^{(2)}) = \Delta_{j-1}(T_u^{(2)}) d_{j-1}^*(T_u^{(2)})$. we obtain

$$\begin{aligned}
\tau(d_j^* \nu_u d_j \Delta_j^{-2} e^{-t\Delta_j^{-1}}) &= \tau(\nu_u d_j \Delta_j^{-2} e^{-t\Delta_j^{-1}} d_j^*) = \tau(\nu_u d_j \Delta_j^{-2} d_j^* e^{-t\Delta_{j+1}^{-1}}) \\
&= \tau(\nu_u d_j d_j^* \Delta_{j+1}^{-2} e^{-t\Delta_{j+1}^{-1}}), \\
\tau(d_{j-1} \nu_u d_{j-1}^* \Delta_j^{-2} e^{-t\Delta_j^{-1}}) &= \tau(\nu_u d_{j-1}^* d_{j-1} \Delta_{j-1}^{-2} e^{-t\Delta_{j-1}^{-1}}), \\
\tau(d_{j-1} d_{j-1\circ}^* \nu_u \Delta_j^{-2} e^{-t\Delta_j^{-1}}) &= \tau(\nu_u d_{j-1} d_{j-1}^* \Delta_j^{-2} e^{-t\Delta_j^{-1}}).
\end{aligned}$$

Here and below, d_j, d_j^* and Δ_j stand for $d_j(T^{(2)}), d_j^*(T^{(2)})$ and $\Delta_j(T_u^{(2)})$. Therefore,

$$\begin{aligned}
\frac{d}{du}\tau(e^{-t\Delta_j(T_u^{(2)})^{-1}}) &= t\tau(\frac{d}{du}(\Delta_j(T_u^{(2)}))\Delta_j(T_u^{(2)})^{-2} e^{-t\Delta_j(T_u^{(2)})^{-1}}) \\
&= t(-\tau(\nu_u d_j^* d_j \Delta_j^{-2} e^{-t\Delta_j^{-1}}) + \tau(\nu_u d_j d_j^* \Delta_{j+1}^{-2} e^{-t\Delta_{j+1}^{-1}}) \\
&- \tau(\nu_u d_{j-1}^* d_{j-1} \Delta_{j-1}^{-2} e^{-t\Delta_{j-1}^{-1}}) + \tau(\nu_u d_{j-1} d_{j-1}^* \Delta_j^{-2} e^{-t\Delta_j^{-1}})).
\end{aligned}$$

As a consequence, we get

$$\begin{aligned}
\frac{d}{du}\log \tau_{\mathcal{A}}(T_u^{(2)}) &= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j (\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\frac{d}{du}\tau(e^{-t(\Delta_j(T_u^{(2)}))^{-1}})dt)_{s=0} \\
&+ \int_1^\infty t^{-1}\frac{d}{du}\tau(e^{-t(\Delta_j(T_u^{(2)}))^{-1}})dt)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}(\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^1 t^s\tau(\nu_u\Delta_j^{-1}e^{-t(\Delta_j)^{-1}}))_{s=0} \\
&+ \int_1^\infty \tau(\nu_u\Delta_j^{-1}e^{-t(\Delta_j)^{-1}})dt) \\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j}(\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^1 t^s\frac{d}{dt}\tau(\nu_u e^{-t(\Delta_j)^{-1}})dt)_{s=0} \\
&+ \int_1^\infty \frac{d}{dt}\tau(\nu_u e^{-t(\Delta_j)^{-1}})dt) \\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j}(-\frac{d}{ds}(\frac{s}{\Gamma(s)}\int_0^1 t^{s-1}\tau(\nu_u e^{-t(\Delta_j)^{-1}})dt)_{s=0} \\
&+ \frac{d}{ds}(\frac{1}{\Gamma(s)}\tau(\nu_u e^{-(\Delta_j)^{-1}}))_{s=0} - \tau(\nu_u e^{-(\Delta_j)^{-1}})) \\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}\frac{d}{ds}(\frac{s}{\Gamma(s)}\int_0^1 t^{s-1}\tau(\nu_u e^{-t(\Delta_j)^{-1}})dt)_{s=0} \\
&= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\tau(\nu_u e^{-t(\Delta_j)^{-1}})dt)_{s=0}.
\end{aligned}$$

Here we used the facts that $\lim_{t\to\infty}\tau(\nu_u e^{-t(\Delta_j)^{-1}}) = 0$ and $\lim_{t\to\infty} t^s\tau(\nu_u e^{-t(\Delta_j)^{-1}}) = 0$, which follow from the assumption. Hence We prove (2.7). The formula for ν_u follows from $i_u^* = u + (1-u)(U^*)^{-1}U^{-1}$. Q.E.D.

The following result concerns the vanishing of the L^2-analytic torsion.

Proposition 2.7 *Let $T \in D(\mathcal{A})$ be such that $[T_i^*, T_j] = 0, i, j = 1, \ldots, n$. Then for $n > 1$, $\log\tau_{\mathcal{A}}(T) = 0$.*

Proof. The assumption implies that $\Delta_j(T) = \sum_{i=1}^n T_iT_i^*, j = 0, \ldots, n$. More precisely, since $dimC_j(T) = m_j, \Delta_j(T) = \oplus_{l=1}^{m_j}(\sum_{i=1}^n T_iT_i^*)$ on $C_j(T)$. Using Proposition 2.2, we have for $n > 1$,

$$\log D_\zeta(\Delta_j'(T)) = m_j \log D_\zeta((\sum_{i=1}^n T_iT_i^*)'),$$

and then

$$\log\tau_{\mathcal{A}}(T) = \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j \log D_\zeta(\Delta_j'(T))$$

$$= \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j m_j \log D_\zeta((\sum_{i=1}^{n} T_i T_i^*)') = 0,$$

since $m_j = \binom{n}{j}$ and $\sum_{j=0}^{n}(-1)^{j+1} j m_j = 0$ for $n > 1$. Q.E.D.

2.3 L^2-Analytic Torsion Functions

In the previous section we used the ζ-determinant to discuss the L^2-analytic torsion $\tau_{\mathcal{A}}(T)$ for n-tuples T of commuting elements in $\mathcal{A}$. As we noted before, we require $\Delta_j'(T)$ to satisfy $\zeta^{\infty}_{\Delta_j'(T)}(0) < \infty$. We now utilize the ζ-determinant function to avoid this assumption and then define the L^2-analytic torsion functions.

Let $D(\mathcal{A}(\lambda))$ be the set of all n-tuples T of commuting elements in $\mathcal{A}$ such that $\Delta_j'(T)$ satisfies (2.3) – (2.5).

Definition 2.4 Let $\in D(\mathcal{A}(\lambda))$. The L^2-analytic torsion function $\tau_{\mathcal{A}}(T, \lambda)$ of T is defined by

$$\log \tau_{\mathcal{A}}(T, \lambda) = \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1} j \log D_\zeta(\Delta_j'(T), \lambda), \quad \Re\lambda > a,$$

where $a = \max_{0 \le j \le n}\{a_j\}$ and a_j are the constants in (2.5) corresponding to $\Delta_j'(T)$.

Since $D_\zeta(\Delta_j'(T), 0) = D_\zeta(\Delta_j'(T))$ for $T \in D(\mathcal{A})$, we see $\tau_{\mathcal{A}}(T, 0) = \tau_{\mathcal{A}}(T)$. Thus $\tau_{\mathcal{A}}(T, \lambda)$ is a natural generalization of $\tau_{\mathcal{A}}(T)$. Indeed, $\tau_{\mathcal{A}}(T, \lambda)$ shares most properties of $\tau_{\mathcal{A}}(T)$.

Proposition 2.8 *Let $T \in D(\mathcal{A}(\lambda))$, $z \in \mathbb{C}, z \neq 0$.*

(1) $\log \tau_{\mathcal{A}}(zT, \lambda) = \log \tau_{\mathcal{A}}(T, |z|^2\lambda) + \sum_{j=0}^{n}(-1)^{j+1} j \zeta_{\Delta_j'(T)}(0, |z|^2\lambda) \ln |z|$.

(2) $\log \tau_{\mathcal{A}}(T,\lambda) = (-1)^{n+1}\log \tau_{\mathcal{A}}(T^*)$.

(3) $\log \tau_{\mathcal{A}}(T^{(1)}\oplus T^{(2)},\lambda) = \log \tau_{\mathcal{A}}(T^{(1)},\lambda) + \log \tau_{\mathcal{A}}(T^{(2)},\lambda)$ *for* $T^{(i)} \in D(\mathcal{A}(\lambda))$.

Proof. (1) Since $\Delta'_j(zT) = |z|^2\Delta'_j(T)$,

$$\begin{aligned}\zeta_{\Delta'_j(zT)}(s,\lambda) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\lambda t}\tau(e^{-t(\Delta'_j(zT))^{-1}})dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty |z|^{2s}t^{s-1}e^{-\lambda|z|^2t}\tau(e^{-t(\Delta'_j(T))^{-1}})dt \\ &= |z|^{2s}\zeta_{\Delta'_j(T)}(s,\lambda|z|^2).\end{aligned}$$

Hence,

$$\begin{aligned}\log D_\zeta(\Delta'_j(zT),\lambda) &= \frac{d}{ds}(\zeta_{\Delta'_j(zT)}(s,\lambda))_{s=0} = \frac{d}{ds}(|z|^{2s}\zeta_{\Delta'_j(T)}(s,\lambda|z|^2))_{s=0} \\ &= \zeta_{\Delta'_j(T)}(0,\lambda|z|^2)\ln|z|^2 + \log D_\zeta(\Delta'_j(T),\lambda|z|^2).\end{aligned}$$

This verifies part (1).

Parts (2) and (3) follows from the proof of Proposition 2.6. Q.E.D.

Proposition 2.9 *Let* $T^{(i)} \in D(\mathcal{A}(\lambda)), i = 1,2$. *If* $T^{(2)} = UT^{(1)}U^{-1}$ *for some invertible* $U \in \mathcal{A}$ *and the cohomology of* $\{C_*(T^{(1)}), d_*(T^{(1)})\}$ *is trivial, then*

$$\begin{aligned}\log \tau_{\mathcal{A}}(T^{(2)},\lambda) &= \log \tau_{\mathcal{A}}(T^{(1)},\lambda) + \frac{1}{2}\sum_{j=0}^n(-1)^j(\int_0^1 du\int_0^\infty \lambda e^{-\lambda t}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}})dt \\ &- (\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-\lambda t}\int_0^1 du\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}}))_{s=0}).\end{aligned}$$

Proof. The notations are the same as in Theorem 2.1. Using the integration by parts, we have

$$\begin{aligned}\frac{d}{du}\tau_{\mathcal{A}}(T_u^{(2)},\lambda) &= \frac{1}{2}\sum_{j=0}^n(-1)^j\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty dt t^s e^{-\lambda t}\frac{d}{dt}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}}))_{s=0} \\ &= \frac{1}{2}\sum_{j=0}^n(-1)^{j+1}(\frac{d}{ds}(\frac{s}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-\lambda t}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}}))_{s=0}\end{aligned}$$

$$
\begin{aligned}
&- \quad (\frac{1}{\Gamma(s)}\lambda \int_0^\infty dt t^s e^{-\lambda t}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}}))_{s=0}) \\
&= \quad \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}((\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} e^{-\lambda t}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}}))_{s=0} \\
&- \quad \int_0^\infty dt \lambda e^{-\lambda t}\tau(\nu_u e^{-t(\Delta_j(T_u^{(2)}))^{-1}})).
\end{aligned}
$$

Taking the integral on both sides of this identity, we obtain the result. Q.E.D.

We now use the measure defined by L. Brown [Brow] and Proposition 2.4 to obtain a mapping formula of the L^2-analytic torsion function.

Theorem 2.2 *Let $T \in L^p(\mathcal{A}), 1 \le p < \infty$, and f be an analytic function near the spectrum $Sp(T)$ of T such that $f(T)$ is injective and nonnegative. If f vanishes at the zero to order at least $p/2$ and $f(T) \in D(\mathcal{A}(\lambda))$, then*

$$
\begin{aligned}
\log \tau_{\mathcal{A}}(f(T),\lambda) \quad &= \quad \frac{1}{2}\log D_\zeta(f(T)^2) - \frac{1}{2}\int_{Sp(T)} \log(1+\lambda f(z)^2)d\mu(z) \\
&+ \quad \frac{1}{2}(c\sum_{j=1}^{[-l_0]}\alpha_{-j}(f)\frac{(-\lambda)^j}{j!} - c\alpha_0(f)), \qquad (2.8)
\end{aligned}
$$

where $\mu(z)$ is the measure introduced by Brown [Brow] and $\alpha_{-j}(f)$ are the coefficients in (2.4) corresponding to $f(T)^2$.

Proof. Since $f(T)$ is injective and nonnegative, Proposition 2.4 implies

$$
\begin{aligned}
\log \tau_{\mathcal{A}}(f(T),\lambda) \quad &= \quad \frac{1}{2}\log D_\zeta(f(T)f(T)^*,\lambda) = \frac{1}{2}(\log D_\zeta(f(T)^2) \\
&- \quad \log \Delta(I+\lambda f^2(T)) + c\sum_{j=1}^{[-l_0]}\alpha_{-j}(f)\frac{(-\lambda)^j}{j!} - c\alpha_0(f)).
\end{aligned}
$$

In view of the formula [Brow],

$$
\log \Delta(I+\lambda f^2(T)) = \int_{Sp(T)} \log(1+\lambda f^2(z))d\mu(z).
$$

We get (2.8). Q.E.D.

Let us remark that if all $\alpha_{l_j}(f)$ are not equal to negative integers, then (2.8) reduces to

$$\log \tau_{\mathcal{A}}(f(T),\lambda) = \frac{1}{2}\log D_\zeta(f(T)^2) - \frac{1}{2}\int_{Sp(T)} \log(1+\lambda f(z)^2)d\mu(z).$$

2.4 Determinant Functions on Riemann Surfaces

In this section we will compute the ζ-determinant and L^2-analytic torsion functions for Laplace-like operators on a compact Riemann surface M of constant curvature -1. Let X^n be the space of tensors $\{f(z)(dz)^n\}$ on M for n integers or half integers. For $n = \frac{1}{2}$ we can consider $X^{1/2}$ as the space of spinors on M. The covariant derivative ∇ from X^n to $X^n \otimes (X^1 \oplus \bar{X}^1)$ can be decomposed as $\nabla = \nabla_z^n \oplus \nabla_n^z$, where $\nabla_z^n : X^n \to X^{n+1}$ and $\nabla_n^z : X^n \to X^{n-1}$ are defined by $\nabla_z^n = \rho^n \partial \rho^{-n}$ and $\nabla_n^z = \rho^{-1}\bar{\partial}$, with $\rho(z)$ the coefficient of metric $ds^2 = \rho(z)|dz|^2$ of M in terms of local conformal variable z. Using the metric ds^2, we endow a Hilbert space structure on X^n. Then $(\nabla_z^n)^* = -\nabla_n^z$ and the Laplace operators on X^n are $\Delta_n^+ = -\nabla_n^z \nabla_z^n$ and $\Delta_n^- = -\nabla_z^n \nabla_n^z$. In particular, $\Delta_0^\pm$ is the Laplacian on functions and $\Delta_{1/2}^-$ is the Laplacian associated with the Dirac operator $\nabla_{1/2}^z$. We can use the uniformization theorem to realize X^n and $\Delta_n^\pm$ more concretely. In fact, let $\mathbb{H}^2$ be the upper half plane with the metric $ds^2 = y^{-2}dzd\bar{z}$. Then $M = \mathbb{H}^2/\Gamma$ for some discrete subgroup $\Gamma \subset PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$. Let $\tilde{\Gamma}$ be the lifting of Γ to $SL(2,\mathbb{R})$ and $\chi : \tilde{\Gamma} \to \{\pm 1\}$ be the character such that $\chi(-1) = -1$. Then a spin structure on M corresponds to a choice of such ξ. Let $S(2n)$ be the space of automorphic forms f on $\mathbb{H}^2$ such that

$$f(\tilde{\nu} z) = (\chi(\tilde{\nu}))^{2n} (\frac{cz+d}{|cz+d|})^{2n} f(z), \quad \tilde{\nu} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \tilde{\Gamma}.$$

$S(2n)$ is a Hilbert space with the inner product $< \cdot, \cdot >$,

$$< f_1, f_2 >= \int_{\mathbb{H}^2} f_1(z) \bar{f}_2(z) \frac{dxdy}{y^2}.$$

We identify X^n with $S(2n)$ via the isometry φ given by $\varphi(f)(z) = y^n f(z)$. Then Δ_n^+ and Δ_n^- are conjugate with $-D_{2n} + n(n+1)$ and $-D_{2n} + n(n-1)$, resp., where D_{2n} is a selfadjoint operator in $S(2n)$ defined by $D_{2n} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 2niy\frac{\partial}{\partial x}$.

Note that $\Delta_n^\pm$ and D_{2n} have the discrete spectra. Choose a negative number λ_0 such that $\lambda_0 \notin Sp(\Delta_n^\pm)$ and $\lambda_0 \pm n(n \pm 1) \notin Sp(D_{2n})$. We want to compute the ζ-determinant and L^2- analytic torsion functions for the operator $T_n^\pm$, $T_n^\pm = (\Delta_n^\pm - \lambda_0)^{-1} = (-D_{2n} + n(n \pm 1) - \lambda_0)^{-1}$. We have $|T_n^\pm| = T_n^\pm$ and $|T_n^\pm|^{-1} = \Delta_n^\pm - \lambda_0 = -D_{2n} + n(n \pm 1) - \lambda_0$. It is thus sufficient to consider D_{2n}.

By using the Selberg trace formula [Hej], we get (cf. [HoP])

$$Tr(e^{tD_{2n}}) = K_e^n(t) + K^n(t),$$

where

$$\begin{aligned}
K_e^n(t) &= -\frac{\chi(M)}{2} \sum_{0 \le j < n-1/2} (2n-2j-1) e^{(n-j)(n-j-1)t} \\
&- 2\pi\chi(M) \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty du \frac{u e^{-\frac{u^2}{4t}}}{\sinh(u/2)} \cosh((n-[n])u), \\
K^n(t) &= \sum_{\gamma \, prime} \sum_{p=1}^{\infty} \chi(\gamma)^{2np} \frac{l}{\sinh(pl/2)} \frac{e^{-t/4}}{4\sqrt{\pi t}} e^{-\frac{p^2 l^2}{4t}}, \quad \forall n \in \mathbb{Z},
\end{aligned}$$

with l given by $\cosh(l/2) = \frac{|Tr(\gamma)|}{2}$. Let $N_n^{\pm} = \lim_{t\to\infty} Tr(e^{-t\Delta_n^{\pm}})$. We have $N_0^{\pm} = 1, N_n^{+} = 0$ for $n \geq 1/2$ half integers.

$$\begin{aligned}
\zeta_{\Delta_n^{\pm}}(s,\lambda) &= \frac{1}{\Gamma(s)}\int_0^{\infty} dt t^{s-1} e^{-\lambda t} e^{-tn(n\pm 1)} Tr(e^{tD_{2n}}) \\
&= \frac{1}{\Gamma(s)}\left(\int_0^{\infty} dt t^{s-1} e^{-t(\lambda+n(n\pm 1))} K_e^n(t) + \int_0^{\infty} dt t^{s-1} e^{-t(\lambda+n(n\pm1))} K^n(t)\right) \\
&\stackrel{\text{def}}{=} \zeta_{n,e}^{\pm}(s,\lambda) + \zeta_{n,\gamma}^{\pm}(s,\lambda).
\end{aligned}$$

Here for n nonnegative half integers,

$$\begin{aligned}
\zeta_{n,\gamma}^{+}(s,\lambda) &= \frac{1}{\Gamma(s)}\int_0^{\infty} dt t^{s-1} e^{-t(\lambda+n(n+1))} K^n(t) \\
&= \frac{1}{\Gamma(s)}\int_0^{\infty} dt t^{s-1} e^{-t(\lambda+n(n+1))} \sum_{\gamma\ prime}\sum_{p=1}^{\infty} \chi(\gamma)^{2np} \frac{l}{\sinh(pl/2)} \frac{e^{-t/4}}{4\sqrt{\pi t}} e^{-\frac{p^2 l^2}{4t}} \\
&= \frac{1}{\Gamma(s)} \sum_{\gamma\ prime}\sum_{p=1}^{\infty} \chi(\gamma)^{2np} \frac{l}{\sinh(pl/2)} \frac{1}{2\sqrt{\pi}} K_{s-1/2}(pl\sqrt{\lambda+(n+1/2)^2}) \\
&\cdot \left(\frac{pl}{2\sqrt{\lambda+(n+1/2)^2}}\right)^{s-1/2},
\end{aligned}$$

where $K_s(t)$ is the modified Bessel function.

$$\begin{aligned}
-\frac{d}{ds}(\zeta_{n,\gamma}^{+}(s,\lambda))_{s=0} &= -\sum_{\gamma\ prime}\sum_{p=1}^{\infty} \chi(\gamma)^{2np} \frac{l}{\sinh(pl/2)} \frac{1}{2\sqrt{\pi}} \\
&\cdot\ K_{-1/2}(pl\sqrt{\lambda+(n+1/2)^2})\left(\frac{pl}{2\sqrt{\lambda+(n+1/2)^2}}\right)^{-1/2} \\
&= \ln Z_{n-[n]}\left(\frac{1}{2} + \sqrt{\lambda+(n+1/2)^2}\right), \qquad (2.9)
\end{aligned}$$

where $Z_n(s)$ is the Selberg Zeta function,

$$Z_n(s) = \prod_{\gamma\ prime}\prod_{j=0}^{\infty}(1-\chi(\gamma)^{2n} e^{-(j+s)l}), \quad n = 0, 1/2.$$

We now compute $\zeta_{n,e}^{+}(s,\lambda)$.

$$\zeta_{n,e}^{+}(s,\lambda) = \frac{1}{\Gamma(s)}\int_0^{\infty} dt t^{s-1} e^{-t(\lambda+n(n+1))} K_e^n(t)$$

$$
\begin{aligned}
&= \frac{-1}{\Gamma(s)}\int_0^\infty dt t^{s-1} e^{-t(\lambda+n(n\pm1))}(\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)e^{(n-j)(n-j-1)t} \\
&+ 2\pi\chi(M)\frac{e^{-t/4}}{(4\pi t)^{3/2}}\int_0^\infty du \frac{e^{-\frac{u^2}{4t}}}{\sinh(u/2)}\cosh((n-[n])u)) \\
&= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)(\lambda+(2n-j)(j+1))^{-s} \\
&- \frac{-2\pi\chi(M)}{(4\pi)^{3/2}}\int_0^\infty du \frac{u\cosh((n-[n])u)}{\sinh(u/2)}(\frac{u}{2\sqrt{\lambda+(n+1/2)^2}})^{s-3/2} \\
&\cdot \frac{2}{\Gamma(s)}K_{s-3/2}(u\sqrt{\lambda+(n+1/2)^2}).
\end{aligned}
$$

Let $I_{n,e}(s,\lambda)$ denote the second term of this identity. The above relation implies

$$
\begin{aligned}
&-\frac{d}{ds}(\zeta^+_{n,e}(s,\lambda))_{s=0} = \frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)\ln(\lambda+(2n-j)(j+1))^{-1} \\
&+ \frac{2\pi\chi(M)}{(4\pi)^{3/2}}\int_0^\infty du \frac{u\cosh((n-[n])u)}{\sinh(u/2)}(\frac{u}{2\sqrt{\lambda+(n+1/2)^2}})^{-3/2}2 \\
&\cdot K_{-3/2}(u\sqrt{\lambda+(n+1/2)^2}). \qquad (2.10)
\end{aligned}
$$

Combining (2.9) and (2.10), we get

$$
\begin{aligned}
-\frac{d}{ds}(\zeta_{\Delta_n^+}(s,\lambda))_{s=0} &= -\frac{d}{ds}(\zeta^+_{n,e}(s,\lambda))_{s=0} - \frac{d}{ds}(\zeta^+_{n,\gamma}(s,\lambda))_{s=0} \\
&= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)\ln(\lambda+(2n-j)(j+1)) \\
&+ \ln Z_{n-[n]}(\frac{1}{2}+\sqrt{\lambda+(n+1/2)^2}) + \frac{d}{ds}(I_{n,e}(s,\lambda))_{s=0}. \quad (2.11)
\end{aligned}
$$

Similarly for $\zeta_{\Delta_n^-}(s,\lambda)$ with $n\ge 1/2$,

$$
\begin{aligned}
\zeta_{\Delta_n^-}(s,\lambda) &= \zeta^-_{n,e}(s,\lambda) + \frac{1}{\Gamma(s)}\sum_{\gamma\ prime}\sum_{p=1}^\infty \chi(\gamma)^{2np}\frac{l}{\sinh(pl/2)}\frac{1}{2\sqrt{\pi}} \\
&\cdot K_{s-1}(pl\sqrt{\lambda+(n-1/2)^2})(\frac{pl}{2\sqrt{\lambda+(n-1/2)^2}})^{s-1/2},
\end{aligned}
$$

where

$$\begin{aligned}
\zeta^-_{n,e}(s,\lambda) &= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} e^{-t(\lambda+n(n-1))} K^n_e(t) \\
&= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)(\lambda+(2n-j+1)j)^{-s} \\
&- \frac{2\pi\chi(M)}{(4\pi)^{3/2}}\int_0^\infty du \frac{u\cosh((n-[n])u)}{\sinh(u/2)}\left(\frac{u}{2\sqrt{\lambda+(n-1/2)^2}}\right)^{s-3/2} \\
&\cdot \frac{2}{\Gamma(s)} K_{s-3/2}(u\sqrt{\lambda+(n-1/2)^2}).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
-\frac{d}{ds}(\zeta_{\Delta^-_n}(s,\lambda))_{s=0} &= -\frac{d}{ds}(\zeta^-_{n-1,e}(s,\lambda))_{s=0} + \ln Z_{n-[n]}(\frac{1}{2}+\sqrt{\lambda+(n-1/2)^2}) \\
&= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)\ln(\lambda+(2n-j+1)j) \\
&+ \ln Z_{n-[n]}(\frac{1}{2}+\sqrt{\lambda+(n-1/2)^2}) - \frac{d}{ds}(I_{n-1,e}(s,\lambda))_{s=0} \qquad (2.12)
\end{aligned}$$

We now use a result of Sarnak to compute $\frac{d}{ds}(I_{n,e}(s,\lambda))_{s=0}$. Let $n = 1/2$ and $\lambda - 1/4 = u(u-1)$. We get $\lambda = (u-1/2)^2$. By Sarnak's theorem [Sar],

$$\begin{aligned}
&- \frac{d}{ds}(\zeta_{-\Delta^-_{1/2}}(s,\lambda))_{s=0} = -\frac{d}{ds}(\zeta_{D_1+u(u-1)}(s))_{s=0} \\
&= \log Z_{1/2}(u) + (2g-2)(c_0 - u(u-1) + 2\log\Gamma_2(u+1/2) + u\log(2\pi)),
\end{aligned}$$

where $c_0 = -1/4 - \frac{1}{2}\log(2\pi) + 2\zeta'(-1)$ with the Riemann Zeta function $\zeta(u)$, and $\Gamma_2(u)$ is the Barnes double gamma function defined by

$$\frac{1}{\Gamma_2(u+1)} = (2\pi)^{u/2} e^{-(u+(c+1)u^2)/2}\prod_{j=1}^\infty (1+\frac{u}{j})^j e^{-u+\frac{u^2}{2j}}.$$

Also for $n = 0, \sqrt{\lambda+\frac{1}{4}} = u - \frac{1}{2}, \lambda = u(u-1)$,

$$\begin{aligned}
&- \frac{d}{ds}(\zeta_{-\Delta^-_0}(s,\lambda))_{s=0} = -\frac{d}{ds}(\zeta_{D_0+u(u-1)}(s))_{s=0} \\
&= \log Z_0(u) + (2g-2)(c_0 - u(u-1) + \log\frac{\Gamma_2^2(u)}{\Gamma(u)} + u\log(2\pi)).
\end{aligned}$$

On the other hand, we know by (2.12)

$$-\frac{d}{ds}(\zeta_{\Delta^-_{1/2}}(s,\lambda))_{s=0} = \log Z_{1/2}(u) - \frac{d}{ds}(I_{-1/2,e}(s,(u-\frac{1}{2})^2))_{s=0},$$

and

$$-\frac{d}{ds}(\zeta_{\Delta^-_0}(s,\lambda))_{s=0} = \log Z_0(u) - \frac{d}{ds}(I_{-1,e}(s,u(u-1)))_{s=0}.$$

Comparing these identities, we get

$$\frac{d}{ds}(I_{-\frac{1}{2},e}(s,u(u-1)))_{s=0} = \chi(M)(c_0 - u(u-1) + \log\Gamma_2^2(u) + u\log(2\pi)), \quad (2.13)$$

and

$$\frac{d}{ds}(I_{-1,e}(s,(u-\frac{1}{2})^2))_{s=0} = \chi(M)(c_0 - u(u-1) + \log\frac{\Gamma_2^2(u)}{\Gamma(u)} + u\log(2\pi)). \quad (2.14)$$

Now replacing $u-1/2$ by $\sqrt{\lambda+(n-\frac{1}{2})^2}$ in (2.13), we obtain the formula for general $n, n-[n]=\frac{1}{2}$,

$$\begin{aligned} \frac{d}{ds}(I_{n-1,e}(s,\lambda))_{s=0} &= \chi(M)(c_0 - (\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})(\sqrt{\lambda+(n-\frac{1}{2})^2}-\frac{1}{2}) \\ &+ \; 2\log\Gamma_2(\sqrt{\lambda+(n-\frac{1}{2})^2}+1) + (\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})\log(2\pi)). \quad (2.15) \end{aligned}$$

Similarly for $n=[n]$ we use (2.14) to express $\frac{d}{ds}(I_{n-1,e}(s,\lambda))_{s=0}$ by (2.15) except replacing $2\log\Gamma_2(\sqrt{\lambda+(n-\frac{1}{2})^2}+1)$ by $\log\frac{\Gamma_2^2(\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})}{\Gamma(\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})}$.

We have proved the following theorem by combining (2.11), (2.12) and (2.15) together with

$$-\log D_\zeta(T_n^\pm,\lambda) = -\frac{d}{ds}(\zeta_{\Delta_n^\pm}(s,\lambda-\lambda_0))_{s=0}.$$

Theorem 2.3 *Let $T_n^\pm = (\Delta_n^\pm - \lambda_0)^{-1}$. Then for $\lambda > 0$ and n positive half integers,*

$$-\log D_\zeta(T_n^+,\lambda) \;=\; \ln Z_{n-[n]}(\frac{1}{2}+\sqrt{\lambda-\lambda_0+(n+1/2)^2})$$

$$
\begin{aligned}
& - \frac{\chi(M)}{2} \sum_{0 \le j < n-1/2} (2n-2j-1)\ln(\lambda-\lambda_0+(2n-j)(j+1)) \\
& - \chi(M)(c_0 - (\sqrt{\lambda-\lambda_0+(n+\frac{1}{2})^2}+\frac{1}{2})(\sqrt{\lambda-\lambda_0+(n+\frac{1}{2})^2} \\
& - \frac{1}{2}) + 2\log\Gamma_2(\sqrt{\lambda-\lambda_0+(n+\frac{1}{2})^2}+1) \\
& + (\sqrt{\lambda-\lambda_0+(n+\frac{1}{2})^2}+\frac{1}{2})\log(2\pi)), \quad n \ge 1/2.
\end{aligned}
$$

$$
\begin{aligned}
-\log D_\zeta(T_n^-,\lambda) &= \ln Z_{n-[n]}(\frac{1}{2}+\sqrt{\lambda-\lambda_0+(n-1/2)^2}) \\
&- \frac{\chi(M)}{2} \sum_{0 \le j < n-1/2} (2n-2j-1)\ln(\lambda-\lambda_0+(2n-j+1)j) \\
&- \chi(M)(c_0 - (\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2}+\frac{1}{2})(\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2} \\
&- \frac{1}{2}) + 2\log\Gamma_2(\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2}+1) \\
&+ (\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2}+\frac{1}{2})\log(2\pi)).
\end{aligned}
$$

If we replace $2\log\Gamma_2(\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2}+1)$ *by* $\log\frac{\Gamma_2^2(\sqrt{\lambda-\lambda_0+(n-\frac{1}{2})^2}+\frac{1}{2})}{\Gamma(\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})}$*, then the above formulas hold for positive integers.*

$$
\begin{aligned}
-\log D_\zeta(T_0^\pm,\lambda) &= \ln Z_0(\frac{1}{2}+\sqrt{\lambda-\lambda_0+1/4}) - \chi(M)(c_0-(\lambda-\lambda_0) \\
&+ \log\frac{\Gamma_2^2(\sqrt{\lambda-\lambda_0+1/4}+\frac{1}{2})}{\Gamma(\sqrt{\lambda-\lambda_0+\frac{1}{4}}+\frac{1}{2})} + (\sqrt{\lambda-\lambda_0+1/4}+\frac{1}{2})\log(2\pi)).
\end{aligned}
$$

Note that the formulas above are valid also for $\lambda = 0$. We can then get the determinants $\log D_\zeta(T_n^\pm)$. Since $T_n^\pm$ are selfadjoint and injective, $\log D_\zeta((T_n^\pm)^2) = 2\log D_\zeta(T_n^\pm)$. Using Theorem 2.2 for $f(z)=z$, we get the formula for $\log\tau_{\mathcal{A}}(T_n^\pm,\lambda)$.

Let $\tilde{\Delta}_n^\pm$ and $\tilde{D}_{2n}$ be the lifting of $\Delta_n^\pm$ and D_{2n} to $\mathbb{H}^2$. Denote

$$
\log det(\tilde{\Delta}_n^\pm,\lambda) = -\frac{d}{ds}(\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}e^{-\lambda t}Tr_\Gamma(e^{-t\tilde{\Delta}_n^\pm})dt)_{s=0},
$$

$$\log det'(\tilde{\Delta}_n^{\pm}, \lambda) \;=\; -\frac{d}{ds}\Big(\frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1} e^{-\lambda t} Tr_\Gamma(e^{-t(\tilde{\Delta}_n^{\pm})'})dt\Big)_{s=0},$$

where Tr_Γ is the natural trace on the finite von Neumann algebra generated by a discrete group Γ. We have

$$\log det(\tilde{\Delta}_n^{\pm}, \lambda) = -\frac{d}{ds}(\zeta_{n,e}^{\pm}(s,\lambda))_{s=0}.$$

As a corollary of the proof of Theorem 2.3, we obtain

Corollary 2.1 *For n positive half integers,*

$$\begin{aligned}
\log det(\tilde{\Delta}_n^{+}, \lambda) &= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)\ln(\lambda+(2n-j)(j+1)) \\
&- \chi(M)(c_0-(\lambda+(n+\frac{1}{2})^2)-\frac{1}{4}+2\log\Gamma_2(\sqrt{\lambda+(n+\frac{1}{2})^2}+1) \\
&+ (\sqrt{\lambda+(n+\frac{1}{2})^2}+\frac{1}{2})\log(2\pi)), \\
\log det(\tilde{\Delta}_n^{-}, \lambda) &= -\frac{\chi(M)}{2}\sum_{0\le j<n-1/2}(2n-2j-1)\ln(\lambda-\lambda_0+(2n-j+1)j) \\
&- \chi(M)(c_0-(\lambda+(n-\frac{1}{2})^2)-\frac{1}{4}+2\log\Gamma_2(\sqrt{\lambda+(n-\frac{1}{2})^2}+1) \\
&+ (\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})\log(2\pi)).
\end{aligned}$$

Same formulas hold for integers n provided we replace $2\log\Gamma_2(\sqrt{\lambda+(n-\frac{1}{2})^2}+1)$ by $\log\frac{\Gamma_2^2(\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})}{\Gamma(\sqrt{\lambda+(n-\frac{1}{2})^2}+\frac{1}{2})}$.

Proof. See (2.10), (2.12) and (2.15). Q.E.D.

Observe that the above determinants do not involve the Selberg Zeta function at all. We finally remark that it is quite difficult to compute the ζ-determinant function of the Laplacian on compact Riemann surfaces of genus $g=1$ due to the occurrence of the modified Besssel function $K_1(s)$ in the trace of the heat kernel.

Chapter 3

K-Theory Torsions for Finite von Neumann Algebras

In the previous chapters we studied the numerical torsions for geometric and operator situations. We will now introduce a K-theory torsion invariant for n-tuples of commuting bounded $\mathcal{A}$-operators on a finitely generated Hilbert $\mathcal{A}$-module H with $\mathcal{A}$ a finite von Neumann algebra. This torsion invariant takes value in weak K-theory group $K_1^w(\mathcal{A})$ of $\mathcal{A}$. The advantage of the K-theory torsion is that unlike L^2 -analytic torsions on non compact manifolds we do not need any restrictions on commuting bounded $\mathcal{A}$ operators and we can more closely look at the relation between the torsion and operators.

This chapter is arranged as follows. In section 3.1 we define the torsion invariant for an n-tuple of commuting bounded $\mathcal{A}$-operators on finitely generated Hilbert $\mathcal{A}$-module H, and recall some useful facts. The properties of the torsion invariant will be discussed in section 3.2. We show that the torsion invariant respects nicely the operator operations. A trivially-embedding theorem and vanishing theorem are also proved. We study the torsion invariant of single operator in section 3.3. In that case, the torsion and the absolute value of the adjoint operator are determined with each other up to conjugation by isomorphism and a direct sum factor of weak isomorphisms. We calculate the torsion invariant for finite abelian von Neumann algebras.

3.1 K-Theory Torsions for Tuples of Commuting Elements

Let $\mathcal{A}$ be a finite von Neumann algebra. This means that there is a normal and faithful finite trace τ on $\mathcal{A}$. Let $l^2(\mathcal{A})$ be the completion of $\mathcal{A}$ under the pre-Hilbert inner product $< a_1, a_2 >= \tau(a_2^* a_1)$. Let H be a Hilbert space. Suppose that we can endow a continuous left $\mathcal{A}$-module structure on H such that H is isometrically isomorphic into a closed subspace of $l^2(\mathcal{A}) \otimes H_1$ for some Hilbert space H_1. H is called a finitely generated Hilbert $\mathcal{A}$-module if there is a surjective $\mathcal{A}$-map from $\oplus_{j=1}^m l^2(\mathcal{A})$ onto H for some positive integer m. Note that in our special case, a finitely generated Hilbert $\mathcal{A}$-module H is projective, i.e., there is a finitely generated Hilbert $\mathcal{A}$- module $\tilde{H}$ such that $H \oplus \tilde{H}$ is isometrically isomorphic to $\oplus_{j=1}^m l^2(\mathcal{A})$ as Hilbert $\mathcal{A}$-modules. Throughout all Hilbert $\mathcal{A}$-modules will be assumed to be finitely generated unless specifically stated.

Let us recall algebraic K-theory group of finite von Neumann algebra $\mathcal{A}$. $K_0(\mathcal{A})$ is defined to be abelian group generated by all isomorphism classes of finitely generated Hilbert $\mathcal{A}$-modules with the relation $[H_1 \oplus H_2] = [H_1] \oplus [H_2]$. $K_1(\mathcal{A})$ (resp. $K_1^w(\mathcal{A})$) is the abelian group generated by conjugation classes of automorphisms (resp. weak automorphisms) of finitely generated $\mathcal{A}$-modules, subject to the relations

(1) if $0 \to (H_1, f_1) \to (H_2, f_2) \to (H_3, f_3) \to 0$ is an exact sequence of automorphisms f_i (resp. weak automorphisms) on finitely generated Hilbert $\mathcal{A}$-modules H_i, then $[(H_2, f_2)] = [(H_1, f_1)] + [(H_3, f_3)]$,

(2) if f_1 and f_2 are two automorphisms (resp. weak automorphisms) on finitely generated Hilbert $\mathcal{A}$-module H, then $[(H, f_1 f_2)] = [(H, f_1)] + [(H, f_2)]$,

(3) $[(H, I)] = 0$ for the identity map I on finitely generated Hilbert $\mathcal{A}$-module H.

Here $f : H_1 \to H_2$ is a weak isomorphism if f is injective and has a dense range in H_2. Clearly, $f : H \to H$ is a weak automorphism iff f is injective, since H is finitely generated and f is an $\mathcal{A}$-map. Thus $f_1 f_2$ is a weak automorphism if f_1 and f_2 are weak automorphisms on H. We know $K_0(\mathcal{A}) = \pi_1(\mathcal{U})$, where $\mathcal{U}$ is the unitary group of $\mathcal{A}$. Since $K_1(\mathcal{A})$ can be identified with the abelianization of the general linear group $\mathcal{G}L(\mathcal{A})$, $K_1(\mathcal{A}) \overset{\det}{\simeq} \mathbb{R}_+^\times$ via the Fuglede-Kadison determinant provided $\mathcal{A}$ is a II_1-factor. Also if $\mathcal{A}$ is a finite abelian von Neumann algebra $L^\infty(X, \mu)$ with X compact and second countable, then $K_1(\mathcal{A}) \overset{\det}{\simeq} L^\infty(X, \mu)^\times$ and $K_1^w(\mathcal{A}) \overset{\det}{\simeq} F(X, \mu)^\times$, where $F(X, \mu)^\times$ is the group of all almost everywhere invertible measurable functions on X. Another useful fact is that $[(H_1, f_1)] = [(H_2, f_2)]$ if $f_i : H_i \to H_i$ are weak isomorphisms satisfying $u f_1 = f_2 u$ for some weak isomorphism $u : H_1 \to H_2$. For the application of the algebraic K-theory in operator theory we refer the reader to the paper of Carey and Pincus [CaP 2-4] where the algebraic K_2-theory was successfully used to explore the deep relation between the Lefschetz numbers.

Let H be a Hilbert $\mathcal{A}$-module. Suppose that $T = (T_1, \ldots, T_n)$ is an n-tuple of commuting bounded $\mathcal{A}$ operators on H, i.e., $T_iT_j = T_jT_i$, $i, j = 1, \ldots, n$, and each T_i commutes with $\mathcal{A}$-action on H. Throughout we always assume that $\mathcal{A}$-operators are bounded. Using the notations of Chapter 2, we let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be the n-indeterminants and $\wedge^j[\sigma]$ the exterior space of complex space $\mathbb{C}[\sigma] = \oplus_{j=1}^n \mathbb{C}\sigma_j$. Then $C_j(T) = H \otimes \wedge^j[\sigma]$ is a finitely generated Hilbert $\mathcal{A}$-module and $d_j(T) = \sum_{j=1}^n T_jS_j : C_j(T) \to C_{j+1}(T)$ is an $\mathcal{A}$-operator. We have $d_{j+1}(T)d_j(T) = 0$. $\{C_*(T), d_*(T)\}$ is called the Koszul complex of finitely generated Hilbert $\mathcal{A}$-modules associated with T and H. The Laplace operator $\Delta_j(T) : C_j(T) \to C_j(T)$ is defined by $\Delta_j(T) = d_j^*(T)d_j(T) + d_{j-1}(T)d_{j-1}^*(T)$. We can write $\Delta_j(T)$ in terms of T_j and S_j as follows.

Lemma 3.1

$$\begin{aligned}
\Delta_0(T) &= d_0^*(T)d_0(T) = \sum_{l=1}^n T_l^*T_l, \\
\Delta_n(T) &= d_{n-1}(T)d_{n-1}^*(T) = \sum_{l=1}^n T_lT_l^*, \\
\Delta_j(T) &= \sum_{l=1}^n((T_l^*T_l - T_lT_l^*)S_l^*S_l + T_lT_l^*) \\
&+ \sum_{l=1}^{n-1}\sum_{k=l+1}^n ((T_k^*T_l - T_lT_k^*)S_k^*S_l + (T_kT_l^* - T_l^*T_k)S_kS_l^*), \quad 0 < j < n.
\end{aligned}$$

Proof. This follows from a trivial computation and the fact that $S_iS_j^* + S_j^*S_i = 1$ for $i = j$, and 0 for $i \neq j$. Q.E.D.

We have $C_j(T) = Ker\Delta_j(T) \oplus (Ker\Delta_j(T))^\perp$. Both $Ker\Delta_j(T)$ and $(Ker\Delta_j(T))^\perp$ are finitely generated $\mathcal{A}$-modules. Let $\Delta_j'(T) = \Delta_j(T)|_{(Ker\Delta_j(T))^\perp}$. It is injective and hence is a weak isomorphism. $\Delta_j'(T)$ defines an element $[\Delta_j'(T)]$ in $K_1^w(\mathcal{A})$.

Definition 3.1 The K-theory torsion invariant $\tau(T,H)$ of T is defined by

$$\tau(T,H) = \sum_{j=0}^{n}(-1)^{j+1}j[\Delta'_j(T)] \in K_1^w(\mathcal{A}).$$

Remark 3.1 (1) *The adjoint of $\mathcal{A}$-operators on finitely generated Hilbert $\mathcal{A}$-modules induces an involution $*$ on $K_1^w(\mathcal{A})$ by $*[(H,f)] = [(H,f^*)]$ for weak automorphism $f: H \to H$. Since $\Delta_j(T)$ and $\Delta'_j(T)$ are selfadjoint, we have $*\tau(T,H) = \tau(T,H)$. Hence $\tau(T,H)$ is in $K_1^w(\mathcal{A})^{\mathbb{Z}/2}$, the fixed points of $*$ in $K_1^w(\mathcal{A})$.*

(2) *$\tau(T,H)$ differs from the torsion invariant in the previous chapters by constant $\frac{1}{2}$. But it is equal to that [LüR]. In fact, for any bounded complex $\{C_j, d_j\}_{j=-k}^{n}$ of Hilbert $\mathcal{A}$-modules, let $\Delta_j = d_j^* d_j + d_{j-1}d_{j-1}^*$ be the Laplace operators and $\Delta'_j = \Delta_j|_{(Ker\Delta_j(T))^\perp}$. We can define the torsion invariant of $\{C_*, d_*\}$ as*

$$\tau(C_*) = \sum_{j=-k}^{n}(-1)^{j+1}j[\Delta'_j].$$

(3) *It is easy to see that $Ker\Delta_j(T) = \frac{Ker d_j}{im d_{j-1}} = \mathcal{H}^j(C_*(T))$. We say $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is in weak resolvent set $\rho_w(T)$ if $\mathcal{H}^j(C_*(T-z)) = 0$ for all $j = 0, 1, \ldots, n$, where $T - z = (T_1 - z_1, \ldots, T_n - z_n)$. For such a point z, we have $\tau(T-z, H) = \sum_{j=0}^{n}(-1)^{j+1}j[\Delta_j(T-z)] \in K_1^w(\mathcal{A})^{\mathbb{Z}/2}$.*

We now define a relative torsion for an $\mathcal{A}$-operator between two Hilbert $\mathcal{A}$-modules. Let $H^{(i)}$ be two Hilbert $\mathcal{A}$-modules and $T^{(i)} = (T_1^{(i)}, \ldots, T_n^{(i)})$ n-tuples of commuting $\mathcal{A}$ -operators on $H^{(i)}, i = 1, 2$. Suppose $U: H^{(1)} \to H^{(2)}$ is an $\mathcal{A}$-operator which interwines $T^{(i)}$, $UT_j^{(1)} = T_j^{(2)}U, j = 1, \ldots, n$. This implies that $d_*^{(2)}U = Ud_*^{(1)}$ for the differentials $d_*^{(i)}$ of the Koszul complexes $C_*(T^{(i)})$ associated with $T^{(i)}$. Hence U induces a morphism from $C_*(T^{(1)})$ to $C_*(T^{(2)})$.

Form the mapping cone $C_*(U)$ of U by $C_j(U) = C_{j+1}(T^{(1)}) \oplus C_j(T^{(2)})$ with differential $d_j(U) : C_j(U) \to C_{j+1}(U)$

$$d_j(U) = \begin{bmatrix} -d^{(1)}_{j+1} & 0 \\ U & d^{(2)}_j \end{bmatrix}, \quad j = -1, 0, \ldots, n.$$

$\{C_*(U), d_*(U)\}$ is a complex of finitely generated $\mathcal{A}$-modules. Let $\Delta_j(U) = d_j^*(U)d_j(U) + d_{j-1}(U)d_{j-1}^*(U)$ and $\Delta'_j(U) = \Delta_j(U)|_{(Ker\Delta_j)^\perp}$.

Definiton 3.2 The torsion invariant $\tau(U, T^{(i)})$ of U relative to $(H^{(i)}, T^{(i)})$ is defined by

$$\tau(U, T^{(i)}) = \sum_{j=-1}^{n} (-1)^{j+1} j[\Delta'_j(U)] \in K_1^w(\mathcal{A})^{\mathbb{Z}/2}.$$

Remark 3.2 *If* $T_j^{(i)} = 0, j = 1, \ldots, n$, $d_j(U) = \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix} : C_j(U) \to C_{j+1}(U)$ *can be thought of a map from* $H^{(1)} \oplus H^{(2)}$ *to* $H^{(1)} \oplus H^{(2)}$. *We have*

$$\Delta_j(U) = U^*U \oplus UU^*, \ \ j = 0, \ldots, n-1, \ \ \Delta_{-1}(U) = U^*U \oplus 0, \ \ \Delta_n = 0 \oplus UU^*.$$

Hence,

$$\begin{aligned}
\tau(U, 0) &= -[(U^*U)^\perp] + \sum_{j=0}^{n-1} (-1)^{j+1}(m_{j+1}[(U^*U)^\perp] \\
&+ m_j[(UU^*)^\perp]) + (-1)^{n+1} n[(UU^*)^\perp] \\
&= \sum_{j=-1}^{n-1} (-1)^{j+1} j(m_{j+1}[(U^*U)^\perp] + m_j[(UU^*)^\perp]) = 0, \ \ n > 1,
\end{aligned}$$

since $\sum_{j=0}^{n} (-1)^j j m_j = 0$ *for* $n > 1$ *and* $m_j = \left(\begin{smallmatrix} n \\ j \end{smallmatrix} \right)$, *and* $\sum_{j=0}^{n} (-1)^j m_j = 0$.

Here $T^\perp$ *denotes* $T|_{(KerT)^\perp}$ *for an operator* T. *We also get for* $n = 1$,

$$\tau(U, 0) = [(U^*U)^\perp] - [(UU^*)^\perp].$$

This shows that $\tau(U,0)$ *is not equal to the torsion of operator* $U_1 = \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix}$ *on* $H^{(1)} \oplus H^{(2)}$. *In fact* $\tau(U_1, H^{(1)} \oplus H^{(2)}) = [(UU^*)^\perp]$.

We can also write down a formula for $\Delta_j(U)$ in terms of $\Delta_j(T^{(i)}), d_j^{(i)}$, U and U^*.

Lemma 3.2 $\Delta_{-1}(U) = (\Delta_0(T^{(1)}) + U^*U) \oplus 0, \Delta_n(U) = 0 \oplus (\Delta_n(T^{(2)}) + UU^*)$,

$$\Delta_j(U) = \begin{bmatrix} \Delta_{j+1}(T^{(1)}) + U^*U & U^*d_j^{(2)} - d_j^{(1)}U^* \\ (d_j^{(2)})^*U - U(d_j^{(1)})^* & \Delta_j(T^{(2)}) + UU^* \end{bmatrix}.$$

Proof. The proof is a direct calculation which we omit. Q.E.D.

Definition 3.3 Let $U : H^{(1)} \to H^{(2)}$ interwine $T^{(k)}$. U is called a weak cohomology equivalence relative to $T^{(i)}$ if the cohomology $\mathcal{H}^*(C_*(U))$ of the mapping cone $C_*(U)$ is zero. U is weakly simple cohomology equivalence if $\mathcal{H}^*(C_*(U)) = 0$ and $\tau(U, T^{(i)}) = 0$.

A necessary condition for U to be a weakly simple cohomology equivalence relative to $T^{(i)}$ is that the weak resolvent sets $\rho_w(T^{(i)})$ of $T^{(i)}$ are equal. This can be seen from the following long weakly exact sequence [ChGr] associated with short exact sequence

$$0 \to C_*(T^{(2)}) \xrightarrow{i} C_*(U) \xrightarrow{\pi} \Sigma C_*(T^{(1)}) \to 0,$$

$$\ldots \xrightarrow{U} \mathcal{H}^j(\Sigma C_*(T^{(1)})) \xrightarrow{\pi^*} \mathcal{H}^j(C_*(U)) \xrightarrow{i^*} \mathcal{H}^j(C_*(T^{(2)})) \xrightarrow{U} \ldots, \tag{3.1}$$

where $\Sigma C_*(T^{(1)})$ is the suspension of $C_*(T^{(1)})$ defined by $(\Sigma C_*(T^{(1)}))_j = C_{j+1}(T^{(1)})$ with differential $\Sigma d_j^{(1)} = -d_{j+1}^{(1)}$.

Proposition 3.1 *Let $H^{(i)}$ be two Hilbert $\mathcal{A}$-modules and $T^{(i)}$ two tuples of commuting $\mathcal{A}$-operators on $H^{(i)}$, $i = 1, 2$. Suppose $U : H^{(1)} \to H^{(2)}$ is an invertible $\mathcal{A}$-operator such that U interwines both $T^{(i)}$ and $(T^{(i)})^*$. If*

$$\sum_{j=0}^{n}(-1)^{j+1}([(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] + [UU^*|_{Ker\Delta_j(T^{(1)})}]) = 0$$

in $K_1^w(\mathcal{A})$, then U is a weakly simple cohomology equivalence.

Proof. Using Lemma 3.2, we have

$$\begin{aligned}
\Delta_j'(U) &= ((\Delta_{j+1}(T^{(1)}) + U^*U)|_{(Ker\Delta_{j+1}(T^{(1)}))^\perp} \oplus (U^*U)|_{Ker\Delta_{j+1}(T^{(1)})}) \\
&\oplus ((\Delta_j(T^{(2)}) + UU^*)|_{(Ker\Delta_j(T^{(2)}))^\perp} \oplus (UU^*)|_{Ker\Delta_j(T^{(2)})}). \\
\tau(U, T^{(i)}) &= \sum_{j=-1}^{n-1}(-1)^{j+1}j([(\Delta_{j+1}(T^{(1)}) + U^*U)|_{(Ker\Delta_{j+1}(T^{(1)}))^\perp}] + [(U^*U)|_{Ker\Delta_{j+1}(T^{(1)})}]) \\
&+ \sum_{j=0}^{n}(-1)^{j+1}j([(\Delta_j(T^{(2)}) + UU^*)|_{(Ker\Delta_j(T^{(2)}))^\perp}] + [(UU^*)|_{Ker\Delta_j(T^{(2)})}]) \\
&= \sum_{j=0}^{n}(-1)^{j}j([(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] + [(U^*U)|_{Ker\Delta_j(T^{(1)})}] \\
&- [(\Delta_j(T^{(2)}) + UU^*)|_{(Ker\Delta_j(T^{(2)}))^\perp}] - [(UU^*)|_{Ker\Delta_j(T^{(2)})}]) \\
&+ \sum_{j=0}^{n}(-1)^{j+1}([(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] + [(U^*U)|_{Ker\Delta_j(T^{(1)})}]).
\end{aligned}$$

We claim that $[(\Delta_j(T^{(2)})+UU^*)|_{(Ker\Delta_j(T^{(2)}))^\perp}] = [(\Delta_j(T^{(1)})+U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}]$ and $[(UU^*)|_{Ker\Delta_j(T^{(2)})}] = [(U^*U)|_{Ker\Delta_j(T^{(1)})}]$. In fact, $Ud_j^{(1)} = d_j^{(2)}U$ and $U(d_j^{(1)})^* = (d_j^{(2)})^*U$. We get $\Delta_j(T^{(2)})U = U\Delta_j(T^{(1)})$. Hence $U|_{Ker\Delta_j(T^{(1)})}$ and $U|_{Ker\Delta_j(T^{(1)}))^\perp}$ are invertible. The claim follows from $U(\Delta_j(T^{(1)})+U^*U) = (\Delta_j(T^{(2)})+UU^*)U$ on $(Ker\Delta_j(T^{(1)}))^\perp$ and $U(U^*U) = (UU^*)U$ on $Ker\Delta_j(T^{(1)})$ and the remark at the beginning of this section. Thus we obtain

$$\tau(U, T^{(i)}) = \sum_{j=0}^{n}(-1)^{j+1}([(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] + [UU^*|_{Ker\Delta_j(T^{(1)})}]).$$

By assumption, $\tau(U, T^{(i)}) = 0$.

It remains to show that the cohomology of the mapping cone of U is zero. It suffices to check by long weakly exact sequence (3.1) that $U : Ker\Delta_j(T^{(1)}) \to Ker\Delta_j(T^{(2)})$ is an isomorphism, which is obviously true. Q.E.D.

Remark 3.3 *The above proof implies that* $\tau(T^{(1)}, H^{(1)}) = \tau(T^{(2)}, H^{(2)})$ *if* $U : H^{(1)} \to H^{(2)}$ *is invertible and interwines both* $T^{(i)}$ *and* $(T^{(i)})^*$. *We call this the weak similarity property of the torsion invariant. In particular, the torsion is a unitary invariant.*

We give some examples that satisfy the conditions of propositions 3.1.

Example 3.1 (1) *If* $T_j^{(i)} = 0, j = 1, \ldots, n$, $\Delta_j(T^{(i)}) = 0, (Ker\Delta_j(T^{(i)}))^\perp = 0$, *then for invertible* $U : H^{(1)} \to H^{(2)}$,

$$\sum_{j=0}^{n}(-1)^{j+1}[(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] = 0,$$

and since $\sum_{j=0}^{n}(-1)^j m_j = 0$ *for* $n > 1$,

$$\begin{aligned}\sum_{j=0}^{n}(-1)^{j+1}[UU^*|_{Ker\Delta_j(T^{(1)})}] &= \sum_{j=0}^{n}(-1)^{j+1}[U^*U|_{C_j(T^{(1)})}] \\ &= \sum_{j=0}^{n}(-1)^{j+1}m_j[U^*U|_{H^{(1)}}] = 0.\end{aligned}$$

Hence U *is a weakly simple cohomology equivalence relative to* $T^{(i)} = 0$.

(2) *More generally, suppose that* $[T_i^{(1)}, (T_j^{(1)})^*] = 0, i, j = 1, \ldots, n$. *Then by Lemma* 3.1, $\Delta_j(T^{(i)}) = \sum_{j=1}^{n} T_j^{(i)}(T_j^{(i)})^*$,

$$Ker\Delta_j(T^{(i)}) = Ker(\sum_{j=1}^{n} T_j^{(i)}(T_j^{(i)})^*) \otimes \wedge^j[\sigma],$$

$$[(U^*U)|_{Ker\Delta_j(T^{(1)})}] = m_j[(U^*U)|_{Ker(\sum_{j=1}^{n} T_j^{(1)}(T_j^{(1)})^*)}],$$

and

$$[(\Delta_j(T^{(1)})+U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] = m_j[(\sum_{j=1}^{n} T_j^{(1)}(T_j^{(1)})^*+U^*U)|_{(Ker\sum_{j=1}^n T_j^{(1)}(T_j^{(1)})^*)^\perp}].$$

We get

$$\begin{aligned} & \sum_{j=0}^{n}(-1)^{j+1}([(\Delta_j(T^{(1)}) + U^*U)|_{(Ker\Delta_j(T^{(1)}))^\perp}] + [UU^*|_{Ker\Delta_j(T^{(1)})}]) \\ = \ & \sum_{l=0}^{n}(-1)^{l+1}m_l([(\sum_{j=1}^{n} T_j^{(1)}(T_j^{(1)})^* + U^*U)|_{(Ker\sum_{j=1}^n T_j^{(1)}(T_j^{(1)})^*)^\perp}] \\ + \ & [U^*U|_{Ker\sum_{j=1}^n T_j^{(1)}(T_j^{(1)})^*}]) = 0 \end{aligned}$$

Therefore, if $U : H^{(1)} \to H^{(2)}$ *is invertible and interwines both* $T^{(i)}$ *and* $(T^{(i)})^*$*, then* U *is a weakly simple cohomology equivalence relative to* $T^{(i)}$*.*

Remark 3.4 *The condition* $[T_i, T_j^*] = 0$ *is satisfied when each* T_j *is selfadjoint or* $T_j = (S)^j$ *for some fixed normal operator* S, $j = 1, \dots, n$*. Moreover, a minimal normal extension* $\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_n)$ *of subnormal n-tuple* $T = (T_1, \dots, T_n)$ *satisfies* $[\tilde{T}_i, \tilde{T}_j^*] = 0$.

3.2 Property of Torsion Invariants

The main properties of the torsion invariant are given in Theorems 3.1 – 3.3. We first prove the following lemma.

Lemma 3.3 *Let* $\{C_j, d_j\}_{j=-k}^{n}$ *be a complex of finitely generated Hilbert* $\mathcal{A}$*-modules and* $\Delta_j(C_*) = d_j^*d_j + d_{j-1}d_{j-1}^*$ *its Laplace operators. Then for* $\Delta_j'(C_*) = \Delta_j(C_*)|_{(Ker\Delta_j(C_*))^\perp}$*,*

$$\sum_{j=-k}^{n}(-1)^j[\Delta_j'(C_*)] = 0.$$

Proof. Using the decomposition $C_j = \overline{imd_{j-1}} \oplus Ker\Delta_j(C_*) \oplus \overline{imd_j^*}$, we have $\Delta_j'(C_*) = d_j^* d_j|_{\overline{imd_j^*}} \oplus d_{j-1}d_{j-1}^*|_{\overline{imd_{j-1}}}$. Since $d_{j-1} : \overline{imd_{j-1}^*} \to \overline{imd_{j-1}}$ is a weak isomorphism and $d_{j-1}(d_{j-1}^* d_{j-1}) = (d_{j-1}d_{j-1}^*)d_{j-1}$ on $\overline{imd_{j-1}^*}$, we obtain $[d_{j-1}^* d_{j-1}|_{\overline{imd_{j-1}^*}}] = [d_{j-1}d_{j-1}^*|_{\overline{imd_{j-1}}}]$. Hence,

$$\sum_{j=-k}^{n}(-1)^j[\Delta_j'(C_*)] = \sum_{j=-k}^{n}(-1)^j([d_j^* d_j|_{\overline{imd_j}}] + [d_{j-1}^* d_{j-1}|_{\overline{imd_{j-1}^*}}]) = 0.$$

Q.E.D.

Theorem 3.1 *Let $H^{(i)}$ be two Hilbert $\mathcal{A}$-modules and $T^{(i)} = (T_1^{(i)}, \ldots, T_n^{(i)})$ two n-tuples of commuting $\mathcal{A}$-operators on $H^{(i)}, i = 1, 2$.*

(1) If $\tilde{T}^{(1)} = (T_{\alpha(1)}^{(1)}, \ldots, T_{\alpha(n)}^{(1)})$ is obtained by permuting $(T_1^{(1)}, \ldots, T_n^{(1)})$, then $\tau(\tilde{T}^{(1)}, H^{(1)}) = \tau(T^{(1)}, H^{(1)})$,

(2) $\tau(T^{(1)} \oplus T^{(2)}, H^{(1)} \oplus H^{(2)}) = \tau(T^{(1)}, H^{(1)}) + \tau(T^{(2)}, H^{(2)})$,

(3) $\tau(T^{(1)}\otimes I + I \otimes T^{(2)}, H^{(1)}\otimes H^{(2)}) = \chi(C_*(T^{(1)}))\otimes\tau(T^{(2)}, H^{(2)}) + \tau(T^{(1)}, H^{(1)})\otimes \chi(C_*(T^{(2)}))$, where $\chi(C_*(T^{(i)})) = \sum_{j=0}^{n}(-1)^j[Ker\Delta_j(T^{(i)})] \in K_0(\mathcal{A})$,

(4) $\tau(T^{(1)}, H^{(1)}) = (-1)^{n+1}\tau((T^{(1)})^*, H^{(1)})$.

Proof. (1) is immediate since the Laplace operators associated with $\tilde{T}^{(1)}$ and $T^{(1)}$ are unitarily conjugate.

(2) Let $\Delta_j(T)$ be the Laplace operators associated with $T = T^{(1)} \oplus T^{(2)}$. Then $\Delta_j'(T) = \Delta_j'(T^{(1)}) \oplus \Delta_j'(T^{(2)})$. Hence, $[\Delta_j'(T)] = [\Delta_j'(T^{(1)})] + [\Delta_j'(T^{(2)})]$. This implies (2).

(3) Clearly, the pairing $K_0(\mathcal{A}) \otimes K_1^w(\mathcal{A}) \to K_w(\mathcal{A})$ given by $[H_0] \otimes [(H_1, f)] = [(H_0 \otimes H_1, I \otimes f)]$ is well defined. Let $T = T^{(1)} \otimes I + I \otimes T^{(2)} = (T_1^{(1)} \otimes I + I \otimes T_1^{(2)}, \ldots, T_n^{(1)} \otimes I + I \otimes T_n^{(2)})$.

$$C_j(T) = (H^{(1)} \otimes H^{(2)}) \otimes \oplus_{p+q=j} \wedge^p [\sigma] \wedge \wedge^q[\sigma]$$

$$= \oplus_{p+q=j} C_p(T^{(1)}) \otimes C_q(T^{(2)}).$$

With this identification, we get

$$\begin{aligned} d_j(T)(\xi) &= \sum_{l=1}^{n} (T_l^{(1)} \otimes I + I \otimes T_l^{(2)}) \sigma_l \wedge \xi \\ &= \oplus_{p+q=j} (d_p^{(1)} \otimes I + (-1)^p I \otimes d_q^{(2)})(\xi). \end{aligned}$$

Hence $\{C_*(T), d_*(T)\}$ is isometrically isomorphic to the tensor product of complexes $\{C_*(T^{(i)}), d_*(T^{(i)})\}$. By Lemma 7.14 [LüR] we get the result.

(4) Let $\hat{d}_j^{(1)} = \sum_{l=1}^{n} T_l^{(1)} S_l^* : C_j(T^{(1)}) \to C_{j-1}(T^{(1)})$ and $\hat{\Delta}_j(T^{(1)}) = (\hat{d}_j^{(1)})^* \hat{d}_j^{(1)} + \hat{d}_{j+1}^{(1)} (\hat{d}_{j+1}^{(1)})^*$ on $C_j(T^{(1)})$. Using the unitary operator $\sharp : C_*(T^{(1)}) \to C_*(T^{(1)})$ defined in Chapter 2, we have $\sharp d_*^{(1)} = \hat{d}_*^{(1)} \sharp$ and then

$$\Delta_j(T^{(1)}) = \sharp^* ((\hat{d}_{n-j}^{(1)})^* \hat{d}_{n-j}^{(1)} + \hat{d}_{n-j+1}^{(1)} (\hat{d}_{n-j+1}^{(1)})^*) \sharp = \sharp^* \hat{\Delta}_{n-j}(T^{(1)}) \sharp \qquad (3.2)$$

Let $\tilde{d}_*^{(1)}$ be the differential of the Koszul complex $C_*((T^{(1)})^*)$ associated with $(T^{(1)})^*$. Then $(\hat{d}_j^{(1)})^* = \tilde{d}_{j-1}^{(1)}$ and $\hat{d}_j^{(1)} = (\tilde{d}_{j-1}^{(1)})^*$. This implies $\hat{\Delta}_j(T^{(1)}) = \tilde{\Delta}_j((T^{(1)})^*)$. By (3.2), we get $\Delta_j(T^{(1)}) = \sharp^* \tilde{\Delta}_{n-j}((T^{(1)})^*) \sharp$. Therefore, by Lemma 3.3,

$$\begin{aligned} \tau(T^{(1)}, H^{(1)}) &= \sum_{j=0}^{n} (-1)^{j+1} j [(\Delta'_{n-j}((T^{(1)})^*)] = (-1)^{n+1} \tau((T^{(1)})^*, H^{(1)}) \\ &+ (-1)^{n+1} n \sum_{j=0}^{n} (-1)^j [\Delta'_j((T^{(1)})^*)] = (-1)^{n+1} \tau((T^{(1)})^*, H^{(1)}). \end{aligned}$$

Q.E.D.

We know in Chapter 1 (cf. [RaS 1]) that the analytic torsions of even dimensional (compact) Riemannian manifolds are zero. The following theorem is an analogue of this result for the operator case.

Theorem 3.2 *Let H be a Hilbert $\mathcal{A}$-module and $T = (T_1, \ldots, T_n)$ an n-tuple of commuting $\mathcal{A}$-operators on H. Suppose $[\tilde{d}_j \tilde{d}_j^*|_{\overline{im\tilde{d}_j}}] = [d_j d_j^*|_{\overline{imd_j}}], j = 0, \ldots, n-1$, where $\tilde{d}_j$ is the differential of the Koszul complex associated with the adjoint n-tuple T^* of T. Then $\tau(T, H) = 0$ for n even. Furthermore, if $T_i T_j^* = T_j^* T_i$, $j = 1, \ldots, n$, then for $n > 1, \tau(T, H) = 0$.*

Proof. We know already from the proof of Lemma 3.3 that

$$[d_{j-1} d_{j-1}^*|_{\overline{imd_{j-1}}}] = [d_{j-1}^* d_{j-1}|_{\overline{imd_{j-1}^*}}].$$

$$\begin{aligned}
\tau(T, H) &= \sum_{j=0}^{n} (-1)^{j+1} j ([d_j^* d_j|_{\overline{imd_j^*}}] + [d_{j-1} d_{j-1}^*|_{\overline{imd_{j-1}}}]) \\
&= \sum_{j=0}^{n-1} (-1)^j [d_j^* d_j|_{\overline{imd_j^*}}]. \qquad (3.3)
\end{aligned}$$

The assumption and (3.3) show that $\tau(T, H) = \tau(T^*, H)$. But Theorem 3.1 implies that for n even, $\tau(T, H) = -\tau(T^*, H)$. We get $\tau(T, H) = 0$.

If $[T_i, T_j^*] = 0$, then by Lemma 3.1, $\Delta_j(T) = \sum_{j=1}^n T_j^{(i)} (T_j^{(i)})^*$. $[\Delta_j'(T)] = m_j[(\sum_{j=1}^n T_j^{(i)} (T_j^{(i)})^*)^\perp]$. This together with the fact that $\sum_{j=0}^n (-1)^j j m_j = 0$ for $n > 1$ proves

$$\tau(T, H) = \sum_{j=0}^{n} (-1)^{j+1} j m_j [(\sum_{l=1}^{n} T_l^{(i)} (T_l^{(i)})^*)^\perp] = 0.$$

Q.E.D.

By Theorem 3.2 we obtain immediately the following.

Corollary 3.1 *Let $\mathcal{A} = L^\infty(X, \mu)$ be a finite abelian von Neumann algebra with X compact second countable and μ a measure on X. Suppose T_j is the operator on $L^2(X, \mu)$ defined by the multiplication by $\varphi_j \in L^\infty(X, \mu), j = 1, \ldots, n$. Then for $n > 1$ and $T = (T_1, \ldots, T_n)$, $\tau(T, L^2(X, \mu)) = 0$.*

We now consider the torsion $\tau(U, T^{(i)})$ of $U : H^{(1)} \to H^{(2)}$ relative to $T^{(i)}$. The following result is an analogue of Theorem 3.1 for $\tau(U, T^{(i)})$.

Proposition 3.2 *Let $H^{(i)}$ and $\tilde{H}^{(i)}$ be Hilbert $\mathcal{A}$-modules and $T^{(i)}$ and $\tilde{T}^{(i)}$ n-tuples of commuting $\mathcal{A}$-operators on $H^{(i)}$ and $\tilde{H}^{(i)}$, resp., $i = 1, 2$. If $U : H^{(1)} \to H^{(2)}$ and $\tilde{U} : \tilde{H}^{(1)} \to \tilde{H}^{(2)}$ are weak cohomology equivalences relative to $T^{(i)}$ and $\tilde{T}^{(i)}$, resp.. Then $\tau(-U, T^{(i)}) = (-1)^n \tau(U^*, (T^{(i)})^*)$.*

Proof. Let us figure out the relation between $d_*(U)$ and $d_*(U^*)$, the differentials of the mapping cones $C_*(U)$ and $C_*(U^*)$ of U and U^*, resp.. Let $\tilde{d}_*^{(i)}$ be the differential of the Koszul complex associated with $(T^{(i)})^*$. We get from the proof of Theorem 3.1 that $d_j^{(i)} = \sharp^* (\tilde{d}_{n-j-1}^{(i)})^* \sharp$. Under the identification $C_j(U^*) = C_{j+1}(T^{(2)}) \oplus C_j(T^{(1)}) \simeq C_j(T^{(1)}) \oplus C_{j+1}(T^{(2)})$, we have

$$
\begin{aligned}
d_j(U^*) &= \begin{bmatrix} -\tilde{d}_{j+1}^{(2)} & 0 \\ U^* & \tilde{d}_j^{(1)} \end{bmatrix} \simeq \begin{bmatrix} \tilde{d}_j^{(1)} & U^* \\ 0 & -\tilde{d}_{j+1}^{(2)} \end{bmatrix} \\
&= \begin{bmatrix} \sharp^* & 0 \\ 0 & \sharp^* \end{bmatrix} \begin{bmatrix} (d_{n-j-1}^{(1)})^* & U^* \\ 0 & -(d_{n-j-2}^{(2)})^* \end{bmatrix} \begin{bmatrix} \sharp & 0 \\ 0 & \sharp \end{bmatrix} \\
&= -\begin{bmatrix} \sharp^* & 0 \\ 0 & \sharp^* \end{bmatrix} d_{n-j-2}^*(-U) \begin{bmatrix} \sharp & 0 \\ 0 & \sharp \end{bmatrix}. \\
\Delta_j(U^*) &\simeq \begin{bmatrix} \sharp^* & 0 \\ 0 & \sharp^* \end{bmatrix} \Delta_{n-j-1}(-U) \begin{bmatrix} \sharp & 0 \\ 0 & \sharp \end{bmatrix}.
\end{aligned}
$$

Therefore by Lemma 3.3,

$$\begin{aligned}\tau(U^*,(T^{(i)})^*) &= \sum_{j=0}^{n}(-1)^{j+1}j[(\Delta'_{n-j-1}(-U))] \\ &= (-1)^n\tau(-U,T^{(i)}).\end{aligned}$$

Q.E.D.

Finally we consider the following embedding problem. Let $T=(T_1,\ldots,T_n)$ be an n-tuple of commuting $\mathcal{A}$-operators on H. Suppose $\tilde{T}=(T_{j_1},\ldots,T_{j_m})$ is a sub-m-tuple of T for $m<n$. How is $\tau(\tilde{T},H)$ related to $\tau(T,H)$? A special case is discussed below.

Theorem 3.3 *Let H be a Hilbert $\mathcal{A}$-module and $T=(T_1,\ldots,T_n)$ an n-tuple of commuting $\mathcal{A}$-operators on H. Let $\tilde{T}=(T_1,\ldots,T_n,0)$ be an $n+1$-tuple. Then $\tau(\tilde{T},H)=0$.*

Proof. Let $C_j(\sigma_{n+1})=C_{j-1}(T)\wedge\sigma_{n+1}$ with differentials $d_j(\sigma_{n+1})=\sum_{l=1}^{n}T_lS_l$, $j=0,1,\ldots,n+1$. Then $C_j(\tilde{T})=C_j(T)\oplus C_j(\sigma_{n+1})$ and the differential $d_*(\tilde{T})$ of the Koszul complex associated with $\tilde{T}$ is equal to that of $C_*(T)$. Let $\Delta_*(\sigma_{n+1})$ be the Laplace operator of $\{C_*(\sigma_{n+1}),d_*(\sigma_{n+1})\}$. We get $\Delta_j(\tilde{T})=\Delta_j(T)\oplus\Delta_j(\sigma_{n+1})$ and then $\tau(\tilde{T},H)=\tau(T,H)+\tau(C_*(\sigma_{n+1}))$. But $\Delta_j(\sigma_{n+1})=\Delta_{j-1}(T)$. By Lemma 3.3 we obtain

$$\begin{aligned}\tau(\tilde{T},H) &= \sum_{j=0}^{n+1}(-1)^{j+1}j([\Delta'_j(T)]+[\Delta'_j(\sigma_{n+1})]) \\ &= \sum_{j=0}^{n+1}(-1)^{j+1}j([\Delta'_j(T)]+[\Delta'_{j-1}(T)]) \\ &= \sum_{j=0}^{n+1}(-1)^{j}[\Delta'_j(T)]=0.\end{aligned}$$

Q.E.D.

The following example is different from Theorem 3.3.

Example 3.2 *Let $T = (T_1, I)$ on H . Then $\Delta_0(T) = T_1^*T_1 + I, \Delta_1(T) = (T_1T_1^* + I) \oplus (T_1^*T_1 + I)$ and $\Delta_2(T) = T_1T_1^* + I$.*

$$\begin{aligned} \tau(T, H) &= ([(T_1T_1^* + I)^\perp] + [(T_1^*T_1 + I)^\perp]) - 2[(T_1T_1^* + I)^\perp] \\ &= [(T_1^*T_1 + I)] - [(T_1T_1^*)] = 0, \end{aligned}$$

Since $Ker(T_1^*T_1 + I) = 0, [(H, I)] = 0$, $T_1(T_1^*T_1 + I) = (T_1T_1^* + I)T_1$ and $T_1 : (KerT_1^*T_1)^\perp \to (KerT_1T_1^*)^\perp = \overline{imT_1}$ is a weak isomorphism. But in general, $\tau(T_1, H) \neq 0$.

3.3 Torsion for a Single Operator

Let H be a Hilbert $\mathcal{A}$-module and T an $\mathcal{A}$-operator on H. By Lemma 3.1, the Laplace operators associated with T are $\Delta_0(T) = T^*T$ and $\Delta_1(T) = TT^*$. Theorem 3.2 implies

$$\tau(T, H) = -[(TT^*)^\perp] = -[(T^*T)^\perp]. \tag{3.4}$$

The theorem below shows that the torsion is a week complete unitary invariant of operators in the sense that the torsion determines and is determined by the operators up to conjugation by isomorphism and a direct sum factor of weak isomorphisms.

Theorem 3.4 *Let $H^{(i)}$ be two Hilbert $\mathcal{A}$-modules and $T^{(i)}$ two $\mathcal{A}$-operators on $H^{(i)}$, resp., $i = 1, 2$. Then $\tau(T^{(1)}, H^{(1)}) = \tau(T^{(2)}, H^{(2)})$ iff there exist finitely generated Hilbert $\mathcal{A}$-modules H_j and weak isomorphisms T_j on $H_j, j = 1, 2,$*

such that $[(H_1, T_1)] = [(H_2, T_2)]$ *and* $(T^{(1)}(T^{(1)})^*)^\perp \oplus T_1$ *and* $(T^{(2)}(T^{(2)})^*)^\perp \oplus T_2$ *are conjugate by an isomorphism.*

Proof. Let $\mathcal{C}(\mathcal{A})$ be the category whose objects are all pairs (H, T) with H a finitely generated Hilbert $\mathcal{A}$-module and T a weak automorphism on H. A morphism $U : (H_1, T_1) \to (H_2, T_2)$ is an isomorphism from H_1 onto H_2 such that $UT_1 = T_2U$. Define product and composition operations on $\mathcal{C}(\mathcal{A})$ by $(H_1, T_1) \oplus (H_2, T_2) = (H_1 \oplus H_2, T_1 \oplus T_2)$ and $(H, T_1) \cdot (H, T_2) = (H, T_1T_2)$. Then $\mathcal{C}(\mathcal{A})$ is a category with product and composition. Let $\tilde{K}_1^w(\mathcal{A}) = K_0(\mathcal{C}(\mathcal{A}), +, 0)$, the Grothendieck group of the category $\mathcal{C}(\mathcal{A})$ [Sil]. $\tilde{K}_1^w(\mathcal{A})$ is an abelian group with generators $[(H, T)]$ subject to the following relations.

(1) $[(H_1, T_1)] = [(H_2, T_2)]$ if there is an isomorphism $U : H_1 \to H_2$ with $UT_1 = T_2U$;

(2) $[(H_1 \oplus H_2, T_1 \oplus T_2)] = [(H_1, T_1)] + [(H_2, T_2)]$;

(3) $[(H, T_1T_2)] = [(H, T_1)] + [(H, T_2)]$;

(4) $[(H, I)] = 0$.

Comparing this with $K_1^w(\mathcal{A})$, we see that $\tilde{K}_1^w(\mathcal{A})$ differs from $K_1^w(\mathcal{A})$ only by (2) which is $[(H_1, T_1)] + [(H_3, T_3)] = [(H_2, T_2)]$ in $K_1^w(\mathcal{A})$ associated with a short exact sequence

$$0 \to (H_1, T_1) \xrightarrow{i} (H_2, T_2) \xrightarrow{\pi} (H_3, T_3) \to 0. \tag{3.5}$$

Hence there is a surjective map from $\tilde{K}_1^w(\mathcal{A})$ to $K_1^w(\mathcal{A})$. The kernel of this map is the set generated by $[(H_1, T_1)] + [(H_3, T_3)] - [(H_2, T_2)]$ associated with short nonsplit exact sequence (3.5). Furthermore, if we forget the composition operation in $\mathcal{C}(\mathcal{A})$ and consider the Grothendieck group $K_0(\mathcal{C}(\mathcal{A}),+)$ of $\mathcal{C}(\mathcal{A})$. Then $K_0(\mathcal{C}(\mathcal{A}),+)$ is an abelian group generated by $[(H,T)]$ with relations $(1),(2)$ and (4) above. Then the natural map from $K_0(\mathcal{C}(\mathcal{A}),+)$ to $\tilde{K}_1^w(\mathcal{A})$ is also surjective. Its kernel is generated by $[(H,T_1)]+[(H,T_2)]-[(H,T_1T_2)]$. Hence the kernel of the map $K_0(\mathcal{C}(\mathcal{A}),+)\to\tilde{K}_1^w(\mathcal{A})\to K_1^w(\mathcal{A})$ is generated by $[(H_1,T_1)]+[(H_3,T_3)]-[(H_2,T_2)]$ and $[(H,T_1)]+[(H,T_2)]-[(H,T_1T_2)]$.

With these preparations we can proceed the proof of the theorem. Suppose $\tau(T^{(1)},H^{(1)})=\tau(T^{(2)},H^{(2)})$ in $K_1^w(\mathcal{A})$. We have in $K_0(\mathcal{C}(\mathcal{A}),+)$,

$$
\begin{aligned}
& \tau(T^{(1)},H^{(1)})+\sum_i[(H_i,T_{i_1}T_{i_2})]+\sum_k[(H_{k_2},T_{k_2})]\\
+ & \sum_j([\tilde{H}_j,\tilde{T}_{j_1})]+[(\tilde{H}_j,\tilde{T}_{j_2})])+\sum_l([(\tilde{H}_{l_1},\tilde{T}_{l_1})]+[(\tilde{H}_{l_3},\tilde{T}_{l_3})])\\
= & \tau(T^{(2)},H^{(2)})+\sum_i([(H_i,T_{i_1})]+[(H_i,T_{i_2})])\\
+ & \sum_k([(H_{k_1},T_{k_1})]+[(H_{k_3},T_{k_3})])+\sum_j[(\tilde{H}_j,\tilde{T}_{j_1}\tilde{T}_{j_2})]+\sum_l[(\tilde{H}_{l_2},\tilde{T}_{l_2})].
\end{aligned}
$$

We simplify the above identity as

$$
\begin{aligned}
& \tau(T^{(1)},H^{(1)})+[(H_1,T_{11}T_{12})]+[(\tilde{H}_1,\tilde{T}_{11})]+[(\tilde{H}_1,\tilde{T}_{12})]\\
+ & [(H_{22},T_{22})]+[(\tilde{H}_{31},\tilde{T}_{31})]+[(\tilde{H}_{33},\tilde{T}_{33})]\\
= & \tau(T^{(2)},H^{(2)})+[(H_1,T_{11})]+[(H_1,T_{12})]+[(\tilde{H}_1,\tilde{T}_{11}\tilde{T}_{12})]\\
+ & [(H_{21},T_{21})]+[(H_{23},T_{23})]+[(\tilde{H}_{32},\tilde{T}_{32})],
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
H_1 &= \oplus_i H_i,\ \ T_{1j}=\oplus_i T_{ij},\ \ j=1,2,\ \ on\ \ H_1,\\
\tilde{H}_1 &= \oplus_j \tilde{H}_j,\ \ \tilde{T}_{1i}=\oplus_j\tilde{T}_{ji},\ \ i=1,2,\ \ on\ \ H_2'\\
H_{2k} &= \oplus_j H_{jk},\ \ T_{2k}=\oplus_j T_{jk},\ \ k=1,2,3,\ \ on\ \ H_{2k},\\
\tilde{H}_{2l} &= \oplus_j \tilde{H}_{jl},\ \ \tilde{T}_{2l}=\oplus_j\tilde{T}_{jl},\ \ l=1,2,3,\ \ on\ \ \tilde{H}_{2l}.
\end{aligned}
$$

All Hilbert $\mathcal{A}$-modules here are finitely generated and all maps involved are weak isomorphisms. Note that $0 \to H_{21} \to H_{22} \to H_{23} \to 0$ and $0 \to \tilde{H}_{21} \to \tilde{H}_{22} \to \tilde{H}_{23} \to 0$ are exact sequences. Since identity (4.6) holds in $K_0(\mathcal{C}(\mathcal{A}), +)$, we obtain that there exists (H_3, T_3) in $\mathcal{C}(\mathcal{A})$ such that

$$\begin{aligned} & (T^{(1)}(T^{(1)})^*)^\perp \oplus T_3 \oplus (T_{11}T_{12}) \oplus \tilde{T}_{11} \oplus \tilde{T}_{12} \oplus T_{22} \oplus \tilde{T}_{31} \oplus \tilde{T}_{33} \\ \cong \; & (T^{(2)}(T^{(2)})^*)^\perp \oplus T_3 \oplus T_{11} \oplus T_{12} \oplus (\tilde{T}_{11}\tilde{T}_{22}) \oplus T_{21} \oplus T_{23} \oplus \tilde{T}_{22}. \end{aligned}$$

We have thus proved one direction of the theorem.

Conversely, if there are Hilbert $\mathcal{A}$-modules H_j and weak isomorphisms T_j on H_j such that $[(H_1, T_1)] = [(H_2, T_2)]$ and

$$(T^{(1)}(T^{(1)})^*)^\perp \oplus T_1 \cong (T^{(2)}(T^{(2)})^*)^\perp \oplus T_2.$$

Then by the definition of $K_1^w(\mathcal{A})$ we get

$$[(T^{(1)}(T^{(1)})^*)^\perp] = [(T^{(2)}(T^{(2)})^*)^\perp]$$

in $K_1^w(\mathcal{A})$. By (3.4) this is $\tau(T^{(1)}, H^{(1)}) = \tau(T^{(2)}, H^{(2)})$. Q.E.D.

Corollary 3.2 *Let $H^{(i)}$ be two Hilbert $\mathcal{A}$-modules and $T^{(i)} = (T_1^{(i)}, \ldots, T_n^{(i)})$ two n-tuples of commuting $\mathcal{A}$-operators on $H^{(i)}$, resp., $i = 1, 2$. Then $\tau(T^{(1)}, H^{(1)}) = \tau(T^{(2)}, H^{(2)})$ iff there exist finitely generated Hilbert $\mathcal{A}$-modules H_j and weak isomorphisms T_j on H_j such that $[(H_1, T_1)] = [(H_2, T_2)]$, and $\oplus_{j\ odd}(\Delta_j'(T^{(1)}))^j \oplus \oplus_{k\ even}(\Delta_k'(T^{(2)}))^k \oplus T_1$ and $\oplus_{j\ odd}(\Delta_j'(T^{(2)}))^j \oplus \oplus_{k\ even}(\Delta_k'(T^{(1)}))^k \oplus T_2$ are conjugate by an isomorphism.*

Proof. Suppose $\tau(T^{(1)}, H^{(1)}) = \sum_{j=0}^n (-1)^{j+1} j [\Delta_j'(T^{(1)})] = \tau(T^{(2)}, H^{(2)}) = \sum_{j=0}^n (-1)^{j+1} j [\Delta_j'(T^{(2)})]$ We obtain

$$[\oplus_{j\ odd}(\Delta_j'(T^{(1)}))^j \oplus \oplus_{k\ even}(\Delta_k'(T^{(2)}))^k] = [\oplus_{j\ odd}(\Delta_j'(T^{(2)}))^j \oplus \oplus_{k\ even}(\Delta_k'(T^{(1)}))^k].$$

We then proceed as the proof of Theorem 3.4 to get the result. Q.E.D.

We close this section by the following example:

Example 3.3 *Let $\mathcal{A} = L^\infty(X,\mu)$ be a finite abelian von Neumann algebra with X compact second countable and μ a measure on X. If $H = L^2(X,\mu)$ and T_j are the multiplication operators determined by $\varphi_j \in L^\infty(X,\mu), j = 1,\ldots,n$, then we know by Corollary 3.1 that $\tau(T,H) = 0$ for $n > 1$. By (3.4) $\tau(T,H) = [(T_1T_1^*)^\perp]$ for $n = 1$. Recall that the determinant det from $K_1^w(\mathcal{A})$ onto $F(X,\mu)^\times$ is an isomorphism. Since $KerT_1T_1^* = KerT_1^* = \{f \in H : f = 0 \ \text{on} \ X\setminus(Ker\varphi_1)\}$ and $(KerT_1T_1^*)^\perp = \{f \in H : f = 0 \ \text{on} \ Ker\varphi_1\}$, we have*

$$\begin{aligned} det(\tau(T,H)) &= (det((T_1T_1^*)^\perp \oplus I|_{KerT_1T_1^*}))^{-1} \\ &= (|\varphi_1|^2|_{\varphi_1\neq 0} + \chi_{\varphi_1=0})^{-1} \in F(X,\mu)^\times, \end{aligned}$$

where $\chi_{\varphi_1\neq 0}$ is the characteristic function of set $\{\varphi_1 = 0\}$ in X.

More generally, if $H = \oplus_{l=1}^m L^2(X,\mu)$ and T_j are the operators determined by the multiplication of matrices $\varphi_j^{(1)} \oplus \ldots \oplus \varphi_j^{(m)}, j = 1,\ldots,n$. Then for $T = (T_1,\ldots,T_n)$ and $n > 1$, $det(\tau(T,H)) = 1$, and

$$det(\tau(T,H)) = (\prod_{l=1}^m (|\varphi_1^{(l)}|^2 + \chi_{\varphi_1^{(l)}\neq 0}))^{-1} \in F(X,\mu)^\times, \ \ n = 1.$$

Chapter 4

Equivariant Bivariant Cyclic Theory

This chapter arises in the desire of extending Jones-Kassel bivariant cyclic theory to the equivariant case. We first face the problem about how to define equivariant bivariant cyclic theory for topological algebras. Keeping in mind the application of cyclic (co)homology to index theory, we choose to define the equivariant bivariant cyclic theory which is slightly different from equivariant cyclic cohomology discused in [Gong 2] and in the next chapter. The advantage of this definition is that we can use it to construct equivariant bivariant Chern characters. As one should expect, the properties of Jones-Kassel bivariant cyclic theory are not always extended to our situation due to topological structure of algebras. Our aim is to prove the excision property and construct the Chern characters.

This chapter is arranged as follows. In section 4.1 we define equivariant (entire) bivariant cyclic and Hochschild theories of Fréchet locally $m-$convex G-algebras for Lie group G. Some basic properties are discussed. We use the results in [Gong 2] to prove the excision property in section 4.2. A six-term exact sequence of equivariant periodic bivariant cyclic theory is obtained. Our main result in this chapter is to construct the equivariant bivariant Chern characters which extend the result [Nis] to the equivariant case. Our approach is different from those in ([Kas], [Wang]) but is related to that in [Nis].

4.1 Equivariant Bivariant Cyclic Theory: Definitions

Let $\mathcal{A}$ be a Fréchet locally m-convex algebra over $\mathbb{C}$. This means that $\mathcal{A}$ is a complete topological algebra with topology determined by a sequence of seminorms $\{\rho_n\}_{n=1}^{\infty}$ satisfying $\rho_n(a_1a_2) \leq \rho_n(a_1)\rho_n(a_2)$ for $a_1, a_2 \in \mathcal{A}$ and $n \in \mathbb{N}$. Suppose that G is a locally compact Lie group acting on $\mathcal{A}$ by smooth automorphisms $\alpha : G \to Aut(\mathcal{A})$ such that $\rho_n(ga) \leq \rho_n(a)$ for $g \in G$ and $a \in \mathcal{A}$ and if $\mathcal{A}$ is unital $\alpha_g(1) = I$. Here $Aut(\mathcal{A})$ is the space of continuous automorphisms of $\mathcal{A}$. Thus for each $a \in \mathcal{A}$ $\alpha_g(a)$ is a smooth map of g. Let

$$C_c^{\infty}(G, \mathcal{A}) = \{\varphi \in C^{\infty}(G, \mathcal{A}) : Supp(\varphi) \textit{ is compact in } G\}$$

with the multiplication

$$(f_1 * f_2)(g) = \int_G f_1(h)\alpha_h f_2(h^{-1}g)dh, \ f_i \in C_c^{\infty}(G, \mathcal{A}).$$

$C_c^{\infty}(G, \mathcal{A})$ is isomorphic to the projective tensor product $C_c^{\infty}(G)\hat{\otimes}\mathcal{A}$ of $C_c^{\infty}(G)$ and $\mathcal{A}$. Since the projective tensor product of Fréchet spaces is Fréchet, we see that $\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^{\infty}(G)$ is a Fréchet space for $n \in \mathbb{N}$. We follow [Gong 2] to define the equivariant cyclic (co)chain complex. Let $d_n^i : \mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^{\infty}(G) \to \mathcal{A}^{\hat{\otimes}(n)}\hat{\otimes}C_c^{\infty}(G)$ and $t_n : \mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^{\infty}(G) \to \mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^{\infty}(G)$ be defined by

$$d_n^i(a_0,a_1,\ldots,a_n,f)(g)=\begin{cases}(a_0\alpha_g(a_1),a_2,\ldots,a_n,f)(g),\ i=0,\\ \\ (a_0,\ldots,a_ia_{i+1},\ldots,a_n,f)(g),\ 1\le i\le n-1,\\ \\ (a_na_0,a_1,\ldots,a_{n-1},f)(g),\ i=n\end{cases}$$

and

$$t_n(a_0,a_1,\ldots,a_n,f)(g)=(a_n,\alpha_g^{-1}(a_0),a_1,\ldots,a_{n-1},f)(g),n=0,1,\ldots,$$

where $(a_0,a_1,\ldots,a_n,f)$ stands for $a_0\hat{\otimes}a_1\hat{\otimes}\ldots,\hat{\otimes}a_n\hat{\otimes}f$. d_n^i and t_n are continuous. One can easily verify [Gong 2]

$$\begin{aligned} d_{n-1}^i d_n^j &= d_{n-1}^{j-1}d_n^i,\quad i<j,\\ d_n^i t_n &= \begin{cases} t_{n-1}d_n^{i-1}, & 1\le i\le n;\\ \\ d_n^n, & i=0.\end{cases}\end{aligned}$$

and

$$t_n^{n+1}(a_0,a_1,\ldots,a_n,f)(g)=(\alpha_g^{-1}(a_0),\ldots,\alpha_g^{-1}(a_n))f(g).$$

Define a G-action on $\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^\infty(G)$ by

$$\rho_h(a_0,a_1,\ldots,a_n,f)(g)=(\alpha_h^{-1}(a_0),\ldots,\alpha_h^{-1}(a_n))f(hgh^{-1}),\quad \forall h\in G.$$

Clearly, $t_n\rho_h=\rho_h t_n$ and $d_n^i\rho_h=\rho_h d_n^i$ for $h\in G$. In fact, as in [Gong 2],

$$\begin{aligned} t_n\rho_h(a_0,a_1,\ldots,a_n,f)(g) &= t_n(\alpha_h^{-1}(a_0),\ldots,\alpha_h^{-1}(a_n),h\cdot f)(g)\\ &= (\alpha_h^{-1}(a_n),\alpha_g^{-1}(\alpha_h^{-1}(a_0)),\ldots,\alpha_h^{-1}(a_{n-1}))(h\cdot f)(g)\end{aligned}$$

$$
\begin{aligned}
&= \alpha_h^{-1}(a_n, \alpha_{hgh^{-1}}^{-1}(a_0), a_1, \ldots, a_{n-1}) f(hgh^{-1}) \\
&= \rho_h(a_n, \alpha_{(\cdot)}^{-1}(a_0), a_1, \ldots, a_{n-1}, f(\cdot))(g) \\
&= \rho_h t_n(a_0, a_1, \ldots, a_n, f)(g),
\end{aligned}
$$

where $h \cdot f(g) = f(hgh^{-1})$, and

$$
\begin{aligned}
& d_n^i \rho_h(a_0, a_1, \ldots, a_n, f)(g) = d_n^i(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n), h \cdot f)(g) \\
&= \begin{cases}
(\alpha_h^{-1}(a_0)\alpha_g(\alpha_h^{-1}(a_1)), \alpha_h^{-1}(a_2), \ldots, \alpha_h^{-1}(a_n)) f(hgh^{-1}), i = 0, \\
(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_i a_{i+1}), \ldots, \alpha_h^{-1}(a_n)) f(hgh^{-1}), 1 \le i \le n-1, \\
(\alpha_h^{-1}(a_n a_0), \alpha_h^{-1}(a_1), \ldots, \alpha_h^{-1}(a_{n-1})) f(hgh^{-1}), i = n
\end{cases} \\
&= \begin{cases}
\alpha_h^{-1}(a_0, \alpha_{hgh^{-1}}(a_1), a_2, \ldots, a_n) f(hgh^{-1}), i = 0, \\
\alpha_h^{-1}(a_0, \ldots, a_i a_{i+1}, \ldots, a_n) f(hgh^{-1}), 1 \le i \le n-1, \\
\alpha_h^{-1}(a_n a_0, \ldots, a_{n-1}) f(hgh^{-1}), i = n
\end{cases} \\
&= \rho_h d_n^i(a_0, a_1, \ldots, a_n, f)(g).
\end{aligned}
$$

Let b and b' be the morphisms on $\mathcal{A}^{\hat{\otimes}(n+1)} \hat{\otimes} C_c^\infty(G)$ defined by

$$
b = \sum_{i=0}^{n} (-1)^i d_n^i, \quad b' = \sum_{i=0}^{n-1} (-1)^i d_n^i.
$$

Then b and b' are equivariant and continuous. Let T and $J : \mathcal{A}^{\hat{\otimes}(n+1)} \hat{\otimes} C_c^\infty(G) \to \mathcal{A}^{\hat{\otimes}(n+1)} \hat{\otimes} C_c^\infty(G)$ be defined by

$$
T = (-1)^n t_n, \quad J(a_0, a_1, \ldots, a_n, f)(g) = (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}, f)(g).
$$

We proved in [Gong 2] the following facts on $\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^\infty(G)$:

(1) $T^{n+1}=\rho$, where

$$\rho(a_0,a_1,\ldots,a_n,f)(g)=(\alpha_g^{-1}(a_0),\ldots,\alpha_g^{-1}(a_n))f(g),$$

(2) $T^iJT^{(i+1)n}=(-1)^i\rho^{i+1}d_n^i$,

(3) $b=\sum_{i=0}^n \rho^{-(i+1)}T^iJT^{(i+1)n}$, $\;b'=\sum_{i=0}^{n-1}\rho^{-(i+1)}T^iJT^{(i+1)n}$.

Let $C_n^G(\mathcal{A})=\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}_{G,\rho}C_c^\infty(G)\stackrel{\text{def}}{=}(\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^\infty(G))/M$, where M is the closure of linear span $\{\rho_h(x\hat{\otimes}f)-x\hat{\otimes}f,\rho(x\hat{\otimes}f)-x\hat{\otimes}f,x\in\mathcal{A}^{\hat{\otimes}(n+1)},f\in C_c^\infty(G)\}$. Then b,b' and T descent to $C_n^G(\mathcal{A})$. We have

Lemma 4.1

$$\begin{aligned} b &= \sum_{i=0}^n T^iJT^{(i+1)n}=\sum_{i=0}^n T^iJT^{-(i+1)},\\ b' &= \sum_{i=0}^{n-1}T^iJT^{-(i+1)},\;\; T^{n+1}=I,\;\; b(1-T)=(1-T)b',\;\; b'N=Nb \end{aligned}$$

and $b^2=(b')^2=0$ *on* $C_n^G(\mathcal{A})$.

Proof. By the above discussion, we need only to check $\rho d_n^i=d_n^i\rho$ and $t_n\rho=\rho t_n$ on $\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^\infty(G)$. In fact,

$$\begin{aligned} \rho d_n^i(a_0,a_1,\ldots,a_n,f)(g) &= \begin{cases} \alpha_g^{-1}(a_0\alpha_g(a_1),a_2,\ldots,a_n)f(g), & i=0,\\ \alpha_g^{-1}(a_0,\ldots,a_ia_{i+1},\ldots,a_n,f)(g), & 1\le i\le n,\end{cases}\\ &= d_n^i\rho(a_0,a_1,\ldots,a_n,f)(g)\\ t_n\rho(a_0,a_1,\ldots,a_n,f)(g) &= (\alpha_g^{-1}(a_n),\alpha_g^{-1}(a_0)\alpha_g^{-2}(a_1),\ldots,\alpha_g^{-1}(a_{n-1}),f)(g)\\ &= \alpha_g^{-1}(a_n,a_0\alpha_g^{-1}(a_1),a_2,\ldots,a_{n-1},f)(g)\\ &= \rho t_n(a_0,a_1,\ldots,a_n,f)(g). \end{aligned}$$

The proof that $b^2 = 0$ and $b'^2 = 0$ is a routine computation which we omit.

Q.E.D.

If $\mathcal{A}$ is unital, we define $S' : C_n^G(\mathcal{A}) \to C_{n+1}^G(\mathcal{A})$ by

$$S'(a_0, a_1, \ldots, a_n, f)(g) = (1, \alpha_g(a_0), a_1, \ldots, a_n, f)(g).$$

Lemma 4.2 *S' commutes with ρ_h and ρ, and $b'S' + S'b = I$.*

Proof. The proof is given in [Gong 2]. Q.E.D.

We now form the double complex $C_{*,*}^G(\mathcal{A})$ by

$$C_{m,n}^G(\mathcal{A}) = C_n^G(\mathcal{A}), \quad m, n \geq 0,$$

with the vertical and horizontal differentials δ_1 and δ_2 given by

$$\delta_1(x_n q^m) = \begin{cases} b(x_n)q^m, & m \ \ even, \\ \\ -b'(x_n)q^m, & m \ \ odd, \end{cases}$$

$$\delta_2(x_n q^m) = \begin{cases} N(x_n)q^{m-1}, & m \ \ even, \\ \\ (1-T)(x_n)q^{m-1}, & m \ \ odd, \end{cases}$$

where q is an indeterminate of degree 1 and the element in $C_{m,n}^G(\mathcal{A})$ is written as $x_n q^m$ for $x_n \in C_n^G(\mathcal{A})$. Then $(\delta_i)^2 = 0 = \delta_1\delta_2 + \delta_2\delta_1$. Define S on $C_{**}^G(\mathcal{A})$ by

$$S(x_n q^m) = x_n q^{m-2}, \quad m, n \geq 0.$$

S commutes with δ_i. Denote by $T_*(C_*^G(\mathcal{A}))$ the total complex of double complex $C_{**}^G(\mathcal{A})$, namely,

$$T_k(C_*^G(\mathcal{A})) = \bigoplus_{m+n=k} C_{m,n}^G(\mathcal{A})$$

with differential $d = \delta_1 + \delta_2$. $T_k(C_*^G(\mathcal{A}))$ is a Fréchet space since each $C_n^G(\mathcal{A})$ is Fréchet.

Suppose that $\mathcal{B}$ is another Fréchet locally m-convex G-algebra over $\mathbb{C}$. Let $Hom(T_i(C_*^G(\mathcal{A})), T_j(C_*^G(\mathcal{B})))$ be the space of all continuous multilinear maps from $T_i(C_*^G(\mathcal{A}))$ to $T_j(C_*^G(\mathcal{B}))$. Let $\{Hom(T_*(C_G^*(\mathcal{A})), T_*(C_G^*(\mathcal{B}))), \underline{d}\}$ be the complex given by

$$Hom(T_*(C_G^*(\mathcal{A})), T_*(C_G^*(\mathcal{B})))_n = \prod_{i \geq 0} Hom(T_i(C_G^*(\mathcal{A})), T_{i+n}(C_G^*(\mathcal{B}))), n \in \mathbb{Z},$$

with the differential $\underline{d}, \underline{d}f = df - (-1)^{|f|} f d$ for $f \in Hom(T_i(C_G^*(\mathcal{A})), T_{i+n}(C_G^*(\mathcal{B})))$, $|f| = n$ is the degree of f. Define $ad(S)$ on $Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{A})))$ by

$$ad(S)(f) = Sf - fS.$$

$ad(S)$ clearly commutes with $\underline{d}$, Let

$$Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))) = \{f \in Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))) : ad(S)(f) = 0\}$$

and

$$Ker(S, T_*(C_*^G(\mathcal{A}))) = Ker\{T_*(C_*^G(\mathcal{A})) \xrightarrow{S} T_*(C_*^G(\mathcal{A}))[2]\},$$

where $X[k]$ is the shifted complex of a complex X by k: $(X[k])_n = X_{n-k}$, with differential $\tilde{d} = (-1)^k d$. Then $\{Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))), \underline{d}\}$ and $\{Hom(Ker(S, T_*(C_*^G(\mathcal{A}))), Ker(S, T_*(C_*^G(\mathcal{B})))), \underline{d}\}$ are complexes.

Definition 4.1 (1) The equivariant bivariant cyclic theory of $\mathcal{A}$ and $\mathcal{B}$ is

$$HC_G^n(\mathcal{A}, \mathcal{B}) = H_{-n}(Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B})))), \quad n \geq 0,$$

(2) The equivariant bivariant Hochschild theory of $\mathcal{A}$ and $\mathcal{B}$ is

$$HH_G^n(\mathcal{A}, \mathcal{B}) = H_{-n}(Ker(S, T_*(C_*^G(\mathcal{A}))), Ker(S, T_*(C_*^G(\mathcal{B})))), \quad n \geq 0,$$

Clearly, the map S on $Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B})))$ given by $S(f) = Sf = fS$ induces a morphism from $HC_G^n(\mathcal{A}, \mathcal{B})$ to $HC_G^{n+2}(\mathcal{A}, \mathcal{B})$.

Definition 4.2 The equivariant periodic bivariant cyclic theory of $\mathcal{A}$ and $\mathcal{B}$ is

$$PHC_G^*(\mathcal{A}, \mathcal{B}) = \varinjlim_{n,S} HC_G^{*+2n}(\mathcal{A}, \mathcal{B}).$$

Observe that there is a short exact sequence

$$0 \to Ker(S, T_*(C_*^G(\mathcal{A}))) \to T_*(C_*^G(\mathcal{A})) \xrightarrow{S} T_*(C_*^G(\mathcal{A}))[2] \to 0. \tag{4.1}$$

Thus if $\mathcal{A}$ is nuclear, $T_*(C_*^G(\mathcal{A}))$ and $Ker(S, T_*(C_*^G(\mathcal{A})))$ are nuclear. We get the following commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 \to & & Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B})))[-2] & \to & Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B})))[-2] \\
 & & \downarrow S & & \downarrow Hom(S, I) \\
0 \to & & Hom_S(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))) & \to & Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))) \\
 & & \downarrow I & & \downarrow I \\
0 \to & & Hom(Ker(S, T^*(C_G^*(\mathcal{A}))), Ker(S, T^*(C_G^*(\mathcal{B})))) & \to & Hom(Ker(S, T^*(C_G^*(\mathcal{A}))), T^*(C_G^*(\mathcal{B}))) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

$$\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
\xrightarrow{ad(S)} & Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B}))) & \to 0 \\
 & \downarrow Hom(S, I) & \\
\xrightarrow{ad(S)} & Hom(T_*(C_*^G(\mathcal{A})), T_*(C_*^G(\mathcal{B})))[2] & \to 0 \\
 & \downarrow I & \\
\xrightarrow{ad(S)} & Hom(Ker(S, T^*(C_G^*(\mathcal{A}))), T^*(C_G^*(\mathcal{B})))[2] & \to 0 \\
 & \downarrow & \\
 & 0 &
\end{array}$$

where each row is obviously exact. The exactness of last two columns follows from (4.1) and the nuclearity of $\mathcal{A}$. By the 3×3-Lemma, we obtain the exactness of the first column which yields the equivariant Connes exact sequence

$$\to HC_G^{n-2}(\mathcal{A},\mathcal{B})\xrightarrow{S}HC_G^n(\mathcal{A},\mathcal{B})\xrightarrow{I}HH_G^n(\mathcal{A},\mathcal{B})\to HC_G^{n-1}(\mathcal{A},\mathcal{B})\xrightarrow{S}HC_G^{n+1}(\mathcal{A},\mathcal{B})\to \quad (4.2)$$

provided $\mathcal{A}$ is nuclear.

We can also define equivariant cyclic theory in terms of other complexes for some special algebras. To this aim, let us introduce the notion of strong equivariant H-unitality.

Defintion 4.3 Let $\mathcal{A}$ be a Fréchet locally m-convex G-algebra. $\mathcal{A}$ is called strongly equivariant H-unital if there is an equivariant map $S' : C_*^G(\mathcal{A}) \to C_{*+1}^G(\mathcal{A})$ such that $b'S' + S'b' = I$.

By Lemma 4.1 , we see that $\mathcal{A}$ is strongly equivariant H-unital provided $\mathcal{A}$ is unital. For strongly equivariant H-unital algebra $\mathcal{A}$ we can define a morphism $B : C_{**}^G(\mathcal{A}) \to C_{**+1}^G(\mathcal{A})$ by $B = (I - T)S'N$. Obviously, $B^2 = 0 = bB + Bb$. We then form a new complex $\{T_*^G(\mathcal{A}), b + B\}$ by

$$T_n^G(\mathcal{A}) = C_n^G(\mathcal{A}) \oplus C_{n-2}^G(\mathcal{A}) \oplus C_{n-4}^G(\mathcal{A}) \oplus \dots,$$

with diferential $b + B$. The map S is also a morphism on $T_*^G(\mathcal{A})$ given by

$$S(x_n, x_{n-2}, x_{n-4}, \dots) = (x_{n-2}, x_{n-4}, \dots).$$

As before, let

$$Hom_S(T_*^G(\mathcal{A}), T_*^G(\mathcal{B})) = \{f \in Hom(T_*^G(\mathcal{A}), T_*^G(\mathcal{B})) : ad(S)(f) = 0\}$$

with the differential $\underline{d}$.

Proposition 4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be strongly equivariant H-unital. Then*

$$HC_G^*(\mathcal{A},\mathcal{B}) = H_{-*}(Hom_S(T_*^G(\mathcal{A}), T_*^G(\mathcal{B}))),$$

and

$$HH_G^*(\mathcal{A},\mathcal{B}) = H_{-*}(Hom(Ker(S, T_*^G(\mathcal{A})), Ker(S, T_*^G(\mathcal{B})))).$$

Hence,

$$PHC_G^*(\mathcal{A},\mathcal{B}) = \lim_{\vec{n,S}} H_{-(*+2n)}(Hom_S(T_*^G(\mathcal{A}), T_*^G(\mathcal{B}))).$$

Proof. The proof is to construct explicit isomorphisms. Introducing an indeterminate u of degree 2, we can write

$$T_n^G(\mathcal{A}) = \bigoplus_{k+2l=n} C_k^G(\mathcal{A})u^l.$$

Define $I_1 : T_n^G(\mathcal{A}) \to T_n^G(C_*^G(\mathcal{A}))$ and $I_2 : T_n^G(C_*^G(\mathcal{A})) \to T_n^G(\mathcal{A})$ by

$$I_1(xu^p) = xq^{2p} + \tilde{S}'N(x)q^{2p-1}$$

and

$$I_2(xq^{2p}) = xu^p, \;\; I_2(xq^{2p+1}) = (I-T)\tilde{S}'(x)u^p,$$

where $\tilde{S}'$ is a morphism on $C_*^G(\mathcal{A})$ to be determined which satisfies $\tilde{S}'b' + b'\tilde{S}' = I$ and $(\tilde{S}')^2 = 0$. One can easily show [Kass]

$$I_2I_1(xu^p) = xu^p + (I-T)(\tilde{S}')^2N(x)u^{p-1} = I(xu^p)$$

and

$$I_1I_2 = I + d\varphi + \varphi d,$$

where $\varphi : C_{**}^G(\mathcal{A}) \to C_{**}^G(\mathcal{A})$ is defined by

$$\varphi(xq^{2p}) = 0, \;\; \varphi(xq^{2p+1}) = \tilde{S}'(x)q^{2p+1}.$$

Hence I_1I_2 and I_2I_1 are homotopic to the identities which imply the result. It remains to find $\tilde{S}'$. Let $\tilde{S}' = S'b'S'$. We get

$$\tilde{S}'b' + b'\tilde{S}' = (S' - S'b')b' + b'(S' - b'S') = S'b' + b'S' = I,$$

and $(\tilde{S}')^2 = S'(S')^2(b')^2S' = 0$, since $S'(b'S' + S'b') = S' = (b'S' + S'b')S'$ and then $(S')^2b' = b'(S')^2$. Q.E.D.

We now come to the entire version of equivariant cyclic theory. Let $\mathcal{A}$ be a unital Fréchet locally m-convex G-algebra whose topology is determined by a sequence $\{\rho_j\}_{j=1}^{\infty}$ of seminorms satisfying $\rho_j(xy) \leq \rho_j(x)\rho_j(y)$ and $\rho_j(gx) \leq \rho_j(x), \forall g \in G, x \in \mathcal{A}$. We assume $\rho_j \leq \rho_k$ for $j \leq k$. Let $\mathcal{A}_j$ be the Banach algebra defined by the completion of $\mathcal{A}/Ker(\rho_j)$ with respect to ρ_j and $\beta_j : \mathcal{A} \to \mathcal{A}_j$ be the continuous quotient map. $\mathcal{A}_j$ is a G-algebra and β_j is equivariant. We have a map from $\mathcal{A}/Ker(\rho_k)$ to $\mathcal{A}/Ker(\rho_j)$ for $j \leq k$. Let $\alpha_{jk} : \mathcal{A}_k \to \mathcal{A}_j$ be the corresponding extension of this map. Then $\{\mathcal{A}_j, \alpha_{jk}, \beta_j\}$ is a projective system and $\mathcal{A} = \lim_{\to j} \mathcal{A}_j$. To define the seminorms on the projective tensor product $\mathcal{A}^{\hat{\otimes}(n+1)}$, we map $\mathcal{A}^{\hat{\otimes}(n+1)}$ to $\mathcal{A}_j^{\hat{\otimes}(n+1)}$ via the map β_j and then for $x_n \in \mathcal{A}^{\hat{\otimes}(n+1)}$ let

$$\|x_n\|_j = \|\beta_j(x_n)\|_{j\pi}$$

for the projective tensor product norm $\|..\|_{j\pi}$ on Banach algebra $\mathcal{A}_j$. We choose an increasing sequence $\{q_j\}$ of seminorms on $C_c^{\infty}(G)$ and tensor them with $\|..\|_j$ to get a sequence of seminorms on $\mathcal{A}^{\hat{\otimes}(n+1)}\hat{\otimes}C_c^{\infty}(G)$. Denote also by $\|..\|_j$ the quotient of these seminorms on $C_n^G(\mathcal{A})$. Let $T_{ev}^G(\mathcal{A})$ (resp. $T_{odd}^G(\mathcal{A})$) be the space of all sequence $\{x_{2n}\}_{n=0}^{\infty}$ (resp. $\{x_{2n+1}\}_{n=0}^{\infty}$) such that for each j

and $r \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \frac{r^n \|x_{2n}\|_j}{(2n)!} < \infty \ (resp. \ \sum_{n=0}^{\infty} \frac{r^n \|x_{2n+1}\|_j}{(2n+1)!} < \infty),$$

where $x_n \in C_n^G(\mathcal{A})$. One can check that b and B map $T_{\nu_1}^G(\mathcal{A})$ to $T_{\nu_2}^G(\mathcal{A})$ for $\nu_1 \neq \nu_2, \nu_i$ even or odd. (cf. for instance, Chapter 5). We may consider the periodic complex $\{T_{e,*}^G(\mathcal{A}), b+B\}$ given by $T_{e,j}^G(\mathcal{A}) = T_{ev}^G(\mathcal{A})$ for j even and $T_{odd}^G(\mathcal{A})$ for j odd, with an obvious operator S of degree 2.

Definition 4.4 Let $\mathcal{A}$ and $\mathcal{B}$ be strongly equivariant H-unital. The entire equivariant bivariant cyclic theory of $\mathcal{A}$ and $\mathcal{B}$ is

$$HC_{G,l}^*(\mathcal{A},\mathcal{B}) = H_{-*}(Hom_S(T_{e,*}^G(\mathcal{A}), T_*^G(\mathcal{B}))),$$

$$HC_{G,r}^*(\mathcal{A},\mathcal{B}) = H_{-*}(Hom_S(T_*^G(\mathcal{A}), T_{e,*}^G(\mathcal{B}))).$$

Since S acts on $HC_{G,*}^*(\mathcal{A},\mathcal{B})$, we can define the periodic version of the entire equivariant bivariant cyclic theory.

Definition 4.5 The periodic entire equivariant bivariant cyclic theory of $\mathcal{A}$ and $\mathcal{B}$ is

$$PHC_{G,i}^*(\mathcal{A},\mathcal{B}) = \lim_{\overrightarrow{S}} HC_{G,i}^{*+2n}(\mathcal{A},\mathcal{B}), i = l, r.$$

There are also normal versions of the above cyclic theories for unital algebras. Let $\bar{C}_n^G(\mathcal{A}) = (\mathcal{A}\hat{\otimes}(\mathcal{A}/\mathbb{C})^{\hat{\otimes}(n)})\hat{\otimes}_{G,\rho}C_c^\infty(G)$. Here $\mathcal{A}/\mathbb{C}$ is the quotient of $\mathcal{A}$ by $\mathbb{C}$. Replacing $C_n^G(\mathcal{A})$ by $\bar{C}_n^G(\mathcal{A})$ through the discussion above, we get $T_*^G(\bar{C}_*^G(\mathcal{A}))$, $\bar{T}_*^G(\mathcal{A})$ and then the normal version of the equivariant cyclic theory, denoted by $\overline{HC}_G^*$, $\overline{HC}_{G,*}^*$ and so on. But we do not know whether in our case $\overline{HC}_G^*$ and HC_G^* are isomorphic due to the lack of explicit formula of strong homotopy. For $\mathcal{A} = \mathbb{C}$ and G compact Lie group, one can construct a strong homotopy from $T_*^G(\mathbb{C})$ to $\bar{T}_*^G(\mathbb{C}) = \mathbb{C}[u] \otimes R^\infty(G)$ [Kass]. Define $I_3 : \bar{T}_*^G(\mathbb{C}) \to T_*^G(\mathbb{C})$ and $I_4 : T_*^G(\mathbb{C}) \to \bar{T}_*^G(\mathbb{C})$ by

$$I_3(u^p) = \sum_{i\geq 0}(-1)^i \frac{(2i)!}{i!} e_{2i}u^{p-i},$$

$$I_4(\sum_{i\geq 0} u^i \otimes x_{n-2i}) = \begin{cases} 0, n\ odd, \\ \\ u^{[\frac{n}{2}]} \otimes x_0, \quad n\ even, \end{cases}$$

where $e_{2i} = \overbrace{1\hat{\otimes}\ldots\hat{\otimes}1}^{2i+1}$. Let $\psi : T_*^G(\mathbb{C}) \to T_*^G(\mathbb{C})$ be the map of degree 1 defined by

$$\begin{aligned} \psi(u^p e_{2n}) &= 0 \\ \psi(u^p e_{2n-1}) &= \sum_{i\geq 0}(-1)^{i+1} \frac{(2n+2i)!n!}{(2n)!(n+i)!} u^{p-i} e_{2n+2i}. \end{aligned}$$

Then $I_4 I_3 = I$, $I_3 I_4 = I + d\psi + \psi d$. This proves that I_3 and I_4 are strongly homotopic inverse with each other. Hence we have obtained the following

Proposition 4.2 *Let $\mathcal{A}$ be strongly equivariant H-unital and G compact. Then*

$$\begin{aligned} HC_G^*(\mathcal{A}, \mathbb{C}) &\simeq H_{-*}(Hom_S(T_*^G(\mathcal{A}), \bar{T}_*^G(\mathbb{C}))), \\ HC_G^*(\mathbb{C}, \mathcal{A}) &\simeq H_{-*}(Hom_S(\bar{T}_*^G(\mathbb{C}), T_*^G(\mathcal{A}))). \end{aligned}$$

Note that $\bar{T}_*^G(\mathbb{C}) \simeq R^\infty(G)[u]$, where $R^\infty(G)$ is the space of G-invariant smooth functions f on G, namely, f is smooth and $f(ghg^{-1}) = f(h)$ for $g, h \in G$.

$HC_G^*(\mathcal{A}, \mathbb{C})$ is in general different from $HC_G^*(\mathcal{A})$ in [Gong 2]. The reason to define $HC_G^*(\mathcal{A}, \mathcal{B})$ as above is that this equivariant bivariant cyclic theory is suitable for the study of the Chern characters. Of course, we may define $HC_G^*(\mathcal{A})$ as $HC_G^*(\mathcal{A}, \mathbb{C})$ and study its application to equivariant index theory which is left to the interested reader.

Proposition 4.2 enables us to compute $HC_G^*(\mathbb{C}, \mathbb{C})$ for compact Lie group G as follows.

$$\begin{aligned} HC_G^*(\mathbb{C}, \mathbb{C}) &= H_{-*}(Hom_S(R^\infty(G)[u], R^\infty(G)[u])) \\ &= H_{-*}(Hom(R^\infty(G), R^\infty(G))) \\ &= \begin{cases} Hom(R^\infty(G), R^\infty(G)), & * = 2n, \\ 0, \ * = 2n+1, \ n \geq 0. & \end{cases} \end{aligned}$$

We now consider the reduced version of equivariant bivariant cyclic theory. For a unital Fréchet locally m-convex G-algebra $\mathcal{A}$ we let $\tilde{C}_*^G(\mathcal{A}) = \bar{C}_*^G(\mathcal{A})/\bar{C}_*^G(\mathbb{C})$ and define $\tilde{T}_*^G(\mathcal{A})$ as that of $T_*^G(\mathcal{A})$ with $C_*^G(\mathcal{A})$ replaced by $\tilde{C}_*^G(\mathcal{A})$. Then $\{\tilde{T}_*^G(\mathcal{A}), b+B\}$ is a complex. Using $\tilde{T}_*^G(\mathcal{A})$ and $\tilde{T}_*^G(\mathcal{B})$ instead of $T_*^G(\mathcal{A})$ and $T_*^G(\mathcal{B})$ in Definition 4.1, we get the reduced version of equivariant bivariant cyclic theory. For a general $\mathcal{A}$, we can compare the reduced complex $\tilde{T}_*^G(\mathcal{A}^+)$ with $T_*^G(\mathcal{A})$, where $\mathcal{A}^+$ is the algebra obtained by adjoining an identity to $\mathcal{A}$. In fact, let

$$\Phi : C_{p,q}^G(\mathcal{A}) \oplus C_{p+1,q-1}^G(\mathcal{A}) = C_q^G(\mathcal{A}) \oplus C_{q-1}^G(\mathcal{A}) \to \tilde{C}_q^G(\mathcal{A})$$

be defined by

$$\Phi(x_0, x_1, \ldots, x_q, f) = (x_0, x_1, \ldots, x_q, f), \Phi(x_1, \ldots, x_q, f) = (1, x_1, \ldots, x_q, f).$$

Clearly, Φ is an isomorphism and defines a morphism of the complexes [LoQ].Hence,

$$\begin{aligned} T_n^G(\mathcal{A}) &= \bigoplus_{p+q=n} C_{p,q}^G(\mathcal{A}) = \bigoplus_{q=0}^{n} C_q^G(\mathcal{A}) \\ &= (C_n^G(\mathcal{A}) \oplus C_{n-1}^G(\mathcal{A})) \oplus (C_{n-2}^G(\mathcal{A}) \oplus C_{n-3}^G(\mathcal{A})) \oplus \ldots \\ &\simeq \tilde{C}_n^G(\mathcal{A}) \oplus \tilde{C}_{n-2}^G(\mathcal{A}) \oplus \ldots = \tilde{T}_n^G(\mathcal{A}^+). \end{aligned}$$

We have obtained the following.

Proposition 4.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be Fréchet locally m-convex G-algebras and $\widetilde{HC}_G^*(\mathcal{A}^+, \mathcal{B}^+)$ be the reduced equivariant bivariant cyclic theory of $\mathcal{A}^+$ and $\mathcal{B}^+$. Then*

$$HC_G^*(\mathcal{A}, \mathcal{B}) \simeq \widetilde{HC}_G^*(\mathcal{A}^+, \mathcal{B}^+).$$

4.2 Excision Property

In this section we will associate a long exact sequence of equivariant bivariant cyclic theory with the following short exact sequence of Fréchet locally m-convex G-algebras

$$0 \longrightarrow \mathcal{A} \xrightarrow{\ i\ } \mathcal{E} \underset{j}{\overset{\pi}{\rightleftarrows}} R \longrightarrow 0. \tag{4.3}$$

Here i and π are topological homomorphisms. j is a continuous linear map such that $\pi j = I$. The method is to use the spectral sequence and equivariant H-unitality as in ([Wod 1, 2], [Kass] [Wang]) and [Gong 1,2]. The long exact sequence of equivariant bivariant cyclic theory will follow from the results in [Gong 2]. Let us first recall the construction of spectral sequences in [Gong 2]. Consider the Bar complex

$$B_n^G(\mathcal{E}) = (\mathcal{E})^{\hat{\otimes}(n)} \hat{\otimes}_{G,\rho} C_c^\infty(G)$$

with the differential b'. let

$$BK_*^G = Ker\{B_*^G(\mathcal{E}) \xrightarrow{\pi_*} B_*^G(\mathcal{R})\},$$

where π_* is induced by π. Identify $\mathcal{A}$ with $i(\mathcal{A})$ and $\mathcal{R}$ with $j(\mathcal{R})$ in $\mathcal{E}$ as topological spaces. We have that $\mathcal{A}$ is an ideal in $\mathcal{E}$ and $B_*^G(\mathcal{A}) \subset B_*^G(\mathcal{E})$. Filter $\{BK_*^G, b'\}$ by

$$F_pBK_{p+q}^G = \overline{Span}\{[e_1, e_2, \dots, e_{p+q}, f] \in B_{p+q}^G(\mathcal{E}) : \textit{ at least } q\ e_i's \in \mathcal{A}\},$$

$$0 \subset B_n^G(\mathcal{A}) = F_0BK_n^G \subset \dots \subset F_{n-1}BK_n^G = BK_n^G.$$

The spectral sequence corresponding to this filtration converges to the homology of $\{BK_*^G, b'\}$. Its E_0-terms are

$$BE_{p,q}^{G,0} = \overline{Span}\{[e_1, \dots, e_{p+q}, f] \in B_{p+q}^G(\mathcal{E}) : \textit{ exactly } q\ e_i's \in \mathcal{A}\}$$

with differential $d_{p,q}^0$ induced by b'. we write elements of $BK_{p,q}^G$ as the following.
Case I: $[a_{(k_1)}, r_{(n_2)}, \dots, a_{(k_l)}, r_{(n_{l+1})}, f],\ n_j, k_i \in \mathbb{Z}^+,\ 2 \leq j \leq l,\ 1 \leq i \leq l,\ n_{l+1} \in \mathbb{N}$;
Case II: $[r_{(n_1)}, a_{(k_1)}, \dots, a_{(k_l)}, r_{(n_{l+1})}, f],\ n_j, k_i \in \mathbb{Z}^+,\ 1 \leq i, j \leq l, n_{l+1} \in \mathbb{N}$, where

$$\begin{aligned} a_{(k_i)} &= (a_{k_{i-1}+1}, a_{k_{i-1}+2}, \dots, a_{k_{i-1}+k_i}, f),\ 1 \leq i \leq l,\ a_i \in \mathcal{A}, \\ r_{(n_j)} &= (r_{n_{j-1}+1}, r_{n_{j-1}+2}, \dots, r_{n_{j-1}+n_j}, f),\ 1 \leq j \leq l+1,\ r_j \in \mathcal{R}. \end{aligned}$$

Then $d_{p,q}^0$ acts on these two kinds of elements as follows.
Case I: $d_{p,q}^0[a_{(k_1)}, r_{(n_2)}, \dots, a_{(k_l)}, r_{(n_{l+1})}, f] = [b'(a_{(k_1)}), r_{(n_2)}, \dots, a_{(k_l)}, r_{(n_{l+1})}, f] +$

$$+ \sum_{j=2}^{l} (-1)^{\sum_{i=1}^{j-1}(k_i + n_{i+1})} [a_{(k_1)}, r_{(n_2)}, \dots, \tilde{b}'(a_{(k_j)}), r_{(n_{j+1})}, \dots, r_{(n_{l+1})}, f]$$

Here, $\tilde{b}'(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$.

Case II: $d^0_{p,q}[r_{(n_1)}, a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f]$

$$= \sum_{j=1}^{l}(-1)^{\sum_{i=1}^{j-1}(n_i+k_i)+n_j}[r_{(n_1)}, a_{(k_1)}, \ldots, r_{(n_j)}, \tilde{b}'(a_{(k_j)}), \ldots, a_{(k_l)}, r_{(n_{l+1})}, f].$$

We now introduce two complexes depending on the form of elements.

Case I: For $l \in \mathbb{Z}^+$ let $\mathcal{O}_1 = (n_2, n_3, \ldots, n_{l+1})$ with $|\mathcal{O}_1| = \sum_{i=2}^{l+1} n_i$ and $l(\mathcal{O}_1) = l, n_i > 0, 2 \leq i \leq l$ and $n_{l+1} \geq 0$. Let $BT^G_*(\mathcal{O}_1)$ be the total complex of the $(l+1)$-tuple of the complexes

$$(B_*(\mathcal{A})\hat{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\hat{\otimes}\ldots\hat{\otimes}\tilde{B}_*(\mathcal{A})[n_l]\hat{\otimes}\mathcal{R}^{\hat{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1| + n_{l+1}]\hat{\otimes}C^\infty_c(G))\hat{\otimes}_{G,\rho}\mathbb{C},$$

where $B_*(\mathcal{A})$ and $\tilde{B}_*(\mathcal{A})$ are equal to $\mathcal{A}^{\hat{\otimes}(*)}$ with differentials b' and $\tilde{b}'$, resp., and $\mathcal{R}^{\hat{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1| + n_{l+1}]$ concerntrates in dimension $-|\mathcal{O}_1| + n_{l+1}$ with trivial differential.

Case II: Let $\mathcal{O}_2 = (n_1, n_2, \ldots, n_{l+1}), n_i > 0, 1 \leq i \leq l, n_{l+1} \geq 0$. Let $BT^G_*(\mathcal{O}_2)$ be the total complex of the following

$$(\tilde{B}_*(\mathcal{A})[n_1]\hat{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\hat{\otimes}\ldots\hat{\otimes}\tilde{B}_*(\mathcal{A})[n_l]\hat{\otimes}\mathcal{R}^{\hat{\otimes}|\mathcal{O}_2|}[-|\mathcal{O}_2|+n_{l+1}]\hat{\otimes}C^\infty_c(G))\hat{\otimes}_{G,\rho}\mathbb{C}.$$

Then we have

$$\{BE^{G,0}_{p,*}, d^0_{p,*}\} \xrightarrow{F} \tilde{\bigoplus_{\substack{\mathcal{O}_1 : |\mathcal{O}_1| = p \\ l(\mathcal{O}_1) \geq 1}}} BT^G_*(\mathcal{O}_1) \tilde{\bigoplus} \tilde{\bigoplus_{\substack{\mathcal{O}_2 : |\mathcal{O}_2| = p \\ l(\mathcal{O}_2) \geq 1}}} BT^G_*(\mathcal{O}_2),$$

where F is defined by

$$F[r_{(n_1)}, a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f] = [a_{(k_1)}, a_{(k_2)}, \ldots, a_{(k_l)}, r_{(n_1)}, r_{(n_2)}, \ldots, r_{(n_{l+1})}, f].$$

F is one-to-one and onto continuous map. As a result, the $BE^{G,1}_{p,*}$- terms of the spectral sequence are isomorphic to

$$\bigoplus_{all\ \mathcal{O}_1:\,|\mathcal{O}_1|=p} H_*(BT^G_*(\mathcal{O}_1), d^0_{p,*}) \bigoplus \bigoplus_{all\ \mathcal{O}_2:\,|\mathcal{O}_2|=p} H_*(BT_*(\mathcal{O}_2), d^0_{p,*}), \quad p \geq 1. \ (4.4)$$

To guarantee that the spectral sequence $BE^{G,k}_{p,*}$ stops at $k = 2$ for $p \geq 1$, we introduce the equivariant H-unitality.

Definition 4.6 Let X be a Fréchet G-space. Let

$$\begin{aligned} B^G_n(\mathcal{A}, X) &= (\mathcal{A}^{\tilde{\otimes}(n)} \tilde{\otimes} X \tilde{\otimes} C^\infty_c(G)) \tilde{\otimes}_{G,\rho} \mathbb{C} \\ \tilde{B}^G_n(\mathcal{A}, X) &= (\mathcal{A}^{\tilde{\otimes}(n)} \tilde{\otimes} X) \tilde{\otimes}_{G,\rho} \mathbb{C}, \quad n \geq 1. \end{aligned}$$

If $H_*(B^G_*(\mathcal{A}, X), b') = 0$ and $H_*(\tilde{B}^G_*(\mathcal{A}, X), \tilde{b}') = 0, * \geq 1$ for any Fréchet G-space X, then $\mathcal{A}$ is called equivariant H-unital.

Note that if $\mathcal{A}$ is unital , then $\mathcal{A}$ is equivariant H-unital by Lemma 4.2. It is likely that the strongly equivariant H-unitality implies the equivariant H-unitality.

Now if $\mathcal{A}$ is equivariant H-unital, by (4.4), the inclusion $B^G_*(\mathcal{A}) \xrightarrow{i_*} BK^G_*$ induces an isomorphism in homology. See [Gong 2] for details. Similarly, we can show that the inclusions $Ker(S, T_*(C^G_*(\mathcal{A}))) \xrightarrow{i_*} Ker(S, CK^G_*)$ and $T^G_*(C^G_*(\mathcal{A})) \hookrightarrow CK^G_* \stackrel{\text{def}}{=} Ker\{T^G_*(C^G_*(\mathcal{E})) \xrightarrow{\pi_*} T^G_*(C^G_*\mathcal{R}))\}$ induce isomorphisms in homology.

Theorem 4.1 ([*Gong* 2]) *Let $\mathcal{A}$ be equivariant H-unital.*

(1) $B^G_*(\mathcal{A}) \xrightarrow{i_*} BK^G_*$, $Ker(S, T_*(C^G_*(\mathcal{A}))) \xrightarrow{i_*} Ker(S, CK^G_*)$ and $T^G_*(C^G_*(\mathcal{A})) \hookrightarrow CK^G_*$ induce isomorphisms in homology.

(2) *There are the following long exact sequences associated with* (4.3),

$$\dots \to HH_n^G(\mathcal{A}) \xrightarrow{i_*} HH_n^G(\mathcal{E}) \xrightarrow{\pi_*} HH_n^G(\mathcal{R}) \xrightarrow{\partial} HH_{n-1}^G(\mathcal{A}) \to \dots,$$

$$\dots \to HC_n^G(\mathcal{A}) \xrightarrow{i_*} HC_n^G(\mathcal{E}) \xrightarrow{\pi_*} HC_n^G(\mathcal{R}) \xrightarrow{\partial} HC_{n-1}^G(\mathcal{A}) \to \dots.$$

The equivariant H-unitality is guaranteed by Theorem 7 in [Gong 2].

We now use Theorem 4.1 to prove the excision of equivariant bivariant cyclic theory. Let us first prove the following crucial lemma.

Lemma 4.3 *Let X_* and Y_* be two complexes of Fréchet G-spaces. Suppose $\varphi : X_* \to Y_*$ induces an isomorphism in homology. Then for any complex Z_* of nuclear Fréchet G-spaces, $Hom(I,\varphi) : Hom(Z_*, X_*) \to Hom(Z_*, Y_*)$ induces an isomorphism in homology.*

Proof. We divide the proof into three steps.

Step 1. Let $C(X_*, Y_*)$ be the mapping cone of φ given by $C_n(X_*, Y_*) = Y_n \oplus X_{n-1}$, with differential $d(y_n, x_{n-1}) = (dy_n + \varphi(x_{n-1}), -d(x_{n-1}))$. We have a short exact sequence

$$0 \to Y_* \to C_*(X_*, Y_*) \to X_*[1] \to 0.$$

Hence, we get a long exact sequence

$$\to H_n(Y_*) \to H_n(C_*) \to H_{n-1}(X_*) \xrightarrow{\varphi_*} H_{n-1}(Y_*) \to H_{n-1}(C_*) \to .$$

Since φ_* is an isomorphism, we obtain $H_n(C_*) = 0, n \geq 0$. This implies that $C_*(X_*, Y_*)$ is an exact complex.

Step 2. We now show that $Hom(Z_*, C_*(X_*, Y_*))$ is exact. In fact, if $f \in Hom(Z_*, C_*)_n$ is a cocycle, i.e., $df - (-1)^n fd = 0$, then in particular, $df(z_0) =$

$(-1)^n f(dz_0) = 0, z_0 \in Z_0$, i.e., $f(z_0) \in C_n$ is a cycle. Since Z_0 is nuclear, we get a long exact sequence [Tay]

$$\leftarrow Hom(Z_0, C_n) \xleftarrow{d} Hom(Z_0, C_{n+1}) \xleftarrow{d} Hom(Z_0, C_{n+2}) \leftarrow .$$

Thus there exists an $f_0' \in Hom(Z_0, C_{n+1})$ such that $df_0' = f$ on Z_0. Consider $f + (-1)^{n+1} f_0' d$ on Z_1. We get $d(f + (-1)^{n+1} f_0' d) = 0$. This shows that $f + (-1)^{n+1} f_0' d$ is a cocycle on Z_1. Hence we can find $f_1' \in Hom(Z_1, C_{n+2})$ such that $df_1' = f + (-1)^{n+1} f_0' d$ on Z_1. We proceed this way to find $f_i' \in Hom(Z_i, C_{i+n+1})$ such that $df_i' = f + (-1)^{n+1} f_{i-1}' d, i \geq 0$. We have thus obtained an element $f' = \{f_i'\} \in Hom(Z_*, C_*)_{n+1}$ such that $\underline{d}f' = df' + (-1)^{n+1} f' d = f$.

Step 3. Let φ_* be the morphism on $Hom(Z_*, X_*)$ induced by φ. Let $\tilde{C}_*$ be the mapping cone of φ_*,

$$\tilde{C}_n = Hom(Z_*, Y_*)_n \oplus Hom(Z_*, X_*)_n[1],$$

with differential $\tilde{\underline{d}}(f_n, f_{n-1}) = (\underline{d}f_n + \varphi_*(f_{n-1}), -\underline{d}(f_{n-1}))$. Clearly, $\tilde{C}_* = Hom(Z_*, C_*(X_*, Y_*))$. We get

$$0 \to Hom(Z_*, Y_*) \to \tilde{C}_* \to Hom(Z_*, X_*)[1] \to 0.$$

Hence we obtain a long exact sequence

$$\to H_n(Z_*, Y_*) \to H_n(\tilde{C}_*) \to H_{n-1}(Hom(Z_*, X_*)) \xrightarrow{\varphi_*} H_{n-1}(Hom(Z_*, Y_*)) \to .$$

To show that φ_* is an isomorphism, it is enough to check

$$H_*(\tilde{C}_*) = H_*(Hom(Z_*, C_*(X_*, Y_*))) = 0,$$

which is true by Step 2. Q.E.D.

Theorem 4.2 *Let $\mathcal{B}$ be a nuclear Fréchet locally m-convex G-algebra. Suppose $\mathcal{A}$ is equivariant H-unital. Then there are long exact sequences of equivariant bivariant Hochschild and cyclic theories associated with short exact sequence* (4.3) *of Fréchet locally m-convex G-algebras,*

$$\ldots \to HH^n_G(\mathcal{B},\mathcal{A}) \xrightarrow{i_*} HH^n_G(\mathcal{B},\mathcal{E}) \xrightarrow{\pi_*} HH^n_G(\mathcal{B},\mathcal{R}) \xrightarrow{\partial} HH^{n+1}_G(\mathcal{B},\mathcal{A}) \to \ldots \quad (4.5)$$

and

$$\ldots \to HC^n_G(\mathcal{B},\mathcal{A}) \xrightarrow{i_*} HC^n_G(\mathcal{B},\mathcal{E}) \xrightarrow{\pi_*} HC^n_G(\mathcal{B},\mathcal{R}) \xrightarrow{\partial} HC^{n+1}_G(\mathcal{B},\mathcal{A}) \to \ldots . \quad (4.6)$$

Proof. We know already by Theorem 4.1 that the inclusions

$$i_* : Ker(S, T_*(C^G_*(\mathcal{A}))) \to Ker(S, CK^G_*) \;\; and \;\; i_* : T^G_*(C^G_*(\mathcal{A})) \to CK^G_*$$

induce isomorphisms in homology. By Lemma 4.3,

$$Hom(Ker(S,T_*(C^G_*(\mathcal{B}))), Ker(S,T_*(C^G_*(\mathcal{A})))) \xrightarrow{Hom(I,i_*)} Hom(Ker(S,T_*(C^G_*(\mathcal{B}))), Hom(S, CK^G_*))$$

and

$$Hom(I, i_*) : Hom(T_*(C^G_*(\mathcal{B})), T_*(C^G_*(\mathcal{A}))) \to Hom(T_*(C^G_*(\mathcal{B})), CK^G_*)$$

induce isomorphisms in homology. Since

$$0 \to Ker(S, CK^G_*) \to Ker(S, T_*(C^G_*(\mathcal{E}))) \xrightarrow{\pi_*} Ker(S, T_*(C^G_*(\mathcal{R}))) \to 0$$

and

$$0 \to CK^G_* \to T_*(C^G_*(\mathcal{E})) \xrightarrow{\pi_*} T_*(C^G_*(\mathcal{R})) \to 0$$

are short exact, we obtain the following short exact sequences by nuclearity of $\mathcal{B}$,

$$\begin{aligned} 0 \;&\to\; Hom(Ker(S,T_*(C^G_*(\mathcal{B}))), Ker(S,CK^G_*)) \to Hom(Ker(S,T_*(C^G_*(\mathcal{B}))), Ker(S,T_*(C^G_*(\mathcal{E})))) \\ &\xrightarrow{\pi_*}\; Hom(Ker(S,T_*(C^G_*(\mathcal{B}))), Ker(S,T_*(C^G_*(\mathcal{R})))) \to 0, \end{aligned} \quad (4.7)$$

and

$$0 \to Hom(T_*(C_*^G(\mathcal{B})), CK_*^G) \to Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{E}))) \xrightarrow{\pi_*} Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{R}))) \to 0.$$

(4.5) then follows from (4.7). Since in the following commutative diagram

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \uparrow & & \uparrow & \\
0 \to & Hom(T_*(C_*^G(\mathcal{B})), CK_*^G)[2] & \to & Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{E})))[2] & \xrightarrow{\pi_*} \\
 & \uparrow ad(S) & & \uparrow ad(S) & \\
0 \to & Hom(T_*(C_*^G(\mathcal{B})), CK_*^G) & \to & Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{E}))) & \xrightarrow{\pi_*} \\
 & \uparrow & & \uparrow & \\
0 \to & Hom_S(T_*(C_*^G(\mathcal{B})), CK_*^G) & \to & Hom_S(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{E}))) & \xrightarrow{\pi_*} \\
 & \uparrow & & \uparrow & \\
 & 0 & & 0 &
\end{array}$$

$$\begin{array}{cc}
0 & \\
\uparrow & \\
Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{R})))[2] & \to 0 \\
\uparrow ad(S) & \\
Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{R}))) & \to 0 \\
\uparrow & \\
Hom_S(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{R}))) & \to 0 \\
\uparrow & \\
0 &
\end{array} \tag{4.8}$$

each column is exact and the first two rows are exact, we get that the third row is exact by the 3×3 Lemma. Also from the commutative diagram

$$\begin{array}{ccccc} 0 \to Hom_S(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{A}))) & \to & Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{A}))) \\ \downarrow Hom_S(I, i_*) & & \downarrow Hom(I, i_*) \\ 0 \to \quad Hom_S(T_*(C_*^G(\mathcal{B})), CK_*^G) & \to & Hom(T_*(C_*^G(\mathcal{B})), CK_*^G) \end{array}$$

$$\begin{array}{cc} \stackrel{ad(S)}{\to} Hom(T_*(C_*^G(\mathcal{B})), T_*(C_*^G(\mathcal{A}))) & \to 0 \\ \downarrow Hom(I, i_*) & \\ \stackrel{ad(S)}{\to} \quad Hom(T_*(C_*^G(\mathcal{B})), CK_*^G) & \to 0 \end{array}$$

and the long exact sequences associated with both exact rows above, we have that $Hom_S(I, i_*)$ induces an isomorphism in homology. (4.6) then follows from the third row of (4.8). Q.E.D.

Theorem 4.3 *With the assumptions of Theorem* 4.2, *there is a six-term exact sequence of periodic equivariant bivariant cyclic theory associated with* (4.3),

$$\begin{array}{ccccc} PHC_G^0(\mathcal{B}, \mathcal{A}) & \xrightarrow{i_*} & PHC_G^0(\mathcal{B}, \mathcal{E}) & \xrightarrow{\pi_*} & PHC_G^0(\mathcal{B}, \mathcal{R}) \\ \uparrow & & & & \downarrow \\ PHC_G^1(\mathcal{B}, \mathcal{R}) & \xleftarrow{\pi_*} & PHC_G^1(\mathcal{B}, \mathcal{E}) & \xleftarrow{i_*} & PHC_G^1(\mathcal{B}, \mathcal{A}). \end{array}$$

Proof. The result follows from the facts that the maps in (4.6) commute with the map S and that the direct limit preserves the exactness. Q.E.D.

We should point out that the excision perpoty of the first variable in $HC_G^*(.,.)$ and $HH_G^*(.,.)$ is unknown due to the lack of Lemma 4.3 in this situation.

4.3 Equivariant Bivariant Chern Characters

Let G be a compact Lie group and $\mathcal{A}$ and $\mathcal{B}$ be Fréchet locally m-convex G-algebras. Recall that equivariant KK-theory of $\mathcal{A}$ and $\mathcal{B}$ can be defined as follows. We call

$$\Phi = (\varphi, \psi) : \mathcal{A} \to \mathcal{E} \rhd \mathcal{J} \xrightarrow{\mu} \mathcal{B}$$

an equivariant quasi-homomorphism if

(1) $\mathcal{E}$ is a Fréchet locally m-convex G-algebra, $\mathcal{J} \subset \mathcal{E}$ is an equivariant ideal in $\mathcal{E}$,

(2) φ and ψ are equivariant homomorphisms from $\mathcal{A}$ to $\mathcal{E}$ such that $\mathcal{E}$ is generated by the images of φ and ψ and $\mathcal{J}$ is generated by the image of $\varphi - \psi$ as an ideal in $\mathcal{E}$,

(3) $\mathcal{J}$ is an essential ideal in $\mathcal{E}$, i.e., each nonzero closed ideal in $\mathcal{E}$ has a nonzero intersection with $\mathcal{J}$,

(4) μ is injective.

Two equivariant quasi-homomorphisms $\Phi_0 = (\varphi_0, \psi_0)$ and $\Phi_1 = (\varphi_1, \psi_1)$ are G-homotopic if there is a family$\{\tilde{\Phi}_t\}$ of equivariant quasi-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$, $t \in [0,1]$, such that $\tilde{\Phi}_0 = \Phi_0$ and $\tilde{\Phi}_1 = \Phi_1$ and the maps $t \to D_{\tilde{\Phi}_t}(a)$ and $t \to Q_{\tilde{\Phi}_t}(a, b)$ are continuous for all $a, b \in \mathcal{A}$, where $D_\Phi(a) = \mu(\varphi(a) - \psi(a))$ and $Q_\Phi(a,b) = \mu(\psi(a)(\varphi(b) - \psi(b)))$ for $\Phi = (\varphi, \psi)$. The G-homotopy is an equivalent relation on the set of equivariant quasi-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$. Then $KK_G(\mathcal{A}, \mathcal{B})$ is defined to be the set of G-homotopy classes of equivariant quasi-homomorphisms from $\mathcal{A}$ to $\mathcal{B} \otimes \mathcal{K}$. Here $\mathcal{K}$ is the compact operators on $l^2 \otimes l^2(G)$ with inner G-action $\nu_g(a) = u_g a u_g^{-1}$ for $u_g(\xi \otimes \eta) =$

$\xi \otimes \lambda_g(a)$. λ_g is the left regular representation of G on $L^2(G)$. See [Phi].

Our aim in this section is to define a character from some special elements, called p-quasi-homomorphisms, in $KK_G(\mathcal{A},\mathcal{B})$ to $PHC^{ev}_G(\mathcal{A},\mathcal{B})$. $(\varphi,\psi):\mathcal{A}\to\mathcal{E}\rhd\mathcal{J}\xrightarrow{\mu}\mathcal{K}\otimes\mathcal{B}$ is called an equivariant p-quasi-homomorphism if $\varphi(a)-\psi(a)\in L^p\otimes\mathcal{B},\forall a\in\mathcal{A}$, where L^p is the p-Schatten class operators on $l^2\otimes l^2(G)$.

Let $\mathcal{A}$ be unital and $D:\mathcal{A}\to\mathcal{A}$ be an equivariant continuous linear map. Define $e_D: C^G_n(\mathcal{A})\to C^G_{n-1}(\mathcal{A}), E_D: C^G_n(\mathcal{A})\to C^G_{n+1}(\mathcal{A})$ and $L_D: C^G_n(\mathcal{A})\to C^G_n(\mathcal{A})$ by

$$
\begin{aligned}
e_D(a_0,\ldots,a_n,f)(g) &= (a_0\alpha_g(D(a_1)),a_2,\ldots,a_n,f)(g),\\
E_D(a_0,\ldots,a_n,f)(g) &= \sum_{i=1}^{n}(1,\alpha_g^{-1}(a_0),a_1,\ldots,D(a_i),\ldots,a_n,f)(g)+\\
&+\sum_{j=2}^{n}(-1)^{nj}\sum_{i=1}^{j-1}(1,\alpha_g^{-1}(a_{n-j+1}),\ldots,\alpha_g^{-1}(a_n),\alpha_g^{-1}(a_0),\\
&\quad a_1,\ldots,a_i,\ldots,a_{n-j},f)(g),\\
L_D(a_0,\ldots,a_n,f)(g) &= \sum_{i=0}^{n}(a_0,\ldots,D(a_i),\ldots,a_n,f)(g).
\end{aligned}
$$

Let $J_0: C^G_n(\mathcal{A})\to C^G_{n-2}(\mathcal{A})$, $J_1: C^G_n(\mathcal{A})\to C^G_n(\mathcal{A})$ and $J_2:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{A}$ be the morphisms defined by

$$J_2(a_1,a_2)=D(a_1a_2)-a_1D(a_2)-D(a_1)a_2,$$

$$
\begin{aligned}
J_0(a_0,\ldots,a_n,f)(g) &= (a_0\alpha_g(J_2(a_1,a_2)),a_3,\ldots,a_n,f)(g),\\
J_1(a_0,\ldots,a_n,f)(g) &= \sum_{i=1}^{n-1}(-1)^{i+1}(1,\alpha_g^{-1}(a_0),\ldots,J_2(a_i,a_{i+1}),\ldots,a_n,f)(g)\\
&+\sum_{j=2}^{n-1}\sum_{i=1}^{j-1}(-1)^{i+(n-1)j+1}(1,\alpha_g^{-1}(a_{j+1}),\ldots,\alpha_g^{-1}(a_n),\alpha_g^{-1}(a_0),\\
&\quad a_1,\ldots,J_2(a_i,a_{i+1}),\ldots,a_j,f)(g).
\end{aligned}
$$

Let $h_D = I \otimes e_D + S \otimes E_D, L'_D = I \otimes L_D$ and $J_D = I \otimes J_0 + S \otimes J_1$ be the morphisms on $\bar{T}^G_*(\mathcal{A})$.

Proposition 4.4 $\underline{d}h_D = SL'_D - J_D$.

Proof. Note that for a unital Fréchet locally m-convex algebra $\mathcal{B}$, $h \in Hom_S(\bar{T}^G_*(\mathcal{A}), \bar{T}^G_*(\mathcal{B}))$ is given by a sequence of homomorphisms $\{h_i\}_{i\geq 0}$, $h_i : \bar{C}^G_*(\mathcal{A}) \to \bar{C}^G_{*+n+2i}(\mathcal{B})$ is of degree $n+2i$ such that

$$h(u^p \otimes x) = \sum_{i=0}^{p} u^{p-i} \otimes h_i(x),$$

or formally, $h = \sum_{i\geq 0} S^i \otimes h_i$. Then $\underline{d}h = \sum_{i\geq 0} S^i \otimes g_i$, where $g_0 = [b, h_0], g_i = [B, h_{i-1}] + [b, h_i], i \geq 1$. To show $\underline{d}h_D = SL'_D - J_D = S \otimes L_D - I \otimes J_0 - S \otimes J_1 = -I \otimes J_0 + S \otimes (L_D - J_1)$, we need only to check

$$[b, e_D] = -J_0, \quad [B, e_D] + [b, E_D] = L_D - J_1 \quad \textit{and} \quad [B, E_D] = 0.$$

We have

$$\begin{aligned}
be_D(a_0, a_1, \ldots, a_n, f)(g) &= (a_0\alpha_g(D(a_1))\alpha_g(a_2), a_3, \ldots, a_n, f)(g) \\
&+ \sum_{i=2}^{n-1}(-1)^{i-1}(a_0\alpha_g(D(a_1)), \ldots, a_i a_{i+1}, \ldots, a_n, f)(g) \\
&+ (-1)^{n-1}(a_n a_0 \alpha_g(D(a_1)), a_2, \ldots, a_{n-1}, f)(g), \\
e_D b(a_0, a_1, \ldots, a_n, f)(g) &= (a_0\alpha_g(a_1)\alpha_g(D(a_2)), a_3, \ldots, a_n, f)(g) \\
&- (a_0\alpha_g(D(a_1 a_2), a_3, \ldots, a_n, f)(g) \\
&+ \sum_{i=2}^{n-1}(-1)^{i}(a_o\alpha_g(D(a_1)), a_2, \ldots, a_i a_{i+1}, \ldots, a_n, f)(g) \\
&+ (-1)^{n}(a_n a_0 \alpha_g(D(a_1)), a_2, \ldots, a_{n-1}) f(g).
\end{aligned}$$

Therefore,

$$[b, e_D](a_0, \ldots, a_n, f)(g) = (be_D + e_D b)(a_0, \ldots, a_n, f)(g)$$

$$= (a_0\alpha_g(D(a_1))\alpha_g(a_2) + a_0\alpha_g(a_1)\alpha_g(D(a_2)) - a_0\alpha_g(D(a_1a_2)), a_3, \ldots, a_n, f)(g)$$
$$= -J_0(a_0, \ldots, a_n, f)(g).$$

This proves $[b, e_D] = -J_0$. Let us check $[B, E_D] = BE_D + E_DB = 0$. The formulas for B and E_D show $BE_D(a_0, \ldots, a_n, f)(g) = 0$ since 1 is between 1 and n positions in $BE_D(a_0, \ldots, a_n, f)(g)$, which is zero in $\bar{C}^G_*(\mathcal{A})$. Also

$$E_DB(a_0, \ldots, a_n, f)(g) = \sum_{i=0}^{n}(-1)^{ni}E_D((1, \alpha_g^{-1}(a_{n-i+1}), \ldots, \alpha_g^{-1}(a_n), \alpha_g^{-1}(a), a_1, \ldots, a_{n-i}), f)(g) = 0.$$

Hence, $[B, E_D] = 0$. The proof is thus complete by the following lemma.

Lemma 4.4 $[B, e_D] + [b, E_D] = L_D - J_1$.

Proof. We have [Gong 2]

$$b'S' + S'b' = I, B = S'N, L_D = \sum_{i=0}^{n} D_i,$$

where $D_i(a_0, \ldots, a_n, f)(g) = (a_0, \ldots, D(a_i), \ldots, a_n, f)(g)$.

(a) $e_D = d_0D_1$:

$$\begin{aligned} e_D(a_0, \ldots, a_n, f)(g) &= (a_0\alpha_g(D(a_1)), a_2, \ldots, a_n, f)(g) \\ &= d_0(a_0, D(a_1), \ldots, a_n, f)(g) = d_0D_1(a_0, \ldots, a_n, f)(g). \end{aligned}$$

(b) $Be_D = S'\sum_{i=0}^{n-1} T^i d_0D_1 = S'\sum_{i=0}^{n-1} T^{-i}d_0D_1$ on $\bar{C}^G_*(\mathcal{A})$.

(c) $e_DS' = D_0$:

$$\begin{aligned} e_DS'(a_0 \ldots, a_n, f)(g) &= (\alpha_g(D(\alpha_g^{-1}(a_0))), a_1, \ldots, a_n, f)(g) \\ &= (D(a_0), a_1, \ldots, a_n, f)(g) \\ &= D_0(a_0, \ldots, a_n, f)(g). \end{aligned}$$

(*d*) $e_D B = e_D S' N = D_0 N = \sum_{i=0}^n D_0 T^i = \sum_{i=0}^n D_0 T^{-i}$.

(*e*) $bS' + S'b' = I - T$: Since $b'S' + S'b' = I$ and $b = b' + d_n$ on $C_n^G(\mathcal{A})$,

$$\begin{aligned} d_{n+1} S'(a_0, \ldots, a_n, f)(g) &= (-1)^{n+1}(a_n, \alpha_g^{-1}(a_0), a_1, \ldots, a_n, f)(g) \\ &= -T(a_0, \ldots, a_n, f)(g). \end{aligned}$$

Hence, $bS' + S'b' = (b' + d_{n+1})S' + S'b' = b'S' + S'b' + d_{n+1}S' = I - T$.

(*f*) $T^{n-i+1} D_i = D_0 T^{n-i+1}$:

$$\begin{aligned} & T^{n-i+1} D_i(a_0, \ldots, a_n, f)(g) \\ = \;& (-1)^{n(n-i)} T(\alpha_g^{-1}(a_{i+1}), \ldots, \alpha_g^{-1}(a_n), \alpha_g^{-1}(a_0), a_1, \ldots, D(a_i), f)(g) \\ = \;& (-1)^{n(n-i+1)}(D(a_i), \alpha_g^{-1}(a_{i+1}), \ldots, \alpha_g^{-1}(a_n), \alpha_g^{-1}(a_0), a_1, \ldots, a_{i-1}, f)(g) \\ = \;& D_0 T^{n-i+1}(a_0, \ldots, a_n, f)(g). \end{aligned}$$

(*g*) Let $E'_D = \sum_{j=2}^n \sum_{i=1}^{j-1} T^{-j} D_i + \sum_{i=1}^n D_i$ on $\bar{C}_*^G(\mathcal{A})$. Then $bE_D = L_D - \sum_{j=0}^n T^{-j} D_j - S'b'E'_D$ and $E_D = S'E'_D$. In fact,

$$E_D = S'(\sum_{j=2}^{n} \sum_{i=1}^{j-1} T^{-j} D_i + \sum_{i=1}^{n} D_i) = S'(\sum_{j=2}^{n+1} \sum_{i=1}^{j-1} T^{-j} D_i) = S'E'_D,$$

and by (*e*),

$$\begin{aligned} bE_D &= bS'E'_D = (I - T - S'b')E'_D = (I - T)\sum_{j=2}^{n+1} \sum_{i=1}^{j-1} T^{-j} D_i - S'b'E'_D \\ &= \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} T^{-j} D_i - \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} T^{-j+1} D_i - S'b'E'_D \\ &= \sum_{i=1}^{n} D_i + \sum_{j=2}^{n} \sum_{i=1}^{j-1} T^{-j} D_i - \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} T^{-j+1} D_i - S'b'E'_D \\ &= L_D - \sum_{j=0}^{n} T^{-j} D_i - S'b'E'_D. \end{aligned}$$

Furthermore, by (f) and $T^{n+1} = I, T^{-j}D_j = T^{n-j+1}D_j = D_0 T^{n-j+1} = D_0 T^{-j}$. We get $bE_D = L_D - \sum_{j=0}^{n} T^{-j}D_i - S'b'E'_D = L_D - \sum_{j=0}^{n} D_0 T^{-j} - S'b'E'_D$. Hence in view of (b) and (d),

$$\begin{aligned}
[B, e_D] + [b, E_D] &= S' \sum_{j=0}^{n-1} T^{-j} d_0 D_1 + \sum_{j=0}^{n} D_0 T^{-j} + L_D - \sum_{j=0}^{n} D_0 T^{-j} \\
&- S'b'E'_D + S'E'_D b \\
&= S' \sum_{i=0}^{n-1} T^{-i} d_0 D_1 + L_D - S'(S'E'_D - E_D b).
\end{aligned}$$

This implies that $[B, e_D] + [b, E_D] = L_D + J_1$ iff

$$-J_1 = S'(\sum_{j=0}^{n-1} T^{-j} d_0 D_1 - S'E'_D + E'_D b) \tag{4.9}$$

We need therefore to check (4.9). Now $b'E'_D = \sum_{l=0}^{n-1} \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} d_l T^{-j} D_i$ on $\bar{C}_n^G(\mathcal{A})$. Note that

$$d_l T^{-j} = \begin{cases} T^{-j} d_{l+j}, & l + j \leq n, \\ \\ T^{-(j-1)} d_{l+j-n-1}, & l + j \geq n + 1. \end{cases} \tag{4.10}$$

In fact, since

$$d_l T = \begin{cases} T d_{l-1}, & 0 \leq l \leq n-1, \\ \\ d_n, & l = 0, \end{cases} \qquad d_l T^{-1} = \begin{cases} T^{-1} d_{l+1}, & 0 \leq l \leq n-1, \\ \\ d_0, & l = n. \end{cases}$$

$d_l T^{-j} = d_l T^{-1} T^{-(j-1)} = T^{-j} d_{l+j}, l + j \leq n$, and $d_l T^{-j} = T^{-j} d_{l+j-n-1}, l + j \geq n + 1$. Using (4.10), we get

$$\begin{aligned}
b'E'_D &= \sum_{l=0}^{n-1} \sum_{j=2}^{n-l} \sum_{i=1}^{j-1} d_l T^{-j} D_i + \sum_{l=0}^{n-1} \sum_{j=n-l+1}^{n+1} \sum_{i=1}^{j-1} d_l T^{-j} D_i \\
&= \sum_{l=2}^{n} \sum_{j=2}^{l} \sum_{i=1}^{j-1} T^{-j} d_l D_i + \sum_{l=0}^{n-1} \sum_{j=l+1}^{n} \sum_{i=1}^{j} T^{-j} d_l D_i.
\end{aligned} \tag{4.11}$$

Clearly,

$$\begin{aligned} E'_D b &= \sum_{k=0}^{n}\sum_{j=2}^{n}\sum_{i=1}^{j-1} T^{-j} D_i d_k \\ &= \sum_{j=2}^{n}\sum_{i=1}^{j-1} T^{-j} D_i d_0 + \sum_{j=2}^{n}\sum_{i=1}^{j-1} T^{-j} D_i d_1 + \sum_{k=2}^{n}(\sum_{j=2}^{k} + \sum_{j=k+1}^{n})\sum_{i=1}^{j-1} T^{-j} D_i d_k. \end{aligned}$$

Let us check for $k \geq 2$,

$$D_i d_k = \begin{cases} d_k D_i, \ \ 1 \leq i \leq k-1, \\ \\ d_k(D_k + D_{k+1}) + m_k, \ \ i = k, \\ \\ d_k D_{i+1}, \ \ i \geq k+1, \end{cases} \tag{4.12}$$

where $m_k(a_0, \ldots, a_n, f)(g) = (-1)^k (a_0, \ldots, J_2(a_k, a_{k+1}), \ldots, a_n, f)(g)$. In fact,

$$\begin{aligned} D_i d_k(a_0, \ldots, a_n, f)(g) &= (-1)^k (a_0, \ldots, D(a_i), \ldots, a_k a_{k+1}, \ldots, a_n, f)(g) \\ &= d_k D_i(a_0, \ldots, a_n, f)(g). \end{aligned}$$

Thus, $D_i d_k = d_k D_i, 1 \leq i \leq k-1$.

$$D_k d_k(a_0, \ldots, a_n, f)(g) = (-1)^k (a_0, \ldots, D(a_k a_{k+1}), a_{k+2}, \ldots, a_n, f)(g)$$

$$\begin{aligned} & (d_k(D_k + D_{k+1}) + m_k)(a_0, \ldots, a_n, f)(g) = d_k((a_0, \ldots, D(a_k), \ldots, a_n, f)(g) \\ + \ \ & (a_0, \ldots, a_k, D(a_{k+1}), \ldots, a_n, f)(g)) + (-1)^k (a_0, \ldots, J_2(a_k, a_{k+1}), \ldots, a_n, f)(g) \\ = \ \ & (-1)^k (a_0, \ldots, D(a_k a_{k+1}), \ldots, a_n, f)(g) = D_k d_k(a_0, \ldots, a_n, f)(g). \end{aligned}$$

Also,

$$\begin{aligned} D_i d_k(a_0, \ldots, a_n, f)(g) &= (-1)^k (a_0, \ldots, a_k a_{k+1}, \ldots, D(a_i), \ldots, a_n, f)(g) \\ &= d_k D_{i+1}(a_0, \ldots, a_n, f)(g), \ \ i \geq k+1. \end{aligned}$$

In particular, $D_i d_0 = d_0 D_{i+1},\ \ i \geq 1$. Using (4.12), we obtain

$$
\begin{aligned}
E'_D b &= \sum_{j=2}^{n}\sum_{i=1}^{j-1} T^{-j} d_0 D_{i+1} + \sum_{j=2}^{n}(T^{-j} D_1 d_i + \sum_{i=2}^{j-1} T^{-j} d_1 D_{i+1}) \\
&+ \sum_{k=2}^{n}(\sum_{j=2}^{k}\sum_{i=1}^{j-1} T^{-j} d_k D_i + \sum_{j=k+1}^{n}(\sum_{i=1}^{k-1} T^{-j} d_k D_i + T^{-j}(d_k(D_k + D_{k+1}) + m_k) \\
&+ \sum_{i=k+1}^{j-1} T^{-j} d_k D_{i+1})) \\
&= \sum_{j=2}^{n}\sum_{i=2}^{j} T^{-j} d_0 D_i + \sum_{j=2}^{n} T^{-j}(d_1 D_i + d_1 D_2 + m_1) + \sum_{j=2}^{n}\sum_{i=2}^{j-1} T^{-j} d_1 D_{i+1} \\
&+ \sum_{k=2}^{n}\sum_{j=2}^{k}\sum_{i=1}^{j-1} T^{-j} d_k D_i + \sum_{k=2}^{n}\sum_{j=k+1}^{n}(\sum_{i=1}^{k} T^{-1} d_k D_i + \sum_{i=k}^{i-1} T^{-j} d_k D_{i+1} + T^{-j} m_k) \\
&= \sum_{j=2}^{n}\sum_{i=2}^{j} T^{-j} d_0 D_i + \sum_{k=2}^{n}\sum_{j=2}^{k}\sum_{i=1}^{j-1} T^{-j} d_k D_i + \sum_{k=1}^{n}\sum_{j=k+1}^{n} T^{-j} m_k \\
&+ \sum_{k=1}^{n}\sum_{j=k+1}^{n}\sum_{i=1}^{j} T^{-j} d_k D_i .
\end{aligned}
$$

Consequently by (4.11),

$$
\begin{aligned}
S'(b'E'_D - E'_D b - \sum_{j=0}^{n-1} T^{-j} d_0 D_1) &= S'(\sum_{j=1}^{n}\sum_{i=1}^{j} T^{-j} d_k D_i - \sum_{j=2}^{n}\sum_{i=2}^{j} T^{-j} d_0 D_i \\
&- \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} T^{-j} m_k - \sum_{j=0}^{n-1} T^{-j} d_0 D_1) \\
&= S'(-\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} T^{-j} m_k).
\end{aligned}
$$

Hence to prove (4.9) we need only to check $J_1 = -S' \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} T^{-j} m_k$. In fact, this is why we define J_1 at the beginning. We have

$$
\begin{aligned}
&T^{-j}(a_0, \ldots, a_n, f)(g) \\
&= (-1)^{jnn}(\alpha^{-1}_{g^{j-1}}(a_j), \alpha^{-1}_{g^j}(a_{j+1}), \ldots, \alpha^{-1}_{g^j}(a_0), \alpha^{-1}_{g^{j-1}}(a_1), \ldots, \alpha^{-1}_{g^{j-1}}(a_{j-1}), f)(g).
\end{aligned}
$$

$$
- \quad S' \sum_{k=1}^{n-1}\sum_{j=k+1}^{n} T^{-j} m_k(a_0, \ldots, a_n, f)(g) =
$$

$$\begin{aligned}
&- S'\sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^{j(n-1)(n-1)+k}(\alpha_{g^{j-1}}^{-1}(a_{j+1}),\alpha_{g^j}^{-1}(a_{j+2}),\ldots,\alpha_{g^j}^{-1}(a_n),\alpha_{g^j}^{-1}(a_0),\\
&\quad \alpha_{g^{j-1}}^{-1}(a_1),\ldots,\alpha_{g^{j-1}}^{-1}(J_2(a_k,a_{k+1})),\ldots,\alpha_{g^{j-1}}^{-1}(a_j),f)(g)\\
&= \sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^{(n-1)(n-1)+k+1}(1,\alpha_{g^j}^{-1}(a_{j+1}),\alpha_{g^j}^{-1}(a_{j+2}),\ldots,\alpha_{g^j}^{-1}(a_n),\alpha_{g^j}^{-1}(a_0),\\
&\quad \alpha_{g^{j-1}}^{-1}(a_1),\ldots,\alpha_{g^{j-1}}^{-1}(J_2(a_k,a_{k+1})),\ldots,\alpha_{g^{j-1}}^{-1}(a_j),f)(g)\\
&= J_1(a_0,\ldots,a_n,f)(g),\ \ on\ \ \bar{C}_*^G(\mathcal{A}).
\end{aligned}$$

Q.E.D.

We now use Proposition 4.4 to construct the Chern character. For a Fréchet locally m-algebra $\mathcal{A}$ let $Q\mathcal{A}$ be the universal algebra generated by symbols $a, q(a)$ with $a \in \mathcal{A}$ satisfying $q(a_1a_2) = a_1q(a_2) + q(a_1)a_2 - q(a_1)q(a_2)$ [Cun] and $q\mathcal{A}$ be the ideal in $Q\mathcal{A}$ generated by $q(a), a \in \mathcal{A}$. G acts naturally on $Q\mathcal{A}$ and $q\mathcal{A}$. Let I_5 and $I_6 : \mathcal{A} \to Q\mathcal{A}$ be the canonical equivariant embeddings, $I_5(a) = a, I_6(a) = a - q(a)$. Then $\mathcal{A} \simeq I_5(\mathcal{A})$. We also have

$$Q\mathcal{A} \xrightarrow{\varphi} \Omega(\mathcal{A}) = \mathcal{A} \oplus \oplus_{n>0}(\mathcal{A}^+\hat{\otimes}\mathcal{A}^{\hat{\otimes}n}),$$

where φ is an isomorphism given by $\varphi(a_0qa_1\ldots qa_n) = a_0da_1\ldots da_n$ and $\varphi(qa_1\ldots qa_n) = da_1\ldots da_n$, and $d : \Omega(\mathcal{A}) \to \Omega(\mathcal{A})$ is the differential. With this identification, the multiplication on $Q\mathcal{A}$ is given by

$$\omega_1 * \omega_2 = \begin{cases} \omega_1\omega_2, & |\omega_1|\ \ even, \\ \omega_1\omega_2 + \omega_1 d\omega_2, & |\omega|\ \ odd. \end{cases}$$

Here $|\omega|$ is the degree of $\omega \in Q\mathcal{A}$. Note that the degree $|qa|$ of qa is -1. To use Proposition 4.4, we define $D : Q\mathcal{A}^+ \to Q\mathcal{A}^+$ by $D\omega = -|\omega|\omega$ for homogeneous ω. Then for $a_1, a_2 \in \mathcal{A}, |a_1| = 0 \bmod (2)$, $J_2(a_1, a_2) = D(a_1a_2) - D(a_1)a_2 - D(a_1)a_2 = 0$ and for $|a_1| = 1 \bmod (2)$, $J_2(a_1, a_2) = D(a_1 \cdot a_2 + a_1 da_2) - a_1 D(a_2) - D(a_1)a_2 = a_1 da_2$. Hence, $J_2(a_1, a_2) \in q\mathcal{A}$ and J_2 is of degree -1. Denote also by I_5 and I_6 the extensions of I_5 and I_6 to $\mathcal{A}^+$, which induce morphisms $\bar{T}_*(I_i) : \bar{T}_*^G(\mathcal{A}^+) \to \bar{T}_*^G(Q\mathcal{A}^+)$ and $\bar{T}_*(I_1) = \bar{T}_*(I_2)$ on $\bar{T}_*^G(Q\mathbb{C})$. This implies that $\bar{T}_*(I_5) - \bar{T}_*(I_6)$ descend to $\tilde{T}_*^G(\mathcal{A}^+) = \bar{T}_*^G(\mathcal{A}^+)/\bar{T}_*^G(\mathbb{C}), i = 5, 6$. Let $F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ be the filtration of $Ker\{\pi_* : \tilde{C}_*^G(Q\mathcal{A}^+) \to \tilde{C}_*^G(q\mathcal{A}^+)\}$,

$$F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)_k = \{[a_0, \ldots, a_k, f] \in Ker(\tilde{C}_*^G(Q\mathcal{A}^+) \xrightarrow{\pi_*} \tilde{C}_*^G(q\mathcal{A}^+)) : -\sum_{i=0}^{k} |a_i| \geq n\}$$

for $k \geq n-1$, and for $k < n-1, F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)_k = 0$. The differential d preserves $F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)$. Since $q(a) = I_5(a) - I_6(a)$ and $|q(a)| = -1$, we see $(\bar{T}_*(I_5) - \bar{T}_*(I_6))(\tilde{T}_*^G(\mathcal{A}^+)) \subset F_1^G(Q\mathcal{A}^+, q\mathcal{A}^+)$. Note that J_0 and J_1 defined at the biginning by using D and J_2 map $F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ to $F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ since $J_2(a_1, a_2) \in q(\mathcal{A})$. Clearly,

$$\begin{aligned} L_D(a_0, \ldots, a_k, f)(g) &= \sum_{i=0}^{k} (a_0, \ldots, D(a_i), \ldots, a_k, f)(g) \\ &= -\sum_{i=0}^{k} |a_i| (a_0, \ldots, a_k, f)(g). \end{aligned}$$

Thus on $F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+) \setminus F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ $L_D = -nI$. Because of this, we can define $S_n : F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+) \to F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ by

$$S_n = S - \frac{1}{n} \underline{d} h_D = S - \frac{1}{n} S L_D' - \frac{1}{n} J_D.$$

S_n is well defined since $J_D(F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+)) \subset F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+)$ and $S - \frac{1}{n} S L_D' = S - S = 0$ on $F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+) \setminus F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+)$. Obviously, S_n commutes with S. Hence $S_n \in Hom_S(F_n^G(Q\mathcal{A}^+, q\mathcal{A}^+), F_{n+1}^G(Q\mathcal{A}^+, q\mathcal{A}^+))_{-2}$

and $\underline{d}S_n = 0$. $[S_n] = [S - \frac{1}{n}\underline{d}h_D] = [S]$. Let $i_{n+1} : F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+) \to F^G_n(Q\mathcal{A}^+, q\mathcal{A}^+)$ be the embedding. Then $i_{n+1}S_n \in Hom_S(F^G_n(Q\mathcal{A}^+, q\mathcal{A}^+), F^G_n(Q\mathcal{A}^+, q\mathcal{A}^+))$ and $S_n i_{n+1} \in Hom_S(F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+), F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+))$ which imply that $[i_{n+1}][S_n] = [S] = [S_n][i_{n+1}]$.

Definition 4.7 The Chern character $Ch^{2n}_G(\mathcal{A})$ is defined by

$$Ch^{2n}_G(\mathcal{A}) = [S_n][S_{n-1}]\ldots[S_1]ch^0_G(\mathcal{A}) \in HC^{2n}_G(\tilde{T}^G_*(\mathcal{A}^+), \tilde{F}^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+)),$$

where $Ch^0_G(\mathcal{A}) = [\bar{T}_*(I_5) - \bar{T}_*(I_6)] \in HC^0_G(\tilde{T}^G_*(\mathcal{A}^+), \tilde{F}^G_1(Q\mathcal{A}^+, q\mathcal{A}^+))$.

Observe that

$$\begin{aligned} [i_{n+1}]Ch^{2n}_G(\mathcal{A}) &= [i_{n+1}][S_n][S_{n-1}]\ldots[S_1]ch^0_G(\mathcal{A}) \\ &= [S][S_{n-1}]\ldots[S_1]Ch^0_G(\mathcal{A}) = [S]Ch^{2n-2}_G(\mathcal{A}). \end{aligned}$$

We now focus on the following special situation. Let $(\varphi, \psi) : \mathcal{A} \to \mathcal{E} \rhd \mathcal{J} \xrightarrow{\mu} \mathcal{K}\hat{\otimes}\mathcal{B}$ be an equivariant p-quasi-homomorphism. $\varphi(a) - \psi(a) \in L^p\hat{\otimes}\mathcal{B}, \forall a \in \mathcal{A}$, where L^p is the p-Schatten classes. Thus

$$\mathcal{A} \underset{\psi}{\overset{\varphi}{\longrightarrow}} \mathcal{L}(H)\hat{\otimes}\mathcal{B} \rhd L^p\hat{\otimes}\mathcal{B} \to \mathcal{K}\hat{\otimes}\mathcal{B}.$$

Since $Q\mathcal{A}$ is universal for the quasi-homomorphisms [Cun], there exists an equivariant homomorphism $Q(\varphi, \psi) : Q\mathcal{A} \to \mathcal{L}(H)\hat{\otimes}\mathcal{B}$ which extends φ and ψ, i.e., $Q(\varphi, \psi)(I_5(x)) = \varphi(x), Q(\varphi, \psi)(I_6(x)) = \psi(x)$. This implies that the restriction $q(\varphi, \psi)$ of $Q(\varphi, \psi)$ to $q\mathcal{A}$ maps $q\mathcal{A}$ to $L^p\hat{\otimes}\mathcal{B}$.

We define a map $F_G : F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+) \to \tilde{T}^G_*(\mathcal{B}^+), n+1 \geq p$, by

$$F_G(m_0 \otimes a_0, \ldots, m_k \otimes a_k, f)(g) = Tr(m_0\mu_g m_1 \ldots m_k)(a_0, \ldots, a_k, f)(g).$$

Since there are $n+1(\geq p)$ $m_i's \in L^p, m_0\mu_g m_1 \dots m_k \in L^1$. F_G is well defined (see Chapter 5 for details). Now the map

$$Q(\varphi,\psi)_* : F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+) \to F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+)$$

is induced by $Q(\varphi,\psi)$ and commutes with S. Hence $[Q(\varphi,\psi)_*] \in HC^0_G(F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+), F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+))$. Also $[F_G] \in HC^0_G(F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+), \tilde{T}^G_*(\mathcal{B}^+))$. We can therefore use the composition of the elements to define the Chern character $CH^{2n}_G(\varphi,\psi)$ of equivariant p-quasi-homomorphism (φ,ψ).

Definition 4.8 The Chern character $CH^{2n}_G(\varphi,\psi)$ of (φ,ψ) is defined by

$$\begin{aligned}
CH^{2n}_G(\varphi,\psi) &= [F_G][Q(\varphi,\psi)_*]Ch^{2n}_G(\mathcal{A}) \\
&\in HC^{2n}_G(\tilde{T}^G_*(\mathcal{A}^+), \tilde{F}^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+)) \\
&\otimes HC^0_G(F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+), F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+)) \\
&\otimes HC^0_G(F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+), \tilde{T}^G_*(\mathcal{B}^+)) \\
&\to \widetilde{HC}^{2n}_G(\mathcal{A}^+, \mathcal{B}^+) \simeq HC^{2n}_G(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

Theorem 4.4 *Let (φ,ψ) be an equivariant p-quasi-homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

(1) $S(CH^{2n}_G(\varphi,\psi)) = CH^{2n+2}_G(\varphi,\psi), n \geq p-1$,

(2) $\{CH^{2n}_G(\varphi,\psi)\}_{n\geq p-1}$ *defines an element* $CH^{ev}_G(\varphi,\psi)$ *in* $PHC^{ev}_G(\mathcal{A},\mathcal{B})$,

(3) *If two equivariant p-quasi-homomorphisms (φ_0,ψ_0) and (φ_1,ψ_1) of $\mathcal{A}$ and $\mathcal{B}$ are connected by a smooth path of equivariant p-quasi-homomorphisms, then* $CH^{ev}_G(\varphi_0,\psi_0) = CH^{ev}_G(\varphi_1,\psi_1)$.

(4) *If $\varphi \equiv \psi$ on a dense subalgebra $\mathcal{A}_\infty$ of $\mathcal{A}$, then* $CH^{2n}_G(\varphi,\psi) = 0$.

Proof. (1) We have the commutative diagram

$$\begin{tikzcd}
F^G_{n+2}(Q\mathcal{A}^+, q\mathcal{A}^+) \arrow[r, "i_{n+2}"] \arrow[dr, "Q(\varphi,\psi)_*"'] & F^G_{n+1}(Q\mathcal{A}^+, q\mathcal{A}^+) \arrow[r, "Q(\varphi,\psi)_*"] & F^G_{n+1}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+) \arrow[r, "F_G"] & \tilde{T}^G_*(\mathcal{B}^+) \\
& F^G_{n+2}(\mathcal{L}(H)\hat{\otimes}\mathcal{B}^+, L^p\hat{\otimes}\mathcal{B}^+)) \arrow[urr, "F_G"'] & &
\end{tikzcd}$$

Hence, $[F_G][Q(\varphi,\psi)_*][i_{n+2}] = [F_G][Q(\varphi,\psi)_*]$.

$$\begin{aligned}
CH^{2n+2}_G(\varphi,\psi) &= [F_G][Q(\varphi,\psi)_*]Ch^{2n+2}_G(\mathcal{A}) = [F_G][Q(\varphi,\psi)_*][i_{n+2}]Ch^{2n+2}_G(\mathcal{A}) \\
&= [F_G][Q(\varphi,\psi)_*][S]Ch^{2n}_G(\mathcal{A}) \\
&= [S][F_G][Q(\varphi,\psi)_*]Ch^{2n}_G(\mathcal{A}) = S(CH^{2n}_G(\varphi,\psi)).
\end{aligned}$$

(2) is trivial by (1).

(3) Let (φ_t,ψ_t) be a smooth path of equivariant p-quasi-homomorphisms of $\mathcal{A}$ and $\mathcal{B}$ connecting (φ_0,ψ_0) and (φ_1,ψ_1). Let $e_t : C_{[0,1]}\hat{\otimes}\mathcal{B} \to \mathcal{B}$ be the evaluation map at $t, t \in [0,1]$. We have $(\varphi_i,\psi_i) = (e_i)_*(\varphi_t,\psi_t)$, and $CH^{2n}_G(\varphi_i,\psi_i) = [(e_i)_*]CH^{2n}_G(\varphi_t,\psi_t), n \geq p-1$, where $CH^{2n}_G(\varphi_t,\psi_t) \in HC^{2n}_G(\mathcal{A}, C_{[0,1]}\hat{\otimes}\mathcal{B})$ is defined by (φ_t,ψ_t). Using Proposition 4.4 for $J_D = 0$, we get $S([(e_1)_*] - [(e_0)_*]) = 0$. By part (1),

$$\begin{aligned}
CH^{2n+2}_G(\varphi_1,\psi_1) - CH^{2n}_G(\varphi_0,\psi_0) &= S(CH^{2n}_G(\varphi_1,\psi_1) - CH^{2n}_G(\varphi_0,\psi_0)) \\
&= S([(e_1)_*] - [(e_0)_*])CH^{2n}_G(\varphi_t,\psi_t) = 0.
\end{aligned}$$

(4) is clear, since $Q(\varphi,\psi)(q(x)) = Q(\varphi,\psi)(I_5(x) - I_6(x)) = 0$ for $x \in \mathcal{A}_\infty$.

Q.E.D.

Theorem 4.4 shows that the Chern character $CH^{ev}_G(\varphi,\psi)$ is well defined for the equivalent classes in the sense of (3) and (4) of the theorem. One may notice that the above construction of the Chern character does not involves the excision property of equivariant bivariant cyclic theory which is crucial in ([Kass], [Wang]). But it is not clear how to construct the odd Chern character without using the excision property.

Chapter 5

Chern Characters in Entire Equivariant Cyclic (Co-)homology

In the previous chapter we constructed the Chern character for equivariant p-quasi-homomorphisms. It is natural to try to define the Chern character for a more general situation. As a partial solution to this question we will obtain the Chern character for equivariant θ-summable Fredholm modules which extend p-summable Fredholm modules, the analogue of p-quasi-homomorphisms in one variable case. The main purpose of the present chapter is to construct Chern characters in (entire) equivariant cyclic (co-)homology and to apply this construction to the equivariant index theorem. In section 5.1 we recall the definitions of (entire) equivariant cyclic (co-)homology for a unital Banach algebra $\mathcal{A}$ with compact group G action. Some basic morphisms on the (entire) equivariant cyclic (co-)homology are defined, which are useful for the later sections. In section 5.2 we construct the Chern characters from equivariant K-theory to the normal (entire) equivariant cyclic homology. In particular, we study the character from odd equivariant K-theory $K_1^G(\mathcal{A})$ to the odd normal (entire) equivariant cyclic homology. In section 5.3 we obtain the pairings of (entire) equivariant cyclic cohomology with equivariant K-theory. In section 5.4 we define the Chern character from even equivariant $\theta-$summable Fredholm modules to the normal entire equivariant cyclic cohomology. Finally, the results in the preceding sections are used in section 5.5 to get the equivariant index theorem of even equivariant $\theta-$summable Fredholm modules. Another possible application of the Chern characters and pairings defined in this chapter would be the proof of the equivariant Novikov conjecture for some special cases (cf. Chapters 6 and 8). Our formulas for the Chern characters and pairings in the even case are inspired by the ordinary ones except for the group twisting ([Con 1, 2],[GeS]). We will thus have to reconstruct many identities whose proofs are complicated. It should be pointed out that the formulas here are quite different from those in ([KKL], [KlL]), which deal with finite group actions. On may also consult [BlG] for the Chern Characters.

5.1 Equivariant Cyclic (Co-)homology

Throughout we assume that G is a compact group and that $\mathcal{A}$ is a unital $G-$Banach algebra over $\mathbb{C}$. This means that G acts on $\mathcal{A}$ by continuous automorphisms $\alpha : G \to Aut(\mathcal{A})$ such that α_g is unital for each $g \in G$. In this section we recall various equivariant cyclic (co-)homologies [Gong 2]. Let $C(G)$ be the space of all continuous functions on G and $\tilde{\otimes}$ be the projective tensor product. Let

$$C_n^G(\mathcal{A}) = (\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbb{C},$$

the quotient of $\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$ by $\overline{Span}\{(\rho_h - I)(x), (\rho - I)(x) : x \in \mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)\}$,

and

$$\begin{aligned} C_G^n(\mathcal{A}) &= Hom_G(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G)) \\ &= \{f \in Hom(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G)) : \tilde{\rho}_h(f) = f, h \in G\}, \end{aligned}$$

where $\mathcal{A}^{\tilde{\otimes}(n+1)}$ and $Hom(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$ are the $(n+1)-$tuple projective tensor product of $\mathcal{A}$ and the space of all continuous multi-linear maps from $\mathcal{A}^{\tilde{\otimes}(n+1)}$ to $C(G)$, resp.. G acts on $\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$ and $Hom(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$ by ρ_h and $\tilde{\rho}_h$, respectively,

$$\begin{aligned} \rho_h(a_0, a_1, \ldots, a_n, f)(g) &= (\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n), h \cdot f)(g) \\ &= (\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n))f(hgh^{-1}), \quad g \in G \end{aligned}$$

for $(h \cdot f)(g) = f(hgh^{-1}), f \in C(G), a_i \in \mathcal{A}$, and

$$(\tilde{\rho}_h f)(a_0, a_1 \ldots, a_n)(g) = f(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n))(h^{-1}gh);$$

ρ is defined by

$$\rho(a_0, a_1, \ldots, a_n, f)(g) = (\alpha_g^{-1}(a_0), \alpha_g^{-1}(a_1), \ldots, \alpha_g^{-1}(a_n))f(g).$$

Here $(a_0, a_1, \ldots, a_n, f)$ stands for $(a_0\tilde{\otimes}a_1\tilde{\otimes}\ldots\tilde{\otimes}a_n\tilde{\otimes}f)$ in $\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$.

Define differentials b and B on $\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$ by

$$b = \sum_{i=0}^{n}(-1)^i d_n^i \ \ and \ \ B = (1-T)S'N,$$

where d_n^i,T and S' were defined in Chapter 4. It is easy to see that d_n^i,T and S' are continuous equivariant maps [Gong 2]. We also define b and B on $Hom(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$ to be the dual version of b and B above [Con 1]. The b

and B descend to $C_n^G(\mathcal{A})$ and $C_G^n(\mathcal{A})$ and satisfy $b^2 = B^2 = bB + Bb = 0$ by Lemmas 1 and 2 in [Gong 2]. Let

$$T_n^G(\mathcal{A}) = C_n^G(\mathcal{A}) \oplus C_{n-2}^G(\mathcal{A}) \oplus C_{n-4}^G(\mathcal{A}) \oplus \dots,$$

and

$$T_G^n(\mathcal{A}) = C_G^n(\mathcal{A}) \oplus C_G^{n-2}(\mathcal{A}) \oplus C_G^{n-4}(\mathcal{A}) \oplus \dots$$

be the complexes with differential $b + B$,

$$(b+B)(x_{2n+i}, x_{2n-2+i}, \dots, x_i) = (bx_{2n+i} + Bx_{2n-2+i}, bx_{2n-2+i} + Bx_{2n-4+i}, \dots),$$

$x_k \in C_k^G(\mathcal{A}), i = 0, 1$, and

$$(b+B)(f_{2n+i}, f_{2n-2+i}, \dots, f_i) = (bf_{2n+i}, bf_{2n-2+i} + Bf_{2n+i}, \dots).$$

Definition 5.1 The equivariant cyclic (co-)homology of $\mathcal{A}$ is defined by

$$HC_n^G(\mathcal{A}) = H_n(T_n^G(\mathcal{A}), b + B),$$

$$HC_G^n(\mathcal{A}) = H^n(T_G^n(\mathcal{A}), b + B),$$

for $n \geq 0$.

To define the entire equivariant cyclic (co-)homology, we need to estimate the norms of b and B. Note that $\mathcal{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ has the projective tensor product norm and $C_n^G(\mathcal{A})$ has the quotient norm of $\mathcal{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$. The norm on $Hom(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$ is given by

$$\|f\|_n = max_{g \in G, \|a_i\| \leq 1} |f(a_0, a_1, \dots, a_n)(g)|.$$

Then

$$\|b(a_0, a_1, \dots, a_n, f)\|_{n-1} \leq (n+1)\|f\| \prod_{i=0}^{n} \|a_i\|,$$

$$\|S'(a_0, a_1, \ldots, a_n, f)\|_{n+1} \leq \|f\| \prod_{i=0}^{n} \|a_i\|,$$

and

$$\|T(a_0, a_1, \ldots, a_n, f)\| \leq \|f\| \prod_{i=0}^{n} \|a_i\|.$$

Hence $\|b\|_n \leq (n+1), \|S'\|_n \leq 1, \|T\|_n \leq 1, \|N\|_n \leq (n+1)$ and $\|B\|_n \leq 2(n+1)$ on $\mathcal{A}^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$ and then on $C_n^G(\mathcal{A})$. Similarly, $\|b\|_n \leq (n+2), \|S'\|_n \leq 1, \|T\|_n \leq 1, \|N\| \leq n+1$ and $\|B\|_n \leq 2(n+2)$ on $C_G^n(\mathcal{A})$.

Let $T_{ev}^G(\mathcal{A})$ (resp.$T_{odd}^G(\mathcal{A})$) be the space of all sequences $\{x_{2n}\}_{n=0}^{\infty}$ (resp.$\{x_{2n+1}\}_{n=0}^{\infty}$) such that for any $r \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \frac{r^n \|x_{2n}\|_{2n}}{(2n)!} \leq \infty (resp. \sum_{n=0}^{\infty} \frac{r^n \|x_{2n+1}\|_{2n+1}}{(2n+1)!} \leq \infty),$$

where $x_k \in C_k^G(\mathcal{A})$. Let $T_G^{ev}(\mathcal{A})$ (resp. $T_G^{odd}(\mathcal{A})$) be the space of all sequences $\{f_{2n}\}_{n=0}^{\infty}$ (resp.$\{f_{2n+1}\}_{n=0}^{\infty}$), $f_k \in C_G^k(\mathcal{A})$, such that for any $r \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} r^n n! \|f_{2n}\|_{2n} \leq \infty \; (resp. \sum_{n=0}^{\infty} r^n n! \|f_{2n+1}\|_{2n+1} \leq \infty).$$

Then it follows from the estimation of the norms of b and B that b and B map $T_{\gamma_1}^G(\mathcal{A})$ and $T_G^{\gamma_1}(\mathcal{A})$ to $T_{\gamma_2}^G(\mathcal{A})$ and $T_G^{\gamma_2}(\mathcal{A})$, respectively, for $\gamma_1, \gamma_2 = ev \text{ or } odd$ and $\gamma_1 \neq \gamma_2$.

Definition 5.2 With these notations, the entire equivariant cyclic (co-)homology of $\mathcal{A}$ is

$$HC_{\gamma_1}^G(\mathcal{A}) = \frac{Ker((b+B)|T_{\gamma_1}^G(\mathcal{A}))}{Im((b+B)|T_{\gamma_2}^G(\mathcal{A}))}$$

and

$$HC_G^{\gamma_1}(\mathcal{A}) = \frac{Ker((b+B)|T_G^{\gamma_1}(\mathcal{A}))}{Im((b+B)|T_G^{\gamma_2}(\mathcal{A}))}.$$

Remark 5.1 (1) *The entire equivariant cyclic homology here differs slightly from the ordinary one ([Con 2], [GeS]) by the coefficients in the decay conditions of* $T^G_{\gamma_1}(\mathcal{A})$*, since we want to deal with the Chern character in* $HC^G_{odd}(\mathcal{A})$.

(2) *We will need the normal version of (entire) equivariant cyclic (co-)homology, which is defined as follows. Let* $\bar{\mathcal{A}} = \mathcal{A}/\mathbb{C}$ *be the quotient of* $\mathcal{A}$ *by* $\mathbb{C}$ *and* $\bar{C}^G_n(\mathcal{A}) = (\mathcal{A}\tilde{\otimes}\bar{\mathcal{A}}^{\tilde{\otimes}(n)}\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbb{C}$ *and* $\bar{C}^n_G(\mathcal{A}) = \{f \in Hom(\mathcal{A}\tilde{\otimes}(\bar{\mathcal{A}})^{\tilde{\otimes}(n)}, C(G)):$ $\tilde{\rho}_h(f) = f, h \in G\}$*. Replacing* $C^G_*(\mathcal{A})$ *and* $C^*_G(\mathcal{A})$ *by* $\bar{C}^G_*(\mathcal{A})$ *and* $\bar{C}^*_G(\mathcal{A})$ *in* $T^G_*(\mathcal{A}), T^G_{\gamma_1}(\mathcal{A}), T^*_G(\mathcal{A})$ *and* $T^{\gamma_1}_G(\mathcal{A})$*, we get the complexes* $\bar{T}^G_*(\mathcal{A}), \bar{T}^G_{\gamma_1}(\mathcal{A}), \bar{T}^*_G(\mathcal{A})$ *and* $\bar{T}^{\gamma_1}_G(\mathcal{A})$ *with differential* $b+B$*. Then the normal (entire) equivariant cyclic (co-)homology is the (co-)homology of* $\bar{T}^G_*(\mathcal{A}), \bar{T}^G_{\gamma_1}(\mathcal{A}), \bar{T}^*_G(\mathcal{A})$ *and* $\bar{T}^{\gamma_1}_G(\mathcal{A})$*, resp..*

Note that in the normal case, B has the following simple formula

$$B(a_0,a_1,\ldots,a_n,f)(g) = \sum_{i=0}^{n}(-1)^{in}(1,\alpha_g^{-1}(a_{n-i+1}),\ldots,\alpha_g^{-1}(a_n),\alpha_g^{-1}(a_0),a_1,\ldots,a_{n-i})f(g) \quad (5.1)$$

Let now (V,β) be a finite dimensional unitary representation of the compact group G, and $End(V)$ be the space of all continuous linear maps on V. G acts on $End(V)$ by defining the action of ν_g,

$$\nu_g(m) = \beta_g m \beta_g^{-1}, \quad m \in End(V), \quad g \in G.$$

$\mathcal{A}\tilde{\otimes}End(V)$ has diagonal $G-$action. Let F_G and F^*_G,

$$C^G_*(\mathcal{A}\tilde{\otimes}End(V)) \xrightarrow{F_G} C^G_*(\mathcal{A})$$

and

$$C^*_G(\mathcal{A}) \xrightarrow{F^*_G} C^*_G(\mathcal{A}\tilde{\otimes}End(V)),$$

be the morphisms defined by

$$F_G(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1,\ldots,a_n\tilde{\otimes}m_n, f)(g) = (a_0, a_1,\ldots,a_n)f(g)Tr(m_0\beta_g m_1\ldots m_n) \quad (5.2)$$

on $(\mathcal{A}\tilde{\otimes}End(V))^{\tilde{\otimes}(n+1)}\tilde{\otimes}C(G)$, and for $f \in C^*_G(\mathcal{A})$,

$$(F^*_G f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1,\ldots,a_n\tilde{\otimes}m_n)(g) = f(a_0, a_1,\ldots,a_n)(g)Tr(m_0\beta_g m_1\ldots m_n) \quad (5.3)$$

Here Tr is the natural trace on $End(V)$.

Lemma 5.1 *Let $\mathcal{A}$ be a unital $G-$Banach algebra and (V,β) a finite dimensional unitary representation of G. Then F_G and F^*_G induce morphisms on (entire) equivariant cyclic (co-)homology,*

$$HC^G_*(\mathcal{A}\tilde{\otimes}End(V)) \xrightarrow{F_G} HC^G_*(\mathcal{A})$$

and

$$HC^*_G(\mathcal{A}) \xrightarrow{F^*_G} HC^*_G(\mathcal{A}\tilde{\otimes}End(V))$$

for $* = n$, and for both cases of even and odd entire cyclic cohomology (for short "ev" and "odd"). Similar results hold for the normal case.

Proof. The proof is divided into several steps.

(1). $\rho_h F_G = F_G(\rho_h \otimes \nu_h^{-1})$ and $(\tilde{\rho}_h \otimes \nu_h^{-1})F^*_G = F^*_G\tilde{\rho}_h$.

$$\begin{aligned}
&(\rho_h F_G)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g) \\
&= \rho_h(a_0, a_1, \ldots, a_n, f(\cdot)Tr(m_0\beta_{(\cdot)}m_1\ldots m_n)(g) \\
&= (\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n))f(hgh^{-1})Tr(m_0\beta_{hgh^{-1}}m_1\ldots m_n) \\
&= F_G(\alpha_h^{-1}(a_0)\tilde{\otimes}\nu_h^{-1}(m_0), \ldots, \alpha_h^{-1}(a_n)\tilde{\otimes}\nu_h^{-1}(m_n), h\cdot f)(g) \\
&= F_G(\rho_h \otimes \nu_h^{-1})(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g),
\end{aligned}$$

and

$$((\tilde{\rho}_h \otimes \nu_h^{-1})F^*_G f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g)$$

$$= (F_G^* f)(\alpha_h^{-1}(a_0)\tilde{\otimes}\nu_h^{-1}(m_0), \ldots, \alpha_h^{-1}(a_n)\tilde{\otimes}\nu_h^{-1}(m_n))(h^{-1}gh)$$

$$= f(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n))(h^{-1}gh)Tr(m_0\beta_g m_1 \ldots m_n)$$

$$= F_G^*(\tilde{\rho}_h f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g).$$

It is easy to see that $\rho F_G = F_G\rho$. Hence, F_G and F_G^* are equivariant.

(2). $d_n^i F_G = F_G d_n^i$ and $d_n^i F_G^* = F_G^* d_n^i$.

$$d_n^0 F_G(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g)$$

$$= d_n^0(a_0, a_1, \ldots, a_n, f(\cdot)Tr(m_0\beta_{(\cdot)}m_1 \ldots m_n)(g)$$

$$= (a_0\alpha_g(a_1), a_2, \ldots, a_n)f(g)Tr(m_0\beta_g m_1 \ldots m_n)$$

$$= F_G(a_0\alpha_g(a_1)\tilde{\otimes}m_0\nu_g(m_1), a_2\tilde{\otimes}m_2, \ldots, a_n\tilde{\otimes}m_n)f(g))$$

$$= F_G d_n^0(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g).$$

Clearly, $d_n^i F_G = F_G d_n^i, 1 \leq i \leq n$, and $d_n^i F_G^* = F_G^* d_n^i, 1 \leq i \leq n+1$.

$$(d_n^0 F_G^* f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_{n+1}\tilde{\otimes}m_{n+1})(g)$$

$$= (F_G^* f)(a_0\alpha_g(a_1)\tilde{\otimes}m_0\nu_g(m_1), a_2\tilde{\otimes}m_2, \ldots, a_{n+1}\tilde{\otimes}m_{n+1})(g)$$

$$= f(a_0\alpha_g(a_1), a_2, \ldots, a_{n+1})(g)Tr(m_0\beta_g m_1 \ldots m_{n+1})$$

$$= d_n^0 f(a_0, a_1 \ldots, a_{n+1})(g)Tr(m_0\beta_g m_1 \ldots m_{n+1})$$

$$= (F_G^* d_n^0 f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_{n+1}\tilde{\otimes}m_{n+1})(g).$$

(3). $TF_G = F_G T$ and $TF_G^* = F_G^* T$.

$$(TF_G)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g)$$

$$= T(a_0, a_1, \ldots, a_n, f(\cdot)Tr(m_0\beta_{(\cdot)}m_1 \ldots m_n)(g)$$

$$= (-1)^n(a_n, \alpha_g^{-1}(a_0), a_1, \ldots, a_{n-1})f(g)Tr(m_0\beta_g m_1 \ldots m_n)$$

$$= F_G((-1)^n(a_n\tilde{\otimes}m_n, \alpha_g^{-1}(a_0)\tilde{\otimes}\nu_g^{-1}(m_0), a_1\tilde{\otimes}m_1, \ldots, a_{n-1}\tilde{\otimes}m_{n-1}, f))(g)$$

$$= F_G T(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g),$$

and

$$(TF_G^*f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g)$$

$$= (-1)^n(F_G^*f)(a_n\tilde{\otimes}m_n, \alpha_g^{-1}(a_0)\tilde{\otimes}\nu_g^{-1}(m_0), a_1\tilde{\otimes}m_1, \ldots, a_{n-1}\tilde{\otimes}m_{n-1})(g)$$

$$= f(a_n, \alpha_g^{-1}(a_0), a_1, \ldots, a_{n-1})(g)Tr(m_n\beta_g\nu_g^{-1}(m_0)m_1\ldots m_{n-1})$$

$$= (F_G^*Tf)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g).$$

(4). $S'F_G = F_GS'$ and $S'F_G^* = F_G^*S'$.

$$S'F_G(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g)$$

$$= S'(a_0, a_1, \ldots, a_n, f(\cdot)Tr(m_0\beta_{(\cdot)}m_1\ldots m_n)(g)$$

$$= (1, \alpha_g^{-1}(a_0), a_1, \ldots, a_n)f(g)Tr(m_0\beta_g m_1\ldots m_n)$$

$$= F_GS'(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n, f)(g),$$

and

$$(S'F_G^*f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g)$$

$$= (F_G^*f)(1, \alpha_g^{-1}(a_0)\tilde{\otimes}\nu_g^{-1}(m_0), a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g)$$

$$= f(1, \alpha_g^{-1}(a_0), a_1, \ldots, a_n)(g)Tr(1\beta_g\nu_g^{-1}(m_0)m_1\ldots m_n)$$

$$= (S'f)(a_0, a_1, \ldots, a_n)(g)Tr(m_0\beta_g m_1\ldots m_n)$$

$$= (F_G^*S'f)(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \ldots, a_n\tilde{\otimes}m_n)(g).$$

Therefore, F_G and F_G^* commute with b and B.

(5). $F_G(T_*^G(\mathcal{A}\tilde{\otimes}End(V))) \subseteq T_*^G(\mathcal{A})$ and $F_G^*(T_G^*(\mathcal{A}) \subseteq T_G^*(\mathcal{A}\tilde{\otimes}End(V))$.

In fact, with the norm $\|m\| = max_{1\leq i,j\leq q}|m_{ij}|$ for $m = (m_{ij})_{i,j=1}^{q} \in End(V)$ and $q = dim(V)$, we have $|Tr(m)| \leq q\|m\|$,

$$\|F_G(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \dots, a_n\tilde{\otimes}m_n, f)\|_n = max_{g\in G}\|(a_0, \dots, a_n)f(g)$$
$$\cdot \quad Tr(m_0\beta_g m_1 \dots m_n)\| \leq q\prod_{i=0}^{n}(\|a_i\|\|m_i\|)\|f\|\|\beta\|$$

for $\|\beta\| = max_{g\in G}\|\beta_g\|$, and

$$max_{g\in G,\|a_i\tilde{\otimes}m_i\|\leq 1}|F_G^*f(a_0\tilde{\otimes}m_0, a_1\tilde{\otimes}m_1, \dots, a_n\tilde{\otimes}m_n)(g)|$$
$$\leq max_{\|a_i\tilde{\otimes}m_i\|\leq 1}q\|f\|\prod_{i=0}^{n}(\|a_i\|\|m_i\|)\|\beta\| \leq q\|f\|_n.$$

Hence, $\|F_G\|_n \leq q\|\beta\|$ and $\|F_G^*\|_n \leq q\|\beta\|$. The same arguments work for the normal case. Q.E.D.

The maps F_G and F_G^* will be used later on in the construction of Chern characters. A similar map for entire equivariant cyclic cohomology is also defined in [KKL].

To consider other useful maps, let us define $L_1(G, \mathcal{A})$ to be the space of all functions $x : G \to \mathcal{A}$ such that $\int_G \|x(g)\|dg \leq \infty$. Here dg is the normalized Haar measure on G. $L_1(G, \mathcal{A})$ is a unital Banach algebra with multiplication given by

$$(x_1 * x_2)(g) = \int_G x_1(h)\alpha_h(x_2(h^{-1}g))dh.$$

Let $R(G)$ be the space of all continuous central functions on G, i.e., $\varphi \in R(G)$ if $\varphi \in C(G)$ and $\varphi(h^{-1}gh) = \varphi(g), \forall h \in G$. For $\varphi \in R(G)$ define morphisms F_φ and F_φ^*,

$$C_*(L_1(G, \mathcal{A})) \xrightarrow{F_\varphi} C_*^G(\mathcal{A})$$

$$C_G^*(\mathcal{A}) \xrightarrow{F_\varphi^*} C^*(L_1(G,\mathcal{A})),$$

by the following formulas

$$\begin{aligned} &F_\varphi(x_0, x_1, \ldots, x_n)(g) \\ &= \int_{G^{n+1}:h_0..h_n=g} (x_0(h_0), \alpha^{-1}_{h_1..h_n}(x_1(h_1)), \ldots, \alpha^{-1}_{h_n}(x_n(h_n)))\varphi(h_0 \ldots h_n) d^n h \end{aligned} \quad (5.4)$$

and

$$\begin{aligned} &(F_\varphi^* f)(x_0, x_1, \ldots, x_n) \\ &= \int_{G^{n+1}} f(x_0(h_0), \alpha^{-1}_{h_1..h_n}(x_1(h_1)), \ldots, \alpha^{-1}_{h_n}(x_n(h_n)))(h_0 \ldots h_n)\varphi(h_0 \ldots h_n) d^{n+1} h. \end{aligned} \quad (5.5)$$

Here, $C_n(L_1(G,\mathcal{A})) = (L_1(G,\mathcal{A}))^{\tilde{\otimes}(n+1)}$, $C^n(L_1(G,\mathcal{A})) = Hom((L_1(G,\mathcal{A}))^{\tilde{\otimes}(n+1)}, \mathbb{C})$. It is straightforward to check that $\|F_\varphi\|_n \leq \|\varphi\|$ and $\|F_\varphi^*\|_n \leq \|\varphi\|$.

The following proposition can also be used to construct the Chern characters. But the proof consists of rather lengthy computation. Since we will not use it in this dissertation, we omit the details of the proof.

Proposition 5.1 *With the above notations, F_φ and F_φ^* induce morphisms in (entire) cyclic (co-)homology*

$$HC_*(L_1(G,\mathcal{A})) \xrightarrow{F_\varphi} HC_*^G(\mathcal{A}),$$

and

$$HC_G^*(\mathcal{A}) \xrightarrow{F_\varphi^*} HC^*(L_1(G,\mathcal{A}))$$

for $* = n$, "ev" or "odd" and $\varphi \in R(G)$.

5.2 Chern Characters in Equivariant Cyclic Homology

Recall that the equivariant K-theory group $K_0^G(\mathcal{A})$ of a unital G-Banach algebra $\mathcal{A}$ consists of the equivalence classes of all equivariant idempotents in $\mathcal{A}\tilde{\otimes}End(V)$ for any finite dimensional unitary representation (V,β) of compact group G. Here an equivariant idempotent $p' \in \mathcal{A}\tilde{\otimes}End(V)$ is equivalent to an equivariant idempotent $p'' \in \mathcal{A}\tilde{\otimes}End(V_1)$ if there are equivariant elements $p \in \mathcal{A}\tilde{\otimes}L(V,V_1)$ and $q \in \mathcal{A}\tilde{\otimes}L(V_1,V)$ such that $pq = p'', qp = p'$, where (V_1,β_1) is another finite dimensional unitary representation of G; $L(V,V_1)$ is the space of all continuous linear maps from V to V_1. Note that $\mathcal{A}\tilde{\otimes}End(V)$, considered as endomorphisms on $\mathcal{A}\tilde{\otimes}V$, has diagonal $G-$action $h\cdot(a\tilde{\otimes}m) = \alpha_h^{-1}a\alpha_h\tilde{\otimes}\beta_h^{-1}m\beta_h$, i.e., $h\cdot(a\tilde{\otimes}m)(a_1\tilde{\otimes}m_1) = (\alpha_h^{-1}a\alpha_h\tilde{\otimes}\beta_h^{-1}m\beta_h)(a_1\tilde{\otimes}m_1) = \alpha_h^{-1}(a)a_1\tilde{\otimes}(\beta_h^{-1}m\beta_h)(m_1)$. Thus, the G-action on $\mathcal{A}\tilde{\otimes}End(V)$ is the same as $\alpha_h^{-1}\tilde{\otimes}\nu_h^{-1}$ in section 5.1. See [Phi] for details.

Let $p \in \mathcal{A}\tilde{\otimes}End(V)$ be an equivariant idempotent. Define elements $[p_n]$ and $[\tilde{p}_n] \in \tilde{C}_n^G(\mathcal{A}\tilde{\otimes}End(V))$ by

$$p_n(g) = (\overbrace{p,p,\ldots,p}^{n+1},1), \quad n \geq 0,$$

$$\tilde{p}_n(g) = (1,\overbrace{p,p,\ldots,p}^{n},1), \quad n > 0,$$

and

$$\tilde{p}_0(g) = 0.$$

Here the $1 \in C(G)$ in the last position is the constant map with value 1. Then since $(\alpha_h^{-1}\tilde{\otimes}\nu_h^{-1})(p) = p$ and $p^2 = p$, we have

$$(bp_n)(g) = \begin{cases} p_{n-1}(g), & n \text{ } even; \\ 0, & n \text{ } odd, \end{cases} \tag{5.6}$$

$$(b\tilde{p}_n)(g) = \begin{cases} (2p_{n-1} - \tilde{p}_{n-1})(g), n \text{ } even; \\ 0, \quad n \text{ } odd, \end{cases} \tag{5.7}$$

$$(Bp_n)(g) = \begin{cases} (n+1)\tilde{p}_{n+1}(g), & n \text{ } even; \\ 0, & n \text{ } odd, \end{cases} \tag{5.8}$$

and

$$(B\tilde{p}_n)(g) = 0, n \geq 0. \tag{5.9}$$

Let

$$\tilde{Ch}^G_{2n}(p) = \{(-1)^k \frac{(2k)!}{k!}(p_{2k} - \frac{1}{2}\tilde{p}_{2k})\}^n_{k=0}$$

and

$$\tilde{Ch}^G_{ev}(p) = \{(-1)^k \frac{(2k)!}{k!}(p_{2k} - \frac{1}{2}\tilde{p}_{2k})\}^\infty_{k=0}.$$

Then $\tilde{Ch}^G_*(p) \in \bar{T}^G_*(\mathcal{A}\tilde{\otimes}End(V)), * = 2n$ or "ev".

Theorem 5.1 *let $[p] \in K^G_0(\mathcal{A})$ be represented by the equivariant idempotent $p \in \mathcal{A}\tilde{\otimes}End(V)$ for some finite dimensional unitary representation (V, β) of G. Then $Ch^G_*(p) = F_G(\tilde{Ch}^G_*(p))$ define maps from $K^G_0(\mathcal{A})$ to $\bar{HC}^G_*(\mathcal{A})$ for $* = 2n$ or "ev", where F_G is given by (5.2).*

Proof. It follows easily from identities (5.6)$-$(5.9) that $Ch^G_*(p)$ are cycles, i.e., $(b+B)Ch^G_*(p) = 0$ for $* = 2n$ or "ev". To show that $Ch^G_*(p)$ are independent of the choice of p, let us note that if (V_1, β_1) is another finite dimensional representation of G and $p \oplus 0 = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}\tilde{\otimes}End(V \oplus V_1)$, then

$$F_G((p \oplus 0)_{2n}) = F_G(p_{2n})$$

and

$$F_G((p \tilde{\oplus} 0)_{2n}) = F_G(\tilde{p}_{2n}).$$

Hence $Ch_*^G(p \oplus 0) = Ch_*^G(p)$ for $* = 2n$ or "ev". Thus, if $\tilde{p} \in \mathcal{A}\tilde{\otimes}End(V_1)$ is another idempotent such that $\tilde{p}$ is equivalent to p, we can consider both p and $\tilde{p}$ as elements in $\mathcal{A}\tilde{\otimes}End(V \oplus V_1)$ by using $p \oplus 0$ and $0 \oplus \tilde{p}$. As a consequence, we can assume that two equivalent idempotents p and $\tilde{p}$ are in $\mathcal{A}\tilde{\otimes}End(V)$ and we need only to consider the case when there is an equivariant path $p_t, t \in [0, 1]$ such that $p_0 = p$ and $p_1 = \tilde{p}$ [Phi]. Furthermore, we can assume that p_t is differentiable with respect to t. Thus we need to show that the $Ch_*^G(p_t)$ are independent of t. It suffices to show that the $\frac{d}{dt}\tilde{Ch}_*^G(p_t)$ are boundaries. To this aim, let us define for the equivariant element $y \in \mathcal{A}\tilde{\otimes}End(V)$ two maps

$$L_1(y) : \bar{C}_n^G(\mathcal{A}\tilde{\otimes}End(V)) \longrightarrow \bar{C}_{n+1}^G(\mathcal{A}\tilde{\otimes}End(V))$$

and

$$L_2(y) : \bar{C}_n^G(\mathcal{A}\tilde{\otimes}End(V)) \longrightarrow \bar{C}_n^G(\mathcal{A}\tilde{\otimes}End(V))$$

by

$$L_1(y)(y_0, y_1, \ldots, y_n, f) = \sum_{i=0}^{n}(y_0, \ldots, y_i, y, y_{i+1}, \ldots, y_n, f)$$

and

$$L_2(y)(y_0, y_1, \ldots, y_n, f) = \sum_{i=0}^{n}(y_0, \ldots, y_{i-1}, [y, y_i], y_{i+1}, \ldots, y_n, f)$$

for $y_i \in \mathcal{A}\tilde{\otimes}End(V), f \in C(G)$. The formulas of $L_i(y)$ are the same as those in ([Con 1],[GeS]). But we have to check that the $L_i(y)$ satisfy the following

$$[b, L_1(y)] = bL_1(y) + L_1(y)b = L_2(y), \tag{5.10}$$

$$[B, L_1(y)] = BL_1(y) + L_1(y)B = 0. \tag{5.11}$$

Hence, $[b + B, L_1(y)] = L_2(y)$. The proof of these identities is a tedious computation which we omit here.

We now get by (5.10) – (5.11)

$$\begin{aligned} \frac{d}{dt}(\tilde{Ch}_*^G(p_t)) &= \frac{d}{dt}\{(-1)^k\frac{(2k)!}{k!}(p_{2k}^t - \frac{1}{2}\tilde{p}_{2k}^t)\} = L_2(y_t)\tilde{Ch}_*^G(p_t) \\ &= [b + B, L_1(y_t)]\tilde{Ch}_*^G(p_t) = (b + B)(L_1(y_t)\tilde{Ch}_*^G(p_t)), \end{aligned}$$

since $(b + B)\tilde{Ch}_*^G(p_t) = 0$. Here $y_t = \frac{d}{dt}(p_t) \cdot (2p_t - 1)$ is equivariant. Clearly, $L_1(y_t)\tilde{Ch}_{ev}^G(p_t) \in \bar{T}_{odd}^G(\mathcal{A}\tilde{\otimes}End(V))$. Q.E.D.

Theorem 5.1 is an equivariant analogue of a result in [GeS]. The reason why the ordinary Chern character formulas work for the equivariant case is that the idempotents are equivariant, thus the group twisting does not cause trouble. We should expect that this is true for the Chern characters from $K_1^G(\mathcal{A})$ to $\bar{HC}_{2n+1}^G(\mathcal{A})$ and $\bar{HC}_{odd}^G(\mathcal{A})$. From this point of view, we first consider the ordinary case, i.e., $G = \{e\}$ is a trivial group.

Note that by definition K-theory $K_1(\mathcal{A}) = \lim_{n\to\infty}(GL_n(\mathcal{A})/GL_n(\mathcal{A})_0)$, where $GL_n(\mathcal{A})$ is the group of invertible $n \times n$ matrices with entries from $\mathcal{A}$ and $GL_n(\mathcal{A})_0$ is the connected component of the identity in $GL_n(\mathcal{A})$ [Bla]. Let $[u] \in K_1(\mathcal{A})$ with representative $u \in GL_r(\mathcal{A})$. Define $u_{2k+1} \in \bar{C}_{2k+1}(\mathcal{A})$ by

$$u_{2k+1} = Tr(\overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2k+2}), k = 0, 1, \ldots$$

Here, if we write $u_{2k+1} = Tr(a_0, a_1, \ldots, a_{2k+1})$, then $a_{2j} = u^{-1}, a_{2j+1} = u, 0 \leq j \leq k$.

Lemma 5.2 *let*

$$Ch_{2n+1}(u) = \{(-1)^k k! u_{2k+1}\}_{k=0}^{n} \in \bar{T}_{2n+1}(\mathcal{A})$$

and

$$Ch_{odd}(u) = \{(-1)^k k! u_{2k+1}\}_{k=0}^{\infty} \in \bar{T}_{odd}(\mathcal{A}).$$

Then the $Ch_*(u)$ are cycles, i.e., $(b+B)Ch_*(u) = 0, * = 2n+1$ or "odd".

Proof. Clearly, $Ch_{odd}(u) \in \bar{T}_{odd}(\mathcal{A})$. We have

$$bu_{2k+1} = Tr(1, \overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2k}) - Tr(1, \overbrace{u, u^{-1}, \ldots, u, u^{-1}}^{2k})$$

in $\bar{C}_{2k+1}(\mathcal{A})$, and

$$T^{2i+1}u_{2k+1} = -Tr(\overbrace{u, u^{-1}, u, u^{-1}, \ldots, u, u^{-1},}^{2k+2}), i = 0, 1, \ldots, k,$$

$$T^{2i}u_{2k+1} = Tr(\overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2k+2}), i = 0, 1, \ldots, k,$$

$$Nu_{2k+1} = (k+1)(Tr(\overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2k+2}) - Tr(\overbrace{u, u^{-1}, u, u^{-1}, \ldots, u, u^{-1}}^{2k+2})),$$

and

$$Bu_{2k+1} = (k+1)(Tr(1, \overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2k+2}) - Tr(1, \overbrace{u, u^{-1}, \ldots, u, u^{-1}}^{2k+2})).$$

Therefore,

$$(b+B)Ch_{2n+1}(u) = ((-1)^n n! bu_{2n+1} + (-1)^{n-1}(n-1)! Bu_{2n-1}, \ldots, bu_3 + Bu_1, bu_1) = 0.$$

This also proves $(b+B)Ch_{odd}(u) = 0$. Q.E.D.

Theorem 5.2 *The Chern characters from $K_1(\mathcal{A})$ to $HC_*(\mathcal{A})$ defined by $Ch_*(u)$ for $* = 2n+1$ or "odd" are multiplicative.*

Proof. The multiplicativity of $Ch_*(u)$ means that $Ch_*(uu') = Ch_*(u) + Ch_*(u')$ for $[u], [u'] \in K_1(\mathcal{A})$ with representatives u and u' in $GL_r(\mathcal{A})$ and $GL_s(\mathcal{A})$, respectively. Note that $Ch_*(u \oplus I) = Ch_*(u)$ in $\bar{HC}_*(\mathcal{A})$ for $* = 2n+1$ or "odd". We can assume that both u and u' are in $GL_r(\mathcal{A})$. Let

$$u_0' = uu' \oplus I \;\; and \;\; u_1' = u \oplus u'$$

in $GL_{2r}(\mathcal{A})$. $Ch_*(u_0') = Ch_*(uu')$ and $Ch_*(u_1') = Ch_*(u) + Ch_*(u')$. Then

$$u_t = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} sint & -cost \\ cost & sint \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u' \end{bmatrix} \begin{bmatrix} sint & cost \\ -cost & sint \end{bmatrix}$$

defines a path in $GL_{r+s}(\mathcal{A})$ connecting u_0' to u_1' [Con 1]. Thus it suffices to check that the $\frac{d}{dt} Ch_*(u_t)$ are boundaries.

To this aim, let

$$\begin{aligned} u_{2k}(t) &= (-1)^k (k-1)! \sum_{i=0}^{k-1} u_{2k}^i(t), k = 1, 2, \ldots, \\ u_0(t) &= Tr(u_t' u_t^{-1}), \end{aligned}$$

where

$$u_{2k}^i = Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2i}, u_t^{-1}, u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i-1)}),$$

and $u_t' = \frac{d}{dt}(u_t)$. Then $\{u_{2k}(t)\}_{k=0}^{\infty} \in \bar{T}_{ev}(\mathcal{A})$,

$$bu_{2k}^0(t) = Tr(u_t^{-1} u_t' u_t^{-1}, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2(k-1)}, u_t)$$

$$
\begin{aligned}
& - \; Tr(u_t^{-1}, u_t' u_t^{-1}, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-1)}) \\
& + \; Tr(1, u_t' u_t^{-1}, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2(k-1)}), \\
bu_{2k}^i(t) \; = \; & Tr(1, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(i-1)}, u_t^{-1}, u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i-1)}) \\
& + \; (-1)^{2i} Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2i}, u_t^{-1} u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i-1)}) \\
& + \; (-1)^{2i+1} Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2i}, u_t^{-1}, u_t', \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i-1)}) \\
& + \; (-1)^{2k} Tr(1, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}, u_t}^{2i-1}, u_t^{-1}, u_t' u_t^{-1}, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2(k-i-1)}), \\
Nu_{2k}^i(t) \; = \; & \sum_{j=0}^{k} Tr(\overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2j}, u_t' u_t^{-1}, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2(k-j)}) \\
& + \; \sum_{j=1}^{k} Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(j-1)}, u_t^{-1}, u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-j)}),
\end{aligned}
$$

and

$$
\begin{aligned}
Bu_{2k}(t) \; = \; & (-1)^k k! (\sum_{j=0}^{k} Tr(1, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2j}, u_t' u_t^{-1}, \overbrace{u_t, u_t^{-1}, \ldots, u_t, u_t^{-1}}^{2(k-j)}) \\
& + \; \sum_{j=1}^{k} Tr(1, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(j-1)}, u_t^{-1}, u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-j)})).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
bu_{2k+2} + Bu_{2k} \; = \; & (-1)^{k+1} (k)! \{ \sum_{i=1}^{k} Tr(1, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(i-1)}, u_t^{-1}, u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i)}) \\
& + \; \sum_{i=0}^{k} [Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2i}, u_t^{-1} u_t' u_t^{-1}, u_t, \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i)}) \\
& - \; Tr(\overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2i}, u_t^{-1}, u_t', \overbrace{u_t^{-1}, u_t, \ldots, u_t^{-1}, u_t}^{2(k-i)})]
\end{aligned}
$$

$$+\sum_{i=0}^{k} Tr(1,\overbrace{u_t,u_t^{-1},\ldots,u_t,u_t^{-1},u_t,u_t^{-1}}^{2i},u_t'u_t^{-1},\overbrace{u_t,u_t^{-1},\ldots,u_t,u_t^{-1}}^{2(k-i)})\}+Bu_{2k}$$

$$= (-1)^{k+1}k![\sum_{j=0}^{k} Tr(\overbrace{u_t^{-1},u_t,\ldots,u_t^{-1},u_t}^{2j},u_t^{-1}u_t'u_t^{-1},u_t,\overbrace{u_t^{-1},u_t,\ldots,u_t^{-1},u_t}^{2(k-j)})$$

$$- \sum_{j=0}^{k} Tr(\overbrace{u_t^{-1},u_t,\ldots,u_t^{-1},u_t}^{2j},u_t^{-1},u_t',\overbrace{u_t^{-1},u_t,\ldots,u_t^{-1},u_t}^{2(k-j)})]$$

$$= (-1)^k k!\frac{d}{dt}u_{2k+1}(t), k>0,$$

since $\frac{d}{dt}(u_t^{-1}) = -u_t^{-1}u_t'u_t^{-1}$. Obviously, $bu_2(t)+Bu_0(t) = \frac{d}{dt}u_1(t)$. This proves $\frac{d}{dt}Ch_*(u_t) = (b+B)(\{u_{2k}(t)\})$.

To show that $Ch_*(u)$ is well defined, let us note that $GL_r(\mathcal{A})_0$ is path connected. We need to check that if $u_t \in GL_r(\mathcal{A})_0$ is a path connecting the identity Id to u_1. Then $Ch_*(u_1) = 0, * = 2n+1$ or "odd". This follows from the proof of the last paragraph because $Ch_*(Id) = 0$ in $\bar{HC}_*(\mathcal{A})$. Q.E.D.

Consider now the general case with nontrivial compact group G. We adopt the following odd equivariant K-theory $K_1^G(\mathcal{A})$,

$$K_1^G(\mathcal{A}) = \lim_{\substack{\to \\ V\in R_+(G)}} (GL^G(\mathcal{A}\tilde{\otimes}End(V))/(GL^G(\mathcal{A}\tilde{\otimes}End(V))_0),$$

where $R_+(G)$ is the set of equivalent classes of finite dimensional unitary representations of G; $GL^G(\mathcal{A}\tilde{\otimes}End(V))$ is the group of all equivariant invertible elements in $\mathcal{A}\tilde{\otimes}End(V)$; $GL^G(\mathcal{A}\tilde{\otimes}End(V))_0$ is the connected component of the identity in $GL^G(\mathcal{A}\tilde{\otimes}End(V))$ and the direct limit is with respect to the order relation $\leq$ in $R_+(G)$. V and $V_1 \in R_+(G)$ are $V \leq V_1$ if $V_1 \simeq V \oplus V_0$ for some $V_0 \in R_+(G)$. Then $u \in GL^G(\mathcal{A}\tilde{\otimes}End(V))$ can be considered as an element $u \oplus 1 \in GL^G(\mathcal{A}\tilde{\otimes}End(V_1))$. Note that this is not the definition of $K_1^G(\mathcal{A})$ in [Phi].

let $[u] \in K_1^G(\mathcal{A})$ be represented by $u \in GL^G(\mathcal{A}\tilde{\otimes}End(V))$. Define $\tilde{Ch}_*(u) \in \tilde{T}_*^G(\mathcal{A}\tilde{\otimes}End(V))$ with the representative $\{(-1)^k k!(\overbrace{u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u}^{2(k+1)}, 1)\}$ for $* = 2n+1$ or "odd". As before, 1 stands for the constant function on G with value 1. Since u and u^{-1} are equivariant, b and B acting on $\tilde{Ch}_*(u)$ have the same formulas as in the ordinary case. The proof of Theorem 5.2 can be repeated to get the following equivariant version of Theorem 5.2.

Theorem 5.3 *Let $Ch_*^G : K_1^G(\mathcal{A}) \longrightarrow \bar{HC}_*^G(\mathcal{A})$ be defined by*

$$Ch_*^G(u) = F_G(\tilde{Ch}_*^G(u)) \; for \; * = 2n+1 \; or \; \text{“odd”}.$$

Then $Ch_*^G(u)$ are multiplicative maps. Here the morphism F_G is given by (5.2).

5.3 Pairings of Equivariant Cyclic Cohomology with K-theory

To study the pairing of (entire) equivariant cyclic cohomology with equivariant K-theory, we first consider normalized equivariant cocycles ([Con 2],[KKL]). Let $\{f_{2k+i}\} \in T_G^*(\mathcal{A})$ for $* = 2n, 2n+1$, "ev" or "odd" and $i = 0, 1$. $\{f_{2k+i}\}$ is a normalized cocycle if it is a cocycle and

$$B_0\{f_{2k+i}\} = AB_0\{f_{2k+i}\},$$

where $B_0 f_k = S(1-T)f_k$ and $Af_k = \frac{1}{k+1}Nf_k, k \geq 0$. Clearly, $\{f_{2k+i}\}$ is a normalized cocycle iff $(1-T)B_0\{f_{2k+i}\} = 0$. As shown in [KKL], each (entire) equivariant cocycle $\{f_{2k+i}\}$ is cohomologous to the normalized one $\{\tilde{f}_{2k+i}\}$

given by

$$\tilde{f}_{2k+i} = f_{2k+i} - (b+B)DB_0 f_{2k+i}, \tag{5.12}$$

where $D = \frac{-1}{n+1}\sum_{i=0}^{n} iT^i$ on $C_G^n(\mathcal{A})$.

Let now $\psi = \{f_{2k}\} \in T_G^*(\mathcal{A})$ be a cocycle and $\tilde{\psi} = \{\tilde{f}_{2k}\}$ be the normalization of ψ for $* = 2n$ or "ev". For any $[p] \in K_0^G(\mathcal{A})$ with representative $p \in \mathcal{A}\tilde{\otimes}End(V)$, we define

$$< [p], \psi >_* (g) = \begin{cases} \sum_{k=0}^{n}(-1)^k \frac{(2k)!}{k!}(F_G^* \tilde{f}_{2k})(p, p, \ldots, p)(g), & * = 2n; \\ \\ \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{k!}(F_G^* \tilde{f}_{2k})(p, p, \ldots, p)(g), & * = \text{“}ev\text{”}. \end{cases} \tag{5.13}$$

Here F_G^* is given by (5.3). Also for $\psi = \{f_{2k}\} \in \bar{T}_G^*(\mathcal{A}), * = 2n$ or "ev", we define, using p_{2k} and $\tilde{p}_{2k}$ in section 5.2,

$$< [p], \psi >_*' (g) = \begin{cases} \sum_{k=0}^{n}(-1)^k \frac{(2k)!}{k!}(F_G^* f_{2k})(p_{2k} - \frac{1}{2}\tilde{p}_{2k})(g), & * = 2n; \\ \\ \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{k!}(F_G^* f_{2k})(p_{2k} - \frac{1}{2}\tilde{p}_{2k})(g), & * = \text{“}ev\text{”}. \end{cases} \tag{5.14}$$

Here F_G^* is given by (5.3).

Theorem 5.4 *The character-valued pairings of $HC_G^*(\mathcal{A})$ and $\bar{HC}_G^*(\mathcal{A})$ with $K_0^G(\mathcal{A})$ given by $(5.13) - (5.14)$ are additive in the variable $[p]$ and linear in ψ for $* = 2n$ or "ev".*

Proof. Evidently, the series in (5.13) and (5.14) are convergent. The proof that the pairings $<, >_*$ are independent of the choice of the representative of ψ is identical to that in [Con 2]. Identities $(5.6) - (5.9)$ can be easily used to show $< [p], (b+B)\psi >_*' = 0$ for $* = 2n$ or "ev".

The pairings are also independent of the choice of the representative of $[p]$. In fact, for the pairing $< [p], \psi >'_*$ this follows from the proof of Theorem 5.1, and for the pairing $< [p], \psi >_*$ this reduecs to the case just mentioned, since we can assume that all $f_{2k}(a_0, a_1, \ldots, a_{2k})(g)$ vanish if some $a_i = 1$ for $i > 0$ by considering f'_{2k} on $(\tilde{\mathcal{A}})^{\bar{\otimes}(2k+1)}$,

$$f'_{2k}(a_0+\lambda_0, \ldots, a_{2k}+\lambda_{2k})(g) = f_{2k}(a_0, a_1, \ldots, a_{2k})(g) + \lambda_0 B_0 f_{2k}(a_1, \ldots, a_{2k})(g),$$

where $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}, \lambda_i \in \mathbb{C}$ with $G-$action $h \cdot (a_0 + \lambda_0) = \alpha_h(a_0) + \lambda_0$. The additivity of F^*_G and the multi-linearity of f_{2k} imply that the pairings are additive in the variable $[p]$. Q.E.D.

Consider now the pairings of $\bar{HC}^{2n+1}_G(\mathcal{A})$ and $\bar{HC}^{odd}_G(\mathcal{A})$ with $K^G_1(\mathcal{A})$. Let $\psi = \{f_{2k+1}\} \in \bar{T}^*_G(\mathcal{A})$ for $* = 2n+1$ or "odd", and $[u] \in K^G_1(\mathcal{A})$ be represented by equivariant invertible element $u \in GL^G(\mathcal{A}\bar{\otimes}End(V))$. Define

$$< [u], \psi >'_* = \begin{cases} \sum_{k=0}^{n} (-1)^k k! (F^*_G f_{2k+1})(\overbrace{u^{-1}, u, \ldots, u^{-1}, u}^{2k+2})(g), * = 2n+1; \\ \sum_{k=0}^{\infty} (-1)^k k! (F^*_G f_{2k+1})(\overbrace{u^{-1}, u, \ldots, u^{-1}, u}^{2k+2})(g), * = \text{"}odd\text{"}. \end{cases} \tag{5.15}$$

The series in (5.15) is convergent by the definition of $\bar{T}^{odd}_G(\mathcal{A})$.

Theorem 5.5 *The character-valued pairings $< [u], \psi >'_*$ of $HC^*_G(\mathcal{A})$ with $K^G_1(\mathcal{A})$ are multiplicative in $[u]$ and linear in ψ for $* = 2n+1$ or "odd".*

Proof. We first show that if $\psi \in \bar{T}^{odd}_G(\mathcal{A})$ is a coboundary then $< [u], \psi >'_* = 0$. In fact, let $\psi = (b+B)\psi'$ with $\psi' = \{f'_{2k}\} \in \bar{T}^{2n}_G(\mathcal{A})$ or $\bar{T}^{ev}_G(\mathcal{A})$, i.e., $f_{2k+1} = bf'_{2k} + Bf'_{2k+2}, k = 0, 1, \ldots, (f_{2n+1} = bf'_{2n}$ if $\psi \in \bar{T}^{2n+1}_G(\mathcal{A}))$. Using the

G-invariance of u, we get by the proof of Lemma 5.2,

$$\begin{aligned} F_G^* b f'_{2k}(\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k+2}) &= (F_G^* f'_{2k})(1,\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k}) \\ &- (F_G^* f'_{2k})(1,\overbrace{u,u^{-1},\ldots,u,u^{-1}}^{2k}) \end{aligned}$$

and

$$\begin{aligned} F_G^* B f'_{2k+2}(\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k+2}) &= (k+1)((F_G^* f'_{2k+2})(1,\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k+2}) \\ &- (F_G^* f'_{2k+2})(1,\overbrace{u,u^{-1},\ldots,u,u^{-1}}^{2k+2})). \end{aligned}$$

Hence,

$$\begin{aligned} <[u],\psi>'_{2n+1} &= \sum_{k=0}^{n-1}(-1)^k k!(F_G^*(bf'_{2k}) + F_G^*(Bf'_{2k+2}))(\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k+2}) \\ &+ (-1)^n F_G^*(bf'_{2n})(\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2n+2}) \\ &= \sum_{k=0}^{n}(-1)^k k![(F_G^* f'_{2k})(1,\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k}) \\ &- (F_G^* f'_{2k})(1,\overbrace{u,u^{-1},\ldots,u,u^{-1}}^{2k})] \\ &+ \sum_{k=0}^{n-1}(-1)^k k!(k+1)[(F_G^* f'_{2k+2})(1,\overbrace{u^{-1},u,\ldots,u^{-1},u}^{2k+2}) \\ &- (F_G^* f'_{2k+2})(1,\overbrace{u,u^{-1},\ldots,u,u^{-1}}^{2k+2})] = 0. \end{aligned}$$

Similarly, $<[u],\psi>'_{odd} = 0$ if $\psi \in \bar{T}_G^{odd}(\mathcal{A})$ is a coboundary.

The proof of Theorem 5.2 can be also used to show that $<[u],\psi>'_*$ are independent of the choice of the representative of $[u]$. The rest of the proof is clear. Q.E.D.

5.4 Chern Character of Even Equivariant θ-Summable Fredholm Modules

In this section we will construct the Chern character from even equivariant θ-summable Fredholm modules to even entire equivariant cyclic cohomology. Let $\mathcal{A}$ be a unital G-Banach algebra and $\lambda : G \to \mathcal{L}(H)$ be a unitary representation of G on a Hilbert space H. Then G acts on $\mathcal{L}(H)$ by $\tau_g(R) = \lambda_g R \lambda_g^{-1}$. Let ε be an equivariant element in $\mathcal{L}(H)$ such that $\varepsilon^2 = Id$. Suppose that D is an equivariant unbounded selfadjoint operator in H and $\rho' : \mathcal{A} \to \mathcal{L}(H)$ is a representation of $\mathcal{A}$ on H such that ρ' is equivariant (i.e., $\rho'(\alpha_g(a)) = \tau_g(\rho'(a))$) and $\varepsilon\rho' = \rho'\varepsilon$.

Difinition 5.3 We call $(\mathcal{A}, H, G, D)$ an odd equivariant θ-summable Fredholm module if we have

(1) Let $\mathcal{A}(D)$ be the set of all $a \in \mathcal{A}$ such that $[D, \rho'(a)]$ is densely defined and extends to a bounded operator on H. The $\mathcal{A}(D)$ is dense in $\mathcal{A}$ and there is a constant $m(D)$ such that

$$\|\rho'(a)\| + \|[D, \rho'(a)]\| \leq m(D)\|a\|, \; for \; a \in \mathcal{A}(D);$$

(2) $Tr(e^{-tD^2}) < \infty, \forall t > 0$.

$(\mathcal{A}, H, G, \varepsilon, D)$ is an even equivariant θ-summable Fredholm module if besides (1) and (2), D anticommutes with ε: $\varepsilon D = -D\varepsilon$.

Note that this definition is the natural generalization of Connes θ- summable Fredholm modules to the equivariant case [Gong 2]. To construct the Chern character of even equivariant θ-summable Fredholm modules, we will use several formulas in [GeS]. See also [JLO].

Let

$$\Delta_{2k} = \Big\{(t_1, t_2, \dots, t_{2k}) \in \mathbb{R}^{2k} : 0 \le t_1 \le t_2 \le \dots \le t_{2k} \le 1\Big\}$$

be a $2k$-simplex and $\bar{C}h_G^{ev}(D) = \{\bar{C}h_G^{2k}(D)\}_{k=0}^{\infty}$ be defined by

$$\bar{C}h_G^{2k}(D)(a_0, a_1 \dots, a_{2k})(g) = \int_{\Delta_{2k}} Tr(\varepsilon a_0 e^{-t_1 D^2} \lambda_g [D, a_1] e^{-(t_2 - t_1)D^2} [D, a_2] \dots$$
$$e^{-(t_{2k} - t_{2k-1})D^2} [D, a_{2k}] e^{-(1-t_{2k})D^2}) d^{2k} t \stackrel{\text{def}}{=} < a_0, \lambda_g [D, a_1], [D, a_2], \dots, [D, a_{2k}] >_{2k} \quad (5.16)$$

for $a_i \in \mathcal{A}(D)$. Here, for convenience, we write $\rho'(a_i)$ as a_i. Since

$$\begin{aligned} &\bar{C}h_G^{2k}(D)(\alpha_h^{-1}(a_0), \alpha_h^{-1}(a_1), \dots, \alpha_h^{-1}(a_{2k}))(h^{-1}gh) \\ &= \ < \alpha_h^{-1}(a_0), \lambda_{h^{-1}gh}[D, \alpha_h^{-1}(a_1)], [D, \alpha_h^{-1}(a_2)], \dots, [D, \alpha_h^{-1}(a_{2k})] >_{2k} \\ &= \ < a_0, \lambda_g [D, a_1], [D, a_2], \dots, [D, a_{2k}] >_{2k} \\ &= \ \bar{C}h_G^{2k}(D)(a_0, a_1, \dots, a_{2k})(g), \end{aligned}$$

we see $\bar{C}h_G^{2k}(D) \in Hom_G^{2k}(\mathcal{A}(D) \tilde{\otimes} (\bar{\mathcal{A}}(D))^{\tilde{\otimes}(2k)}, C(G))$. By the estimation in [GeS],

$$\begin{aligned} &\max_{g \in G} |\bar{C}h_G^{2k}(D)(a_0, a_1 \dots, a_{2k})(g)| \\ &= \ max_{g \in G} | < a_0, \lambda_g [D, a_1], [D, a_2], \dots, [D, a_{2k}] >_{2k} | \\ &\le \ max_{g \in G} \frac{m(D)^{2k+1} Tr e^{-(1-\delta)D^2}}{(2k)!} \prod_{i=0}^{2k} \|a_i\| \|\lambda_g\| \\ &= \ \frac{m_1 m(D)^{2k+1} Tr e^{-(1-\delta)D^2}}{(2k)!} \prod_{i=0}^{2k} \|a_i\|, \end{aligned}$$

where $0 < \delta < 1$ and $m_1 = max_{g \in G} \|\lambda_g\| < \infty$ in view of the compactness of G. Thus, $\bar{C}h_G^{2k}(D)$ extends to $\mathcal{A} \tilde{\otimes} (\bar{\mathcal{A}})^{\tilde{\otimes}(2k)}$ and

$$\|\bar{C}h_G^{2k}(D)\|_{2k} \le \frac{m_1 m(D)^{2k+1} Tr e^{-(1-\delta)D^2}}{(2k)!},$$

$$\sum_{k=0}^{\infty} r^k k! \|\bar{Ch}_G^{2k}(D)\|_{2k} \leq \sum_{k=0}^{\infty} \frac{m_1 r^k k! m(D)^{2k+1} Tre^{-(1-\delta)D^2}}{(2k)!} < \infty.$$

Therefore, $\bar{Ch}_G^{ev}(D) = \{\bar{Ch}_G^{2k}(D)\} \in \bar{T}_G^{ev}(\mathcal{A})$.

Theorem 5.6 *$\bar{Ch}_G^{ev}(D)$ is an equivariant cocycle in $\bar{T}_G^{ev}(\mathcal{A})$.*

Proof. This amounts to verifying $(b+B)\bar{Ch}_G^{ev}(D) = 0$, i.e.,

$$(b\bar{Ch}_G^{2k-2}(D) + B\bar{Ch}_G^{2k}(D))(a_0, a_1 \ldots, a_{2k-1}) = 0, k = 1, 2, \ldots.$$

To this aim, let us use the following identities [GeS]:

$$\sum_{i=0}^{n} (-1)^{|A_0|+\ldots+|A_{i-1}|} < A_0, \ldots, A_n >_n = 0, \tag{5.17}$$

$$\begin{aligned} < A_0, \ldots, [D^2, A_i], \ldots, A_n >_n &= < A_0, \ldots, A_{i-1}A_i, A_{i+1}, \ldots, A_n >_{n-1} \\ &- < A_0, \ldots, A_{i-1}, A_iA_{i+1}, \ldots, A_n >_{n-1}, \end{aligned} \tag{5.18}$$

and

$$<A_0, A_1, \ldots, A_n>_n = \sum_{i=0}^{n} (-1)^{(|A_0|+\ldots+|A_{i-1}|)(|A_i|+\ldots+|A_n|)} <1, A_i, \ldots, A_n, A_0, \ldots, A_{i-1}>_{n+1} \tag{5.19}$$

where A_i are operators in H such that (5.17) – (5.19) are well defined and $|A_i| = 0$ if $\varepsilon A_i = A_i \varepsilon$ and $|A_i| = 1$ if $\varepsilon A_i = -A_i \varepsilon$. We have by (5.1),

$$\begin{aligned} &B\bar{Ch}_G^{2k}(D)(a_0, a_1 \ldots, a_{2k-1})(g) \\ &= \sum_{i=0}^{2k-1} (-1)^{i(2k-1)} < 1, \lambda_g[D, \alpha_g^{-1}(a_{2k-i})], \ldots, [D, \alpha_h^{-1}(a_0)], [D, a_1], \ldots, [D, a_{2k-1-i}] >_{2k} \\ &= \sum_{i=0}^{2k-1} (-1)^{i(2k-1)} < 1, [D, a_{2k-i}], \ldots, [D, a_{2k-1}], [D, a_0], \lambda_g[D, a_1], \ldots, [D, a_{2k-1-i}] >_{2k}. \end{aligned}$$

Here we used the fact that $[D,\alpha_h^{-1}(a_i)] = \lambda_h^{-1}[D,a_i]\lambda_h$. Hence, let $A_0 = [D,a_0], A_1 = \lambda_g[D,a_1], A_i = [D,a_i], i \geq 2$ in (5.19). We get

$$\begin{aligned}
&< a_0, \lambda_g[D,a_1],[D,a_2],\ldots,[D,a_{2k-1}] >_{2k-1} \\
&= \sum_{i=0}^{2k-1}(-1)^{i(2k-1)}<1,[D,a_i],\ldots,[D,a_{2k-1}],[D,a_0],\lambda_g[D,a_1],\ldots,[D,a_{i-1}] >_{2k} \\
&= B\bar{C}h_G^{2k}(D)(a_0,a_1\ldots,a_{2k-1})(g), \quad a_i \in \mathcal{A}(D). \qquad (5.20)
\end{aligned}$$

Also since $[D,a_ia_{i+1}] = [D,a_i]a_{i+1} + a_i[D,a_{i+1}]$,

$$\begin{aligned}
&b\bar{C}h_G^{2k}(D)(a_0,a_1\ldots,a_{2k-1})(g) \\
&= < a_0\alpha_g(a_1), \lambda_g[D,a_2],[D,a_3],\ldots,[D,a_{2k-1}] >_{2k-2} \\
&+ \sum_{i=1}^{2k-1}(-1)^i < a_0, \lambda_g[D,a_1],[D,a_2],\ldots,[D,a_ia_{i+1}],\ldots,[D,a_{2k-1}] >_{2k-2} \\
&= < a_0\lambda_g(a_1),[D,a_2],[D,a_3],\ldots,[D,a_{2k-1}] >_{2k-2} \\
&+ \sum_{i=1}^{2k-2}(-1)^i(<a_0,\lambda_g[D,a_1],[D,a_2],\ldots,[D,a_i]a_{i+1},[D,a_{i+2}],\ldots,[D,a_{2k-1}]>_{2k-2} \\
&+ < a_0,\lambda_g[D,a_1],[D,a_2],\ldots,[D,a_{i-1}],a_i[D,a_{i+1}],\ldots,[D,a_{2k-1}] >_{2k-2}) \\
&+ (-1)^{2k-1} < a_{2k-1}a_0,\lambda_g[D,a_1],[D,a_2],\ldots,[D,a_{2k-2}] >_{2k-2}. \qquad (5.21)
\end{aligned}$$

Using the fact that $[D,[D,a_i]] = [D^2,a_i]$ and identity (5.18) for $A_0 = a_0\lambda_g, A_1 = a_1, A_i = [D,a_i], i \geq 2$, we obtain

$$\begin{aligned}
&< a_0,[D,\lambda_g[D,a_1]],[D,a_2],\ldots,[D,a_{2k-1}] >_{2k-1} \\
&= < a_0\lambda_g,[D,[D,a_1]],[D,a_2],\ldots,[D,a_{2k-1}] >_{2k-1} \\
&= <a_0\lambda_g a_1,[D,a_2],\ldots,[D,a_{2k-1}] >_{2k-2} - <a_0\lambda_g, a_1[D,a_2],\ldots,[D,a_{2k-1}] >_{2k-2}
\end{aligned}$$

and

$$(-1)^i < a_0, \lambda_g[D,a_1],[D,a_2],\ldots,[D,[D,a_i]],\ldots,[D,a_{2k-1}] >_{2k-1}$$

$$= (-1)^i(<a_0,\lambda_g[D,a_1],[D,a_2],\dots,[D,a_{i-1}]a_i,[D,a_{i+1}],\dots,[D,a_{2k-1}]>_{2k-2}$$

$$- <a_0,\lambda_g[D,a_1],[D,a_2],\dots,[D,a_{i-1}],a_i[D,a_{i+1}],\dots,[D,a_{2k-1}]>_{2k-2}).$$

Hence, these identities and (5.20) prove

$$\begin{aligned} &< a_0,[D,\lambda_g[D,a_1]],[D,a_2],\dots,[D,a_{2k-1}]>_{2k-1} \\ &+ \sum_{i=0}^{2k-1}(-1)^{i-1} < a_0,\lambda_g[D,a_1],[D,a_2],\dots,[D,[D,a_i]],\dots,[D,a_{2k-1}]>_{2k-1} \\ &= b\bar{C}h_G^{2k-2}(D)(a_0,a_1\dots,a_{2k-1})(g). \end{aligned} \tag{5.22}$$

But by identity (5.17) for $A_0 = a_0, A_1 = \lambda_g[D,a_1], A_i = [D,a_i], i \geq 2$,

$$\begin{aligned} &< [D,a_0],\lambda_g[D,a_1],[D,a_2],\dots,[D,a_{2k-1}]>_{2k-1} \\ &+ < a_0,[D,\lambda_g[D,a_1]],[D,a_2],\dots,[D,a_{2k-1}]>_{2k-1} \\ &+ \sum_{i=0}^{2k-1}(-1)^{i-1}<a_0,\lambda_g[D,a_1],[D,a_2],\dots,[D,[D,a_i]],\dots,[D,a_{2k-1}]>_{2k-1}=0. \end{aligned}$$

This together with (5.20) – (5.22) proves $(b+B)\bar{C}h_G^{ev}(D) = 0$. Q.E.D.

We now consider the invariance of the Chern character $\bar{C}h_G^{ev}(D)$ under the differential homotopy of D. Let us define for any bounded operator $A' \in \mathcal{L}(H)$ or $A' = D$,

$$\bar{C}h_G^n(D,A')(a_0,\dots,a_n)(g) = \sum_{i=0}^{n}(-1)^{i|A'|}<a_0,\lambda_g[D,a_1],\dots,[D,a_i],A',\dots,[D,a_n]>_{n+1}$$

Then as shown in [GeS], $\bar{C}h_G^{ev}(D,A') = \{\bar{C}h_G^{2k}(D,A')\}_{k=0}^{\infty} \in \bar{T}_G^{ev}(\mathcal{A})$ if $A' \in \mathcal{L}(H)$ is even, and if $A' = D$, $\bar{C}h_G^{odd}(D,D) = \{\bar{C}h_G^{2n+1}(D,D)\} \in \bar{T}_G^{odd}(\mathcal{A})$, since

$$\max_{g\in G}|\bar{C}h_G^n(D,A')(a_0,a_1,\dots,a_n)(g)| \leq \frac{m_1(n+1)m(D)^{n+1}Tre^{-(1-\delta)D^2}}{(n+1)!}\prod_{i=0}^{n}\|a_i\|\|A'\|,$$

for $A' \in \mathcal{L}(H)$, and

$$\max_{g\in G}|\bar{C}h_G^n(D,D)(a_0,a_1,\dots,a_n)(g)| \leq \frac{m_1(n+1)m(D)^{n+1}\delta^{-\frac{1}{2}}Tre^{-(1-\delta)D^2}}{(n)!}\prod_{i=0}^{n}\|a_i\|.$$

Proposition 5.2 *let D_s be a one-parameter differential family of equivariant selfadjoint odd operators in H such that either $D_s = sD$ and $Tre^{-tD_s^2} < \infty$ for all $t > 0$, or D_s' is a continuous family of bounded operators on H. Then*

$$\frac{d}{ds}\bar{Ch}_G^{ev}(D_s) = -(b+B)\bar{Ch}_G^{odd}(D_s, D_s').$$

Proof. As in the proof of Theorem 5.6, it follows from (5.19) that

$$\begin{aligned} &B\bar{Ch}_G^{2k+1}(D_s, D_s')(a_0, a_1, \ldots, a_{2k})(g) \\ &= -\sum_{i=0}^{2k}(-1)^i < [D_s,a_0], \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_i], D_s', \ldots, [D_s,a_{2k}] >_{2k+1} \end{aligned} \tag{5.23}$$

Similarly, tedious computation proves

$$\begin{aligned} &b\bar{Ch}_G^{2k-1}(D_s, D_s')(a_0, a_1, \ldots, a_{2k})(g) \\ &= -\sum_{i=0}^{2k} < a_0, \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_{i-1}], [D_s',a_i], [D_s,a_{i+1}], \ldots, [D_s,a_{2k}] >_{2k} \\ &\quad - \sum_{i=0}^{2k}(-1)^i\Big(\sum_{1\le j\le i} (-1)^{j-1} < a_0, \lambda_g[D_s,a_1], \ldots, [D_s^2,a_j], \ldots, [D_s,a_i], D_s', \ldots, [D_s,a_{2k}] >_{2k+1} \\ &\quad + \sum_{j\ge i+1} (-1)^j < a_0, \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_i], D_s', \ldots, [D_s^2,a_j], \ldots, [D_s,a_{2k}] >_{2k+1}\Big) \end{aligned} \tag{5.24}$$

The last term of (5.24) is the second one of the following identity

$$\begin{aligned} &\sum_{i=0}^{2k}(-1)^i < [D_s,a_0], \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_i], D_s', [D_s,a_{i+1}], \ldots, [D_s,a_{2k}] >_{2k+1} \\ &\quad + \sum_{i=0}^{2k}(-1)^i\Big(\sum_{1\le j\le i} (-1)^{j-1} < a_0, \lambda_g[D_s,a_1], \ldots, [D_s^2,a_j], \ldots, [D_s,a_i], D_s', \ldots, [D_s,a_{2k}] >_{2k+1} \\ &\quad + \sum_{j\ge i+1} (-1)^j < a_0, \lambda_g[D_s,a_1], \ldots, [D_s,a_i], D_s', \ldots, [D_s^2,a_j], \ldots, [D_s,a_{2k}] >_{2k+1}\Big) \\ &\quad + \sum_{i=0}^{2k} < a_0, \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_i], [D_s,D_s'], \ldots, [D_s,a_{2k}] >_{2k+1} = 0, \end{aligned} \tag{5.25}$$

which is obtained by summing all of the identities from (5.17) with

$$A_j = \begin{cases} a_0, \ \ j = 0; \\ \lambda_g[D_s, a_1], \ \ j = 1; \\ [D_s, a_j], \ \ j \leq i; \\ D_s', \ \ j = i+1; \\ [D_s, a_{j-1}], \ \ j \geq i+1, \end{cases}$$

over $0 \leq i \leq 2k$. Therefore, combining (5.23), (5.24) with (5.25) and using the following identity [GeS]

$$\frac{d}{ds} < A_0, A_1, \ldots, A_n >_n = -\sum_{i=0}^{n} < A_0, \ldots, [D_s, D_s'], A_{i+1}, \ldots, A_n >_{n+1},$$

we obtain

$$\begin{aligned} &\frac{d}{ds}\bar{C}h_G^{ev}(D_s)(a_0, a_1, \ldots, a_{2k})(g) \\ &= -\sum_{i=0}^{2k} < a_0, \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_i], [D_s,D_s'], [D_s,a_{i+1}], \ldots, [D_s,a_{2k}] >_{2k} \\ &+ \sum_{i=0}^{2k} < a_0, \lambda_g[D_s,a_1], [D_s,a_2], \ldots, [D_s,a_{i-1}], [D_s',a_i], [D_s,a_{i+1}], \ldots, [D_s,a_{2k}] >_{2k} \\ &= \ -(b\bar{C}h_G^{2k-1}(D_s, D_s') + B\bar{C}h_G^{2k+1}(D_s, D_s'))(a_0, a_1, \ldots, a_{2k})(g). \end{aligned}$$

Q.E.D.

5.5 Index Theorem of Even Equivariant Θ-Summable Fredholm Modules

We now apply the Chern characters constructed in the previous sections to the equivariant index theorem. Let $(\mathcal{A}, H, G, \varepsilon, D)$ be an even equivariant

θ-summable Fredholm Module. We write H as $H = H^+ \oplus H^-$ and then

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

with $H^\pm$ being the ± 1-eigenspaces of ε and $(D^+)^* = D^-$. Moreover, since H is a unitary representation space of compact group G, the Peter-Weyl theorem implies

$$H = \oplus_{\sigma \in \hat{G}} H_\sigma,$$

where $\hat{G}$ is the set of equivalence classes of irreducible unitary representations of G, and H_σ is the closure of the direct sum of all irreducible unitary representations spaces which are equivalent to σ. Let P_σ be the orthogonal projection from H onto H_σ and $D_\sigma = P_\sigma D P_\sigma$. Since ε and D commute with $\lambda_g, g \in G$, $H_\sigma = H_\sigma^+ \oplus H_\sigma^-$ with $H_\sigma^\pm = H_\sigma \cap H^\pm$ and $D_\sigma^\pm = P_\sigma D^\pm P_\sigma$ and $D_\sigma^2 = P_\sigma D^2 P_\sigma$. Then

$$Tr(e^{-tD^2}) = \sum_{\sigma \in \hat{G}} Tr(P_\sigma e^{-tD^2} P_\sigma) = \sum_{\sigma \in \hat{G}} Tr(e^{-tD_\sigma^2}) < \infty.$$

Hence, for each $\sigma \in \hat{G}, Tr(e^{-tD_\sigma^2}) < \infty$ and $D_\sigma^+ : H_\sigma^+ \to H_\sigma^-$ is a Fredholm operator. Denote by $Ind(D_\sigma^+)$ the index of D_σ^+.

Definition 5.4 Let $(\mathcal{A}, H, G, \varepsilon, D)$ be an even equivariant θ-summable Fredholm module. The character-valued index of D is defined by

$$Ind_G(D^+)(g) = \sum_{\sigma \in \hat{G}} Ind(D_\sigma^+) \tilde{\chi}_\sigma(g), \tag{5.26}$$

where $\tilde{\chi}_\sigma$ is the normalized character of σ: $\tilde{\chi}_\sigma = (dim V_\sigma)^{-1} \chi_\sigma$, V_σ is the finite dimensional representation space of σ.

Note that this character-valued index is the same as that in [AtS]. In fact, let S_σ be the set $\{1, 2, \ldots, m(\sigma)\}$, where $m(\sigma)$ is the multiplicity of σ in H_σ.

Then $H_\sigma = V_\sigma \otimes L^2(S_\sigma)$ up to isomorphic isometry, and $D_\sigma^\pm = Id_{V_\sigma} \otimes \tilde{D}_\sigma^\pm$. Thus, $Ind(D_\sigma^+) = dim(V_\sigma)Ind(\tilde{D}_\sigma^+)$. $Ind(D_\sigma^+)\tilde{\chi}_\sigma(g) = Ind(\tilde{D}_\sigma^+)\chi_\sigma(g)$. The following lemma extends the McKean-Singer index formula to the equivariant case, and also tells us that the right hand side of (5.26) is well defined.

Lemma 5.3 *Let $(\mathcal{A}, H, G, \varepsilon, D)$ be an even equivariant θ-summable Fredholm module. Then*

$$Ind_G(D^+) = Tr(\varepsilon\lambda_g e^{-tD^2}), \forall g \in G, t > 0.$$

Proof. Clearly, $Tr(\varepsilon\lambda_g e^{-tD^2}) < \infty, \forall g \in G$. We get as in the ordinary case [Gil],

$$\begin{aligned} Tr_{H_\sigma}(\varepsilon\lambda_g e^{-tD_\sigma^2}) &= Tr_{Ker(D_\sigma^2)}(\varepsilon\lambda_g) \\ &= Tr_{Ker(D_\sigma^2)\cap H^+}(\lambda_g) - Tr_{Ker(D_\sigma^2)\cap H^-}(\lambda_g) \\ &= dim(Ker(D_\sigma^2)\cap H^+)\tilde{\chi}_\sigma(g) - dim(Ker(D_\sigma^2)\cap H^-)\tilde{\chi}_\sigma(g) \\ &= Ind(D_\sigma^+)\tilde{\chi}_\sigma(g). \end{aligned}$$

Therefore,

$$Tr(\varepsilon\lambda_g e^{-tD^2}) = \sum_{\sigma\in\hat{G}} Tr_{H_\sigma}(\varepsilon\lambda_g e^{-tD_\sigma^2}) = \sum_{\sigma\in\hat{G}} Ind(D_\sigma^+)\tilde{\chi}_\sigma(g).$$

Q.E.D.

Theorem 5.7 *Let $(\mathcal{A}, H, G, \varepsilon, D)$ be an even equivariant $\theta-$summable Fredholm module. Then for any $[p] \in K_0^G(\mathcal{A})$ represented by the equivariant idempotent $p \in \mathcal{A}\tilde{\otimes}End(V)$,*

$$Ind_G((D\otimes Id_V)_p^+) =< [p], \bar{C}h_G^{ev}(D) >'_{ev}, \tag{5.27}$$

where V is a finite dimensional unitary representation space of G and $(D \otimes Id_V)_p^+ = p(D^+ \otimes Id_V)p : p(H^+ \otimes V) \to p(H^- \otimes V)$.

Proof. It is easy to see that the dense subalgebra $\mathcal{A}(D)$ of $\mathcal{A}$ is stable under the holomorphic functional calculus. This means that if (V, β) is a finite dimensional unitary representation of G and $a \in \mathcal{A}(D)\tilde{\otimes}End(V)$, then $f(a) \in \mathcal{A}(D)\tilde{\otimes}End(V)$ for any function f holomorphic near the spectrum of a in $\mathcal{A}\tilde{\otimes}End(V)$. As in the ordinary case, we have $K_0^G(\mathcal{A}) = K_0^G(\mathcal{A}(D))$.

Let $\tilde{\mathcal{A}}(D) = \{R \in \mathcal{L}(H) : [D, R] \in \mathcal{L}(H)\}$ with the norm $\|R\|' = \|R\| + \|[D, R]\|$. $\tilde{\mathcal{A}}(D)$ is a Banach $*$-algebra and $\tilde{\mathcal{A}}(D) \supseteq \mathcal{A}(D)$. As in section 5.4, we can define the Chern character $\tilde{Ch}_G^{ev}(D)$ in $\bar{T}_G^{ev}(\tilde{\mathcal{A}}(D))$ by the same formula as for $\bar{Ch}_G^{ev}(D)$ and $\tilde{Ch}_G^{ev}(D) = \bar{Ch}_G^{ev}(D)$ on $\mathcal{A}(D)$. By the proof of Theorem 5.1, we can assume that there is an equivariant idempotent $p' \in \mathcal{A}(D)\tilde{\otimes}End(V)$ and a path p_t of equivariant idempotents in $\mathcal{A}\tilde{\otimes}End(V)$ connecting p to p'. Then by Theorem 5.4 and the homotopy invariance of index,

$$< [p], \bar{Ch}_G^{ev}(D) >'_{ev} = < [p'], \bar{Ch}_G^{ev}(D) >'_{ev} = < [p'], \tilde{Ch}_G^{ev}(D) >'_{ev}$$

and

$$Ind_G((D \otimes Id_V)_p^+) = Ind_G((D \otimes Id_V)_{p'}^+).$$

Using the result in [Phi], we can assume that there is a path p'_t of equivariant idempotents in $\tilde{\mathcal{A}}(D)$ such that $p'_0 = p'$ and p'_1 is a projection. Thus it suffices to check that for the equivariant projection p'_1,

$$< [p'_1], \tilde{Ch}_G^{ev}(D) >'_{ev} = Ind_G((D \otimes Id_V)_{p'_1}^+).$$

Note that $< [p'_1], \tilde{Ch}_G^{ev}(D) >'_{ev} = < [p'_1], \tilde{Ch}_G^{ev}(D \otimes Id_V) >'_{ev}$. In fact,

$F_G^*\tilde{Ch}_G^{ev}(D) = \tilde{Ch}_G^{ev}(D \otimes Id_V)$ as shown by the following calculation:

$$\tilde{Ch}_G^{ev}(D \otimes Id_V)(a_0\tilde{\otimes}m_0, \ldots, a_{2k}\tilde{\otimes}m_{2k})(g)$$

$$\begin{aligned}
&= <a_0\tilde{\otimes}m_0, (\lambda_g \otimes \beta_g)[D\otimes Id_V, a_1\tilde{\otimes}m_1], [D\otimes Id_V, a_2\tilde{\otimes}m_2], \ldots, [D\otimes Id_V, a_{2k}\tilde{\otimes}m_{2k}]>_{2k} \\
&= \int_{\Delta_{2k}} Tr(\varepsilon a_0 e^{-t_1 D^2}\lambda_g[D, a_1]e^{-(t_2-t_1)D^2}[D, a_2]\ldots \\
&\quad \ldots e^{-(t_{2k}-t_{2k-1})D^2}[D, a_{2k}]e^{-(1-t_{2k})D^2}\tilde{\otimes}m_0\beta_g m_1 \ldots m_{2k})d^{2k}t \\
&= < a_0, \lambda_g[D, a_1], [D, a_2], \ldots, [D, a_{2k}] >_{2k} Tr(m_0\beta_g m_1 \ldots m_{2k}) \\
&= F_G^*\tilde{Ch}_G^{ev}(D)(a_0\tilde{\otimes}m_0, \ldots, a_{2k}\tilde{\otimes}m_{2k})(g).
\end{aligned}$$

Hence, we reduce (5.27) to the equality

$$< [p_1'], \tilde{Ch}_G^{ev}(D \otimes Id_V) >'_{ev} = Ind_G((D \otimes Id_V)^+_{p_1'}). \tag{5.28}$$

Now choose a homotopy $(D \otimes Id_V)_s = D \otimes Id_V + s(2p_1' - 1)[D \otimes Id_V, p_1']$ of equivariant odd selfadjoint operators which satisfies the condition of Proposition 5.2. It follows from Proposition 5.2 that

$$< [p_1'], \tilde{Ch}_G^{ev}(D \otimes Id_V) >'_{ev} = < [p_1'], \tilde{Ch}_G^{ev}((D \otimes Id_V)_1) >'_{ev} .$$

By the homotopy invariance of index, (5.28) is equivalent to the following

$$< [p_1'], \tilde{Ch}_G^{ev}((D \otimes Id_V)_1) >'_{ev} = Ind_G(((D \otimes Id_V)_1)^+_{p_1'})$$

where p_1' is an equivariant projection in $\mathcal{A}(D)\tilde{\otimes}End(V)$ and $[p_1', (D\otimes Id_V)_1] = [p_1', p_1'(D \otimes Id_V)p_1' + (1 - p_1')(D \otimes Id_V)(1 - p_1')] = 0$. But by Lemma 5.3,

$$\begin{aligned}
&< [p_1'], \tilde{Ch}_G^{ev}((D \otimes Id_V)_1) >'_{ev} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!}{k!}\tilde{Ch}_G^{ev}((D \otimes Id_V)_1)((p_1')_{2k} - \frac{1}{2}(\tilde{p}_1')_{2k})
\end{aligned}$$

$$
\begin{aligned}
&= \; < p_1'(\lambda_g \otimes \beta_g) >_0 \\
&+ \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{k!} < p_1' - \frac{1}{2}, (\lambda_g \otimes \beta_g)[(D \otimes Id_V)_1, p_1], \ldots, [(D \otimes Id_V)_1, p_1'] >_{2k} \\
&= \; Tr((\varepsilon \otimes Id_V) p_1' (\lambda_g \otimes \beta_g) e^{-((D \otimes Id_V)_1)^2}) \\
&= \; Ind_G(((D \otimes Id_V)_1)^+_{p_1'}).
\end{aligned}
$$

Q.E.D.

Chapter 6

Higher Equivariant Analytic Index

In [CoM 2] Connes and Moscovici used the Alexander-Spanier cohomology to construct the higher index. They proved the higher index theorem which is important in their approach to the Novikov conjecture. They also employed cyclic cohomology in a crucial way. It is natural to study the higher equivariant index theory by using the equivariant cyclic cohomology discussed in Chapters 4 and 5. This together with the application of the higher equivariant index theory to the pairing version of the equivariant Novikov conjecture is the main motivation of the present chapter. It turns out that one has to develop certain equivariant Alexander-Spanier cohomology. We will thus define in section 6.1 the equivariant cohomologies of spaces and groups. These cohomologies involve the group twisting at the cochain level and in the coboundary operators and are quite different from usual equivariant cohomologies. The equivariant cohomology of spaces so defined will be called the equivariant Alexander-Spanier cohomology. We will discuss the excision and tautness of this cohomology. In section 6.2 we will use the equivariant Alexander-Spanier cohomology to construct the higher equivariant analytic index map of Riemannian manifolds. This higher index map shares most properties of the usual index map. We will prove in section 6.3 that this higher equivariant index is well defined in the equivariant K-theory of manifolds and satisfies the Atiyah-Singer index axioms. We finally propose a pairing version of the equivariant Novikov conjecture which deserves further investigation.

6.1 Equivariant Alexander-Spanier Cohomology

We assume throughout this section that G is a compact group and M is a locally compact Hausdorff space and G acts on M by homeomorphisms. To define the equivariant Alexander-Spanier cohomology, let $F(G)$ be the space of all functions on G and $M^{q+1} = \overbrace{M \times \ldots \times M}^{q+1}$. Define $C^q_G(M)$ to be the space of all functions $\varphi : M^{q+1} \to F(G)$ such that for each $g \in G, \varphi(x_0, \ldots, x_q)(g)$ is bounded as a function of $x_i \in M$ and satisfies the following

1. $\varphi(gx_0, \ldots, gx_q)(ghg^{-1}) = \varphi(x_0, \ldots, x_q)(h), \ \ \forall g, h \in G,$

2. $\varphi(x_0, \ldots, x_q)(g) = (-1)^q \varphi(x_q, g^{-1}x_0 \ldots, x_{q-1})(g).$

Let $\bar{C}^q_G(M)$ be the subspace of $C^q_G(M)$ consisting of all elements satisfying $\varphi(gx_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_q)(g) = 0$ for some $x_i = x_{i+1}, 0 \le i \le q-1, g \in G.$

Let $\delta : C_G^q(M) \to C_G^{q+1}(M)$ be the coboundary operator defined by

$$(\delta\varphi)(x_0,\ldots,x_{q+1})(g) = \varphi(gx_1,x_2,\ldots,x_{q+1})(g)+\sum_{i=1}^{q+1}(-1)^i\varphi(x_0,\ldots,\hat{x}_i,\ldots,x_{q+1})(g).$$

we have for $\varphi \in C_G^q(M)$,

(a) $(\delta\varphi)(gx_0,\ldots,gx_{q+1})(h) = (\delta\varphi)(x_0,\ldots,x_{q+1})(g^{-1}hg)$:

$$\begin{aligned} & (\delta\varphi)(gx_0,\ldots,gx_{q+1})(h) = \varphi(h(gx_1),gx_2,\ldots,gx_{q+1})(h) \\ + & \sum_{i=1}^{q+1}(-1)^i\varphi(gx_0,\ldots,gx_{i-1},gx_{i+1},\ldots,gx_{q+1})(h) \\ = & (\delta\varphi)(x_0,\ldots,x_{q+1})(g^{-1}hg). \end{aligned}$$

(b) $(\delta\varphi)(x_0,\ldots,x_{q+1})(g) = (-1)^{q+1}(\delta\varphi)(x_{q+1},g^{-1}x_0,x_1,\ldots,x_q)(g)$. In fact,

$$\begin{aligned} & (\delta\varphi)(x_0,\ldots,x_{q+1})(g) = (-1)^q\varphi(x_{q+1},x_1,\ldots,x_q)(g) \\ + & \sum_{i=1}^{q}(-1)^{i+q}\varphi(x_{q+1},g^{-1}x_0,\ldots,\hat{x}_i,\ldots,x_q)(g) + (-1)^{q+1}(\varphi)(x_0,\ldots,x_q)(g) \\ = & (-1)^{q+1}(\delta\varphi)(x_{q+1},g^{-1}x_0,\ldots,x_q)(g). \end{aligned}$$

(c) $\delta^2\varphi = 0$. Indeed,

$$\begin{aligned} & \delta(\delta\varphi)(x_0,\ldots,x_{q+2})(g) = (\delta\varphi)(gx_1,x_2,\ldots,x_{q+2})(g) \\ + & \sum_{i=1}^{q+2}(-1)^i(\delta\varphi)(x_0,\ldots,\hat{x}_i,\ldots,x_{q+2})(g) \\ = & \sum_{i=2}^{q+2}(-1)^i(\sum_{j=2}^{i-1}(-1)^j\varphi(x_0,\ldots,\hat{x}_j,\ldots,\hat{x}_i,\ldots,x_{q+2})(g) \\ + & \sum_{j=i+1}^{q+2}(-1)^{j-1}\varphi(x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{q+2})(g)) = 0. \end{aligned}$$

Clearly, $(\delta\varphi)(gx_0,\ldots,x_{q+1})(g) = 0$ if $\varphi \in \tilde{C}_G^q(M)$ and $x_i = x_{i+1}$ for some $0 \le i \le q$. Hence $\{C_G^*(M),\delta\}$ (resp. $\{\tilde{C}_G^*(M),\delta\}$) is a complex. Let $\mathcal{U} = \{U_i, i \in J\}$ be an open covering of M. We say $\mathcal{U}$ is a G-covering of M if for $U \in \mathcal{U}$ and $g \in G$, $gU \in \mathcal{U}$. We call $\varphi \in C^q_G(M)$ is locally zero if there is an open G-covering $\mathcal{U}$ of M such that $\varphi(gx_0, x_1, \ldots, x_q)(g) = 0$ for $(x_0, \ldots, x_q) \in \cup_{U \in \mathcal{U}} U^{q+1}$ and $g \in G$. Obviously if $\varphi \in C^q_G(M)$ is locally zero so is $\delta\varphi$. All locally zero cochains $\varphi \in C^q_G(M)$ form a subcomplex $\{C^*_{G,0}(M), \delta\}$. Then $\bar{C}^*_G(M) = \{C^*_G(M)/C^*_{G,0}(M), \delta\}$ is called the equivariant Alexander-Spanier complex. Its cohomology $\bar{H}^*_G(M)$ is called the equivariant Alexander-Spanier cohomology of M. We denote by $\hat{H}^*_G(M)$ the cohomology of complex $\hat{C}^*_G = \{\tilde{C}^*_G(M)/\tilde{C}^*_G(M) \cap C^*_{G,0}(M), \delta\}$.

Remark 6.1 (1) *When $G = \{e\}$ is a trivial group, $\bar{H}^*_G(M)$ reduces to the ordinary Alexander-Spanier cohomology of M [CoM 2].*

(2) *$\bar{H}^*_G(M)$ is formally different from equivariant Alexander-Spanier cohomology in the sense of Honkasalo [Hon]. Unlike [Hon] the coboundary operator here involves the group action.*

(3) *If $M = \{pt\}$, then $\bar{C}^*_G(M) = C^*_G(M) \simeq F_c(G) \overset{\text{def}}{=} \{\varphi \in F(G) : \varphi(ghg^{-1}) = \varphi(h), g, h \in G\}$. $\bar{H}^q_G(M) = F_c(G)$ for $q = 0$, and 0 for $q > 0$.*

Let N be a locally compact Hausdorff G-space and $f : N \to M$ be a continuous G-equivariant map. Clearly, f induces homomorphisms $f^* : C^*_G(M) \to C^*_G(N)$ and $f^* : C^*_{G,0}(M) \to C^*_{G,0}(N)$. We have $f^* : \bar{H}^*_G(M) \to \bar{H}^*_G(N)$. Let $M_0 \subset M$ be a G-subspace of M and $i : M_0 \hookrightarrow M$ be the inclusion. $i^* : \bar{C}^*_G(M) \to \bar{C}^*_G(M_0)$ is surjective. Denote by $\bar{C}^*_G(M, M_0)$ the kernel of i^*. Let $\bar{H}^*_G(M, M_0)$ be the cohomology of the complex $\{\bar{C}_G(M, M_0), \delta\}$. We have the following properties of $\bar{H}^*_G(M)$.

Proposition 6.1 (1) *Exactness: there is a long exact sequence*

$$\ldots \to \bar{H}^q_G(M, M_0) \to \bar{H}^q_G(M) \xrightarrow{i_*} \bar{H}^q_G(M_0) \xrightarrow{\partial} \bar{H}^{q+1}_G(M, M_0) \to \ldots,$$

(2) *Excision: let U be a G-subset of $M_0 \subset M$ such that there is an open G-neighborhood W of U with $\bar{W} \subset int(M_0)$. Then the inclusion $i : (M \setminus U, M_0 \setminus U) \to (M, M_0)$ induces an isomorphism $i^* : \bar{C}^*_G(M, M_0) \to \bar{C}^*_G(M \setminus U, M_0 \setminus U)$.*

Similar results hold for $\hat{C}^*_G(M)$ and $\hat{H}^*_G(M)$.

Proof. (1) is from the short exact sequence

$$0 \to \bar{C}^*_G(M, M_0) \to \bar{C}^*_G(M) \to \bar{C}^*_G(M_0) \to 0.$$

To prove (2), let us note that by assumption the rows of the following commutative diagram are exact

$$\begin{array}{ccccccc}
0 \to & C^*_{G,0}(M) & \to & C^*_G(M, M_0) & \xrightarrow{j^*} & \bar{C}^*_G(M, M_0) & \to 0 \\
& \downarrow i^* & & \downarrow i^* & & \downarrow i^* & \\
0 \to & C^*_{G,0}(M \setminus U) & \to & C^*_G(M \setminus U, M_0 \setminus U) & \xrightarrow{j_1^*} & \bar{C}^*_G(M \setminus U, M_0 \setminus U) & \to 0,
\end{array}$$

where $C^*_G(M, M_0)$ is the subcomplex of $C^*_G(M)$ consisting of all $\varphi \in C^*_G(M)$ such that φ is locally zero on M_0. Following [Spa], we need only to show that $j_1^* i^*$ is surjective and $(i^*)^{-1}(C^*_{G,0}(M \setminus U)) = C^*_{G,0}(M)$. Define for $\varphi \in C^q_G(M \setminus U, M_0 \setminus U)$

$$\varphi_1(x_0, \ldots, x_q)(g) = \begin{cases} 0, \ \ if \ \ some \ \ x_i \in W, \ \ 0 \le i \le q, \\ \\ \varphi(x_0, \ldots, x_q)(g), \ \ otherwise. \end{cases}$$

Then φ_1 is a bounded function of x_i and G-equivariant. Also $\varphi_1(x_0, \ldots, x_q)(g) = (-1)^q \varphi_1(x_q, g^{-1}x_0, x_1, \ldots, x_{q-1})(g)$ and $\varphi_1(gx_0, x_1, \ldots, x_q)(g) = 0$ for some $x_i = x_{i+1}$. φ_1 is in $C^q_G(M)$. Let $\mathcal{U} = \{U_i : i \in J\}$ be an open G-covering of $M_0 \setminus U$ such that $\varphi = 0$ on $\mathcal{U}^{q+1}$. Then $\mathcal{U}_1 = \{U_i \cup W : i \in J\}$ is an open G-covering of M_0 such that $\varphi_1 = 0$ on $\mathcal{U}_1^{q+1}$. Hence $\varphi_1 \in C^*_G(M, M_0)$. It follows from the definition of φ_1 that $i^*\varphi_1 - \varphi = 0$ on $\mathcal{U}_2^{q+1}$, where $\mathcal{U}_2 = \{U_i \cap int(M_0) : i \in J\} \cup \{M \setminus \bar{W}\}$. Since $int(M_0)$ and $M \setminus \bar{W}$ are G-equivariant, $\mathcal{U}_2$ is an open G-covering of $M \setminus W$. This proves $j_1^*(i^*\varphi_1 - \varphi) = 0$.

To check $(i^*)^{-1}(C^*_{G,0}(M \setminus U)) = C^*_{G,0}(M)$, let $\varphi \in C^q_G(M, M_0)$ be such that $i^*\varphi \in C^*_{G,0}(M \setminus U)$. There are open G-coverings $\mathcal{U}$ of M_0 and $\mathcal{U}_1$ of $M \setminus U$ such that $\varphi = 0$ on $\mathcal{U}^{q+1}$ and $\mathcal{U}_1^{q+1}$. Take $\mathcal{U}_2 = \{U_i \cap int(M_0) : i \in J\}$, $\mathcal{U}_3 = \{U_i \cap (M \setminus \bar{U}) : i \in J\}$. $\mathcal{U}_4 = \mathcal{U}_2 \cup \mathcal{U}_3$ is an open G-covering of M, since $M \setminus \bar{U}$ is G- equivariant. But $\varphi = 0$ on $\mathcal{U}_4^{q+1}$. $\varphi \in C^*_{G,0}(M)$. Q.E.D.

To obtain the tautness of equivariant Alexander-Spanier cohomology, we need the following lemma.

Lemma 6.1 *Let M be a locally compact Hausdorff G-space and $M_0 \subset M$ a G-subspace. Suppose $\mathcal{U} = \{U_i : i \in J\}$ is an open G-covering of M. Then there is an open G-neighborhood O of M_0 and an equivariant function $f : O \to M_0$ such that*

(1) $f(x) = x$, $\forall x \in M_0$,

(2) *for any $U \in \mathcal{U}, f(U \cap O) \subset U^*$, where $U^* = \cup\{U' \in \mathcal{U} : U' \cap U \neq \emptyset\}$.*

Proof. Let O be the set of all $x \in M$ such that there are an $U \in \mathcal{U}$ and an equivariant map $f_x : Gx \to M_0$ satisfying $x \in U$ and $f_x(x) \in U \cap M_0$. We have

(a) $M_0 \subset O$: take $f_x = I : Gx \to M_0$. Since $\mathcal{U}$ is an open covering of M, there is an $U \in \mathcal{U}$ such that $x \in U$. Then $f_x(x) = x \in U \cap M_0$.

(b) O is open: let $x \in O$, U and $f_x : Gx \to M_0$ as above. By the slice theorem [Bre], there is a slice $O_x \subset U$ at x. Then GO_x is an open neighborhood of x and for each $y \in O_x$ there exists a G-map $\varphi_x : Gy \to Gx$ such that $\varphi_x(y) = x$. It is enough to show $GO_x \subset O$. Let $x_1 \in GO_x, x_1 = gy$ for some $y \in O_x$. $x_1 \in gU$ and $f_x\varphi_{gy} = gf_x\varphi_y : G(gy) \to G(gx) \to gM_0 \subset M_0$. Furthermore, $(f_x\varphi_{gy})(x_1) = g(f_x\varphi_y)(y) \in (gU) \cap M_0$. Hence $x_1 \in O$. O is an open neighborhood of x.

Note that $O \setminus M_0$ is G- equivariant. We define the equivariant map $f : O \to M_0$ by

$$f = \begin{cases} I, & on \ \ M_0, \\ \\ f_x, & on \ \ Gx \subset \mathcal{O} \setminus M_0, \ \ x \in O \setminus M_0. \end{cases}$$

Evidently, $f(x) = x$ for $x \in M_0$. Let $U_0 \in \mathcal{U}$. If $x \in U_0 \cap O$, then there are $U \in \mathcal{U}$ and $f_x : Gx \to M_0$ such that $x \in U$ and $f_x \in U \cap M_0$. Thus $x \in U_0 \cap U, U \in U_0^*$ and $f(x) \in U$. Q.E.D.

Let us point out that $f : \mathcal{O} \to M_0$ may not be continuous. But f is good enough for us to prove the tautness of equivariant Alexander-Spanier cohomology.

Proposition 6.2 *Let M be a locally compact Hausdorff G-space and M_0 a closed G-subset of M. Then the natural homomorphism*

$$\lim_{\vec{O}} \bar{H}^q_G(O) \xrightarrow{i^*} \bar{H}^q_G(M_0)$$

is an isomorphism, where O runs over all of the G-neighborhoods of M_0.

Proof. Surjectivity of i^*: let $\varphi \in C^q_G(M_0)$ be a G-cochain such that $\delta\varphi = 0$ on $\mathcal{W}^{q+1}$ for some open G-covering $\mathcal{W}$ of M_0, Let $\mathcal{U} = \{W \cup (M \setminus M_0) : W \in \mathcal{W}\}$. $\mathcal{U}$ is an open G-covering of M since $M \setminus M_0$ is G-equivariant open. Choose an open star refinement $\mathcal{V}$ of $\mathcal{U}$ (cf. [Hon],p.30), i.e., $\mathcal{V}^* \subset \mathcal{U}$. By Lemma 6.1, there exists an open G-neighborhood O of M_0 and equivariant $f : O \to M_0$ such that $f(V \cap O) \subset V^*$ for any $V \in \mathcal{V}$. Then $\delta(f^*\varphi) = f^*(\delta\varphi) = 0$ on $\mathcal{V}^{q+2} \cap O^{q+2}$. But $f(V \cap O) \subset V^* \subset U$ for some $U \in \mathcal{U}$ since $\mathcal{V}$ is the open star refinement of $\mathcal{U}$. Hence $f(V \cap O) \subset U \cap M_0$ and $\delta(f^*\varphi) = 0$ on $(V \cap O)^{q+2}$. This proves $f^*(\varphi)$ is a cocycle in $\bar{C}^q_G(O)$ and $(f^*\varphi)|_{M_0} = \varphi$.

Injectivity of i^*: let O' be an open G-neighborhood of M_0 and $\varphi \in C^q_G(O')$ be such that $\delta\varphi = 0$ on $\mathcal{W}^{q+2}$ and $\varphi|_{M_0} = \delta\varphi_1$ on $\mathcal{W}_1^{q+1}$ for some open G-coverings $\mathcal{W}$ of O' and $\mathcal{W}_1$ of M_0. Let $\mathcal{U} = \{W_1 \cup (O' \setminus M_0) : W_1 \in \mathcal{W}_1\}$. $\mathcal{U}$ is an open G-covering of O'. Choose an open G-star refinement $\mathcal{V}$ for both $\mathcal{U}$ and $\mathcal{W}$. By Lemm 6.1, there are a G-neighborhood O of M_0 in O' and an equivariant function $f : O \to M_0$ with the properties of Lemma 6.1. Since $f(V \cap O) \subset V^* \subset W_1$ for $V \in \mathcal{V}$ and some $W_1 \in \mathcal{W}_1$, $f^*(\varphi|_{M_0}) = f^*(\delta\varphi_1) = \delta(f^*\varphi_1)$ on $V^{q+1} \cap O^{q+1}$. Hence it suffices to check that $f^*(\varphi|_{M_0})$ is cohomologous to $\varphi|_O$ in $C^q_G(O)$. Let $D : C^q(O) \to C^{q-1}(O)$ be defined by

$$\begin{aligned}(D\psi)(x_0, \ldots, x_{q-1})(g) &= \sum_{i=1}^{q-1}(-1)^i \psi(x_0, \ldots, x_i, f(x_i), \ldots, f(x_{q-1}))(g) \\ &+ \psi(x_0, g^{-1}f(x_0), f(x_1), \ldots, f(x_q))(g).\end{aligned}$$

Clearly, $(D\psi)(gx_0, \ldots, x_i, x_{i+1}, \ldots, x_{q-1})(g) = 0$ for some $x_i = x_{i+1}$ if ψ has such a property. We have

$$(D\psi)(gx_0, \ldots, gx_{q-1})(ghg^{-1}) = \sum_{i=1}^{q-1}(-1)^i \psi(x_0, \ldots, x_i, f(x_i), \ldots, f(x_{q-1}))(h)$$
$$+ \quad \psi(x_0, g^{-1}f(x_0), f(x_1), \ldots, f(x_{q-1}))(h) = (D\psi)(x_0, \ldots, x_{q-1})(h),$$

provided $\psi(gx_0, \ldots, gx_q)(ghg^{-1}) = \psi(x_0, \ldots, x_q)(h)$.

$$(\delta D\psi)(x_0, \ldots, x_q)(g) = \psi(gx_0, g^{-1}f(gx_1), f(x_2), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{i=2}^{q}(-1)^{i-1}\psi(gx_1, x_2, \ldots, x_i, f(x_i), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{i=1}^{q}(-1)^i(\psi(x_0, g^{-1}f(x_0), f(x_1), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{j=1}^{i-1}(-1)^j\psi(x_0, \ldots, x_j, f(x_j), \ldots, f(\hat{x}_i), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{j=i+1}^{q}(-1)^{j-1}\psi(x_0, \ldots, \hat{x}_i, \ldots, x_j, f(x_j), \ldots, f(x_q))(g)),$$
$$(D\delta\psi)(x_0, \ldots, x_q)(g) = \psi(gg^{-1}f(x_0), f(x_1), \ldots, f(x_q))(g)$$
$$- \quad \psi(x_0, f(x_1), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{i=1}^{q}(-1)^{i+1}\psi(x_0, gf(x_0), \ldots, f(\hat{x}_i), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{i=1}^{q}(-1)^i(\psi(gx_1, \ldots, x_i, f(x_i), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{j=1}^{i}(-1)^j\psi(x_0, \ldots, \hat{x}_j, \ldots, x_i, f(x_i), f(x_q))(g)$$
$$+ \quad (-1)^{i+1}\psi(x_0, \ldots, x_i, f(\hat{x}_i), f(x_{i+1}), \ldots, f(x_q))(g)$$
$$+ \quad \sum_{j=i+1}^{q}(-1)^{j+1}\psi(x_0, \ldots, x_i, f(x_i), \ldots, f(\hat{x}_j), \ldots, f(x_q))(g))$$

Therefore,

$$(\delta D\psi + D\delta\psi)(x_0, \ldots, x_q)(g) = \psi(f(x_0), \ldots, f(x_q))(g) - \psi(x_0, \ldots, x_q)(g).$$

Let $\tilde{D} = \frac{1}{q}\sum_{i=0}^{q-1} t^i D : C_G^q(O) \to C_G^{q-1}(O)$. The above arguments show that $\tilde{D}$ is well defined and for $\psi \in C_G^q(O)$,

$$\delta\tilde{D}\psi + \tilde{D}\delta\psi = f^*(\psi|_{M_0}) - \psi.$$

In particular, we have $\delta\tilde{D}\varphi + \tilde{D}\delta\varphi = f^*(\varphi|_{M_0}) - \varphi$. We know already $\delta\varphi = 0$ on $\mathcal{W}$ and $(V\cap O)\cup f(V\cap O) \subset W$ for some $W \in \mathcal{W}$. Hence $\delta\tilde{\varphi} = f^*(\varphi|_{M_0}) - \varphi$ on $V^{q+1}\cap O^{q+1}$, i.e., $f^*(\varphi|_{M_0})$ is cohomologous to φ in $C_G^q(O)$. Q.E.D.

We discuss briefly the equivariant Alexander-Spanier cohomology with compact support. Let M be a locally compact Hausdorff G-space and M_0 a G-subapace of M. Denote by $C_{G,c}^q(M, M_0)$ (resp. $\tilde{C}_{G,c}^*(M, M_0)$) the subspace of $C_G^q(M, M_0)$ (resp. $\tilde{C}_G^*(M, M_0)$) consisting of all $\varphi \in C_G^q(M, M_0)$ such that φ is locally zero on some cobounded G-subset of M. Recall $U \subset M$ is cobounded G-subset if $\overline{M\setminus U}$ is compact and U is G-equivariant. Obviously, δ maps $C_{G,c}^q(M, M_0)$ into $C_{G,c}^{q+1}(M, M_0)$ and $C_{G,0}^q(M) \subset C_{G,c}^q(M, M_0)$. Let $\bar{C}_{G,c}^q(M, M_0) = C_{G,c}^q(M, M_0)/\ C_{G,0}^q(M)$. The cohomology $\bar{H}_{G,c}^*(M, M_0)$ of the complex $\{\bar{C}_{G,c}^*(M, M_0), \delta\}$ is called equivariant Alexander-Spanier cohomology with compact support. Similarly, we can define the complex $\{\hat{C}_{G,c}^*(M, M_0), \delta)$ and its cohomology $\hat{H}_{G,c}^*(M, M_0)$ by using $\tilde{C}_{G,c}^*(M, M_0)$. We can discuss the properties of $\bar{H}_{G,c}^*(M, M_0)$ as above, which we omit.

We now come to the equivariant cohomology of groups. Let Γ be a group. Suppose that compact group G acts on Γ by automorphisms $\alpha : G \to Aut(\Gamma)$ such that $\alpha(g_1g_2) = \alpha(g_1)\alpha(g_2), \alpha(g^{-1}) = \alpha(g)^{-1}$. Let $C_G^q(\Gamma)$ be the space of all functions φ from Γ^{q+1} to $F(G)$ satisfying the following:

(1) $\varphi(\nu_0, \ldots, \nu_q)(g)$ is a bounded function of ν_i for each fixed $g \in G$,

(2) $\varphi(g\nu_0, \ldots, g\nu_q)(h) = \varphi(\nu_0, \ldots, \nu_q)(g^{-1}hg), \forall g, h \in G$,

(3) $\varphi(\nu\nu_0, \ldots, \nu\nu_q)(h) = \varphi(h(\nu)^{-1}\nu\nu_0, \nu_1, \ldots, \nu_q)(h), \forall \nu, \nu_i \in \Gamma, h \in G.$

Define the coboundary operator $\delta : C_G^q(\Gamma) \to C_G^{q+1}(\Gamma)$ by

$$(\delta\varphi)(\nu_0, \ldots, \nu_{q+1})(g) = \varphi(g\nu_1, \nu_2, \ldots, \nu_{q+1})(g) + \sum_{i=1}^{q+1}(-1)^i \varphi(\nu_0, \ldots, \hat{\nu}_i, \ldots, \nu_{q+1})(g).$$

To check that δ is well defined, we can use the calculation of the coboundary operator on $C_G^*(M)$. It suffices to verify

$$(\delta\varphi)(\nu\nu_0, \ldots, \nu\nu_q)(h) = (\delta\varphi)(h(\nu)^{-1}\nu\nu_0, \nu_1, \ldots, \nu_q)(h),$$

which is obtained by a computation. As before, $\delta^2 = 0$. Hence $\{C_G^*(\Gamma), \delta\}$ is a complex. The cohomology $H_G^*(\Gamma)$ of the complex $\{C_G^*(\Gamma), \delta\}$ is called the equivariant cohomology of Γ. Note that $H_G^*(\Gamma)$ reduces to the usual cohomology of Γ when G is trivial. The natural example of the group Γ with a compact group action is the fundamental group $\pi_1(M)$ of Riemannian G-manifold M. Let $g \in G$ have a fixed point in M. Denote by G_1 the subgroup of G generated by g. Then G_1 has a fixed point in M. Hence G_1 acts on $\pi_1(M)$, even though G may not act on $\pi_1(M)$. We will use the cohomology $H_{G_1}^*(\Gamma)$ elsewhere to consider the pairing version of the equivariant Novikov conjecture.

6.2 Higher Equivariant Analytic Index

Assume throughout this section that G is a compact Lie group and M is a complete smooth Riemannian manifold. This section is devoted to defining higher equivariant analytic index for M. Let E and F be two smooth complex G-vector bundles over M. Denote by $\Psi_G^r(M; E, F)$ (resp. $\Psi_G^r(M; E, F)^{-1}$) the space of all equivariant (resp. elliptic) pseudo-differential operators A of order

r, $A : C_c^\infty(M, E \otimes |\wedge|^{\frac{1}{2}}(M)) \to C^\infty(M, F \otimes |\wedge|^{\frac{1}{2}}(M))$, where $|\wedge|^{\frac{1}{2}}(M)$ is the C^∞-line bundle of half densities over M whose fiber at $x \in M$ is defined by $|\wedge|^{\frac{1}{2}}(M)_x = |\wedge|^{\frac{1}{2}}(T_xM)$. Recall that for a finite dimensional space V over $\mathbb{R}$ and the set of all ordered bases $B(V)$ of V, a half density on V is a function from $B(V)$ to $\mathbb{C}$ satisfying $f(\beta_2) = |det(\beta_2, \beta_1)|^{\frac{1}{2}} f(\beta_1)$ for $\beta_i \in B(V)$ and $|\wedge|^{\frac{1}{2}}(V)$ is the $\mathbb{C}$-vector space of all half densities on V. That A is equivariant means $g(A) = A$, namely, $\lambda_g A \lambda_g^{-1} = A$, where λ_g is the operator induced by the G-action on various spaces such as $C_c^\infty(M, E \otimes |\wedge|^{\frac{1}{2}}(M))$, $g \in G$. Let $K(x, y) \in Hom(E_y, F_x) \otimes |\wedge|^{\frac{1}{2}}(T_xM) \otimes |\wedge|^{\frac{1}{2}}(T_yM)$ be the kernel function of A,

$$(Af)(x) = \int_M K(x, y) f(y) dy, \ f \in C_c^\infty(M, E \otimes |\wedge|^{\frac{1}{2}}(M)).$$

Then

$$(\lambda_g A \lambda_g^{-1} f)(x) = \lambda_g \cdot A(\lambda_g^{-1} f)(g^{-1}x) = \int_M \lambda_g \cdot K(g^{-1}x, g^{-1}y) g^{-1} f(y) dy.$$

We see that A is equivariant iff $K(g^{-1}x, g^{-1}y) = \lambda_g^{-1} K(x, y) \lambda_g, \forall g \in G$, where λ_g acts on the fibers of vector bundles E and F. Let $A_i \in \Psi_G^\infty(M; E, F) = \cup_{r \geq 0} \Psi_G^r(M; E, F)$ with kernel $K_i(x, y), 0 \leq i \leq q$. We define for $\varphi \in \hat{C}_{G,c}^q(M)$ $\tau_\varphi : (\Psi_G^r(M; E, E))^{\otimes(q+1)} \to F(G)$ by

$$\tau_\varphi(A_0, \dots A_q)(g) = (-1)^q \int_{M^{q+1}} Tr(\lambda_g \cdot K_0(x_0, x_1) \dots K_q(x_q, gx_0)) \varphi(gx_0, x_1, \dots, x_q)(g),$$

where Tr is the trace on the fibers of vector bundle E. We first prove the following basic lemma.

Lemma 6.2 *Let $\varphi \in \hat{C}_{G,c}^q(M)$. τ_φ satisfies the following*

(1) $\tau_\varphi(A_0, \dots, A_q)(g) = (-1)^q \tau_\varphi(A_q, g^{-1}(A_0), A_1, \dots, A_{q-1})(g)$,

(2) $b\tau_\varphi(A_0,\dots,A_{q+1})(g) = \tau_{\delta\varphi}(A_0,\dots,A_{q+1})(g)$,

(3) $\tau_\varphi(g(A_0),\dots,g(A_q))(h) = \tau_\varphi(A_0,\dots,A_q)(g^{-1}hg)$,

(4) *If* $f_i \in \Psi^0_G(M;E,E)$ *are functions, then*

$$\tau_\varphi(A_0+f_0,\dots,A_q+f_q)(g) = (-1)^q\tau_\varphi(A_0,\dots,A_q)(g).$$

Proof. (1) We calculate

$$\begin{aligned}
& \tau_\varphi(A_q,A_0,\dots,A_{q-1})(g)\\
= \;& (-1)^q\int_{M^{q+1}} Tr(K_q(gx_q,gx_0)\lambda_g K_0(x_0,x_1)\dots K_{q-1}(x_{q-1},gx_q))\\
& \cdot\varphi(gx_q,x_0,\dots,x_{q-1})(g)\\
= \;& (-1)^{2q}\int_{M^{q+1}} Tr(\lambda_g K_0(x_0,x_1)\dots K_q(x_q,gx_0))\varphi(gx_0,x_1,\dots,x_q)(g)\\
= \;& (-1)^q\tau_\varphi(A_1,\dots,A_q)(g).
\end{aligned}$$

Here we used the properties of K_q and φ that $\lambda_g K_q(x_q,x_0)\lambda_g^{-1} = K_q(gx_q,gx_0)$ and $\varphi(x_0,\dots,x_q)(g) = (-1)^q\varphi(x_q,g^{-1}x_0,x_1,\dots,x_{q-1})(g)$.

(2) We have

$$\begin{aligned}
& \tau_\varphi(A_0g(A_1),A_2,\dots,A_{q+1})(g)\\
= \;& (-1)^q\int_{M^{q+2}} Tr(\lambda_g K_0(x_0,x_1)\dots K_{q+1}(x_{q+1},gx_0))\varphi(gx_0,x_2,\dots,x_{q+1})(g),
\end{aligned}$$

$$\begin{aligned}
& \tau_\varphi(A_0,\dots,A_iA_{i+1},\dots,A_{q+1})(g)\\
= \;& (-1)^q\int_{M^{q+2}} Tr(\lambda_g K_0(x_0,x_1)\dots K_{q+1}(x_{q+1},gx_0))\varphi(gx_0,\dots,\hat{x}_i,\dots,x_{q+1})(g),
\end{aligned}$$

and

$$\begin{aligned}
& \tau_\varphi(A_{q+1}A_0,A_1,\dots,A_q)(g)\\
= \;& (-1)^q\int_{M^{q+2}} Tr(\lambda_g K_{q+1}(x_{q+1},x_0)K_0(x_0,x_1)\dots K_q(x_q,gx_{q+1}))
\end{aligned}$$

$$
\begin{aligned}
\cdot\quad & \varphi(gx_{q+1}, x_1, \ldots, x_q)(g) \\
= \quad & (-1)^{2q} \int_{M^{q+2}} Tr(\lambda_g K_0(x_0, x_1) \ldots K_{q+1}(x_{q+1}, gx_0))\varphi(gx_1, x_2, \ldots, x_{q+1})(g).
\end{aligned}
$$

Hence, $(b\tau_\varphi)(A_0, \ldots A_{q+1})(g) = \tau_{\delta\varphi}(A_0, \ldots, A_{q+1})(g)$.

(3) Since $g(A_i) = A_i$, we need only to check

$$\tau_\varphi(A_0, \ldots, A_q)(h) = \tau_\varphi(A_0, \ldots, A_q)(g^{-1}hg).$$

This can be seen by the following computation:

$$
\begin{aligned}
& \tau_\varphi(A_0, \ldots, A_q)(g^{-1}hg) \\
= \quad & (-1)^q \int_{M^{q+1}} (\lambda_{g^{-1}hg} K_0(x_0, x_1) \ldots K_q(x_q, g^{-1}hgx_0))\varphi(hgx_0, gx_1, \ldots, gx_q)(h) \\
= \quad & (-1)^q \int_{M^{q+2}} Tr(\lambda_h K_0(x_0, x_1) \ldots K_q(x_q, hx_0))\varphi(hx_0, x_1, \ldots, x_q)(h) \\
= \quad & \tau_\varphi(A_0, \ldots, A_q)(h),
\end{aligned}
$$

since $\lambda_g K_i(g^{-1}x, g^{-1}y)\lambda_g^{-1} = K_i(x, y)$.

(4) Since $\varphi(gx_0, \ldots, x_i, x_{i+1}, \ldots, x_q) = 0$ for $x_i = x_{i+1}$, we get

$$\tau_\varphi(A_0, \ldots, A_{i-1}, f_i, \ldots, A_q)(g) = 0.$$

This imples (4). Q.E.D.

Now let $a \in Sym^0_G(M; E, F)^{-1}$, the symbol space of $\Psi^0_G(M; E, F)^{-1}$. Choose A and B in $\Psi^0_G(M; E, F)$ such that the principal symbols $\sigma(A)$ and $\sigma(B)$ are a and a^{-1}, resp.. Then as in [CoM 2], let $S_0 = I - BA \in \Psi^{-1}_G(M; E, E)$ and $S_1 = I - AB \in \Psi^{-1}_G(M; F, F)$.

$$L = \begin{bmatrix} S_0 & -B - S_0 B \\ A & S_1 \end{bmatrix} \in \Psi^0_G(M; E \oplus F), L^{-1} = \begin{bmatrix} S_0 & (I + S_0)B \\ S_1 & -A \end{bmatrix}.$$

Define $P = L \begin{bmatrix} I_E & 0 \\ 0 & 0 \end{bmatrix} L^{-1}, e = \begin{bmatrix} 0 & 0 \\ 0 & I_F \end{bmatrix}$ and

$$R_a = P - e = \begin{bmatrix} S_0^2 & S_0(I+S_0)B \\ S_1 A & -S_1^2 \end{bmatrix} \in \Psi_G^{-\infty}(M; E \oplus F, E \oplus F).$$

Let $\varphi \in \hat{C}^q_{G,c}(M), \delta\varphi \in \tilde{C}^{q+1}_{G,0}(M)$.

Definition 6.1 $Ind^G_\varphi : Sym^0_G(M; E, F)^{-1} \to F(G)$ is defined by

$$\begin{aligned} Ind^G_\varphi(a)(g) &= \tau_\varphi(\overbrace{R_a, \ldots, R_a}^{q+1})(g) \\ &= (-1)^q \int_{M^{q+1}} Tr(\lambda_g R_a(x_0, x_1) \ldots R_a(x_q, gx_0))\varphi(gx_0, x_1, \ldots, x_q)(g). \end{aligned}$$

To show that Ind^G_φ is well defined, we need the following lemma.

Lemma 6.3 ([CoM2]) *Let $\{P_t : t \in [0,1]\}$ be a C^1-piecewise family of idempotents in $\Psi^0_G(M; E \oplus F, E \oplus F)$. Then for q even and $T_t = (1 - 2P_t)\frac{dP_t}{dt}$,*

$$\tau_\varphi(P_1, \ldots, P_1)(g) - \tau_\varphi(P_0, \ldots, P_0)(g) = (q+1)\int_0^1 \tau_{\delta\varphi}(T_t, P_t, \ldots, P_t)(g)dt.$$

In fact, since P_t and T_t are equivariant and q is even,

$$\begin{aligned} \frac{d}{dt}\tau_\varphi(P_t, \ldots, P_t) &= \sum_{i=0}^q \tau_\varphi(P_t, \ldots, \frac{d}{dt}P_t, \ldots, P_t)(g) \\ &= \sum_{i=0}^q \tau_\varphi(P_t, \ldots, P_t, [T_t, P_t], P_t, \ldots, P_t)(g) \\ &= (q+1)\tau_\varphi([T_t, P_t], P_t, \ldots, P_t)(g) \\ &= (q+1)b\tau_\varphi(T_t, P_t, \ldots, P_t) = (q+1)\tau_{\delta\varphi}(T_t, P_t, \ldots, P_t). \end{aligned}$$

To show that $Ind^G_\varphi(a)$ is independent of the choice of the idempotent P, let L_0 and L_1 be two such idempotents with supports sufficiently around the diagonal. Let P_i be the idempotents associated with L_i and $R_a(i) = P_i - e$. By Lemma 6.2, $\tau_\varphi(R_a(i), \ldots, R_a(i))(g) = \tau_\varphi(P_i, \ldots, P_i)(g)$. Observe that $\tau_\varphi(P_i, \ldots, P_i)(g) = \tau_\varphi(P_i \oplus 0, \ldots, P_i \oplus 0)(g)$. It suffices to show

$$\tau_\varphi(P_0 \oplus 0, \ldots, P_0 \oplus 0)(g) = \tau_\varphi(P_1 \oplus 0, \ldots, P_1 \oplus 0)(g).$$

This follows from Lemma 6.3 by choosing a path connecting $P_0 \oplus 0$ with $P_1 \oplus 0$,

$$\begin{aligned} P_t &= L_t \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} L_t^{-1}, \\ L_t &= \begin{bmatrix} L_1 L_0^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix} \begin{bmatrix} L_0 L_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi t}{2} & \sin\frac{\pi t}{2} \\ -\sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix}. \end{aligned}$$

$Ind_\varphi^G(a)$ is defined for an equivariant cocycle $\varphi \in \hat{C}^q_{G,c}(M)$. We can show that $Ind^G_{[\varphi]}(a)$ is defined for $[\varphi] \in \hat{H}^q_{G,c}(M)$. Indeed, $Ind^G_\varphi(a) = 0$ for locally zero cochain $\varphi \in \tilde{C}^q_{G,c}(M)$, since the kernel $R_a(x,y)$ is supported around the diagonal. If $\psi \in \tilde{C}^{q-1}_{G,c}(M)$, then for q even,

$$\begin{aligned} Ind^G_{\delta\psi}(a)(g) &= \tau_{\delta\psi}(R_a, \ldots, R_a)(g) = b\tau_\psi(\overbrace{P, \ldots, P}^{q+1}) \\ &= \tau_\psi(\overbrace{P, \ldots, P}^{q})(g) = -\tau_\psi(\overbrace{P, \ldots, P}^{q})(g), \end{aligned}$$

since ψ is antisymmetric and R_a is equivariant. Hence $Ind^G_{\delta\psi}(a)(g) = 0$. Thus $Ind^G_{[\varphi]}$ is well defined.

We remark that a weak version of the McKean-Singer formula holds for $Ind^G_{[\varphi]}(a)$. To state it, we assume M is a compact G-manifold and $D \in$

$\Psi^r_G(M;E,F)^{-1}$ is an equivariant elliptic differential operator. Using the idempotents in [CoM 2],

$$W(D) = \begin{bmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D}(\frac{I-e^{-D^*D}}{D^*D})^{\frac{1}{2}}D^* \\ e^{-\frac{1}{2}DD^*}(\frac{I-e^{-DD^*}}{DD^*})^{\frac{1}{2}}D & I-e^{-DD^*} \end{bmatrix},$$

$$P(D) = \begin{bmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D}(\frac{I-e^{-D^*D}}{D^*D})D^* \\ e^{-\frac{1}{2}DD^*}D & I-e^{-DD^*} \end{bmatrix},$$

and

$$P_s(D) = \begin{bmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D}(\frac{I-e^{-D^*D}}{D^*D})^{\frac{1}{2}+s}D^* \\ e^{-\frac{1}{2}DD^*}(\frac{I-e^{-DD^*}}{DD^*})^{\frac{1}{2}-s}D & I-e^{-DD^*} \end{bmatrix}, \; s \in [0, \frac{1}{2}].$$

We see $P_s(D)$ is a path of idempotents connecting $P(D)$ with $W(D)$. Let $W_1(tD) = W(tD) - e$, $t \geq 0$.

Lemma 6.4 *Let $g \in G$ be sufficiently close to the identity. Then*

$$\lim_{t\to 0} \tau_{[\varphi]}(W_1(tD), \ldots, W_1(tD))(g) = Ind^G_{[\varphi]}(\sigma(D))(g).$$

The proof is the same as that in [CoM 2] which we omit.

6.3 Analytic Index Map in Equivariant K-Theory

In this section we will use $Ind^G_{[\varphi]}$ to define an index map in equivariant K-theory and prove some properties of this index map which are similar to those of the Atiyah-Singer index map.

Proposition 6.3 *Let M be a complete Riemannian G-manifold. There is a pairing of $\hat{H}^{2q}_{G,c}(M)$ with $K^0_G(T^*M, T^*M \setminus M)$ to $F_c(G)$ given by $Ind^G_{[\varphi]}(a)$ for $[\varphi] \in \hat{H}^{2q}_{G,c}(M)$.*

Proof. The proof is the same as that of Theorem 2.4 [CoM 2] except we have to use the equivariant homotopy theorem of equivariant vector bundles. First we define the relative K-theory group $K^0_G(T^*M, T^*M \setminus M)$ as the quotient of $E_G(T^*M, T^*M \setminus M)$ by $E^0_G(T^*M, T^*M \setminus M)$, where $E_G(T^*M, T^*M \setminus M)$ is the set of G-homotopy classes of all triples (σ, E, F) with E and F C^∞-complex vector bundles over T^*M and σ a G-isomorphism from $E|_{T^*M\setminus M}$ onto $F|_{T^*M\setminus M}$, and $E^0_G(T^*M, T^*M \setminus M)$ is the subset of $E_G(T^*M, T^*M \setminus M)$ consisting of (σ, E, F) with σ a G-isomorphism from E onto F. Two triples $(\sigma_i, E_i, F_i), i = 0, 1$, are G-homotopic if there is a triple $(\sigma, E, F) \in E_G(T^*M \times [0,1], (T^*M \setminus M) \times [0,1])$ such that $(\sigma, E, F)|_{T^*M \times \{i\}}$ is G-isomorphic to $(\sigma_i, E_i, F_i), i = 0, 1$. $E_G(T^*M, T^*M \setminus M)$ is equipped with direct sum operation. We can obtain from this description of $K^0_G(T^*M, T^*M \setminus M)$ that each equivariant elliptic principal symbol $a \in Sym^0_G(M; E, F)^{-1}$ defines an element (a, π^*E, π^*F) in $E_G(T^*M, T^*M \setminus M)$, where $\pi : T^*M \to M$ is the canonical projection and σ is homogeneous of degree 0. Let $S_G(T^*M, T^*M \setminus M)$ denote the subset of $E_G(T^*M, T^*M \setminus M)$ consisting of G-homotopy classes of all such elements (a, π^*E, π^*F). The important fact is that $S_G(T^*M, T^*M \setminus M)$ generates the group $K^0_G(T^*M, T^*M \setminus M)$. This is because any (σ, E, F) in $E_G(T^*M, T^*M \setminus M)$ is isomorphic to a triple $(a_\sigma, \pi^*E_0, \pi^*F_0)$ in $S_G(T^*M, T^*M \setminus M)$, where E_0 and F_0 are equal to $E|_M$ and $F|_M$, resp., and a_σ is the extension of $\sigma|_{S^*M}$ to π^*F by homogeneity of degree 0 [Bie], since M is a G-deformation retract of T^*M. Here S^*M is the unit sphere in T^*M.

Let $[\varphi] \in \hat{H}^{2q}_{G,c}(M)$. We know already that $Ind^G_{[\varphi]}(a)$ is defined for the triple (a, π^*E, π^*F) in $S_G(T^*M, T^*M \setminus M)$. It suffices to check that $Ind^G_{[\varphi]}(a)$ is independent of the choice of the representative of G-homotopy class. To this aim, let (a, π^*E, π^*F) be in $S_G(T^*M \times [0,1], (T^*M \setminus M) \times [0,1])$. We obtain that E and F are G-isomorphic to $E_0 \times [0,1]$ and $F_0 \times [0,1]$, resp., where $E_0 = E|_{M \times \{0\}}$ and $F_0 = F|_{M\times\{0\}}$ [Bie], and $a_s = a|_{(T^*M\setminus M)\times\{s\}}$ is in $Sym^0_G(M; E_0, F_0)^{-1}, s \in [0,1]$. Choose C^1-path $\{A_s\}$ (resp. $\{B_s\}$) in $\Psi^0_G(M; E_0, F_0)$ (resp. $\Psi^0_G(M; F_0, E_0)$) such that $\sigma(A_s) = a_s$ (resp. $\sigma(B_s) = a_s^{-1}$) and each A_s (resp. B_s) has the support near the diagonal. As in the previous section, we can produce a C^1-path P_s of G-idempotents. Using Lemma 6.3, we get $\tau_{[\varphi]}(P_0, \dots, P_0)(g) = \tau_{[\varphi]}(P_1, \dots, P_1)(g)$. Q.E.D.

$Ind^G_{[\varphi]} : K^0_G(T^*M, T^*M \setminus M) \to F_c(G)$ is called the higher equivariant analytic index map for $[\varphi] \in \hat{H}^{2q}_{G,c}(M)$. $Ind^G_{[\varphi]}$ shares most properties of the usual index map as we will see below.

Clearly, if $f : N \to M$ is an isometric G-diffeomorphism of complete smooth Riemannian G-manifolds, then for $[\varphi] \in \hat{H}^{2q}_{G,c}(M)$, $f^*([\varphi]) \in \hat{H}^{2q}_{G,c}(N)$ and the diagram

$$\begin{array}{ccc} K^0_G(T^*M, T^*M \setminus M) & \xrightarrow{f^*} & K^0_G(T^*N, T^*N \setminus N) \\ Ind^G_{[\varphi]} \searrow & F_c(G) & \swarrow Ind^G_{[f^*(\varphi)]} \end{array}$$

commutes. The point is that one can use $[\varphi]$ to assume that M and N are compact. The commutativity then follows from the definition of $Ind^G_{[\varphi]}$. Moreover, if $\rho : G_1 \to G$ is a homomorphism of compact Lie Groups, then the diagram

$$\begin{array}{ccc} K^0_G(T^*M, T^*M\setminus M) & \xrightarrow{\rho^*} & K^0_{G_1}(T^*M, T^*M\setminus M) \\ \Big\downarrow Ind^G_{[\varphi]} & & \Big\downarrow Ind^{G_1}_{[\rho^*(\varphi)]} \\ F_c(G) & \xrightarrow{\rho^*} & F_c(G_1) \end{array}$$

also commutes, where $[\rho^*(\varphi)] \in \hat{H}^{2q}_{G_1,c}(M)$.

Observe that if M is a point, then $K^0_G(T^*M, T^*M\setminus M) = R(G)$, the representation space of G, and $\check{H}^0_{G,c}(M) = F_c(G)$. The elliptic operators A and B are just G-linear maps on finite dimensional G- modules E and F: $A : E \to F$ and $B : F \to E$, which are clearly G-compact. We can choose $S_0 = I \in \mathcal{L}(F, F)$ and $S_1 = I \in \mathcal{L}(E, E)$, $L = I \in \mathcal{L}(E \oplus F)$. We get for q even,

$$Ind^G_{[\varphi]}(A) = Tr(\lambda_g \overbrace{R \dots R}^{q+1})\varphi(g) = (Tr(\lambda_g|_E) - Tr(\lambda_g|_F))\varphi(g) = Ind^G(A)\varphi(g).$$

This is the axiom $(A1)$ of Atiyah-Singer in the hgher analytic index case.

We assume for the rest of this chapter that M is a compact G-manifold. Then $K^0_G(T^*M, T^*M\setminus M)$ reduces to $K^0_G(T^*M)$ in the sense of Atiyah-Singer [AtS]. Let U be an open G-subset of M and $i : U \to M$ be the inclusion. i induces a homomorphism $i_* : K^*_G(T^*U) \to K^*_G(T^*M)$. As shown in [AtS], any element $[a] \in K^0_G(T^*U)$ can be represented by the symbol of an elliptic G-operator $A \in \bar{\Psi}^0_G(U; E, F)^{-1}$, where $\bar{\Psi}^0_G$ is the closure of $\Psi^0_G(U; E, F)$ in the space of continuous operators on Sobolev spaces. In fact, there are G-bundle isomorphisms α and β outside a compact subset M_0 of U,

$$\alpha : E|_{U\setminus M_0} \to (U\setminus M_0)\times \mathbb{C}^n,\ \ \beta : F|_{U\setminus M_0} \to (U\setminus M_0)\times \mathbb{C}^n.$$

Then $Au = \beta^{-1}\alpha u$ for u a distributive section of E on $U\backslash M_0$. The symbol $\sigma(A)$ of A is an isomorphism outside a G-compact subset of U and $[\sigma(A)] = [a]$. We can extend E and F trivially to bundles i_*E and i_*F on M via isomorphisms α and β and extend A outside U by formula $i_*(A)u = \beta^{-1}\alpha u$. Evidently, $[\sigma(i_*(A))] = i_*[\sigma(A)] = i_*[a] \in K_G^*(T^*M)$. Similarly, we have an elliptic G-operator $B \in \bar{\Psi}_G^0(U; F, E)^{-1}$ such that B can be extended to $i_*(B)$ on M and $\sigma(B) = \sigma(A)^{-1}$. If we use A and B to construct the idempotent P and R_a as in section 6.2, we have that $\tilde{R}_a$ constructed from $i_*(A)$ and $i_*(B)$ is the extension $i_*(R_a)$ of R_a to M. We also extend $\varphi \in \hat{C}_G^q(U)$ to $i_*(\varphi) \in \hat{C}_G^q(M)$ in a natural way, i.e., $i_*(\varphi)$ is zero outside U^{q+1}. Consequently, we obtain

$$
\begin{aligned}
Ind_{[\varphi]}^G(a) &= \int_{U^{q+1}} (-1)^q Tr(\lambda_g R_a(x_0, x_1) \dots R_a(x_q, gx_0))\varphi(gx_0, x_1, \dots, x_q)(g) \\
&= \int_{M^{q+1}} (-1)^q Tr(\lambda_g \tilde{R}_a(x_0, x_1) \dots \tilde{R}_a(x_q, gx_0))\varphi(gx_0, x_1, \dots, x_q)(g) \\
&= Ind_{[i_*(\varphi)]}^G(i_*(a))(g).
\end{aligned}
$$

This is the higher version of the excision axiom [AtS], i.e., the following diagram commutes

$$
\begin{array}{ccc}
\hat{H}_{G,c}^{ev}(U) \otimes K_G^0(T^*U) & \xrightarrow{i_*} & \hat{H}_{G,c}(M) \otimes K_G^0(T^*M) \\
Ind^G \searrow & F_c(G) & Ind^G \swarrow
\end{array}
$$

Provided U is an open G-subset of M.

To consider other axioms, we first give another form of the analytic equivariant index map.

Proposition 6.4 *Let $\varphi \in \hat{C}^q_G(M)$ be a cocycle and $D \in \Psi^r_G(M; E, F)^{-1}$ a G-elliptic operator. Then*

$$Ind^G_{[\varphi]}(D)(g) = \tau_{[\varphi]}(P(D), \ldots, P(D))(g) - \tau_{[\varphi]}(P(D)', \ldots, P(D)')(g),$$

where $P(D)$ and $P(D)'$ are the projections of $L^2(M, E)$ and $L^2(M, F)$ onto $Ker(D)$ and $Coker(D)$, resp..

Proof. Let $Q(D) = \frac{I - e^{-\frac{1}{2}D^*D}}{D^*D} D^*$, a parametrix of D in $\Psi^{-r}_G(M; F, E)^{-1}$. The idempotent corresponding to $Q(D)$ is denoted by P and $R(D) = P - e$ as in section 6.2. To control the support of $Q(D)$, we introduce a cut function $\alpha \in C^\infty_G(M \times M)$ as in [CoM 2], which is equivariant, nonnegative and supported in a sufficiently small neighborhood of the diagonal. Let $\tilde{Q}_s(D) = \alpha^{s-\frac{1}{2}} Q(D) \alpha^{s-\frac{1}{2}}, \frac{1}{2} \le s \le 1$. Clearly, $\tilde{Q}_s(D) - Q(D) \in \Phi^{-\infty}_G(M; F, E), s \ge \frac{1}{2}$. $\tilde{Q}_s(D)$ is also a parametrix of D. We can then use $\tilde{Q}_s(D)$ to form the idempotent $\tilde{P}_s$ and then $\tilde{R}_s(D) = \tilde{P}_s - e, \frac{1}{2} \le s \le 1$. Note that $\tilde{R}_{\frac{1}{2}} = R$. Then

$$Ind^G_{[\varphi]}(D)(g) = Ind^G_{[\varphi]}(tD)(g) = \tau_{[\varphi]}(\tilde{R}_s(tD), \ldots, \tilde{R}_s(tD))(g), s > \frac{1}{2}.$$

By Lemma 6.3,

$$\begin{aligned} & Ind^G_{[\varphi]}(D)(g) - \tau_{[\varphi]}(R(tD), \ldots, R(tD))(g) \\ = \; & (q+1) \int_{\frac{1}{2}}^1 \tau_{\delta\varphi}(\tilde{T}_s(tD), \tilde{R}_s(tD), \ldots, \tilde{R}_s(tD))(g) ds = 0. \end{aligned}$$

Therefore,

$$Ind^G_{[\varphi]}(D)(g) = \tau_{[\varphi]}(R(tD), \ldots, R(tD))(g), \quad t > 0, g \in G.$$

By the asymptote of the eigenvalues of D on compact manifold M [Gil], we get as $t \to \infty$,

$$e^{-t^2 D^*D} \xrightarrow{\text{strongly}} P(D), e^{-\frac{1}{2}t^2 D^*D} Q(tD) \xrightarrow{\text{strongly}} 0, e^{-\frac{1}{2}t^2 DD^*}(tD) \xrightarrow{\text{strongly}} 0,$$

and $e^{-t^2DD^*} \overset{\text{strongly}}{\longrightarrow} P(D)$. Hence, $R(tD) \overset{\text{strongly}}{\longrightarrow} P(D) \oplus (-P(D)')$ as $t \longrightarrow \infty$. This motivates the desired formulas. In fact, using the iterated heat kernels and the fact that φ is bounded, we obtain

$$
\begin{aligned}
\tau_{[\varphi]}(R(tD), \ldots, R(tD))(g) &\overset{t\to\infty}{\longrightarrow} \tau_{[\varphi]}(P(D), \ldots, P(D))(g) \\
&+ \quad (-1)^{q+1}\tau_{[\varphi]}(P(D)', \ldots, P(D)')(g).
\end{aligned}
$$

Q.E.D.

Proposition 6.4 is useful in deriving the normalization and multiplicative axioms of higher equivariant analytic index map.

Recall that Atiyah and Singer introduced elements $\rho_{S^n} \in K^0_G(TS^n)$ for $n = 1, 2$ and $G = O(1)$ and $SO(2)$ [AtS], which are the symbol classes of the de Rham complexes of exterior differential forms on S^1 and S^2, resp.. Let $D = d + d^* : \oplus_i \Omega^{2i}(S^n) \to \oplus \Omega^{2i+1}(S^n)$ be the differential operator given by the exterior differential and its adjoint d^* with respect to a Riemannian metric on S^n. Then

$$
Ker(D) = \begin{cases} \mathbb{C} \oplus \mathbb{C}w, & n = 2, \\ \\ \mathbb{C}, & n = 1, \end{cases} \qquad Coker(D) = \begin{cases} 0, & n = 2, \\ \\ \mathbb{C}dx, & n = 1, \end{cases}
$$

where w is the volume form in $\Omega^2(S^2)$. Since $SO(2)$ acts trivially on $H^{2i}(S^2)$ and the generator of $O(1)$ acts trivially on $H^0(S^1)$ and changes dx to $-dx$ on

$H^1(S^1)$,

$$P(D)(x,y)=\begin{cases}1\oplus w_x, & n=2,\\ 1, & n=1,\end{cases}\quad P(D)'(x,y)=\begin{cases}0, & n=2,\\ dx, & n=1,\end{cases}$$

We obtain

$$\begin{aligned}&\tau_{[\varphi]}(P(D),\ldots,P(D))(g)-\tau_{[\varphi]}(P(D)',\ldots,P(D)')(g)\\ =&\begin{cases}Tr(\lambda_g|_{Ker(D)})\int_{(S^2)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)w_0\wedge\ldots\wedge w_q,\ n=2,\\ (Tr(\lambda_g|_{Ker(D)})-Tr(\lambda_g|_{Coker(D)}))\int_{(S^1)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)dx_0\wedge\ldots dx_q,\ \ n=1\end{cases}\\ =&\begin{cases}2\int_{(S^2)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)w_0\wedge\ldots\wedge w_q,\ n=2,\\ (1-Tr(\lambda_g|_{Coker(D)}))\int_{(S^1)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)dx_0\wedge\ldots dx_q,\ \ n=1,g\in G.\end{cases}\end{aligned}$$

Also for operator A defined on $S^1=\mathbb{R}/2\pi\mathbb{Z}$ by $Ae^{inx}=e^{i(n+1)x}, n\geq 0$, and $e^{inx}, n<0$, we have that $Ker(A)=0$ and $Coker(A)$ is generated by the constant functions.

$$\begin{aligned}Ind^G_{[\varphi]}(a)(A)(g) &= -\tau_{[\varphi]}(P(A)',\ldots,P(A)')(g)\\ &= -\int_{(S^1)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)dx_0\wedge\ldots dx_q.\end{aligned}$$

But we know that the symbol $\sigma(A)$ of A , given by $\sigma(A)(x,\xi)=e^{ix}$ for $\xi>0$ and 1 for $\xi<0$, determines the element $-J!(1)\in K^0_G(TS^1)$ [AtS]. Hence,

$$Ind^G_{[\varphi]}(J!(1))(g)=\int_{(S^1)^{q+1}}\varphi(gx_0,x_1,\ldots,x_q)(g)dx_0\wedge\ldots dx_q.$$

We now consider the multiplicative axiom of equivariant analytic index map. Let M and N be compact smooth Riemannian G-manifolds. Assume that $a_1 \in K_G^0(T^*M)$ and $a_2 \in K_G^0(T^*N)$ be represented by smooth symbols α and β, resp.. Let $A_1 \in \Psi_G^1(M;E,F)^{-1}$ and $A_2 \in \Psi_G^1(N;E',F')^{-1}$ with $\sigma(A_1) = \alpha$ and $\sigma(A_2) = \beta$. Then the element $a_1a_2 \in K_G^0(T^*(M \times N))$ is represented by the symbol

$$\nu = \begin{bmatrix} \alpha \otimes 1 & -1 \otimes \beta^* \\ 1 \otimes \beta & \alpha^* \otimes 1 \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} A_1 \otimes I & -I \otimes A_2^* \\ I \otimes A_1 & A_1^* \otimes I \end{bmatrix} \in \bar{\Psi}_G^1(M \times N, E \otimes E' \oplus F \otimes F', F \otimes E' \oplus E \otimes F').$$

$\sigma(D) = \nu$. By considering $Ker(DD^*)$ and $Ker(D^*D)$, We get

$$\begin{aligned} Ker(D) &= Ker(A_1^*A_1 \otimes I + I \otimes A_2^*A_2) \oplus Ker(I \otimes A_2A_2^* + A_1A_1^* \otimes I) \\ &= Ker(A_1) \otimes Ker(A_2) \oplus Coker(A_1) \otimes Coker(A_2), \\ Coker(D) &= Ker(A_1A_1^* \otimes I + I \otimes A_2^*A_2) + Ker(I \otimes A_2A_2^* + A_1^*A_1 \otimes I) \\ &= Coker(A_1) \otimes Ker(A_2) \oplus Ker(A_1) \otimes Coker(A_2). \end{aligned}$$

Hence,

$$\begin{aligned} P(D) &= P(A_1) \otimes P(A_2) \oplus P(A_1)' \otimes P(A_2)', \\ P(D)' &= P(A_1)' \otimes P(A_2) \oplus P(A_1) \otimes P(A_2)'. \end{aligned}$$

Now let $\varphi \in \hat{C}^q_G(M)$ be a cocycle. Then $\varphi \otimes 1 \in \hat{C}^q_G(M \times N)$ is a cocycle.

$$\begin{aligned}
Ind^G_{[\varphi\otimes 1]}(D)(g) &= \tau_{[\varphi\otimes 1]}(P(A_1)\otimes P(A_2), \ldots, P(A_1)\otimes P(A_2))(g) \\
&+ \tau_{[\varphi\otimes 1]}(P(A_1)'\otimes P(A_2)', \ldots, P(A_1)'\otimes P(A_2)')(g) \\
&- \tau_{[\varphi\otimes 1]}(P(A_1)'\otimes P(A_2), \ldots, P(A_1)'\otimes P(A_2))(g) \\
&- \tau_{[\varphi\otimes 1]}(P(A_1)\otimes P(A_2)', \ldots, P(A_1)\otimes P(A_2)')(g) \\
&= (\tau_{[\varphi]}(P(A_1), \ldots, P(A_1))(g) - \tau_{[\varphi]}(P(A_1)', \ldots, P(A_1)')(g)) \\
&\quad (\tau_{[1]}(P(A_2), \ldots, P(A_2))(g) - \tau_{[1]}(P(A_2)', \ldots, P(A_2)')(g)) \\
&= Ind^G_{[\varphi]}(A_1)(g) Ind^G_{[1]}(A_2)(g) = Ind^G_{[\varphi]}(a_1)(g) Ind^G_{[1]}(a_2)(g).
\end{aligned}$$

Note that $[1] \in \bar{H}^q_G(M)$. Since $Ind^G_{[\varphi]}(a_1)(g)$ and the left side of the above identity is well defined, $Ind^G_{[1]}(a_2)(g)$ makes sense when $Ind^G_{[\varphi]}(a_1)(g) \neq 0$. Similarly, if $\varphi \in \hat{C}^q_G(N)$ is a cocycle, then $1 \otimes \varphi \in \hat{C}^q_G(M \times N)$ is a cocycle and

$$Ind^G_{[1\otimes\varphi]}(a_1 a_2)(g) = Ind^G_{[1]}(a_1)(g) Ind^G_{[\varphi]}(a_2)(g).$$

To summarize, we have obtained

Theorem 6.1 *Let M and N be compact Riemannian G-manifolds and $[\varphi] \in \hat{H}^q_G(M)$ and $[\psi] \in \hat{H}^q_G(N)$ with q even. The higher equivariant analytic index map satisfies the Atiyah-Singer axioms:*

(1) *If M is a point,*

$$Ind^G_{[\varphi]}(a)(g) = Ind^G(a)(g)\varphi(g), \ \ \forall a \in K^0_G(M) = R(G),$$

(2) *Excision axiom: let $U \xrightarrow{i} M$ be an open G-subset, then for $a \in K^0_G(T^*U)$ and $[\varphi] \in \hat{H}^q_G(U)$, $i_*[\varphi] \in \bar{H}^q_G(M)$, $i_*(a) \in K^0_G(T^*M)$, and*

$$Ind^G_{[\varphi]}(a)(g) = Ind^G_{[i_*(\varphi)]}(i_*(a))(g),$$

(3) *Normalization axiom: for* $[\varphi] \in \hat{H}^q_G(S^n), G = SO(2), O(1), n = 1, 2,$

$$(a)\ Ind^G_{[\varphi]}(\rho_{S^2})(g) = 2\int_{(S^2)^{q+1}} \varphi(gx_0, x_1, \ldots, x_q)(g),\ \ g \in SO(2),$$

$$(b)\ Ind^G_{[\varphi]}(\rho_{S^1})(g) = (1 - tr(\lambda_g|_{H^1(S_1)}))\int_{(S^1)^{q+1}} \varphi(gx_0, x_1, \ldots x_q)(g)dx_0 \wedge \ldots dx_q,$$

$$(c)\ Ind^G_{[\varphi]}(J!(1))(g) = \int_{(S^1)^{q+1}} \varphi(gx_0, x_1, \ldots, x_q)(g)dx_0 \wedge \ldots \wedge dx_q,\ \ g \in O(1),$$

(4) *Multiplicative axiom: for* $a_1 \in K^0_G(T^*M)$ *and* $a_2 \in K^0_G(T^*N)$

$$Ind^G_{[\varphi \otimes 1]}(a_1 a_2)(g) = Ind^G_{[\varphi]}(a_1)(g) Ind^G_{[1]}(a_2)(g)$$

and

$$Ind^G_{[1 \otimes \psi]}(a_1 a_2)(g) = Ind^G_{[1]}(a_1)(g) Ind^G_{[\psi]}(a_2)(g).$$

We close this section by proposing a pairing version of the equivariant Novikov conjecture. Let us recall the Atiyah-Segal-Singer equivariant index theorem [AtS] for signature operators. Suppose G is a compact Lie group and M is a closed oriented Riemannian G-manifold. Let E be a G-vector bundle over M and D_E the signature operator with coefficients in E. Then the equivariant index $Ind^G(D_E)$ of D_E is a character of G given by

$$Ind^G(D_E)(h) = < Ch(i^*(E)) w(h, M^h) \mathcal{L}(M^h), [M^h] >,\ h \in G, \tag{6.1}$$

where M^h is the submanifold of fixed points of h in M, $i : M^h \to M$ is the inclusion, $w(h, M^h)$ is certain element in $H^*(M^h, \mathbb{Q})$, $\mathcal{L}(M^h)$ is the stable Hirzebruch $\mathcal{L}$-class of M^h, and $[M^h]$ is the twisted fundamental class of M^h (see also [Don 1]). We should point out that M^h is in general the disjoint union of compact connected submanifolds of M. The right side of (6.1) is a sum of pairings on the connected submanifolds. Given $h \in G$, let G_1 be the closed subgroup of G generated by h. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & B\pi_1(M) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ M^h & \xrightarrow{f^h} & B\pi_1(M^h), \end{array}$$

where $B\pi_1(M)$ is the classifying space and f is the classifying map. Since $M^h \neq \emptyset$ (otherwise nothing needs to be considered). G_1 has a fixed point in M. Hence G_1 acts on $\pi_1(M)$. We can assume f is G_1-equivariant by averaging over G_1. Then

$$(f^h)^* i^* : H^*_{G_1}(B\pi_1(M), \mathbb{R}) \to H^*_{G_1}(M^h, \mathbb{R}) = H^*(M^h, \mathbb{R}) \otimes R(G_1)$$

is a homomorphism. Here $H^*_{G_1}(B\pi_1(M), \mathbb{R})$ is the usual Borel equivariant cohomology of $B\pi_1(M)$. Let N and M be two closed oriented Riemannian G-manifolds and $f_1 : N \to M$ be a G-pseudo-equivalence, i.e., f_1 is equivariant and is a homotopy equivalence. Let $[\varphi] \in H^*_{G_1}(B\pi_1(M), \mathbb{R})$ and

$$Ind^G([\varphi]) =< ((f^h)^* i^*([\varphi])) w(h, M^h) \mathcal{L}(M^h), [M^h] > .$$

Question: Is $Ind^G([\varphi])$ a G-pseudo-equivalence invariant? Namely, does the equality

$$< (f_1^*(f^h)^* i^*([\varphi])) w(h, N^h) \mathcal{L}(N^h), [N^h] >=< ((f^h)^* i^*([\varphi])) w(h, M^h) \mathcal{L}(M^h), [M^h] >$$

hold?

Note that for $h = 1$ this question is the Novikov conjecture. One possible way to attack the question is to prove the higher equivariant index theorem and use it to bridge $Ind^G([\varphi])$ with the G-pseudo-equivalence invariant which will be proved in Chapter 9. We will investigate this elsewhere.

Chapter 7

Higher Equivariant Index Theorem for Homogeneous Spaces

In the previous chapter we used the ordinary trace to study the higher equivariant analytic index. Such a trace is not suitable for the index problem on homogeneous spaces. We will thus employ the trace on type II- von Neumann algebras to define the higher equivariant analytic index for homogeneous spaces. This is similar to the case of covering spaces where the trace tr_Γ is localized on the fundamental domain. We multiply the kernels on the homogeneous spaces with a cut-off function. We finally reduce the higher equivariant analytic index problem to the one studied by Connes and Moscovici. The reason we proceed this way is that every thing involved is invariant under the whole Lie group action. The group twisting does not cause big trouble in this case. But when we prove the higher equivariant index theorem for general equivariant pseudo-differential operators on homogeneous spaces, we have to be careful about the significant difference between the K-theory groups of homogeneous spaces and the usual manifolds without group structure. We will discuss the higher equivariant analytic index in section 7.1 and prove the higher equivariant index theorem in section 7.2.

7.1 Higher Equivariant Analytic Index for Homogeneous Spaces

We begin with some notations (see also chapter 1). Let G be a unimodular Lie group with countably many connected components, H a compact subgroup of G. Denote by $M = G/H$ the homogeneous space of all left cosets gH for $g \in G$. Suppose that λ is a unitary representation of H on a finite dimensional complex vector space E. H acts on $G \times E$ by diagonal action: $h(g,x) = (gh, \lambda(h)^{-1}x)$. Then $\mathcal{E} = G \times_H E$ is a homogeneous vector bundle over $M = G/H$. Let $C^\infty(M.\mathcal{E})$ (resp. $C_c^\infty(M,\mathcal{E})$) be the space of all C^∞-sections (resp. with compact support) of $\mathcal{E}$. Then $C^\infty(M.\mathcal{E}) = (C^\infty(G) \times E)^H$ and $C_c^\infty(M,\mathcal{E}) = (C_c^\infty(G) \times E)^H$, where H acts on $C^\infty(G)\times E$ by $h(\varphi, x) = (R(h)\varphi,, \lambda(h)x)$ and $(C^\infty(G)\times E)^H$ is the subspace of H-invariant elements in $C^\infty(G) \times E$. As before, R is the right representation of H. Using the Haar measures on G and H, we can define a G-invariant measure on M. Let $L^2(M,\mathcal{E})$ be the completion of $C_c^\infty(M,\mathcal{E})$ with respect to the global inner product given by the G-invariant measure on M and G--nvariant Hermitian structure on $\mathcal{E}$. We have $L^2(M,\mathcal{E}) = (L^2(G) \times E)^H$.

Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are two homogeneous vector bundles over M. Let $\Psi^n(M;\mathcal{E}_1,\mathcal{E}_2)$ denote the space of all pseudo-differential operators P from $C_c^\infty(M,\mathcal{E}_1)$ to $C^\infty(M.\mathcal{E}_2)$ of order n, and $\sigma(P)$ be the principal symbol which is a smooth and positively homogeneous map from $\pi^*\mathcal{E}_1$ to $\pi^*\mathcal{E}_2$, where $\pi : T^*(M)\backslash\{0\} \to M$ is the natural projection. See [CoM 1] for details . Each $P \in \Psi^n(M;\mathcal{E}_1,\mathcal{E}_2)$ has distributional kernel K_P as an element in $(C^{-\infty}(G\times G)\otimes Hom(E_1,E_2))^{H\times H}$ consisting of all elements $K \in C^{-\infty}(G\times G)\otimes Hom(E_1,E_2)$ such that $\lambda_2(h)K(xh,yg)\lambda_1(g)^{-1} = K(x,y)$ for all $g,h \in H$. P is called compactly supported if $SuppK_P$ is compact in $G\times G$, and P is G-compactly supported if $K_P(x,y) \neq 0$ only for $x^{-1}y$ in a compact set in G. We use $\Psi^n_{cc}(M;\mathcal{E}_1,\mathcal{E}_2)$ (resp. $\Psi^n_c(M;\mathcal{E}_1,\mathcal{E}_2)$) to denote all (resp. G-)compactly supported elements in $\Psi^n(M;\mathcal{E}_1,\mathcal{E}_2)$. P is called G-invariant if $L(g)PL(g)^{-1} = P$ for all $g \in G$, where L is the left regular representation of G. The kernel K_P of G-invariant pseudo-differential operator P can be written as $K_P(x,y) = k_P(x^{-1}y), x,y \in G$, where k_P is an element in $(C^{-\infty}(G)\otimes Hom(E_1,E_2))^{H\times H}$ consisting of all $k \in C^{-\infty}(G)\otimes Hom(E_1,E_2)$ such that $k(x) = \lambda_2(h)K(h^{-1}xg)$ $\lambda_1(g)^{-1}$, for all $g,h \in H$. In fact, take $k_P(y) = K_P(1,y)$. From $L(g)Pl(g)^{-1} = P$ we have $K_P(g^{-1}x,g^{-1}y) = K_P(x,y), \forall g \in G$. Hence $k_P(x^{-1}y) = K_P(1,x^{-1}y)$ $= K_P(x,y)$ and $k_P(y) = \lambda_2(h)K_P(h,yg)\lambda_1(g)^{-1} = \lambda_2(h)k_P(h^{-1}xg)\lambda_1(g)^{-1}, h,$ $g \in H$. Let $\Psi^n(M;\mathcal{E}_1,\mathcal{E}_2)^G$ be the space of all G-invariant pseudo-differential operators and $\Psi^n_c(M;\mathcal{E}_1,\mathcal{E}_2)^G = \Psi^n(M;\mathcal{E}_1,\mathcal{E}_2)^G \cap \Psi^n_c(M;\mathcal{E}_1,\mathcal{E}_2)$. There is a general average procedure to produce G-invariant operators $Av(P)$ for $P \in \Psi^n_{cc}(M;\mathcal{E}_1,\mathcal{E}_2)$,

$$(Av(P)u)(x) = \int_G L(g)PL(g)^{-1}u(x)dg.$$

Clearly, $L(g)Av(P)L(g)^{-1} = Av(P), g \in G$. The kernel $K_{Av(P)}$ of $Av(P)$ is given by

$$K_{Av(P)}(x,y) = \int_G K_P(g^{-1}x,g^{-1}y)dg.$$

We can say a little bit more about G-invariant operator P by using cut-off function. $f \in C_c^\infty(G)$ is an H-invariant cut-off function if $f \geq 0, f(gh) = f(g), \forall g \in G$ and $h \in H$, and $\int_G f(g^{-1}x)dg = 1, \forall x \in G$. One can easily see that if $P \in \Psi_c^n(M; \mathcal{E}_1, \mathcal{E}_2)^G$, then $P = Av(fP)$ for any cut-off function f.

To define a higher index map on the algebra $\Psi_c^n(M; \mathcal{E}_1, \mathcal{E}_2)^G$, let us recall the Alexander-Spanier cohomology of a homogeneous space M. As in Chapter 6, let $\bar{C}^*(M)$ (resp. $\bar{C}_c^*(M)$) denote the Alexander-Spanier complex (resp. with compact support), defined by $\bar{C}^q(M) = C^q(M)/C_0^q(M)$, where $C^q(M)$ (resp. $C_0^q(M)$) consists of all (resp. locally zero) functions from M^{q+1} to $\mathbb{R}$. Its cohomology $\bar{H}^*(M)$ (resp. $\bar{H}_c^*(M)$) are called the Alexander-Spanier cohomology (resp. with compact support). Let $\bar{C}_\infty^*(M)$ (resp. $\bar{C}_{\infty,c}^*(M)$) be the subcomplex of $\bar{C}^*(M)$ (resp. $\bar{C}_c^*(M)$) consisting of all C^∞- cochains. We know already that the cohomology $\bar{H}_\infty^*(M)$ (resp. $\bar{H}_c^*(M)$) of $\bar{C}_\infty^*(M)$ (resp. $\bar{C}_{\infty,c}^*(M)$) is isomorphic to $H^*(M)$ (resp. $H_c^*(M)$)., where $H^*(M)$ (resp. $H_c^*(M)$) is the usual cohomology of M given by the de Rham complex $\wedge^*(M)$ (resp. $\wedge_c^*(M)$). The isomorphism is implemented by

$$\wedge^*(M) \underset{\nu}{\overset{\rho}{\rightleftarrows}} \bar{C}_\infty^*(M),$$

where $\rho(\omega)(x_0, \ldots, x_q) = \chi_q(x_0, \ldots, x_q) \int_{S_q[x_0,\ldots,x_q]} \omega, \omega \in \wedge^q(M)$, $S_q[x_0, \ldots, x_q]$ is the C^∞-complex $\Sigma^{q+1} \to M$ given by $S_q[x_0, \ldots, x_q](t_0, \ldots, t_q) = \sum_{i=0}^q t_i x_i$, and χ_q satisfies the following: let $\mathcal{U}$ be a covering of M with special property [CoM 1], $\chi_q \in C^\infty(M^{q+1}), Supp\chi_q \subset \mathcal{U}^{q+1}$ and $\chi_q(x_{\alpha(0)}, \ldots, x_{\alpha(q)}) = \chi_q(x_0, \ldots, x_q), \forall \alpha \in S_{q+1}$, the permutation group of order $(q+1)!$, and for $\varphi = \varphi_0 \otimes \varphi_1 \otimes \ldots \otimes \varphi_q \in C^q(M)$ $\nu(\varphi) = \varphi_0 \partial\varphi_1 \wedge \ldots \wedge \partial\varphi_q$. If G acts on $\wedge^*(M)$

vis the map induced by $L(g) : M \to M$, then ρ is equivariant provided χ_q is:

$$\begin{aligned} g(\rho(\omega)) &= \rho(\omega)(g^{-1}x_0, \ldots, g^{-1}x_q) \\ &= \chi_q(g^{-1}x_0, \ldots, g^{-1}x_q) \int_{S_q[g^{-1}x_0,\ldots,g^{-1}x_q]} \omega \\ &= \chi_q(x_0, \ldots, x_q) \int_{g^{-1}S_q[x_0,\ldots,x_q]} \omega \\ &= \chi_q(x_0, \ldots, x_q) \int_{S_q[x_0,\ldots,x_q]} g\omega = \rho(g\omega)(x_0, \ldots, x_q). \end{aligned}$$

Clearly, ν is G-invariant. Thus if we require all the coverings of M to be equivariant, then G acts on $\bar{C}^*_\infty(M)$ (resp. $\bar{C}^*_{\infty,c}(M)$), and χ_q can be chosen to be equivariant. We get the homomorphisms

$$(\wedge^*(M))^G \underset{\nu}{\overset{\rho}{\rightleftarrows}} (\bar{C}^*_\infty(M))^G,$$

which satisfy $\nu\rho = I$.

From now on we assume that all Alexander-Spanier cochains are smooth. Let

$$C^q_{\alpha,\infty}(M) = \{\varphi \in C^q_\infty(M) : \varphi(x_{\alpha(0)}, \ldots, x_{\alpha(q)}) = sign(\alpha)\varphi(x_0, \ldots, x_q), \forall \alpha \in S_{q+1}\}$$

and $\bar{C}^q_{\alpha,\infty}(M) = C^q_{\alpha,\infty}(M)/(C^q_{\alpha,\infty}(M) \cap C^q_0(M))$. Then $H^*(\bar{C}^q_{\alpha,\infty}(M)) \simeq \bar{H}^*(M)$.

Note that the evaluation map at the identity of G gives an isomorphism from $\wedge^*(M)^G$ onto $C^*(\mathcal{G}, \mathcal{H}, \mathbb{R})$, where $\mathcal{G}$ and $\mathcal{H}$ are the Lie algebras of G and H resp., and

$$C^*(\mathcal{G}, \mathcal{H}, \mathbb{R}) = \{\varphi \in \wedge^*\mathcal{G}^* : i_x\varphi = 0, \Theta_x\varphi = 0, \forall x \in \mathcal{H}\}$$

with the differential $d : C^q(\mathcal{G}, \mathcal{H}, \mathbb{R}) \to C^{q+1}(\mathcal{G}, \mathcal{H}, \mathbb{R})$ given by

$$\begin{aligned} d\varphi(x_0, \ldots, x_q) &= \sum_{1 \le i < j \le q+1} (-1)^{i+j} \varphi([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}) \\ &+ \sum_i (-1)^i x_i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_q). \end{aligned}$$

Here $i_x : C^q(\mathcal{G},\mathcal{H},\mathbb{R}) \to C^{q-1}(\mathcal{G},\mathcal{H},\mathbb{R})$ and $\Theta_x : C^q(\mathcal{G},\mathcal{H},\mathbb{R}) \to C^q(\mathcal{G},\mathcal{H},\mathbb{R})$ are given by

$$(i_x\varphi)(x_1,\ldots,x_{q-1}) = \varphi(x,x_1,\ldots,x_{q-1})$$

and

$$(\Theta_x\varphi)(x_1,\ldots,x_q) = \sum_i \varphi(x_1,\ldots,[x_i,x],\ldots,x_q) + x\varphi(x_1,\ldots,x_q)$$

Then the relative Lie cohomology group $H^*(\mathcal{G},\mathcal{H},\mathbb{R})$ is defined to be the cohomology of $C^*(\mathcal{G},\mathcal{H},\mathbb{R})$. One has $H^*(\mathcal{G},\mathcal{H},\mathbb{R}) \simeq H^*(\wedge^*(M)^G)$. See [BoW].

We now define the higher equivariant analytic index. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ be a homogeneous vector bundle over M. Assume $\varphi \in C^q_{\alpha,\infty}(M)^G$. Thus $\varphi(gx_0,\ldots,gx_q) = \varphi(x_0,\ldots,x_q), \forall g \in G$. For $P_i \in \Psi_c^\infty(M,\mathcal{E})^G$ with kernel $K_i, i = 0,1,\ldots,q$ we define

$$\tau_\varphi(P_0,\ldots,P_q) = (-1)^q \int_{G^{q+1}} Tr(f(x_0)K_0(x_0,x_1)\ldots K_q(x_q,x_0))\varphi(x_0,\ldots,x_q)dx^{q+1},$$

where f is a cut-off function. This integral is well defined since f and K_i are G-compact supported. Note that we do not need φ to be compactly supported. This is different from the situation in the previous chapter.

Lemma 7.1 (1) $\tau_\varphi(P_0,\ldots,P_q) = (-1)^q\tau_\varphi(P_q,P_0,\ldots,P_{q-1})$.
(2) $b\tau_\varphi(P_0,\ldots,P_{q+1}) = \tau_{\delta\varphi}(P_0,\ldots,P_{q+1})$.
(3) *If* $\psi_i \in \Psi_c^0(M,\mathcal{E})^G, 0 \le i \le q$, *then* $\tau_\varphi(P_0+\psi_0,\ldots,P_q+\psi_q) = \tau_\varphi(P_0,\ldots,P_q)$.
(4) τ_φ *is independent of the choice of the cut-function* f.

Proof. (1) Using $K_i(gx,gy) = K_i(x,y)$ and equivariance of φ, we have

$$\tau_\varphi(P_q,\ldots,P_q)$$

$$\begin{aligned}
&= (-1)^q \int_{G^{q+1}} Tr(f(x_q)K_q(1, x_q^{-1}x_0)K_0(x_q^{-1}x_0, x_q^{-1}x_1)\dots K_{q-1}(x_q^{-1}x_{q-1}, 1)) \cdot \\
&\quad \cdot \varphi(1, x_q^{-1}x_0, \dots, x_q^{-1}x_{q-1})dx^{q+1} \\
&= (-1)^{2q} \int_{G^q} Tr(K_0(x_0, x_1)\dots K_{q-1}(x_{q-1}, 1)K_q(1, x_0)) \\
&\quad \cdot \varphi(x_0, \dots, x_{q-1}, 1)dx^q \int_G f(x_q)dx_q \\
&= (-1)^{2q} \int_{G^q} Tr(K_0(1, x_1)\dots K_q(x_q, 1))\varphi(1, x_1, \dots, x_q)dx^q \\
&= (-1)^q \tau_\varphi(P_0, \dots, P_q).
\end{aligned}$$

Hence, (1) and (4) are true.

(2) The proof is the same as that of Lemma 6.2.

(3) Since $\varphi(x_{\alpha(0)}, \dots, x_{\alpha(q)}) = sign(\alpha)\varphi(x_0, \dots, x_q)$, we have $\varphi(x_0, \dots, x_i, x_i, \dots, x_q)$ $= 0$. Using this we get (3). Q.E.D.

Note that any $P \in \Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$ defines a bounded G-invariant operator from $L^2(M, \mathcal{E}_1)$ to $L^2(M, \mathcal{E}_2)$. Let $\Psi_G^*(M, \mathcal{E})$ be the norm closure in $\mathcal{B}(L^2(M, \mathcal{E}))$ of $\Psi_c^0(M, \mathcal{E})^G$ and $C_G^*(M, \mathcal{E})$ the norm closure of $\Psi_c^{-\infty}(M, \mathcal{E})^G$, where

$$\Psi_c^{-\infty}(M, \mathcal{E})^G = \{R(\psi) \in \mathcal{B}(L^2(M, \mathcal{E})) : \psi \in (C_c^\infty(G) \otimes End(\mathcal{E}))^H\}$$

with $(R(\psi)u)(x) = \int \psi(x^{-1}y)u(y)dy, u \in C_c^\infty(M, \mathcal{E})$. Denote by $C_H(S(V), E)$ the sup-norm closure of $(C^\infty(S(V), End(\mathcal{E})))^H$ for $V = \mathcal{H}^\perp = \{\xi \in \mathcal{G}^* : \xi|_{\mathcal{H}} = 0\} \simeq T_0^*M$ and $S(V)$ the unit sphere of V. Then we have the following exact sequence of separable C^*-algebras:

$$0 \to C_G^*(M, \mathcal{E}) \to \Psi_G^*(M, \mathcal{E}) \xrightarrow{\sigma_0} C_H(S(V), E) \to 0,$$

where for $P \in \Psi^0(M, \mathcal{E})^G$ $\sigma(P)$ is the principal symbol defined by

$$\sigma(P)(x, d\psi(x))u(x) = \lim_{t\to\infty} e^{-t\psi(x)}P(e^{it\psi}u)(x), \ \ \psi \in C^\infty(M),$$

$d\psi(x) \neq 0$, $u \in C_c^\infty(M, \mathcal{E})$ and $\sigma_0(P)(\xi) = \sigma(P)(o, \xi) \in End(\mathcal{E}), \xi \neq 0 \in T_0^*M \simeq V$. $\sigma(P)$ is determined by $\sigma_0(P)$ due to the G-invariance of P. $\sigma_0(P)$ can be considered as an element in $(C^\infty(S(V), End(E))^H$ in view of homogeneity $\sigma_0(P)(t\xi) = \sigma_0(P)(\xi), t > 0$.

Let us now recall the K-theory group $K_H(V)$ which is a substitute for $K^0(T^*M, T^*M \setminus M)$ in the homogeneous case. $K_H(V)$ can be described as follows. Let $\mathcal{E}(M)$ be the set of all homotopy classes of H-invariant maps $\alpha : S(V) \to Iso(E_1, E_2)$, where as before $S(V)$ is the unit sphere of V with respect to the $Ad^*(H)$- invariant metric and E_1 and E_2 are finite dimensional unitary H-modules. Two α_i in $C^\infty(S(V), Iso(E_i, F_i))^H$,$i = 0, 1$, are isomorphic if there exist $\psi_0 \in (Iso(E_0, E_1))^H$ and $\psi_2 \in (Iso(F_0, F_1))^H$ such that $\psi_1\alpha_0(\xi) = \alpha_1(\xi)\psi_0, \forall\xi \in S(V)$, and α_0 is homotopic to α_1 if there exists $\alpha \in (C^\infty(S(V) \times [0,1], Iso(E, F))^H$ such that $\alpha|_{S(V)\times\{0\}} = \alpha_0$ and $\alpha|_{S(V)\times\{1\}} = \alpha_1$. Here H acts on $[0, 1]$ trivially, Let $\mathcal{E}_0(M)$ be the subset of $\mathcal{E}(M)$ consisting of all classes represented by a constant map $\alpha_0, \alpha_0(\xi) \equiv \varphi \in (Iso(E_0, F_0))^H, \forall\xi \in S(V)$. Then $K_H(V) = \mathcal{E}(M)/\mathcal{E}_0(M)$. See [CoM 1] for details.

Now choose a map $\alpha \in C^\infty(S(V), Iso(E_1, E_2))^H$. As in the case of usual compact manifolds, there exist $P_\alpha \in \Psi_c^0(M, \mathcal{E}_1, \mathcal{E}_2)^G$ and $Q_\alpha \in \psi_c^0(M, \mathcal{E}_2, \mathcal{E}_1)^G$ such that $\alpha = \sigma_0(P_\alpha)$ and $\alpha^{-1} = \sigma_0(Q_\alpha)$. Then $S_\alpha^0 = I - Q_\alpha P_\alpha \in \Psi_c^{-\infty}(M, \mathcal{E}_1, \mathcal{E}_1)^G$ and $S_\alpha^1 = I - P_\alpha Q_\alpha \in \Psi_c^{-\infty}(M, \mathcal{E}_2, \mathcal{E}_2)^G$. Define

$$L_\alpha = \begin{bmatrix} S_\alpha^0 & -(I + S_\alpha^0)Q_\alpha \\ P_\alpha & S_\alpha^1 \end{bmatrix} \in \Psi_c^0(M, \mathcal{E}_1 \oplus \mathcal{E}_2)^G.$$

We have

$$L_\alpha^{-1} = \begin{bmatrix} S_\alpha^0 & (I+S_\alpha^0)Q_\alpha \\ S_\alpha^1 & -P_\alpha \end{bmatrix}.$$

Let $U_\alpha = L_\alpha(I_{\mathcal{E}_1}\oplus 0)L_\alpha^{-1}, R_\alpha = U_\alpha - (0\oplus I_{\mathcal{E}_2})$ in $\Psi_c^{-\infty}(M,\mathcal{E}_1\oplus\mathcal{E}_2)^G$. Denote by K_α the kernel of R_α. For $\varphi\in C^{2q}_{\alpha,\infty}(M)^G, \delta\varphi = 0$, we define

$$\begin{aligned} Ind_\varphi(\alpha) &= \tau_\varphi(\overbrace{R_\alpha,\ldots,R_\alpha}^{2q+1}) \\ &= (-1)^{2q}\int_{G^{2q+1}} Tr(f(x_0)K_\alpha(x_0,x_1)\ldots K_\alpha(x_{2q},x_0))\varphi(x_0,\ldots,x_{2q})dx^{2q+1}. \end{aligned}$$

Proposition 7.1 *The map* $Ind : K_H(V)\otimes H^{2q}(\bar{C}^*_{\alpha,\infty}(M)^G)\to\mathbb{R}$ *defined by* $Ind([\alpha],[\varphi]) = Ind_\varphi(\alpha)$ *is a homomorphism of the variable* $[\alpha]$.

Proof. The proof is divided into several steps.

(1) $Ind_\varphi(\alpha)$ is independent of the choice of L_α and of the representative of φ in $[\varphi]$. The proof is exactly the same as that given in section 6.2.

(2) If $\alpha_i\in C^\infty(S(V), Iso(E_i,F_i))^H$ are isomorphic, i=0,1, then $Ind_\varphi(\alpha_0) = Ind_\varphi(\alpha_2)$. In fact, using the notations in the previous paragraph, we have $\psi_1\sigma_0(P_{\alpha_0}) = \sigma_0(P_{\alpha_1})\psi_0$ and then $\sigma_0(P_{\alpha_0}) = \psi_1^{-1}\sigma_0(P_{\alpha_1})\psi_0$. This implies also $\sigma_0(Q_{\alpha_0}) = \sigma_0(P_{\alpha_0})^{-1} = \psi_0^{-1}\sigma_0(Q_{\alpha_1})\psi_1$

$$\begin{aligned} S^0_{\alpha_0} &= I - Q_{\alpha_0}P_{\alpha_0} = I - \psi_0^{-1}Q_{\alpha_1}P_{\alpha_1}\psi_0 = \psi_0^{-1}S^0_{\alpha_1}\psi_0, \\ S^1_{\alpha_0} &= I - P_{\alpha_0}Q_{\alpha_0} = I - \psi_1^{-1}P_{\alpha_1}Q_{\alpha_1}\psi_1 = \psi_1^{-1}S^1_{\alpha_1}\psi_1, \end{aligned}$$

and

$$R_{\alpha_0} = \begin{bmatrix} (S^0_{\alpha_0})^2 & S^0_{\alpha_0}(I+S^0_{\alpha_0})Q_{\alpha_0} \\ S^1_{\alpha_0}P_{\alpha_0} & -(S^1_{\alpha_0})^2 \end{bmatrix}$$

$$= \begin{bmatrix} \psi_0^{-1} & 0 \\ 0 & \psi_1^{-1} \end{bmatrix} \begin{bmatrix} (S_{\alpha_1}^0)^2 & S_{\alpha_1}^0(I + S_{\alpha_1}^0)Q_{\alpha_1} \\ S_{\alpha_1}^1 P_{\alpha_1} & -(S_{\alpha_1}^1)^2 \end{bmatrix} \begin{bmatrix} \psi_0 & 0 \\ 0 & \psi_1 \end{bmatrix}$$

$$= \begin{bmatrix} \psi_0^{-1} & 0 \\ 0 & \psi_1^{-1} \end{bmatrix} R_{\alpha_1} \begin{bmatrix} \psi_0 & 0 \\ 0 & \psi_1 \end{bmatrix}.$$

This proves $Ind_\varphi(\alpha_0) = Ind_\varphi(\alpha_1)$.

(3) If $\alpha(\xi) \equiv \psi \in Iso(E_1, E_2)^H, \xi \in S(V)$, then $(P_\alpha u)(x) = \psi(u(x))$. Then we can choose $S_\alpha^0 = 0 = S_\alpha^1$ and $R_\alpha = 0$, which implies $Ind_\varphi(\alpha) = 0$.

(4) If $\alpha_i \in C^\infty(S(V), Iso(E_i, F_i))^H$ are homotopic, $i = 0, 1$, then $Ind_\varphi(\alpha_0) = Ind_\varphi(\alpha_1)$. To prove this claim, let $\alpha \in C^\infty(S(V) \times [0,1], Iso(E, F))^H$ be a C^1–piecewise homotopy connecting α_0 and α_1. Denote $\alpha_s = \alpha|_{S(V)\times\{s\}}$, $s \in [0,1]$. Note that M is paracompact. We have $E \simeq (E|_{S(V)\times\{0\}}) \times [0,1]$ and $F \simeq (F|_{S(V)\times\{0\}}) \times [0,1]$. We can thus consider $\alpha_s \in C_H(M, \mathcal{E}_0, \mathcal{F}_0)$. Let U_{α_s} be the corresponding idempotents and $R_{\alpha_s} = U_{\alpha_s} - (0 \oplus I)$. By Lemma 6.3,

$$Ind_\varphi(R_{\alpha_1}, \ldots, R_{\alpha_1}) - Ind_\varphi(R_{\alpha_0}, \ldots, R_{\alpha_0}) = (2q+1)\int_0^1 \tau_{\delta\varphi}(T_s, R_{\alpha_s}, \ldots, R_{\alpha_s})ds = 0,$$

since $\delta\varphi$ is locally zero and R_{α_s} is zero near the diagonal.

Therefore, $Ind_{[\varphi]}([\alpha])$ is well defined. Obviously, $Ind_{[\varphi]}([\alpha_1] + [\alpha_2]) = Ind_{[\varphi]}([\alpha_1]) + Ind_{[\varphi]}([\alpha_2])$. Q.E.D.

$Ind_{[\varphi]}([\alpha])$ is called the higher equivariant analytic index of operator $P_\alpha \in \Psi_c^n(M, \mathcal{E}_1, \mathcal{E}_2)^G$ with $\sigma_0(P_\alpha) = \alpha$.

7.2 Higher Equivariant Index Theorem

In this section we will obtain a higher equivariant index theorem which expresses the higher equivariant analytic index of G-elliptic pseudo- differential operators in terms of topological information.

Recall that the usual proof of the general Atiyah-Singer index theorem on compact Riemannian manifolds consists of several steps. First one checks the index formula for the signature operators on even dimensional manifolds. Second using the fact that the K-theory group of the unit sphere of the cotangent bundle is generated by the signature elements modulo the 2-torsion elements and the K-theory of the manifolds, one can deduce the index theorem for general elliptic pseudo-differential operators on even dimensional oriented manifolds. Then the case of nonoriented manifolds can be obtained by the lifting to the double covering spaces. One reduces the case of odd dimensional manifolds to the even dimensional ones by crossing the manifolds with S^1 and modifying the operators appropriately. For the homogeneous space $M = G/H$ the second step is invalid, since $K_H(V)$ is not in general a module over $R(H)$. To deal with this difficulty together with non-spinor case, we need further detailed information about $K_H(V)$.

Since $K_H(V) = 0$ for odd dimensional M and a connected group H, we consider only the case of even dimensional $M = G/H$. To describe $K_H(V)$ in terms of the representation ring $R(H)$ of H, we choose a double covering $\tilde{H}$ of H consisting of all elements $(h, s) \in H \times Spin(V)$ such that $Ad^*(h)$ coincides with orthogonal transformation of V defined by $s \in Spin(V)$, where $Spin(V)$ is the Spinor group of V. Then there are two representations of

$\tilde{H}$ on V, namely, $(h,s) \in \tilde{H} \to Ad^*(h) \in SO(V)$ and $s \in Spin(V)$. As shown in [CoM 1], $K_{\tilde{H}}(V) = K_{\tilde{H}}(V)^0 \oplus K_{\tilde{H}}(V)^1$, $K_{\tilde{H}}(V)^0 = K_H(V)$ and $R(\tilde{H}) = R(\tilde{H})^0 \oplus R(\tilde{H})^1$, the $\pm$ eigenspaces of the irreducible representation ε such that $\varepsilon(u) = (-1)^i I$ for the generator u of the kernel of the covering map $Spin(V) \to So(V), i = 0, 1$. Every element $x \in K_H(V)$ is the form $x = \alpha\tilde{\beta}$ with $\alpha \in R(\tilde{H})^1$ and $\tilde{\beta}$ the Bott class in $K_{\tilde{H}}(V)$ determined by the Dirac complex.

Now for $x = \alpha\tilde{\beta} \in K_H(V)$ with $\alpha \in R(\tilde{H})^1$ we can also use α to denote the finite dimensional unitary representation $\alpha : \tilde{H} \to End(E)$. Let $S^{\pm}$ be the half-spin representations of $Spin(V)$. Then $E^{\pm} = E \otimes S^{\pm}$ are the unitary H-modules, hence induce homogeneous vector bundles $\mathcal{E}^{\pm}$ over M. Let

$$\nabla^{\pm}_{\alpha} : C^{\infty}(M, \mathcal{E}^{\pm}) \to C^{\infty}(M, T^*_c M \otimes \mathcal{E}^{\pm})$$

be G-invariant connections which are compatible with the Hermitian structures of $\mathcal{E}^{\pm}$. Then the Dirac operators $D^{\pm}_{\alpha}$ with coefficients in $\mathcal{E}^{\pm}$ are the composition

$$D^{\pm}_{\alpha} : C^{\infty}_c(M, \mathcal{E}^{\pm}) \overset{\nabla^{\pm}_{\alpha}}{\to} C^{\infty}_c(M, T^*_c M \otimes \mathcal{E}^{\pm}) \overset{c}{\to} C^{\infty}_c(M, \mathcal{E}^{\pm}),$$

where c is the homomorphism induced by the Clifford multiplication. To derive the higher equivariant index theorem, we need also the $\hat{A}$-genus and the Chern character. Denote by $H^*(\mathcal{G}, H, \mathbb{R})$ the relative Lie algebraic cohomology of the complex

$$C^q(\mathcal{G}, H, \mathbb{R}) = \{\varphi \in \wedge^q \mathcal{G}^* : i_x\varphi = 0 \; for \; x \in \mathcal{H}, Ad^*(h)\varphi = \varphi, h \in H\}$$

with the differential $d : C^q(\mathcal{G}, H, \mathbb{R}) \to C^{q+1}(\mathcal{G}, H, \mathbb{R})$ given by

$$d\varphi(x_1, \ldots, x_{q+1}) = \frac{1}{q+1} \sum_{1 \le i < j \le q+1} (-1)^{i+j+1} \varphi([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}).$$

Note that $H^*(\mathcal{G},H,\mathbb{R}) = H^*(\mathcal{G},\mathcal{H},\mathbb{R})^{H/H_0}$, where H_0 is the component of the identity in H. We now choose an $Ad(H)$-invariant spliting of $\mathcal{G}$, $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$. This amounts to fixing a G-invariant connection on the principle bundle $H \to G \to M$ given by the projection $\theta_1 : \mathcal{G} \to \mathcal{H}$ parallel to $\mathcal{P}$ with the curvature form

$$\Theta(x,y) = -\frac{1}{2}\theta_1([x,y]), \quad x,y \in \mathcal{P}.$$

We can also associate a representation $\alpha : \mathcal{H} \to \mathcal{G}L(E)$ with a unitary representation α of $\tilde{H}$ on E. This produces an element $\Theta_\alpha \in \wedge^2\mathcal{P}^* \otimes \mathcal{G}L(E)$ given by

$$\Theta_\alpha(x,y) = \frac{1}{2\pi i}\alpha(\Theta(x,y)), \quad x,y \in \mathcal{P}.$$

Clearly, Θ_α satisfies $\Theta_\alpha(Ad(h)x, Ad(h)y) = \alpha(\tilde{h})\Theta_\alpha(x,y)\alpha(\tilde{h})^{-1}, x,y \in \mathcal{P}$ and $h \in H$, where $\tilde{h}$ is the lifting of h to $\tilde{H}$. Thus $Tr(exp\Theta_\alpha) \in \wedge\mathcal{P}^*$ is H-invariant which implies that

$$(1-\Theta)^*(Trexp\Theta_\alpha) \in \sum_q C^q(\mathcal{G},H,\mathbb{R}) \subset \sum_q \wedge^q\mathcal{G}^*.$$

one can check that this form is closed, hence defines an element in $H^*(\mathcal{G},H,\mathbb{R})$ denoted by $Ch(\alpha)$. $Ch(\alpha)$ is well defined. If we replace the representation space E above by an H-module V and form $\Theta_V \in \wedge^2\mathcal{P}^* \otimes \mathcal{G}L_{\mathbb{C}}(V)$ as above, we get an element

$$\hat{A}(\mathcal{G},H) = det\frac{\Theta_V}{exp(\frac{1}{2}\Theta_V) - exp(-\frac{1}{2}\Theta_V)} = det\frac{\Theta_V/2}{\sinh(\Theta_V/2)}.$$

Using the above procedure for $Ch(\alpha)$, we also have that $\hat{A}$ defines an element in $H^*(\mathcal{G},H,\mathbb{R})$. This is the Hirzebruch $\hat{A}$-genenus for a homogeneous space M.

Similarly, we can define $\mathcal{L}(\mathcal{G}, H) \in H^*(\mathcal{G}, H, \mathbb{R})$ given by the form $det\frac{\frac{1}{2}\Theta_V}{\tan(\frac{1}{2}\Theta_V)}$ $\in \wedge \mathcal{P}^*$. With these notations we can state the higher equivariant index theorem.

Theorem 7.1 *let* $[\varphi] \in H^{2q}(\bar{C}^*_\alpha(M)^G)$ *be such that* $\rho([\varphi]) \in H^{2q}(\mathcal{G}, H, \mathbb{R})$. *Then for any G-elliptic pseudo-differential operator* $P \in \Psi^n_c(M, \mathcal{E})^G$,

$$Ind_{[\varphi]}(P) = \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!} < Ch(\sigma_0(P))\hat{A}(\mathcal{G}, H)\rho([\varphi]), [V] >, \qquad (7.1)$$

where $\sigma_0(P)$ *is the principal symbol of* P *and the right side of* (7.1) *is the scalar of the* m*-component of* $Ch(\sigma_0(P))\hat{A}(\mathcal{G}, H)\rho([\varphi])$ *in* $H^*(\mathcal{G}, H, \mathbb{R})$, *and* $dimM = m = 2n$.

Note that the right side of (7.1) makes sense since $H^m(\mathcal{G}, H, \mathbb{R}) = C^m(\mathcal{G}, H, \mathbb{R})$ $= \wedge^m V$ is one dimensional. Thus the m-component of $Ch(\sigma_0(P))\hat{A}(\mathcal{G}, H)\rho([\varphi])$ can be written as $< Ch(\sigma_0(P))\hat{A}(\mathcal{G}, H)\rho([\varphi]), [V] > \omega$ with a nonzero m-form ω in $\wedge^m V$. As we pointed out before, $[\sigma_0(P)]$ in $K_H(V)$ has the form $[\sigma_0(P)] = \alpha\tilde{\beta}$ with $\alpha \in R(\tilde{H})^1$ and $[\sigma_0(D^+_\alpha)] = \alpha$. Thus it suffices to prove Theorem 7.1 for the Dirac operator D^+_α associated with the unitary representation α.

Theorem 7.2 *Let* $[\varphi] \in H^{2q}(\bar{C}^*_\alpha(M)^G)$ *be such that* $\rho([\varphi]) \in H^{2q}(\mathcal{G}, H, \mathbb{R})$. *Then for* $\alpha \in R(\tilde{H})^1$,

$$Ind_{[\varphi]}(D^+_\alpha) = \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!} < Ch(\sigma_0(P))\hat{A}(\mathcal{G}, H)\rho([\varphi]), [V] > .$$

Proof. The idea of the proof is to reduce this index problem to the one studied in [CoM 1]. The proof is broken up into two steps.

Step 1. Since $Ind_{[\varphi]}(D_\alpha^+$ is defined by the integral of G-compact supported kernel functions and a cut function, we can assume that G is compact and then $M = G/H$ is compact.

Using the notations in Chapter 6, we get by Lemma 6.2

$$Ind_{[\varphi]}(D_\alpha^+) = \lim_{t\to 0} \tau_\varphi(W_1(tD_\alpha^+), \dots, W_1(tD_\alpha^+)).$$

This is the McKean-Singer formula for homogeneous space M (cf. [CoM 2]). Furthermore, if we write $D = D_\alpha = \begin{bmatrix} 0 & D_\alpha^- \\ D_\alpha^+ & 0 \end{bmatrix}$, $\nu = I \oplus (-I)$ and $D_\alpha^* = -D_\alpha$, then

$$Ind_{[\varphi]}(D^+) = \lim_{t\to 0} \tau_\varphi(W_2(tD), \dots, W_2(tD)),$$

where $W_2(tD) = (e^{t^2D^2} + e^{\frac{1}{2}t^2D^2}W(-t^2D^2)tD)\nu$, and $W(x) = (\frac{1-e^{-x}}{x})^{\frac{1}{2}}$ for $x > 0$, and 1 for $x = 0$.

Let $\Omega^q(M)$ be the space generated by elements $\varphi_0\partial\varphi_1 \otimes \dots \otimes \partial\varphi_q$ for $\varphi_i \in C^\infty(M)$. Then in the Fréchet topology $C^q(M)$ is identified with $\Omega^q(M)$ via the map $\varphi_0 \otimes \dots \varphi_q \to \varphi_0\partial\varphi_1 \otimes \dots \partial\varphi_q$. With this identification, we can write φ as

$$\varphi = \sum_{i=1}^{\infty} \varphi_0^{(i)}\partial\varphi_1^{(i)} \otimes \dots \otimes \partial\varphi_{2q}^{(i)} \in \Omega^{2q}(M).$$

Let $\psi^{(i)} = \varphi_0^{(i)}\partial\varphi_1^{(i)}\otimes\dots\otimes\partial\varphi_{2q}^{(i)}$. It suffices thus to consider $\tau_{\psi^{(i)}}(W_2(tD),\dots,W_2(tD))$. We omit the upper index (i) in $\psi^{(i)}$.

$$\begin{aligned}
&\tau_\psi(W_2(tD), \dots, W_2(tD)) \\
&= (-1)^{2q}\int_{M^{2q+1}} Tr(W_2(tD)(x_0,x_1)\dots W_2(tD)(x_{2q},x_0))f\varphi_0\partial\varphi_1 \otimes\dots\otimes \partial\varphi_{2q}dx^{2q+1} \\
&= Tr(W_2(tD)f\varphi_0[W_2(tD),\varphi_1]\dots[W_2(tD),\varphi_{2q}]) \\
&= Tr_s((e^{t^2D^2} - e^{-\frac{1}{2}t^2D^2}W(-t^2D^2)tD)f\varphi_0[W_2(tD),\varphi_1]\dots[W_2(tD),\varphi_{2q}]). \quad (7.2)
\end{aligned}$$

Here we used the facts that $W_2(tD)(xh, yg) = W_2(tD)(x, y), f(xh) = f(x)$ and $\varphi_i(xh) = \varphi_i(x), \forall h \in H, i = 0, 1, \ldots, 2q$. Tr_s means the super trace $Tr \cdot \nu$.

Step 2. We see (7.2) is similar to the higher analytic index in [CoM 2] except for the factor f. We can therefore reduce the computation of (7.2) to the one calculated in [CoM 2]. This means that we can use the Getzler symbolic calculation to compute the limit of the right side of (7.2) as $t \to 0$ as long as we replace T^*M and $\hat{A}(R)$ in the calculation of [CoM 2] by V and $\hat{A}(\mathcal{G}, H)$. Denote by $\prod(t)$ the operator under the super trace in the right side of (7.2). The computation in [CoM 2] can be applied here to get (up to an isomorphism between T_0^*M and V)

$$\begin{aligned}
\tau_\psi(D^+) &= \lim_{t\to 0}(\prod(t)) \\
&= \lim_{t\to 0}\frac{1}{(2\pi)^{2n}}\int_{T^*M} Tr_s(\sigma_{t^{-1}}(\prod(t))(x,\xi))dxd\xi \\
&= \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!}\int_M \hat{A}(\mathcal{G},H)Ch(\alpha)f\varphi_0 d\varphi_1\wedge\ldots\wedge d\varphi_{2q} \\
&= \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!}\int_M f < \hat{A}(\mathcal{G},H)Ch(\alpha)\varphi_0 d\varphi_1\wedge\ldots\wedge d\varphi_{2q},[V] > \\
&= \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!} < \hat{A}(\mathcal{G},H)Ch(\alpha)\varphi_0 d\varphi_1\wedge\ldots\wedge d\varphi_{2q},[V] > \int_M f\omega(M) \\
&= \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!} < \hat{A}(\mathcal{G},H)Ch(\alpha)\varphi_0 d\varphi_1\wedge\ldots\wedge d\varphi_{2q},[V] >,
\end{aligned}$$

since $\int_M f\omega(M) = 1$ for the volume $\omega(M)$ on M. Q.E.D.

In particular, we have the higher index theorem for the signature operator.

Corollary 7.1 *Let $D_\alpha = d + d^*$ on $C_c^\infty(M, \wedge T_c^*M \otimes \mathcal{E})$ be the operator associated with a finite dimensional unitary representation $\alpha : H \to End(E)$ and $D_\alpha = D_\alpha^+ \oplus D_\alpha^-$ be the decomposition corresponding to the ± 1-eigenspaces of the* *operator* $\tau(\xi) = (-1)^{\partial\xi(\partial\xi-1)+n} * (\xi)$ *on the forms. Then for* $[\varphi] \in H^{2q}(\bar{C}^*_\alpha(M)^G)$ *with* $\rho([\varphi]) \in H^{2q}(\mathcal{G}, H, \mathbb{R})$,

$$Ind_{[\varphi]}(D^+_\alpha) = \frac{(-1)^n}{(2\pi i)^q}\frac{q!}{(2q)!} < Ch(\alpha)\mathcal{L}(\mathcal{G}, H)\rho([\varphi]), [V] > .$$

Here $dimM = m = 2n$. As we noted before, $\rho([\varphi])$ is always in $H^{2q}(\mathcal{G}, H, \mathbb{R}) = H^{2q}(\mathcal{G}, \mathcal{H}, \mathbb{R})$ if H is connected.

Chapter 8

A Survey on the Novikov Conjecture

In this chapter we will give a survey of the Novikov conjecture. The goal of this chapter is twofold. First, we give an introduction to the conjecture and review the progress made so far on the conjecture. Second, according to the Gromov principle: any statement claimed to be true for all discrete groups is trivial, hence the Novikov conjecture is unlikely to be valid ([BaC 3], [Kas 2]). To construct any counter-example for the conjecture, it is necessary to know the groups that satisfy the conjecture . We thus present in Section 8.1 several equivalent forms of the original conjecture and of the (equivariant) strong Novikov conjecture. We also include the Novikov conjecture for foliations and the Cohen-Jones conjecture. In Section 8.2 we consider several kinds of discrete groups satisfying the Novikov conjecture. We also point out some discrete groups for which the Novikov conjecture is unknown. This group picture of the conjecture will be used in Section 8.3 to obtain a manifold picture for the conjecture, i.e., to obtain those manifolds whose fundamental groups satisfy the conjecture. In particular, the Novikov conjecture for the fundamental groups of compact 3-dimensional manifolds is related to the Thurston geometrization conjecture. This chapter can be considered as a complement to Weinberger's nice survey on the Novikov conjecture [Wein 3].

8.1 Several Conjectures

(A) The Novikov Conjecture (NC)

Let M be a compact oriented Riemannian manifold of dimension n. Recall that the signature $Sign(M)$ of M is defined to be zero if $n \neq 4k$ and $\rho^+ - \rho^-$ if $n = 4k$, where ρ^+ (resp. ρ^-) is the number of positive (resp. negative) eigenvalues of symmetric bilinear form $(x, y) =< x \cdot y, [M] >$ on $H^{2k}(M)$. As usual, $x \cdot y$ means the cup product of x and y, $[M] \in H_{4k}(M)$ is the fundamental class and $<, >$ is the pairing of $H^{4k}(M)$ with $H_{4k}(M)$. The well-known Hirzebruch index theorem [Hir] says that $Sign(M^{4k})$ can be written as

$$Sign(M^{4k}) =< L_k(M^{4k}), [M] >, \tag{8.1}$$

where $L_k(M^{4k}) \in H^{4k}(M^{4k}, \mathbb{Q})$ is the Hirzebruch L_k- class which is the universal polynomial of rational Pontrjagin classes P_i of M. An important consequence of (8.1) is that $L_k(M^{4k})$ is a topological homotopy invariant. Novikov [Nov 1] extended this to the case where $dim M = 4k + 1$ and conjectured that $< L_k(M^{4k})\varphi_1 \dots \varphi_l, [M] >$ is a homotopy invariant for any $\{\varphi_i\} \in H^1(M, \mathbb{Z})$ and $dim M = 4k + l$. This conjecture was proved by Rohlin [Roh] for $l = 2$ and independently by Farrell and Hsiang (cf. [Hsi]) and Kasparov [Kas 1] for arbitrary l. Novikov then proposed the following well-known conjecture:

The Novikov Conjecture (shortly, NC): Let M be a compact oriented Riemannian manifold, Γ a finitely generated discrete group and $B\Gamma$ its classifying space. If $f : M \rightarrow B\Gamma$ is a continuous map, then for any $\varphi \in H^*(B\Gamma, \mathbb{Q})$ $< L(M)f^*(\varphi), [M] >$ is a homotopy invariant. This means that if N is another compact oriented Riemannian manifold and $h : N \rightarrow M$ is a homotopy equivalence, then

$$< L(M)f^*(\varphi), [M] >=< L(N)(fh)^*(\varphi), [N] >, \tag{8.2}$$

where $L(M)$ is the total Hirzebruch L-class, $L(M) = \prod \frac{x_i}{\tanh x_i}$ with characteristic classes x_i of M.

This conjecture is one of several currently active research problems in topology and functional analysis. To express the NC in terms of homology and K-theory, let us introduce the signature element $[D]$ in K-theory $K_j(M)$ of M for $dimM = 2k + j, j = 0, 1$. Let d be the exterior differential of p-forms $\Omega^p(M)$ on M and d^* the formal adjoint of d with respect to the inner product in $\Omega^p(M)$ induced by the Riemannian metric on M. Using the Hodge operator $* : \Omega^p(M) \rightarrow \Omega^{n-p}(M), 0 \leq p \leq n$, we can define an involution τ on $\Omega^p(M)$ for $n = 2k$ by

$$\tau(\varphi) = i^{p(p-1)+\frac{n}{2}} * \varphi.$$

Then $D = d + d^* : \Omega^*(M) \rightarrow \Omega^*(M)$, and $\tau D = -D\tau$ for n=2k. D defines an element $[D] \in K_j(M)$ by modules $(L^2(\wedge_{\mathbb{C}} T^*M), D(1 + D^2)^{-\frac{1}{2}})$ for $n = 2k$. Also for $n = 2n + 1$ D defines an element $[D] \in K_1(M)$ by considering $K_0(M \times S^1) \simeq K_0(M) \oplus K_1(M)$ which follows from the six-term exact sequence. Now if Q denotes the Poincare duality, $Q(L(M)) = L(M) \cap [M]$, then for

$\varphi \in H^*(B\Gamma, \mathbb{Q})$,

$$<\varphi, f_*(Q(L(M)))>=<f^*(\varphi), Q(L(M))>=<L(M)f^*(\varphi), [M]>$$

and

$$<\varphi, (fh)_*(Q(L(N)))>=<(fh)^*(\varphi), Q(L(N))>=<L(N)(fh)^*(\varphi), [N]> .$$

Thus (8.2) is equivalent to

$$f_*(Q(L(M))) = (fh)_*(Q(L(N))) \in H_*(B\Gamma, \mathbb{Q}). \tag{8.3}$$

This is the homology form of the Novikov conjecture.

Let $\mathcal{L}(M)$ be the modified Hirzebruch L-class which differs from $L(M)$ by a constant c [AtS]. We have $Ch([D]) = c\mathcal{L}(M) \cap [M] = Q(L(M))$. (8.3) is then equivalent to

$$\begin{aligned} Ch(f_*([D_M])) &= f_*(Ch([D_M])) = (fh)_*(Ch([D_N])) \\ &= Ch((fh)_*([D_N])) \in H_*(B\Gamma, \mathbb{Q}). \end{aligned}$$

Since the Chern character $Ch : K_*(M) \otimes \mathbb{Q} \to H_*(M, \mathbb{Q})$ is isomorphism, we obtain that (8.3) is equivalent to

$$f_*([D_M]) = (fh)_*([D_N]) \in K_*(B\Gamma) \otimes \mathbb{Q}, \tag{8.4}$$

where $K_*(B\Gamma) = \lim_{\rightarrow} K_*(Y)$ with the limit taking over all finite CW-subcomplexes $Y \subset B\Gamma$. (8.4) is the K-theory form of the Novikov conjecture.

The significance of the NC is not only its application in topology ([FaJ 1], [Wein 1]), but also the interesting methods of proving it. Successful applications of KK-theory and cyclic cohomology to the NC provide such examples

([Kas 3], [CoM 2]). One can see that the phenomenon of the NC spreads out currently over several research areas such as higher index theory, higher Whitehead torsion and higher η-invariant and so on.

(B)Strong Novikov Conjectures (SNC)

We now come up with some conjectures which imply the Novikov conjecture. To define these conjectures we first recall the KK-theory group $RKK^G(X;\mathcal{A},\mathcal{B}) = \mathcal{R}KK^G(X;\mathcal{A}(X),\mathcal{B}(X))$ of $G-C(X)$ -algebras $\mathcal{A}$ and $\mathcal{B}$ for locally compact group G and G-space X, which is defined to be the quotient of usual Kasparov $G-\mathcal{A}(X)-\mathcal{B}(X)$-bimodules (E,T) with an additional condition that $(fa)cb = ac(fb)$ for $a \in \mathcal{A}(X), b \in \mathcal{B}(X), f \in C(X)$ and $c \in E$ by the usual equivalences [Kas 3]. Here $\mathcal{A}(X) = C_0(X,\mathcal{A})$. In particular, we define

$$RK_G^*(X) = RKK^G(X;\mathbb{C}\otimes C^{*,0},\mathbb{C}) \simeq RKK^G(X;\mathbb{C},\mathbb{C}\otimes C^{0,*}),$$

and

$$RK_*^G(X) = \varinjlim K_*^G(Y) = lim_{\rightarrow} KK_*^G(C(Y),\mathbb{C}) = \varinjlim KK^G(C(Y)\otimes C_{*,0},\mathbb{C}),$$

where the inductive limit is taken over all compact G-subspaces $Y \subset X$ and $C_{p,q}$ is the Clifford algebra of $\mathbb{C}^{p+q}$ with quadratic form $Q_p(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ and $C^{p,q} = C_{q,p}$.

let H be another locally compact group and $X, \mathcal{A}$ and $\mathcal{B}$ have the commutative $H-G$ actions. There is an induction map $i^{H,X,G}$ from $RKK^H(Y;\mathcal{A},\mathcal{B})$ to $\mathcal{R}KK^{G\times H}(Y\times_H X;\mathcal{A}(X)^H,\mathcal{B}(X)^H)$ defined by

$$\begin{aligned} RKK^H(Y;\mathcal{A},\mathcal{B}) &\overset{\sigma_{C(X)}}{\rightarrow} \mathcal{R}KK^{G\times H}(Y\times X;\mathcal{A}(X),\mathcal{B}(X)) \\ &\overset{\lambda^H}{\rightarrow} \mathcal{R}KK^G(Y\times_H X;\mathcal{A}(X)^H,\mathcal{B}(X)^H), \end{aligned}$$

where in terms of bimodules $\sigma_{C(X)}((E,T)) = (E\otimes C(X), T\otimes 1)$ and $\lambda^H((E,T)) = (E^H, T^H), T^H = \int_H h(\tilde{c}T)dh$, and E^H denotes the set of the fixed points of H. Here $\tilde{c}$ is a nonnegative continuous function on X such that $\int_H \tilde{c}(h^{-1}x)dh = 1$ for $x \in X$ and $Supp(\tilde{c}) \cap HY$ is compact for compact $Y \subset X$.

Now for a countable discrete group Γ let $C^*(\Gamma)$ be the group algebra of Γ which is the completion of $l^1(\Gamma)$ with respect to the greatest norm. Let $I_\Gamma \in K_0^\Gamma(C^*(\Gamma))$ be defined by Γ-equivariant $\mathbb{C} - C^*(\Gamma)$-bimodule $(\mathcal{C}^*(\Gamma), 0)$, where $\mathcal{C}^*(\Gamma)$ is equal to $C^*(\Gamma)$ as a Hilbert module over itself with Γ-action given by the product of $C^*(\Gamma)$, $g(a) = g \cdot a, g \in \Gamma$ and $a \in C^*(\Gamma)$, since $g \in \Gamma$ can be considered as an element in $C^*(\Gamma)$. We also consider $C^*(\Gamma)$ with trivial action. Let $[\beta_\Gamma] = i^{\Gamma, E\Gamma, \{1\}}(I_\Gamma) \in RKK(B\Gamma; \mathbb{C}, C^*(\Gamma))$,

$$\begin{aligned}
I_\Gamma \in K_0^\Gamma(C^*(\Gamma)) \quad &= \quad \mathcal{R}KK^\Gamma(\cdot; \mathbb{C}, C^*(\Gamma)) \\
&\overset{i^{\Gamma,E\Gamma,\{1\}}}{\to} \quad [\beta_\Gamma] \in \mathcal{R}KK(\cdot \times_\Gamma E\Gamma; \mathbb{C}(E\Gamma)^\Gamma, C^*(\Gamma)(E\Gamma)^\Gamma) \\
&= \quad \mathcal{R}KK(B\Gamma; \mathbb{C}(B\Gamma), C^*(\Gamma)(B\Gamma)) \\
&= \quad RKK(B\Gamma; \mathbb{C}, C^*(\Gamma)).
\end{aligned}$$

Here $E\Gamma$ is the universal covering space of $B\Gamma$. For compact subspace $Y \subset B\Gamma$ let β_Y^r be the restriction of $[\beta_\Gamma]$ to $K_0(C(Y) \otimes C^*(\Gamma)) \simeq RKK(\cdot; \mathbb{C}, C(Y) \otimes C^*(\Gamma))$. β_Y^r defines a homomorphism $\beta_Y : K_*(Y) \to K_*(C^*(\Gamma))$ by the Kasparov product. The inductive limit β of $\beta_Y : RK_*(B\Gamma) \to K_*(C^*(\Gamma))$ is then dual to the homomorphism $\alpha : K^*(C^*(\Gamma)) \to RK^*(B\Gamma)$ which is defined by the cap product with $[\beta_\Gamma]$. This means that for any compact subspace $Y \overset{i}{\subset} B\Gamma$,

$$\tilde{\alpha}(x) \otimes_{C(Y)} y = x \otimes_{C^*(\Gamma)} \beta(y) \in \mathbb{C}, \forall x \in K^*(C^*(\Gamma)), y \in K_*(Y).$$

Here $\tilde{\alpha} : K^*(C^*(\Gamma)) \xrightarrow{\alpha} RK^*(B\Gamma) \xrightarrow{i^*} K^*(Y)$.

We can now state the strong Novikov conjectures as follows ([Ros], [Kas 3], [KaS]).

$\underline{SNC_{\beta\otimes\mathbb{Q}}}$: $\beta\otimes\mathbb{Q} : RK_*(B\Gamma)\otimes\mathbb{Q} \to K_*(C^*(\Gamma))\otimes\mathbb{Q}$ is injective.

$\underline{SNC_{\beta}}$: $\beta : RK_*(B\Gamma) \to K_*(C^*(\Gamma))$ is split injective.

$\underline{SNC_{\beta}^{isom}}$: $\beta : RK_*(B\Gamma) \to K_*(C^*(\Gamma))$ is an isomorphism.

$\underline{SNC_{\alpha}}$: $\alpha : K^*(C^*(\Gamma)) \to RK^*(B\Gamma)$ is split surjective.

$\underline{SNC_{\alpha}^{\lim}}$: for any compact subset $Y \overset{i}{\hookrightarrow} B\Gamma$,

$$Im(RK^*(B\Gamma) \xrightarrow{i^*} K^*(Y)) = Im(i^*\alpha : K^*(C^*(\Gamma)) \to K^*(Y)).$$

$\underline{SNC_{\alpha}^{coef}}$: for any separable C^*-algebra $\mathcal{A}$ with trivial Γ-action, the homomorphism $\alpha_{\mathcal{A}} : KK_{\Gamma}^*(\mathbb{C},\mathcal{A}) \to RKK^*(B\Gamma;\mathbb{C},\mathcal{A})$ defined by $KK_{\Gamma}^*(\mathbb{C},\mathcal{A}) \xrightarrow{P^*} RKK_{\Gamma}^*(E\Gamma;\mathbb{C},\mathcal{A}) \simeq RKK^*(B\Gamma;\mathbb{C},\mathcal{A})$ is surjective.

$\underline{SNC_{\alpha}^{\lim,\mathbb{Q}}}$: for any C^*-algebra $\mathcal{A}$ with $K_0(\mathcal{A}) = \mathbb{Q}$ and $K_1(\mathcal{A}) = 0$, α has dense range.

In the SNC_{α}^{coef} P is a map from $E\Gamma$ to a point. There are the following links among these strong Novikov conjectures and the NC:

$$\begin{array}{ccccccc}
 & & SNC_{\alpha}^{\lim} & \overset{?}{\searrow} & NC & & \\
 & & & & \Uparrow & & \\
SNC_{\beta}^{isom} & \Longrightarrow & SNC_{\beta} & \Longrightarrow & SNC_{\beta\otimes\mathbb{Q}} & \overset{?}{\Longleftarrow} & SNC_{\alpha} \\
 & & \Uparrow & & \Uparrow & & \\
 & & SNC_{\alpha}^{coef} & & SNC_{\alpha}^{\lim,\mathbb{Q}} & &
\end{array}$$

Note that if $B\Gamma$ is compact, then the $SNC_{\alpha}^{\lim}$ (resp.SNC_{α}) implies the $SNC_{\beta\otimes\mathbb{Q}}$ due to the dual relation between α and β. But this is not true in general.

The implication $SNC_{\beta\otimes\mathbb{Q}} \Longrightarrow NC$ comes from the homotopy invariance of the Miscenko symmetric signature $\sigma(M) \in L_0(\mathbb{C}\Gamma)$ [Mis 1] and the relation $\beta([D_M]) = J_*(\sigma(M))$ for even dimensional manifold M, where $J_* : L_0(\mathbb{C}\Gamma) \simeq K_0(\mathbb{C}\Gamma) \to K_*(C^*(\Gamma))$ is induced by the inclusion $\mathbb{C}\Gamma \xrightarrow{J} C^*(\Gamma)$ and $L_0(\mathbb{C}\Gamma)$ is the Witt groups of equivalence classes of non-singular hermitian forms over involutive ring $\mathbb{C}\Gamma$.

One can also define $\beta : RK_*(B\Gamma) \to K_*(C_r^*(\Gamma))$ with $C^*(\Gamma)$ replaced by $C_r^*(\Gamma)$ above and make similar conjectures about β, where $C_r^*(\Gamma)$ is defined as the completion of $l^1(\Gamma)$ with respect to the norm $\|f\| = \|\lambda(f)\|$ for the left regular representation λ of Γ on $l^2(\Gamma)$. We should point out that all operator K-theory approaches to the NC are based on the Miscenko symmetric signature $\sigma(M)$ except [CGM 1] and [HiS]. But because of the difficulty of dealing with $\sigma(M)$ for the general equivariant case, we will adapt equivariant Hilsum-Skandalis$'$ technique to prove the equivariant Novikov conjecture.

(C) Strong Algebraic Novikov Conjectures (SANC)

We mention briefly some strong algebraic forms of the Novikov conjecture, since we will not be concerned with them in this dissertation.

Let Γ be a discrete group, $\Omega_*(\Gamma)$ and $L_*(\Gamma)$ its cobordism groups and L-groups. The pairing of $\Omega_*(\Gamma)$ with $L_*(\{1\})$ defines an Ω_*-module map $\psi : \Omega_*(\Gamma) \to L_*(\Gamma)$. ψ factors through $H_*(B\Gamma, \mathbb{Q}) \simeq \Omega_*(B\Gamma) \otimes_{\Omega_*} \mathbb{Q}$,

$$\begin{tikzcd} \Omega_*(\Gamma) \otimes_{\Omega_*} \mathbb{Q} \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & L_*(\Gamma) \otimes \mathbb{Q} \\ & H_*(B\Gamma, \mathbb{Q}) \arrow[ur, "8l_\Gamma"'] & \end{tikzcd}$$

to define a map $l_\Gamma : H_*(B\Gamma, \mathbb{Q}) \to L_*(\Gamma) \otimes \mathbb{Q}$ with homology graded by $\mathbb{Z}_4$.

Here φ is given by $\varphi([f]) = f_*([M] \cap L(M))$ for $f : M \to B\Gamma$ a cobordism representative in $\Omega_*(\Gamma)$. l_Γ is called the L-theory assembly map. Since the image of $\psi(M)$ in $L_n(\mathbb{Z}(\frac{1}{2}(\Gamma))$ is equal to $8\sigma(M)$, the injectivity of l_Γ implies the Novikov conjecture [Wall 1]. The following is the Wall L-theory version of the NC.

$\underline{SANC_{l_\Gamma}}$: $l_\Gamma : H_*(B\Gamma, \mathbb{Q}) \to L_*(\Gamma) \otimes \mathbb{Q}$ is injective.

See ([Hsi], [FaH 1,2], [Cap], [Wein 1,2]) for more details about this conjecture.

There are also natural assembly maps in algebraic K-theory $k_\Gamma : H_*(B\Gamma, \mathbb{Q}) \otimes K_*(\mathbb{Z}) \to K_*(\mathbb{Z}(\Gamma)) \otimes \mathbb{Q}$, and in the Waldhausen K-theory of spaces, $A_\Gamma : H_*(B\Gamma, \mathbb{Q}) \otimes \pi_*(A(pt)) \to \pi_*(A(B\Gamma)) \otimes \mathbb{Q}$ [Lod]. Here $A(X)$ is the Waldhausen algebraic K-theory of space X [Wal 2]. Cohen and Jones [CoJ] have proposed the following (Waldhausen) algebraic K-theory version of the Novikov conjecture:

$\underline{SANC_{k_\Gamma}}$: $k_\Gamma : H_*(B\Gamma, \mathbb{Q}) \otimes K_*(\mathbb{Z}) \to K_*(\mathbb{Z}(\Gamma)) \otimes \mathbb{Q}$ is injective.

$\underline{SANC_{A_\Gamma}}$: $A_\Gamma : H_*(B\Gamma, \mathbb{Q}) \otimes \pi_*(A(pt)) \to \pi_*(A(B\Gamma)) \otimes \mathbb{Q}$ is injective.

Since the rational homotopy groups $\pi_*(A(B\Gamma)) \otimes \mathbb{Q}$ is isomorphic to $K_*(\mathbb{Z}(\Gamma)) \otimes \mathbb{Q}$ and this isomorphism is compatible with the assembly maps, we see that the $SANC_{k_\Gamma}$ and $SANC_{A_\Gamma}$ are equivalent.

The important progress on the $SANC_{A_\Gamma}$ has been made by Cohen-Jones [CoJ] and Bökstedt-Hsiang-Madsen [BHM]. Cohen and Jones were partially motivated by the Chern character in cyclic homology to construct a character map $Ch_* : \pi_*(A(pt)) \otimes \mathbb{Q} \to \pi_*(Map(CP^\infty, \mathbb{Q}S^0)) \otimes \mathbb{Q}$, where $\mathbb{Q}S^0 = \lim_{\to} \Omega^n S^n$. They showed that the injectivity of Ch_* implies the $SANC_{A_\Gamma}$ and conjectured the following

<u>$SANC_{Ch}$</u>: $Ch_* : \pi_*(A(pt)) \otimes \mathbb{Q} \to \pi_*(Map(CP^\infty, QS^0)) \otimes \mathbb{Q}$ is injective.

The surprising point is that the group Γ is completely out of the $SANC_{Ch}$ due to some commutative diagrams. Thus unlike the operator K-theory approach to the NC which proceeds each time for only one class of groups, the $SANC_{A_\Gamma}$ could be verified for all discrete groups once the $SANC_{Ch}$ is proved. This seems to contradict the Gromov principle.

Bőkstedt, Hsiang and Madsen have proved the $SANC_{Ch}$ under two conditions. One of the conditions is about homology of the group Γ. We have heard that they might be able to remove this homology condition. This is certainly a very significant result. The question now is to form the operator K-theory version of the $SANC_{Ch}$ and prove it.

There are also other conjectures which imply the Novikov conjecture [FaJ 2].

(D) Equivariant Novikov Conjectures (ENC)

A generalization of the Novikov conjecture to the equivariant case is proposed by Rosenberg and Weinberger [RoW 2] which is stated as follows.

Let compact Lie group G act by isometries on closed oriented connected Riemannian manifolds M and N. As usual, we can define the signature element $[D_M] \in K_*(M)$. Since G does not act in general on the fundamental group $\pi_1(M)$, we use the fundamental groupoid $\pi(M)$ with a natural G-action. There is a "classifying space" $B\pi(M)$ with G-map $f_M : M \to B\pi(M)$ such that for any closed subgroup $H \subset G$ all components of the fixed points $(B\pi(M))^H$ are aspherical and f induces isomorphisms on π_0 of all fixed point sets of H and on π_1 of all components of the fixed point sets of H. f is unique up to

G-homotopy (see May's appendix to [RoW 2]).

Let $h : N \to M$ be an orientation-preserving G-pseudo-equivalence. Namely, h is G-equivariant and is homotopy equivalence as a usual map. Rosenberg and Weinberger conjectured the following:

<u>$ENC_{B\pi}1$</u>: If $K_*^G(B\pi(M))$ is finitely generated over $R(G)$, then

$$h_*(f_N)_*([D_N]) = (f_M)_*([D_M]) \in K_*^G(B\pi(M)), \tag{8.5}$$

where $h_* : K_*^G(B\pi(N)) \to K_*^G(B\pi(M))$ is induced by h.

<u>$ENC_{B\pi}2$</u>: Let $\mathcal{F}$ be some collection of prime ideals of $R(G)$ and the localization $K_*^G(B\pi(M))_{\mathcal{F}}$ be finitely generated over $R(G)_{\mathcal{F}}$. Then (8.5) holds after localizing.

<u>ENC_Y</u>: Let Y be a G-space such that $K_*^G(B\pi(Y))$ is finitely generated over $R(G)$. Then for a commutative diagram of G-maps,

$$\begin{tikzcd} M \arrow[dr, "\varphi"] & & \\ & Y \arrow[r, "f_Y"] & B\pi(Y), \\ N \arrow[uu, "h"] \arrow[ur, "\psi"'] & & \end{tikzcd}$$

$$(f_Y)_*(\varphi_*)_*([D_M]) = (f_Y)_*\psi_*([D_N]) \in K_*^G(B\pi(Y)).$$

There is also a localization version of the ENC_Y. Note that one can ask the equivariant Novikov conjecture for topological manifolds since the relation between smooth and topological equivariant cases is not clear.

Rosenberg and Weinberger have proved the ENC_Y for complete Riemannian manifold Y of nonpositive curvature [RoW 2]. They also verified the

ENC_Y for topological manifolds M and N and the above Y. We will prove in Chapter 10 the ENC_Y for the geometric realization X of an euclidean building without the condition on $K_*^G(Y)$.

If G acts on M trivially, one can then produce a pairing version of the equivariant Novikov conjecture [RoW 1]. The general pairing version of the equivariant Novikov conjecture was proposed at the end of Chapter 6. Baum and Connes recently proposed an equivariant version of the NC (see also [Ogle 1]) by using their universal equivariant classifying space $\tilde{E}\Gamma$. This version is similar to the ENC_Y for $Y = \tilde{E}\Gamma$.

(E) The Novikov Conjecture for Foliations (NCF)

The Novikov conjecture for foliations is similar to the ordinary one. Let (V, F) and (V', F') be two orientable C^∞-foliations with V and V' closed. Suppose $h : V' \to V$ is an orientation-preserving homotopy equivalence, i.e., there exists a leafwise map $h_1 : V \to V'$ such that $h \cdot h_1$ and $h_1 \cdot h$ are leafwise homotopic to I_V and $I_{V'}$, resp.. Let $B\pi(V)$ be the classifying space of the topological groupoid $\pi(V)$ of V and $f : V \to \pi(V)$ be the classifying map. Baum and Connes [BaC 2] asked the following:

$\underline{NCF1}$: $f_*(L(TV) \cap [V]) = h_* f_*(L(TV') \cap [V']) \in H_*(B\pi(V), \mathbb{Q})$.

Suppose $dim(V) = n, dim(F_x) = l, q = n - l$. Let Γ_q be the Haefliger groupoid of all germs of homeomorphisms of $\mathbb{R}^q$ and $f_q : V \to B\Gamma_q$ and $f'_q : V' \to B\Gamma_q$ be the Haefliger classifying maps.

$\underline{NCF2}$: let $\varphi \in H^*(B\Gamma_q; \mathbb{C})$. Then

$$< L(TV) f_q^*(\varphi), [V] > = < L(TV')(f'_q)^*(\varphi), [V'] > .$$

Clearly, the $NCF2$ is a special case of the $NCF1$. The $NCF1$ was proved by Baum and Connes [BaC 2] for nonpositive curved leaves. Recently, Hurder announced a proof of the $NCF1$ for leaves with hyperbolic fundamental groups. Note that if $F = TV$ and $F' = TV'$, then $B\pi(V)$ is homotopy equivalent to $B\pi_1(V)$. In this case the $NCF1$ reduces to the NC. There are also other Baum-Connes conjectures which imply the $NCF1$ [BaC 3]. We finally mention that John Roe proposed recently an exotic cohomology version of the Novikov conjecture [Roe] and Weinberger [Wein 3] generalized the NC to the stratified spaces and to the manifolds with boundary (see also [Lott 1]).

8.2 Group Picture of the Conjecture

We know that the $SNC_{\beta\otimes\mathbb{Q}}$ is true for the following kinds of discrete groups: (1) groups with properties of rapid decay (RD) and of polynomial cohomology (PC) [CoM 2]; (2) subgroups of finite component Lie groups ([Kas 3],[FaH 2]); (3) groups whose classifying spaces can be realized as complete Riemannian manifolds of nonpositive sectional curvature ([Mis 2],[Kas 3], [FaH 1],[Fac], [FeW]); (4) torsion free discrete groups acting properly on locally compact euclidean buildings [KaS]; (5) Cappell class of groups [Cap]. We now discuss these groups more carefully.

Let Γ be a finitely generated discrete group with a length $|\cdot| : \Gamma \to \mathbb{R}$. Define

$$H^\infty(\Gamma) = \{\varphi : \Gamma \to \mathbb{C} : \sum_{g\in\Gamma}^{\infty} |\varphi(g)|^2(1+|g|)^{2k} < \infty, \forall k \in \mathbb{N}\}.$$

Γ has property (RD) if $H^\infty(\Gamma) \subset C_r^*(\Gamma)$. The following groups have property (RD):

(1) groups of polynomial growth; (2) the Gromov hyperbolic groups. The class of groups with property (RD) has the following properties [Jol]:

(a) If $\Gamma_0 \subset \Gamma$ is a subgroup and Γ has property (RD), so does Γ_0;

(b) Let $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ be an extension of groups and Γ_1 be finite. Then Γ_2 has property (RD) iff Γ_3 has;

(c) Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index. If Γ_0 has property (RD), then Γ has property (RD);

(d) If Γ_1 and Γ_2 have property (RD), then $\Gamma_1 \times \Gamma_2$ and $\Gamma_1 * \Gamma_2$ have property (RD); More generally,

(e) The amalgamated product $\Gamma_1 *_A \Gamma_2$ has property (RD) if Γ_1 and Γ_2 have property (RD) and A is finite;

(f) If Γ is amenable, then Γ has property (RD) iff Γ is of polynomial growth. Thus for $n \geq 3$ $SL(n, \mathbb{Z})$ does not have property (RD), since there exists a solvable and non-polynomial growth subgroup $\Gamma_0 \subset SL(n, \mathbb{Z})$. But $SL(2, \mathbb{Z})$ has property (RD). From this one see that the Novikov conjecture is unknown for general amenable discrete groups.

To define the group Γ with polynomial cohomology, let us note that the cohomology $H^*(\Gamma, \mathbb{C})$ of Γ is defined as the cohomology of complex $\{C^*(\Gamma, \mathbb{C}), \delta\}$, where

$$C^*(\Gamma, \mathbb{C}) = \{\varphi : \Gamma^{\otimes(n+1)} \to \mathbb{C} : \varphi(\nu\nu_0, \ldots, \nu\nu_n) = \varphi(\nu_0, \ldots, \nu_n)\}$$

and

$$\delta\varphi(\nu_0, \ldots, \nu_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \varphi(\nu_0, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_{n+1}).$$

Γ has property (PC) if each element in $H^n(\Gamma)$ has a representative φ such that

$$|\varphi(\nu_0, \ldots, \nu_n)| \leq a \prod_{i=0}^{n} (1 + |\nu_i|)^k$$

for some constants $k > 0$ and $a > 0$. Note that any group of polynomial growth has property (PC). This was proved by Connes and Moscovici. Also if Γ is a hyperbolic group, then for $p \geq 2, H^p(\Gamma, \mathbb{C})$ is bounded, hence has property (PC) by Gromov's theorem [Gro]. Using the Künneth formula that $H^*(\Gamma_1 \times \Gamma_2, \mathbb{C}) \simeq H^*(\Gamma_1, \mathbb{C}) \otimes H^*(\Gamma_2, \mathbb{C})$, we see that $\Gamma_1 \times \Gamma_2$ has property (PC) if Γ_1 and Γ_2 have it. We list a few properties of hyperbolic groups below.

(1) Γ is hyperbolic iff the Cayley graph X associated with Γ is hyperbolic metric space, i.e., there exists $\delta_0 > 0$ such that

$$(x, y) \geq min\{(x, z), (y, z))\} - \delta_0, \forall x, y, z \in X,$$

where $(x, y) = \frac{1}{2}(|x|_0 + |y|_0 - |x - y|_0)$ and $|x|_0 = |x - x_0|$, x_0 is a fixed point in X.

(2) If $\Gamma_0 \subset \Gamma$ is a subgroup of finite index, then Γ is hyperbolic iff Γ_0 is.

(3) If Γ is hyperbolic, then there are no subgroups in Γ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

We see that $SL(n, \mathbb{Z})$ is neither hyperbolic nor polynomial growth for $n \geq 3$. There are hyperbolic groups with Kazhdan property (T) ([Gro],p.153). See [HaV] for very nice treatment about (T)-groups. A natural question is of course to prove the Novikov conjecture for (T)-groups.

We now consider the Cappell class of groups. let $\mathcal{J}_0$ be the smallest set of groups satisfying

(a) $\{1\} \in \mathcal{J}_0$;

(b) if Γ_1, Γ_2 and H are in $\mathcal{J}_0$ with $H \subset \Gamma_i$, then $\Gamma_1 *_H \Gamma_2 \in \mathcal{J}_0$;

(c) if $\Gamma_1, \Gamma_2 \in \mathcal{J}_0$ and $\alpha_i : \Gamma_1 \to \Gamma_2$ are monomorphisms, $i = 1, 2$, then the HHN extension $\Gamma_2 *_{\Gamma_1} \{t\}$ of $\Gamma_1 \overset{\alpha_i}{\Longrightarrow} \Gamma_2$ is in $\mathcal{J}_0$, where

$$\Gamma_2 *_{\Gamma_1} \{t\} = \mathbb{Z} * \Gamma_2 / \{< t\alpha_1(\nu)t^{-1}\alpha_2(\nu)^{-1} >, \nu \in \Gamma_1, t \in \mathbb{Z} \textit{ a fixed generator}\}.$$

$\mathcal{J}_0$ is called the Waldhausen class of groups. Each group in $\mathcal{J}_0$ is torsion free. The Cappell class $\mathcal{J}$ of groups is a generalization of the Waldhausen class $\mathcal{J}_0$. It is defined as the smallest set of groups satisfying

1) $\{1\} \in \mathcal{J}$;

2) if $\Gamma_1, \Gamma_2 \in \mathcal{J}, H \in \mathcal{J}_0$ and $H \subset \Gamma_i$, then $\Gamma_1 *_H \Gamma_2 \in \mathcal{J}$;

3) if $\Gamma_1 \in \mathcal{J}_0, \Gamma_2 \in \mathcal{J}, \alpha_i : \Gamma_1 \to \Gamma_2$ are monomorphisms, $i = 1, 2$, then $\Gamma_2 *_{\Gamma_1} \{t\} \in \mathcal{J}$;

4) if $\Gamma_1 \in \mathcal{J}$ and $\Gamma_1 \subset \Gamma$ with $[\Gamma, \Gamma_1]$ finite, then $\Gamma \in \mathcal{J}$.

One has that $\Gamma_1 \times \Gamma_2 \in \mathcal{J}$ if $\Gamma_i \in \mathcal{J}, i = 1, 2$. Cappell also gave a more general class of groups satisfying the $SANC_{l_\Gamma}$ [Cap].

To see how the Cappell class of groups fits into the program of operator K-theory, let us discuss the closedness of various Novikov conjectures under the group operations.

Lemma 8.1 ([*Ros*]) *If* $\Gamma = \lim_{\to} \Gamma_n$ *is the direct limit of groups* Γ_n *and all* Γ_n *satisfy the* SNC_β^{isom} *(resp.* SNC_β, $SNC_{\beta\otimes\mathbb{Q}}$*), then* Γ *also satisfies the* SNC_β^{isom} *(resp.* SNC_β, $SNC_{\beta\otimes\mathbb{Q}}$*).*

Lemma 8.2 ([*Ros*], [*Kas*3]) *Let* $\Gamma_1 \subset \Gamma$ *be a subgroup of finite index such that the* $SNC_{\beta\otimes\mathbb{Q}}$ *is valid for* Γ_1. *Then the* $SNC_{\beta\otimes\mathbb{Q}}$ *is valid for* Γ.

We do not know whether the converse of Lemma 8.2 is true.

Lemma 8.3 ([*Ros*]) *Let* $\Gamma_1 \subset \Gamma$ *be a finite normal subgroup such that the* $SNC_{\beta\otimes\mathbb{Q}}$ *holds for* Γ/Γ_1. *Then the* $SNC_{\beta\otimes\mathbb{Q}}$ *holds for* Γ.

Lemma 8.4 ([*Ros*]) *Let* Γ_1 *and* Γ_2 *be groups satisfying the* $SNC_{\beta\otimes\mathbb{Q}}$ *(resp.* SNC_β^{isom}*). Suppose that* $C^*(\Gamma_1)$ *belongs to the category that the Künneth theorem of operator* K*-theory* [*Sch*] *holds. Then the* $SNC_{\beta\otimes\mathbb{Q}}$ *(resp.* SNC_β^{isom}*) is valid for* $\Gamma_1 \times \Gamma_2$.

Lemma 8.5 ([*Ros*]) *Let* Γ_1 *and* Γ_2 *be two groups satisfying the* $SNC_{\beta\otimes\mathbb{Q}}$ *(resp.* $SNC_\beta, SNC_\beta^{isom}$*). Then the* $SNC_{\beta\otimes\mathbb{Q}}$*(resp.* $SNC_\beta, SNC_\beta^{isom}$*) holds for* $\Gamma_1 * \Gamma_2$.

In general, we have the following.

Lemma 8.6 *Let* Γ *be a torsion free discrete group acting on a tree* X *without inversion. Denote by* Σ^1 *and* Σ^0 *the sets of edges and vertices in* $\Gamma \setminus X$*, resp.. Let* Γ_y *and* Γ_ν *be the stabilizers of the edge* $y \in \Sigma^1$ *and vertex* $\nu \in \Sigma^0$*, resp.. Suppose the* SNC_β^{isom} *holds for all* $\Gamma_y, y \in \Sigma^1$*. If the* SNC_β^{isom}*(resp.* $SNC_\beta, SNC_{\beta\otimes\mathbb{Q}}$*) is valid for all* $\Gamma_\nu, \nu \in \Sigma^0$*, then the* SNC_β^{isom} *(resp.* $SNC_\beta, SNC_{\beta\otimes\mathbb{Q}}$*) holds for* Γ.

Proof (cf. [BaC 3]). Since Γ is torsion free, Baum-Connes' theorem [BaC 1] implies

$$RK_*(B\Gamma) \simeq K^*(\cdot,\Gamma) \stackrel{\text{def}}{=} \lim_{\overrightarrow{\mathcal{C}(\cdot,\Gamma)}} K_*(C_0(TM) \rtimes \Gamma),$$

where $\mathcal{C}(\cdot,\Gamma)$ is the category of all proper Γ-smooth manifolds without boundary. Its morphisms are Γ-equivariant smooth maps $f : M_1 \to M_2$ which induce homomorphisms $f! : K_*(C_0(TM_1) \rtimes \Gamma) \to K_*(C_0(TM_2) \rtimes \Gamma)$. In general, we have that $RK_i(B\Gamma) \to K^i(\cdot,\Gamma)$ is rationally injective and that there is a map $\mu : K^i(\cdot,\Gamma) \to K_i(C_r^*(\Gamma))$ defined by the K-theory index map such that the map $\beta : RK_i(B\Gamma) \to K_i(C_r^*(\Gamma))$ factors through $RK_i(B\Gamma) \to K^i(\cdot,\Gamma) \xrightarrow{\mu} K_i(C_r^*(\Gamma))$ [BaC 1]. Now in view of the exactness of inductive limit, we can apply Pimsner's theorem to $K_*(C_0(TM) \rtimes \Gamma)$ and $K_*(C^*(\Gamma))$ to get the following commutative diagram with exact rows [BaC 3]:

$$\begin{array}{ccccccccc}
\bigoplus_{\nu\in\Sigma^0} RK_{n+1}(B\Gamma_\nu) & \xleftarrow{\sigma} & \bigoplus_{\nu\in\Sigma^1} RK_{n+1}(B\Gamma_y) & \xleftarrow{\partial} & RK_n(B\Gamma) & \xleftarrow{\tau} & \bigoplus_{\nu\in\Sigma^0} RK_n(B\Gamma_\nu) & \xleftarrow{\sigma'} \\
\downarrow \beta_{\Gamma_\nu} & & \downarrow \beta_{\Gamma_y} & & \downarrow \beta_{\Gamma} & & \downarrow \beta_{\Gamma_\nu} & \\
\bigoplus_{\nu\in\Sigma^0} K_{n+1}(C^*(\Gamma_\nu)) & \xleftarrow{\sigma} & \bigoplus_{\nu\in\Sigma^1} K_{n+1}(C^*(\Gamma_y)) & \xleftarrow{\partial} & K_n(C^*(\Gamma)) & \xleftarrow{\tau} & \bigoplus_{\nu\in\Sigma^0} K_n(C^*(\Gamma_\nu)) & \xleftarrow{\sigma}
\end{array}$$

Here the map ∂, τ and σ are defined in [Pim]. The assertion then follows from the Five Lemma. Q.E.D.

One of the Baum-Connes conjectures says that μ is always an isomorphism. We denote this conjecture by BC_μ.

Corollary 8.1 ([$BaC3$]) *Let Γ be a finitely generated discrete group acting on a tree X without inversion. If the BC_μ is true for all Γ_y and Γ_ν, $y \in \Sigma^1, \nu \in \Sigma^0$, then the BC_μ holds for Γ.*

Since the HNN extension and emalgamated free products of groups act on tree X without inversion, we obtain the following corollaries.

Corollary 8.2 ([$BaC3$]) (1) *Let Γ_0 be a subgroup of Γ_1 and Γ_2 such that the BC_μ holds for Γ_0. Let $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$ be the amalgamated free product of Γ_1 and Γ_2 along Γ_0. If the BC_μ holds for Γ_1 and Γ_2, then BC_μ is true for Γ.*

(2) *Let* Γ_0 *be a subgroup of* Γ_1 *and* $\alpha_i : \Gamma_0 \to \Gamma_1$ *be injective homomorphisms,* $i = 1, 2$ *. Let* Γ *be the HNN extension determined by* Γ_0, Γ_1 *and* α_i*. Suppose the* BC_μ *holds for* Γ_0*. If the* BC_μ *holds for* Γ_1*, then the* BC_μ *is true for* Γ.

Corollary 8.3 ([$BaC3$]) *Let free group* F_n *of* n *generators act on* Γ_1 *via automorphisms* $\alpha_i, 1 \le i \le n$*. If* $\Gamma = \Gamma_1 \rtimes F_n$ *is torsion free and the* SNC_β^{isom} *holds for* Γ_1*, then the* SNC_β^{isom} *is valid for* Γ.

The reason for this corollary to be true is that $C_0(TM) \rtimes (\Gamma_1 \rtimes F_n) \simeq (C_0(TM) \rtimes \Gamma_1) \rtimes F_n$ and $C^*(\Gamma_1 \rtimes F_n) \simeq C^*(\Gamma_1) \rtimes F_n$. From the construction of the Waldhausen class $\mathcal{J}_0$ and the above corollaries, we get

Corollary 8.4 *The* BC_μ *holds for the Waldhausen class* $\mathcal{J}_0$ *of groups.*

The question is whether the Cappell class satisfies the $NC_{\beta\otimes\mathbb{Q}}$. There are two possible ways to solve this problem. One may try to prove that if $\Gamma_1 \subset \Gamma$ is a subgroup of finite index and μ is injective for Γ_1 then μ is injective for Γ (see Lemma 8.2). One may also try to remove the torsion freeness condition in Lemma 8.6 and prove:

"Assertion" Let Γ be a finitely generated discrete group acting on a tree X without inversion. With the notations in Lemma 8.6, if the SNC_β^{isom} is valid for all $\Gamma_y, y \in \Sigma^1$ and the $SNC_{\beta\otimes\mathbb{Q}}$ is valid for all $\Gamma_\nu, \nu \in \Sigma^0$, then the $SNC_{\beta\otimes\mathbb{Q}}$ holds for Γ.

Let us point out where the difficulty of proving this "Assertion" comes from. Since the Chern character $Ch : K_i(B\Gamma) \otimes \mathbb{Q} \to \oplus_k H_{2k+i}(B\Gamma, \mathbb{Q})$ is an isomorphism, to show the injectivity of $\beta \otimes \mathbb{Q}$ it suffices to check that

$(\beta \otimes \mathbb{Q}) \cdot Ch^{-1} : \oplus_k H_{2k+i}(B\Gamma, \mathbb{Q}) \to K_i(C^*(\Gamma)) \otimes \mathbb{Q}$ is injective. In view of Serre's theorem $[Ser]$, the following is an exact sequence

$$\to \bigoplus_{y\in\Sigma^1} H_{n+1}(\Gamma_y, \mathbb{Q}) \to \bigoplus_{\nu\in\Sigma^0} H_{n+1}(\Gamma_\nu, \mathbb{Q}) \to H_{n+1}(\Gamma, \mathbb{Q}) \to \bigoplus_{y\in\Sigma^1} H_n(\Gamma_y, \mathbb{Q}) \to .$$

We get the exact sequence

$$\bigoplus_k \bigoplus_{y\in\Sigma^1} H_{2k+i}(\Gamma_y, \mathbb{Q}) \xrightarrow{\sigma} \bigoplus_k \bigoplus_{\nu\in\Sigma^0} H_{2k+i}(\Gamma_\nu, \mathbb{Q}) \xrightarrow{\tau} \bigoplus_k H_{2k+i}(\Gamma, \mathbb{Q}) \xrightarrow{\partial} \bigoplus_k \bigoplus_{y\in\Sigma^1} H_{2k+i-1}(\Gamma_y, \mathbb{Q}) \xrightarrow{\sigma} .$$

This together with Pimsner's theorem shows that there is the following diagram

$$\begin{array}{ccccccc}
\to \oplus_{y\in\Sigma^1}(\oplus_k H_{2k+i}(\Gamma_y, \mathbb{Q})) & \xrightarrow{\sigma} & \oplus_{\nu\in\Sigma^0}(\oplus_k H_{2k+i}(\Gamma_\nu, \mathbb{Q})) & \xrightarrow{\tau} & \oplus_k H_{2k+i}(\Gamma, \mathbb{Q}) & \xrightarrow{\partial} \\
\downarrow (\beta_{\Gamma_y} \otimes \mathbb{Q}) \cdot Ch^{-1} & & \downarrow (\beta_{\Gamma_\nu} \otimes \mathbb{Q}) \cdot Ch^{-1} & & \downarrow (\beta_{\Gamma} \otimes \mathbb{Q}) \cdot Ch^{-1} & \\
\to \oplus_{y\in\Sigma^1} K_i(C^*(\Gamma_y)) \otimes \mathbb{Q} & \xrightarrow{\sigma} & \oplus_{\nu\in\Sigma^0} K_i(C^*(\Gamma_\nu)) \otimes \mathbb{Q} & \xrightarrow{\tau} & K_i(C^*(\Gamma)) \otimes \mathbb{Q} & \xrightarrow{\partial}
\end{array}$$

One can show by the definitions of σ, τ and β that all squares above are commutative except those involving the connecting operators ∂. $\partial's$ are defined via the product with elements determined by two extensions ([Bla], [Pim], [Ser]). It is conceivable that these squares are also commutative. If the diagram is commutative, then the "Assertion" follows from the Five Lemma.

Note that the group Γ in Lemma 8.6 is torsion free provided all $\Gamma_y, y \in \Sigma^1$ and $\Gamma_\nu, \nu \in \Sigma^0$ are torsion free $[Ser]$. In particular, $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$ is torsion free if Γ_i are torsion free, $0 \le i \le 2$. Observe that the SNC_β^{isom} is valid for Γ being one of the following groups:

1) torsion free abelian groups;

2) free groups of finite rank;

3) torsion free discrete subgroups of connected Lie groups which are isomorphic to $H \times SO(n_1, 1) \times \ldots \times SO(n_k, 1)$ for compact Lie group H and positive integers n_i;

4) torsion free discrete subgroups of connected simply-connected solvable Lie groups;

5) countable solvable groups having a composition series with torsion free abelian composition factors [Ros], and from the above discussion,

6) the Waldhausen class $\mathcal{J}_0$ of groups.

See [BaC 3] for the proofs of $1) - 4)$.

Recently, Connes, Gromov and Moscovici [CGM 2] proved the Novikov conjecture for hyperlinear groups by using the Miscenko symmetric signature $\sigma(M)$. A finitely generated discrete group Γ is called hyperlinear if its cohomology is isomorphic to its hyperlinear cohomology [CGM 2]. A hyperbolic group is hyperlinear. But we do not know whether an automatic group is hyperlinear. Ogle's papers [Ogle 1] might be helpful to prove the NC for the automatic groups. Ogle has verified the $SNC_{\beta\otimes\mathbf{Q}}$ for the groups with bounded homotopy property [Ogle 2].

We close this section with the Kasparov-Skandalis' examples [KaS].

(1) Let K_i be locally compact fields, $1 \leq i \leq m$. If $G_i \subset GL_{n_i}(K_i)$ is closed subgroup and G_0 is an almost connected locally compact group, then for every discrete subgroup Γ of $\prod_{i=0}^{m} G_i$ the $SNC_{\beta\otimes\mathbf{Q}}$ is valid. Moreover, if Γ is torsion free, then the SNC_β is valid for Γ.

This extends Solovev's result [Sol].

(2) For every subgroup Γ of $GL_n(\bar{\mathbb{Q}})$ the $SNC_{\beta\otimes\mathbf{Q}}$ is valid. If in adition Γ is

torsion free, then the SNC_{β} holds for Γ, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.

We expect that the work of Connes, Gromov and Moscovici [CGM 1] might be helpful to prove the Novikov conjecture for residually finite groups. These groups include the arithmetic groups, the mapping class groups of Teichmüller spaces, the fundamental groups of Haken manifolds [Hem 1,2] and some knot groups, which is a topic of the next section.

8.3 Manifold Picture of the Conjecture

We now examine several classes of manifolds and their fundamental groups that satisfy the Novikov conjecture. Let us first consider the manifolds of sectional curvature of constant sign.

Lemma 8.7 *let Γ be the fundamental group $\pi_1(M)$ of a complete manifold M of sectional curvature of constant sign. Then the Novikov conjecture holds for Γ.*

Proof. Since we know already that the Novikov conjecture holds for the fundamental groups of manifolds of nonpositive curvature, it is suffices to prove the lemma for $\Gamma = \pi_1(M)$, where M is a complete manifold of positive sectional curvature. Using Cheeger-Gromoll's theorem [ChG 2], one can find a finite subgroup $\Gamma_1 \subset \Gamma$ such that $\Gamma_2 = \Gamma/\Gamma_1$ contains a free abelian normal subgroup Γ_3 of finite rank with index $[\Gamma_2, \Gamma_3] < \infty$. Hence, the NC holds for Γ_2 and then for Γ by Lemmas 8.2 and 8.3. Q.E.D.

The special examples of manifolds of nonpositive sectional curvature are hyperbolic manifolds, locally symmetric manifolds of non compact type. Note that the fundamental groups of manifolds of negative curvature are non amenable [Bro] and that the fundamental groups of complete Riemannian flat manifolds are torsion free and finitely generated and contain an abelian subgroup of finite index [Mil 3].

Thus we need to consider manifolds of sectional curvature of mixed signs. The following is a special result on almost flat manifolds.

Lemma 8.8 *let M be a compact Riemannian manifold of dimension n with diameter d and sectional curvature k_M. Then there exists a $\varepsilon > 0$ such that for $|k_M|d^2 < \varepsilon$ the Novikov conjecture holds for $\pi_1(M)$.*

Proof. This is an immediate consequence of Ruh's theorem [Ruh] which says that M is diffeomorphic to $\Gamma \setminus M_1$, where M_1 is a simply connected nilpotent Lie group and Γ is an extension of a lattice $L \subset M_1$ by a finite group. Since $\Gamma = \pi_1(M)$, the lemma follows from Lemma 8.2 and Kasparov's theorem [Kas 3]. Q.E.D.

We consider next the manifolds of nonnegative Ricci curvature.

Lemma 8.9 (1) *Let M be a complete Riemannian manifold of nonnegative Ricci curvature. Then the Novikov conjecture holds for every finitely generated discrete subgroup Γ of $\pi_1(M)$.*

(2) *Let M be a complete manifold of almost flat Ricci curvature. Then the Novikov conjecture holds for every finitely generated discrete subgroup Γ of $\pi_1(M)$.*

Proof. (1) By Cheeger-Gromoll's theorem [ChG 1], Γ is of polynomial growth. The assertion follows from Connes-Moscovici's theorem [CoM 2].

(2) The same reasoning as in (1) works by using [Wei]. Q.E.D.

The case of manifolds of negative Ricci curvature remains unclear, neither is the case of scalar curvature. Here we provide an example of manifolds of positive scalar curvature.

Lemma 8.10 *Let M be a 3-dimensional compact Riemannian manifold of positive scalar curvature. Then the Novikov conjecture is valid for $\pi_1(M)$.*

Proof. According to Gromov-Lawson' theorem [GrL], there are no $K(\pi_1, 1)$ factors in the prime decomposition of M (see [Hem 1]), i.e., M can be decomposed as connected sums

$$M \simeq M_1 \sharp M_2 \sharp \ldots \sharp M_l \sharp \overbrace{(S^1 \times S^2) \sharp \ldots \sharp (S^1 \times S^2)}^{m},$$

where $\pi_1(M_i)$ is finite. Observe that $\pi_1(M) \simeq \pi_1(M_1) * \ldots * \pi_1(M_l) * \pi_1(S^1 \times S^2) * \ldots * \pi_1(S^1 \times S^2)$. The proof is complete by Lemma 8.5. Q.E.D.

Note that the Novikov conjecture holds for the fundamental groups of Lie groups [Mil 2] and of symmetric spaces [Wolf]. But we do not know in general the truth of the conjecture for the fundamental groups of homogeneous spaces M, though there exists an abstract description of $\pi_1(M)$ [Mos]. One trivial observation is that the Novikov conjecture holds for the fundamental groups of compact homogeneous space G/H if H is closed, connected and locally connected subgroup of G, since in this case $\pi_1(G/H)$ is abelian.

We now focus on the fundamental groups of low-dimensional manifolds. Our starting point is that any finitely presented group can be realized as the fundamental group of a compact 4-dimensional manifold. We will particularly pay attention to the manifolds with geometric stricture and to the 3-dimensional manifolds.

Recall that the "geometry" in the sense of Thurston means a pair (X, G_X) with X a complete, simply-connected Riemannian manifold and G_X a Lie group acting transitively on X by isometries such that G_X contains a discrete subgroup Γ with $\Gamma \setminus X$ of finite volume. A manifold M has a geometric structure of type (X, G_X) if M has an atlas of charts mapping to X with coordinate changes defined by elements of G_X such that $M \simeq \Gamma \setminus X$ for a discrete subgroup Γ of G_X [Wall 2]. Here is a list of geometries of dimensions ≤ 4.

Dim1: There is only one geometry, the euclidean line E^1.

Dim2: There are only three geometries, the sphere S^2, the euclidean space E^2 and the hyperbolic plane H^2.

Dim3: There are only eight geometries, the sphere S^3, the euclidean space E^3, the hyperbolic space H^3, $S^2 \times E^1$, $H^2 \times E^1$, the universal cover $\tilde{SL}_2$ of $SL(2, \mathbb{R})$, the nilpotent group Nil^3 and solvable group Sol^3.

Dim4: There are only twenty geometries which we omit here. See [Wall 2].

The geometries of dim 3 and dim 4 are classified by Thurston [Thu 1,2] and Filipkiewrcz (cf. also [Wall 2]), resp..

Lemma 8.11 *The Novikov conjecture holds for the fundamental groups of manifolds of dimensional ≤ 4 with geometric structure.*

Proof. This is an immediate consequence of Kasparov' theorem [Kas 3], since the fundamental group of a manifold of dim ≤ 4 with geometric structure is a discrete subgroup in some finite component Lie group. Q.E.D.

The examples of the 3-dimensional manifolds with geometric structure are the closed Seifert fibre spaces which are modeled on $S^3, E^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \tilde{SL}_2$ and Nil^3. Recall that a compact Seifert fibre space is a 3-dimensional manifold foliated by circles [Sco]. The Seifert fibre space can not be a connected sum except $\mathbb{R}P^3 \sharp \mathbb{R}P^3$. The universal covering of a Seifert fibre space without boundary is homeomorphic to one of $S^3, \mathbb{R}^3$ and $S^2 \times \mathbb{R}$. Hence a closed Seifert fibre space is aspherical unless it is covered by S^3 or $S^2 \times \mathbb{R}$.

Let us remark that some compact complex surfaces have geometric structures [Wall 2]. Thus the Novikov conjecture holds for the fundamental groups of these compact complex surfaces. But we do not know whether the conjecture is true for the fundamental groups of all compact complex surfaces. Certainly, the Novikov conjecture holds for the fundamental groups of all compact connected Riemann surfaces, since there exist metrics of constant sectional curvature on such surfaces. See [HaK] for 4-dimensional manifolds with finite fundamental groups.

We now consider compact 3-dimensional manifolds. First observe that the Novikov conjecture is valid for the following 3-dimensional groups:

(1) fundamental groups of compact oriented 3-dimensional manifolds whose irreducible summands either have non-empty boundary, or are simply connected, or are sufficiently large (i.e., containing a properly embedded 2-sided 2- manifold in it);

(2) fundamental groups of submanifolds of the 3-sphere.

In fact, these groups are contained in the Waldhausen class. For the fundamental groups of general compact 3-dimensional manifolds, we will see below that the Novikov conjecture is related to the Thurston geometrization conjecture. To state the following proposition, we recall that a compact 3-dimensional manifold M is Haken if it is prime and contains a 2-sided incompressible surface (whose boundary is in ∂M, if any) which is not a 2-sphere. M is irreducible if any embedded 2-sphere in M bounds a ball. See Hempel's book for the terminology [Hem 1].

Proposition 8.1 *The Novikov conjecture is valid for the fundamental groups of all compact* 3*-dimensional manifolds provided that the Thurston geometrization conjecture holds for non-Haken closed* 3*-dimensional irreducible manifolds with infinite fundamental groups.*

One can consult the end of the proof of this proposition for the Thurston geometrization conjecture (see also [Sco]).

Proof. Let M be a compact 3-dimensional manifold. If M is not oriented, we can choose a 2-sheeted covering $\hat{M}$ of M such that $\hat{M}$ is oriented. Since $\pi_1(M)/\pi_1(\hat{M}) \simeq \mathbb{Z}_2$, it follows from Lemma 8.2 that if the Novikov conjecture holds for $\pi_1(\hat{M})$, then it holds also for $\pi_1(M)$. It suffices therefore to prove the proposition for oriented compact 3-dimensional manifold M.

By the prime decomposition theorem ([Mil 1], [Hem 1]), we can decompose M uniquely into prime 3-dimensional manifolds

$$M = M_1 \sharp \dots \sharp M_l \sharp \overbrace{(S^1 \times S^2) \sharp \dots \sharp (S^1 \times S^2)}^{m} \sharp K_1 \sharp \dots \sharp K_r,$$

where $\pi_1(M_i)$ is finite, $i = 1, 2, \ldots, l$, and each K_j is compact oriented irreducible 3-dimensional manifold. Note that K_j is aspherical and then $\pi_1(K_j)$ is torsion free. We have

$$\pi_1(M) \simeq \pi_1(M_1) * \ldots * \pi_1(M_l) * \overbrace{\pi_1(S^1 \times S^2) * \ldots * \pi_1(S^1 \times S^2)}^{m} * \pi_1(K_1) * \ldots * \pi_1(K_r).$$

Hence by Lemma 8.5 to prove the Novikov conjecture for $\pi_1(M)$ it suffices to check it for $\pi_1(K_j)$.

First assume that the boundary ∂K_j is not empty. Then $\pi_1(K_j)$ is the Waldhausen class. The Novikov conjecture holds for $\pi_1(K_j)$. We can assume that K_j is closed. If K_j is a Haken manifold, Thurston has proved that K_j admits a geometric structure [Thu 1,2]. Hence the Novikov conjecture is valid for $\pi(K_j)$. The remaining case is the non-Haken, irreducible manifolds with infinite fundamental groups. But the Thurston geometrization conjecture states that such manifolds are either Seifert fibre spaces or admit hyperbolic structure ([Thu 2], [Sco]). If this conjecture is true, then the Novikov conjecture holds for the fundamental groups of all compact 3-dimensional manifolds. Q.E.D.

We close this chapter by remarking that the higher signatures of oriented spherical space forms are zero. In fact, let M be an oriented spherical space form, $M = S^n/\tau(\Gamma)$, where Γ is a finite group and $\tau : \Gamma \to O(n+1)$ is a fixed point free representation of Γ in $O(n+1)$ for $n \geq 2$. Then $\pi_1(M) = \Gamma$ and since Γ is finite, $H^i(B\Gamma, \mathbb{Q}) = 0$ for $i > 0$. Thus all the higher signatures are zero except $< L(M)f^*(\varphi), [M] >$ for $\varphi \in H^0(B\Gamma, \mathbb{Q})$. But $L(M) = 1$ due to the vanishing of Pontrjagin classes of constant curvature manifolds ($K_M = 1$). Then the assertion follows. In particular, the higher signature of generalized Lens spaces are zero.

Chapter 9

Equivariant Hilsum-Skandalis Technique

This chapter was originally motivated by Connes-Gromov-Moscovici theorem on homotopy invariance of signature with coefficients in almost flat vector bundles [CGM 1]. As we noted in Chapter 8, this theorem has been used to produce the most general results on the Novikov conjecture from operator K-theory point of view. It is certainly interesting to have such a beautiful theorem in the equivariant case. One possible way to obtain the equivariant Connes-Gromov-Moscovici theorem is to use equivariant cyclic cohomology. But there are technical difficulties caused by group actions. Fortunately, we get around the trouble points by using equivariant Hilsum-Skandalis technique [HiS]. We use thus operator K-theory in this chapter instead of cyclic cohomology. The interesting point of the Hilsum-Skandalis approach is that one can obtain the Connes-Gromov-Moscovici theorem for signature with coefficients in general almost flat C^*-algebra modules and that the estimation of the norms of operators involved is not sensitive to group actions. Meanwhile, we show that the equivariant Hilsum-Skandalis approach is a good substitute for the equivariant Miscenko symmetric signature and enables us to avoid the difficulty in dealing with the equivariant Miscenko symmetric signature for general compact Lie group actions. This observation is the essential point of our approach to the equivariant Novikov conjecture. This chapter serves therefore as a technical tool to prove the equivariant Connes-Gromov-Moscovici theorem and to provide a substitute for the equivariant Miscenko symmetric signature. In section 9.1 we will define the so-called equivariant signature type elements in equivariant operator K-theory, which are modeled on the signature element $[D]$ defined in Chapter 8. We will then obtain the main machinery of this chapter. That is to give sufficient conditions for two equivariant signature type elements being equal. In section 9.2 and 9.3 we will verify these sufficient conditions for the signature elements with coefficients in flat and almost flat equivariant C^*-algebra modules. Hence we get our desired results (Theorems 9.1 and 9.2).

9.1 Signature Type Elements in Equivariant K-Theory

This section is largely a generalized version of [HiS] and [KaM] to the equivariant case. Our effort here is to deal with some technical points about group actions.

To begin with, we assume throughout this section that G is a compact group, $\mathcal{A}$ is a $G - C^*$-algebra over $\mathbb{C}$ and $\mathcal{E}$ is a right $G - \mathcal{A}$-module. Here G acts on $\mathcal{A}$ and $\mathcal{E}$ by continuous automorphisms which are compatible with module structure, i.e., $g(xa) = g(x)g(a), g \in G, x \in \mathcal{E}, a \in \mathcal{A}$. If $\mathcal{E}$ is graded with grading operator ε ($\varepsilon^2 = 1$), then we require $g\varepsilon = \varepsilon g, \forall g \in G$. See [Bla] for the terminology.

Defintion 9.1. Suppose that Q is a $\mathbb{C}$-sesquilinear map from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{A}$.

(1) If $Q(\xi,\eta) = Q(\eta,\xi)^*, Q(\xi,\eta a) = Q(\xi,\eta)a$ and $gQ(\xi,\eta) = Q(g\xi, g\eta)$ for $a \in \mathcal{A}, \xi, \eta \in \mathcal{E}, g \in G$, then Q is called an equivariant quadratic form on $\mathcal{E}$.

(2) If $\mathcal{E}$ is graded and $Q(\xi,\eta) = 0$ for $\partial\xi = \partial\eta$, then Q is of degree 1.

(3) Q is regular if there is an equivariant $\mathcal{A}$-linear bijection T on $\mathcal{E}$, $gT = Tg$ such that $\mathcal{E}$ with scalar product $< . >$ defined by $< \xi, \eta >= Q(\xi, T\eta)$ is a Hilbert $G - \mathcal{A}$-module. T is then said to be compatible with Q.

(4) A scalar product $< . >: \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ is compatible with Q if $< \xi, \eta >= Q(\xi, T\eta)$ for some T compatible with Q.

(5) Two Hilbert $G - \mathcal{A}$-scalar products $< . >_1$ and $< . >_2$ on $\mathcal{E}$ are compatible if there exists an equivariant $\mathcal{A}$-linear invertible operator T on $\mathcal{E}$ such that $< \xi, \eta >_2 = < \xi, T\eta >_1$.

Remark 9.1 (a) *Clearly, T in* (5) *is unique, and any T satisfying $< \xi, \eta >_2 = < \xi, T\eta >_1$ is injective.*

(b) *If there are adjoint operators S_1^* and S_2^* of an operator S on $\mathcal{E}$ with respect to the compatible scalar products $< . >_1$ and $< . >_2$, then $S_2 = T^{-1}S_1^*T$. In fact,*

$$< S\xi, \eta >_2 = < S\xi, T\eta >_1 = < \xi, S_1^*T\eta >_1 = < \xi, T^{-1}S_1^*T\eta >_2 = < \xi, S_2^*\eta >_2 .$$

(c) *T in* (5) *is selfadjoint and positive with respect to $< . >_1$ and $< . >_2$:*

$$< T\xi, \eta >_1 = \overline{< \xi, T\eta >_1} = \overline{< \xi, \eta >_2} = < \eta, \xi >_2 = < \eta, T\xi >_1 .$$

Thus, $T_1^* = T$ *and* $T_2^* = T^{-1}T_1^*T = T$. *Clearly,* $0 << \xi, \xi >_2 = < \xi, T\xi >_1$ *and* $< \xi, T\xi >_2 = < T\xi, T\xi >_1 > 0$ *for* $\xi \neq 0$ *in* $\mathcal{E}$. *Hence* T *is positive.*

(*d*) *If two scalar products* $< . >_1$ *and* $< . >_2$ *on* $\mathcal{E}$ *are compatible with* Q, *then* $< . >_1$ *and* $< . >_2$ *are compatible. In fact, let* $< \xi, \eta >_i = Q(\xi, T_i\eta)$. *Then* $< \xi, \eta >_2 = Q(\xi, T_2\eta) = Q(\xi, T_1(T_1^{-1}T_2)\eta) = < \xi, T_1^{-1}T_2\eta >_1$.

(*e*) *If* T *is compatible with* Q, *then* T *is selfadjoint and there exists an operator* T_1 *on* $\mathcal{E}$ *compatible with* Q *such that* $T_1^2 = I$. *In fact,*

$$< T\xi, \eta > = \overline{Q(T\eta, T\xi)} = \overline{< T\eta, \xi >} = < \xi, T\eta > .$$

T *is selfadoint. Hence* T^{-1} *is also selfadjoint. Since* T^2 *is invertible and positive,* $U = |T^2|^{\frac{1}{2}}$ *is well defined, invertible and positive. Let* $T_1 = TU^{-1}$. $T_1^2 = I$, *and* $Q(\xi, T_1\eta) = Q(\xi, TU^{-1}\eta) = < \xi, U^{-1}\eta >$ *is a Hilbert scalar product on* $\mathcal{E}$. T_1 *is compatible with* Q.

(*e*) *If* $\mathcal{E}$ *is endowed with scalar product* $< . >$ *compatible with* Q, $< \xi, \eta > = Q(\xi, T\eta)$ *and* S *is an operator on* $\mathcal{E}$ *such that there is an adjoint* S^* *with respect to* $< . >$, *then* $S' = TS^*T^{-1}$ *is the conjugate of* S *with respect to* Q. *Indeed,*

$$Q(S\xi, \eta) = < S\xi, T^{-1}\eta > = < \xi, T^{-1}(TS^*T^{-1})\eta > = Q(\xi, TS^*T^{-1}\eta).$$

Let $\mathcal{E}$ be a Hilbert $G - \mathcal{A}$-module with scalar product $< . >$. Recall that the space $\mathcal{L}(\mathcal{E})$ of bounded operators on $\mathcal{E}$ consists of all continuous $\mathcal{A}$-linear maps $S : \mathcal{E} \to \mathcal{E}$ such that its adjoint S^* exists and is $\mathcal{A}$-linear. The space $\mathcal{K}(\mathcal{E})$ of compact operators on $\mathcal{E}$ is the ideas of $\mathcal{L}(\mathcal{E})$ generated by $Q_{x,y}, x, y \in \mathcal{E}$, where $Q_{x,y}(z) = x < y, z >$. G acts on $\mathcal{L}(\mathcal{E})$ by $g(S)(\xi) = gS(g^{-1}(\xi))$. Denote by $\mathcal{L}_G(\mathcal{E})$ (resp. $\mathcal{K}_G(\mathcal{E})$) all G-continuous operators S in $\mathcal{L}(\mathcal{E})$ (resp. $\mathcal{K}(\mathcal{E})$), i.e., $g \to g(S)$ is norm continuous. Obviously, if $S \in \mathcal{L}(\mathcal{E})$ is equivariant, then $S \in \mathcal{L}_G(\mathcal{E})$. A regular operator on $\mathcal{E}$ is a densely defined operator S on $\mathcal{E}$ with densely defined adjoint S^* such that $I + S^*S$ has dense range in $\mathcal{E}$. $\mathcal{L}(\mathcal{E}), \mathcal{K}(\mathcal{E})$ (resp. $\mathcal{L}_G(\mathcal{E}), \mathcal{K}_G(\mathcal{E})$) and regular operators do not depend on compatible Hilbert $G-\mathcal{A}$-module scalar products.

Using the convention that 1-graded and 0-graded mean graded and trivially graded , we have the following definition.

Definition 9.2. (1) $\mathcal{L}_G^k(\mathcal{A})$ is the set of all triples $(\mathcal{E}, Q, D)$, where $\mathcal{E}$ is a k-graded $G-\mathcal{A}$-module, Q is a strongly nondegenerate G-quadratic form of degree k, $k = 0, 1$, and $D \in \mathcal{L}_G(\mathcal{E})$ is G-equivariant such that

(*a*) $D + D' \in \mathcal{K}_G(\mathcal{E})$; (*b*) $D^2 \in \mathcal{K}_G(\mathcal{E})$; (*c*) there are equivariant S_1 and S_2 in $\mathcal{L}_G(\mathcal{E})$ such that $S_1D + DS_2 - I \in \mathcal{K}_G(\mathcal{E})$; (*d*) D is of degree k.

(2) $\mathcal{L}_{G,u}^k(\mathcal{A})$ is the set of all triples $(\mathcal{E}, Q, D)$, where $\mathcal{E}$ and $\mathcal{A}$ are the same as in (1), but D is an equivariant regular operator such that

(*a*) $D + D' \in \mathcal{L}_G(\mathcal{E})$; (*b*) $im(D) \subset dom(D)$ and $D^2 \in \mathcal{L}_G(\mathcal{E})$; (*c*) there are equivariant S_1 and S_2 in $\mathcal{K}_G(\mathcal{E})$ with $im(S_2) \subset dom(D), DS_2, S_1D \in \mathcal{L}_G(\mathcal{E})$ and $S_1D + DS_2 - I \in \mathcal{K}_G(\mathcal{E})$; (*d*) D is of degree k.

We will see in section 9.2 that this definition is modeled on the signature element. The following two elementary lemmas will be used to analyze $\mathcal{L}_{G,u}^k(\mathcal{A})$.

Lemma 9.1 ([HiS]) (*a*) *If D is a densely defined adjointable equivariant operator on a Hilbert $G-\mathcal{A}$-module $\mathcal{E}$ and $S \in \mathcal{L}_G(\mathcal{E})$ such that $im(S) \subset dom(D^*)$, then D^*S is in $\mathcal{L}_G(\mathcal{E})$.*

(*b*) *If D is a regular adjointable equivariant operator on $\mathcal{E}$ and $S \in \mathcal{L}_G(\mathcal{E})$ such that $SD \in \mathcal{L}_G(\mathcal{E})$, then $im(S^*) \subset dom(D^*)$ and $D^*S^* = (SD)^*$.*

(*c*) *If* D_1 *and* D_2 *are two regular adjointable equivariant operators on* $\mathcal{E}$ *such that* $D_1 : dom(D_1) \to \mathcal{E}$ *is bijective and* $dom(D_1) \subset dom(D_2)$*, then* $D_2 D_1^{-1}$ *is in* $\mathcal{L}_G(\mathcal{E})$.

Lemma 9.2 ([*HiS*]) *Let* D *be a regular equivariant operator on* $\mathcal{E}$ *such that* $D^2 = 0$*. Then* $D + D^*$ *is selfadjoint and regular on* $\mathcal{E}$ *and* $dom(D + D^*) = dom(D) \cap dom(D^*)$.

We now consider the property of elements in $\mathcal{L}^k_G(\mathcal{A})$ and $\mathcal{L}^k_{G,u}(\mathcal{A})$.

Lemma 9.3 (*a*) *If* $(\mathcal{E}, Q, D) \in \mathcal{L}^k_G(\mathcal{A})$*, then* $D + D^*$ *is invertible modulo* $\mathcal{K}_G(\mathcal{E})$*, where* D^* *is the adjoint of* D *with respect to the scalar product compatible with* Q.
(*b*) *If* $(\mathcal{E}, Q, D) \in \mathcal{L}^k_{G,u}(\mathcal{A})$*, then* $D + D^*$ *is regular and selfadjoint with domain equal to* $dom(D) \cap dom(D^*)$*, and has the resolvent in* $\mathcal{K}_G(\mathcal{E})$.

Proof. We first consider the trivially graded case.
(*a*) The proof is to find the inverse of $D + D^*$ modulo $\mathcal{K}_G(\mathcal{E})$. Let $p = S_1 D, q = DS_2$, where S_i are as in Definition 9.2. Since $D^2 \in \mathcal{K}_G(\mathcal{E})$, $p^2 = S_1 D S_1 D = S_1 D(I - DS_2 + k) = S_1 D + k_1 = p \bmod (\mathcal{K}_G(\mathcal{E}))$, and $q^2 = DS_2 DS_2 = (k' + I - S_1 D)DS_2 = DS_2 + k_1' = q \bmod (\mathcal{K}_G(\mathcal{E}))$ for $k_i, k_i' \in \mathcal{K}_G(\mathcal{E})$. $qDp = DS_2 D(I - DS_2 + k) = DS_2 D - k_2 + DS_2 Dk = (I - S_1 D + k)D - k_2 + DS_2 Dk = D + k_3 = D \bmod (\mathcal{K}_G(\mathcal{E}))$. Take projection (mod $\mathcal{K}_G(\mathcal{E})$) $p' \in \mathcal{L}_G(\mathcal{E})$ [Bla] such that $p'p = p'$ and $pp' = p \bmod(\mathcal{K}_G(\mathcal{E}))$. Since $p + q - I = S_1 D + DS_2 - I \in \mathcal{K}_G(\mathcal{E}), p = I - q \bmod (\mathcal{K}_G(\mathcal{E}))$, and $qDp = (I - p)Dp = D \bmod (\mathcal{K}_G(\mathcal{E}))$. We get

$$(I - p)Dp = (I - p)Dpp' = Dp' \; mod \; (\mathcal{K}_G(\mathcal{E})),$$

$$(I-p')(I-p)Dp = (I-p')Dp' \; mod \, (\mathcal{K}_G(\mathcal{E})).$$

But $(I-p')(I-p)Dp = (I-p'-p+p'p)Dp = (I-p)Dp = D \bmod (\mathcal{K}_G(\mathcal{E}))$. Hence, $D = (I-p')Dp' \bmod(\mathcal{K}_G(\mathcal{E}))$. Let $p_1 = p'S_1(I-p'), p_2 = p'S_2(I-p')$. We have $p_1D = p'S_1D - p'S_1p'D = p'S_1D = p'p = p' \bmod (\mathcal{K}_G(\mathcal{E}))$, since $p'D = p'(I-p')Dp' = 0 \bmod (\mathcal{K}_G(\mathcal{E}))$.

$$\begin{aligned}
Dp_2 &= qDpp'S_2(I-p') = qDpS_2(I-p') = DS_2(I-p') \\
&= (I-p)(I-p') = I-p' \; mod \, (\mathcal{K}_G(\mathcal{E})), \\
p_1 &= p'S_1(I-p')(I-p') = p'S_1(I-p')Dp_2 = p_1Dp_2 \\
&= p'p_2 = p'S_2(I-p') = p_2 \; mod \, (\mathcal{K}_G(\mathcal{E})).
\end{aligned}$$

Hence,

$$\begin{aligned}
(p_1+p_1^*)(D+D^*) &= p' + (I-p')S_1^*p'D + p_1D^* + (Dp_1)^* \\
&= p' + (Dp_2)^* = p' + (I-p') = I \; mod \, (\mathcal{K}_G(\mathcal{E})).
\end{aligned}$$

Here we have used $p'D = p'(I-p')Dp' = 0 \bmod (\mathcal{K}_G(\mathcal{E}))$ and $p_1D^* = p_2D^* = p'S_2(I-p')D^* = p'S_2(I-p')p'D^*(I-p') = 0$. This implies that $(D+D^*)^{-1} = p_1 + p_1^* \bmod (\mathcal{K}_G(\mathcal{E}))$.

(b) We divide the proof into several steps.

Step 1. First we assume $D^2 = 0$. By Lemma 9.2, $D + D^*$ is selfadjoint and regular. Write $S_1D + DS_2 = I - k, S_i, k \in \mathcal{K}_G(\mathcal{E})$. Since $D^2 = 0, (S_1D)^2 = S_1D(I-k-DS_2) = S_1D - S_1Dk = S_1D \bmod (\mathcal{K}_G(\mathcal{E}))$, we can thus choose $p \in \mathcal{L}_G(\mathcal{E})$ such that $p = p^*, p^2 - p \in \mathcal{K}_G(\mathcal{E}), pS_1D - p \in \mathcal{K}_G(\mathcal{E})$ and $S_1Dp - S_1D \in \mathcal{K}_G(\mathcal{E})$ [Bla]. Since $I - (pS_1D + DS_2(I-p)) = I - DS_2 - (pS_1D - DS_2p) = S_1D - (pS_1D - (I-S_1D)p) = S_1D(I-p) - p(S_1D-I) \in \mathcal{K}_G(\mathcal{E})$, $DS_2(I-p) =$

$I - pS_1D = 1 - p \mod (\mathcal{K}_G(\mathcal{E}))$ and $p = pS_1D = p^* = D^*S_1^*p \mod (\mathcal{K}_G(\mathcal{E}))$. Thus, $q \stackrel{\text{def}}{=} I - D^*S_1^*p - DS_2(I-p) = I - (pS_1D + DS_2(I-p)) \in \mathcal{K}_G(\mathcal{E})$. Note that $dom(D+D^*) = dom(D) \cap dom(D^*)$. By Lemma 9.1(c) and $dom(i + D + D^*)^* \subset dom(D)$, $\alpha \stackrel{\text{def}}{=} D((i + D + D^*)^{-1})^* \in \mathcal{L}_G(\mathcal{E})$. This implies that $D = \alpha(i+D+D^*)^*$. Also $D^* = \beta(i+D+D^*)^*$ for some $\beta \in \mathcal{L}_G(\mathcal{E})$ by the same argument. Using the fact that $\alpha^* = (i+D+D^*)^{-1}D^*, \beta^* = D(i+D+D^*)^{-1}$, and S_1^*p, q and $S_2(I-p)$ are all in $\mathcal{K}_G(\mathcal{E})$, we obtain

$$
\begin{aligned}
(i + D + D^*)^{-1} &= (i + D + D^*)^{-1}(q + D^*S_1^*p + DS_2(I-p)) \\
&= (i + D + D^*)^{-1}q + (\alpha^*S_1^*p + \beta^*S_2(I-p)) \in \mathcal{K}_G(\mathcal{E}).
\end{aligned}
$$

This proves that $(D + D^*)$ has the resolvent in $\mathcal{K}_G(\mathcal{E})$ if $D^2 = 0$.

Step 2. Suppose $D = D_0 + a$, where D_0 is regular, $D_0^2 = 0$ and $a \in \mathcal{L}_G(\mathcal{E})$. Since $\|(\lambda i + D_0 + D_0^*)^{-1}\| < \lambda^{-1}$ for sufficiently large $\lambda > 0$, $(\lambda i + D + D^*)^{-1} = (I + (\lambda i + D_0 + D_0^*)^{-1}(a + a^*))^{-1}(\lambda i + D_0 + D_0^*)^{-1}$. This proves that $(\lambda i + D + D^*)^{-1} \in \mathcal{K}_G(\mathcal{E})$ since $(\lambda i + D_0 + D_0^*)^{-1} \in \mathcal{K}_G(\mathcal{E})$.

Step 3. In general, we define for $(\mathcal{E}, Q, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$

$$
D_1 = \begin{bmatrix} D & D^2 \\ -I & -D \end{bmatrix}, D_2 = \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix}.
$$

$D_1^2 = 0, im(D_1) \subset dom(D_1)$. Clearly, we have $(\mathcal{E} \oplus \mathcal{E}, Q \oplus (-Q), D_j) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ and $D_2 = D_1 - \begin{bmatrix} 0 & D^2 \\ -I & 0 \end{bmatrix}$. Steps 1 and 2 show that $D_2 + D_2^*$ is regular with resolvent in $\mathcal{K}_G(\mathcal{E} \oplus \mathcal{E})$. Therefore, $D + D^*$ is regular with resolvent in $\mathcal{K}_G(\mathcal{E})$.

Now for the graded case we repeat the above proof to get (a) and (b), even though D_i in Step 3 are not of degree 1. Q.E.D.

Remark 9.2 *We obtain by the proof of Part (a) that if $D \in \mathcal{L}_G(\mathcal{E})$ is equivariant and $S_1, S_2 \in \mathcal{L}_G(\mathcal{E})$ such that $D^2 = 0$ and $S_1D + DS_2 = I$, then $D + D^*$ is invertible with the inverse $(D + D^*)^{-1} = p'S_1(I - p') + (I - p')S_1^*p'$, where p' was given in the proof.*

To define the maps from $\mathcal{L}_G^k(\mathcal{A})$ and $\mathcal{L}_{G,u}^k(\mathcal{A})$ to $K_G^k(\mathcal{A})$, let us recall the definition of Kasparov KK-theory. Let $\mathcal{A}$ and $\mathcal{B}$ be two $G - C^*$-algebras. $(\mathcal{E}, \varphi, F)$ is called Kasparov G-bimodule if $\mathcal{E}$ is a countably generated graded Hilbert $G - \mathcal{B}$-module, $\varphi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism, $F \in \mathcal{L}_G(\mathcal{E})$ is of degree 1 such that $[F, \varphi(a)], (F - F^*)\varphi(a), (F^2 - I)\varphi(a), (g(F) - F)\varphi(a)$ are in $\mathcal{K}_G(\mathcal{E})$ and $\varphi(a)F \in \mathcal{L}_G(\mathcal{E})$. Two G-bimodules $(\mathcal{E}_j, \varphi_j, F_j)$ are isomorphic if there exists an isomorphism $u : \mathcal{E}_1 \to \mathcal{E}_2$ such that $F_2 = uF_1u^{-1}$. Let $E^G(\mathcal{A}, \mathcal{B})$ be the set of all isomorphic classes of G-bimodules $(\mathcal{E}, \varphi, F)$.

A homotopy between $(\mathcal{E}_0, \varphi_0, F_0)$ and $(\mathcal{E}_1, \varphi_1, F_1)$ is an element $(\mathcal{E}, \varphi, F) \in E^G(\mathcal{A}, C([0,1], \mathcal{B}))$ such that $(\mathcal{E}, \varphi, F)$ generalizes a family $\{(\mathcal{E}_t, \varphi_t, F_t) \in E^G(\mathcal{A}, \mathcal{B})\}$ given by $\mathcal{E}_t = \mathcal{E}\hat{\otimes}_{C([0,1],\mathcal{B})}\mathcal{B}, \varphi_t = \varphi|_t, F_t = F \otimes I$. Then the Kasparov KK-theory group is $KK^G(\mathcal{A}, \mathcal{B}) = E^G(\mathcal{A}, \mathcal{B})/\sim$ and $KK_1^G(\mathcal{A}, \mathcal{B}) = KK^G(\mathcal{A}, \mathcal{B} \otimes C_{2,0}) \simeq KK^G(\mathcal{A} \otimes C_{2,0}, \mathcal{B})$. In particular, $KK_i^G(\mathbb{C}, \mathcal{B}) \simeq K_i^G(\mathcal{B}), i = 0, 1$.

We can also use unbounded modules to define $KK^G(\mathcal{A}, \mathcal{B})$ [BaJ] as follows. $(\mathcal{E}, \varphi, F)$ is an unbounded G-bimodule if $\mathcal{E}$ is a graded Hilbert $G - \mathcal{B}$-module, $\varphi : \mathcal{A} \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism, F is a selfadjoint regular equivariant operator of degree 1 on $\mathcal{E}$ such that

(a) $(I + F^2)^{-1}\varphi(a) \in \mathcal{K}_G(\mathcal{E}), \forall a \in \mathcal{A}$;

(b) $\{a \in \mathcal{A} : [F, \varphi(a)]$ is densely defined and extends to an element in $\mathcal{L}_G(\mathcal{E})\}$ is dense in $\mathcal{A}$.

Let $E_u^G(\mathcal{A},\mathcal{B})$ be the set of all unbounded G-bimodules. There is a map $\psi : E_u^G(\mathcal{A},\mathcal{B}) \to E^G(\mathcal{A},\mathcal{B})$ given by $\psi((\mathcal{E},\varphi,F)) = (\mathcal{E},\varphi,F(I+F^2)^{-\frac{1}{2}})$.

Suppose that $\mathcal{A}$ and $\mathcal{E}$ are trivially graded. Let $(\mathcal{E},Q,F) \in \mathcal{L}_G^0(\mathcal{A})$ and $T \in \mathcal{L}_G(\mathcal{E})$ be associated with Q, $T^2 = I$. Since T is an involution, we can use it to grade $\mathcal{E}$. It follows from $D^* = TD'T$ that $T(D+D^*) = TD + D'T$ and $(D+D^*)T = DT + TD'$. Using $D' = -D$ mod $(\mathcal{K}_G(\mathcal{E}))$, we get $T(D+D^*) = -(DT - TD) = -(D+D^*)T$ mod $(\mathcal{K}_G(\mathcal{E}))$. Hence, $D + D^*$ is of degree 1 mod $(\mathcal{K}_G(\mathcal{E}))$. By Lemma 9.3, $D + D^*$ is selfadjoint and invertible mod $(\mathcal{K}_G(\mathcal{E}))$. Then $D + D^* = F|D+D^*|$ mod$(\mathcal{K}_G(\mathcal{E}))$, $|D+D^*| = ((D+D^*)^2)^{\frac{1}{2}}$ and $F = (D+D^*)|D+D^*|^{-1}$ mod $(\mathcal{K}_G(\mathcal{E}))$. Since $D + D^*$ is of degree 1 mod $(\mathcal{K}_G(\mathcal{E}))$, F is of degree 1 mod$(\mathcal{K}_G(\mathcal{E}))$. We have the following:

(a) $F^* = |D+D^*|^{-1}(D+D^*) = F \; mod \; (\mathcal{K}_G(\mathcal{E}))$,

(b) $F^2 - I = (D+D^*)^2|D+D^*|^{-2} - I = 0 \; mod \; (\mathcal{K}_G(\mathcal{E}))$,

(c) $g(F) - F \in \mathcal{K}_G(\mathcal{E})$: since $gQ(\xi,\eta) = Q(g\xi,g\eta)$ and $gT = Tg$, we have $Q(gD'\xi,g\eta) = Q(D'g\xi,g\eta)$. Hence, $gD' = D'g$, and then $D^*g = gD^*$, since $D^* = TD'T$. It follows that $g(D+D^*) = (D+D^*)g$ and $g|D+D^*| = |D+D^*|g$. From this we get $gF = Fg$ mod $(\mathcal{K}_G(\mathcal{E}))$,

(d) define $\varphi : \mathbb{C} \to \mathcal{L}_G(\mathcal{E})$ by $\varphi(\lambda) = \lambda I$. φ is equivariant.

Therefore, $(\mathcal{E},\varphi,F)$ is a $G - \mathbb{C} - \mathcal{A}$-module.

Let $(\mathcal{E},Q,D) \in \mathcal{L}_{G,u}^0(\mathcal{A})$ and $T \in \mathcal{L}_G(\mathcal{E})$ compatible with Q such that $T^2 = I$. Suppose that the conjugate D' of D with respect to Q is $D' = -D$. Then $D^* = TD'T = -TDT$ and $T(D+D^*) = -(D+D^*)T$. This implies that $D + D^*$ is of degree 1 if $\mathcal{E}$ is graded by T, But we know already by Lemma 9.3 that $D + D^*$ is selfadjoint and regular with resolvent in $\mathcal{K}_G(\mathcal{E})$. Therefore,

$(\mathcal{E}, \varphi, D + D^*)$ is an unbounded $G - \mathbb{C} - \mathcal{A}$-module, where $\varphi(\lambda) = \lambda I$.

Let $\mathcal{A}$ be trivially graded and $(\mathcal{E}, \varphi, D) \in \mathcal{L}_G^1(\mathcal{A})$ be such that $D' = -D$. Then $T(D + D^*) = -(D + D^*)T$, where $T \in \mathcal{L}_G(\mathcal{E})$ is compatible with Q and $T^2 = I$. Let ε be the grading operator of $\mathcal{E}$, $\varepsilon^2 = I$. Since $Q(\xi, \eta) = 0$ if $\partial\xi = \partial\eta$, we must have $\varepsilon T = -T\varepsilon$. Using $D\varepsilon = -\varepsilon D$ and $D^*\varepsilon = -\varepsilon D^*$, we see that $\varepsilon(D+D^*) = -(D+D^*)\varepsilon$. Let $P = \frac{I+iT\varepsilon}{2}, \mathcal{E}_1 = P\mathcal{E}$ and $D_1 = P(D+D^*)P$ on $\mathcal{E}_1$. Then $P^2 = P, P^* = P, T\varepsilon P = PT\varepsilon$ and $P(D + D^*) = (D + D^*)P$. As for $\mathcal{L}_G^0(\mathcal{A})$, we can construct F from $(\mathcal{E}_1, Q, D_1)$ such that $F^* = F, F^2 = I$ mod $(\mathcal{K}_G(\mathcal{E}))$. Then $(\mathcal{E}_1, F)$ determines an element in $K_1^G(\mathcal{A})$ (cf. [Bla], p. 185).

Similarly, we can work out an unbounded $G-\mathbb{C}-\mathcal{A}$-module from $(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^1(\mathcal{A})$ with $D' = -D$ and $\mathcal{A}$ trivially graded, which is denoted by $(\mathcal{E}_1, Q, P(D+D^*)P)$.

To summarize, we have the following:

Definition 9.3. (1) $\psi_0 : \mathcal{L}_G^0(\mathcal{A}) \to K_0^G(\mathcal{A})$ is given by $\psi_0((\mathcal{E}, Q, D)) = [(\mathcal{E}, F)]$,
(2) Let $\mathcal{L}_{G,u}^{0'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^0(\mathcal{A}) : D' = -D\}$. $\psi_{0,u} : \mathcal{L}_{G,u}^{0'}(\mathcal{A}) \to K_0^G(\mathcal{A})$ is defined by $\psi_{0,u}((\mathcal{E}, Q, D)) = [(\mathcal{E}, F)]$, where $F = (D + D^*)(1 + (D + D^*)^2)^{-\frac{1}{2}}$.
(3) Let $\mathcal{A}$ be trivially graded and $\mathcal{L}_G^{1'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_G^1(\mathcal{A}) : D = -D'\}$. $\psi_1 : \mathcal{L}_G^{1'}(\mathcal{A}) \to K_1^G(\mathcal{A})$ is defined by $\psi_1((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)]$.
(4) Let $\mathcal{A}$ be trivially graded and $\mathcal{L}_{G,u}^{1'}(\mathcal{A}) = \{(\mathcal{E}, Q, D) \in \mathcal{L}_{G,u}^1(\mathcal{A}) : D = -D'\}$. $\psi_{1,u} : \mathcal{L}_{G,u}^{1'}(\mathcal{A}) \to K_1^G(\mathcal{A})$ is defined by $\psi_1((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)]$, where $F = P(D + D^*)P(I + (P(D + D^*))^2)^{-\frac{1}{2}}$.

Remark 9.3 *$\psi_0 : \mathcal{L}_G^0(\mathcal{A}) \to K_0^G(\mathcal{A})$ is independent of the choice of F and T.*

In fact, if T_1 and T_2 are compatible with Q, $T_i^2 = I$. Let $U = T_2T_1T_2$. $U^2 = I$ and $Q(\xi, U\eta) = Q(T_2\xi, T_1T_2\eta) =< T_2\xi, T_2\eta >_1$. U is compatible with Q. By the equivariant stability theorem, we can assume $\mathcal{E} = H_{\mathcal{A}}$, the universal Hilbert $\mathcal{A}$-module. Using the invertibility of $U^{\frac{1}{2}}$ and contractibility of $GL(\mathcal{L}(H_{\mathcal{A}}))$, we get a homotopy h_t in $GL(\mathcal{L}(H_{\mathcal{A}}))$ connecting $U^{\frac{1}{2}}$ to I. Let $\tilde{h}_t = h_t^* h_t$. Then $\tilde{h}_t$ is a selfadjoint homotopy between U and I. Hence we obtain a homotopy H_t connecting T_1 and T_2 such that $H_t^2 = I$ and H_t is compatible with Q. Thus, ψ_0 is independent of the choice of T. Clearly, ψ_0 is independent of the choice of F since two such choices differ by an element in $\mathcal{K}_G(\mathcal{E})$.

Similar remark applies to ψ_1.

The following lemma gives a link between $\mathcal{L}^{0'}_{G,u}(\mathcal{A})$ and $\mathcal{L}^0_G(\mathcal{A})$.

Lemma 9.4 ([*HiS*]) *Let $(\mathcal{E}, Q, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ be such that $D' = -D$. Let $U = D(I + D^*D)^{-\frac{1}{2}}$. Then for $\mathcal{E}$ with the scalar product,*

$$\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_0((\mathcal{E}, Q, U)).$$

Lemma 9.5 *Let $\mathcal{A}$ be trivially graded. Suppose that $(\mathcal{E}, \varphi, D) \in \mathcal{L}^1_{G,u}(\mathcal{A})$ is such that $D' = -D$. Let $U = D(I + D^*D)^{-\frac{1}{2}}$. Then for $\mathcal{E}$ with the scalar product associated with Q,*

$$\psi_{1,u}((\mathcal{E}, Q, D)) = \psi_1((\mathcal{E}, Q, U)).$$

Proof. Note that $\psi_{1,u}((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F)], F = P(D + D^*)(I + (P(D + D^*))^2)^{-\frac{1}{2}}$, where $\mathcal{E}_1 = P\mathcal{E}, P = \frac{I+iT_\varepsilon}{2}$, and $\psi_1((\mathcal{E}, Q, D)) = [(\mathcal{E}_1, F_1)]$ with $F_1 \in \mathcal{L}_G(\mathcal{E}_1)$ such that $P(U + U^*)P - F_1|P(U + U^*)P| \in K_G(\mathcal{E}_1)$. The proof is the same as Lemma 9.4. Q.E.D.

We now examine when the images of two elements in $\mathcal{L}_G^0(\mathcal{A})$ under ψ_0 are equal.

Lemma 9.6 *Let* $(\mathcal{E}, \varphi, D) \in \mathcal{L}_G^0(\mathcal{A})$ *(resp.* $\mathcal{L}_{G,u}^0(\mathcal{A})$*).*
(a) If $Ker(D) = im(D)$ *and* $D' = -D$*, then* $\psi_0((\mathcal{E}, Q, D)) = 0$.
(b) $\psi_0((\mathcal{E}, -Q, -D)) = -\psi_0((\mathcal{E}, Q, D))$.
The same results hold for $\psi_{0,u}$.

Proof. (a) By Lemma 9.4, it suffices to prove the assertion for the bounded case. By the definition of ψ_0, it is enough to check $F^2 = I$ and $F^* = F$, which is guaranteed by the invertibility of $D + D^*$. In fact, if $D + D^*$ is invertible, then $((D + D^*)^2)^{\frac{1}{2}}$ is invertible with the inverse $((D + D^*)^2)^{-\frac{1}{2}}$ and $F^2 = ((D + D^*)((D + D^*)^2)^{-\frac{1}{2}})^2 = I$.

To show that $D + D^*$ is invertible, let us first note that for two Hilbert modules $\mathcal{E}$ and $\mathcal{E}_1$ and $S \in \mathcal{L}(\mathcal{E}, \mathcal{E}_1)$ SS^* is invertible, provided S is surjective. Indeed, by the open mapping theorem, there is $k > 0$ such that $SS^* \geq k^{-2}$. We take $\mathcal{E}_1 = Ker(D) = im(D)$ and $S = D : \mathcal{E} \to im(D)$. S is surjective. Then SS^* is invertible. This implies that the zero is at worst an isolated spectral point of $S^*S = D^*D$, since $Spec(SS^*) \setminus \{0\} = Spec(S^*S) \setminus \{0\}$. We can then choose a submodule $\mathcal{E}_2$ in $\mathcal{E}$ which is orthogonal to $\mathcal{E}_1$ such that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and DD^* is invertible on $\mathcal{E}_1$ and D^*D is invertible on $\mathcal{E}_2$. It follows that $DD^* + D^*D$ is invertible on $\mathcal{E}$. But $D^2 = 0$. $(D + D^*)^2 = DD^* + D^*D$ is invertible on $\mathcal{E}$. Hence $D + D^*$ is invertible in $\mathcal{L}(\mathcal{E})$ and then in $\mathcal{L}_G(\mathcal{E})$, since $D + D^*$ is equivariant.

(b) If we change Q to $-Q$ and D to $-D$, then T is altered to $-T$, $D^* = TD'T$ to $-D^*$ and then F to $-F$. Hence, we change $(\mathcal{E}, F)$ to $(\mathcal{E}, -F)$, i.e.,

$\psi_0((\mathcal{E},-Q,-D)) = [(\mathcal{E},-F)] = -[(\mathcal{E},F)] = -\psi_0((\mathcal{E},Q,D)).$ Q.E.D.

Lemma 9.7 *Let $\mathcal{A}$ be trivially graded and $(\mathcal{E},Q,D) \in \mathcal{L}^1_G(\mathcal{A})$ (resp. $\mathcal{L}^1_{G,u}(\mathcal{A})$).*
(a) If $Ker(D) = im(D)$ and $D' = -D$, then $\psi_1((\mathcal{E},Q,D)) = 0$,
(b) $\psi_1((\mathcal{E},-Q,-D)) = -\psi_1((\mathcal{E},Q,D))$, where $-\mathcal{E}$ is equal to $\mathcal{E}$ graded by $-\varepsilon$. The same results hold for $\psi_{1,u}$.

Proof. (a) Note that $(D+D^*)$ commutes with $P = \frac{I+iT\varepsilon}{2}$. The proof of Lemma 9.6 implies that $(D+D^*)$ is invertible in $\mathcal{L}_G(\mathcal{E})$. Hence $P(D+D^*)$ is invertible in $\mathcal{L}_G(\mathcal{E}_1)$. As a result, $\psi_1((\mathcal{E},Q,D)) = 0$.

(b) Observe that if Q and D are changed to $-Q$ and $-D$, resp., then T is changed to $-T$, ε to $-\varepsilon$, D^* to $-D^*$, P to P, and hence $P(D+D^*)$ to $-P(D+D^*)$. The result then follows easily. Q.E.D.

9.2 Flat Bundle Case

We now use the machinery in section 9.1 to prove the homotopy invariance of the signature elements with coefficients in flat bundles. This result will give a substitute for the Miscenko symmetric signature for the equivariant case.

We assume throughout this section that G is a compact Lie group, $\mathcal{A}$ is a $G - C^*$-algebra over $\mathbb{C}$, M is a compact oriented Riemannian G-manifold and E is a smooth G-bundle over M whose fiber is a finitely generated projective Hilbert $G - \mathcal{A}$-module E_0. Let $\mathcal{E}_E = L^2(E)$ be the completion of $C^\infty(M, \Lambda_{\mathbb{C}} T^*M \otimes E)$ with respect to the scalar product

$$< \xi, \eta > = \int_M < \xi(x), \eta(x) > .$$

Here, we use equivariant Riemannian structure on M to get a Hilbert $G-\mathcal{A}$-module structure on $\wedge_{\mathbf{C}} T_x^* M \otimes E_x, x \in M$

Let $\xi, \eta \in \wedge_{\mathbf{C}} T_x^* M \otimes E_x$. Define $\xi^* \wedge \eta \in \wedge_{\mathbf{C}} T_x^* M \otimes \mathcal{A}$ by first letting

$$(\alpha \otimes e_1)^* \wedge (\beta \otimes e_2) = \bar{\alpha} \wedge \beta \otimes < e_1, e_2 >, \alpha, \beta \in \wedge_{\mathbf{C}} T_x^* M, e_i \in E_x,$$

$< e_1, e_2 > \in \mathcal{A}$, and then extending this to general elements ξ, η, where $< . >$ is the scalar product on $E_x = E_0$.

We now define a signature element in $\mathcal{L}^k_{G,u}(\mathcal{A})$. Let Q be the quadratic form on $\mathcal{E}_E$ given by

$$Q(\xi, \eta) = i^{\partial\xi(n-\partial\xi)} \int_M \xi_x^* \wedge \eta_x$$

for $\xi \in \mathcal{E}_E$ with homogeneous degree $\partial\xi$. Here n is the dimension of M. Q is equivariant since the metric and the scalar product $< . >$ on E_0 are equivariant. Let us check that Q is regular, i.e., there exists equivariant $\mathcal{A}$-bijection $T \in \mathcal{L}(\mathcal{E}_E)$ such that $< \xi, \eta >= Q(\xi, T\eta)$ is a scalar product. Let

$$(T\xi)_x = i^{-\partial\xi(n-\partial\xi)}(* \otimes I_E)(\xi_x), \ \ \xi \in \mathcal{E}_E.$$

Here $*$ is the Hodge operator, $*^2\alpha = (-1)^{\partial\alpha(n-\partial\alpha)}\alpha, \alpha \in \wedge_{\mathbf{C}} T^* M$. $T^2\xi = i^{-2\partial\xi(n-\partial\xi)}(*^2 \otimes I_E)\xi = \xi$. Hence $T^2 = I$. Clearly, T is equivariant $\mathcal{A}$-bijective. The following

$$< \xi, \eta >= Q(\xi, T\eta) = i^{\partial\xi(n-\partial\xi)} \int_M \xi_x^* \wedge (T\eta)_x = \int_M \xi_x^* \wedge (* \otimes I_E)(\eta_x)$$

is evidently a scalar product. Q also is quadratic form: $Q(\xi, \eta) = Q(\eta, \xi)^*$ and $Q(\xi, \eta a) = Q(\xi, \eta)a, a \in \mathcal{A}$.

To use $\mathcal{E}_E$ and Q to define an element in $\mathcal{L}^k_{G,u}(\mathcal{A})$, we need an operator D satisfying the conditions in Definition 9.2. Let ∇ be an antisymmetric equivariant connection on E. Define operator D_E by

$$D_E(\xi) = i^{\partial\xi}\nabla\xi, \quad \xi \in \mathcal{E}.$$

In fact, let $\mathcal{H}_\mathcal{A} = L^2(G) \otimes_{\mathbf{C}} H_\mathcal{A}$ be the universal $G - \mathcal{A}$-module [Bla]. By the equivariant stability theorem, E_0 can be embedded in $\mathcal{H}_\mathcal{A}$. Then E can be constructed via a family of differential equivariant projections $P = \{P_x\}_{x \in M}$ on $\mathcal{H}_\mathcal{A}$. Here each P_x has a finitely generated projective range. In other words, $P : M \to \mathcal{L}(\mathcal{H}_\mathcal{A})$ is differential and the fiber $E_x = P_x(\mathcal{H}_\mathcal{A})$. Then let

$$(\nabla\xi)_x = (I_{\wedge_{\mathbf{C}} T^*_x M} \otimes P_x)((d \otimes I_{\mathcal{H}_\mathcal{A}})\xi)_x, \quad \xi \in C^\infty(M, \wedge_{\mathbf{C}} T^*M \otimes E).$$

The following lemma implies that D_E satisfies the conditions in Definition 9.2.

Lemma 9.8 *With the above notations,*

(1) $\forall \omega \in C^\infty(M, \wedge_{\mathbf{C}} T^*M), \xi \in dom(D_E), \omega \wedge \xi \in dom(D_E)$ *and* $D_E(\omega \wedge \xi) = i^{\partial\xi + \partial\omega} d\omega \wedge \xi + i^{-\partial\omega}\omega \wedge D_E\xi$;

(2) *let* D'_E *be the conjugate of* D_E *with respect to* Q, *then* $D'_E = -D_E$;

(3) $im(D_E) \subset dom(D_E)$ *and* $D^2_E \in \mathcal{L}_G(\mathcal{E}_E)$;

(4) *there are equivariant* S_1 *and* S_2 *in* $\mathcal{K}_G(\mathcal{E}_E)$ *such that* $im(S_2) \subset dom(D_E)$, $D_E S_2, S_1 D_E \in \mathcal{L}_G(\mathcal{E}_E)$ *and* $S_1 D_E + D_E S_2 - I \in \mathcal{K}_G(\mathcal{E}_E)$.

Proof. (1) Let $\omega \in C^\infty(M, \wedge_{\mathbf{C}} T^*M), \xi \in dom(D_E)$. Assume first that $\xi = \alpha \otimes a, \alpha \in C^\infty(M, \wedge_{\mathbf{C}} T^*M)$ and $a \in E_0, P_x a = a$.

$$D_E(\omega \wedge \xi)_x \;=\; i^{\partial\omega + \partial\alpha}\nabla(\omega \wedge \alpha \otimes a)_x$$

$$
\begin{aligned}
&= i^{\partial\omega+\partial\alpha}[(I\otimes P_x)((d\omega\wedge\alpha+(-1)^{\partial\omega}\omega\wedge d\alpha)\otimes a)_x] \\
&= i^{\partial\omega+\partial\alpha}[(d\omega\wedge\alpha\otimes P_x a)+(-1)^{\partial\omega}\omega\wedge(I\otimes P_x)(d\alpha\otimes a)_x] \\
&= (i^{\partial\omega+\partial\alpha}d\omega\wedge\xi+i^{-\partial\omega}\omega\wedge D_E\xi)_x.
\end{aligned}
$$

Similarly, we can treat the general case.

(2) To show $D'_E=-D_E$, it is enough to check $Q(D_E\xi,\eta)=-Q(\xi,D_E\eta)$. Let $\xi=\alpha\otimes a,\eta=\beta\otimes b,\alpha,\beta\in C^\infty(M,\wedge_{\mathbf{C}}T^*M),a,b\in E_0,P_xa=a,P_xb=b,D_E\xi=i^{\partial\alpha}(d\alpha\otimes Pa),D_E\eta=i^{\partial\beta}(d\beta\otimes Pb)$,

$$
\begin{aligned}
Q(D_E\xi,\eta) &= i^{3\partial\xi+(\partial\alpha+1)\partial\beta}\int_M(\overline{d\alpha}\wedge\beta)\otimes<Pa,b> \\
&= -i^{(\partial\beta+1)\partial\alpha}i^{\partial\beta}\int_M(\bar\alpha\wedge d\beta)\otimes<a,Pb> \\
&= -Q(\xi,D_E\eta).
\end{aligned}
$$

Similarly, one can check $Q(D_E\xi,\eta)=-Q(\xi,D_E\eta)$ for the general case.

(3) Since the projection map $x\to P_x$ is differentiable, $im(D_E)\subset dom(D_E)$,

$$
\begin{aligned}
D_E^2(\alpha\otimes a) &= i^{2\partial\alpha+1}(I\otimes P)(d(d\alpha\otimes Pa)) \\
&= i^{2\partial\alpha+1}(I\otimes P)(d^2\alpha\otimes Pa+(-1)^{\partial\alpha+1}d\alpha\otimes dPa) \\
&= -i(d\alpha\otimes PdPa).
\end{aligned}
$$

In general, the curvature ∇^2 is in $\wedge^2(T^*M,End(E))$. Hence D_E^2 is in $\mathcal{L}_G(\mathcal{E}_E)$.

(4) Take $S_1=S_2=D_E^*(I+D_E^*D_E+D_ED_E^*)^{-1}$. Then

$$
\begin{aligned}
S_1D_E+D_ES_2-I &= D_E^*(I+D_E^*D_E+D_ED_E^*)^{-1}D_E \\
&+ D_ED_E^*(I+D_E^*D_E+D_ED_E^*)^{-1}-I \\
&= (D_ED_E^*+D_E^*D_E)(I+D_E^*D_E+D_ED_E^*)^{-1}-I \\
&= -(I+D_E^*D_E+D_ED_E^*)^{-1}.
\end{aligned}
$$

Hence it suffices to check S_i and $(I + D_E^* D_E + D_E D_E^*)^{-1}$ are in $\mathcal{K}_G(\mathcal{E}_E)$.

Let E be constructed by a family of equivariant $\mathcal{A}$-projections $P = \{P_x\}_{x \in M}$ with finitely generated projective ranges: $E_x = P_x F_0$, where F_0 is a finitely generated $G-\mathcal{A}$-projective Hilbert module. Then $\nabla_E = P D_F P, D_F$ is the flat exterior differential on the trivial Hilbert $G-\mathcal{A}$-bundle $M \otimes F_0 = F$. As in the ordinary case [Hil], $(I + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$ and $D_F(I + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$ for $\mathcal{F} = L^2(F)$. Since the orthogonal complement $E^\perp$ of E in F is given by $(I-P)F$, and its connection $D_{E^\perp}$ is $(I-P)D_F(I-P)$, we have

$$\begin{aligned} D_E \oplus D_{E^\perp} &= P D_F - P D_F(I-P) + (I-P)D_F(I-P) \\ &= D_F - (I-P)D_F - (2P-I)D_F(I-P) = D_F + A, \end{aligned}$$

where $A = -(I-P)D_F - (2P-I)D_F(I-P)$. Hence it is enough to check $B_1 = (D_F + A)(I + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$ and $B_2 = (I + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$. Now for λ sufficiently large,

$$(\lambda + (D_F + A)^*(D_F + A) + (D_F + A)(D_F + A)^*)^{-1} =$$

$$= (I + (\lambda + D_F^* D_F + D_F D_F^*)^{-1}(D_F^* A + A D_F^* + A^* D_F + D_F A^*))^{-1}(\lambda + D_F^* D_F + D_F D_F^*)^{-1},$$

since the products of A, A^* with D_F and D_F^* are bounded operators (by means of the argument of Sobolev spaces) and since the norm of $B_3 = (\lambda + D_F^* D_F + D_F D_F^*)^{-1}(D_F^* A + A D_F^* + A^* D_F + D_F A^*)$ is less than 1 for sufficiently large λ. This proves that $B_2 \in \mathcal{K}_G(\mathcal{F})$. Since A is bounded, $A B_2 \in \mathcal{K}_G(\mathcal{F})$ and $D_F B_2 = D_F(I + B_3)^{-1}(\lambda + D_F^* D_F + D_F D_F^*)^{-1} \in \mathcal{K}_G(\mathcal{F})$. Q.E.D.

Therefore, we have constructed an element $(\mathcal{E}_E, Q, D_E) \in \mathcal{L}^{0'}_{G,u}(\mathcal{A})$ for $dim(M) = 2k$. If $dim(M) = 2k+1$, we proceed as follows.

Let $\varepsilon(\xi) = (-1)^{\partial\xi}\xi$ for homogeneous $\xi \in \mathcal{E}_E$. Then ε extends to a grading on $\mathcal{E}_E$. Clearly, $\varepsilon T = -T\varepsilon$ (this is not true for $dim(M) = 2k$), $\varepsilon D_E = -D_E\varepsilon$. We get $(\mathcal{E}_E, Q, D_E) \in \mathcal{L}^{1'}_{G,u}(\mathcal{A})$.

We now consider the central problem of this chapter. Let E be a smooth $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated Hilbert $G-\mathcal{A}$-module E_0. As usual, E defines an element $[E] \in K_0^G(\mathcal{A} \otimes C(M))$. Let $[D_M]$ be the signature element in $K^*_G(C(M))$. Then

$$\begin{aligned} \psi(E, M) &\stackrel{\text{def}}{=} [E] \otimes_{C(M)} [D_M] \in K_0^G(\mathcal{A} \otimes C(M)) \otimes_{C(M)} KK^G_*(C(M), \mathbb{C}) \\ &\rightarrow KK^G_*(\mathbb{C}, \mathcal{A}) = K^G_*(\mathcal{A}), \ \ * = 0, 1. \end{aligned}$$

Question: Is $\psi(E, M)$ a G-pseudo-equivalence invariant ? Namely, let $h : N \to M$ be an orientation-preserving G-pseudo-equivalence of connected closed Riemannian G-manifolds. Is $\psi(h^*(E), N)$ equal to $\psi(E, M)$?

Recall that h is a G-pseudo-equivalence if h is a homotopy equivalence and G-equivariant. The following observation is important: if h is equivariantly homotopic to an orientation-preserving equivariant map $h_1 : N \to M$, then $h^*(E)$ is isomorphic to $h_1^*(E)$. Hence $[h^*(E)] = [h_1^*(E)] \in K_0^G(\mathcal{A} \otimes C(N))$ (cf. [Hus], Thm 4.7, p. 29). According to G-smooth approximation theorem ([Bre], Thm 4.2,p. 317), h is G-homotopic to a smooth orientation preserving G-pseudo-equivalence $h_1 : N \to M$. Therefore, it suffices to check $\psi(h_1^*(E), N) = \psi(E, M)$.

The following lemma then reduces the question to the G-pseudo-equivalent invariance of elements in $\mathcal{L}^{k'}_{G,u}(\mathcal{A})$.

Lemma 9.9 *Let $\mathcal{A}$ be unital. Suppose that E is a smooth $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert $G-\mathcal{A}$-module E_0. Let $(\mathcal{E}_E, Q, D_E)$ be as in Lemma* 9.8. *Then*

$$\psi_{i,u}((\mathcal{E}_E, Q, D_E)) = [E] \otimes_{C(M)} [D_M] \in K_i^G(\mathcal{A}).$$

Proof. Let $dim(M) = n$ be even. Recall that $[E] \in KK_0^G(\mathbb{C}, \mathcal{A} \otimes C(M))$ is defined by the Kasparov module $(\Gamma(E) \oplus 0, I, 0)$ and $[D_M] \in KK_0^G(C(M), \mathbb{C})$ is given by $(L^2(M, T^*M), \varphi, (D_M + D_M^*)(I + (D_M + D_M^*)^2)^{-\frac{1}{2}})$, where $\Gamma(E)$ is the space of continuous sections of E and φ is the multiplication by elements in $C(M)$. Assume first that E is trivial, $E = M \times E_0$. Then $D_E = D_M \otimes I$ and $D_E + D_E^* = (D_M + D_M^*) \otimes I$. The Kasparov product of $[E]$ and $[D_M]$ is given by

$$(((C(M) \otimes E_0) \oplus 0) \otimes_{C(M)\otimes\mathcal{A}} (L^2(M, T^*M) \otimes \mathcal{A}), I, (D_M + D_M^*)(I + (D_M + D_M^*)^2)^{-\frac{1}{2}} \otimes I).$$

where $((C(M) \otimes E_0) \oplus 0) \otimes_{C(M)\otimes\mathcal{A}} (L^2(M, T^*M) \otimes \mathcal{A}) \simeq L^2(M, T^*M \otimes E)$ with inner product $< (x_1, x_2), (y_1, y_2) >=< x_2, < x_1, y_1 >_1 y_2 >_2$ equivalent to the inner product on $L^2(M, T^*M \otimes E)$. This proves the assertion for the trivial bundle case. In general, $E = PF$, where F is a trivial smooth $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert $G-\mathcal{A}$ module F_0 and $P : M \to \mathcal{L}(F_0)$ is a smooth family of projections. Then $D_E = PD_FP, D_E + D_E^* = P(D_F + D_F^*)P$. Hence, $(D_E + D_E^*)(I + (D_E + D_E^*)^2)^{-\frac{1}{2}} = P(D_F + D_F^*)(I + (D_F + D_F^*)^2)^{-\frac{1}{2}}P$ is a $(D_F + D_F^*)(I + (D_F + D_F^*)^2)^{-\frac{1}{2}}$-connection of $\mathcal{E}_E$ (cf. [Bla], Prop. 18.3.3, P. 206). The result follows easily from the definition of the Kasparov product.

Now if $dim(M) = n$ is odd, then $[D_M]$ is given by $(P_1L^2(M,T^*M),\varphi, P_1(D_M+D_M^*)(I+(D_M+D_M^*)^2)^{-\frac{1}{2}}P_1)$, where $P_1 = \frac{I+iT\varepsilon}{2}$ with $\varepsilon(\xi) = (-1)^{\partial\xi}\xi$. The argument of the last paragraph works also in this case. Q.E.D.

Remark 9.4 *If $\mathcal{A}$ is non unital, then $K_0^G(\mathcal{A}\otimes C(M)) = Ker\{K_0(C(M)\otimes \mathcal{A}^+) \xrightarrow{i^*} K_0^G(\mathbb{C})\}$ and $[E] = [((\Gamma(E)\oplus 0), I, 0)] \in Ker(i^*)$, since $\Gamma(E)\otimes_{C(M)\otimes\mathcal{A}} \mathbb{C} = 0$, where $\mathcal{A}^+$ is the $G-C^*$-algebra obtained by adjoining an identity to $\mathcal{A}$. Note also that $\mathcal{E}_E$ can be considered as an $\mathcal{A}^+$-module and $[(\mathcal{E}_E, Q, D_E)]$ formed by the same way as $\psi_{0,u}((\mathcal{E}_E,Q,D_E))$ is in $Ker(i^*)$, i.e., this element is $\psi_{0,u}((\mathcal{E}_E,Q,D_E))$. With this in mind, we see that Lemma 9.9 holds for non unital $\mathcal{A}$.*

Clearly, $\psi_{k,u}((\mathcal{E}_E,Q,D_E))$ is independent of the connection on E, since any two connections on E differ by a bounded operator in $\mathcal{L}_G(\mathcal{E}_E)$.

To show the G-pseudo-equivalent invariance of $\psi_{k,u}((\mathcal{E}_E,Q,D_E))$, we need the following technical proposition.

Proposition 9.1 *Let $(\mathcal{E}_i,Q_i,D_i) \in \mathcal{L}^0_{G,u}(\mathcal{A}), i = 0,1$ and $R \in \mathcal{L}_G(\mathcal{E}_1,\mathcal{E}_2)$ satisfy*

(a) $D_i' = -D_i, D_i^2 = 0$;

(b) $R(dom(D_1)) \subset dom(D_2), RD_1 = D_2R$;

(c) $R : \frac{Ker(D_1)}{im(D_1)} \to \frac{Ker(D_2)}{im(D_2)}$ is an isomorphism;

(d) there is equivariant $S \in \mathcal{L}(\mathcal{E}_1)$ such that $S(dom(D_1)) \subset dom(D_1)$ and $I - R'R = D_1S + SD_1$;

(e) there is equivariant $\varepsilon \in \mathcal{L}(\mathcal{E}_1)$ such that $\varepsilon(dom(D_1)) \subset dom(D_1), \varepsilon D_1 = -D_1\varepsilon, \varepsilon^2 = I, \varepsilon' = \varepsilon$ and $\varepsilon(I-R'R) = (I-R'R)\varepsilon$.

Then $\psi_{0,u}((\mathcal{E}_1,Q_1,D_1)) = \psi_{0,u}((\mathcal{E}_2,Q_2,D_2))$.

Proof. We can assume that $S' = -S$. In fact, for $\xi, \eta \in dom(D_1)$,

$$Q_1(S'\xi, D_1\eta) = Q_1(\xi, SD_1\eta) = Q_1(\xi, (I - R'R)\eta) - Q_1(D_1\xi, S\eta).$$

It follows that $S'\xi \in dom(D_1), S'(dom(D_1)) \subset dom(D_1)$ and $I - R'R = D_1S' + S'D_1$. Take $\tilde{S} = \frac{S-S'}{2}$. We have $\tilde{S}(dom(D_1)) \subset dom(D_1)$ and $I - R'R = D_1\tilde{S} + \tilde{S}D_1$. $\tilde{S}$ is obviously equivariant. Thus we can consider $\tilde{S}$ with $\tilde{S}' = -\tilde{S}$.

Let $(\mathcal{E}, Q, D) = (\mathcal{E}_1 \oplus \mathcal{E}_2, Q_1 \oplus (-Q_2), D_1 \oplus (-D_2))$. By Lemma 9.6, $\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) + \psi_{0,u}((\mathcal{E}_2, -Q_2, -D_2)) = \psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) - \psi_{0,u}((\mathcal{E}_2, Q_2, D_2))$. Thus it suffices to show $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$.

For $t \in [0,1]$ we define $B_t(\xi, \eta) = Q(T_t\xi, T_t\eta), C_t(\xi, \eta) = Q(L_t\xi, \eta)$, where

$$T_t = \begin{bmatrix} I & 0 \\ itR\varepsilon & I \end{bmatrix}, L_t = \begin{bmatrix} I - R'R & (i\varepsilon + tS)R' \\ R(i\varepsilon + tS) & I \end{bmatrix}, \nabla_t = \begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix}.$$

We have the following:

(1) $DT_t = T_tD$, since $\varepsilon D_1 = -D_1\varepsilon$ and $RD_1 = D_2R$.

(2) $L_t\nabla_t = -\nabla_tL_t$. This follows from $I - R'R = D_1S + SD_1, D_1^2 = 0$ and $D_1R' = R'D_2$. Note that by $D_i' = -D_i$,

$$\nabla_t' = \begin{bmatrix} -D_1 & 0 \\ -tR & D_2 \end{bmatrix}.$$

(3) $C_t(\nabla_t\xi, \eta) = Q(-\nabla_t'L_t\xi, \eta) = -C_t(\xi, \nabla_t\eta)$. Hence ∇_t is antiselfadjoint with respect to $C_t(.,.)$.

(4) Since ε commutes with $R'R$, $L_0 = T_1'T_1$ and

$$T_t' = \begin{bmatrix} I & it\varepsilon R' \\ 0 & I \end{bmatrix}.$$

Let us check $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

(a) D is antisymmetric with respect to B_t. Indeed, by (1), $DT_t = T_tD$ and $D' = -D$, $B_t(D\xi, \eta) = Q(DL_t\xi, L_t\eta) = Q(L_t\xi, -DL_t\eta) = B_t(\xi, -D\eta)$.

(b) Clearly, $D^2 \in \mathcal{L}_G(\mathcal{E})$.

(c) Obviously, there are equivariant R_1 and R_2 in $\mathcal{L}_G(\mathcal{E})$ such that $R_1D + DR_2 - I \in \mathcal{K}_G(\mathcal{E})$, since D_i have the corresponding property.

(d) B_t is associated with the invertible operator $T_t \in \mathcal{L}_G(\mathcal{E})$, hence is a strongly nondegenerated quadratic form. In fact, the inverse of T_t is

$$T_t^{-1} = \begin{bmatrix} I & 0 \\ -tR\varepsilon & I \end{bmatrix}.$$

B_t is also regular with compatible operator $T_t^*TT_t$, where T is the operator compatible with Q, $T^2 = I$. Indeed,

$$< \xi, \eta >_{B_t} = B_t(\xi, T_t^*TT_t\eta) = Q(T_t\xi, T_t(T_t^*TT_t\eta) = < T_t^*TT_t\xi, T_t^*TT_t\eta > .$$

Therefore, $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ for $0 \le t \le 1$. In other words, $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(C([0,1], \mathcal{A}))$.

Let us now check $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ for t near 0. Since T_t is invertible and $L_0 = T_1'T_1$, L_0 is also invertible. It follows that L_t is invertible for t near 0, say $0 \le t \le t_0$. By (3), ∇_t is antisymmetric with respect to C_t and

$$\nabla_t^2 = \begin{bmatrix} D_1^2 & 0 \\ 0 & tD_2^2 \end{bmatrix},$$

since $D_1R' = R'D_2$. Let $S_i, \tilde{S}_i \in \mathcal{K}_G(\mathcal{E}_i)$ be such that $S_iD_i + D_i\tilde{S}_i - I \in$

$\mathcal{K}_G(\mathcal{E}_i), S_iD_i \in \mathcal{L}_G(\mathcal{E}_i), im(\tilde{S}_i) \subset dom(D_i), D_i\tilde{S}_i \in \mathcal{L}_G(\mathcal{E}_i), i = 0, 1$. Then

$$\begin{bmatrix} S_1 & 0 \\ 0 & -S_2 \end{bmatrix}\begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix} + \begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix}\begin{bmatrix} \tilde{S}_1 & 0 \\ 0 & -\tilde{S}_2 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} =$$

$$= \begin{bmatrix} S_1D_1 + D_1\tilde{S}_1 - I & tS_1R' - tR'\tilde{S}_2 \\ 0 & S_2D_2 + D_2\tilde{S}_2 - I \end{bmatrix} \in \mathcal{K}_G(\mathcal{E}),$$

since $S_1 \in \mathcal{K}_G(\mathcal{E}_1), \tilde{S}_2 \in \mathcal{K}_G(\mathcal{E}_2)$ and $R' \in \mathcal{L}_G(\mathcal{E}_1, \mathcal{E}_2)$. Note that $< \xi, \eta >_{C_t} \stackrel{\text{def}}{=} C_t(\xi, TL_t\eta) = Q(L_t\xi, TL_t\eta) =< L_t\xi, L_t\eta >$. We have that C_t is strongly nondegenerate and regular quadratic form. Thus we conclude that $(\mathcal{E}_t, C_t, \nabla_t)$ is in $\mathcal{L}^0_{G,u}(\mathcal{A})$ for $0 \le t \le t_0$.

Now $C_0(\xi, \eta) = Q(L_0\xi, \eta) = Q(T_1'T_1\xi, \eta) = B_1(\xi, \eta)$ and $\nabla_0 = D$. $(\mathcal{E}, B_1, D) = (\mathcal{E}, C_0, \nabla_0)$. Hence,

$$\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}, B_0, D)) = \psi_{0,u}((\mathcal{E}, B_1, D)) = \psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})).$$

From this we get that it suffices to show $\psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})) = 0$. By Lemma 9.6, we need only to check $im(\nabla_{t_0}) = ker(\nabla_{t_0})$.

Obviously, $\nabla^2_{t_0} = 0, im(\nabla_{t_0}) \subset \ker(\nabla_{t_0})$. To show $Ker(\nabla_{t_0}) \subset im(\nabla_{t_0})$, let $(\xi_1, \xi_2) \in \mathcal{E}$ such that $\nabla_{t_0}(\xi_1, \xi_2) = 0$. By condition (c), $R' : \frac{Ker(D_2)}{im(D_2)} \to \frac{Ker(D_1)}{im(D_1)}$ is bijective. Then there exists $\xi_2' \in \mathcal{E}_2$ such that $\xi_2 = -D_2\xi_2'$, Since $-\alpha R'\xi_2 = D_1\xi_1 = 0$ in $\frac{Ker(D_1)}{im(D_1)}$. This imples that $D_1\xi_1 = -\alpha R'\xi_2 = \alpha R'D_2\xi_2' = \alpha D_1R'\xi_2', D_1(\xi_1 - \alpha R'\xi_2') = 0$. Hence $RD_1(\xi_1 - \alpha R'\xi_2') = D_2R(\xi_1 - \alpha R'\xi_2') = 0$, i.e., $R(\xi_1 - \alpha R'\xi_2') \in Ker(D_2)$. Using $Ker(D_2) = im(D_2)$, we can find ξ_2'' such that $R(\xi_1 - \alpha R'\xi_2') = D_2\xi_2''$. Since R is injective on $\frac{Ker(D_1)}{im(D_1)}$ and

$R(\xi_1 - \alpha R'\xi_2') = D_2\xi_2''$ in $\frac{Ker(D_2)}{im(D_2)}$, there exists ξ_1' such that $\xi_1 - \alpha R'\xi_2' = D_1\xi_1'$, i.e., $\xi_1 = D_1\xi_1' + \alpha R'\xi_2'$. We have thus found $(\xi_1', \xi_2') \in \mathcal{E}$ such that $\nabla_{t_0}(\xi_1', \xi_2') = (\xi_1, \xi_2)$. Therefore, $im(\nabla_{t_0}) = ker(\nabla_{t_0})$. Q.E.D.

Corollary 9.1 *Let $\mathcal{A}$ be a trivially graded $G - C^*$-algebra. Assume that $(\mathcal{E}_i, Q_i, D_i)$ are in $\mathcal{L}^1_{G,u}(\mathcal{A})$ and $R \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ is equivariant, $i = 1, 2$. Let ε_i be the grading operator of $\mathcal{E}_i$ and T_i be the operator compatible with Q_i, $T_i^2 = I, i = 1, 2$. Suppose*

(a) $D_i' = -D_i, D_i^2 = 0, \varepsilon_i D_i = -D_i\varepsilon_i$;

(b) $R(dom(D_1)) \subset dom(D_2), RD_1 = D_2R, \varepsilon_2 R = R\varepsilon_1, T_2R = RT_1$;

(c) $R : \frac{Ker(D_1)}{im(D_1)} \to \frac{Ker(D_2)}{im(D_2)}$ is an isomorphism;

(d) there is equivariant $S \in \mathcal{L}(\mathcal{E}_1)$ such that $S(dom(D_1)) \subset dom(D_1)$, $-\varepsilon_1 S = S\varepsilon_i, -T_1S = ST_1$ and $I - R'R = D_1S + SD_1$;

(e) there is equivariant $\varepsilon \in \mathcal{L}(\mathcal{E}_1)$ such that $\varepsilon(dom(D_1)) \subset dom(D_1), \varepsilon D_1 = -D_1\varepsilon, \varepsilon^2 = I, \varepsilon' = \varepsilon$, $-\varepsilon_1\varepsilon = \varepsilon\varepsilon_1$, and $\varepsilon(I - R'R) = (I - R'R)\varepsilon$.

Then $\psi_{1,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{1,u}((\mathcal{E}_2, Q_2, D_2))$.

Proof. With the notations as in Proposition 9.1, it suffices to check that $(\mathcal{E}, B_t, D) \in \mathcal{L}^1_{G,u}(\mathcal{A}), 0 \leq t \leq 1, (\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^1_{G,u}(\mathcal{A}), 0 \leq t \leq t_0$. By the proof of Proposition 9.1, we need only to verify that B_t, D, C_t and ∇_t are of degree 1.

(1) Since $\varepsilon\varepsilon_1 = -\varepsilon_1\varepsilon$ and $\varepsilon_2 R = R\varepsilon_1$, $\tilde{\varepsilon}T_t = T_t\tilde{\varepsilon}$, where $\tilde{\varepsilon} = \varepsilon_1 \oplus (-\varepsilon_2)$ is the grading operator on $\mathcal{E}$. Thus, B_t is of degree 1.

(2) $\tilde{\varepsilon}L_t = L_t\tilde{\varepsilon}$. This follows also from $\varepsilon\varepsilon_1 = -\varepsilon_1\varepsilon$ and $\varepsilon_2 R = R\varepsilon_1$. Hence C_t is of degree 1.

Clearly, $\nabla_t\tilde{\varepsilon} = -\tilde{\varepsilon}\nabla_t$. Q.E.D.

We now use Proposition 9.1 to show the G-pseudo-equivalent invariance of $\psi_{0,u}(\mathcal{E}, Q, D)$ for a flat Hilbert $G-\mathcal{A}$-bundle E. The rest of this section is to devote to checking the conditions of Proposition 9.1.

Let $\mathcal{A}$ be a $G-C^*$-algebra, N and M be two compact oriented Riemannian G-manifolds and E a Hilbert $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert $G-\mathcal{A}$-module E_0. E is said to be G-unitary flat if E is furnished with an equivariant scalar product and antisymmetric flat equivariant connection. Let $f : N \to M$ be an equivariant smooth map. It is clear that $f^*(E)$ is an equivariant unitary flat bundle if E is. If f and f_1 are two equivariant smooth maps from N into M which are G-homotopic via G-homotopy H, then $H^* : f^*(E) \to f_1^*(E)$ induces an isomorphism. Suppose that ∇ is the antisymmetric flat equivariant connection on E and $D_E\xi = i^{\partial\xi}\nabla\xi$, then $D_{f^*(E)} = D_{f_1^*(E)}$. Observe that the map $f^* : C^\infty(M, \wedge_{\mathbb{C}}T^*M \otimes E) \to C^\infty(N, \wedge_{\mathbb{C}}T^*N \otimes f^*(E))$ may not be extended to a bounded operator from $\mathcal{E}_E = L^2(E)$ to $\mathcal{E}_{f^*(E)} = L^2(f^*(E))$, since f^* is not closed in general. To get around this problem, we follow [HiS] to use the embedding.

Let $\omega \in \wedge_{\mathbb{C}}T^*N$. Define $e_\omega : \wedge_{\mathbb{C}}T^*N \to \wedge_{\mathbb{C}}T^*N$ by $e_\omega(\alpha) = \omega \wedge \alpha$.

Lemma 9.10 ([*His*]) *Let M and N be compact oriented Riemannian G-manifolds.*

(a) Suppose $h : N \to M$ is a smooth equivariant map and $\omega \in \wedge_{\mathbb{C}}T^*N$ is equivariant. Let N_0 be the support of ω in N and h be a G-submersion from a neighborhood of N_0 into M. Then $e_\omega h^* : \mathcal{E}_E \to \mathcal{E}_{h^*(E)}$ is an equivariant operator in $\mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$, where E is a Hilbert $G-\mathcal{A}$-unitary flat bundle over M, and the norm of $e_\omega h^*$ is independent of E.

(b) Assume $h : N \times [0,1] \to M$ is a smooth equivariant map and $\omega \in \wedge_{\mathbb{C}} T^*N$ is a closed equivariant form on N, where G acts trivially on $[0,1]$. Suppose h (resp. h_0, h_1) is a G-submersion from a neighborhood of $N_0 \times [0,1]$ (resp. N_0) into M. If ∇ is an equivariant unitary flat connection on Hilbert $G - \mathcal{A}$-unitary flat bundle E over M, then there exists an equivariant operator $S \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$ such that

$$e_\omega h_1^* - e_\omega h_0^* = h_0^*(\nabla)S + S\nabla.$$

Proof. (a) Since $e_\omega h^*$ is equivariant, it suffices to show the boundedness of $e_\omega h^*$. The proof is the same as that of Lemma 3.2 in [HiS]. We provide a little detail below.

Using the partition of the unity, we have that the assertion is a local statement. Thus let $N = O \times M$ and E be trivial. $h : N \to M$ is then a projection. Let $\alpha dy_I \otimes \xi \in \wedge_{\mathbb{C}} T^*M \otimes E_0$.

$$h^*(\alpha dy_I \otimes \xi)_{(x,y)} = \alpha(h(x,y))dh_I \otimes h^*(\xi)_{(x,y)} = \alpha(y)dy_I \otimes \xi,$$

$$e_\omega h^*(\alpha dy_I \xi)_{(x,y)} = \omega(x,y) \wedge \alpha(y)dy_I \otimes \xi.$$

$$\begin{aligned} \|e_\omega h^*(\alpha dy_I \otimes \xi)\|^2 &= \| \int_N \overline{(\alpha(y)\omega(x,y) \wedge dy_I)} \wedge *(\alpha(y)\omega(x,y) \wedge dy_I) \otimes < \xi, \xi > \|^2 \\ &= \|\omega \wedge \alpha dy_I\|^2 \| < \xi, \xi > \|^2 \leq \|\omega\|^2 \|\alpha dy_I \otimes \xi\|^2 \end{aligned}$$

Hence, $\|e_\omega h^*\| \leq \|\omega\|$.

(b) Recall that the contraction $i_{\frac{\partial}{\partial t}}$ is defined by

$$i_{\frac{\partial}{\partial t}}(\alpha dx_I \wedge dt) = (-1)^{|I|}\alpha dx_I.$$

$i_{\frac{\partial}{\partial t}}$ is equivariant since G acts trivially on $[0,1]$. Define for $\omega_1 \in \mathcal{E}_E$,

$$S(\omega_1) = \int_0^1 i_{\frac{\partial}{\partial t}}(e_\omega h^*(\omega_1))dt \in L^2(N, \wedge_{\mathbb{C}} T^*N \otimes h_0^*(E)).$$

Here ω is considered as a smooth form on $N \times [0,1]$. By part (a), $e_\omega h^*$ is bounded, and $i_{\frac{\partial}{\partial t}}$ is bounded. S is thus bounded. Clearly, S is equivariant.

Since $e_\omega h_1^* - e_\omega h_0^* = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*)dt$, it is enough to show locally

$$h_0^*(\nabla)S + S\nabla = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*)dt.$$

Let $\omega_1 = \sum_I \alpha_I dy_I \otimes \xi$, $dy_I = dy_{i_1} \wedge \ldots \wedge dy_{i_I}$,

$$\begin{aligned} e_\omega h^*(\omega_1) &= \omega \wedge h^*(\omega_1) = \omega(z,y,t) \wedge h^*(\omega_1) \\ &= \sum_I \omega(z,y,t) \wedge \alpha_I(h(z,y,t))(\sum_k h^I_{z_k} dz^I_k + \sum_j h^I_{y_j} dy^I_j + h^I_t dt) \otimes \xi, \end{aligned}$$

where $h : O \times M \times [0,1] \to M$, z and y are the variables of O and M resp., and $h^I_{z_k}, h^I_{y_j}$ and h^I_t are the derivatives of h^I with respect to z_k, y_j and t, resp..

$$\frac{\partial}{\partial t}(e_\omega h^*(\omega_1))dt = \frac{\partial}{\partial t}[\sum_I \omega \wedge \alpha_I(h)(\sum_k h^I_{z_k} dz^I_k + \sum_j h^I_{y_j} dy^I_j)]dt \otimes \xi. \qquad (9.1)$$

Since ∇ is flat, we can assume $\nabla = d \otimes p$, p is locally a constant projection.

$$\nabla(\omega_1) = d(\sum_I \alpha_I dy_I) \otimes \xi = \sum_{I,s} \alpha_{I,s} dy_s \wedge dy_I \otimes \xi,$$

$$h^*\nabla(\omega_1) = \sum_{I,s} \alpha_{I,s}(h(z,y,t)) dh_s \wedge dh^I \otimes \xi,$$

$$\begin{aligned} e_\omega h^*\nabla(\omega_1) &= \sum_I \omega(z,y,t) \wedge [\sum_s \alpha_{I,s}(h(z,y,t))(\sum_k h_{s,z_k} dz_k + \sum_j h_{s,y_j} dy_j + h_{s,t} dt) \\ &\wedge \; (\sum_k h^I_{z_k} dz^I_k + \sum_j h^I_{y_j} dy^I_j + h^I_t dt) \otimes \xi], \end{aligned}$$

$$\int_0^1 [i_{\frac{\partial}{\partial t}}(e_\omega h^*\nabla(\omega_1))]dt = \int_0^1 \sum_I \omega(z,y,t) \wedge \{\sum_s \alpha_{I,s}(h(z,y,t))$$
$$(-\sum_k h_{s,z_k} h_t^I dz_k - \sum_j h_{s,y_j} h_t^I dy_j + \sum_k h_{s,t} h_{z_k}^I dz_k^I + \sum_j h_{s,t} h_{y_j}^I dy_j)\} dt \otimes \xi. \quad (9.2)$$

Now since $d\omega = 0$,

$$\begin{aligned} h_0^*(\nabla)(\int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega h^*(\omega_1)))dt) &= \int_0^1 \nabla(\omega \wedge \sum_I \alpha_I(h) h_t^I \otimes \xi)dt \\ = \quad & \int_0^1 \{((d\omega) \wedge \sum_I \alpha_I h_t^I)dt \otimes \xi + \omega \wedge \sum_I (\sum_s \alpha_{I,s}(h(z,y,t)) h_t^I dh_s + \\ + \quad & \sum_k \alpha_I(h(z,y,t)) h_{t,z_k} dz_k^I + \sum_j \alpha_I(h(z,y,t)) h_{t,y_j}^I dy_j^I)dt \otimes \xi\} \\ = \quad & \int_0^1 \omega \wedge \sum_I \{\sum_s \alpha_{I,s} h_t^I (\sum_k h_{s,z_k} dz_k + \sum_j h_{s,y_j} dy_j) + \\ + \quad & \sum_k \alpha_I h_{t,z_k}^I dz_k^I + \sum_j \alpha_I h_{t,y_j}^I dy_j^I\} dt \otimes \xi. \end{aligned} \quad (9.3)$$

Adding (9.2) – (9.3) together and using (9.1), we get

$$(h_0^*(\nabla)S + S\nabla)(\omega_1) = \int_0^1 \frac{\partial}{\partial t}(e_\omega h^*(\omega_1))dt.$$

Q.E.D.

Let $h_{!,\omega}$ be the conjugate of $e_\omega h^*$ with respect to Q, where $h : N \to M$ is a submersion in a neighborhood of the support of ω. Locally, we can find a formula for $h_{!,\omega}$. Let $h : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ be the projection. If ω is a continuous function with compact support in $\mathbb{R}^m \times \mathbb{R}^k$, then for $\alpha dx_I \otimes \xi \in \wedge_{\mathbb{C}} T^*\mathbb{R}^m$, $e_\omega h^*(\alpha dx_I \otimes \xi) = \omega \wedge \alpha(x) dx_I \otimes \xi$. We can calculate for $|J_1| = k$

$$Q(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta, e_\omega h^*(\alpha dx_I \otimes \xi)) = i^{(m-|I_1|)k} Q(h_{!,\omega}(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta), \alpha dx_I \otimes \xi),$$

with

$$h_{!,\omega}(\beta dx_{I_1} \wedge dy_{J_1} \otimes \eta) = i^{(m-|I_1|)|J_1|} dx_{I_1} \int \bar{\omega}(x,y) \beta(x,y) dy_{J_1} \otimes \eta,$$

and $h_{!,\omega} = 0$ for $|J_1| \neq k$.

let G act on $\mathbb{R}^k$ by isometries and B^k be the equivariant unit open ball in $\mathbb{R}^k$. Suppose that $P : N \times B^k \to M$ is an equivariant submersion and ν_k is an equivariant volume form of mass 1 on B^k. Let $q : N \times B^k \to N$ and $\pi_k : N \times B^k \to B^k$ be the projections. $\omega = \pi_k^*(\nu_k)$ and $P_0 = P|_{N \times \{0\}}$. Define

$$R_{p,\nu_k} = q_! e_\omega P^* \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{P_0^*(E)}),$$

where $q_! : \mathcal{E}_{P^*(E)} \to \mathcal{E}_{P_0^*(E)}$ is the pushforward. The following lemma is crucial in verifying the conditions of Proposition 9.1.

Lemma 9.11 *(a) Let $P' : N \times B^l \to M$ be another equivariant submersion and ν_l be an equivariant volume form of mass 1 on B^l. If the maps $P(x, o)$ and $P'(x, o)$ are equivariant homotopic, then there exists $S \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{P_0^*(E)})$ such that*

$$D_E S + S D_E = R_{P,\nu_k} - R_{P'\nu_l}.$$

(b) Let $P' : O \times B^l \to N$ be an equivariant submersion and ν_l an equivariant volume form of mass 1 on B^l, where O is a compact oriented Riemannian G-manifold. If $P'' : O \times B^l \times B^k \to M$ is given by $P''(z, s, t) = P(P'(z, s), t)$ and $\nu = \nu_l \times \nu_k$, then $R_{p',\nu_l} R_{p,\nu_k} = R_{p'',\nu}$.

Proof. Since the volume forms ν_k, ν_l and ν have the compact supports, there exists equivariant $\varphi \in C_c^\infty(B^k)$ such that $\varphi\nu_k = \nu_k$. As a consequence, $R_{p,\nu_k} = q_{!,\varphi} e_\omega P^*$.

(a) Let $\tilde{P} : N \times B^k \times B^l \times B^r \times [0, 1] \to M$ be an equivariant submersion such that $\tilde{P}(x, s, t, u, o) = P(x, s)$ and $\tilde{P}(x, s, t, u, 1) = P'(x, t)$. This $\tilde{P}$ exists since P and P' are submersions and homotopic. Let ν_r be a volume form of mass 1 on B^r with compact support. Define $\tilde{P}_i : N \times B^k \times B^l \times B^r \to M$ by $\tilde{P}_i(z) = \tilde{P}(z,i), z \in N \times B^k \times B^l \times B^r, i = 0,1$. Then $R_{p,\nu_k} = R_{\tilde{p}_0,\nu_k \times \nu_l \times \nu_r} = q_!^{k,l,r} e_{\tilde{\omega}} \tilde{P}_0^*$ and $R_{p',\nu_l} = R_{\tilde{p}_1,\nu_k \times \nu_l \times \nu_r} = q_!^{k,l,r} e_{\tilde{\omega}} \tilde{P}_1^*$, where $q^{k,l,r} : N \times B^k \times B^l \times B^r \to N, \pi_{k,l,r} : N \times B^k \times B^l \times B^r \to B^k \times B^l \times B^r$ are projections and $\tilde{\omega} = (\pi_{k,l,r})^*(\nu_k \times \nu_l \times \nu_r)$. By Lemma 9.10, there is $S_1 \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{\tilde{P}_0^*(E)})$ such that

$$q_!^{k,l,r}(e_{\tilde{\omega}} \tilde{P}_1^* - e_{\tilde{\omega}} \tilde{P}_0^*) = q_!^{k,l,r}(\tilde{P}_0^*(D) S_1 + S_1 D) = D(q_!^{k,l,r} S_1) + (q_!^{k,l,r} S_1) D.$$

Hence, $R_{p',\nu_l} - R_{p,\nu_k} = DS + SD, \ S = q_!^{k,l,r} S_1$.

(*b*) Let $\bar{P} : O \times B^k \times B^l \to N \times B^k, \bar{q} : O \times B^k \times B^l \to O \times B^l$ and $q' : O \times B^l \to O$ be defined by

$$\bar{P}(z,s,t) = (P'(z,t),s), \ \ \bar{q}(z,s,t) = (z,t), \ \ q'(z,t) = z.$$

Let $\pi_{k,l} : O \times B^k \times B^l \to B^k \times B^l$ and $\pi_l : O \times B^k \times B^l \to B^l$ be the projections and $\tilde{\omega}_1 = (\pi_{k,l})^*(\nu_k \times \nu_l), \omega_1 = (\pi_l)^*(\nu_l)$. Then for the projection $q^0 : O \times B^k \times B^l \to O$,

$$R_{p'',\nu} = q_!^0 e_{\tilde{\omega}_1} (P'')^* = q'_! \bar{q}_! e_{\omega_1} \bar{P}^* e_\omega P^*.$$

This identity is illustrated by the following picture:

$$\begin{tikzcd}
q_!^0(P'')^*(E) \arrow[d] & (P'')^*(E) \arrow[l, "q_!^0"'] \arrow[d] & (P'')^*(E) \arrow[l, "e_{\tilde{\omega}_1}"'] \arrow[dl] & E \arrow[l, "(P'')^*"'] \arrow[dl] \\
O & O \times B^k \times B^l \arrow[l, "q^0"'] \arrow[r, "P''"] & M &
\end{tikzcd}$$

$$\begin{tikzcd}
q'_! \bar{q}_*(\bar{P}^* P^*(E)) \arrow[d] & \bar{q}_*(\bar{P}^* P^*(E)) \arrow[l, "q'_!"'] \arrow[d] & \bar{P}^* P^*(E) \arrow[l, "\bar{q}_! e_{\omega_1}"'] \arrow[d] & P^*(E) \arrow[l, "\bar{P}^* e_\omega"'] \arrow[d] & E \arrow[l, "P^*"'] \arrow[d] \\
O & O \times B^l \arrow[l, "q'"'] & O \times B^k \times B^l \arrow[l, "\bar{q}"'] \arrow[r, "\bar{P}"] & N \times B^k \arrow[r, "P"] & M.
\end{tikzcd}$$

Here the following identities are used:

$$P''(z,s,t) = P(P'(z,s),t) = P\bar{P}(z,s,t),$$

$$q^0 = q'\bar{q} : O \times B^k \times B^l \xrightarrow{\bar{q}} O \times B^l \xrightarrow{q'} O,$$

$$\tilde{\omega}_1 = (\pi_l)^*(\nu_l) \wedge (\tilde{\pi})^*(\pi_k^*(\nu_k)) = \omega_1 \wedge (\tilde{\pi})^*(\omega),$$

$$\tilde{\pi} : O \times B^k \times B^l \to O \times B^k,\ \pi_k : O \times B^k \to B^k,\ e_{(\tilde{\pi})^*(\omega)}\bar{P}^* = \bar{P}^* e_\omega.$$

But $\bar{q}_! e_{\omega_1}\bar{P}^* = e_{\omega_1'}\bar{q}_!\bar{P}^* = e_{\omega_1'}P^{*'}q_!$ with $\omega_1' = (\pi_l)^*(\nu_l), \omega_1 = (\pi^l)^*(\nu_l)$, where $\pi_l : O \times B^l \to B^l$ and $\pi^l : O \times B^k \times B^l \to B^l$ are the projections. This can be seen from the following pictures:

$$\begin{array}{ccccc}
\bar{q}_*(\bar{P}^*P^*(E)) & \overset{\bar{q}_! e_{\omega_1}}{\longleftarrow} & \bar{P}^*P^*(E) & \overset{\bar{P}^*}{\longleftarrow} & P^*(E) \\
\downarrow & & \downarrow & & \downarrow \\
O \times B^l & \overset{\bar{q}}{\longleftarrow} & O \times B^k \times B^l & \overset{\bar{P}}{\longrightarrow} & O \times B^k,
\end{array}$$

$$\begin{array}{ccccccc}
P'^*q_*P^*(E) & \overset{e_{\omega_1'}}{\longleftarrow} & (P')^*q_*P^*(E) & \overset{(P')^*}{\longleftarrow} & q_*P^*(E) & \overset{q_!}{\longleftarrow} & P^*(E) \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & O \times B^l & \overset{P'}{\longrightarrow} & O & \overset{q}{\longleftarrow} & O \times B^k,
\end{array}$$

where as before, $\bar{P}(z,s,t) = (P'(z,t),s), q(z,s) = z, \bar{q}(z,s,t) = (z,t)$, and $P'(z,t) = z$. Therefore,

$$\begin{aligned}
R_{p'',\nu} &= q_!'\bar{q}_! e_{\omega_1}\bar{P}^* e_\omega P^* = q_! e_{\omega_1'}(P')^* q_! e_\omega P^* \\
&= R_{p',\nu_l} R_{p,\nu_k}.
\end{aligned}$$

Q.E.D.

We can now prove one of the main theorems in this chapter.

Theorem 9.1 *Let M and N be two compact oriented Riemannian G-manifolds of even dimensions. Suppose that $h : N \to M$ is an orientation preserving homotopy equivalence which is equivariant. Then for each unitary flat Hilbert $G - \mathcal{A}$-bundle E over M,*

$$\psi_{0,u}((\mathcal{E}_E, Q, D_E)) = \psi_{0,u}((\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})).$$

Proof. We need only to check that $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$ satisfy the conditions of Proposition 9.1. By the observation preceding Lemma 9.9, we can assume that h is smooth.

Let $J : M \to \mathbb{R}^k$ be an equivariant embedding ($k \equiv 0(4)$), O an equivariant tubular neighborhood of $J(M)$ in $\mathbb{R}^k$ and $\pi : O \to M$ be the associated equivariant projection (see [Bre]). Suppose $J(M) + B^k \subset O$. Define a submersion $P : N \times B^k \to M$ by $P(x,t) = \pi(h(x)+t), x \in N, t \in B^k$. Let ν_k be an equivariant volume form of mass 1 on B^k and φ be an equivariant smooth map such that $\varphi\nu_k = \nu_k$. With the notations as in the paragraph preceding Lemma 9.11, we divide the proof into several steps.

Step 1. Condition (a) of Proposition 9.1 is valid already since $D^2 = 0$ by the flatness. To check condition (b) of Proposition 9.1, we define R to be $R_{p,\nu_k} \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$, i.e., $R = q_{!,\varphi} e_\omega P^*$. Note that $P_0(x) = P(x,o) = \pi(h(x)) = h(x)$. Clearly, $q_{!,\varphi}, e_\omega$ and P^* preserve the domain of D and commute with D. In fact, since P is homotopic to hq,

$$\begin{aligned} (e_\varphi q^*)' e_\omega P^* \nabla &= (e_\varphi q^*)' e_\omega P^*(\nabla) P^* = (e_\varphi q^*)' P^*(\nabla) e_\omega P^* \\ &= h^*(\nabla)(e_\varphi q^*)' e_\omega P^*. \end{aligned}$$

Hence, $R(dom(D)) \subset dom(D)$ and $RD_E = D_{h^*(E)} R$.

Let $\varepsilon \in \mathcal{L}_G(\mathcal{E}_E)$ be defined by $\varepsilon(\xi) = (-1)^{\partial\xi}\xi$. Then $\varepsilon^2 = I, \varepsilon' = \varepsilon, \varepsilon D = -D\varepsilon, \varepsilon P^* = P^*\varepsilon, \varepsilon e_\omega = (-1)^k e_\omega \varepsilon$ and $\varepsilon(e_\varphi h)' = (-1)^k (e_\varphi h^*)'\varepsilon$. Hence $\varepsilon(I - R'R) = (I - R'R)\varepsilon$. This verifies condition (e) of Proposition 9.1.

Step 2. We use Lemma 9.11 to verify condition (c) of Proposition 9.1, i.e., $R : \frac{Ker(D_E)}{im(D_E)} \to \frac{Ker(D_{h^*(E)})}{im(D_{h^*(E)})}$ is an isomorphism. By Lemma 9.11(a), the map $R = R_{p,\nu}$ from $\frac{Ker(D_E)}{im(D_E)}$ to $\frac{Ker(D_{h^*(E)})}{im(D_{h^*(E)})}$ is independent of the submersion P and volume form ν, i.e., $R_{p,\nu} = R_{p',\nu'}$ on $\frac{Ker(D_E)}{im(D_E)}$ as long as $P'(*, o)$ is homotopic to $P(*, o)$. Let $h' : M \to N$ be the smooth map such that hh' and $h'h$ are (non equivariantly) homotopic to I_M and I_N, resp.. As we construct the submersion $P : N \times B^k \to M$ from h, we can find a submersion $P' : M \times B^l \to N$ from h', namely, $P'(x, t) = \pi'(h'(x) + t)$ and $\pi' : O' \to N$ is a projection with O' a tubular neighborhood of $J'(N) \subset \mathbb{R}^l$. Then define $P'' : M \times B^l \times B^k \xrightarrow{P'} N \times B^k \xrightarrow{P} M$ by $P''(x, s, t) = P(P'(x, s), t)$. Clearly, $P''(x, o, o) = P(h'(x), o) = \pi(h(h'(x)) + o) = h(h'(x)) \overset{\text{homotopic}}{\rightsquigarrow} I(x)$. Hence $R_{p'',\nu''} = I$ on $\frac{Ker(D_E)}{im(D_E)}$ by Lemma 9.11(a). Furthermore, using Lemma 9.11(b), we obtain $R_{p',\nu'}R_{p,\nu} = R_{p'',\nu''} = I$ on $\frac{Ker(D_E)}{im(D_E)}$. This proves that $R_{p,\nu}$ is injective. Same reasoning shows by switching the role of $R_{p',\nu'}$ and $R_{p,\nu}$ that $R_{p,\nu}$ is surjective in $\frac{Ker(D_E)}{im(D_E)}$. Therefore, $R = R_{p,\nu}$ is an isomorphism. Note that this step does not require the equivariance of various maps.

Step 3. We now check condition (d) of Proposition 9.1, i.e., there is equivariant $S \in \mathcal{L}_G(\mathcal{E}_E)$ such that $S(dom(D_E)) \subset dom(D_E)$ and $I - R'R = D_E S + SD_E$.

Let $q_i : N \times B^k \times B^k \to N \times B^k$ be the projections, $q_i(x, t_1, t_2) = (x, t_i), i = 1, 2$, and $q : N \times B^k \to N$ and $\pi_k : N \times B^k \to B^k$ be the projections and $\omega = \pi_k^*(\nu_k)$. To find the conjugate of $e_{q_1^*(\omega)}q_2^*$, we compute for $\tilde{\eta} = \alpha dx_I \wedge dt_{J_2} \otimes \eta$

and $\tilde{\xi} = \beta dx_{I_1} \wedge dt_{J_1} \wedge dt_{J_2'} \otimes \xi$,

$$Q(\tilde{\xi}, (e_{q_1^*(\omega)} q_2^*)(\tilde{\eta})) = i^{k(|I_1|+|J_2'|)} Q((e_{q_1^*(\omega)} q_2^*)'(\tilde{\xi}), \tilde{\eta})$$

with

$$(e_{q_1^*(\omega)} q_2^*)'(\tilde{\xi}) = i^{k(|I_1|+|J_2'|)} (\int_{B^k} \beta \overline{q_1^*(\omega)}) dx_{I_1} \wedge dT_{J_2'}$$

for $|J_1| = 0$, and 0 otherwise. In particular, if $\tilde{\xi} = e_{q_2^*(\omega)} q_1^*(\tilde{\eta}) = i^{2k|I_1|} \alpha(q_1) dx_{I_1} \wedge q_2^*(\omega) \wedge dt_{J_1}$, then

$$(e_{q_1^*(\omega)} q_2^*)'(e_{q_2^*(\omega)} q_1^*)(\alpha dx_I \wedge dt_{J_1} \otimes \eta) = i^{k(3|I_1|+k)} \int_{B^k} \alpha(q_1) q_1^*(\omega) dx_{I_1} \wedge q_2^*(\omega) \otimes \eta$$

$$= i^{-k|I_1|+k^2} (\int_{B^k} \alpha(q_1) q_1^*(\omega)) dx_{I_1} \wedge q_2^*(\omega) \otimes \eta, \quad |J_1| = 0, \tag{9.4}$$

and it is zero for $|J_1| \neq 0$. On the other hand,

$$\begin{aligned} (q_! e_\omega)(\alpha dx_{I_1} \wedge dt_{J_1}) \otimes \eta &= q_!(\omega \wedge dx_{I_1} \wedge dt_{J_1} \otimes \eta) \\ &= i^{k(n+|I_1|)} (\int_{B^k} \varphi \alpha \omega) dx_{I_1} \otimes \eta. \end{aligned}$$

If $\tilde{\xi} = \beta dx_{I_1'} \otimes \xi$, then

$$\begin{aligned} Q(\tilde{\xi}, q_! e_\omega(\alpha dx_{I_1} \wedge dt_{J_1})) &= i^{kn} i^{(|I_1'|+k)(n+k-|I_1'|-k)} \int_{N \times B^k} (\bar{\beta} \varphi \omega \wedge dx_{I_1'}) \wedge \alpha dx_{I_1} \\ &\otimes < \xi, \eta > = i^{kn} Q((q_! e_\omega)'(\tilde{\xi}), \tilde{\eta}). \end{aligned}$$

Consequently, $(q_! e_\omega)'(\tilde{\xi}) = i^{kn} \beta \omega \wedge dx_{I_1'} \otimes \xi$. Using this identity, we have

$$\begin{aligned} (q_! e_\omega)'(q_! e_\omega)(\alpha dx_{I_1} \otimes \eta) &= i^{kn} i^{k(n+|I_1|)} (\int_{B^k} \alpha \omega) \omega \wedge dx_{I_1} \otimes \eta) \\ &= i^{2kn-k|I_1|} (\int_{B^k} \alpha \omega) dx_{I_1} \wedge \omega \otimes \eta. \end{aligned}$$

Comparing this with (9.4), we get

$$(q_! e_\omega)'(q_! e_\omega) = i^{2kn-k^2} (e_{q_1^*(\omega)} q_2^*)'(e_{q_2^*(\omega)} q_1^*). \tag{9.5}$$

Furthermore, $(e_{q_2^*(\omega)}q_2^*)(\alpha dx_I \wedge dt_{J_2} \otimes \eta) = q_2^*(\omega) \wedge \alpha(q_2) dx_I \wedge dt_{J_2} \otimes = q_2^*(\omega) \wedge \alpha dx_I \otimes \eta, |J_2| = 0,$

$$\begin{aligned}(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_2^*)(\tilde{\eta}) &= i^{2k|I|+k(|I|+k)}(\int_{B^k} \alpha\overline{q_1^*(\omega)})dx_I \wedge q_2^*(\omega) \otimes \eta \\ &= i^{k^2-k|I|}(\int_{B^k} \overline{q_1^*(\omega)})\alpha dx_I \wedge q_2^*(\omega) \otimes \eta.\end{aligned}$$

But $e_\omega(\alpha dx_I \otimes \eta) = i^{2k|I|}\alpha dx_I \wedge \otimes \eta$. Therefore,

$$e_\omega = i^{-k^2-k|I|}(e_{q_1^*(\omega)}q_2^*)'(e_{q_2^*(\omega)}q_2^*). \tag{9.6}$$

This together with (9.5) proves $e_\omega = i^{-k(2n+|I|)}(q_! e_\omega)'(q_! e_\omega)$ modulo the boundary $D_E S + S D_E$ by Lemma 9.10. This implies that $R'R = (P^*)'(q_! e_\omega)'(q_! e_\omega)P^* = P_!(q_! e_\omega)'(q_! e_\omega)P^* = P_! e_\omega P^*$ modulo the boundary $D_E S + S D_E$. Hence it suffices to show that $P_! e_\omega P^* = I$ modulo the boundary $D_E S + S D_E$.

Let $P_1 : M \times B_k \to M, P_2 : M \times B^k \to B^k$ be the projections and $\tilde{\pi} : M \times B_k \to M$ be an equivariant submersion, $\tilde{\pi}(x,t) = \pi(x+t)$ and $\omega' = P_2^*(\nu_k)$. Let $P : N \times B^k \to M$ be the equivariant submersion, $P(x,t) = \pi(h(x)+t)$.

(1) $(P_1)_! e_{\omega'} P_1^*$ is the identity:

$$\begin{aligned}(P_1)_! e_{\omega'} P_1^*(\alpha dx_I \otimes \eta) &= (P_1)_! e_{\omega'} \alpha(P_1) dx_I \otimes \eta = i^{2k|I|}(P_1)_{!,\varphi}(\alpha(P_1)dx_I \wedge \omega') \\ &= i^{(n+|I|)k}(\int \varphi\omega')\alpha dx_I \otimes \eta = \alpha dx_I \otimes \eta, k \equiv 0(4).\end{aligned}$$

(2) $P_! e_\omega P^* = \tilde{\pi}_! e_{\omega'} \tilde{\pi}^*$:

$$\begin{aligned}P_! e_\omega P^*(\alpha dx_I \otimes \eta) &= i^{2|I|k} P_!(\alpha(P) dP_I \wedge \omega \otimes \eta) \\ &= i^{2|I|k+(n-|I|)k} \int_{N\times B^k} \alpha(P)\omega dP_I \otimes \eta \\ &= i^{k(n+|I|)} \int_{N\times B^k} \alpha(\pi(h(x)+t))\nu_k(\pi_k(x,t))d(\pi(h(x)+t))_I \otimes \eta \\ &= i^{k(n+|I|)} \int_{M\times B^k} \alpha(\pi(y+t))\nu_k(t)d\pi_I \otimes \eta \\ &= i^{2k|I|}\tilde{\pi}_!(\alpha(\tilde{\pi})d\pi_I\omega' \otimes \eta) = (\tilde{\pi}_! e_{\omega'}\tilde{\pi}^*)(\alpha dx_I \otimes \eta).\end{aligned}$$

Finally, $\tilde{\pi}_! e_{\omega'} \tilde{\pi}^* = (P_1)_! e_{\omega'} P_1^*$ modulo the boundary $D_E S + S D_E$ by Lemma 9.11. Therefore, $R'R = I$ modulo the boundary $D_E S + S D_E$. Q.E.D.

Theorem 9.1 will play a crucial role in proving the equivariant Novikov conjecture. It provides a substitute for the equivariant Miscenko symmetric signature as we pointed out early.

9.3 Equivariant Connes-Gromov-Moscovici Theorem

Let $\mathcal{A}$ be a $G - C^*$-algebra, E_0 a finitely generated projective Hilbert $G - \mathcal{A}$-module, and M a compact oriented Riemannian G-manifold. Let E be a Hilbert $G - \mathcal{A}$-bundle over M whose fiber is E_0. Suppose ∇ is a unitary connection of E and $\Theta = \nabla^2 \in C^\infty(M, \wedge_{\mathbb{C}} T^*M \otimes E)$ is its curvature. Let $\nu > 0$. ∇ is said to be ν-flat if

$$\|\Theta\| = \max_{x \in M}\{\|\Theta_x\|\} < \nu,$$

where $\|\Theta_x\|$ is the norm of operator Θ_x on $\wedge_{\mathbb{C}} T^*M$,

$$\|\Theta_x\| = \max_{X,Y \in TM, \|X\|, \|Y\| \leq 1} \|\Theta_x(X,Y)\|.$$

The norm of $\|\Theta\|$ is equal to the norm of Θ as an operator on Hilbert $G - \mathcal{A}$-module $\mathcal{E}_E$. E is called almost flat if for any $\nu > 0$ there is a ν-flat unitary connection on E.

To prove our main theorem of this section, we need a proposition which is the generalization of Proposition 9.1. We first have the following lemma.

Lemma 9.12 *(a) Let $\nu, k > 0$ be such that $6\sqrt{2}\nu k^2 < 1$. Suppose that U is an equivariant regular operator on Hilbert $G-\mathcal{A}$-module $\mathcal{E}$ such that $im(U) \subset dom(U), \|U^2\| < \nu^2$. If there are equivariant $R, S \in \mathcal{L}_G(\mathcal{E})$, S invertible such that $R(dom(U)) \subset dom(U), RU + UR = S, \|R\| \leq k$ and $\|S^{-1}\| \leq k$, then $U + U^*$ has a bounded inverse with the norm*

$$\|(U+U^*)^{-1}\| \leq \frac{2k^2}{1-6\sqrt{2}\nu k^2}.$$

(b) Let $\nu, k > 0$ be such that $2\sqrt{2}\nu k^4(k+2\sqrt{k}) < 1$. Suppose that $(\mathcal{E}, Q, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$, and $T \in \mathcal{L}_G(\mathcal{E})$ is invertible and equivariant such that $Q(\xi, \eta) =< \xi, T\eta >$ for the scalar product $< .,. >$ on $\mathcal{E}$. If $\|D^2\| \leq \nu^2, \|D' + D\| \leq \nu$ and there are R and $S \in \mathcal{L}_G(\mathcal{E})$ equivariant with S invertible such that $R(dom(D)) \subset dom(D)$ and $DR \in \mathcal{L}_G(\mathcal{E})$, $DR + RD = S, \|R\| \leq k, \|S^{-1}\| \leq k$ and $\|T\|\|T^{-1}\| \leq k^2$, then $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$.

Proof. We first prove part (a). Let

$$U_1 = \begin{bmatrix} U & U^2/\nu \\ -\nu & -U \end{bmatrix}, R_1 = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \text{ and } S_1 = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}.$$

Then $U_1^2 = 0$, and $R_1U_1 + U_1R_1 - S_1 = \begin{bmatrix} 0 & (RU^2 - U^2R)/\nu \\ 0 & 0 \end{bmatrix}$. Since $\|R_1U_1 - U_1R_1\| = \|(RU^2 - U^2R)/\nu\| \leq \frac{2}{\nu}\|R\|\|U^2\| \leq 2k\nu < 1$, and $\|S_1^{-1}(R_1U_1 + U_1R_1 - S_1)\| \leq \|S_1^{-1}\|\|(R_1U_1 + U_1R_1 - S_1)\| \leq \sqrt{2}k(2k\nu) = 2\sqrt{2}k^2\nu < 1$, $R_1U_1 + U_1R_1$ has the inverse

$$(R_1U_1 + U_1R_1)^{-1} = (I + S_1^{-1}(R_1U_1 + U_1R_1 - S_1))^{-1}S_1^{-1},$$

and

$$\|(R_1U_1+U_1R_1)^{-1}\| \leq \|(I+S_1^{-1}(R_1U_1+U_1R_1-S_1))^{-1}S_1^{-1}\| \leq \frac{1}{1-2\sqrt{2}k^2\nu}\sqrt{2}k.$$

Now

$$U_1^* = \begin{bmatrix} U^* & -\nu \\ U^{2*}/\nu & -U^* \end{bmatrix}, \quad U_1 + U_1^* = \begin{bmatrix} U+U^* & U^2/\nu - \nu \\ -\nu + U^*/\nu & -(U+U^*) \end{bmatrix}.$$

We show that $U_1 + U_1^*$ is invertible and $\|(U_1 + U_1^*)^{-1}\| \leq \frac{2k^2}{1-2\sqrt{2}k^2\nu}$. In fact, U_1 commutes with $U_1R_1 + R_1U_1$ since $U_1^2 = 0$. Let $W = (R_1U_1 + U_1R_1)^{-1}$. Then U_1 also commutes with W and $U_1WR_1 + WR_1U_1 = W(R_1U_1 + U_1R_1) = I$. Since $U_1R_1 = (R_1U_1 + U_1R_1)WU_1R_1 = U_1R_1WU_1R_1$ and $R_1U_1 = R_1U_1W(R_1U_1 + U_1R_1) = R_1U_1WR_1U_1$, $WU_1R_1 = WU_1R_1U_1R_1$ and $WR_1U_1 = WR_1U_1WR_1U_1$, i.e., WU_1R_1 and WR_1U_1 are idempotents. Thus, we can find a projection P such taht $P(WU_1R_1) = P$ and $(WU_1R_1)P = WU_1R_1$. By Remark 9.2, $(U_1 + U_1^*)$ has the inverse $(U_1 + U_1^*)^{-1} = P(WR_1)(I-P) + (I-P)(WR_1)^*P$. Hence, $\|(U_1 + U_1^*)^{-1}\| \leq \|WR_1\| \leq \frac{\sqrt{2}k}{1-2\sqrt{2}k^2\nu}\sqrt{2}k = \frac{2k^2}{1-2\sqrt{2}k^2\nu} \stackrel{\text{def}}{=} k_1$. We get from this estimation that for

$$Z \stackrel{\text{def}}{=} (U \oplus (-U)) + (U \oplus (-U))^* - (U_1 + U_1^*) = \begin{bmatrix} 0 & \nu - U^2/\nu \\ \nu - U^2/\nu & 0 \end{bmatrix},$$

$\|Z\| \leq \sqrt{2}\|\nu - U^2/\nu\| = 2\sqrt{2}\nu$ and

$$((U\oplus(-U))+(U\oplus(-U))^*)^{-1} = (Z+(U_1+U_1^*))^{-1} = (I+((U_1+U_1^*))^{-1}Z)^{-1}(U_1+U_1^*))^{-1},$$

since $\|(U_1 + U_1^*)^{-1}Z\| \leq \frac{2k^2}{1-2\sqrt{2}k^2\nu}2\sqrt{2}\nu < 1$, i.e., $6\sqrt{2}\nu k^2 < 1$. Then

$$\begin{aligned} \|((U \oplus (-U)) + (U \oplus (-U))^*)^{-1}\| &= \|(I + ((U_1 + U_1^*))^{-1}Z)^{-1}\|\|(U_1 + U_1^*))^{-1}\| \\ &= \frac{1}{1 - \frac{2k^2}{1-2\sqrt{2}k^2\nu}2\sqrt{2}\nu}\frac{2k^2}{1-2\sqrt{2}k^2\nu} = \frac{2k^2}{1-6\sqrt{2}\nu k^2}. \end{aligned}$$

We now prove part (2). Define a new scalar product on $\mathcal{E}$ by $(\xi,\eta) = <\xi,|T|\eta>$. If $Z \in \mathcal{L}(\mathcal{E})$, then

$$\begin{aligned} \|Z\|_{(.)}^2 &= \max_{\|\xi\|_{(.)}\leq 1} \|(Z\xi,Z\xi)\| \\ &= \max_{\||T|^{\frac{1}{2}}\xi\|\leq 1} \| < Z|T|^{-\frac{1}{2}}(|T|^{\frac{1}{2}}\xi), |T|Z|T|^{-\frac{1}{2}}(|T|^{\frac{1}{2}}\xi) > \| \\ &= \max_{\||T|^{\frac{1}{2}}\xi\|\leq 1} \||T|\|\|Z\|^2\||T|^{-\frac{1}{2}}\|^2\||T|^{\frac{1}{2}}\xi\|^2 \\ &= \|Z\|^2\||T|^{-\frac{1}{2}}\|^2\||T|\|, \end{aligned}$$

namely, $\|Z\|_{(.)} \leq k\|Z\|$. With the scalar product (.) on $\mathcal{E}$, we have $\|D^2\|_{(.)} \leq k\|D\| \leq k\nu^2, \|D'+D\|_{(.)} \leq k\|D'+D\| \leq k\nu, \|R\|_{(.)} \leq k\|R\| \leq k^2$ and $\|S_1\|_{(.)} \leq k\|S^{-1}\| \leq k^2$. Using part (a) with k and ν replaced by k^2 and $\sqrt{k}\nu$, we obtain that $D+D^*$ is invertible and $\|(D+D^*)^{-1}\| \leq \frac{2(k^2)^2}{1-6\sqrt{2}\sqrt{k}\nu(k^2)^2}$, since $6\sqrt{2}\sqrt{k}\nu(k^2)^2 = 6\sqrt{2}\nu\sqrt{k}k^4 < 1$ by the assumption. Thus we are done for $D' = -D$.

Note that $(\xi,\eta) =< \xi, |T|\eta >= Q(\xi, T^{-1}|T|\eta)$. Let $T_1 = T^{-1}|T|, T_1^2 = I$. T_1 is compatible with Q. Let $P_1 = \frac{I+T_1}{2}, P_1^2 = P_1 = P_1T_1$ and $(I-P_1)T_1 = T_1 - P_1 = -(I-P_1)$. Then $(D+D^*)^{(0)} = P_1(D+D^*)P_1 + (I-P_1)(D+D^*)(I-P_1) = P_1(D+D')P_1 + (I-P_1)(D+D')(I-P_1)$. It yields that $\|(D+D^*)^{(0)}\| \leq \sqrt{2}\|D+D'\|$ and then

$$\begin{aligned} \|(D+D^*)-(D+D^*)^{(1)}\|\|(D+D^*)^{-1}\| &= \|(D+D^*)^{(0)}\|\|(D+D^*)^{-1}\| \\ &\leq \sqrt{2}k\nu\frac{2(k^2)^2}{1-6\sqrt{2}\sqrt{k}\nu(k^2)^2} < 1, \end{aligned}$$

since $2\sqrt{2}\nu k^4(k+3\sqrt{k}) < 1$. Therefore $(D+D^*)^{-1}$ is invertible:

$$\begin{aligned} ((D+D^*)^{(1)})^{-1} &= ((D+D^*) - ((D+D^*)-(D+D^*)^{(1)})^{-1} \\ &= (I-(D+D^*)^{-1}((D+D^*)-(D+D^*)^{(1)}))^{-1}(D+D^*)^{-1}. \end{aligned}$$

Observe that the map $\psi_{u,0}$ in Definition 9.3 can be defined for $D' \neq -D$, in which one replaces $(D + D^*)$ by $(D + D^*)^{(1)}$ in the formula of F. We have shown that $(D + D^*)^{(1)}$ is invertible. Hence, $\psi_{u,0}((\mathcal{E}, Q, D)) = 0$. Q.E.D.

Proposition 9.2 *Let $(\mathcal{E}_i, Q_i, D_i) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ be such that $D_i' = -D_i$ and $\mathcal{E}_i$ have the scalar products compatible with $Q_i, i = 0, 1$. If there are equivariant $R \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2), \nu, k > 0$ satisfying*

(a) $\|D_i^2\| \leq \nu, \|R\| \leq k, R(dom(D_1)) \subset dom(D_1)$ and $\|RD_1 - D_2R\| \leq \nu^2$;

(b) there is equivariant $S \in \mathcal{L}(\mathcal{E}_1)$ such that $\|S\| \leq k, S(dom(D_1)) \subset dom(D_1)$, and $\|I - R'R - D_1S - SD_1\| \leq \nu^2$;

(c) there is equivariant $\varepsilon \in \mathcal{L}(\mathcal{E}_1)$ such that $\varepsilon(dom(D_1)) \subset dom(D_1), \varepsilon^2 = I, \varepsilon' = \varepsilon$ and $\varepsilon(I - R'R) = (I - R'R)\varepsilon$;

(d) there is $Z \in \mathcal{L}(\mathcal{E}_2)$ equivariant such that $\|Z\| \leq k, Z(dom(D_2)) \subset dom(D_2)$ and $\|I - RR' - D_2Z - ZD_2\| \leq \nu^2$,

then for $64\nu(k + 4\sqrt{2}k^3(1 + k)^2)^5 < 1, \nu < \frac{1}{2\sqrt{2}2^{\frac{1}{4}}}$ and $k \geq 1$,

$$\psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{0,u}((\mathcal{E}_2, Q_2, D_2)).$$

Furthermore, if (d) is replaced by (d'):

(d') there are equivariant $W \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$ and $Z \in \mathcal{L}_G(\mathcal{E}_2)$ such that $\|Z\| \leq k, \|W\| \leq k, W(dom(D_2)) \subset dom(D_2), Z(dom(D_2)) \subset dom(D_2), \|WD_2 - D_1W\| \leq \nu^2$ and $\|I - RW - D_2Z - ZD_2\| \leq \nu^2$, then for $64\nu\sqrt{1 + 6k^2}\{k(1 + 2k^2) + 4\sqrt{2}k^3(1 + 2k^2)^3(1 + k(1 + 2k^2))^2\}^5 < 1$, and $\nu < \frac{1}{2\sqrt{2}2^{\frac{1}{4}}\sqrt{1+6k^2}}$ and $k \geq 1$,

$$\psi_{0,u}((\mathcal{E}_1, Q_1, D_1)) = \psi_{0,u}((\mathcal{E}_2, Q_2, D_2)).$$

Proof. The proof is divided into several steps.

Step 1. As in the proof of Proposition 9.1, we can assume $S = -S'$ and $Z = -Z'$ by considering $\frac{S-S'}{2}$ and $\frac{Z-Z'}{2}$ that satisfy (b) and (d).

Let $(\mathcal{E}, Q, D) = (\mathcal{E}_1 \oplus \mathcal{E}_2, Q_1 \oplus (-Q_2), D_1 \oplus (-D_2))$. By Lemma 9.7, it suffices to show $\psi_{0,u}((\mathcal{E}, Q, D)) = 0$. The proof is similar to that of Proposition 9.1.

Define for $t \in [0,1]$

$$T_t = \begin{bmatrix} I & 0 \\ itR\varepsilon & I \end{bmatrix}, L_t = \begin{bmatrix} I - R'R & (i\varepsilon + tS)R' \\ R(i\varepsilon + tS) & I \end{bmatrix}, \nabla_t = \begin{bmatrix} D_1 & tR' \\ 0 & -D_2 \end{bmatrix}.$$

These operators are all equivariant. Let $B_t(\xi, \eta) = Q(T_t\xi, T_t\eta)$ and $C_t(\xi, \eta) = Q(L_t\xi, \eta)$. We check that $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

If D'_t denotes the conjugate of D with respect to B_t, then as before, $D'_t = (T'_tT_t)^{-1}D'(T'_tT_t) = -(T'_tT_t)^{-1}D(T'_tT_t)$. since

$$T_tD - DT_t = \begin{bmatrix} 0 & 0 \\ -it(RD_1 - D_1R)\varepsilon & 0 \end{bmatrix},$$

$T_tD - DT_t$ is in $\mathcal{L}_G(\mathcal{E})$ by condition (a). This implies that $D + D'_t \in \mathcal{L}_G(\mathcal{E})$ since $D + D'_t = (T'_tT_t)^{-1}(T'_t(T_tD - DT_t) + (T'_tD - DT'_t)T_t)$. Now

$$\|[T_t, D]\| \leq t\|RD_1 - D_2R\| \leq t\nu^2,$$

$$\|[T'_t, D]\| \leq t\|R'D_2 - D_1R'\| \leq t\nu^2,$$

and $\|T_t\| \leq (\|R\|t + 1)\sqrt{2} \leq \sqrt{2}(1 + tk), \|T'_t\| \leq \sqrt{2}(1 + tk)$. Here we endow the norm $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ on $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Clearly,

$$T_t^{-1} = \begin{bmatrix} I & 0 \\ -itR\varepsilon & I \end{bmatrix}, (T'_t)^{-1} = \begin{bmatrix} I & -it\varepsilon R' \\ 0 & I \end{bmatrix}, (T'_tT_t)^{-1} = \begin{bmatrix} I & -it\varepsilon R' \\ -itR\varepsilon & t^2R\varepsilon R' + I \end{bmatrix}.$$

We have $\|(T_t'T_t)^{-1}\| \leq \sqrt{2}(1+t^2\|RR'\|+t\|R\|) \leq \sqrt{2}(1+tk^2+tk)$, and

$$\begin{aligned}\|D_1+D_t'\| &= \|(T_t'T_t)^{-1}\|(\|T_t\|\|[T_t,D]\|+\|[T_t',D]\|\|T_t\|)\\ &= 4t\nu^2(1+tk)(1+t^2k+tk).\end{aligned}$$

This proves $D+D_t' \in \mathcal{L}_G(\mathcal{E})$. Obviously, $D^2 \in \mathcal{L}_G(\mathcal{E})$ and as in the proof of proposition 9.1, we can find equivariant R_1 and $S_1 \in \mathcal{L}_G(\mathcal{E})$ such that $DS_1+R_1D-I \in \mathcal{K}_G(\mathcal{E})$. we have thus verified $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

Step 2. We now check that $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A})$ for t near 0. Since $L_0 = T_1'T_1$, $\|L_0^{-1}\| \leq \|T_1^{-1}\|\|(T_1')^{-1}\| \leq (\sqrt{2}(1+k))^2 = 2(1+k)^2$. Note that

$$L_t = \begin{bmatrix} I-R'R & i\varepsilon R' \\ iR\varepsilon & I \end{bmatrix} + \begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix} = L_0(I+L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix}).$$

we see that if $\|L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix}\| \leq 2(1+k)^2\sqrt{2}\|S\|\|R\|t \leq 2\sqrt{2}(1+k)^2k^2t < 1$, then L_t is invertible. Let $t_0 = \frac{1}{4\sqrt{2}(1+k)^2k^2}$.

$$\|L_t^{-1}\| \leq \|(I+L_0^{-1}\begin{bmatrix} 0 & tSR' \\ tRS & 0 \end{bmatrix})^{-1}\|\|L_0^{-1}\| \leq \frac{1}{1-2\sqrt{2}(1+k^2)^2k^2t}(2(1+k)^2).$$

In particular, for $t=t_0$, $\|L_{t_0}^{-1}\| \leq 4(1+k)^2$. Since $C_t(\xi, \nabla_t\eta) = Q(L_t\xi, \nabla_t\eta) = C_t(L_t^{-1}\nabla_t'L_t\xi, \eta)$, the conjugate ∇_{t,C_t}' of ∇_t with respect to C_t is $\nabla_{t,C_t}' = L_t^{-1}\nabla_t'L_t$. We have

$$L_t\nabla_t = \begin{bmatrix} (I-R'R)D_1 & t(I-R'R)R'-(i\varepsilon+tS)R'D_2 \\ R(i\varepsilon+tS)D_1 & tR(i\varepsilon+tS)R'-D_2 \end{bmatrix},$$

$$\nabla'_t L_t = \begin{bmatrix} D'_1(I - R'R) & D'_1(i\varepsilon + tS)R' \\ -tR(I - R'R) - D'_2 R(i\varepsilon + tS) & -tR(i\varepsilon + tS)R' - D_2 \end{bmatrix}.$$

This implies that

$$L_t\nabla_t + \nabla'_t L_t = \begin{bmatrix} b & u_t \\ v_t & 0 \end{bmatrix},$$

where

$$\begin{aligned} b &= (I - R'R)D_1 - D_1(I - R'R) = (D_1R' - R'D_2)R + R'(D_2R - RD_1), \\ v_t &= R(i\varepsilon + tS)D_1 - tR(I - R'R) + D_2R(i\varepsilon + tS) \\ &= (D_2R - RD_1)(i\varepsilon + tS) + tR(D_1S + SD_1 - (I - R'R)), \end{aligned}$$

and

$$\begin{aligned} u_t &= -D_1(i\varepsilon + tS)R' + t(I - R'R)R' - (i\varepsilon + tS)R'D_2 \\ &= -(i\varepsilon + tS)(R'D_2 - D_1R') - (D_1S + SD_1 - (I - R'R))tR'. \end{aligned}$$

By assumptions (a) and (b), b, u_t and v_t are in $\mathcal{L}_G(\mathcal{E})$. Thus $\nabla_t + \nabla'_{t,C_t} = L_t^{-1}(L_t\nabla_t + \nabla'_t L_t) \in \mathcal{L}_G(\mathcal{E})$, $t \in [0, t_0]$. One can easily check that $(\mathcal{E}, C_t, \nabla_t)$ satisfies other conditions. Hence $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A})$.

Step 3. We have shown that $(\mathcal{E}, B_t, D) \in \mathcal{L}^0_{G,u}(\mathcal{A}[0,1])$ and $(\mathcal{E}, C_t, \nabla_t) \in \mathcal{L}^0_{G,u}(\mathcal{A}[0, t_0])$. Note that $\nabla_0 = D$ and $C_0 = B_1, (\mathcal{E}, B_1, D) = (\mathcal{E}, C_0, \nabla_0)$. This implies that $\psi_{0,u}((\mathcal{E}, Q, D)) = \psi_{0,u}((\mathcal{E}, B_0, D)) = \psi_{0,u}((\mathcal{E}, B_1, D)) = \psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0}))$. Consequently, it suffices to show $\psi_{0,u}((\mathcal{E}, C_{t_0}, \nabla_{t_0})) = 0$.

We now use Lemma 9.12(b).

(1) $\|\nabla_{t_0}^2\| \le \nu_1^2 : \nabla_t^2 = \begin{bmatrix} D_1^2 & t(D_1R' - R'D_2) \\ 0 & D_2^2 \end{bmatrix}$,

$$\|\nabla_{t_0}^2\| \le \|D_1^2\| + t_0\|D_1R' - R'D_2\| + \|D_2^2\| \le (2\nu)^2,\ \nu_1 = 2\nu.$$

(2) Since $2k\nu_{t_0} = 2k\frac{1}{4\sqrt{2}(1+k)^2k^2} < 1$ and $12\nu(1+k)^3 < 1$,

$$\begin{aligned} \|\nabla_{t_0} + \nabla'_{t,C_{t_0}}\| &= \|L_{t_0}^{-1}\|\|L_{t_0}\nabla_{t_0} + \nabla'_{t,C_{t_0}}L_{t_0}\| \le \|L_{t_0}^{-1}\|(\|b\| + \|u_{t_0}\| + \|v_{t_0}\|) \\ &= 4(1+k)^2((\nu^2k + k\nu^2) + 2\nu^2\|i\varepsilon + tS\| + 2t_0k\nu^2) \\ &= 8(1+k)^2\nu^2(k + 2 + 2k\nu) \le 8(1+k)^2\nu^2(3+k) < \nu_1. \end{aligned}$$

(3) Take $X = \begin{bmatrix} S & 0 \\ R/t_0 & -Z \end{bmatrix}$.

$$\nabla_{t_0}X + X\nabla_{t_0} = \begin{bmatrix} D_1S + SD_1 + R'R & t_0(SR' - R'Z) \\ -\frac{1}{t_0}(D_2R - RD_1) & D_2Z + ZD_2 + RR' \end{bmatrix}.$$

We obtain

$$\begin{aligned} \|\nabla_{t_0}X + X\nabla_{t_0} - I\| &= \sqrt{2}\max\{\|D_1S + SD_1 + R'R - I\| + \|t_0(SR' - R'Z)\|, \\ &\qquad \|\frac{1}{t_0}(D_2R - RD_1)\| + \|D_2Z + ZD_2 + RR' - I\|\} \\ &\le \sqrt{2}\max\{\nu^2 + 2t_0k^2, \frac{\nu^2}{t_0} + \nu^2\} \\ &= \sqrt{2}\max\{\nu^2 + \frac{1}{2\sqrt{2}(1+k)^2}, \nu^2(4\sqrt{2}(1+k)^2k^2 + 1)\} \\ &< \frac{1}{2}, \end{aligned}$$

since $\nu^2 + \frac{1}{2\sqrt{2}(1+k)^2} < \frac{1}{2\sqrt{2}}$ and $\nu^2(4\sqrt{2}(1+k)^2k^2 + 1) < \frac{1}{2\sqrt{2}}$. Therefore, $\nabla_{t_0}X + X\nabla_{t_0}$ is invertible and

$$\|(\nabla_{t_0}X + X\nabla_{t_0})^{-1}\| \le \frac{1}{1 - \|\nabla_{t_0}X + X\nabla_{t_0} - I\|} \le 2.$$

(4) The norm of X can be estimated by

$$\begin{aligned}\|X\| &= \sqrt{2}\max\{\|S\|, \|Z\| + \|R/t_0\|\} \leq \sqrt{2}(k + \frac{k}{t_0}) \\ &= \sqrt{2}(k + 4\sqrt{2}(1+k)^2k^3) \stackrel{\text{def}}{=} k_1.\end{aligned}$$

(5) $\mathcal{E}$ has already the scalar product $< \xi, \eta >= Q(\xi, T\eta)$ with $T^2 = I$. Now $C_{t_0}(\xi, \eta) = Q(\xi, L'_{t_0}\eta) =< \xi, TL'_{t_0}\eta >$. But

$$\begin{aligned}\|L'_{t_0}\| &\leq \sqrt{2}\max\{\|I - R'R\| + \| - i\varepsilon + t_0S'\|\|R'\|, \|R'(-i\varepsilon + t_0S')\| + 1\} \\ &\leq \sqrt{2}(1 + k^2 + (1 + t_0k)k) = \sqrt{2}(1 + k + k^2 + \frac{1}{4\sqrt{2}(1+k)^2}).\end{aligned}$$

$$\begin{aligned}\|(TL'_{t_0})\|\|(TL'_{t_0})^{-1}\| &= \sqrt{2}(1 + k + k^2 + \frac{1}{4\sqrt{2}(1+k)^2})4(1+k)^2 \\ &= 4\sqrt{2}((1+k)^3 + k^2(1+k)^2 + \frac{1}{4\sqrt{2}}) \leq k_1^2.\end{aligned}$$

(6) Finally, we check $2\sqrt{2}\nu_1 k_1^4(k_1 + 3k_1^{\frac{1}{2}}) < 1$, where $\nu_1 = 2\nu, k_1 = \sqrt{2}(k + 4\sqrt{2}(1+k)^2k^3)$. In fact,

$$\begin{aligned}2\sqrt{2}\nu_1 k_1^4(k_1 + 3k_1^{\frac{1}{2}}) &= 32\nu(k + 4\sqrt{2}(1+k)^2k^3)^5(1 + \frac{3}{\sqrt{\sqrt{2}k + 8(1+k)^2k^3}}) \\ &\leq 32\nu(k + 4\sqrt{2}(1+k)^2k^3)^5 2 = 64\nu(k + 4\sqrt{2}(1+k)^2k^3)^5 \\ &< 1.\end{aligned}$$

Hence, the conditions of Lemma 9.12 are satisfied. The proof of the first part of the proposition is complete.

Step 4: We now prove the assertion with (d) replaced by (d'). Let $Z_1 = Z - RR'Z + RSW, X_1 = I - RW - D_2Z - ZD_2, X_2 = I - R'R - D_1S - SD_1$.

$$\|Z_1\| \leq \|Z\| + \|R\|\|R'\| + \|R\|\|S\|\|W\| \leq k + 2k^3,$$

$$\begin{aligned} RSWD_2 + D_2RSW &= (D_2R - RD_1)SW + R(D_1S + SD_1)W \\ &- RS(D_1W - WD_2) \\ &= (D_2R - RD_1)SW + R(I - R'R)W \\ &- RX_2W - RS(D_1W - WD_2). \end{aligned}$$

It follows that $\|RSWD_2 + D_2RSW\| \leq \nu^2k^2 + k(1+k^2)k + k\nu^2k + k^2\nu^2$. Also,

$$RR'ZD_2 + D_2RR'Z = (D_2R - RD_1)R'Z + R(D_1R' - R'D_2)Z + RR'(D_2Z + ZD_2)$$

It yields $\|RR'ZD_2 + D_2RR'Z\| \leq k^2\nu^2 + k^2\nu^2 + k^2(1 + k^2 + \nu^2)$. Now

$$RR' - I + Z_1D_2 + D_2Z_1 =$$

$$\begin{aligned} &= RR' - I + D_2Z + ZD_2 + (RSW - RR'Z)D_2 + D_2(RSW - RR'Z) \\ &= (RR' - I + ZD_2 + D_2Z) + (RSWD_2 + D_2RSW) - (RR'ZD_2 + D_2RR'Z) \\ &= (RR' - I)(I - RW - D_2Z - ZD_2) + RR'(D_2Z + ZD_2) + \\ &+ (D_2R - RD_1)SW - RX_2W - RS(D_1W - WD_2) \\ &- (D_2R - RD_1)R'Z - R(D_1R' - R'D_2)Z - RR'(I - RW - X_1) \\ &= (RR' - I)X_1 - (D_2R - RD_1)(R'Z - SW) - R(D_1R' - R'D_2) - \\ &- RX_2W - RS(DW - WD_2). \end{aligned}$$

We get

$$\|RR' - I + Z_1D_2 + D_2Z_1\| \leq (1 + k^2)\nu^2 + \nu^2(2k^2) + 3k^2\nu^2 = \nu^2(1 + 6k^2).$$

Therefore, replacing ν and k in the first part of this proposition by $\nu\sqrt{1+6k^2}$ and $k(1+2k^2)$, we obtain that if

$$64\nu\sqrt{1+6k^2}\{k(1+2k^2) + 4\sqrt{2}k^3(1+2k^2)^3(1+k(1+2k^2))^2\}^5 < 1,$$

then the conclusion holds. Q.E.D.

Let M and N be two compact oriented Riemannian G-manifolds and $h : N \to M$ be a G-pseudo-equivalence. Let E be an almost flat Hilbert $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert $G-\mathcal{A}$-module E_0. Let ∇ be an almost flat connection on E. Then we have two elements $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$ in $\mathcal{L}^0_{G,u}(\mathcal{A})$. The rest of this section is to show that these elements satisfy the conditions of Proposition 9.2. Hence, $\psi_{u,0}((\mathcal{E}_E, Q, D_E)) = \psi_{u,0}((\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)}))$.

Lemma 9.13 ([HiS]) *With the above notations, if $f : N \times [0,1] \to M$ is an equivariant smooth map, then there is $k > 0$ such that for any unitary connection ∇ on E, $\|f_0^*(\nabla) - f_1^*(\nabla)\| \leq k\|\nabla^2\|$.*

Proof. This lemma is independent of the group action since we are concerned only with the norms. See the proof of Lemma 4.3 [HiS]. Q.E.D.

The following extends Lemmas 9.10 and 9.11 to the nonflat case.

Lemma 9.14 *Let M and N be two compact oriented Riemannian G-manifolds and E be a Hilbert $G-\mathcal{A}$-bundle over M whose fiber is a finitely generated projective Hilbert $G-\mathcal{A}$-module E_0.*

(a) Let $h : N \to M$ be an equivariant smooth map and ω an equivariant smooth form on N. Denote by N_0 the support of ω and suppose that h is a submersion near a G-neighborhood of N_0. Then $e_\omega h^* : \mathcal{E}_E \to \mathcal{E}_{h^*(E)}$ is a bounded equivariant operator. Moreover, the norm of $e_\omega h^*$ can be estimated by a number which is independent of E. Let ∇ be a connection of E and ∇' a connection of $h^*(E)$. Then $e_\omega h^*(dom(D_E)) \subset dom(\nabla')$ and $\nabla'(e_\omega h^*) - (e_\omega h^*)\nabla \in \mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$.

(b) Let $h : N \times [0,1] \to M$ be an equivariant smooth map and ω an equivariant smooth form on N. Let N_0 be the support of ω. Suppose that h (resp. h_0, h_1) is a G-submersion near a neighborhood of $N_0 \times [0,1]$ (resp. N_0). Then there is $k > 0$ such that for any connection ∇ of E there exists $R \in \mathcal{L}(\mathcal{E}_E, \mathcal{E}_{h_0^*(E)})$, $\|R\| \leq k$ and

$$\|e_\omega h_1^* - e_\omega h_0^* - h_0^*(\nabla)R - R\nabla\| \leq k\|\nabla^2\|.$$

Proof. (a) Note that the difference of two connections is a bounded operator and the domain of the connection is independent of the connection itself. By treating first $N = O \times M$ and taking E trivial, we can quickly prove (a), since the result is local.

To prove part (b), let

$$R(\tilde{\xi}) = \int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega h^*(\tilde{\xi})))dt, \ \tilde{\xi} \in \mathcal{E}_E.$$

Then the norm of R can be estimated by the number k which is independent of h and the structures on M and N. As in the proof of Lemma 9.10,

$$(h_0^*(\nabla)R + R\nabla)(\tilde{\xi}) = (e_\omega h_1^* - e_\omega h_0^*)(\tilde{\xi}) + \int_0^1 (i_{\frac{\partial}{\partial t}}(e_\omega(h^*(\nabla) - h'^*(\nabla))(\tilde{\xi})))dt,$$

where $h'(x,t) = h(x,o) : N \times [0,1] \to M$. Let R_1 be defined by the second term of the above identity. Then

$$e_\omega h_1^* - e_\omega h_0^* = h_0^*(\nabla)R + R\nabla + R_1.$$

To estimate the norm of R_1, let $H(x,t,\lambda) = h(x,\lambda t)$. $H(x,t,0) = h'(x,t)$, and $H(x,t,1) = h(x,t)$. Using Lemma 9.13, we get $\|R_1\| \leq k\|\nabla^2\|$. Q.E.D.

Let $P : N \times B^k \to M$ be a G-submersion, ν a G-smooth form of mass 1 on B^k. Suppose that $q : N \times B^k \to N$ and $r : N \times B^k \to B^k$ are the projections, $\omega = r^*(\nu)$, and $h : N \to M$ is the restriction to $N \times \{o\}$ of P. Then $hq : N \times B^k \to M$ is homotopic to P via the homotopy $H(x,t,\lambda) = P(x, \lambda t)$. For this reason, we identify $P^*(E)$ with $q^*h^*(E)$. Let $R_{P,\nu} = R_{P,\nu}(E, \nabla) = q_! e_\omega P^* \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$.

Lemma 9.15 *(a) Let $P' : N \times B^l \to M$ be another G-submersion and ν' a G-smooth form of mass 1 on B^l. Suppose $h(x) = P(x, o)$ and $h'(x) = P'(x, o)$ are homotopic. Then there exists $k > 0$ such that for each pair (E, ∇) as above there is $R \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$ satisfying $\|R\| \leq k$ and*

$$\|R\nabla + \nabla R + R_{P,\nu}(E,\nabla) - R_{P',\nu'}(E,\nabla)\| \leq k\|\nabla^2\|.$$

(b) If $\tilde{P} : O \times B^l \to N$ is a G-submersion and ν' a G-smooth form of mass 1 on B^l. Let $P'' : O \times B^l \times B^k \to M$ be defined by $P''(x,s,t) = P(\tilde{P}'(x,s),t)$ and $\nu'' = \nu' \times \nu$. Then there exists $k > 0$ such that for each pair (E, ∇),

$$\|R_{P',\nu'}(h^*(E), h^*(\nabla))R_{P,\nu}(E,\nabla) - R_{P'',\nu''}(E,\nabla)\| \leq k\|\nabla^2\|.$$

Proof. (a) By Lemma 9.14, the proof is the same as that of Lemma 9.11.
(b) Define $f : O \to M$ by $f(x) = P''(x, o, o)$ and let $q'' : O \times B^k \times B^l \to O$ be the projection. Define two homotopies H and $\bar{H}$ between P'' and fq'' from $O \times B^l \times B^k \times [0,1]$ to M by

$$H(x,s,t,\lambda) = P''(x, \lambda s, \lambda t),$$

and

$$\bar{H}(x,s,t,\lambda) = P''(x, \sup(0, 2\lambda - 1)s, \inf(1, 2\lambda)t).$$

$H(x,s,t,o) = (fq'')(x,s,t), H(x,s,t,1) = P''(x,s,t)$. We can pull back the unitary connection ∇ on E to a unitary connection $H(\nabla)$ on $H^*(E)$ via the smooth map H, and then by the parallel transport along $[0,1]$ we define a unitary operator $U \in \mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{(fq'')^*(E)})$. Same reasoning shows that $\bar{H}$ produces unitary operators in $\mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{H_{\frac{1}{2}})^*(E)})$ and $\mathcal{L}_G(\mathcal{E}_{H^*_{\frac{1}{2}}(E)}, \mathcal{E}_{(fq'')^*(E)})$, hence a unitary operator $\bar{U} \in \mathcal{L}_G(\mathcal{E}_{P''^*(E)}, \mathcal{E}_{(fq'')^*(E)})$. By the definition, $R_{P'',\nu''} = q''_! U e_{\omega''}(P'')^*$ for $\omega'' = \pi^*_{l,k}(\nu'')$, where $\pi_{l,k} : O \times B^l \times B^k \to B^l \times B^k$ is the projection. Let $\bar{R}_{P'',\nu''} = q''_! \bar{U} e_{\omega''}(P'')^*$. Then as in the proof of Lemma 9.11(b), $\bar{R}_{P'',\nu''} = R_{P',\nu'} R_{P,\nu}$. the result follows easily from [HiS]

$$\|\bar{R}_{P'',\nu''} - R_{P'',\nu''}\| \leq \|q''_!(\bar{U} - U)e_{\omega''}(P'')^*\| \leq K_1\|\bar{U} - U\| \leq k_1 m\|\nabla^2\|,$$

where k_1 and m are constants independent of ∇ and E. Q.E.D.

Finally, we can prove the equivariant Connes-Gromov-Moscovici theorem.

Theorem 9.2 *Let G be a compact Lie group , M and N be two compact oriented Riemannian G-manifolds of even dimensions. If $h : N \to M$ is an orientation preserving G-pseudo-equivalence, then there exists $\nu > 0$ such that for each $G - C^*$-algebra $\mathcal{A}$ and ν-flat Hilbert $G - \mathcal{A}$-bundle E over M whose fiber is a finitely generated projective Hilbert $G - \mathcal{A}$-module,*

$$\psi_{0,u}((\mathcal{E}_E, Q, D_E)) = \psi_{0,u}((\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})).$$

Proof. The proof is to use Proposition 9.2 for $(\mathcal{E}_E, Q, D_E)$ and $(\mathcal{E}_{h^*(E)}, Q, D_{h^*(E)})$.

Let $J : M \to \mathbb{R}^k$ be a G-embedding, O a G-tubular neighborhood of $J(M)$ in $\mathbb{R}^k$ and $\pi : O \to M$ be the corresponding G-projection. Suppose $J(M) + B^k \subset O$. Let $P : N \times B^k \to M$ be the G-submersion given by

$P(x,t) = \pi(h(x)+t)$, and ν a G-volume form of mass 1 on B^k. Take $R = R_{P,\nu} \in \mathcal{L}_G(\mathcal{E}_E, \mathcal{E}_{h^*(E)})$. We now check the conditions of Proposition 9.2.

To check condition (a), let $\nu_1^2 \geq \|\nabla^2\|$. $\|D_E^2\| = \|\nabla^2\| \leq \nu_1^2$, and $\|D_{h^*(E)}\| = \|\nabla^2\| \leq \nu_1^2$. Take a G-smooth function φ with compact support such that $\varphi\omega = \omega$ on $N \times B^k$. By Lemma 9.14, we can estimate $\|e_\omega P^*\|$ and $\|q_{!,\varphi}\| = \|(e_\omega P^*)'\| = \|e_\omega P^*\|$. Clearly, $P^*\nabla = P^*(\nabla)P^*, q^*h^*(\nabla) = (hq)^*(\nabla)q^*$ and $e_\omega P^*(\nabla) = P^*(\nabla)e_\omega$ for the closed form ω. Then $(e_\varphi q^*)'e_\omega P^*\nabla = (e_\varphi q^*)'P^*(\nabla)e_\omega P^*$, and

$$\begin{aligned}
\|(e_\varphi q^*)'e_\omega P^*\nabla - h^*(\nabla)(e_\varphi q^*)'e_\omega P^*\| &= \|(e_\varphi q^*)'P^*(\nabla)e_\omega P^* - h^*(\nabla)(e_\varphi q^*)'e_\omega P^*\| \\
&\leq \|(e_\varphi q^*)'(P^*(\nabla) - (hq)^*(\nabla))e_\omega P^*\| \\
&+ \|h^*(\nabla)(e_\varphi q^*)' - (e_\omega q^*)'(hq)^*(\nabla))e_\omega P^*\| \\
&\leq m\|\nabla^2\|,
\end{aligned}$$

by Lemmas 9.13 and 9.14 and the fact that $h^*(\nabla)(e_\varphi q^*)' = (e_\varphi q^*)'(hq)^*(\nabla)$. We have $R(dom(D_E)) \subset dom(D_{h^*(E)})$ and $\|RD_E - D_{h^*(E)}R\| \leq m_1\|\nabla^2\|$ for some $m_1 > 0$ independent of (E, ∇).

To check condition (b), let $q_i : N \times B^k \times B^k \to N \times B^k$ be given by $q_i(x, t_1, t_2) = (x, t_i)$, $P_1 : M \times B^k \to M$ be the projection and $\tilde{\pi} : M \times B^k \to M$ be the G-submersion given by $\tilde{\pi}(x,t) = \pi(x+t)$. Then we can argue as in the proof of Theorem 9.1 by using Proposition 9.2 to get $S \in \mathcal{L}_G(\mathcal{E})$ such that $\|S\| \leq m_2, S(dom(D_E)) \subset dom(D_E)$ and

$$\|I - R'R - D_ES - SD_E\| \leq m_3\|\nabla^2\|$$

for constants $m_i > 0$ independent of (E, ∇). The rest is the same as in Theorem 9.1.

To check condition (c), take $\varepsilon(\tilde{\xi}) = (-1)^{\partial\tilde{\xi}}\tilde{\xi}$. Then clearly $\varepsilon(dom(D_E)) \subset dom(D_E), \varepsilon D_E = -D_E\varepsilon, \varepsilon = \varepsilon', \varepsilon^2 = I$ and $\varepsilon(I - R'R) = (I - R'R)\varepsilon$, See the corresponding part of Theorem 9.1.

Finally we verify condition (d') of Proposition 9.2. Indeed, let $P_1 : M \times B^l \to N$ be a submersion such that $q_1 : M \to N$ given by $q_1(x) = P_1(x, o)$ is a homotopic inverse of h, i.e., $q_1 h$ and hq_1 are homotopic to the identities, resp.. Note that P_1 and q_1 may not be equivariant, and homotopy between $q_1 h$ and the identity may not be equivariant either. But we can still identify E with $(hq_1)^*(E)$ non-equivariantly. Let ν' be a volume form of mass 1 on B^l and $W_1 = R_{P_1,\nu'} \in \mathcal{L}_G(\mathcal{E}_{h^*(E)}, \mathcal{E}_E)$. then as before, there are constants $m_4, m_5 > 0$ such that $\|W_1\| \leq m_5, W_1(dom(D_E)) \subset dom(D_E)$ and

$$\|W_1 D_{h^*(E)} - D_E W_1\| \leq m_4 \|\nabla^2\|.$$

By Lemma 9.15 and Step 3 of the proof of Theorem 9.1, there exists $Z_1 \in \mathcal{L}_G(\mathcal{E}_{h^*(E)})$ satisfying $\|Z_1\| \leq m_6, Z_1(dom(D_{h^*(E)})) \subset dom(D_{h^*(E)})$ and

$$\|I - RW_1 - D_{h^*(E)} Z_1 - Z_1 D_{h^*(E)}\| \leq m_7 \|\nabla^2\|,$$

where m_6 and m_7 are positive constants independent of (E, ∇). Let $W = \int_G g(W_1)dg, Z = \int_G g(Z_1)dg$. We get the required operators. Let

$$\nu^2 = \max\{\|\nabla^2\|, m_1\|\nabla^2\|, m_3\|\nabla^2\|, m_4\|\nabla^2\|, m_7\|\nabla^2\|\}$$

and $k = \max\{m_0, m_2, m_5, m_6\}$. Then we have verified the conditions of Proposition 9.2 as long as $\|\nabla^2\|$ is sufficiently small. Q.E.D.

Chapter 10

Equivariant Novikov Conjecture for Groups Acting on Buildings

This chapter is devoted to the proof of the equivariant Novikov conjecture for groups acting on buildings. The case we deal with is the most general one so far which the operator K-theory version of the equivariant Novikov conjecture is known to be true. In fact, our result implies also that the equivariant Novikov conjecture holds for groups acting on the manifolds of nonpositive curvature which was treated by Rosenberg and Weinberger [RoW 2]. More significantly, we remove a crucial condition on $K_*^G(Y)$ in Rosenberg-Weinberger's theorem. Our method is to use Theorem 9.1 instead of the equivariant Miscenko symmetric signature for compact group actions, which has not yet been defined. This approach is quite different from the usual method for proving the Novikov conjecture in which the Miscenko symmetric signature plays always an important role. Since We have already devoted a long chapter to proving the homotopy invariance of the (higher) equivariant signature with coefficients in (almost) flat vector bundles, the present chapter is short. We need only to employ the results of Kasparov-Skandalis [KaS] to verify the injectivity of the map β. This is done in section 10.1 where we set up the notations and construct the map β. In section 10.2 we complete the proof of the homotopy invariance of the higher equivariant signature by using Theorem 9.1.

10.1 Euclidean Buildings and Map β

Let M be a compact oriented Riemannian manifold and G a compact Lie group acting on M by isometries. Denote by $\pi(M)$ the fundamental groupoid of M which is defined by the equivalent classes of all paths in M. $\pi(M)$ can be also given by $\pi(M) = \frac{\tilde{M}\times\tilde{M}}{\pi_1(M)}$, where $\tilde{M}$ is the universal covering space of M. G acts naturally on $\pi(M)$. Let $B\pi(M)$ be the equivariant classifying space of $\pi(M)$ (see May's appendix for [RoW 2]) and $f_M : M \to B\pi(M)$ be the equivariant classifying map. If $h : N \to M$ is a G-pseudo-equivalence for another compact oriented Riemannian G-manifold N, i.e., h is G-invariant and is a homotopy equivalence, then h induces maps $h_* : \pi(N) \to \pi(M)$ and $h_* : B\pi(N) \to B\pi(M)$. Hence the following diagram of equivariant K-homology groups is commutative:

$$\begin{array}{ccc} K_*^G(M) & \overset{(f_M)_*}{\to} & K_*^G(B\pi(M)) \\ \uparrow h_* & & \uparrow h_* \\ K_*^G(N) & \overset{(f_N)_*}{\to} & K_*^G(B\pi(N)). \end{array} \qquad (10.1)$$

Let D_M be the equivariant signature operator on M. Then D_M defines an element $[D_M]$ in $K_*^G(M)$, which is in $K_0^G(M)$ for even dimensional M and in $K_1^G(M)$ for odd dimensional M. One can expect in view of (10.1) that $h_*(f_N)_*([D_N]) = (f_M)_*([D_M])$. More generally, given a G-equivariant commutative diagram

$$\begin{array}{ccc} M & & \\ & \overset{\varphi}{\searrow} & \\ \uparrow h & & Y \xrightarrow{f_Y} B\pi(Y), \\ & \overset{\psi}{\nearrow} & \\ N & & \end{array} \tag{10.2}$$

where Y is a G-space, h is a G-pseudo-equivalence and φ and ψ are G-equivariant maps, one could conjecture the following (see Chapter 8)
$\underline{ENC_Y}$ ([RoW 2]): $(f_Y)_*\varphi_*([D_M]) = (f_Y)_*\psi_*([D_N])$ in $K_*^G(B\pi(Y))$, provided $K_*^G(B\pi(Y))$ is finitely generated over $R(G)$.

Rosenberg and Weinberger have proved this conjecture for Y a complete Riemannian manifold of nonpositive curvature, provided $K_*^G(Y)$ is a finitely generated module over the representation ring $R(G)$ of G. The condition on $K_*^G(Y)$ is crucial in the proof of the conjecture due to the lack of the equivariant Miscenko symmetric signature for general compact Lie group actions. The unsolved problem in this case is to remove the condition on $K_*^G(Y)$ and to prove the existence of the maps φ and ψ in (10.2) for a general manifold M. On the other hand, it is desirable to verify the ENC_Y for those Y whose universal coverings are equivariantly isomorphic to the geometric realization of euclidean buildings, since the euclidean buildings are the natural analogue of complete Riemannian manifolds of nonpositive curvature. The recent work of Kasparov-Skandalis [KaS] and Gromov-Schoen [GrS] on euclidean buildings enables us to verify the ENC_Y for such Y. In fact, the existence of the maps φ and ψ in (10.2) may follow from the results of Gromov and Schoen. The goal of this chapter is to carry out the proof of the equivariant Novikov conjecture for the above mentioned Y. Meanwhile, we will also get rid of the assumption on $K_*^G(Y)$.

We now recall the definition of euclidean buildings (cf. [Bro],[KaS],[Tits] for more details). Let X be a simplicial complex of dim n and B its geometric realization. X can be considered as a set of its faces, $X = \bigcup_{0\leq k\leq n} X^k$. X is said to be typed if there is a map $\nu : X^0 \to \{0, 1, \ldots, , n\}$ such that for any simplex $x \in X$, the images under ν of the vertices of x are pairwise different. ν is called a type of X. There is typed simplicial complex X_1 associated with a given simplicial complex X of dim n such that X_1 and X have the same geometric realization. We use the notations that chambers are the simplices of dim n; walls are the simplices of dim $n-1$, and apartments are some subcomplexes of X determined by the Weyl system. (X, B) is called an euclidean building if
(1) B has a metric such that the apartments are affinely isometric to euclidean space $\mathbb{R}^n$;
(2) any pair of simplices of X is contained in an apartment;
(3) the intersection $S \cap S'$ of any two apartments S and S' is convex and there is simplicial isometry $j : S \to S'$ such that j is an identity map on $S \cap S'$ and preserves the type, i.e., $\nu(j(x)) = \nu(x), \forall x \in S$;
(4) for any two chambers σ and σ' of an apartment S there is a type-preserving

simplicial isometry $j : S \to S$ mapping σ to σ'.

This definition of the euclidean building is slightly different from that in [Tits]. In fact, every Bruhat-Tits euclidean building is an euclidean building in the above sense. But comparing to the euclidean buildings given in [Bro], we require an extra condition (4).

Example 10.1 (1) *Let F be a field with a discrete valuation and $SL_n(F)$ the group of $n \times n$ matrices on F with determinant* 1. *Then we can associate with $SL_n(F)$ an euclidean building. In particular, this building for $SL_2(\mathbb{Q})$ is a tree [Bro].*

(2) *The universal covering space of a complete Riemannian manifold of nonpositive curvature can be considered as a geometric realization of some topological buildings [BuS]. For more examples of euclidean buildings we refer to [Bro].*

Note that the geometric realization B of a euclidean building X is contractible [Bro]. Let Γ_0 be a discrete group which acts properly and freely on the building (X, B) by type permuting isometries (we will take Γ_0 to be the fundamental group of M). Note that the action of Γ_0 is called the type permuting if there is a group homomorphism $\pi : \Gamma_0 \to S_n$, the permutation group of $\{0, 1, \ldots, n\}$, such that $\nu(g(x)) = \pi(g)(x), \forall g \in \Gamma_0, x \in X$. The action is called the type preserving if $\pi(g) = I, \forall g \in \Gamma_0$. Then the universal covering space of $Y = B/\Gamma_0$ is B. We assume that G acts on Y by cellular isometries via a homomorphism of G into $Isom(Y)$ such that the lifting of G-action to B is the type permuting. More precisely, there is a locally compact group Γ in $Isom(B)$ such that Γ is the group extension of Γ_0 by G,

$1 \to \Gamma_0 \to \Gamma \to G \to 1$, and acts on B by type permuting isometries. The importance of Γ is that $C_r^*(\Gamma)$ is strongly Morita equivalent to $C_r^*(\pi(Y)) \rtimes G$ [RoW 2]. Hence $K_*^G(C_r^*(\pi(Y))) \simeq K_*(C_r^*(\pi(Y)) \rtimes G) \simeq K_*(C_r^*(\Gamma))$. To consider the ENC_Y for $Y = B/\Gamma_0$ we need a non-Hausdorff smooth manifold M_X associated with (X, B). M_X is given by a Γ-atlas $(U_i, U_{ij}, \varphi_{ij})_{i,j\in J}$ satisfying the following properties:

(1) Γ acts on the index set J and U_i such that $g(U_i) = U_{g(i)}, i \in J, g \in \Gamma$:

(2) each U_i is a Hausdorff manifold:

(3) $U_{ii} = U_i$ and for $i, j \in J, U_{ij}$ is an open subset in U_i:

(4) $\varphi_{ij} : U_{ji} \to U_{ij}$ is a diffeomorphism such that $\varphi_{ii} = I_{U_i}$ and $\varphi_{ij}(gx) = g\varphi_{ij}(x), g \in \Gamma$.

Let U^0 be the disjoint union of $U_i, i \in J$. Then M_X is the quotient of U^0 by the equivalent relation "$x \sim \varphi_{ij}(x)$", $x \in U_{ji}$. M_X is in general non-Hausdorff. It is Hausdorff iff the maps $(r, s) : U_{ij} \to U_i \times U_j$ given by $r(x) = x$ and $s(x) = r(\varphi_{ji}(x))$ for $x \in U_{ij}$ are proper. More specifically, let E be the affine euclidean space $E = \{t = (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1\}$ and $\Sigma = \{t \in E : t = (t_0, \ldots, t_n), t_i \geq 0, \forall i\}$. Since Σ is a convex set in E, we can define a continuous map $q : E \to E$ by the formula $\|q(t) - t\| = \inf\{\|t - s\| : s \in \Sigma\}$. For a subset O of $\{0, 1, \ldots, n\}$ let F_O be the face of Σ defined by $F_O = \{t \in \Sigma : t = (t_0, \ldots, t_n) : i \in O \ if \ t_i \neq 0\}$. Denote by Ω_O the interior in E of $q^{-1}(F_O)$. Obviously, $F_\emptyset = \Omega_\emptyset = \emptyset$. Then the Γ-atlas $\{U_i, U_{ij}, \varphi_{ij}\}$ is given by $U_x = E, x \in X, U_{x,y} = \Omega_{\nu(x\cap y)}$ for $x \neq y$ in X and $\varphi_{x,y} = I$. Thus $M_X = U^0/\sim$ is a non-Hausdorff Γ-smooth manifold. M_X is endowed with a Γ-invariant Riemannian metric. The tangent bundle of M_X is trivial whose fiber is the space of tangent vectors to E. The crucial property of M_X is that M_X is Γ- equivalent to B, i.e, there are Γ-equivariant maps $f_1 : M_X \to B$ and $f_2 : B \to M_X$ such that $f_1 f_2$ and $f_2 f_1$ are homotopic to the identity maps, reps.. We have the diagram

$$\begin{array}{ccc} B & \overset{f_2}{\rightleftharpoons}_{f_1} & M_X \\ \downarrow P & & \downarrow P \\ Y = B/\Gamma_0 & \overset{f_4}{\rightleftharpoons}_{f_3} & Y_X = M_X/\Gamma_0, \end{array}$$

where P is the natural projection, f_3 and f_4 are defined by the commutativity of the diagram. Then Y is G-homotopic to Y_X via the maps f_3 and f_4. It follows from the isomorphism of $(f_4)_*$ ([Bla], p. 194) that we need only to show $(f_4\varphi)_*([D_M]) = (f_4\psi)_*([D_N])$ in $K_*^G(Y_X)$. The advantage of introducing Y_X is that one can then use various KK-theory information of non-Hausdorff special manifolds. We now reduce the problem to the following:

(1) to construct an injective map $\beta : K_*^G(Y_X) \to K_*^G(C_r^*(\pi(Y_X)))$,

(2) to show the G-pseudo-equivalence of $\beta((f_4\varphi)_*([D_M]))$.

To define the map β, let V_X be the Hilbert $G - C_r^*(\pi(Y_X))$-bundle over Y_X whose fiber V_y is the completion of $C_c(r^{-1}(y), \Omega^{\frac{1}{2}})$, where $r : \pi(Y_X) \to Y_X$ is the range map and $\Omega^{\frac{1}{2}}$ is the $\frac{1}{2}$ -densities of $T^*(Y_X)$. As shown in [RoW 2], V_X is a flat $G - C_r^*(\pi(Y_X))$-bundle. Note that V_X restricted to any G -compact subset $W \subset Y_X$ gives a $G - C_r^*(\pi(Y_X))$-bundle over W, and hence defines an element $[\beta_W]$ in $K_0^G(C^*(W) \otimes C_r^*(\pi(Y_X)))$. Let β_W^* be the homomorphism from $K_*^G(W)$ to $K_*^G(C_r^*(\pi(Y_X)))$ given by the Kasparov product with $[\beta_W]$. β_W^* respects with the direct limit to give a map $\beta : K_*^G(Y_X) = \lim_{\to W \subset Y_X} K_*^G(W) \to K_*^G(C_\tau^*(\pi(Y_X)))$. Furthermore, let U_X be the completion of the space of all continuous sections of V_X with compact support in Y_X. Then $(U_X, \varphi, 0)$ is a Kasparov $G - C(Y_X) - \mathcal{A} \otimes C(Y_X)$ module with $\mathcal{A} = C_\tau^*(\pi(Y_X))$, and defines an element $[\beta_X] \in KK^G(C(Y_X), \mathcal{A} \otimes C(Y_X))$. We can consider $[\beta_W]$ as a restriction of $[\beta_X]$ to $W \subset Y_X$.

We are now going to interpret the map β as a Dirac element on M_X essentially. Let $C^*(U)$ be the G–C^*-algebra of groupoid $U = \bigcup_{x,y \in X} U_{x,y}$ associated with the covering $\{U_x, U_{x,y}, I\}$ of M_X. $C^*(U)$ is the completion of $C_c(U) = \bigoplus_{x,y \in X} C_c(U_{x,y})$ with the norm given by $\|f\| = \sup\{\|f_s\|, s \in M_X\}$ for $f = \oplus_{x,y} f_{x,y}$ in $C_c(U)$ and $\|f_s\|$ is the operator norm on $H_s = l^2(\{x \in X; s \in U_x\})$, since f defines a finite rank operator f_s by $< e_x, f_s(e_y) >= f(x,y,s) = f_{x,y}(s)$. The product of $C^*(U)$ is the convolution given by $(f_1 f_2)_{x,y} = \sum_z f_{1,x,z} f_{2,z,y}$ for $f_i = \oplus_{x,y} f_{i,x,y}$ in $C_c(U)$. As pointed out in [KaS], $C^*(U)$ is independent of the covering up to the Morita equivalence. Let $C_\tau^*(U) = C^*(U) \hat{\otimes} Cliff(E)$. The following facts were proved in [KaS]:

(1) Let $P : M_X \to pt$ be the trivial map. Then $P_!$ defines a Gysin element $P_! \in KK^\Gamma(C_\tau^*(U), \mathbb{C})$ which is called the Dirac element D_X;

(2) There is an element $\eta_X \in KK^\Gamma(\mathbb{C}, C_\tau^*(U))$ such that $D_X \otimes_{\mathbb{C}} \eta_X = I_{C_\tau^*(U)} \in KK^\Gamma(C_\tau^*(U), C_\tau^*(U))$;

(3) There is an element $\Theta_X \in RKK^\Gamma(B; \mathbb{C}, C_\tau^*(U))$ such that (a) $\Theta_X \otimes_{C_\tau^*(U)} D_X = I \in RK^\Gamma(B)$, (b) $\sigma_{B, C_\tau^*(U)}(\Theta_X) \otimes_{C_\tau^*(U)} D_X = I_{C_\tau^*(U)} \in KK^\Gamma(C_\tau^*(U), C_\tau^*(U))$, (c) $\sigma_{B, C_\tau^*(U)}(\Theta_X)$ is invariant under the automorphism $(s_1, s_2) \to (s_2, s_1)$ of $C_\tau^*(U \times U)$, where $\sigma_{B, C_\tau^*(U)} : RKK^\Gamma(B; \mathbb{C}, C_\tau^*(U)) \to KK^\Gamma(C_\tau^*(U), C_\tau^*(U \times U))$

is given by $(\mathcal{E}, T) \to (\mathcal{E} \otimes C_r^*(U), T \otimes I)$ at the Kasparov module level. In fact, $\Theta_X = (f, I)_!$, where $f : B \to M_X$ is a Γ-homotopy equivalence and $(f, I)_!$ is the Gysin element induced by the map $(f, I) : B \to M_X \times B$. But the construction of η_X is complicated. We refer to [KaS] for details.

Proposition 10.1 *Let (X, B) be a locally finite euclidean building and Γ_0 a finitely generated discrete group acting on (X, B) properly and freely. Suppose G is a compact Lie group acting on $Y = B/\Gamma_0$ with a fixed point. Let Γ be the extension of Γ_0 by G and M_X be the non-Hausdorff smooth Riemannian manifold associated with (X, B). Then $\beta : K_*^G(Y_X) \to K_*^G(C_r^*(\pi(Y_X)))$ is injective.*

Proof. As shown by Kasparov ([Kas 3], Thm 4.10, 6.6 and 6.7), the facts $(1)-(3)$ above imply that the map β is the composition

$$
\begin{array}{rcl}
K_*^G(Y_X) & \overset{\text{Poincare duality}}{\simeq} & K_*^G(C_r^*(U/\Gamma_0)) \\
 & \overset{\text{Green-Julg Thm}}{\simeq} & K_*(C_r^*(U/\Gamma_0) \rtimes G) \\
 & \overset{\text{Morita equiv}}{\simeq} & K_*(C_r^*(U) \rtimes \Gamma) \\
 & \overset{j^\Gamma[D_X]}{\longrightarrow} & K_*(C_r^*(\Gamma)) \simeq K_*^G(C_r^*(\pi(Y_X)),
\end{array}
$$

where $j^\Gamma : KK^\Gamma(\mathcal{A}, \mathcal{B}) \to KK(\mathcal{A} \otimes C_r^*(\Gamma), \mathcal{B} \otimes C_r^*(\Gamma))$ is the reduction map. See also [RoW 2]. The Morita equivalence above follows from the fact that $K_*(C_r^*(U/\Gamma_0) \rtimes G) \simeq K_*(C_r^*(U)^{\Gamma_0} \rtimes G) \overset{\text{Morita equiv}}{\simeq} K_*((C_r^*(U) \rtimes \Gamma_0) \rtimes G) \simeq K_*(C_r^*(U) \rtimes \Gamma)$ by Theorem 1 [CMW] and Theorem 3.13 [Kas 3], since Gacts on Γ_0 and $\Gamma = \Gamma_0 \rtimes G$. The Poincare duality follows from Theorem 6.8 [KaS] and the fact that for non-Hausdorff manifold Y_X and its classifying space $\tilde{Y_X}$,

$K_*^G(Y_X) \simeq K_*^G(\tilde{Y_X})$. Since $j^\Gamma[D_X] \otimes_{C_r^*(\Gamma)} j^\Gamma[\eta_X] = I_{C_r^*(U)\rtimes\Gamma}$, we have that β is injective. Q.E.D.

10.2 Homotopy Invariance of Higher Equivariant Signatures

We now prove the homotopy invariance of higher equivariant signature for groups acting on euclidean buildings. Note that the action of a locally compact group on the building is given by a homomorphism from the group to the isometry group of the building.

Theorem 10.1 *Let (X, B) ba a locally finite euclidean building, Γ_0 a discrete group acting on (X, B) properly and freely by type permuting isometries. Suppose that compact Lie group G acts on $Y = B/\Gamma_0$ by cellular isometries with a fixed point. Then for G-equivariant commutative diagram* (10.2), $\varphi_*([D_M]) = \psi_*([D_N])$.

Proof. As we remarked in the previous section, we need only to check $(f_4)_*\varphi_*([D_M]) = (f_4)_*\psi_*([D_N])$ in $K_*^G(Y_X)$, where $f_4 : Y \to Y_X$ induces an isomorphism in equivariant K-theory. By Proposition 10.1, we know that $\beta : K_*^G(Y_X) \to K_*^G(C_r^*(\pi(Y_X)))$ is injective. It is thus sufficient to show $\beta((f_4)_*\varphi_*([D_M])) = \beta((f_4)_*\psi_*([D_N]))$. Hence, the problem reduces to the G-pseudo-equivalence of the invariant $\beta((f_4)_*\varphi_*([D_M]))$. In view of the construction of β above, the KK- index theorem [RoW 1] shows that $Ind_G(D_{(f_4\varphi)^*(V_X)}) = \beta((f_4)_*\varphi_*([D_M]))$, where V_X is the $G - C_r^*(\pi(Y_X))$-bundle over Y_X constructed in the definition of β. Theorem 9.1 implies that $Ind_G(D_{(f_4\varphi)^*(V_X)})$ is a G-pseudo-equivalence invariant. Q.E.D.

Combining Theorem 9.1 with the proof above (see also [RoW 2]), We can remove the condition that $K_*^G(Y)$ is finitely generated over $R(G)$ in Rosenberg-Weinberger's theorem.

Theorem 10.2 *Let Y be a complete Riemannian G-manifold of nonpositive curvature in the commutative diagram* (10.2)*.Suppose G acts on M with a fixed point. Then $\varphi_*([D_M]) = \psi_*([D_N])$ in $K_*^G(Y)$.*

The condition that $K_*^G(Y)$ is finitely generated over $R(G)$ enables one to reduce the compact group action to the finite group action and then the McClure theorem is available.

We should point out that the condition on the fixed point of G is only to guarantee the Morita equivalence in the proof of Proposition 10.1 which was also used in [RoW 2]. The full generality of this Morita equivalence is not clear to us at the moment.

Open Problems

The previous chapters lead us to many interesting problems. We list some of them as follows.

(1) The L^2-analytic torsion for covering spaces is built on the positivity condition of the Novikov-Shubin invariants. W. Lück told us about his conjecture that this condition is always satisfied. More generally, he conjectures the positivity of the Novikov-Shubin invariants for any $n \times n$ matrices with coefficients in $\mathbb{Z}\Gamma$ as an operator on $\oplus_{i=1}^{n} l^2(\Gamma)$, when Γ is a finitely generated discrete group (cf. also [Sun]). One can easily see that this is not true for general finite von Neumann algebras.

(2) One can also define the L^2-Reidemeister torsion for covering spaces provided the Novikov-Shubin invariants are positive. The generalized Ray-Singer conjecture is that the L^2-Reidemeister and analytic torsions are equal. The recent work of Müller [Mül] and Bismut-Zheng [BiZ] might be helpful for this problem.

(3) We are developing the K-theory torsion invariants for any C^*-algebras which differ from the torsions for tuples of commuting elements. The new K-theory is needed to handle this problem.

(4) Furthermore, we are studying [Gong 3] torsion invariants in non commutative geometry. This interesting problem deserves a full investigation,

since it may stimulate the study of other problems, such as index problems in non commutative geometry.

The following two open problems concern the application of (bivariant) cyclic cohomology to the index problem.

(5) How can one use the bivariant Chern characters to study the index problem? This problem is also addressed in [Dou].

(6) Can one obtain the index formula for (equivariant) odd θ-summable Fredholm modules (cf. Chapter 5 and [DHK])? We should point out that D is not a Fredholm operator without mild assumptions. Connes has also pointed out this problem.

(7) Three interesting problems are to study the q-analogues of cyclic (co-)homology, operator algebra cohomology and group cohomology. These new research directions are under consideration. One point that is not clear to us is the q-analogue of K-theory. Perhaps one may first deal with the q-analogue of homotopy groups.

(8) We would like very much to obtain the higher equivariant index theorem.

There are many conjectures in Chapter 8. We stress the following questions.

(9) Is an automatic group a hyperlinear group (cf. Chapter 8)? Does the Strong Novikov conjecture hold for automatic groups?

(10) We are in particular interested in the Novikov conjecture for virtually residual discrete groups and amenable discrete groups. One expects that the bivariant version of [BHM] is very complicated and interesting.

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