Differentiable Sphere Theorems for Ricci Curvature

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Abstract of the Dissertation

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We use a compactness theorem of Anderson-Cheeger to prove a differentiable diameter sphere theorem for Ricci curvature and as a corollary we have an differentiable eigenvalue sphere theorem.

...and yet to me, what is this quintessence of dust?

Contents

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Chapter 1

Introduction.

A basic problem in Riemannian geometry is the study of relations between the topological and the Riemannian structures of a complete Riemannian manifold. More precisely, under certain geometric bounds in M (e.g. bounds on the sectional curvature, Ricci curvature, volume, diameter) describe topological properties of M. For example:

Myer's Theorem: *If (M^n, g) is a Riemannian n-manifold whose Ricci curvature $Ric_{(M^n,g)}$ satisfies $Ric_{(M^n,g)} \geq (n-1)\delta > 0$, then:*

1) The diameter of M $diam_{(M^n,g)}$ satisfies $diam_{(M^n,g)} \leq \frac{\pi}{\sqrt{\delta}}$

2) The universal cover $\tilde{M}$ of M is compact and the fundamental group $\pi_1(M)$ of M is finite. Where $Ric_{(M^n,g)}$ is the Ricci curvature of (M^n, g).

The underlying philosophy is: Which topological properties of a manifold M can be deduced when geometric bounds are imposed ?

One of the aspects of these relations is the so-called topological rigidity, (respectively metric rigidity), i.e. bounds on the geometry of a manifold M, restricts M to a finite class of manifolds up to homeomorphism, diffeomorphism (respectively isometry). One of the most striking examples is the *Classical Sphere Theorem* due to Rauch, Klingenberg, Berger ([Kl], [Be]).

Classical Sphere Theorem: *If (M^n, g) is a complete, connected and simply connected n-manifold whose sectional curvature $K_{(M,g)}$ satisfies $1/4 < K_{(M,g)} \leq 1$, then M is homeomorphic to the n-sphere S^n.*

Thus the condition on the sectional curvature provides a uniqueness of topological types in this class of manifolds. There are other examples in which a class of Riemannian n-manifolds is metrically rigid . For example:

Cheng Maximal Diameter Theorem ([Ch]): *If (M^n, g) is a Riemannian manifold and satisfies $Ric_{(M^n,g)} \geq (n-1)g$, $diam_{(M^n,g)} = \pi$ then (M^n, g) is isometric to $(S^n(1), g_{can})$.*

This thesis is concerned with some rigidity phenomena under Ricci curvature bounds. It is shown that a Riemannian manifold (M, g) satisfying certain geometric bounds is diffeomorphic to the canonical sphere $S^n(1)$, (Theorems A, B). To introduce the particular problem considered let (M^n, g) be a compact n-dimensional Riemannian manifold satisfying $Ric_{(M,g)} \geq (n-1)g$, where $Ric_{(M,g)}$ is the Ricci curvature of (M^n, g). By Myer's Theorem, the diameter $diam_{(M,g)}$ of M satisfies $diam_{(M,g)} \leq \pi$ and if $diam_{(M,g)} = \pi$, then (M, g) is isometric to $S^n(1)$. Similarly, a theorem of Lichnerowicz ([L]) , implies that $\lambda_1(M, g) \geq n$, while Obata ([Ob]) proved that $\lambda_1(M, g) = n$ implies that (M, g) is isometric to $(S^n(1), g_{can})$. Here $\lambda_1(M, g)$ is the first non-zero eigenvalue of (M^n, g) for the Laplace operator. Note that in the above results (Cheng's Maximal Diameter Theorem, Lichnerowicz-Obata Theorem) the isometries occur when the diameter assume the maximum value (respectively the minimum value for the first non-zero eigenvalue for the Laplace operator). This raises the following questions:

1) If $Ric_{(M,g)} \geq (n-1)g$ and the diameter is close to π, is M homeomorphic (diffeomorphic) to $S^n(1)$?

2) If $Ric_{(M,g)} \geq (n-1)g$ and $\lambda_1(M^n, g)$ is close to n is M homeomorphic (diffeomorphic) to $S^n(1)$?

The answer to both questions is no. Anderson, (c.f. [A2]) constructed for $n \geq 4$, a family of n-manifolds (M, g_ϵ) satisfying $Ric_{(M,g_\epsilon)} \geq (n-1)g_\epsilon$, $vol(M, g_\epsilon) \geq v$ and $diam_{(M,g_\epsilon)} \geq (\pi - \epsilon)$ for any $\epsilon > 0$ and some v independent of ϵ, which are not homotopy equivalent to $S^n(1)$. These manifolds (M, g_ϵ) also satisfy $\lambda_1(M, g_\epsilon) \leq n + \epsilon$. It should be remarked that independently Otsu answered the first question for $n \geq 5$ (c.f. [Ot]). Thus extra hypotheses are needed to have a diameter or eigenvalue sphere theorem for Ricci curvature.

Our main result is the following theorem:

Theorem A: *Given an integer $n \geq 2$ and $\rho_0 > 0$ there exists an $\epsilon = \epsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$Ric_{(M,g)} \geq (n-1)g, \quad inj_{(M,g)} \geq \rho_0, \quad diam_{(M,g)} \geq \pi - \epsilon,$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\epsilon' = \epsilon'(\epsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature $+1$ on $S^n(1)$.

Remark: Theorem A gives an affirmative answer to the first question in the case that the injectivity radius is bounded from below. Note that in Anderson's examples (as well as in Otsu's) one can not replace the lower bound on the injectivity radius by a lower bound on the volume in dimensions bigger than or equal to 4. (A lower bound on the volume is a weaker hypothesis that a lower bound on the injectivity radius).

Again in Croke's Theorem the condition in the lower bound on the sectional curvature can not be replaced by $Ric_{(M,g)} \geq (n-1)g$ without imposing an extra condition as Anderson's examples show.

As a corollary of the Theorem A we have the following result, which answer affirmatively the second question in the case that the injectivity radius is bounded from below. In some extent, Theorem B extends Croke's result to the diffeomorphism case.

Theorem B *Given $n \geq 2$ and $\rho_0 > 0$ there exists an $\epsilon = \epsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$Ric_{(M,g)} \geq (n-1)g, \quad inj_{(M,g)} \geq \rho_0, \quad \lambda_1(M,g) \leq n + \epsilon,$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\epsilon' = \epsilon'(\epsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature $+1$ on $S^n(1)$.

In chapter 2 we are going to give present a (not complete) survey of sphere theorems to put our results in context in Riemannian geometry and in chapter 3 we present the proof of Theorems A and B.

Chapter 2

Sphere Theorems.

In this chapter we begin stating some theorems relating bounds on sectional curvature, Ricci curvature, volume, and diameter and first non-zero eigenvalue for the Laplace operator of an n-dimensional Riemannian manifold (M, g) and conditions for (M, g) be isometric (respectively homeomorphic, diffeomorphic) to the canonical sphere (S^n, g_{can})

2.1 Isometric Conditions

To study relations between topological and Riemannian structures of complete Riemannian manifolds, it is natural to seek in first place sufficient conditions to provide rigidity results. For example the following theorem

Theorem 2.1 *If (M^n, g) is a complete, connected, simply connected n-manifold with constant sectional curvature equal to 1, then (M^n, g) is isometric to (S^n, g_{can}).*

naturally led to the study of the following class $\{(M^n, g) \mid \delta \leq K_{(M,g)} \leq 1\}$ of manifolds. The Classical Sphere Theorem, states that for $\delta \in (1/4, 1]$ this class has one element up to homeomorphism. Cheeger took one step further and proved a finiteness theorem in a larger class of Riemannian manifolds, which is known as Cheeger Finiteness Theorem ([JC]).

Theorem 2.2 (Cheeger Finiteness Theorem) *Given $\Lambda, v, D > 0$ the class of Riemannian n-manifolds $\mathcal{M}(\Lambda, v, D) = \{(M, g); \ | K_{(M,g)} | \leq \Lambda, vol(M, g) \geq v, diam_{(M,g)} \leq D\}$ has finitely many diffeomorphism types.*

Now we start with some theorems providing sufficient conditions for (M, g) be isometric to S^n without an upper bound on the sectional curvature and in some cases with a weaker hypothesis as a lower bound on Ricci curvature.

Theorem 2.3 *If (M, g) satisfies $K_{(M,g)} \geq 1$ then $vol(M, g) \leq vol(S^n, g_{can})$. Here equality holds if and only if (M, g) is isometric to (S^n, g_{can}).*

Theorem 2.4 (Toponogov) *If (M, g) satisfies $K_{(M,g)} \geq 1$ then $diam_{(M,g)} \leq \pi$. Here equality holds if and only if (M, g) is isometric to (S^n, g_{can}).*

Theorems (2.3, 2.4) are true under weaker hypotheses as stated below. For references see ([CE], [Sh2]).

Theorem 2.5 (Bishop, [BC]) *If (M, g) satisfies $Ric_{(M,g)} \geq (n-1)g$ then $vol(M, g) \leq vol(S^n, g_{can})$. Here equality holds if and only if (M, g) is isometric to (S^n, g_{can}).*

Theorem 2.6 (Cheng, [Ch]) *If* (M, g) *satisfies* $Ric_{(M,g)} \geq (n-1)g$ *then the* $diam_{(M,g)} \leq \pi$. *Here equality holds if and only if* (M, g) *is isometric to* (S^n, g_{can}).

Theorem 2.7 (Lichnerowicz-Obata Theorem, [L], [Ob]) *If* (M, g) *satisfies* $Ric_{(M,g)} \geq (n-1)g$ *then* $\lambda_1(M, g) \geq n$. *Here equality holds if and only if* (M, g) *is isometric to* (S^n, g_{can}).

Here $\lambda_1(M, g)$ is the first non-zero eigenvalue of (M, g) for the Laplace operator.

Remark: Again, note that the isometries in the results above occur when the volume, diameter assume maximum values (respectively the first non-zero eigenvalue for the Laplace operator the minimum value), therefore is reasonable to expect that some homeomorphism (diffeomorphism) may be obtained near the maximum (respectively minimum) values. Furthermore whenever $Ric_{(M,g)} \geq (n-1)g$ and the volume of (M, g) is close to the volume of the canonical sphere (S^n, g_{can}) the diameter of (M, g) is close to the diameter of $S^n(1)$.

Guided by the isometric conditions above, many authors have shown that there exists some topological rigidity near the maximum values of volume and diameter (respectively minimum value for the first non-zero eigenvalue for the Laplace operator) whenever some extra condition is imposed. For example: a lower bound on the sectional curvature or injectivity radius, (c.f. [Es], [Nk], [Sh], [P], [GP], [Cr2]). In the next section we are going to present some of these topological rigidity theorems.

2.2 Topological Sphere Theorems

Using the fact that if a compact manifold M is covered by two non-overlapping closed disks is homeomorphic to S^n. Shiohama proved the following theorem:

Theorem 2.8 ((Shiohama) [Sh]) *Given $n \geq 2$ and $k > 0$ there exists $\epsilon = \epsilon(n,k) > 0$ such that if a manifold M admits a metric g satisfying $K_{(M,g)} \geq -k^2$, $Ric_{(M,g)} \geq (n-1)g$, $vol(M,g) \geq (1-\epsilon)vol(S^n, g_{can})$ then M is homeomorphic to S^n.*

Eschemburg and Nakamura ([Es], [Nk]) independently extended the result of Shiohama for a diameter sphere theorem imposing a lower bound on the injectivity radius. In fact they proved:

Theorem 2.9 ((Eschenburg-Nakamura) [Es], [Nk]) *Given $n \geq 2$ and ρ_0, $k > 0$, there exists an $\epsilon = \epsilon(n,k,\rho_0) > 0$ such that if M admits a metric g satisfying $K_{(M,g)} \geq -k^2$, $Ric_{(M,g)} \geq (n-1)g$, $inj_{(M,g)} \geq \rho_0$ and $diam_{(M,g)} \geq (\pi - \epsilon)$, then M is homeomorphic to S^n.*

Two possible ways to extend Eschemburg-Nakamura's result to a larger class of manifolds are:

1) Removing the lower bound on the sectional curvature.

2) Replacing the lower bound on the injectivity radius by a weaker condition like a lower bound on the volume.

Petersen ([P]) was able to extend Eschemburg-Nakamura's result removing the lower bound on the sectional curvature, and Grove-Petersen ([GP]) replacing the injectivity radius by a lower bound on the volume. The statements are as follows:

Theorem 2.10 ((Petersen) [P]) *Given an integer $n \geq 2$ and $\rho_0 > 0$ there exists an $\epsilon = \epsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying $Ric_{(M,g)} \geq (n-1)g$, $inj_{(M,g)} \geq \rho_0$ and $diam_{(M,g)} \geq (\pi - \epsilon)$ then M is a twisted sphere.*

Theorem 2.11 ((Grove-Petersen) [GP]) *Given an integer $n \geq 2$ and $k, v > 0$ there exists an $\epsilon = \epsilon(n, k, v) > 0$ such that if M admits a metric g satisfying $K_{(M,g)} \geq k$, $Ric_{(M,g)} \geq (n-1)g$, $vol(M, g) \geq v$ and $diam_{(M,g)} \geq (\pi - \epsilon)$ then M is a twisted sphere.*

On the other hand Croke gave an affirmative answer for the second question under stronger hypothesis (i.e. a lower bound on the sectional curvature), in fact he proved the following result:

Theorem ([Cr1]): *If M is a compact n-dimensional Riemannian manifold with sectional curvature $K_M \geq 1$, then there is a constant $C(n) > 1$ such that if $C(n) \cdot n > \lambda_1(M) \geq n$ then M is homeomorphic to $S^n(1)$.*

It is known that for $n \geq 7$ there exists manifolds which are homeomorphic to but not diffeomorphic to the n-sphere with its standard differentiable structure (see [Mi], [GM]). In fact Gromoll-Meyer construct examples of exotic 7-spheres with non-negative sectional curvature and positive Ricci curvature. The existence of such manifolds give rise to the problem of finding conditions to single out the differentiable structures of S^n.

2.3 Differentiable Sphere Theorems

The classical differentiable sphere theorem was first proved by Gromoll ([Gr]), Shikata ([Sk]) and Calabi (not published). They proved that a n-dimensional Riemannian manifold (M^n, g) satisfying $0 < \delta(n) \leq K_{(M,g)} \leq 1$ is diffeomorphic to S^n with its canonical differentiable structure. Shiohama-Sugimoto ([SS]) proved the independence of the dimension. It can be stated as follows:

Theorem: ([Gr], Calabi, [Sk], [SS]) *There exists $\delta \in (1, 1/4]$ such that if a simply connected n-dimensional Riemannian manifold M admits a metric g satisfying $1 \geq K_{(M,g)} > \delta$ then M is diffeomorphic to the n-sphere S^n with the the standard differentiable structure.*

The most recent estimate for δ is $\delta = 0.681$ (see [Su]), but is generally believed that the Classical Differentiable Sphere Theorem holds for $\delta = 1/4$.

The first result on the differentiable sphere theorem without assuming an upper bound for the sectional curvature was proved by Otsu-Shiohama-Yamaguchi ([OSY]).

Theorem 2.12 ([OSY]) *For a given $n \geq 2$ there exists an $\epsilon = \epsilon(n) > 0$ such that if a manifold M admits a metric g satisfying $K_{(M,g)} \geq 1$, $vol(M, g) \geq (1 - \epsilon)vol(S^n, g_{can})$, then M is diffeomorphic to n-sphere S^n with its canonical standard differentiable structure.*

Yamaguchi extended Theorem (2.8) to a differentiable sphere theorem, (c.f [Sh]). The first differentiable sphere theorem without bounding sectional curvature was proved by Anderson (c.f. [A1]).

Theorem 2.13 (Anderson, [A1]) *Given $n \geq 2$ and $C \geq (n-1)$ there exists an $\epsilon = \epsilon(n, C) > 0$ such that if an n-manifold admit a metric g satisfying*

$$C \geq Ric_{(M,g)} \geq (n-1), \quad vol(M,g) \geq (1-\epsilon)vol(S^n, g_{can}),$$

then M is diffeomorphic to canonical sphere of sectional curvature +1 on $S^n(1)$.

This upper bound in Theorem (2.13) is a technical hypothesis necessary in the proof. It remains an open question whether the same result holds without this upper bound.

Finally we remark that the our main result (Theorem A) extends Theorem (2.10) to a differentiable sphere theorem and as a corollary we have a differentiable eigenvalue sphere theorem. We restate Theorems A and B for sake of completeness of this section.

Theorem A: *Given an integer $n \geq 2$ and $\rho_0 > 0$ there exists an $\epsilon = \epsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$Ric_M \geq (n-1)g, \quad inj_M \geq \rho_0, \quad diam_M \geq \pi - \epsilon,$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\epsilon' = \epsilon'(\epsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature +1 on $S^n(1)$.

Theorem B: *Given $n \geq 2$ and $\rho_0 > 0$ there exists an $\epsilon = \epsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$Ric_M \geq (n-1)g, \quad inj_M \geq \rho_0, \quad \lambda_1(M) \leq n + \epsilon,$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\epsilon' = \epsilon'(\epsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature $+1$ on $S^n(1)$.

Chapter 3

Proof of Theorems A and B

We start this chapter giving the definition of convergency of sequences of Riemannian manifolds in C^α-topology.

Definition 3.1 *A sequence (M_i, g_i) of Riemannian manifolds is said to converge in the C^α-topology for a fixed $\alpha < 1$ to a C^α-Riemannian manifold (M_∞, g_∞) if M_∞ is a C^∞ manifold with a C^α metric tensor g_∞ and there is a sequence of $C^{1,\alpha}$ diffeomorphisms $f_i : M_\infty \to M_i$ for i sufficiently large, such that the metrics $f_i^* g_i$ converge to g_∞ in the C^α-topology on M_∞. Here the $C^{1,\alpha}$ structure is defined with respect to some fixed $C^{1,\alpha}$ atlas compatible with its C^∞ structure.*

If a sequence (M_i, g_i) of Riemannian manifolds converges in the C^α-topology to a C^α-Riemannian manifold (M_∞, g_∞), we may assume that all metrics g_i are defined on a C^∞ fixed manifold M and the metrics g_i converges to g_∞ in the C^α-topology. The following theorem (C^α-Compactness, see [AC]) which is a generalization of Cheeger-Gromov Compactness Theorem is the main tool used in the proof of Theorem A, besides elliptic regularity.

Theorem 3.2 (**C^α-compactness (Anderson-Cheeger)**) *The space of compact Riemannian n-manifolds (M,g) such that*

$$Ric_{(M,g)} \geq -\lambda, \quad inj_{(M,g)} \geq \rho_0 > 0, \quad vol_{(M,g)} \leq V \tag{3.3}$$

is pre-compact in the C^α topology for any $\alpha < 1$. More precisely, given any sequence of n-manifolds $\{(M_i, g_i)\}$ satisfying the bounds (3.3) and given any fixed $\alpha < 1$, there is a convergent subsequence in the C^α topology. The limit manifold (M_∞, g_∞) admits an atlas of $C^{1,\alpha}$ harmonic coordinates charts $F_\mu : U_\mu \to \mathbf{R}^n$ having the following property:

(1) The domains U_μ are of the form $U_\mu = F_\mu^{-1}(B(r_h))$, where $B(r_h) \subset \mathbf{R}^n$, with radius satisfying $r_h \geq c(n, \rho_0, \alpha, Q)$, $\alpha < 1$ and $Q > 1$ fixed. The domains $F_\mu^{-1}(B(r_h/2))$ cover M_∞.

(2) The overlaps $F_{\mu\nu} = F_\mu \circ F_\nu^{-1}$ are controlled in the $C^{1,\alpha}$-topology, i.e. $\|F_{\mu\nu}\|_{C^{1,\alpha}} \leq c(n, \rho_0, \alpha, Q)$

(3) The metric $g_{\infty ij} = g_\infty(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ in the charts F_μ are controlled in the C^α-topology in the sense that $Q^{-1} \cdot \delta_{ij} \leq g_{\infty ij} \leq Q \cdot \delta_{ij}$ and $r_h^\alpha \|g_{\infty ij}\|_{C^\alpha} \leq Q - 1$

3.1 Proof of Theorem A

To start let us consider a sequence of Riemannian manifolds $\{(M_i, g_i)\}$ satisfying

$$Ric_{(M_i,g_i)} \geq (n-1)g_i, \quad inj_{(M_i,g_i)} \geq \rho_0, \quad diam_{(M_i,g_i)} \geq (\pi - \epsilon_i) \tag{3.4}$$

where $\lim \epsilon_i = 0$. By theorem (3.2), $(M_i, g_i) \to (M_\infty, g_\infty)$ in the C^α topology, where (M_∞, g_∞) is a C^α Riemannian manifold. In particular for i sufficiently large all (M_i, g_i) are diffeomorphic to (M_∞, g_∞). Then we show that the first eigenvalue for the Dirichilet problem in any ball of radius $\pi/2$, $\lambda_1(B_i(x, \pi/2))$ converge to $\lambda_1(B_{S^n}(\pi/2)) = n$. From this we show that g_∞ is a weak $L^{1,p}$ ($p > n$) solution of Ricci equation ($Ric_{(M,g_\infty)} = (n-1)g_\infty$) in harmonic coordinates. By use of elliptic regularity g_∞ is smooth, has Ricci curvature $Ric_{(M,g_\infty)} = (n-1)g_\infty$. Since diameter $diam_{(M,g_\infty)} = \pi$ we can apply Cheng's maximal diameter theorem to conclude that (M, g_∞) is isometric to (S^n, can).

From now on, we assume that all the metrics are defined on a fixed C^∞ manifold M and the metrics g_i converge to g_∞ in C^α topology.

Proposition 3.5 *Let $\{(M, g_i)\}$ be a sequence of Riemannian manifolds satisfying:*

$$Ric_{(M,g_i)} \geq (n-1)g_i, \quad inj_{(M,g_i)} \geq \rho_0, \quad diam_{(M,g_i)} \geq \pi - \epsilon_i, \tag{3.6}$$

where $\lim_{i\to\infty} \epsilon_i = 0$,. *Then passing to a subsequence if necessary,*

$$\lambda_1(B_i(x, \pi/2), g_i) \to \lambda_1(B_{S^n}(\pi/2), g_{can}). \tag{3.7}$$

Here $\lambda_1(B_i(x, \pi/2), g_i)$ is the first non-zero eigenvalue for the Dirichlet problem for the ball $B_i(x, \pi/2)$ in the metric g_i.

To prove this proposition we need some lemmas.

Lemma 3.8 (Cheng's Lemma ([Ch])) *Suppose (M, g) is a complete Riemannian n-manifold satisfying $Ric_{(M,g)} \geq (n-1)g$. Then for any $x \in (M, g)$ and any $r \in [0, diam_{(M,g)}]$ we have:*

$$\lambda_1(B_g(x, r), g) \leq \lambda_1(B_{S^n}(r), g_{can})$$

with equality holding iff $(B_g(x,r), g)$ *is isometric to* $(B_{S^n}(r), g_{can})$.

Lemma 3.9 *Let* $\{(M, g_i)\}$ *be a sequence of Riemannian manifolds satisfying the bounds (3.6), then passing to a subsequence if necessary,*

$$\lambda_1(B_i(x,\pi/2), g_i) \to \lambda_1(B_\infty(x,\pi/2), g_\infty). \tag{3.10}$$

The proof of this is going is to be shown in 3 steps.

Step 1

Let $\{g_i\}$ be a sequence of metrics on a compact manifold M converging to g_∞ in C^0-topology and let D be a domain in M. Then,

$$\lambda_1(D, g_i) \to \lambda_1(D, g_\infty) \tag{3.11}$$

Proof:

Let g and g_0 be two metrics on a closed manifold M such that $a^2 g_0 \leq g \leq b^2 g_0$ with $0 < a^2 < b^2$. Then for any k

$$\frac{a^n}{b^{n+2}} \lambda_k(M, g_0) \leq \lambda_k(M, g) \leq \frac{b^n}{a^{n+2}} \lambda_k(M, g_0) \tag{3.12}$$

The proof of this fact is in [GHL] page 179, and only uses the max-min principle. The same proof also works for a compact domain $D \subset M$. Hence, if a sequence $\{g_i\}$ of metrics on M converges to g_∞ in the C^0-topology then $\lambda_1(D, g_i) \to \lambda_1(D, g_\infty)$, since we can find sequences of real numbers $\{a_i\}$ and $\{b_i\}$ satisfying $0 < a_i^2 < b_i^2$, $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = 1$ and $a_i^2 g_i \leq g_\infty \leq b_i^2 g_i$.

Step 2

Let $\{g_i\}$ be a sequence of metrics on a compact manifold M converging in C^0-topology to g_∞. Given $\delta > 0$ there exists $i_0 = i_0(\delta) > 0$ whenever $i \geq i_0$ then

$$B_\infty(x, \pi/2 - \delta) \subset B_i(x, \pi/2) \subset B_\infty(x, \pi/2 + \delta) \tag{3.13}$$

Proof :

From Theorem (3.2) part (1), we know that M has coordinate charts of the form $F_\mu^{-1}(B(r_h)$ and the domains $F_\mu^{-1}(B(r_{h/2}))$ cover M. Then $(g_{kj})_\infty$ are bounded on the closure of $F_\mu^{-1}(B(r_{h/2}))$

Given $\epsilon > 0$ there exists $i_0 > 0$ such that for all $i \geq i_0$ and for each chart

$$\sup_y \mid (g_{kj})_i - (g_{kj})_\infty \mid \leq \epsilon \tag{3.14}$$

Now fix $i \geq i_0$ and consider $y \in B_i(x, \pi/2)$. Let $\gamma : [0,1] \to M$ be a minimal geodesic with respect to g_i joining x to y. Assume that $\gamma([0,1])$ is contained in the closure of $F_\mu^{-1}(B(r_{h/2})$ for some chart. The general case is done by breaking γ in pieces such that this condition is satisfied.

In a local chart $\gamma' = \sum_k v_k \partial/\partial x_k$.

$$\begin{aligned} \mid l_{g_i}(\gamma) - l_{g_\infty}(\gamma) \mid &\leq \int_0^1 \mid (\sqrt{\sum (g_{kj})_i v_k v_j} - \sqrt{\sum (g_{kj})_\infty v_k v_j}) \mid dt \\ &\leq \int_0^1 \mid \frac{\sum((g_{kj})_i - (g_{kj})_\infty) v_k v_j}{\sqrt{\sum((g_{kj})_\infty v_k v_j)}} \mid dt \end{aligned}$$

Using twice Cauchy-Schwarz inequality we get

$$| l_{g_i}(\gamma) - l_{g_\infty}(\gamma) | \leq \frac{\epsilon n \mid v \mid}{\sqrt{\lambda}}, \tag{3.15}$$

where $\mid v \mid^2 = \sum v_k^2$ and λ is the smallest number such that $\sum((g_{kj})_\infty v_k v_j) \geq \lambda \mid v \mid^2 > 0$ in all charts $F_\mu^{-1}(B(r_{h/2})$ covering the manifold. (Note, there are only finite number of them.) In other words

$$d_\infty(x,y) \leq d_i(x,y) + \frac{\epsilon n \mid v \mid}{\sqrt{\lambda}} \tag{3.16}$$

Now we have to show that $\mid v \mid$ has an upper bound independent of i and y. Since v is a tangent vector of a minimal geodesic $\gamma : [0,1] \to M$ in a metric g_i, the length $\| v \|_{g_i} \leq diam_{M,g_i} \leq \pi$. Then $\pi^2 \geq \mid \sum (g_{kj})_i v_k v_j \mid = \mid \sum((g_{kj})_i - (g_{kj})_\infty) v_k v_j + \sum (g_{kj})_\infty v_k v_j \mid$

$\geq \mid \sum (g_{kj})_\infty v_k v_j \mid - \mid \sum((g_{kj})_i - (g_{kj})_\infty) v_k v_j \mid \geq \lambda \mid v \mid^2 - n\epsilon \mid v \mid^2$

Therefore we have that $\mid v \mid \leq \frac{\pi}{\sqrt{\lambda - n\epsilon}}$. This shows that $B_i(x, \pi/2) \subset B_\infty(x, \pi/2 + \frac{n\epsilon\pi}{\sqrt{\lambda(\lambda - n\epsilon)}})$,

$\forall i \geq i_0$. Set $\delta = \frac{n\epsilon\pi}{\sqrt{\lambda(\lambda - n\epsilon)}}$ and solve for ϵ, this gives (by the above) $i_0 = i_0(\epsilon(\delta)) > 0$ satisfying (3.13). The other inclusion is proved similarly.

Step 3

$$\lim_{i \to \infty} \lambda_1(B_i(x, \pi/2), g_i) = \lambda_1(B_\infty(x, \pi/2), g_\infty)$$

Proof:

From (3.13) we have for $i \geq i_0$:

$$\lambda_1(B_\infty(x, \pi/2 + \delta), g_i) \leq \lambda_1(B_i(x, \pi/2), g_i) \leq \lambda_1(B_\infty(x, \pi/2 - \delta), g_i) \tag{3.17}$$

Therefore

$$\lim_{i\to\infty}\lambda_1(B_\infty(x,\pi/2+\delta),g_i)\leq\lim_{i\to\infty}\lambda_1(B_i(x,\pi/2),g_i)\leq\lim_{i\to\infty}\lambda_1(B_\infty(x,\pi/2-\delta),g_i) \tag{3.18}$$

From (3.11) the left hand side limit is

$$\lambda_1(B_\infty(x,\pi/2+\delta),g_\infty)$$

and the right hand side is

$$\lambda_1(B_\infty(x,\pi/2-\delta),g_\infty)$$

Letting $\delta\to 0$ we have that

$$\lim_{i\to\infty}\lambda_1(B_i(x,\pi/2),g_i)=\lambda_1(B_\infty(x,\pi/2),g_\infty) \tag{3.19}$$

Remark: It is clear in the proof of lemma (3.9) that given $\epsilon>0$ there exists $i_0=i_0(\epsilon)>0$ depending only on ϵ such that $\forall i\geq i_0$ and any $x\in M$, $|\lambda_1(B_\infty(x,\pi/2))-\lambda_1(B_i(x,\pi/2))|\leq\epsilon$.

Lemma 3.20 *Let $\{(M,g_i)\}$ be a sequence of Riemannian manifolds satisfying:*

$$Ric_{(M,g_i)}\geq(n-1)g_i,\quad inj_{(M,g_i)}\geq\rho_0,\quad diam_{(M,g_i)}\geq\pi-\epsilon_i, \tag{3.21}$$

where $\lim_{i\to\infty}\epsilon_i=0$. Then passing to a subsequence if necessary,

$$Vol(M,g_i)\to Vol(S^n,g_{can}). \tag{3.22}$$

Proof :

By the Bishop-Gromov volume comparison theorem ([G], [BC], [CGT]), the functions

$$f_i(r) = \frac{Vol(\partial B_i(x,r))}{Vol(\partial B_{S^n}(r))}, \quad \hat{f}_i(r) = \frac{Vol(B_i(x,r))}{Vol(B_{S^n}(r))} \tag{3.23}$$

are non-increasing as function of r for any fixed x.

Here $B_i(x,r)$ is a geodesic ball in M of center x and radius r in the metric g_i.

Claim:

$$Vol(\partial B_i(x,r), g_i) \to Vol(\partial B_\infty(x,r), g_\infty).$$

For almost all r in $[0,\pi]$ and fixed x.

Since $g_i \to g_\infty$ in the C^α topology we have

$$Vol(B_k(x,r), g_i) \to Vol(B_k(x,r), g_\infty)$$

The proof of this fact is trivial.

Now given $\delta > 0$ $\exists i_0 = i_0(\delta) > 0$ such that $\forall i \geq i_0$ we have (see (3.13)):

$$Vol(B_\infty(x, r-\delta, g_i)) \leq Vol(B_i(x,r), g_i) \leq Vol(B_\infty(x, r+\delta), g_i).$$

Then

$$\lim_{i\to\infty} Vol(B_\infty(x,r-\delta), g_i) \leq \lim_{i\to\infty} Vol(B_i(x,r), g_i) \leq \lim_{i\to\infty} Vol(B_\infty(x,r+\delta), g_i).$$

Therefore

$$Vol(B_\infty(x,r-\delta),g_\infty) \leq \lim_{i\to\infty} Vol(B_i(x,r),g_\infty) \leq Vol(B_\infty(x,r+\delta),g_\infty).$$

Letting $\delta \to 0$ we have:

$$\lim_{i\to\infty} Vol(B_i(x,r),g_i) = Vol(B_\infty(x,r),g_\infty).$$

Thus $\hat{f}_\infty(r) = \frac{Vol(B_\infty(x,r),g_\infty)}{Vol(B_{S^n}(r))}$ is non-increasing as a function of r for any fixed x. The sequence of functions defined by $s \mapsto Vol(\partial B_i(x,s),g_i)$ is uniformly bounded, just note that $f_i(r)$ is non-increasing (see (3.23)).

Then

$\int_0^r \lim_{i\to\infty} Vol(\partial B_i(x,s),g_i)ds = \lim_{i\to\infty} \int_0^r Vol(\partial B_i(x,s),g_i)ds = \lim_{i\to\infty} Vol(B_i(x,r),g_i)$
$= Vol(B_\infty(x,r),g_\infty)$

Therefore

$$\int_0^r [\lim_{i\to\infty} Vol(\partial B_i(x,s),g_i) - Vol(\partial B_\infty(x,s),g_\infty)]ds = 0$$

for all r in $[0,\pi]$.

This implies that $\lim_{i\to\infty} Vol(\partial B_i(x,r),g_i) = Vol(\partial B_\infty(x,r),g_\infty)$ for almost all r in $[0,\pi]$.

Thus $f_\infty(r) = \frac{Vol(\partial B_\infty(x,r))}{Vol(\partial B_{S^n}(r))}$ is non-increasing as function of r almost everywhere in $[0,\pi]$ for any fixed x.

Before we proceed we need to recall a definition.

Definition 3.24 *Given two points p,q in a compact Riemannian manifold (M,g) we define the excess function $e_{p,q}: M \to \mathbb{R}$ as*

$$e_{p,q}(x) = dist_g(p,x) + dist_g(q,x) - dist_g(p,q)$$

and the excess of a manifold as $e(M,g) = min_{p,q}\{\, max_x\, e_{p,q}(x)\,\}$

We are going to present a proof that (M, g_∞) has excess zero due to Grove-Petersen (see [GP]).

Given two points p, q in (M, g_∞) realizing the diameter, i.e. $dist_{g_\infty}(p,q) = \pi$, we just need to show that for any $\delta > 0$, we have:

$$\partial B_\infty(p, \pi - \delta) = \partial B_\infty(q, \delta) \tag{3.25}$$

because given any x in (M, g_∞), suppose $dist_{g_\infty}(p,x) = \pi - \delta$. Then $x \in \partial B_\infty(p, \pi-\delta) = \partial B_\infty(q,\delta)$. Thus there exists minimal geodesics γ_1, γ_2 from p to x and from x to q respectively such that the lengh $l(\gamma_1) = \pi - \delta$ and $l(\gamma_2) = \delta$. Then $e_{p,q}(x) = dist_{g_\infty}(p,x) + dist_{g_\infty}(x,q) - dist_{g_\infty}(p,q) = l(\gamma_1) + l(\gamma_2) - dist_{g_\infty}(p,q) = 0$.

Now to prove (3.25) suppose that is not true, so there is $x \in M$ such that $dist_{g\infty}(p,x) > \pi - \delta + \eta_1$ and $dist_{g\infty}(q,x) > \delta + \eta_2$.

Then the balls $B_\infty(p, \pi - \delta)$, $B_\infty(q,\delta)$ and $B_\infty(x, \eta_3)$ are pairwise disjoint, for $\eta_3 = 1/2\, min\{\eta_1, \eta_2\}$.

Since $\hat{f}_\infty(r)$ is non-increasing for every r, we have:

$$Vol(M, g_\infty) \geq Vol(B_\infty(p, \pi - \eta)) + Vol(B_\infty(q,\delta)) + Vol(B_\infty(x, \eta_3))$$

$$\geq \frac{Vol(M, g_\infty)}{Vol(S^n, g_{can})}(Vol(B_{S^n}(\pi - \delta) + Vol(B_{S^n}(\delta) + Vol(B_{S^n}(\eta_3)) > Vol(M, g_\infty)$$

A contradiction.

Since (M, g_∞) is a smooth manifold with a C^α metric, we have (see [CGT]):

$$\lim_{r\to 0} \frac{Vol(\partial B_\infty(p,r))}{Vol(\partial B_{S^n}(r))} = 1$$

Note that the set $A = \{\ s \in [0,\pi]\ \mid\ \lim_{i\to\infty} Vol_{g_i}(\partial B_i(x,s)) = Vol_{g_\infty}(\partial B_\infty(x,s))\ \}$ is dense in $[0,\pi]$ and $\mu(A) = \mu([0,\pi])$.

By (3.23) we have:

$$1 = \lim_{r\to 0} \frac{Vol(\partial B_\infty(p,r))}{Vol(\partial B_{S^n}(r))} \geq \frac{Vol(\partial B_\infty(p,\pi-\delta))}{Vol(\partial B_{S^n}(\pi-\delta))} = \frac{Vol(\partial B_\infty(q,\delta))}{Vol(\partial B_{S^n}(\delta))} \to 1 \tag{3.26}$$

as $\delta \to 0$.

Thus $Vol(\partial B_\infty(p,r)) = Vol(\partial B_{S^n}(r))\ \forall r \in A$. This implies that

$$Vol(M, g_\infty) = \int_0^\pi Vol_{g_\infty}(\partial B_\infty(x,s))ds = \int_0^\pi Vol(\partial B_{S^n}(s))ds = Vol(S^n, g_{can})$$

Proof of Proposition (3.5)

By lemma (3.10) we need to show that $\forall x$, $\lambda_1(B_\infty(x,\pi/2)) = \lambda_1(B_{S^n}(\pi/2))$.

Since $Vol(M, g_\infty) = Vol(S^n, g_{can})$ (lemma (3.22)) and because $\hat{f}_\infty(r)$ is a non-increasing function of r it is easy to see that given any point x in (M, g_∞) there exists a point $y = h(x)$ such that $dist_{g_\infty}(x,y) = \pi$. By the same argument showing that (M, g_∞) has excess zero is clear that the excess is realized by any two points realizing the diameter. In particular, x and $h(x)$ realize the excess $e(M, g_\infty) = 0$ of (M, g_∞). Moreover, given x, $h(x)$ is unique, for if there exists $z \neq h(x)$ such that $dist_{g_\infty}(x, z) = \pi$, we have a minimal geodesic γ from x to $h(x)$, passing through z since x and $h(x)$ realize the excess of (M, g_∞), $e(M, g_\infty) = 0$. Thus $l(\gamma) = dist_{g_\infty}(x, z) + dist_{g_\infty}(z, h(x)) = dist_{g_\infty}(x, h(x)) = \pi$. It contradicts the fact that $z \neq h(x)$. This defines a function $h : (M, g_\infty) \to (M, g_\infty)$. By the previous discussion h is well defined and injective.

Claim:

$$h \text{ is a } C^{1,\alpha} \text{ isometry of } (M, g_\infty).$$

Proof:

Given any two points in (M, g_∞), say x and y. There are minimal geodesics γ_1 and γ_2 from x to $h(x)$ passing through y and $h(y)$ respectively. So $dist_{g_\infty}(x, h(x)) = dist_{g_\infty}(x, y) + dist_{g_\infty}(y, h(x)) = dist_{g_\infty}(x, h(y)) + dist_{g_\infty}(h(y), h(x))$. Suppose $dist_{g_\infty}(x, y) = \epsilon$, then $dist_{g_\infty}(y, h(x)) = \pi - \epsilon$. By triangle inequality we have:

$$\pi = dist_{g_\infty}(y, h(y)) \leq dist_{g_\infty}(y, h(x)) + dist_{g_\infty}(h(x), h(y)) = \pi - \epsilon + dist_{g_\infty}(h(x), h(y))$$

Then $dist_{g_\infty}(h(x), h(y)) \geq \epsilon$.

Also we have:

$$\pi = dist_{g_\infty}(y, h(y)) \leq dist_{g_\infty}(y, x) + dist_{g_\infty}(x, h(y)) = \epsilon + dist_{g_\infty}(x, h(y))$$

Thus $dist_{g_\infty}(x, h(y)) \geq \pi - \epsilon$. Since $dist_{g_\infty}(x, h(x)) = dist_{g_\infty}(x, h(y)) + dist_{g_\infty}(h(y), h(x))$,

we have that $dist_{g_\infty}(h(y), h(x)) = \epsilon$ i.e. h is a distance preserving function.

Theorem(Calabi-Hartman ([CH])). Let (M, g) be a connected n-dimensional C^α Riemannian manifold, $0 < \alpha \leq 1$. Then any isometry ϕ of $(M, dist_g)$ is of

$C^{1,\alpha}$ regularity. Furthermore ϕ satisfies

$$\phi^* g = g$$

By this result of Calabi-Hartmam, h is a $C^{1,\alpha}$ isometry.

Note that $\partial B_\infty(x,\pi/2) = \partial B_\infty(h(x),\pi/2)$ see (3.25). Then if $z \in \partial B_\infty(x,\pi/2)$ by triangle inequality $h(z) \in \partial B_\infty(x,\pi/2)$. Thus $h(B_\infty(x,\pi/2)) = B_\infty(h(x),\pi/2)$. Therefore,

$$\lambda_1(B_\infty(x,\pi/2), g_\infty) = \lambda_1(B_\infty(h(x),\pi/2), g_\infty). \tag{3.27}$$

Now we are in position to show that $\lambda_1(B_\infty(x,\pi/2), g_\infty) = \lambda_1(B_{S^n}(\pi/2), g_{can})$, $\forall x \in (M, g_\infty)$. Given $x \in (M, g_\infty)$ the balls $B_\infty(x,\pi/2)$, $B_\infty(h(x),\pi/2)$ are disjoint, $(dist_{g_\infty}(x, h(x)) = \pi)$. By Cheng's lemma ((3.8), [Ch]) we have:

$$\lambda_1(B_i(x,\pi/2), g_i) \leq \lambda_1(B_{S^n}(\pi/2), g_{can}) \quad \forall i. \tag{3.28}$$

Hence from lemma (3.9)

$$\lambda_1(B_\infty(x,\pi/2), g_\infty) \leq \lambda_1(B_{S^n}(\pi/2), g_{can}). \tag{3.29}$$

By Lichnerowicz's formula, (see [L], [BGM]) $\lambda_1(M, g_i) \geq n$, thus $\lambda_1(M, g_\infty) \geq n$.

Therefore,

$$\begin{aligned} n &\leq \lambda_1(M, g_\infty) \leq max\{\lambda_1(B_\infty(x,\pi/2), g_\infty), \lambda_1(B_\infty(h(x),\pi/2), g_\infty)\} \\ &\leq \lambda_1(B_{S^n}(\pi/2), g_{can}) = n \end{aligned}$$

But by (3.27), $\lambda_1(B_\infty(x,\pi/2),g_\infty) = \lambda_1(B_\infty(h(x),\pi/2),g_\infty)$. Then the proposition (3.5) is proved.

Proposition 3.30 *The limit metric g_∞ is smooth and satisfies $Ric_{(M,g_\infty)} = (n-1)g_\infty$.*

Proof:

The proof will be in 3 steps. Let φ be the nonnegative first eigenfunction of $B_{S^n}(\pi/2)$. It is known that φ is $\cos(r)$. Let ρ_i be the distance function w.r.t. the point x in M and the metric g_i. Let us fix i and work on the manifold (M, g_i)

The function $\cos \circ \rho_i$, satisfies the boundary conditions of the Dirichlet problem.

Hence:

$$\lambda_1(B_i(x,\pi/2)) \leq \frac{\int(d\cos\circ\rho_i, d\cos\circ\rho_i)}{\int(\cos\circ\rho_i)^2} \tag{3.31}$$

Step1:

Given $\epsilon > 0$ there is $i_0 > 0$ such that if $i > i_0$ then we have the following estimate:

$$0 \leq -\int_\xi \int_0^{a(\xi)} \cos(t)\cdot\sin(t)\cdot\theta_i(t\xi)\cdot\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi \leq \epsilon\cdot c, \tag{3.32}$$

where $d\xi$ is the canonical measure of S^{n-1}, $\theta_i(t\xi)$ is $\sqrt{det(g_i)_{k,l}}\cdot t^{n-1}$ w.r.t. normal coordinates, $a(\xi)$=min($\pi/2$, dist. of cut point in the direction ξ) and c is a constant independent of i.

Proof :

From (3.31), integrating on the tangent space of x we have:

$$\lambda_1(B_i(x,\pi/2))\cdot\int_\xi\int_0^{a(\xi)}\cos(t)^2\cdot\theta_i(t\xi)dtd\xi \leq \int_\xi\int_0^{a(\xi)}\sin(t)^2\cdot\theta_i(t\xi)dtd\xi \quad (3.33)$$

Now integrating the right hand side of (3.33) by parts we have:

$$\begin{aligned}
&\int_\xi\int_0^{a(\xi)}\sin(t)^2\theta_i(t\xi)dtd\xi = \int_\xi\cos(a(\xi))(-\sin(a(\xi)))\theta_i(a(\xi)\xi)d\xi - \\
-\ &\int_\xi\int_0^{a(\xi)}\cos(t)\theta_i(t\xi)[-\ cos(t)+(\frac{n-1}{t}-\frac{n-1}{t}+ \\
+\ &\frac{\theta_i'(t\xi)}{\theta_i(t\xi)}-\frac{[\sin(t)^{(n-1)}]\prime}{\sin(t)^{n-1}}+\frac{[\sin(t)^{(n-1)}]\prime}{\sin(t)^{n-1}})(-\sin(t))]dtd\xi \\
=\ &\int_\xi\cos(a(\xi))(-\sin(a(\xi)))\theta_i(a(\xi)\xi)d\xi \\
-\ &\int_\xi\int_0^{a(\xi)}\cos(t)(-\sin(t))\theta_i(t\xi)\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi \\
+\ &\int_\xi\int_0^{a(\xi)}\cos(t)^2\theta_i(t\xi)\lambda_1(B_{S^n}(\pi/2)dtd\xi) \qquad (3.34) \\
+\ &\int_\xi\int_0^{a(\xi)}\frac{(n-1)}{t}\cos(t)(-\sin(t))\theta_i(t\xi)dtd\xi.
\end{aligned}$$

Then from (3.33) and from (3.34) we have:

$$\begin{aligned}
&\lambda_1(B_i(x,\pi/2))\cdot\int_\xi\int_0^{a(\xi)}\cos(t)^2\cdot\theta_i(t\xi)dtd\xi \leq \int_\xi\cos(a(\xi))(-\sin(a(\xi)))\theta_i(a(\xi)\xi)d\xi \\
&+\int_\xi\int_0^{a(\xi)}\cos(t)\sin(t)\theta_i(t\xi)\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi \qquad (3.35) \\
&+\int_\xi\int_0^{a(\xi)}\varphi(t)^2\theta_i(t\xi)\lambda_1(B_{S^n}(\pi/2))+\int_\xi\int_0^{a(\xi)}\frac{(n-1)}{t}\cos(t)(-\sin(t))\theta_i(t\xi)dtd\xi.
\end{aligned}$$

The first and the fourth term of the right hand side of (3.35) are negative.

Then

$$\lambda_1(B_i(x,\pi/2))\cdot\int_\xi\int_0^{a(\xi)}\cos(t)^2\cdot\theta_i(t\xi)dtd\xi$$

$$\leq \int_\xi \int_0^{a(\xi)} \cos(t)\sin(t)\theta_i(t\xi)\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi \quad (3.36)$$
$$+ \int_\xi \int_0^{a(\xi)} \cos(t)^2\theta_i(t\xi)\lambda_1(B_{S^n}(\pi/2))dtd\xi$$

By proposition (3.5) and Cheng's lemma (3.8) given ϵ there exists an $i_0 > 0$ such that if $i \geq i_0$ then $\forall x$

$$0 \leq \lambda_1(B_{S^n}(\pi/2)) - \lambda_1(B_i(x,\pi/2)) \leq \epsilon. \quad (3.37)$$

From (3.37) and (3.36) it is then easy to see that

$$0 \leq \int_\xi \int_0^{a(\xi)} \cos(t)(\sin(t))\theta_i(t\xi)\ln'(\tfrac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi$$

$$+ \int_\xi \int_0^{a(\xi)} \cos(t)^2\theta_i(t\xi)[\lambda_1(B_{S^n}(\pi/2)) - \lambda_1(B_i(x,\pi/2)]dtd\xi \quad (3.38)$$

From the proof of Bishop-Gromov inequality ([BC]), $\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}}) < 0$, then the first term of (3.38) is negative. Therefore

$$0 \leq \int_\xi \int_0^{a(\xi)} \cos(t)(-\sin(t)\theta_i(t\xi)\ln'(\tfrac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi$$

$$\leq \int_\xi \int_0^{a(\xi)} \cos(t)^2\theta_i(t\xi)[\lambda_1(B_{S^n}(\pi/2)) - \lambda_1(B_i(x,\pi/2)]dtd\xi \leq \epsilon \cdot c, \quad (3.39)$$

where

$$c = \int_\xi \int_0^{\pi/2} \cos(t)^2(\frac{\sin(t)}{t})^n dtd\xi = \int_{B_{S^n}(\pi/2)} \cos(t)^2 dx. \quad (3.40)$$

Then for $i \geq i_0$ we have

$$0 \leq -\int_\xi \int_0^{a(\xi)} \cos(t)\cdot(\sin(t))\cdot\theta_i(t\xi)\cdot\ln'(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}})dtd\xi \leq \epsilon \cdot c \quad (3.41)$$

Remark: Note that the integrand in (3.41) is positive hence the same estimate is true if the integral is taken from 0 to $r \leq a(\xi)$ instead.

Step 2:

$$\int_M | Ric_{g_i} - (n-1)g_i | dv_{g_i} \to 0 \tag{3.42}$$

Proof :

Now let us consider $r_0 < \frac{\rho_0}{2}$ and fix it from now on.

Let $S_i(x,r)$ be the geodesic sphere in M_i of radius $r \leq r_0$, and let $H_i(r,\xi)$ be the mean curvature vector at $exp_x(r\xi)$ in M_i. It is well-known that $H_i(r,\xi) = \frac{\theta_i'(r\xi)}{\theta_i(r\xi)}$.

By the second variational formula,

$$H_i(r,\xi) = \sum_{k=1}^{n-1} \int_0^r | \{\nabla_T J_k |^2 - R_i(T, J_k)\} dt \tag{3.43}$$

where J_k are Jacobi fields vanishing at x, forming an orthonormal basis at $T_{exp_x(r\xi)} S_i(X,r)$ and T is the tangent vector of the radial geodesic $t \to exp_x(t\xi)$.

Now by the proof of Bishop-Gromov theorem, (see [A1] [BC]), we have :

$H_i(r,\xi) \leq \int_0^r \{(n-1)f'^2 - f^2 Ric_{g_i}(T,T)\} dt$

$$= (n-1)\int_0^r \{f'^2 - f^2\} dt - \int_0^r [Ric_{g_i}(T,T) - (n-1)] f^2 dt \tag{3.44}$$

where $f(0) = 0$, $f(r) = 1$. Choosing $f(s) = \frac{\sin(s)}{\sin(r)}$ one obtains:

$$\frac{\theta_i'(r\xi)}{\theta_i(r\xi)} = H_i(r,\xi) \leq \ln'(\sin^{n-1}(r)) - \frac{1}{\sin^2(r)} \cdot \int_0^r [Ric_{g_i}(T,T) - (n-1)] \sin^2(s) ds \tag{3.45}$$

Hence

$$\ln'\left(\frac{\theta_i(r\xi)}{\sin^{n-1}(r)}\right) \leq -\frac{1}{\sin^2(r)} \cdot \int_0^r [Ric_{g_i}(T,T) - (n-1)] \sin^2(s) ds \leq 0 \tag{3.46}$$

Now multiplying both sides of the expression (3.46) by $\cos(r)\cdot(-\sin(r)\cdot\theta_i(r\xi)$ we obtain:

$$\cos(r)\cdot(-\sin(r))\cdot\theta_i(r\xi)\cdot\ln'(\tfrac{\theta_i(r\xi)}{\sin(r)^{n-1}})$$

$$\geq \frac{\cos(r)\cdot\sin(r)\cdot\theta_i(r\xi)}{\sin^2(r)}\cdot\int_0^r[Ric_{g_i}(T,T)-(n-1)]\sin^2(s)ds \geq 0 \qquad (3.47)$$

Integrating twice over S_x^{n-1}, once over M_i, (the underlying manifold M with metric g_i) and over $[0,r_0]$,

we obtain:

$$\int_{M_i}\int_{S_x^{n-1}}\int_{S_x^{n-1}}\int_0^{r_0}\cos(r)(-\sin(r))\theta_i(r\xi)\ln'(\tfrac{\theta_i(r\xi)}{\sin(r)^{n-1}})drd\xi dTdx \geq$$

$$\int_{M_i}\int_{S_x^{n-1}}\int_{S_x^{n-1}}\int_0^{r_0}\tan(r)\cdot\theta_i(r\xi)\cdot\int_0^r[Ric_{g_i}(T,T)-(n-1)]\sin^2(s)dsdrd\xi dTdx \geq 0 \qquad (3.48)$$

Since $r < \frac{\varrho_0}{2}$, we have by a proposition of Croke ([Cr1] proposition 14):

$$\int_{S_x^{n-1}}\theta_i(r\xi)d\xi = vol(S_i(x,r)) \geq 2^{n-1}\frac{(\omega_{n-1})^n}{(\omega_n)^{n-1}}\cdot r^{n-1} = c(n,r) > 0, \qquad (3.49)$$

where ω_{n-1} and ω_n are respectively the volumes of the standard $(n-1)$ and n spheres.

From (3.49) we have the following:

$$\int_{M_i}\int_{S_x^{n-1}}\int_0^{r_0}\cos(r)(-\sin(r))\theta_i(r\xi)\ln'(\tfrac{\theta_i(r\xi)}{\sin(r)^{n-1}})drdTdx$$

$$\geq \int_0^{r_0}\tan(r)\cdot c(n,r)\int_0^r[\int_{\$M_i}[Ric_{g_i}(T,T)-(n-1)g_i]dTdx]\sin^2(s)dsdr \geq 0. \qquad (3.50)$$

The left hand side of (3.50) goes to zero as $i\to\infty$. The right hand side of

(3.50) is the following:

$$\int_0^{r_0} \tan(r) \cdot c(n,r) \int_0^r \int_M \mid s_{g_i} - (n-1)n \mid \sin^2(s) dx ds dr, \tag{3.51}$$

where s_{g_i} is the scalar curvature of M_{g_i}. This implies that $\int_M \mid s_{gi} - (n-1)n \mid \to 0$ as $i \to \infty$.

Claim:

$$\int_M \mid Ric_{gi} - (n-1)g_i \mid \to 0 \tag{3.52}$$

Let us suppose the contrary, there then exists a unit vector field Y on an open set Ω with $\mu(\Omega) > 0$, such that $\int_\Omega \mid Ric_{g_i}(Y,Y) - (n-1)g_i(Y,Y) \mid \geq c > 0$. Complete Y to an orthonormal frame $\{Y, e_1, ..e_{n-1}\}$. Hence $\int_\Omega \sum_{k=1}^{n-1} Ric_{gi}(e_k, e_k) - (n-1)g_i(e_k, e_k) + \int_\Omega Ric_{g_i}(Y,Y) - (n-1)g_i(Y,Y) = \int_\Omega \mid s_{g_i} - (n-1)n \mid$. Since all terms are positive it is clear that $\int_\Omega \mid s_{g_i} - (n-1)n \mid$ will not converge to zero. Therefore we have a contradiction.

Step 3

The limit metric g_∞ is real-analytic Einstein metric

Proof :

First we show that g_∞ is a weak $L^{1,p}$ solution of Einstein equation $Ric_g = (n-1)g$.

Let $h \in L^{1,p}$ with $\| h \|_{L^{1,p}} = 1$. By Hölder inequality we have:

$$\int_M < Ric_{g_i} - (n-1)g_i, h > dx \leq \| Ric_{g_i} - (n-1)g_i \|_{L^1} \| h \|_{L^\infty}$$

$$\leq K(n,p) \| Ric_i - (n-1)g_i \|_{L^1} \| h \|_{L^{1,p}} = K(n,p) \| Ric_i - (n-1)g_i \|_{L^1}, \tag{3.53}$$

since $L^{1,p} \subset L^\infty$ for large p. We have the right hand side of (3.53) goes to zero as $i \to \infty$. This implies for all p large that g_∞ is a weak $L^{1,p}$ solution of Einstein's equation . To show that g_∞ is real analytic, recall that Einstein's equation in harmonic coordinates is given by the elliptic system (we write $g_\infty = g$ for simplicity of notation),

$$g^{kj}\frac{\partial^2 g_{rs}}{\partial x_k \partial x_j} + Q(g, \partial g) = (Ric_g)_{rs} = (n-1)g_{rs}, \tag{3.54}$$

where Q is a quadratic term in g and the 1^{st} derivatives of g, (see [DeK]). The equation (3.54) is a uniformly elliptic system for which we have, locally, uniform $C^{0,\alpha}$ bounds on the coefficients g^{kj} and $L^{p/2}$ bounds on the terms Q and $(n-1)g_{rs}$, for any $p < \infty$.

Elliptic regularity ([Mo], Theorem 6.2.6) gives uniform bounds on $\| g \|_{L^{2,p/2}}$, for any $p < \infty$, so that $g \in L^{2,p/2} \cap C^{1,\alpha}$. Continuing in this process, elliptic regularity implies that $g = g_\infty$ is real analytic in harmonic coordinates.

Since (M, g_∞) satisfies $Ric_{(M,g_\infty)} = (n-1)g_\infty$, $diam_{(M,g_\infty)} = \pi$ and g_∞ is smooth, by Cheng's maximal diameter theorem [Ch], (M, g_∞) is isometric to the n-sphere $S^n(1)$.

3.2 Proof of Theorem B

Theorem 3.55 (Croke([Cr2])) *Let M be a compact n-dimensional Riemannian manifold satisfying $Ric_{(M,g)} \geq (n-1)g$ and $diam_M \leq D < \pi$. Then there is a constant $C(n, D) > 1$ such that $\lambda_1(M, g) \geq C(n, D) \cdot n$*

The proof of Theorem B is as follows: choose $\tilde{\epsilon}$ in Theorem A and choose $C(n, \pi - \tilde{\epsilon}) > 1$ in Theorem (3.55). If $\lambda_1(M) \leq C(n, \pi - \tilde{\epsilon}) \cdot n$, then the diameter of M, $diam_{(M,g)} \geq \pi - \tilde{\epsilon}$. Now let $\epsilon = C(n, \pi - \tilde{\epsilon}) \cdot n - n$.

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